Existence and uniqueness of ∞-harmonic functions under assumption of ∞-Poincaré inequality

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Abstract

Given a complete metric measure space whose measure is doubling and supports an ∞-Poincaré inequality, and a bounded domain Ω in such a space together with a Lipschitz function $f : \partial\Omega \to \mathbb{R}$, we show the existence and uniqueness of an ∞-harmonic extension of f to Ω. To do so, we show that there is a metric that is bi-Lipschitz equivalent to the original metric, such that with respect to this new metric the metric space satisfies an ∞-weak Fubini property and that a function which is ∞-harmonic in the original metric must also be ∞-harmonic with respect to the new metric. We also show that if the metric on the metric space satisfies an ∞-weak Fubini property, then the notion of ∞-harmonic functions coincide with the notion of AMLEs proposed by Aronsson. The notion of ∞-harmonicity is in general distinct from the notion of strongly absolutely minimizing Lipschitz extensions found in [13, 25, 26], but coincides when the metric space supports a p-Poincaré inequality for some finite $p \geq 1$.

Key words: ∞-Poincaré inequality, ∞-harmonic, AMLE, metric measure spaces.

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1 Introduction

Since the pioneering work of Aronsson [2], the notions of absolute minimizing Lipschitz extensions (AMLEs) and ∞-harmonic functions in Euclidean domains have been extensively studied in connection with a variety of applications. We refer to the survey [3] for general information on this subject. Recent applications of these notions include image processing and inpainting or brain and surface warping. The articles [6] and [29] give a good overview of such applications.

The idea behind AMLEs is simple. The *Lipschitz constant* of a Lipschitz function $f : Y \to \mathbb{R}$ for a set $Y \subset \mathbb{R}^n$ is denoted $\mathrm{LIP}(f, Y)$. Then we can construct at least two Lipschitz extensions

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$F : \mathbb{R}^n \to \mathbb{R}$ of f to $\mathbb{R}^n$ with the *same* Lipschitz constant, that is, $\mathrm{LIP}(f, Y) = \mathrm{LIP}(F, \mathbb{R}^n)$ as follows. We can set:

$$F(x) = \sup\{f(y) - \mathrm{LIP}(f, Y)d(x, y) \,:\, y \in Y\}$$

for all $x \in \mathbb{R}^n$ or, we can set:

$$F(x) = \inf\{f(y) + \mathrm{LIP}(f, Y)d(x, y) \,:\, y \in Y\}$$

for all $x \in \mathbb{R}^n$. These two extensions were first studied by McShane [32]. Note that the quantity $\mathrm{LIP}(F, \mathbb{R}^n)$ does not care about the local behavior of F, only the global behavior. Aronsson sought to take into account also the local behavior. More precisely, given a domain $\Omega \subset \mathbb{R}^n$ and a Lipschitz function f on $Y := \partial\Omega$, Aronsson looked for a Lipschitz extension $F : \overline{\Omega} \to \mathbb{R}$ of f to Ω such that in addition to the above requirement that $\mathrm{LIP}(f, \partial\Omega) = \mathrm{LIP}(F, \Omega)$, F also supports $\mathrm{LIP}(F, \partial V) = \mathrm{LIP}(F, V)$ for all subdomains $V \subset \Omega$. Functions F that satisfy this condition are called *absolutely minimizing Lipschitz extensions*, or AMLEs for short. In [2], existence of such a function was demonstrated using a variant of the Perron method. Note that such F would equivalently satisfy the condition that whenever $V \subset \Omega$ is a subdomain and $\varphi : \overline{V} \to \mathbb{R}$ such that $\varphi = F$ on ∂V, we must have $\mathrm{LIP}(F, V) \le \mathrm{LIP}(\varphi, V)$. Thus the local nature of minimizing Lipschitz constant is established for AMLEs. It was also shown in [2] and [24] that AMLEs F in Euclidean domains are ∞-harmonic in the sense that they satisfy $\Delta_\infty F = 0$, where

$$\Delta_\infty F = \sum_{i,j=1}^{n} \frac{\partial F}{\partial x_i} \frac{\partial F}{\partial x_j} \frac{\partial^2 F}{\partial x_i \partial x_j}.$$

In fact, a function on an Euclidean domain is an AMLE if and only if it is ∞-harmonic. In the Euclidean setting, one can construct ∞-harmonic functions via p-harmonic approximations, that is, p-harmonic functions in Ω that take on the value f on $\partial\Omega$ approximate the ∞-harmonic functions as $p \to \infty$. While the definition of AMLEs requires only the metric d, the definition of ∞-harmonicity requires in addition the knowledge of measure on the space as well (for the notion of weak partial derivatives). The interested reader is referred to [3] for further information.

In applications to image processing, ∞-harmonic extensions are used for image inpainting. In image inpainting an image with a patch of loss is corrected by "painting in" the lost image. Usually it is preferable to make the extension of the image into the lost patch as smooth as possible. For each $1 \le p < \infty$ the p-harmonic extension is the extension F whose p-th energy $J_p(F) := \int_\Omega |\nabla F|^p \, d\mathcal{L}^n$ is minimal amongst all Sobolev functions with the same boundary (outside image) data. When $p = 1$, the corresponding minimizer preserves edges found in the image (see for example [1]); as $p \to \infty$, the corresponding processed image becomes smoother, with $p = \infty$ corresponding to Lipschitz smoothness. See [34] for a survey on this subject. By the local nature of J_p, if F minimizes the energy J_p, then it does so locally as well. This is not the case for $p = \infty$. Thus in requiring minimization of ∞-energy, we require the minimizers to do so locally as well; this is in keeping with the behavior of Euclidean solutions to the equation

$\Delta_\infty u = 0$. In keeping with the nomenclature that minimizers of J_p are called p-harmonic, we call the global-local minimizers of ∞-energy ∞-harmonic.

In the abstract setting of separable length spaces, the existence of AMLEs with given Lipschitz boundary data was studied in [25] using Perron's method. The existence and uniqueness of AMLEs in general length spaces is obtained in [35] using random games. Thanks to the development of a Sobolev theory in the setting of metric measure spaces, the notion of p-harmonic function has been considered as well (see [23] and [7]). In [26], for doubling metric measure spaces supporting a p-Poincaré inequality for some finite $p \geq 1$, it was shown that the limit (as $p \to \infty$) of p-harmonic solutions to the Dirichlet problem on the domain, with a given Lipschitz boundary data, yields a so-called *strongly absolutely minimizing Lipschitz extension.* It was also shown there that when X satisfies a "weak Fubini property" of exponent p, a function is an AMLE if and only if it is a strongly absolutely minimizing Lipschitz extension. This latter notion coincides with our notion of ∞-harmonic functions in the metric setting when the metric space supports a p-Poincaré inequality for some finite $p \geq 1$. While strongly absolutely minimizing Lipschitz extensions minimize (with respect to the L^∞-norm), both locally and globally, the local Lipschitz constant function $\operatorname{Lip} u$ associated with the Lipschitz function u, the ∞-harmonic functions minimize the minimal ∞-weak upper gradient of u (see Definition 2.5). It was shown in [12] that when the metric space supports a p-Poincaré inequality for some finite p, the minimal p-weak upper gradient of a Lipschitz function u agrees almost everywhere with $\operatorname{Lip} u$. Since in our setting the metric space may *not* support any p-Poincaré inequality for any finite $p > 1$, the Euclidean notion of ∞-harmonicity is more naturally related to our notion of minimizing ∞-weak upper gradients; hence this is the object we study in this paper.

In [20] it was shown that there are complete metric measure spaces whose measure is doubling and supports an ∞-Poincaré inequality but not supporting any p-Poincaré inequality for finite $p \geq 1$. The examples in [20] can still be addressed using the techniques in [26] since the domain in consideration is a bounded domain, and the failure of p-Poincaré inequality occurs only at large scales. However, the sphericalization of the examples in [20], using the procedure described in [31], also supports an ∞-Poincaré inequality but does not support any p-Poincaré inequality for finite p, see [18] and [19], and the techniques of [26] fail for domains in this sphericalized space that contain the image of infinity from the original space of [20].

In light of these examples we are interested in knowing whether, given a bounded domain in a doubling metric measure space supporting an ∞-Poincaré inequality, and given a Lipschitz function defined on the boundary of the domain, there is an ∞-harmonic function on the domain with the prescribed boundary data. Our main result is the following:

Theorem 1.1 *Let (X, d, μ) be a complete metric measure space with μ doubling and supporting an ∞-Poincaré inequality, and let $\Omega \subset X$ be a bounded domain such that $X \setminus \Omega$ has positive measure. Given a Lipschitz function $f : \partial\Omega \to \mathbb{R}$, there is a unique Lipschitz function $u : \overline{\Omega} \to \mathbb{R}$*

such that $u = f$ on $\partial\Omega$ and u is ∞-harmonic in Ω.

The problem of *existence* of ∞-harmonic functions is studied in Section 3, and the corresponding result is given in Theorem 3.3. The standard technique of considering p-harmonic extensions of the Lipschitz boundary data and letting p tend to ∞ does *not* work in our setting as in the absence of p-Poincaré inequality for finite p we do not have control of the behavior of p-harmonic functions. Instead, we consider a different minimization problem for each finite p, and the family of solutions to this problem is shown to have the desirable limit as $p \to \infty$.

The question of *uniqueness* is related to the equivalence between AMLEs and ∞-harmonic functions. In [26], in order to obtain this equivalence, a p-weak Fubini property with $1 < p < \infty$ is needed for showing that one can neglect zero measure sets when computing the Lipschitz constant of a function. In Section 4, we prove the equivalence between AMLEs and ∞-harmonic functions under the weaker hypothesis of ∞-weak Fubini property (see Definition 4.1). This is the content of the second main result of this paper, Theorem 1.2 below.

Theorem 1.2 *Let (X, d, μ) be a complete metric measure space with μ doubling and satisfying an ∞-weak Fubini property. Consider a bounded domain $\Omega \subset X$ such that $X \setminus \Omega$ has positive measure and a Lipschitz function $f : \partial\Omega \to \mathbb{R}$. A Lipschitz function $u : \overline{\Omega} \to \mathbb{R}$ is ∞-harmonic in Ω if and only if it is an* AMLE *of f to Ω.*

In the Euclidean setting uniqueness of AMLEs for a given boundary data was established via the tool of viscosity solutions in [24], and an alternate proof using viscosity solutions and tug-of-war games was provided in [35]. A simpler proof of this uniqueness is given in [4]. In the setting of Heisenberg groups, uniqueness was demonstrated in [5]. Uniqueness for AMLEs in metric spaces that are length spaces was established in [35, Theorem 1.4], see also [4]. In the Euclidean setting the notion of AMLEs coincide with the notion of ∞-harmonic functions, but in the metric setting this is not the case.

Proposition 4.2 gives a simple metric characterization of ∞-weak Fubini property. It shows that the link between ∞-weak Fubini property and the measure μ is only via μ-null sets. Note that the hypotheses of Theorem 1.1 do not guarantee that the space satisfies a weak Fubini property. Hence, to prove Theorem 1.1, we will show that under the hypotheses of this theorem there is a bi-Lipschitz equivalent metric $\widehat{d}$ on X such that $(X, \widehat{d}, \mu)$ satisfies an ∞-weak Fubini property, see Proposition 4.4. We then show that a function that is ∞-harmonic with respect to the original metric is also ∞-harmonic with respect to $\widehat{d}$, and as $\widehat{d}$ does satisfy a weak Fubini property, we then know that the function is an AMLE *with respect to the metric $\widehat{d}$.* Finally invoking the uniqueness result of [35, 4], we have uniqueness of functions that are ∞-harmonic solutions with respect to the metric $\widehat{d}$ and hence with respect to the original metric d. Observe that Theorem 1.1 deals with ∞-harmonic functions; we do not know uniqueness of AMLEs with respect to the original metric d as (X, d) need not be a length space.

We also provide an example of a (length) space that does not satisfy any ∞-weak Fubini property, and for which uniqueness of solutions to ∞-harmonic Dirichlet problem fails, see Example 4.12. Given the uniqueness of AMLEs, this example also shows that there are ∞-harmonic functions that are not AMLEs when we do not have ∞-weak Fubini property.

In the final section of this paper we study the issue of stability of ∞-harmonic functions, and show that uniform limits of ∞-harmonic functions are ∞-harmonic.

2 Notation and definitions

In this paper we will assume that (X, d, μ) is a complete metric measure space. That is, (X, d) is a complete metric space equipped with a Borel measure μ which is positive and finite on each ball. We also require that the measure μ is *doubling* on X, that is, there is a constant $C_D \geq 1$ such that whenever $x \in X$ and $r > 0$, $\mu(B(x, 2r)) \leq C_D\, \mu(B(x, r))$.

Given a set $A \subset X$ and a Lipschitz function $u : A \to \mathbb{R}$, we set for $x \in A$,

$$\operatorname{Lip} u(x) := \limsup_{x \neq y \to x} \frac{|u(x) - u(y)|}{d(x, y)}, \qquad \text{and} \qquad \operatorname{LIP}(u, A) := \sup_{x,y \in A, x \neq y} \frac{|u(x) - u(y)|}{d(x, y)}.$$

We say that u is L-Lipschitz on A if $\operatorname{LIP}(u, A) \leq L$. The class of all bounded Lipschitz functions on X is denoted $\operatorname{LIP}^\infty(X)$. This class is equipped with the norm

$$\|u\|_{\operatorname{LIP}^\infty(X)} := \sup_{x \in X} |u(x)| + \operatorname{LIP}(u, X).$$

We refer the reader to [23] for an exposition on path integrals in metric spaces. A metric space (X, d) is a *length space* if for each pair $x, y \in X$, $d(x, y) = \inf_\gamma \ell(\gamma)$, the infimum being over curves with end points x, y. The metric space X is *C-quasiconvex*, or quasiconvex for some $C \geq 1$, if for each pair $x, y \in X$ there is a curve γ connecting x and y with $\ell(\gamma) \leq C d(x, y)$.

In the setting of non-smooth metric measure spaces, the role of derivatives is taken on by the upper gradients (see [22]). Given a function $u : X \to \overline{\mathbb{R}}$, we say that a Borel-measurable function $g : X \to [0, \infty]$ is an *upper gradient* of f if

$$|u(y) - u(x)| \leq \int_\gamma g\, ds \tag{1}$$

whenever γ is a non-constant compact rectifiable curve in X connecting the points x and y, and that $\int_\gamma g\, ds = \infty$ if at least one of $u(x)$, $u(y)$ is not finite. We refer the interested reader to [23] for more on the theory of upper gradients. Henceforth, in this paper we will assume all rectifiable curves to be compact and non-constant; for such curves γ the arc-length integral $\int_\gamma g\, ds$ is independent of re-parametrization of γ. In places where we need the curves to be parametrized by arc-length, we will explicitly state so.

Definition 2.1 *Given a family* Γ *of curves in* X*, set* $F(\Gamma)$ *to be the family of all Borel measurable functions* $\rho : X \to [0,\infty]$ *such that* $\int_\gamma \rho\, ds \geq 1$ *for all* $\gamma \in \Gamma$*. We define the* ∞-modulus of Γ *by*

$$\mathrm{Mod}_\infty(\Gamma) = \inf_{\rho \in F(\Gamma)} \|\rho\|_{L^\infty(X)},$$

and for $1 \leq p < \infty$ *the* p*-modulus of* Γ *is*

$$\mathrm{Mod}_p(\Gamma) = \inf_{\rho \in F(\Gamma)} \int_X \rho^p \, d\mu.$$

In this paper we are only concerned with whether, given a family Γ of curves in X, we have $\mathrm{Mod}_\infty(\Gamma)$ is positive or zero; if it is positive, its precise value is not needed here. In particular, we will use the following characterization.

Remark 2.2 Given a family Γ of curves in X, we have $\mathrm{Mod}_\infty(\Gamma) = 0$ if and only if there is a non-negative Borel function ρ that is zero a.e. in X such that $\int_\gamma \rho\, ds = \infty$ for each $\gamma \in \Gamma$, see [15, Lemma 5.7].

Definition 2.3 *A non-negative Borel measurable function* g *on* X *is said to be a* p-weak upper gradient *of a function* $u : X \to \overline{\mathbb{R}}$ *if the collection* Γ *of all non-constant rectifiable curves* γ *in* X *for which the inequality* (1) *fails has zero* p*-modulus.*

The Newton-Sobolev space $N^{1,p}(X)$ $(1 \leq p \leq \infty)$ is defined as follows. First consider the class $\widetilde{N}^{1,p}(X)$ of all functions in $L^p(X)$ that have a p-weak upper gradient in $L^p(X)$. For $u_1, u_2 \in \widetilde{N}^{1,p}(X)$ we say that $u_1 \sim u_2$ if $\|u_1 - u_2\|_{L^p(X)} + \inf_g \|g\|_{L^p(X)} = 0$, where the infimum is taken over all p-weak upper gradients g of $u_1 - u_2$. The relation $\sim$ is an equivalence relation on the vector space $\widetilde{N}^{1,p}(X)$, and we set $N^{1,p}(X)$ to be the collection of all equivalence classes of $\widetilde{N}^{1,p}(X)$. If $A \subset X$ is a measurable set, we can consider A to be endowed with the metric $d|_A$ and the measure $\mu|_A$, and consider the space $N^{1,p}(A)$.

From Remark 2.2 we have the following lemma.

Lemma 2.4 *Let* $u \in N^{1,\infty}(X)$*. Every* ∞*-weak upper gradient* g *of* u *can be modified on a set of measure zero so that the modification* $\widetilde{g}$ *is an upper gradient of* u*.*

From [33, Lemma 4.1], we know that if g_1, g_2 are ∞-weak upper gradients of a function $u \in N^{1,\infty}(X)$, then $g = \min\{g_1, g_2\}$ is also an ∞-weak upper gradient of u. In fact, we know from [33, Theorem 4.6] that for each $u \in N^{1,\infty}(X)$ there is an ∞-weak upper gradient $g_u \in L^\infty(X)$ which is *minimal* in the sense that whenever $g \in L^\infty(X)$ is an ∞-weak upper gradient of u, we have that $g_u \le g$ a.e. in X. Furthermore, g_u is unique up to sets of measure zero. By Lemma 2.4 we can also assume that g_u is an upper gradient of u.

We now define ∞-harmonic functions as follows. By a *domain* in a metric space we mean a non-empty connected open subset.

Definition 2.5 *Let X be a metric measure space, and Ω a bounded domain in X such that $X \setminus \Omega$ has positive measure. We say that a function $u : \Omega \to \mathbb{R}$ is ∞-harmonic in Ω if it admits an extension, also denoted u, to X such that $u \in N^{1,\infty}(X)$ and whenever $V \subset \Omega$ is an open set and $v \in N^{1,\infty}(X)$ such that $v = u$ on $X \setminus V$, we have*

$$\|g_u\|_{L^\infty(V)} \le \|g_v\|_{L^\infty(V)}. \tag{2}$$

Furthermore, we say that $u \in N^{1,\infty}(X)$ is ∞-harmonic in Ω with boundary data $f \in N^{1,\infty}(X)$ if u is ∞-harmonic in Ω and $u = f$ on $X \setminus \Omega$.

Remark 2.6 If $N^{1,\infty}(X) = L^\infty(X)$, then for each $x \in X$ and $r > 0$ the function $\chi_{B(x,r)} \in L^\infty(X) = N^{1,\infty}(X)$; so $\chi_{B(x,r)}$ is absolutely continuous on ∞-modulus almost every curve in X. Hence the collection of all rectifiable curves that intersect both $B(x,r)$ and $X \setminus \overline{B}(x,r)$ has zero ∞-modulus. Recall that μ is doubling and supported on X; hence X is separable. As the collection of all non-constant compact rectifiable curves in X is the union of the family $\Gamma(B(x_i, r_j))$ of all rectifiable curves in X intersecting both $B(x_i, r_i)$ and $X \setminus \overline{B}(x_i, r_j)$, with $\{x_i\}_i$ a countable dense subset of X and $\{r_i\}_i$ is the set of positive rational numbers, we must have by the countable subadditivity of modulus that the ∞-modulus of the collection of all non-constant compact rectifiable curves is zero and zero is an ∞-weak upper gradient of each $u \in L^\infty(X)$. Thus the following three conditions are equivalent:
1. $N^{1,\infty}(X) = L^\infty(X)$;
2. With $\Gamma(X)$ the collection of all non-constant rectifiable curves in X, $\text{Mod}_\infty(\Gamma(X)) = 0$;
3. For each $u \in L^\infty(X)$, $g \equiv 0$ is an ∞-weak upper gradient of u.
For X that supports any of the above three conditions, zero is an ∞-weak upper gradient of each $u \in N^{1,\infty}(X)$, and so each $u \in N^{1,\infty}(X) = L^\infty(X)$ is ∞-harmonic, and hence uniqueness of solutions to the Dirichlet problem for ∞-harmonic functions fails here.

There are many metric measure spaces where the triviality $N^{1,\infty}(X) = L^\infty(X)$ does not happen. For example, if X supports an ∞-Poincaré inequality, then $N^{1,\infty}(X) \ne L^\infty(X)$, see [15, 16]. Of such spaces, there is a collection of metric spaces that do not support a p-Poincaré inequality for any finite $p > 1$, and in such a setting the currently known approaches of constructing ∞-harmonic functions fail. Thus in this paper we focus on giving a construction of ∞-harmonic functions that does not rely on the existence of p-Poincaré inequality for any finite $p > 1$.

In the Euclidean setting, ∞-harmonic functions u are precisely those which satisfy the equation $\Delta_\infty u = 0$, see for example [13] or [3, Theorem 4.13]. This notion depends intrinsically on the measure μ as well as the metric d. The following related notion, due to Aronsson [2] (see also [3]), relies only on the metric d. Under certain conditions on the metric measure space X we show that both these notions coincide; see also [26] for a discussion in the metric setting, where a stronger assumption on the metric measure space was required. See the beginning of this section for the definition of $\mathrm{LIP}(u, V)$.

Definition 2.7 *Let (X, d) be a metric space, Ω a domain in X and $f : \partial\Omega \to \mathbb{R}$ a Lipschitz function. We say that a Lipschitz function u defined on the closure $\overline{\Omega}$ is an absolutely minimizing Lipschitz extension* (AMLE *for short*) *of f to Ω if $f = u$ on $\partial\Omega$ and whenever $V \subset \Omega$ is an open set and $v : \overline{V} \to \mathbb{R}$ is a Lipschitz function with $v = u$ on ∂V, we have*

$$\mathrm{LIP}(u, V) \le \mathrm{LIP}(v, V).$$

If u is an $N^{1,\infty}(\overline{\Omega})$-function that has a minimal ∞-weak upper gradient g_u on $\overline{\Omega}$ such that $g_u \le L$ a.e. in Ω, and f is a Lipschitz function on $X \setminus \Omega$ such that L is an upper gradient of f and $u = f$ on $\partial\Omega$, then u has an extension $\widehat{u} = f$ to $X \setminus \Omega$ such that the extension $\widehat{g_u}$ of g_u to $X \setminus \overline{\Omega}$ by the constant L is an ∞-weak upper gradient of $\widehat{u}$, see [7, Proposition 2.39]. As a consequence, we see that if $u \in N^{1,\infty}(\overline{\Omega})$ has an ∞-weak upper gradient that is a.e. in $\overline{\Omega}$ bounded by L and $u = f$ on $\partial\Omega$, then u has an extension $\widehat{u} \in N^{1,\infty}(X)$ to X that has an ∞-weak upper gradient dominated a.e. in X by L.

Lemma 2.8 *Let Ω, G be two non-empty open subsets of X, $G \subset \Omega$ with $dist(G, X \setminus \Omega) > 0$, and $u \in N^{1,p}(\Omega)$, $f \in N^{1,\infty}(X)$. If $u = f$ on ∂G, then the function $\widehat{u}$ given by*

$$\widehat{u}(x) = \begin{cases} u(x) & \text{if } x \in G, \\ f(x) & \text{if } x \in X \setminus G \end{cases}$$

is in $N^{1,p}_{loc}(X)$.

Proof. To prove the lemma, it suffices to show that $\widehat{u}$ has a p-weak upper gradient in $L^p_{loc}(X)$. Note that, since $f \in N^{1,\infty}(X)$, it has an upper gradient in $L^\infty(X)$, and in particular in $L^p_{loc}(X)$. We set $u_0 = \widehat{u} - f$, and then it suffices to show that $u_0 \in N^{1,p}_{loc}(X)$. Let $g \in L^p_{loc}(\Omega)$ be an upper gradient of $u - f$ in Ω, and let g_0 be the zero extension of g to $X \setminus \Omega$. We wish to show that g_0 is a p-weak upper gradient of u_0 in X.

Let γ be a non-constant compact rectifiable curve in X, and let x, y denote the two end points of γ. It suffices to consider only γ for which $x \in G$ and $y \in X \setminus G$. In this case we have that $u_0(y) = 0$. Then, with $\gamma : [a, b] \to X$ and $\gamma(a) = x$, there is some $t_0 \in (a, b]$ such that $\gamma((a, t_0)) \subset G$. Let t_0 be the largest such number in $(a, b]$. Note that as $y \notin G$, we must have

$\gamma(t_0) \in \partial G$ and $u_0(\gamma(t_0)) = u(\gamma(t_0)) - f(\gamma(t_0)) = 0$. From the facts that $g_0 \circ \gamma = g \circ \gamma$ on $[a, t_0)$ and g is an upper gradient of $u - f$, we can infer that

$$|u_0(x) - u_0(y)| = |u(x) - f(x) - (u(\gamma(t_0)) - f(\gamma(t_0)))| \le \int_{\gamma|_{[a,t_0]}} g_0 \, ds.$$

It now follows that g_0 is a p-weak upper gradient of u_0 and so $u_0 \in N^{1,p}_{loc}(X)$. $\square$

We next introduce the notion of p-Poincaré inequalities, which play a main role in this paper.

Definition 2.9 *Given $1 \le p \le \infty$, we say that a metric measure space X supports a p-Poincaré inequality if there are positive constants C, λ such that whenever $B = B(x,r)$ is a ball in X and g is an upper gradient of u,*

$$\fint_B |u - u_B| \, d\mu \le C\, r \left(\fint_{\lambda B} g^p \, d\mu \right)^{1/p},$$

where the right-hand side of the above is replaced with $Cr\|g\|_{L^\infty(\lambda B)}$ when $p = \infty$. Here $u_B := \mu(B)^{-1} \int_B u \, d\mu =: \fint_B u \, d\mu$ is the average of u on the ball B, and $\lambda B := B(x, \lambda r)$.

By Hölder's inequality, we know that every metric measure space supporting a p-Poincaré inequality for some $1 \le p < \infty$ must necessarily support an ∞-Poincaré inequality. The converse need not hold true, as demonstrated in [20].

The following geometric characterization of ∞-Poincaré inequality was established in [16, 17].

Theorem 2.10 ([17, Theorem 3.1]) *Let (X, d, μ) be a complete metric measure space with μ be doubling. Then the following are equivalent:*
(1) X supports an ∞-Poincaré inequality.
(2) There exists a constant $C \ge 1$ such that if $u \in N^{1,\infty}(X)$ with an ∞-weak upper gradient $g \in L^\infty(X)$, then u is $C\|g\|_{L^\infty(X)}$-Lipschitz continuous on X.
(3) There is a constant $C \ge 1$ such that whenever $N \subset X$ with $\mu(N) = 0$ and $x, y \in X$ with $x \ne y$, then there is a rectifiable curve γ with end points x, y such that $\ell(\gamma) \le Cd(x,y)$ and $\mathcal{H}^1(\gamma^{-1}(N)) = 0$.

Note that each of the criteria listed above imply that X is connected.

Example 2.11 Let (X, d, μ) be the Sierpiński carpet equipped with the Euclidean metric and the corresponding Hausdorff measure. Then X does not support an ∞-Poincaré inequality (see

[16, Example 4.14]). From the discussion in [8], we know the existence of a set $\widehat{N} \subset [0,1]$ such that, with the Hausdorff measure on X denoted by μ, the "first coordinate projection" $\Pi_1\mu$ of μ to $[0,1]$ given by $\Pi_1\mu(A) = \mu(\Pi_1^{-1}(A))$ for Borel sets $A \subset [0,1]$ sees $\widehat{N}$ as of measure zero but $\mathcal{H}^1(\widehat{N}) = 1$. Let $N = (\Pi_1^{-1}(\widehat{N}) \cup \Pi_2^{-1}(\widehat{N}))$. Here Π_1 and Π_2 are the first coordinate and the second coordinate projection maps from X to the interval $[0,1]$. Note that $\mu(N) = 0$. Given any non-constant curve γ in X, by breaking the curve up into two sub-curves if necessary, we can assume that its end points x, y satisfy $(x_1, x_2) = x \neq y = (y_1, y_2)$. Then

$$\begin{aligned}\mathcal{H}^1(\gamma^{-1}(N)) \geq \mathcal{H}^1(\gamma \cap N) &\geq \max\{\mathcal{H}^1(\Pi_1 \circ \gamma(\gamma^{-1}(N))), \mathcal{H}^1(\Pi_2 \circ \gamma(\gamma^{-1}(N)))\} \\ &\geq \max\{|x_1 - y_1|, |x_2 - y_2|\} > 0.\end{aligned}$$

Hence $\mathrm{Mod}_\infty(\Gamma(X)) = 0$ where $\Gamma(X)$ is the collection of all non-constant rectifiable curves in the carpet X. By Remark 2.6, we obtain $N^{1,\infty}(X) = L^\infty(X)$. Moreover, for *each* distinct pair $x, y \in X$ and *each* rectifiable curve γ connecting x to y we must have that $\mathcal{H}^1(\gamma^{-1}(N)) > 0$, with N independent of x, y. Therefore in some situations where ∞-Poincaré inequality fails, we might have a universal choice of null set N (that is, independent of x, y) that violates the third condition of the above theorem.

We end this section with a technical lemma that will be needed in Section 3, showing a locality property of minimal ∞-weak upper gradients of functions in $N^{1,\infty}(X)$. This lemma follows from [7, Theorem 2.18], [33, Lemma 4.1] and Lemma 2.4.

Lemma 2.12 *Let X be a metric measure space and E a measurable subset of X. Suppose that $u, v \in N^{1,\infty}(X)$ are such that $u = v$ on E. Then $g_u = g_v$ a.e. on E.*

3 Existence of ∞-harmonic functions

In this section we show the existence of an ∞-harmonic function on a domain $\Omega \subset X$ with prescribed Lipschitz boundary data. To do so, we solve a variational (minimization) problem corresponding to each exponent $p > 1$ and then let $p \to \infty$ to obtain the solution. A similar technique was employed in [26] where the variational problem was to minimize the L^p-energy and obtain a p-harmonic function for each finite p; however, without a p-Poincaré inequality for some finite value of p, we have no control over the behavior of p-harmonic functions, and hence the variational problem we consider is different.

Standing Assumptions: Throughout this section we assume that (X, d, μ) is a complete metric measure space with μ doubling and supporting an ∞-Poincaré inequality. We fix a bounded domain $\Omega \subset X$ and we assume that $\mu(X \setminus \Omega) > 0$ in order to avoid trivial statements.

Definition 3.1 *Given $L > 0$, let $N_L^{1,\infty}(X)$ be the collection of all functions u in $N^{1,\infty}(X)$ that have an upper gradient g with $\|g\|_{L^\infty(X)} \le L$. For $u \in N_L^{1,\infty}(X)$ we set $D_L(u)$ to be the collection of all upper gradients g of u such that $\|g\|_{L^\infty(X)} \le L$.*

Let $f : X \to \mathbb{R}$ with $f \in N^{1,\infty}(X)$. In this section we establish the existence of a function $u \in N^{1,\infty}(X)$ that is ∞-harmonic in Ω with boundary data f. Since X is complete and supports an ∞-Poincaré inequality, by Theorem 2.10 we know that every function in $N_L^{1,\infty}(X)$ is CL-Lipschitz on X, so as $N^{1,\infty}(X) = \bigcup_{L\ge 0} N_L^{1,\infty}(X)$, f is Lipschitz. Let $L > 0$ such that $D_L(f) \ne \emptyset$.

Definition 3.2 *Fix $1 < p < \infty$. For $u \in N_L^{1,\infty}(X)$ we set*

$$I_L^p(u) := \int_\Omega g_u^p \, d\mu = \inf\left\{ \int_\Omega g^p \, d\mu \, : \, g \in D_L(u) \right\}, \tag{3}$$

$$J_f^p := \inf\left\{ I_L^p(u) \, : \, u \in N_L^{1,\infty}(X);\ u = f \text{ on } X \setminus \Omega \right\}. \tag{4}$$

Theorem 3.3 *Given a Lipschitz function $f : X \to \mathbb{R}$, there is a Lipschitz function $\varphi : \overline{\Omega} \to \mathbb{R}$ such that $\varphi = f$ on $\partial\Omega$ and φ is ∞-harmonic in Ω.*

If $f : \partial\Omega \to \mathbb{R}$ is an L-Lipschitz function, then using the McShane extension theorem [32] (see Section 1 of this paper), we can extend f to a bounded Lipschitz function defined on X. Hence in the above theorem it suffices to prescribe f only on $\partial\Omega$. The remainder of this section is devoted to the proof of this theorem. The proof is divided into different steps:

Step 1. Fix $L > 0$ such that f is L-Lipschitz on X, and note that the constant function $g = L$ is an upper gradient of f. For every $1 < p < \infty$, we will show that there is a Lipschitz solution u_p of the variational problem defined in (4) such that $u_p = f$ on $X \setminus \Omega$.

Note that $J_f^p \le I_L^p(f) \le L^p \mu(\Omega) < \infty$, and hence we can find a sequence $\{u_k\}_k \subset N_L^{1,\infty}(X)$ such that $u_k = f$ on $X \setminus \Omega$ and $\lim_k I_L^p(u_k) = J_f^p$. Since each u_k is CL-Lipschitz, the family $\{u_k\}_k$ is equicontinuous on X, and since $u_k = f$ on $X \setminus \Omega$ with Ω bounded, it follows that the family is also equibounded on X. Thus an invocation of the Arzela-Ascoli theorem leads us to conclude that, passing to a subsequence if necessary, there is a CL-Lipschitz function u_p on X such that $\{u_k\}_k \to u_p$ uniformly on X.

Lemma 3.4 *For each $1 < p < \infty$ we have that $u_p \in N_L^{1,\infty}(X)$, $u_p = f$ on $X \setminus \Omega$, and*

$$J_f^p = I_L^p(u_p) = \int_\Omega (g_{u_p})^p \, d\mu. \tag{5}$$

Proof. Since $\{u_k\}_k \to u_p$ uniformly on X, we only need to consider upper gradients of u_p now. By passing to a subsequence if needed, for each k we can find an upper gradient g_k of u_k such that $g_k \leq L$ a.e. on X and $\int_\Omega g_k^p \, d\mu \leq J_f^p + 1/k$.

Fix a bounded domain Ω_0 in X such that $\Omega \Subset \Omega_0$. Thus $\{g_k\}_k$ is a bounded sequence in $L^p(\Omega_0)$. By the reflexivity of $L^p(\Omega_0)$, taking a further subsequence we may assume that $\{g_k\}_k$ is weakly convergent in $L^p(\Omega_0)$ to a non-negative Borel function $g_p \in L^p(\Omega_0)$. By Mazur's lemma, there is a convex combination subsequence $\{h_k\}_k$ (with $h_k = \sum_{j=k}^{N(k)} \lambda_{k,j} g_k$) such that $\{h_k\}_k \to g_p$ both in $L^p(\Omega_0)$ and pointwise outside a set $E \subset \Omega_0$ with $\mu(E) = 0$. From [27, Lemma 3.1] we know that g_p is a p-weak upper gradient of u_p on Ω_0. Note that g_p is defined only on Ω_0. On the other hand, since $u_k = f$ on $\Omega_0 \setminus \Omega$, the extension of each u_k by f to $X \setminus \Omega_0$ is also in $N^{1,\infty}(X)$ with the extension of g_p by L to $X \setminus \Omega_0$ a p-weak upper gradient of u_p on X, see Lemma 2.8. Because each $g_{u_k} \leq L$ a.e. in X, we have that $g_p \leq L$ on $X \setminus (E \cup \bigcup_k E_k)$, where each $E_k = \{g_k > L\}$; and note that by assumption on g_k, we have $\mu(E_k) = 0$. However, we do not know that g_p is an *upper gradient* of u_p. Thus we need to modify g_p suitably as follows.

Setting $F = E \cup \bigcup_k E_k$, we have $\mu(F) = 0$. Let Γ_F^+ denote the collection of all non-constant rectifiable (arc-length parametrized) curves γ in X such that $\mathcal{H}^1(\gamma^{-1}(F)) > 0$. Then, by considering $\rho = \infty \cdot \chi_F$ in Remark 2.2, we obtain that $\mathrm{Mod}_\infty(\Gamma_F^+) = 0$. For rectifiable non-constant curves γ in X that do *not* belong to Γ_F^+ we know that $\{h_k \circ \gamma\}_k \to g_p \circ \gamma$ $\mathcal{H}^1$-a.e. on the domain of γ, and that a.e. there we also have each $h_k \leq L$ and $g_p \leq L$. Therefore by the Lebesgue dominated convergence theorem, $\lim_k \int_\gamma h_k \, ds = \int_\gamma g_p \, ds$. Denoting the endpoints of γ by x, y, and noting that the convex combination sequence $v_k = \sum_{j=k}^{N(k)} \lambda_{k,j} u_k$, with $N(k), \lambda_{k,j}$ as in the choice of h_k, converges uniformly to u_p as well on X, we have that

$$|u_p(x) - u_p(y)| = \lim_k |v_k(x) - v_k(y)| \leq \lim_k \int_\gamma h_k \, ds = \int_\gamma g_p \, ds.$$

Therefore g_p is an ∞-weak upper gradient of u_p (this is stronger than saying that g_p is a p-weak upper gradient of u_p), with (1) being satisfied for all rectifiable curves in X that are not in Γ_F^+. Therefore $\widehat{g_p} := g_p + \infty\chi_F$ is an upper gradient of u_p on X such that $\widehat{g_p} \leq L$ on $X \setminus F$. Hence $u_p \in N_L^{1,\infty}(X)$, and by construction, $u_p = f$ on $X \setminus \Omega$. This also means that $I_L^p(u_p) \geq J_f^p$.

Finally, since $h_k \to \widehat{g_p}$ in $L^p(\Omega)$ we have that $\lim_k \int_\Omega h_k^p \, d\mu = \int_\Omega \widehat{g_p}^p \, d\mu$. By the lower continuity of L^p-norms, we deduce that

$$J_f^p \leq I_L^p(u_p) \leq \int_\Omega \widehat{g_p}^p \, d\mu \leq \lim_k \int_\Omega g_k^p \, d\mu \leq J_f^p.$$

Suppose now that $g \in D_L(u)$. Then by the lattice property of ∞-weak upper gradients (see [33, Lemma 4.1]) we have that $\min\{g_p, g\}$ is an ∞-weak upper gradient of u_p. Hence by the minimality of $I_L^p(u_p)$ we must have $g_p \leq g$ a.e. in Ω, that is, $g_p = g_{u_p}$. $\square$

Now let U be a subdomain of Ω, and consider the analogous variational problem on U with boundary data u_p. For $u \in N^{1,\infty}_L(X)$ we set $I^p_{L,U}(u)$ to be as in (3), with Ω replaced with U, and for functions $w \in N^{1,\infty}(X)$, we set

$$J^p_{w,U} := \inf \left\{ \int_U g_u^p \, d\mu \, : \, u \in N^{1,\infty}_L(X); \, u = w \text{ on } \partial U \right\}. \tag{6}$$

The next Lemma shows that u_p solves the minimization problem (6).

Lemma 3.5 *Let $1 < p < \infty$ and $v \in N^{1,\infty}_L(X)$ such that $v = u_p$ on ∂U. Then*

$$\int_U (g_{u_p})^p \, d\mu \le \int_U g_v^p \, d\mu.$$

Proof. Consider the Lipschitz function $w = v \cdot \chi_U + u_p \cdot \chi_{X \setminus U}$. By Lemma 2.12, we have $g_w = g_v$ a.e. on U and $g_w = g_{u_p}$ a.e. on $X \setminus U$. In particular $w \in N^{1,\infty}_L(X)$, and since $w = f$ on $X \setminus \Omega$,

$$\int_\Omega (g_{u_p})^p \, d\mu = J^p_f \le I^p_L(w) \le \int_\Omega g_w^p \, d\mu = \int_U g_v^p \, d\mu + \int_{\Omega \setminus U} (g_{u_p})^p \, d\mu.$$

$\square$

Step 2: In this step we show that the function u_p obtained in Step 1 is unique and satisfies the comparison property. We start with the following lemma, which shows a strong locality property for functions in $N^{1,\infty}(X)$.

The following lemma can be proven by showing with the aid of ∞-Poincaré inequality that such u is locally constant on Ω, and then using the fact that Ω is connected.

Lemma 3.6 *Let $u \in N^{1,\infty}(X)$ and suppose that $g \in L^\infty(X)$ is an upper gradient of u such that $g = 0$ a.e. in Ω. Then u is constant on Ω.*

Next we show uniqueness of u_p.

Lemma 3.7 *Let $1 < p < \infty$. If v_p is another minimizer of J^p_f, then $v_p = u_p$.*

Proof. The proof of this follows exactly as in [12, Theorem 7.14] (see [7, Theorem 7.2] for a more detailed proof, considering the obstacle $\psi = -\infty$ there), upon noticing that $D_L(u)$ is a convex subset of $L^p(X)$ (since Ω is bounded, we may without loss of generality assume that $\mu(X) < \infty$), and by the proof of Lemma 3.4, $D_L(u)$ is closed in $L^p(X)$ as well. Now invoking

Lemma 3.6 we obtain the desired result. $\square$

The next lemma yields the desired comparison theorem for functions u_p.

Lemma 3.8 *Let $1 < p < \infty$. Let f, F be two bounded functions in $N_L^{1,\infty}(X)$ such that $f \le F$ on $X \setminus \Omega$, and let u_p, U_p be the two respective minimizers of J_f^p and J_F^p. Then $u_p \le U_p$ on Ω.*

Proof. Since both u_p and U_p are Lipschitz continuous on X, and since $u_p = f \le F = U_p$ on $X \setminus \Omega$, it follows that $W := \{x \in X : u_p(x) > U_p(x)\}$ is an *open* subset of Ω with $u_p = U_p$ on ∂W. Suppose that W is non-empty (if W is empty, then the claim of the lemma follows). Then $u_p = U_p$ on ∂W and hence has a common L-Lipschitz extension Ψ to $X \setminus W$. It follows from the local nature of the L^p-norm that both u_p and U_p solve the minimization problem J_Ψ^p on W, and hence by Lemma 3.7 we must have $u_p = U_p$ in W, which contradicts the choice of W. Thus W must be empty. This concludes the proof of the lemma. $\square$

Step 3. In this step we fix a monotone increasing sequence $\{p_k\}_k$ with $1 < p_k < \infty$ and $\{p_k\}_k \to \infty$, and for each k let u_{p_k} be the function constructed in Step 1. Note that $\{u_{p_k}\}$ is an equicontinuous and equibounded sequence of CL-Lipschitz functions on X. So, by passing to a subsequence if necessary, and noting that each $u_{p_k} = f$ in $X \setminus \Omega$ with $\overline{\Omega}$ compact, by the Arzela-Ascoli theorem we can assume that $\{u_{p_k}\}$ converges uniformly on X to a Lipschitz function φ on X, with $\varphi = f$ on $X \setminus \Omega$.

Lemma 3.9 *The function φ is ∞-harmonic in Ω.*

Proof. For each $k \in \mathbb{N}$, we will denote for simplicity by g_k the minimal ∞-weak upper gradient $g_{u_{p_k}}$ of u_{p_k}. Now for each fixed $k_0 \in \mathbb{N}$ we have that $\int_\Omega g_k^{p_{k_0}}\, d\mu \le L^{p_{k_0}}\, \mu(\Omega)$, and so $\{g_k\}_{k \ge k_0}$ forms a bounded sequence in $L^{p_{k_0}}(\Omega)$. An appeal to reflexivity of $L^{p_{k_0}}(\Omega)$ and to Mazur's lemma gives us a convex combination subsequence of the sequence $\{g_k\}_{k \ge k_0}$ that converges both in $L^{p_{k_0}}(\Omega)$ and pointwise a.e. in Ω (and hence in X) to some non-negative Borel function ρ_{k_0}. Since each $g_k \le L$ a.e. in X, by a repeat of the proof of Lemma 3.4 we see that a modification of ρ_{k_0} on a set of measure zero makes it an upper gradient of φ with $\rho_{k_0} \le L$ a.e. in X.

To check that φ is ∞-harmonic on Ω, consider $v \in N^{1,\infty}(X)$ such that $v = f$ on $X \setminus \Omega$ and let g_v be its minimal ∞-weak upper gradient. If $\|g_v\|_{L^\infty(\Omega)} > L$ then as $\|\rho_{k_0}\|_{L^\infty(\Omega)} \le L$, we have the comparison (2). Therefore, without loss of generality, we assume that $g_v \le L$ a.e. in Ω. Since $v = f$ on $X \setminus \Omega$, we have by the pasting lemma [7, Theorem 2.18] together with the lattice property that the extension of g_v by L to $X \setminus \Omega$ is an ∞-weak upper gradient of v. Thus we have $g_v \le L$ a.e. in X. That is, $g_v \in D_L(v)$.

For each $k \in \mathbb{N}$ we know from Lemma 3.4 that

$$I_L^{p_k}(u_{p_k}) = \int_\Omega g_k^{p_k}\, d\mu \le \int_\Omega g_v^{p_k}\, d\mu.$$

Therefore, using Hölder's inequality, for each $k_0 \in \mathbb{N}$ and each $k \ge k_0$, we have that

$$\left(\fint_\Omega g_k^{p_{k_0}}\, d\mu \right)^{1/p_{k_0}} \le \left(\fint_\Omega g_k^{p_k}\, d\mu \right)^{1/p_k} \le \left(\fint_\Omega g_v^{p_k}\, d\mu \right)^{1/p_k} \le \|g_v\|_{L^\infty(\Omega)}.$$

As pointed out above, $\rho_{k_0} \le L$ a.e. in X. An argument analogous to the one given in the proof of Lemma 3.4 also tells us that ρ_{k_0} is an ∞-weak upper gradient of φ. Therefore $g_\varphi \le \rho_{k_0}$ a.e. in X. Since ρ_{k_0} is a weak limit of $\{g_{p_k}\}_{k \ge k_0}$ in $L^{p_{k_0}}(\Omega)$, it follows by letting $k \to \infty$ that

$$\left(\fint_\Omega g_\varphi^{p_{k_0}}\, d\mu \right)^{1/p_{k_0}} \le \left(\fint_\Omega \rho_{k_0}^{p_{k_0}}\, d\mu \right)^{1/p_{k_0}} \le \|g_v\|_{L^\infty(\Omega)}.$$

Now letting $k_0 \to \infty$ we obtain

$$\|g_\varphi\|_{L^\infty(\Omega)} \le \|g_v\|_{L^\infty(\Omega)}. \tag{7}$$

We now need to prove the above inequality for every open subset $V \subset \Omega$ rather than just Ω, and for every $v \in N^{1,\infty}(V)$ such that $v = \varphi$ on $X \setminus V$. To do so, consider first a connected component U of V. Note that, because of the quasiconvexity of X, each connected component of V is an open set. Furthermore, since Ω is connected and $U \subset \Omega$, it follows that ∂U is non-empty and we have $v = \varphi$ on ∂U. Thus the extension of v by φ to $X \setminus U$ is a test function for checking ∞-harmonicity of φ in U. Now for each $k \in \mathbb{N}$ consider the problem of minimizing the functional $I_{L,U}^{p_k}(\cdot)$ considered in (6) over all $u \in N_L^{1,\infty}(X)$ for which $u = \varphi$ on ∂U. As in Lemma 3.4, for each $k \in \mathbb{N}$ we obtain a minimizing function $w_{p_k} \in N_L^{1,\infty}(X)$ such that $J_{\varphi,U}^{p_k} = I_{L,U}^{p_k}(w_{p_k})$. See (6) for the definition of $J_{\varphi,U}^{p_k}$. As before, $\{w_{p_k}\}_k$ is an equicontinuous and equibounded sequence of Lipschitz functions on X. Then, there is a subsequence $\{w_{p_k}\}_k$ that converges uniformly on X to some Lipschitz function ψ (in the same manner that we have obtained φ). Then as in (7), for every $u \in N^{1,\infty}(U)$ such that $u = \varphi$ on $X \setminus U$, $\|g_\psi\|_{L^\infty(U)} \le \|g_u\|_{L^\infty(U)}$. In particular,

$$\|g_\psi\|_{L^\infty(U)} \le \|g_v\|_{L^\infty(U)}.$$

Since $\{u_{p_k}\}_k$ converges uniformly to φ in X, for each $\varepsilon > 0$ there is some $k_\varepsilon \in \mathbb{N}$ such that whenever $k \in \mathbb{N}$ with $k \ge k_\varepsilon$,

$$w_{p_k} - \varepsilon = \varphi - \varepsilon < u_{p_k} < \varphi + \varepsilon = w_{p_k} + \varepsilon \text{ on } X \setminus U.$$

From Lemma 3.5 we know that u_{p_k} is a minimizer $J_{u_{p_k},U}^{p_k}$. Now by Lemma 3.8, applied to the pair of functions $w_{p_k} - \varepsilon$ and u_{p_k} on U, and again to the pair of functions u_{p_k} and $w_{p_k} + \varepsilon$ on U,

$$w_{p_k} - \varepsilon \le u_{p_k} \le w_{p_k} + \varepsilon \text{ on } U.$$

Thus, letting $k \to \infty$, we obtain that $\psi - \varepsilon \leq \varphi \leq \psi + \varepsilon$ on V whenever $\varepsilon > 0$, that is, $\psi = \varphi$ on U. Thus from Lemma 2.12 we have that $g_\psi = g_\varphi$ a.e. on U. Then

$$\|g_\varphi\|_{L^\infty(U)} = \|g_\psi\|_{L^\infty(U)} \leq \|g_v\|_{L^\infty(U)} \leq \|g_v\|_{L^\infty(V)}.$$

To complete the proof, note that, since X is complete and μ doubling, we have that X is a *proper* metric space, that is, every closed ball in X is compact (see, e.g. pg. 102 in [23]). In particular X is separable, and the open set V has at most a countable number of connected components. Then we obtain that $\|g_\varphi\|_{L^\infty(V)} \leq \|g_v\|_{L^\infty(V)}$ as required. □

The above three steps together complete the proof of Theorem 3.3.

4 Coincidence of ∞-harmonicity and AMLEs under the assumption of ∞-weak Fubini property

In this section we compare the notions of ∞-harmonicity and AMLE. We show that if X satisfies an ∞-weak Fubini property, then the two notions coincide.

In [26] it was shown that if the metric measure space supports a p-Poincaré inequality for some finite $p \geq 1$ and satisfies a notion of weak Fubini property associated with the index p, then a function is an AMLE if and only if it is ∞-harmonic. In our paper we only require X to satisfy an ∞-weak Fubini property (see below). Note that ∞-weak Fubini property implies that X supports an ∞-Poincaré inequality. However, the satisfaction of a weak Fubini property as in [26] *does not* imply the support of a p-Poincaré inequality, but *does* imply the satisfaction of ∞-weak Fubini property, which in turn implies the support of an ∞-Poincaré inequality. As described in Section 2, there are metric measure spaces equipped with a doubling measure and supporting an ∞-Poincaré inequality, but supporting no p-Poincaré inequality, $1 \leq p < \infty$. Recall the definition of $\mathrm{Mod}_\infty(\Gamma)$ of a family Γ of curves from Definition 2.1.

Definition 4.1 *We say that (X, d, μ) satisfies an ∞-weak Fubini property if there exist constants $C > 0$ and $\tau_0 > 0$ such that, for every $0 < \tau < \tau_0$ and for every pair of balls B_1, B_2 in X with $dist(B_1, B_2) > \tau \cdot \max\{diam(B_1), diam(B_2)\}$, we have that $\mathrm{Mod}_\infty(\Gamma(B_1, B_2, \tau)) > 0$, where $\Gamma(B_1, B_2, \tau)$ denotes the family of all paths γ from B_1 to B_2 with $\ell(\gamma) \leq dist(B_1, B_2) + C\tau$.*

Given a subset N of a metric measure space X, we say that a curve γ is *transversal to N* if $\mathcal{H}^1(\gamma^{-1}(N)) = 0$. The terminology of transversality is from [9, 10, 11]. The next characterization of ∞-weak Fubini property will be useful to us.

Proposition 4.2 *The space (X, d, μ) satisfies an ∞-weak Fubini property if and only if for every set $N \subset X$ with $\mu(N) = 0$ and every $\varepsilon > 0$, for each pair of distinct points $x, y \in X$, there is a rectifiable curve γ transversal to N, with end points x, y and such that $\ell(\gamma) \le d(x, y) + \varepsilon$. Moreover, if X satisfies an ∞-weak Fubini property, then X supports an ∞-Poincaré inequality.*

Proof. Note first that the support of ∞-Poincaré inequality is a consequence of ∞-weak Fubini property, and this can be seen by following the proof of $(b) \Rightarrow (f)$ given in [17, Theorem 3.1.].

Suppose first that for every null set $N \subset X$ and $\varepsilon > 0$, for each $x, y \in X$ there is a transversal curve γ with end points x, y and $\ell(\gamma) \le d(x, y) + \varepsilon$. Let B_1, B_2 be as in Definition 4.1 with $\tau = \varepsilon$. If, with $C = 2$, we have $\mathrm{Mod}_\infty\, \Gamma(B_1, B_2, \varepsilon) = 0$, then there is a non-negative Borel measurable function ρ such that $\rho = 0$ μ-a.e. in X and for all $\gamma \in \Gamma(B_1, B_2, \varepsilon)$ we have $\int_\gamma \rho\, ds = \infty$ (see Remark 2.2). Let $N = \{x \in X : \rho(x) > 0\}$. We choose $x_1 \in B_1$ and $x_2 \in B_2$ such that

$$d(x_1, x_2) \le \mathrm{dist}(B_1, B_2) + \varepsilon.$$

Then by assumption of ρ we have $\mu(N) = 0$ and so there is a transversal curve γ_0 connecting x_1 and x_2 such that $\ell(\gamma_0) \le d(x_1, x_2) + \varepsilon$. But then we have $\int_{\gamma_0} \rho\, ds = 0 < \infty$, and $\ell(\gamma_0) \le \mathrm{dist}(B_1, B_2) + 2\varepsilon$, which means that $\gamma_0 \in \Gamma(B_1, B_2, \varepsilon)$, contradicting the choice of ρ. Thus we must have $\mathrm{Mod}_\infty\,(\Gamma(B_1, B_2, \varepsilon)) > 0$, that is, an ∞-weak Fubini property is satisfied.

Conversely, suppose X satisfies an ∞-weak Fubini property. Let $N \subset X$ with $\mu(N) = 0$, $\varepsilon > 0$, and $x, y \in X$ be two distinct points. Choose $\varepsilon > 0$ such that $\tau < \min\{\varepsilon, \tau_0, d(x, y)\}/(10C)$. Let B_1, B_2 be the balls of radius τ, centered at x and y respectively. These balls satisfy the hypotheses in Definition 4.1, and so $\mathrm{Mod}_\infty\, \Gamma(B_1, B_2, \tau) > 0$. Thus we can find $x_\tau \in B_1$, $y_\tau \in B_2$ and a transversal rectifiable curve γ_τ with end points x_τ, y_τ such that $\ell(\gamma_\tau) \le \mathrm{dist}(B_1, B_2) + C\tau$.

By choosing τ to be small enough, we can ensure that $\ell(\gamma_\tau) \le d(x, y) + \frac{\varepsilon}{2}$. Note that $d(x, x_\tau) < \tau$ and $d(y, y_\tau) < \tau$, and so by the ∞-Poincaré inequality (a consequence of the ∞-weak Fubini property as noted above), there exist curves β_τ connecting x to x_τ and α_τ connecting y to y_τ such that $\ell(\beta_\tau) < C\tau$ and $\ell(\alpha_\tau) < C\tau$, with $\mathcal{H}^1(\beta_\tau^{-1}(N) \cup \alpha_\tau^{-1}(N)) = 0$. The concatenation $\gamma = \alpha_\tau * \gamma_\tau * \beta_\tau$ is a transversal rectifiable curve connecting x to y with

$$\ell(\gamma) \le d(x, y) + \tfrac{\varepsilon}{2} + 2C\tau.$$

By choosing τ small enough so that we also have $2C\tau < \varepsilon/2$, we obtain the result. □

Now, we define a geodesic distance on the metric measure space by using the notion of transversality for a given null set. This distance has been used in [9, 10, 11].

Definition 4.3 *Let X be a metric measure space. For each null set N in X we define*

$$\widehat{d}_N(x, y) = \inf\{\ell(\gamma) : \gamma \text{ is a curve transversal to } N \text{ and connecting } x \text{ to } y\}.$$

It is easily seen that for null sets $N \subset X$, $\widehat{d}_N$ is an *extended* metric on X, in the sense that $\widehat{d}_N$ can possibly take infinite values (since the infimum of the empty set is ∞). Furthermore, if X supports an ∞-Poincaré inequality, then by Theorem 2.10 there exists $C \geq 1$ such that for each null set $N \subset X$,

$$d(x,y) \leq \widehat{d}_N(x,y) \leq C d(x,y). \tag{8}$$

The next result shows that, if a metric measure space X supports an ∞-Poincaré inequality, then there is a bi-Lipschitz equivalent length metric on X that makes X satisfy the ∞-weak Fubini property. In the proof, we use some ideas from [10].

For $x, y \in X$ we set

$$\widehat{d}(x,y) = \sup\{\widehat{d}_N(x,y) \,:\, N \text{ null set in } X\}. \tag{9}$$

Proposition 4.4 *Let (X,d,μ) be a complete metric measure space with μ doubling and supporting an ∞-Poincaré inequality. Then the following properties are satisfied:*

(a) *There exists $C \geq 1$ such that $d(x,y) \leq \widehat{d}(x,y) \leq C d(x,y)$ whenever $x, y \in X$.*

(b) *$\widehat{d}$ is a length metric on X and $(X, \widehat{d}, \mu)$ satisfies an ∞-weak Fubini property.*

(c) *(X,d,μ) satisfies an ∞-weak Fubini property if and only if $d = \widehat{d}$.*

(d) *For every domain Ω in X, a function u on Ω is ∞-harmonic in Ω with respect to $\widehat{d}$ and μ if and only if it is ∞-harmonic with respect to d and μ.*

Proof. From the discussion preceding (8) we have that $\widehat{d}$ is a metric on X and also that (a) holds.

In order to complete the proof we will need several claims.

Claim 1. For every $x, y \in X$ and every $\varepsilon > 0$, there exists a null set $E \subset X$ so that $\widehat{d}(x,y) = \widehat{d}_E(x,y)$. Indeed, for each $j \in \mathbb{N}$ there exists a null set $E_j \subset X$ such that

$$\widehat{d}(x,y) \leq \widehat{d}_{E_j}(x,y) + \frac{1}{j}.$$

It suffices now to consider $N := \bigcup_{j=1}^{\infty} E_j$. Then for every j we have $\widehat{d}_{E_j} \leq \widehat{d}_E$ and so

$$\widehat{d}(x,y) \leq \widehat{d}_E(x,y) + \frac{1}{j} \leq \widehat{d}(x,y) + \frac{1}{j}.$$

Letting $j \to \infty$, we obtain that $\widehat{d}(x,y) = \widehat{d}_E(x,y)$ as desired.

Our next Claim follows at once, taking into account that, since X is complete and μ is doubling, X is separable.

Claim 2. Fix a countable dense subset D of X. There exists a null set $M \subset X$ such that $\widehat{d}(p,q) = \widehat{d}_M(p,q)$ for every $p, q \in D$.

Indeed, for every $x, y \in X$, Claim 1 provides a null set E_{xy} (depending on x and y) such that $\widehat{d}(x,y) = \widehat{d}_{E_{xy}}(x,y)$. If we choose $M = \bigcup_{x,y\in D} E_{xy}$, then for $p, q \in D$ we have $\widehat{d}(p,q) = \widehat{d}_{E_{pq}}(p,q) \le \widehat{d}_M(p,q) \le \widehat{d}(p,q)$, and so Claim 2 follows.

Now denote by ℓ and $\widehat{\ell}$ the corresponding length functionals associated to d and $\widehat{d}$, respectively.

Claim 3. For every curve γ in X transversal to the set M, we have that $\ell(\gamma) = \widehat{\ell}(\gamma)$.

In order to prove Claim 3, first note that, since $d \le \widehat{d}$, we have $\ell \le \widehat{\ell}$. Let $\gamma : [a,b] \to X$ be transversal to M. To obtain the reverse inequality, fix $\varepsilon > 0$ and choose a subdivision $P = \{a = t_0 < t_1 < \cdots < t_n = b\}$ of $[a,b]$ such that

$$\widehat{\ell}(\gamma) \le \sum_{i=1}^{n} \widehat{d}(\gamma(t_{i-1}), \gamma(t_i)) + \varepsilon.$$

For each $i = 1, 2, \cdots, n$ we obtain an estimate of $\widehat{d}(\gamma(t_{i-1}), \gamma(t_i))$ as follows. First choose $p_i, q_i \in D$ such that $\widehat{d}(p_i, \gamma(t_{i-1})) < \varepsilon/n$ and $\widehat{d}(q_i, \gamma(t_i)) < \varepsilon/n$. Since X supports an ∞-Poincaré inequality, by [17, Theorem 3.1] there exist a curve β from p_i to $\gamma(t_{i-1})$ and a curve α from $\gamma(t_i)$ to q_i, both transversal to M, such that $\ell(\alpha) < C_1\varepsilon/n$ and $\ell(\beta) < C_1\varepsilon/n$, with the constant C_1 depending only on the doubling constant and the constants related to the Poincaré inequality. Then the concatenation $\sigma = \alpha * (\gamma|_{[t_{i-1},t_i]}) * \beta$ is a curve connecting p_i and q_i, transversal to M, and from Claim 2 and the definition of $\widehat{d}_M$ we have $\widehat{d}(p_i, q_i) = \widehat{d}_M(p_i, q_i) \le \ell(\sigma)$. Thus, for each $i = 1, 2, \cdots, n$,

$$\begin{aligned} \widehat{d}(\gamma(t_{i-1}), \gamma(t_i)) \le \varepsilon/n + \widehat{d}(p_i, q_i) &\le 2\varepsilon/n + \ell(\alpha) + \ell(\gamma|_{[t_{i-1},t_i]}) + \ell(\beta) \\ &\le [1 + C_1]\varepsilon/n + \ell(\gamma|_{[t_{i-1},t_i]}). \end{aligned}$$

Summing up, we obtain that

$$\widehat{\ell}(\gamma) - \varepsilon \le \sum_{i=1}^{n} \widehat{d}(\gamma(t_{i-1}), \gamma(t_i)) \le 2[1 + C_1]\varepsilon + \sum_{i=1}^{n} \ell(\gamma|_{[t_{i-1},t_i]}) = [1 + C_1]\varepsilon + \ell(\gamma).$$

Letting $\varepsilon \to 0^+$, we see that $\widehat{\ell}(\gamma) \le \ell(\gamma)$.

Claim 4. For every $x, y \in X$, every $\varepsilon > 0$ and every null set $N \subset X$, there exists a curve γ from x to y, transversal to $M \cup N$, such that $\widehat{d}(x,y) \le \ell(\gamma) \le \widehat{d}(x,y) + \varepsilon$.

To see this, choose $p, q \in D$ such that $\widehat{d}(p,x) < \varepsilon_1$ and $\widehat{d}(q,y) < \varepsilon_1$, where $\varepsilon_1 > 0$ is to be chosen below. As X supports an ∞-Poincaré inequality, we can find as before a curve β from x to p and a curve α from q to y, both transversal to $M \cup N$, such that $\ell(\alpha) < C_1\varepsilon_1$ and $\ell(\beta) < C_1\varepsilon_1$. On the other hand, since $p, q \in D$, from Claim 2 above we have

$$\widehat{d}(p,q) = \widehat{d}_M(p,q) \leq \widehat{d}_{M\cup N}(p,q) \leq \widehat{d}(p,q).$$

Therefore, there exists a curve σ from p to q, transversal to $M \cup N$, such that $\ell(\sigma) \leq \widehat{d}(p,q) + \varepsilon_1$. Consider then the concatenation $\gamma = \alpha * \sigma * \beta$, which is a curve from x to y transversal to $M \cup N$, and satisfies

$$\widehat{d}(x,y) \leq \ell(\gamma) = \ell(\alpha) + \ell(\sigma) + \ell(\beta) \leq 2C_1\varepsilon_1 + \ell(\sigma) \leq 2[1+C_1]\varepsilon_1 + \widehat{d}(p,q) \leq 2[2+C_1]\varepsilon_1 + \widehat{d}(x,y).$$

We can choose $\varepsilon_1 = \varepsilon/[4 + 2C_1]$ to conclude the proof of Claim 4.

Note that, as a consequence, we have that $\widehat{d}$ is a length metric. Furthermore, using Proposition 4.2 again, we obtain the second part of Claim (b) as well as the Claim (c).

We next prove Claim (d). First note that, by Claim 3, the arc-length parametrization of every curve γ in X transversal to the set M coincides for (X,d) and $(X,\widehat{d})$. Now, if we denote by Γ_M^+ the family of curves in X which are *not* transversal to M, we know that $\mathrm{Mod}_\infty(\Gamma_M^+) = 0$ (because we can assume that M is a Borel set, and then see that $\infty\chi_M$ is admissible for computing $\mathrm{Mod}_\infty(\Gamma_M^+)$, see [15, Lemma 5.8]). Thus if $\rho : X \to [0,\infty]$ is a Borel function, the path integral $\int_\gamma \rho\, ds$ coincides for (X,d,μ) and $(X,\widehat{d},\mu)$ for Mod_∞-almost every curve γ. This means that given a function u on X, a function g is an ∞-weak upper gradient of u with respect to d if and only if it is an ∞-weak upper gradient of u with respect to $\widehat{d}$. In particular, the corresponding Newton-Sobolev spaces coincide: $N^{1,\infty}(X,d,\mu) = N^{1,\infty}(X,\widehat{d},\mu)$ isometrically. Now the result follows from the definition of ∞-harmonicity. $\square$

The following example shows that Claim (b) of the above proposition is not true without the hypothesis of ∞-Poincaré inequality.

Example 4.5 Without ∞-Poincaré inequality $\widehat{d}$ may possibly take infinite values, and in particular it may not be equivalent to d. The Sierpiński carpet X from Example 2.11 does not support an ∞-Poincaré inequality and hence cannot satisfy any ∞-weak Fubini property. Since the length metric on this carpet is bi-Lipschitz equivalent to the Euclidean metric, it follows that the above statement holds also when X is equipped with the length metric. To see that $\widehat{d}$ is not equivalent to d in this case, we consider the set N constructed in Example 2.11. Observe that $\widehat{d}_N(x,y) = \infty$, and so $\widehat{d}$ is not equivalent to d in X.

Lemma 4.6 *Suppose that (X, d, μ) is a complete metric measure space with μ doubling and satisfying an ∞-weak Fubini property. Then for each $u \in \mathrm{LIP}^\infty(X) = N^{1,\infty}(X)$,*

$$\mathrm{LIP}(u, X) = \sup_{x \in X} \mathrm{Lip}\, u(x) = \|\mathrm{Lip}\, u\|_{L^\infty(X)} = \|g_u\|_{L^\infty(X)}. \tag{10}$$

Furthermore, if $V \subset X$ is a non-empty open set, then for each $u \in N^{1,\infty}(V)$ (noting that such functions are necessarily locally Lipschitz continuous in V),

$$\sup_{x \in V} \mathrm{Lip}\, u(x) = \|\mathrm{Lip}\, u\|_{L^\infty(V)} = \|g_u\|_{L^\infty(V)}. \tag{11}$$

Proof. Note that as $\mathrm{Lip}\, u$ is an upper gradient of u and g_u is the minimal ∞-weak upper gradient of u, we have that $g_u \le \mathrm{Lip}\, u$ a.e. in X.

Let $u \in \mathrm{LIP}^\infty(X)$, and define $N = \{x \in X : \mathrm{Lip}\, u(x) > \|\mathrm{Lip}\, u\|_{L^\infty(X)}\}$. Now, fix $x, y \in X$. Given $\varepsilon > 0$ take γ in X connecting x and y that is transversal to N, parametrized by the arc-length, such that $\ell(\gamma) \le d(x, y) + \varepsilon$. Then

$$|u(x) - u(y)| \le \int_0^{\ell(\gamma)} \mathrm{Lip}\, u(\gamma(t))dt \le \|\mathrm{Lip}\, u\|_{L^\infty(X)} \ell(\gamma) \le \|\mathrm{Lip}\, u\|_{L^\infty(X)}[d(x, y) + \varepsilon].$$

Now, let $\varepsilon \to 0$ and then take the supremum over $x, y \in X$ to obtain $\mathrm{LIP}(u, X) \le \|\mathrm{Lip}\, u\|_{L^\infty(X)}$.

Replacing the role of $\mathrm{Lip}\, u$ in the above with g_u and noting that the collection Γ of curves for which the function-upper gradient inequality does not hold has ∞-modulus zero, there must be a set $N \subset X$ with $\mu(N) = 0$ such that for each $\gamma \in \Gamma$ we must have $\mathcal{H}^1(\gamma^{-1}(N)) > 0$, which gives the last equality in the first claim.

Let $V \subset X$ be open and non-empty set, and $u \in N^{1,\infty}(V)$ with $B(x, 2r) \subset V$ Fix $r > 0$ such that $B(x, 2r) \subset V$, and $0 < \varepsilon < r/2$. Let $x \in V$ and $N = \{y \in B(x, r) : g_u(y) > \|g_u\|_{L^\infty(B(x,r))}\}$. Then $\mu(N) = 0$. Note in the above inequality that for each $y \in B(x, r/2)$ there is a rectifiable curve γ with end points x, y such that $\ell(\gamma) \le d(x, y) + \varepsilon$, γ is transversal to N, and

$$\frac{|u(x) - u(y)|}{d(x, y)} \le \frac{\ell(\gamma)}{d(x, y)} \fint_{[0,\ell(\gamma)]} g_u \circ \gamma\, ds \le \frac{\ell(\gamma)}{d(x, y)} \|g_u\|_{L^\infty(B(x,r))}.$$

By the choice of r and ε, $\gamma \subset V$. It follows that $\mathrm{Lip}\, u(x) \le \lim_{r \to 0^+} \|g_u\|_{L^\infty(B(x,r))}$. From the previous inequality we also have that whenever $V \subset X$ is a non-empty open set, then

$$\|g_u\|_{L^\infty(V)} \le \|\mathrm{Lip}\, u\|_{L^\infty(V)}.$$

On the other hand, for each $\varepsilon > 0$ there exists $z_0 \in V$ such that

$$\|\mathrm{Lip}\, u\|_{L^\infty(V)} - \varepsilon \le \mathrm{Lip}\, u(z_0) \le \lim_{r \to 0^+} \|g_u\|_{L^\infty(B(z_0,r))} \le \|g_u\|_{L^\infty(V)}.$$

Therefore $\|g_u\|_{L^\infty(V)} = \|\mathrm{Lip}\, u\|_{L^\infty(V)}$ for any non-empty open set $V \subset X$. $\square$

Remark 4.7 A converse of the above lemma also holds. Suppose that $\mathrm{LIP}^\infty(X) = N^{1,\infty}(X)$ and that (10) holds for each $u \in N^{1,\infty}(X)$. Then X satisfies an ∞-weak Fubini property. To see this, note that under the above hypotheses, by Theorem 2.10 we know that X supports an ∞-Poincaré inequality. Fix a set $N \subset X$ with $\mu(N) = 0$ and consider $\widehat{d}_N$ as in Definition 4.3. It follows from Theorem 2.10 that there is a constant $C \geq 1$ with $\widehat{d}_N(z, w) \leq C\, d(z, w)$ whenever $z, w \in X$. We fix $y \in X$, $R > 1$, and consider the function

$$u(x) = \min\left\{R, \inf_\gamma \int_\gamma [1 + \infty \cdot \chi_N]\, ds\right\},$$

where the infimum is over all rectifiable curves γ connecting x to y. Note that $u(x) \leq \widehat{d}_N(x, y)$ for each $x \in X$, and so $u \in N^{1,\infty}(X)$. Furthermore, $g = 1 + \infty\chi_N \in L^\infty(X)$ is an upper gradient of u, and so by the hypothesis we have $\mathrm{LIP}(u, X) = \|g_u\|_{L^\infty(X)} \leq \|g\|_{L^\infty(X)} = 1$, that is, u is 1-Lipschitz on X. Hence for each $x \in X$ and $\varepsilon > 0$ we can find a curve γ connecting x to y that is transversal to N and with $\ell(\gamma) \leq d(x, y) + \varepsilon$. Therefore, from Proposition 4.2, X satisfies an ∞-weak Fubini property.

Under the ∞-weak Fubini property (which implies the ∞-Poincaré inequality), we know that $\mathrm{LIP}^\infty(X) = N^{1,\infty}(X)$, see Theorem 2.10. Hence the property of every $u \in N^{1,\infty}(X)$ satisfying (10) characterizes complete metric measure spaces that satisfy an ∞-weak Fubini property. The property (10) is crucial in understanding the connections between AMLEs and ∞-harmonic functions, see for example [26] and [14].

Example 4.8 In this example we construct a metric measure space $X \subset \mathbb{R}^2$ where the measure is doubling and supports an ∞-Poincaré inequality, and a function u for which

$$\sup_X \mathrm{Lip}\, u > \|\,\mathrm{Lip}\, u\|_{L^\infty(X)}.$$

We start with the interval $[0, 1]$, and for each $n \in \mathbb{N}$ we replace $[1/(n+1), 1/n]$ with the union of the two line segments in $\mathbb{R}^2$, one joining $(1/n, 0)$ to $P_n \in \mathbb{R}^2$ and the other joining $(1/(n+1), 0)$ to P_n, where P_n is a point such that

$$\|P_n - (1/n, 0)\| = \|P_n - (1/(n+1), 0)\| = 1/[n(n+1)] = \|(1/(n+1), 0) - (1/n, 0)\|.$$

By doing this we obtain X, equipped with the restriction of the Euclidean metric from $\mathbb{R}^2$ to X, and with the measure $\mu = \mathcal{H}^1$. Consider the function u on X given by

$$u(x, y) = \mathcal{H}^1(X \cap \{(s, t) \in \mathbb{R}^2 \,:\, s \geq x\}).$$

Note that $g_u = 1$ is a minimal ∞-weak upper gradient of u, and that for $X \ni (x, y) \neq (0, 0)$ we have $\mathrm{Lip}\, u(x, y) = 1$. On the other hand,

$$\mathrm{Lip}\, u(0, 0) \geq \limsup_{n\to\infty} \frac{u(0, 0) - u(1/n, 0)}{1/n} = \limsup_{n\to\infty} \frac{\sum_{k=n}^\infty \frac{2}{k(k+1)}}{1/n} = \limsup_{n\to\infty} 2 = 2.$$

It follows that $\sup_X \operatorname{Lip} u \geq 2 > 1 = \|\operatorname{Lip} u\|_{L^\infty(X)}$.

Lemma 4.9 [26, Lemma 5.4] *If X is proper (that is, every closed ball in X is compact) and is a length space, then whenever $V \subset X$ is a non-empty open set, we have*

$$\operatorname{LIP}(u, V) = \max\left\{ \operatorname{LIP}(u, \partial V),\, \sup_{z\in V} \operatorname{Lip} u(z) \right\}.$$

We are now ready to prove the first main theorem of this paper, Theorem 1.1.

Proof. [Proof of Theorem 1.1] The existence of ∞-harmonic extensions is obtained in Theorem 3.3. Recall that the notion of ∞-harmonicity yields the same class of functions under each of the metrics d and $\widehat{d}$, see Proposition 4.4 (d). By Proposition 4.4 (b) we have that $(X, \widehat{d})$ is a length space, $(X, \widehat{d}, \mu)$ satisfies an ∞-weak Fubini property, and the function $u := \varphi$ given by Theorem 3.3 is ∞-harmonic in Ω for $(X, \widehat{d}, \mu)$. Also, since $(X, \widehat{d})$ is complete and μ doubling, we have that $(X, \widehat{d})$ is a proper metric space.

By Lemma 4.6 and by Lemma 4.9, if $V \subset \Omega$ is a non-empty open set and if $v : \overline{V} \to \mathbb{R}$ is such that $v = u$ on ∂V, then by (11),

$$\begin{aligned}\operatorname{LIP}(u, V) = \max\left\{ \operatorname{LIP}(v, \partial V),\, \|g_u\|_{L^\infty(V)} \right\} &\leq \max\left\{ \operatorname{LIP}(v, \partial V),\, \|g_v\|_{L^\infty(V)} \right\} \\ &= \max\left\{ \operatorname{LIP}(v, \partial V),\, \sup_{z\in V} \operatorname{Lip} v(z) \right\} = \operatorname{LIP}(v, V).\end{aligned}$$

Note that the above is with respect to the metric $\widehat{d}$. It follows that u is AMLE in Ω for $(X, \widehat{d})$. Finally, by [35, Theorem 1.4] AMLEs are unique; hence the uniqueness of u. $\square$

The proof of Theorem 1.1 also shows that, under the ∞-weak Fubini property, every ∞-harmonic function is an AMLE. The converse is also true, as the following shows.

Theorem 4.10 *Let X be a complete metric measure space with the measure μ be a doubling measure satisfying an ∞-weak Fubini property. Let Ω be a bounded domain in X with $\partial\Omega$ non-empty. If $u : \overline{\Omega} \to \mathbb{R}$ is an AMLE in Ω, then u is ∞-harmonic in Ω.*

Proof. Under the hypotheses of the theorem, we know that X is a proper length space. The result [26, Proposition 4.1] together with [26, Proposition 5.8] shows that if X is a proper length space, then AMLEs on a domain $\Omega \subset X$ are of strong-AMLE class. The proof of [26, Proposition 5.8] would work even if their notion of weak Fubini property is replaced with our weaker notion of ∞-weak Fubini property. The notion of strong-AMLE of [26] agrees with our notion of ∞-harmonicity under our hypotheses on X, see Lemma 4.6 above (more specifically, equation (11)). Therefore we know that AMLEs are ∞-harmonic. $\square$

Combining Theorem 4.10 with Theorem 1.1 we have a proof of Theorem 1.2.

Remark 4.11 If (X, d, μ) is a complete metric measure space with μ a doubling measure and $\widehat{d} < \infty$ one can also guarantee the existence of an ∞-harmonic function. Indeed, by Proposition 4.4(b), $(X, \widehat{d})$ is a length space and $(X, \widehat{d}, \mu)$ satisfies an ∞-weak Fubini property. By [25] we can always find an AMLE in $(X, \widehat{d})$ and by Theorem 4.10 they are ∞-harmonic with respect to $\widehat{d}$ and therefore with respect to d.

In the absence of ∞-Poincaré inequality, an ∞-harmonic function need *not* be an AMLE even if X is a geodesic space, as the next example shows.

Example 4.12 As in Example 4.5 and Example 2.11, consider the Sierpiński Carpet X endowed with its length metric and the corresponding Hausdorff measure. Then X is a geodesic space, but by Remark 2.6 every $u \in L^\infty(X)$ is ∞-harmonic, but if it is not Lipschitz continuous on the carpet then it cannot be an AMLE.

The next example shows that ∞-weak Fubini property is crucial for Theorem 1.2. This example can also be found in [35, Page 171], but for the reader's convenience we give the details here.

Example 4.13 Let $X = \{0\} \times [0, \infty) \cup [0, \infty) \times \{0\} \subset \mathbb{R}^2$ be equipped with the metric obtained as the restriction of the Euclidean metric on $\mathbb{R}^2$ to X, and with the measure $\mu = \mathcal{H}^1|_X$. With $\Omega = \{0\} \times [0, 1) \cup [0, 1) \times \{0\}$, we set $u : X \to \mathbb{R}$ by

$$u(x, y) = \begin{cases} x & \text{if } y = 0, \\ -y & \text{if } x = 0. \end{cases}$$

It is not difficult to see that u is ∞-harmonic on Ω (by noting for example that $(X, \widehat{d})$ is isometric to $\mathbb{R}$), but fails to be AMLE in Ω. To see that u is not an AMLE, we argue as follows. For $0 < \varepsilon < 1$ let $V_\varepsilon = \{0\} \times [0, \varepsilon) \cup [0, 1) \times \{0\}$, note that

$$\mathrm{LIP}(u, V_\varepsilon) \geq \frac{u(\varepsilon, 0) - u(0, \varepsilon)}{\sqrt{2}\,\varepsilon} = \sqrt{2},$$

whereas

$$\mathrm{LIP}(u, \partial V_\varepsilon) = \frac{1 + \varepsilon}{\sqrt{1 + \varepsilon^2}} < \sqrt{2} \text{ for sufficiently small } \varepsilon.$$

Therefore u is not AMLE on Ω with the boundary values $u(1,0) = 1$, $u(0,1) = -1$ (observe that any AMLE of this boundary function must be linear on each arm of Ω, and symmetry considerations together with uniqueness of AMLEs would then tell us that if such AMLE exists then it must be the above function $u(x,y)$). Note that (X,d,μ) is Ahlfors 1-regular and supports an ∞-Poincaré inequality, but does *not* satisfy any ∞-weak Fubini property. Here we have the existence of unique ∞-harmonic extension but no AMLE extension.

5 Stability of ∞-harmonic functions

In this section we consider sequences of ∞-harmonic functions on a complete metric measure space X equipped with a doubling measure supporting an ∞-Poincaré inequality. It is known that if X supports a p-Poincaré inequality for some $1 < p < \infty$, then a locally uniformly bounded sequence of p-harmonic functions on a fixed domain have a locally uniformly convergent subsequence that converges to a p-harmonic function on the domain. See [30] and [37]. This property is known as the *stability property* of p-harmonic functions. This is in general not true for ∞-harmonic functions, given the lack of Caccioppoli-type (or De Giorgi type) inequality that controls the local energy of the ∞-harmonic function in terms of its local bound. But we have the following weaker stability.

Consider a sequence of ∞-harmonic functions, $\{u_i\}_i$, of ∞-harmonic functions on Ω such that each u_i is L-Lipschitz continuous on X. Then by the Arzela-Ascoli theorem, there is a subsequence, also denoted $\{u_i\}_i$, and a Lipschitz function u_0 on X such that $u_i \to u_0$ locally uniformly in X (and hence uniformly on the bounded domain Ω). We now show that u_0 is ∞-harmonic in Ω. To see this, fix $\varepsilon > 0$ and note that there is some $N_\varepsilon \in \mathbb{N}$ such that whenever $i \geq N_\varepsilon$, we have $u_i - \varepsilon \leq u_0 \leq u_i + \varepsilon$ on Ω. Let w be the unique ∞-harmonic function on Ω such that $w = u_0$ on $X \setminus V$, as promised by Theorem 1.1. Given the uniqueness of ∞-harmonic solutions and given Lemma 3.8, we have a comparison theorem for ∞-harmonic functions as well in the manner of Lemma 3.8. Therefore on Ω we have $u_i - \varepsilon \leq w_V \leq u_i + \varepsilon$, and so

$$w_V \leq u_0 + 2\varepsilon \leq w_V + 4\varepsilon.$$

As the above holds for all $\varepsilon > 0$, we see that $u_0 = w$ on Ω, that is, u_0 is ∞-harmonic in Ω. Thus we have the following proposition.

Proposition 5.1 *If $\{u_i\}_i$ is a bounded sequence of L-Lipschitz functions on X such that each u_i is ∞-harmonic in Ω, then there is a subsequence that converges locally uniformly in X to an L-Lipschitz function u_0 such that u_0 is ∞-harmonic in Ω.*

Example 5.2 Let $\Omega = (0,1) \times (0,1) \subset \mathbb{R}^2$ and for each $k \in \mathbb{N}$ let F_k be the sawtooth function

$F_k : \mathbb{R} \to \mathbb{R}$ given as the periodic extension of the function $\varphi_k : [0, 2/k] \to \mathbb{R}$:

$$\varphi_k(t) = \begin{cases} kt & \text{when } t \in [0, 1/k], \\ -kt + 2 & \text{when } t \in [1/k, 2/k]. \end{cases}$$

Then F_k is k-Lipschitz continuous and is bounded by 1, that is, $F_k \in N^{1,\infty}(X)$. Let $f_k : \mathbb{R}^2 \to \mathbb{R}$ be given by $f_k(x, y) = F_k(x)$, and let u_k be the ∞-harmonic extension of f_k to Ω. Then each u_k is bounded by 1 on $\mathbb{R}^2$, but by Lemma 4.9, we know that u_k has no locally uniformly convergent subsequence that can converge to a Lipschitz function on Ω.

On the other hand, we have the following stability theorem.

Theorem 5.3 *For each $k \in \mathbb{N}$ let $f_k \in N^{1,\infty}(X)$ and let $f \in N^{1,\infty}(X)$ such that $f_k \to f_0$ in $N^{1,\infty}(X)$. Let u_k be the ∞-harmonic extension of f_k to Ω. Then u_k converges locally uniformly in X to a function $u_0 \in N^{1,\infty}(X)$ such that u_0 is the ∞-harmonic extension of f_0 to Ω.*

Proof. Since $f_k \to f$ in $N^{1,\infty}(X)$, there is some $L > 0$ such that each f_k and f_0 is L-Lipschitz on X, and f_k converges uniformly to f_0 on X. By the above proposition, we know that every subsequence of $\{u_k\}_k$ has a further subsequence that converges uniformly to the *unique* function that is the ∞-harmonic extension of f_0 to Ω. Therefore the entire sequence $\{u_k\}_k$ converges uniformly in X to a Lipschitz function u_0 that is the ∞-harmonic extension of f_0 to Ω. $\square$

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