Existence and uniqueness of ∞-harmonic functions under assumption of ∞-Poincaré inequality

Estibalitz Durand-Cartagena, Jesús A. Jaramillo, and Nageswari Shanmugalingam *

Abstract

Given a complete metric measure space whose measure is doubling and supports an ∞-Poincaré inequality, and a bounded domain Ω in such a space together with a Lipschitz function $f : \partial\Omega \to \mathbb{R}$ we show the existence and uniqueness of an ∞-harmonic extension of f to Ω. We also show that in the event that the metric on the metric space has an ∞-weak Fubini property, the notion of ∞-harmonic functions coincide with the notion of AMLEs proposed by Aronsson. As an auxiliary tool we show that given that the measure on the metric space is doubling and supports an ∞-Poincaré inequality, one can construct a metric bi-Lipschitz equivalent to the original one, with respect to which the metric space has an ∞-weak Fubini property. The notion of ∞-harmonicity is in general distinct from the notion of strongly absolutely minimizing Lipschitz extensions found in [11, 23, 24], but coincides when the metric space supports a p-Poincaré inequality for some finite $p \geq 1$.

Key words: ∞-Poincaré inequality, ∞-harmonic, AMLE, metric measure spaces.

Mathematics Subject Classification (2010): Primary: 31E05; Secondary: 31C45, 31C05, 54C20.

1 Introduction

Since the pioneering work of Aronsson [2], the notions of absolute minimizing Lipschitz extensions (AMLEs) and ∞-harmonic functions in Euclidean domains have been extensively studied in connection with a variety of applications. We refer to the survey paper [3] for general information on this subject. Recent applications of these notions include image processing and

*N.S.'s research was partially supported from grant # DMS-1500440 of NSF (U.S.A.). The research of J.J. and E.D-C. are partially supported by grant MTM2012-34341 (Spain). E.D-C. was also partially supported from 2016-MAT09 (Apoyo Investigación Matemática Aplicada, ETSI Industriales, UNED). Part of the research was conducted during N.S.'s visit to ICMAT (Madrid, Spain) in Spring 2015, and during the visit of E.D-C. and J.J. to the University of Cincinnati in Spring 2016. The authors thank these institutions for their kind hospitality. The authors also wish to thank Riikka Korte for interesting discussions regarding moduli of curve families.

inpainting or brain and surface warping. The articles [5] and [26] give a good overview of such applications.

The idea behind AMLEs is simple. For a set $A \subset \mathbb{R}^n$ and a Lipschitz function $f : A \to \mathbb{R}$, we denote

$$\mathrm{LIP}(f, A) := \sup_{x,y \in A, x \neq y} \frac{|f(x) - f(y)|}{d(x, y)}.$$

Now given a Lipschitz function $f : Y \to \mathbb{R}$ with $Y \subset \mathbb{R}^n$, we can construct at least two Lipschitz extensions $F : \mathbb{R}^n \to \mathbb{R}$ of f to $\mathbb{R}^n$ with the *same* Lipschitz constant, that is, $\mathrm{LIP}(f, Y) = \mathrm{LIP}(F, \mathbb{R}^n)$ as follows. We can set:

$$F(x) = \sup\{f(y) - \mathrm{LIP}(f, Y)d(x, y) \, : \, y \in Y\}$$

for all $x \in \mathbb{R}^n$ or, we can set:

$$F(x) = \inf\{f(y) + \mathrm{LIP}(f, Y)d(x, y) \, : \, y \in Y\}$$

for all $x \in \mathbb{R}^n$. These two extensions were first studied by McShane [28]. Note that the quantity $\mathrm{LIP}(F, \mathbb{R}^n)$ does not care about the local behavior of F, only the global behavior. Aronsson sought to take into account also the local behavior. More precisely, given a domain $\Omega \subset \mathbb{R}^n$ and a Lipschitz function f on $Y := \partial\Omega$, Aronsson looked for a Lipschitz extension $F : \overline{\Omega} \to \mathbb{R}$ of f to Ω such that in addition to the above requirement that $\mathrm{LIP}(f, \partial\Omega) = \mathrm{LIP}(F, \Omega)$, F also satisfies $\mathrm{LIP}(F, \partial V) = \mathrm{LIP}(F, V)$ for all subdomains $V \subset \Omega$. Functions F that satisfy this condition are called *absolutely minimizing Lipschitz extensions*, or AMLEs for short. In [2], existence of such a function was demonstrated using a variant of the Perron method. Note that such F would equivalently satisfy the condition that whenever $V \subset \Omega$ is a subdomain and $\varphi : \overline{V} \to \mathbb{R}$ such that $\varphi = F$ on ∂V, we must have $\mathrm{LIP}(F, V) \leq \mathrm{LIP}(\varphi, V)$. Thus the local nature of minimizing Lipschitz constant is established for AMLEs. It was also shown in [2] and [22] that AMLEs F in Euclidean domains are ∞-harmonic in the sense that they satisfy $\Delta_\infty F = 0$, where

$$\Delta_\infty F = \sum_{i,j=1}^{n} \frac{\partial F}{\partial x_i} \frac{\partial F}{\partial x_j} \frac{\partial^2 F}{\partial x_i \partial x_j}.$$

In fact, a function on an Euclidean domain is an AMLE if and only if it is ∞-harmonic. One can construct ∞-harmonic functions via p-harmonic approximations, that is, p-harmonic functions in Ω that take on the value f on $\partial\Omega$ approximate the ∞-harmonic functions as $p \to \infty$. While the definition of AMLEs requires only the metric d, the definition of ∞-harmonicity requires in addition the knowledge of measure on the space as well (for the notion of weak partial derivatives). The interested reader is referred to [3] for further information on this topic.

In applications to image processing, ∞-harmonic extensions are used for image inpainting. In image inpainting an image with a patch of loss is corrected by “painting in” the lost image.

To do so, usually it is preferable to make the extension of the image into the lost patch as smooth as possible. For each $1 \leq p < \infty$ the p-harmonic extension is the extension F whose p-th energy $J_p(F) := \int_\Omega |\nabla F|^p \, d\mathcal{L}^n$ is minimal amongst all Sobolev functions with the same boundary (outside image) data. When $p = 1$, the corresponding minimizer preserves edges found in the image (see for example [1]); as $p \to \infty$, the corresponding processed image becomes smoother, with $p = \infty$ corresponding to Lipschitz smoothness. See [30] for a survey on this subject.

In the abstract setting of separable metric spaces that are length spaces, the existence of AMLEs with given Lipschitz boundary data was studied in [23] using a Perron's method approach. The existence of AMLEs in general length spaces is obtained in [31] using random games. On the other hand, thanks to the development of a Sobolev theory in the setting of metric measure spaces, the notion of p-harmonic function has been considered as well (see [21] and [6]). In [24], for doubling metric measure spaces satisfying a p-Poincaré inequality for some finite $p \in [1, \infty)$, it was shown that the limit (as $p \to \infty$) of p-harmonic solutions to the Dirichlet problem on the domain, with a given Lipschitz boundary data, yields a so-called strongly absolutely minimizing Lipschitz extension. It was also shown there that when X satisfies a "weak Fubini property" of exponent p, a function is an AMLE if and only if it is a strongly absolutely minimizing Lipschitz extension. The notion of strongly absolutely minimizing Lipschitz extensions coincides with our notion of ∞-harmonic functions in the metric setting when the metric space supports a p-Poincaré inequality for some finite $p \geq 1$. While strongly absolutely minimizing Lipschitz extensions minimize (with respect to the L^∞-norm), both locally and globally, the local Lipschitz constant function $\operatorname{Lip} u$ associated with the Lipschitz function u, the ∞-harmonic functions minimize the minimal ∞-weak upper gradient of u (see Definition 2.4). It was shown in [10] that when the metric space supports a p-Poincaré inequality for some finite p, the minimal p-weak upper gradient of a Lipschitz function agrees almost everywhere with the local Lipschitz constant function associated with the Lipschitz function. Since in this paper we do *not* know that the metric space supports a p-Poincaré inequality for any finite $p > 1$, the Euclidean notion of ∞-harmonicity is more naturally related to our notion of minimizing ∞-weak upper gradients; hence this is the object we study in this paper.

In [18] it was shown that there are complete metric measure spaces whose measure is doubling and supports an ∞-Poincaré inequality but not supporting any p-Poincaré inequality for finite $p \geq 1$. The examples in [18] can still be addressed using the techniques in [24] since the domain in consideration is a bounded domain, and the failure of p-Poincaré inequality occurs only at large scales. However, the sphericalization of the examples in [18], using the procedure described in [27], also supports an ∞-Poincaré inequality but does not support any p-Poincaré inequality for finite p, see [16] and [17], and the techniques of [24] fail for domains in this sphericalized space that contain the image of infinity from the original space of [18].

In light of these examples we are interested in knowing whether, given a bounded domain in a doubling metric measure space supporting an ∞-Poincaré inequality, and given a Lipschitz function defined on the boundary of the domain, there is an ∞-harmonic function on the domain with the prescribed boundary data. Since we do not assume p-Poincaré inequality for any finite $p \geq 1$, the challenge is to construct such an ∞-harmonic function without Aronsson's prescription of first constructing p-harmonic functions. This is one of the principal foci of the current paper. Our main result is the following:

Theorem 1.1 *Let (X, d, μ) be a complete metric measure space with μ doubling and supporting an ∞-Poincaré inequality, and let $\Omega \subset X$ be a bounded domain such that $X \setminus \Omega$ has positive measure. Given a Lipschitz function $f : \partial\Omega \to \mathbb{R}$, there is a unique Lipschitz function $u : \overline{\Omega} \to \mathbb{R}$ such that $u = f$ on $\partial\Omega$ and u is ∞-harmonic in Ω.*

The problem of *existence* of ∞-harmonic functions is studied in Section 3, and the corresponding result is given in Theorem 3.3. The question of *uniqueness* is related to the equivalence between AMLEs and ∞-harmonic functions. In [24], in order to obtain this equivalence, a p-weak Fubini property with $1 < p < \infty$ is needed for showing that one can neglect zero measure sets when computing the Lipschitz constant of a function. In this paper, we prove the equivalence between AMLEs and ∞-harmonic functions under the weaker hypothesis of ∞-weak Fubini property (see Definition 4.1). Proposition 4.2 gives a simple metric characterization of ∞-weak Fubini property. This characterization shows that the link between ∞-weak Fubini property and the measure μ is weak, and depends only on the collection of μ-null sets. We will also show that under the hypotheses of Theorem 1.1, there is a bi-Lipschitz equivalent metric $\widehat{d}$ on X such that $(X, \widehat{d}, \mu)$ satisfies an ∞-weak Fubini property, see Proposition 4.4.

The second main result of this work is the following.

Theorem 1.2 *Let (X, d, μ) be a complete metric measure space with μ doubling and satisfying an ∞-weak Fubini property. Consider a bounded domain $\Omega \subset X$ such that $X \setminus \Omega$ has positive measure and a Lipschitz function $f : \partial\Omega \to \mathbb{R}$. A Lipschitz function $u : \overline{\Omega} \to \mathbb{R}$ is ∞-harmonic in Ω if and only if it is an AMLE of f to Ω.*

In the Euclidean setting uniqueness of AMLEs for a given boundary data was established via the tool of viscosity solutions in [22], and an alternate proof using viscosity solutions and tug-of-war games was provided in [31]. In the setting of Heisenberg groups, uniqueness was demonstrated in [4]. Uniqueness for AMLEs in metric spaces that are length spaces was established in [31, Theorem 1.4]. Combining this with some tools we develop here, in Theorem 4.9 we obtain uniqueness of ∞-harmonic functions with prescribed Lipschitz boundary data in the setting of complete doubling metric measure spaces satisfying an ∞-Poincaré inequality. We also provide an example of a (length) space that does not satisfy any ∞-weak Fubini property, for which uniqueness of solutions to ∞-harmonic Dirichlet problem fails, see Example 4.12. Given the uniqueness of AMLEs, this example also shows that there are ∞-harmonic functions that are not AMLEs when we do not have ∞-weak Fubini property.

2 Notation and definitions

In this paper we will assume that (X, d, μ) is a complete metric measure space. That is, (X, d) is a complete metric space equipped with a Borel measure μ which is positive and finite on each ball. We say that the measure μ is *doubling* on X if there is a constant $C_D \geq 1$ such that whenever $x \in X$ and $r > 0$,

$$\mu(B(x, 2r)) \leq C_D \, \mu(B(x, r)).$$

Given a set $A \subset X$ and a Lipschitz function $u : A \to \mathbb{R}$, we set for $x \in A$,

$$\operatorname{Lip} u(x) := \lim_{r \to 0^+} \sup_{y \in A \cap B(x,r), y \neq x} \frac{|u(x) - u(y)|}{d(x, y)}$$

and

$$\operatorname{LIP}(u, A) := \sup_{x,y \in A, x \neq y} \frac{|u(x) - u(y)|}{d(x, y)}.$$

We say that u is L-Lipschitz on A if $\operatorname{LIP}(u, A) \leq L$. The class of all bounded Lipschitz functions on X is denoted $\operatorname{LIP}^\infty(X)$. This class is equipped with the norm

$$\|u\|_{\operatorname{LIP}^\infty(X)} := \sup_{x \in X} |u(x)| + \operatorname{LIP}(u, X).$$

By a *curve* in X we mean a continuous function $\gamma : I \to X$ defined on some compact interval $I \subset \mathbb{R}$. The *length* of a curve $\gamma : I \to X$ is defined by

$$\ell(\gamma) := \sup_{t_0 < t_1 < \cdots < t_n} \sum_{j=1}^{n} d(\gamma(t_{j-1}), \gamma(t_j)),$$

where the supremum is taken over all finite subdivisions $t_0 < t_1 < \cdots < t_n$ of the interval I. The curve γ is said to be *rectifiable* if it has finite length. Note that every rectifiable curve can be re-parametrized so that it is arc-length parametrized, that is, $I = [0, \ell(\gamma)]$ and for each $s \in I$, if we denote $I_s := \{t \in I : t \leq s\}$ we have that

$$\ell(\gamma|_{I_s}) = s.$$

Henceforth in the paper we will assume all rectifiable curves, unless otherwise indicated, to be arc-length parametrized as above. The integral of a Borel function $\rho : X \to [0, \infty]$ over an arc-length parametrized curve γ is defined as

$$\int_\gamma \rho\, ds := \int_0^{\ell(\gamma)} \rho(\gamma(t))\, dt.$$

A metric space (X, d) is said to be a *length space* if for each pair of points $x, y \in X$ the distance $d(x, y)$ coincides with the infimum of all lengths of curves in X connecting x with y. The metric space X is C-*quasiconvex*, or quasiconvex, for some $C \geq 1$ if for each pair of points $x, y \in X$, there exists a curve γ connecting x and y with $\ell(\gamma) \leq Cd(x, y)$.

In the setting of non-smooth metric measure spaces, the role of derivatives is taken on by the upper gradients (see [20]). Given a function $u : X \to \overline{\mathbb{R}}$, we say that a Borel-measurable function $g : X \to [0, \infty]$ is an *upper gradient* of f if

$$|u(y) - u(x)| \leq \int_\gamma g\, ds \tag{1}$$

whenever γ is a non-constant compact rectifiable curve in X connecting the points x and y. The above inequality should be interpreted to mean that $\int_\gamma g\, ds = \infty$ if at least one of $u(x)$, $u(y)$ is not finite.

Note that a function with an almost-everywhere finite upper gradient will have more than one upper gradient, since the sum of an upper gradient and any non-negative Borel measurable function will also be an upper gradient. The set of all upper gradients of a given function f is a convex set. We refer the reader to [21] and [6] for more on the properties of upper gradients. In the Euclidean setting, the function $|\nabla u|$ of a smooth function f acts as its upper gradient (and in fact, no function that is smaller than $|\nabla u|$ on a set of positive measure can act as an upper gradient). However, $|\nabla u|$ of a more general Sobolev function $u \in W^{1,p}(\mathbb{R}^n)$ is not in general an upper gradient of u, for the inequality (1) may fail for a few curves γ, but $|\nabla u|$ will be a p-weak upper gradient of u in the sense that the upper gradient inequality (1) fails for at most a negligible family of curves γ in the following sense of p-modulus.

Definition 2.1 Given a family Γ of curves in a metric measure space X, set $\mathcal{A}(\Gamma)$ to be the family of all Borel measurable functions $\rho : X \to [0, \infty]$ such that

$$\int_\gamma \rho\, ds \geq 1 \quad \text{for all} \quad \gamma \in \Gamma.$$

In current literature on analysis in metric spaces the functions in $\mathcal{A}(\Gamma)$ are said to be *admissible* for Γ.

We define the ∞-*modulus of* Γ by

$$\mathrm{Mod}_\infty(\Gamma) = \inf_{\rho \in \mathcal{A}}(\Gamma) \, \|\rho\|_{L^\infty(X)},$$

and for $1 \le p < \infty$ the p-modulus of Γ is

$$\mathrm{Mod}_p(\Gamma) = \inf_{\rho \in \mathcal{A}}(\Gamma) \int_X \rho^p \, d\mu.$$

A non-negative Borel measurable function g on X is said to be a *p-weak upper gradient* of a function $u : X \to \overline{\mathbb{R}}$ if the collection Γ of all non-constant rectifiable curves γ in X for which the inequality (1) fails has zero p-modulus.

It can be shown (see e.g. [21]) that a family Γ of curves in X has zero p-modulus if, and only if, there is a non-negative Borel measurable function $\rho \in L^p(X)$ such that $\int_\gamma \rho \, ds = \infty$ for each $\gamma \in \Gamma$. When $p = \infty$, one can even require that $\rho = 0$ almost everywhere:

Lemma 2.2 [13, Lemma 5.7] *Let Γ be a family of curves in a metric measure space X. Then $\mathrm{Mod}_\infty(\Gamma) = 0$ if and only if there is a Borel function $\rho \ge 0$ with $\|\rho\|_{L^\infty(X)} = 0$ such that $\int_\gamma \rho \, ds = \infty$ for each $\gamma \in \Gamma$.*

The Newton-Sobolev space $N^{1,p}(X)$ $1 \le p \le \infty)$ is defined as follows. First consider the class $\widetilde{N}^{1,p}(X)$ of all functions in $L^p(X)$ that have an ∞-weak upper gradient in $L^p(X)$. For$u_1, u_2 \in \widetilde{N}^{1,p}(X)$ we say that $u_1 \sim u_2$ if

$$\|u_1 - u_2\|_{L^p(X)} + \inf_g \|g\|_{L^p(X)} = 0,$$

where the infimum is taken over all p-weak upper gradients g of $u_1 - u_2$. The relation $\sim$ is an equivalence relation on the vector space $\widetilde{N}^{1,p}(X)$, and we set $N^{1,p}(X)$ to be the collection of all equivalence classes of $\widetilde{N}^{1,p}(X)$. The space $N^{1,p}(X)$ is equipped with the norm

$$\|u\|_{N^{1,p}(X)} := \|u\|_{L^p(X)} + \inf_g \|g\|_{L^p(X)},$$

the infimum being taken over all ∞-weak upper gradients of u. It was proven in [13] that $N^{1,\infty}(X)$ is a Banach space. If $A \subset X$ is a measurable set, we can consider the space $N^{1,p}(A)$ endowed with the metric $d|_A$ and the measure $\mu|_A$.

Taking into account the following lemma we can, without loss of generality, assume that ∞-weak upper gradients of a function $u \in N^{1,\infty}(X)$ are in fact upper gradients after modifying them on a set of μ-measure zero.

Lemma 2.3 *Let X be a metric measure space, and $u \in N^{1,\infty}(X)$. Every ∞-weak upper gradient g of u can be modified on a set of measure zero such that the modification $\widetilde{g}$ is an upper gradient of u.*

Proof. Let g be an ∞-weak upper gradient of u, and let Γ denote the family of curves in X for which the inequality (1) fails. By Lemma 2.2 there exists a Borel function $\rho \geq 0$ on X with $\|\rho\|_{L^\infty(X)} = 0$ such that $\int_\gamma \rho\, ds = \infty$ for each $\gamma \in \Gamma$. Then $E = \{x \in X : \rho(x) > 0\}$ has zero-measure, and it is easily seen that

$$\widetilde{g} = g + \rho$$

is an upper gradient of u. $\square$

It follows from [29, Lemma 4.1] that if g_1, g_2 are ∞-weak upper gradients of a function $u \in N^{1,\infty}(X)$, then the pointwise minimum $g = \min\{g_1, g_2\}$ is also an ∞-weak upper gradient of u. In fact, we know from [29, Theorem 4.6] that for each $u \in N^{1,\infty}(X)$ there is an ∞-weak upper gradient $g_u \in L^\infty(X)$ which is *minimal* in the sense that whenever $g \in L^\infty(X)$ is an ∞-weak upper gradient of u, we have that $g_u \leq g$ almost everywhere in X. Furthermore, g_u is unique up to sets of measure zero. As we mentioned above, by Lemma 2.3 we can also assume that g_u is an upper gradient of u.

We can now define ∞-harmonic functions as follows. Note that by a *domain* in a metric space we mean a non-empty connected open subset.

Definition 2.4 *Let X be a metric measure space, and Ω a bounded domain in X such that $X \setminus \Omega$ has positive measure. We say that $u \in N^{1,\infty}(X)$ is ∞-harmonic in Ω if whenever $V \subset \Omega$ is an open set and $v \in N^{1,\infty}(X)$ such that $v = u$ on $X \setminus V$, we have*

$$\|g_u\|_{L^\infty(V)} \leq \|g_v\|_{L^\infty(V)}. \tag{2}$$

Furthermore, we say that $u \in N^{1,\infty}(X)$ is ∞-harmonic in Ω with boundary data $f \in N^{1,\infty}(X)$ if u is ∞-harmonic in Ω and $u = f$ on $X \setminus \Omega$.

Remark 2.5 In the case when $N^{1,\infty}(X) = L^\infty(X)$, we have that $g_u = 0$ almost everywhere for every $u \in N^{1,\infty}(X)$, because the ∞-modulus of the collection of all non-constant compact rectifiable curves in X is zero in this instance. To see that the modulus of this collection is zero, note that because for each $x \in X$ and $r > 0$ the function $\chi_{B(x,r)} \in L^\infty(X)$ and hence in $N^{1,\infty}(X)$ by hypothesis, we must have that this function is absolutely continuous on ∞-modulus almost every curve in X. In particular, this means that the collection of all rectifiable curves that intersect both $B(x,r)$ and $X \setminus \overline{B}(x,r)$ has zero ∞-modulus. Since the collection of all non-constant compact rectifiable curves in X is the union of the family $\Gamma(B(x_i, r_j))$ of all rectifiable curves in X intersecting both $B(x_i, r_i)$ and $X \setminus \overline{B}(x_i, r_j)$, with $\{x_i\}_i$ a countable dense subset of X and $\{r_i\}_i$ is the set of positive rational numbers, we must have by the countable subadditivity of modulus that the ∞-modulus of the collection of all non-constant compact rectifiable curves is zero. Therefore given a boundary data function $f \in N^{1,\infty}(X)$, any $u \in N^{1,\infty}(X)$ with $u = f$ on $X \setminus \Omega$ is ∞-harmonic in Ω.

In metric measure spaces where the ∞-modulus of the collection of all non-constant compact rectifiable curves is zero, one can justifiably argue that $N^{1,\infty}(X)$ is the wrong Sobolev type space to use. However, there are many metric measure spaces where the triviality $N^{1,\infty}(X) = L^\infty(X)$ does not happen. For example, if X supports an ∞-Poincaré inequality, then $N^{1,\infty}(X) \neq L^\infty(X)$, see [13, 14]. Of such spaces, there is a collection of metric spaces that do not support a p-Poincaré inequality for any finite $p > 1$, and in such a setting the currently known approaches of constructing ∞-harmonic functions known so far fail. Thus in this paper we focus on giving a construction of ∞-harmonic functions that does not rely on the existence of p-Poincaré inequality for any finite $p > 1$.

Example 2.6 Let (X, d, μ) be the Sierpinski carpet equipped with the Euclidean metric and the corresponding Hausdorff measure. The Sierpinski carpet does not support an ∞-Poincaré inequality (see [14, Example 4.14]). From the discussion in [7], we know the existence of a set $\widehat{N} \subset [0,1]$ such that, with the Hausdorff measure on X denoted by μ, the "first coordinate projection" $\Pi_1\mu$ of μ to $[0,1]$ given by $\Pi_1\mu(A) = \mu(\Pi_1^{-1}(A))$ for Borel sets $A \subset [0,1]$ sees $\widehat{N}$ as of measure zero but $\mathcal{H}^1(\widehat{N}) = 1$. Let $N = (\Pi_1^{-1}(\widehat{N}) \cup \Pi_2^{-1}(\widehat{N}))$. Here Π_1 and Π_2 are the first coordinate and the second coordinate projection maps from X to the interval $[0,1]$. Note that $\mu(N) = 0$, but given any curve γ in X with end points x, y such that $(x_1, x_2) = x \neq y = (y_1, y_2)$, we must have

$$\begin{aligned}\mathcal{H}^1(\gamma^{-1}(N)) \geq& \mathcal{H}^1(\gamma \cap N) \geq \max\{\mathcal{H}^1(\Pi_1 \circ \gamma(\gamma^{-1}(N))), \mathcal{H}^1(\Pi_2 \circ \gamma(\gamma^{-1}(N)))\} \\ \geq& \max\{|x_1 - y_1|, |x_2 - y_2|\} > 0.\end{aligned}$$

Now, let $\rho = \infty \cdot \chi_N$. Observe that ρ is a non-negative Borel function in X and, as shown above, given any $x, y \in X$ and any rectifiable curve γ connecting x and y, $\mathcal{H}^1(\gamma^{-1}(N)) > 0$ and so we have that

$$\int_\gamma \rho ds = \int_0^{\ell(\gamma)} \rho(t) dt \geq \infty \times \mathcal{H}^1(\gamma^{-1}(N)) = \infty.$$

By Lemma 2.2, we obtain that $\mathrm{Mod}_\infty(\Gamma_{xy}) = 0$, where Γ_{xy} denotes the family of rectifiable curves connecting x and y and by the previous remark $N^{1,\infty}(X) = L^\infty(X)$.

In the Euclidean setting, ∞-harmonic functions u are precisely those which satisfy the equation $\Delta_\infty u = 0$, see for example [11] or [3, Theorem 4.13]. This notion depends intrinsically on the measure μ as well as the metric d. The following related notion, due to Aronsson [2] (see also [3]), relies only on the metric d. Under certain conditions on the metric measure space X we show that both these notions coincide; see also [24] for a discussion in the metric setting, where a stronger assumption on the metric measure space was required. See the beginning of this section for the definition of $\mathrm{LIP}(u,V)$.

Definition 2.7 *Let (X,d) be a metric space, Ω a domain in X and $f:\partial\Omega\to\mathbb{R}$ a Lipschitz function. We say that a Lipschitz function u defined on the closure $\overline{\Omega}$ is an absolutely minimizing Lipschitz extension (AMLE for short) of f to Ω if $f=u$ on $\partial\Omega$ and whenever $V\subset\Omega$ is an open set and $v:\overline{V}\to\mathbb{R}$ is a Lipschitz function with $v=u$ on ∂V, we have*

$$\mathrm{LIP}(u,V)\le \mathrm{LIP}(v,V).$$

If u is an $N^{1,\infty}(\overline{\Omega})$-function that has a minimal ∞-weak upper gradient g_u on $\overline{\Omega}$ such that $g_u\le L$ almost everywhere in Ω, and f is a Lipschitz function on $X\setminus\Omega$ such that L is an upper gradient of f and $u=f$ on $\partial\Omega$, then u has an extension $\widehat{u}$ to X such that the extension $\widehat{g_u}$ of g_u to $X\setminus\overline{\Omega}$ by the constant L is an ∞-weak upper gradient of $\widehat{u}$ as well. To see this, for $x\in X$ we set

$$\widehat{u}(x)=\begin{cases} u(x) & \text{if } x\in\overline{\Omega},\\ f(x) & \text{if } x\in X\setminus\Omega.\end{cases}$$

The proof that $\widehat{g_u}$ is an ∞-weak upper gradient of $\widehat{u}$ follows from [6, Proposition 2.39]. As a consequence, we see that if $u\in N^{1,\infty}(\overline{\Omega})$ has an ∞-weak upper gradient that is almost everywhere in $\overline{\Omega}$ bounded by L and $u=f$ on $\partial\Omega$, then u has an extension $\widehat{u}\in N^{1,\infty}(X)$ to X that has an ∞-weak upper gradient dominated almost everywhere in X by L.

Lemma 2.8 *Let Ω, G be two non-empty open subsets of X, $G\subset\Omega$ with dist$(G,X\setminus\Omega)>0$, and $u\in N^{1,p}(\Omega)$, $f\in N^{1,\infty}(X)$. If $u=f$ on ∂G, then the function $\widehat{u}$ given by*

$$\widehat{u}(x)=\begin{cases} u(x) & \textit{if } x\in G,\\ f(x) & \textit{if } x\in X\setminus G\end{cases}$$

is in $N^{1,p}_{loc}(X)$.

Proof. Let u be a function satisfying the hypotheses of the lemma, and let $\widehat{u}$ be the corresponding extension of u to $X\setminus G$. To prove the lemma, it suffices to show that $\widehat{u}$ has a p-weak upper gradient in the class $L^\infty(X)$. We set $u_0=\widehat{u}-f$, and then it suffices to show that $u_0\in N^{1,p}_{loc}(X)$.

Let $g\in L^p(\Omega)$ be an upper gradient of $u-f$ in Ω, and let g_0 be the zero extension of g to $X\setminus\Omega$. Note that $g_0\in L^p(X)$, and as Ω is open and g is Borel, we also have that g_0 is Borel. We wish to show that g_0 is a p-weak upper gradient of u_0 in X.

Let γ be a non-constant compact rectifiable curve in X, and let x, y denote the two end points of γ. If both x and y belong to $X \setminus G$, then trivially we have $|u_0(x) - u_0(y)| \le \int_\gamma g_0\, ds$. So it suffices to consider only γ for which $x \in G$. Because $x \in G$, with $\gamma : [a, b] \to X$ and $\gamma(a) = x$, there is some $t_0 \in (a, b]$ such that $\gamma((a, t_0)) \subset G$. Let t_0 be the largest such number in $(a, b]$. If $t_0 < b$, then $\gamma(t_0) \in \partial G$ and $u_0(\gamma(t_0)) = u(\gamma(t_0)) - f(\gamma(t_0))$. If $t_0 = b$, then again we have that $u_0(\gamma(b)) = u_0(y) = u(y) - f(y)$. In either case, from the facts that $g_0 \circ \gamma = g \circ \gamma$ on $[a, t_0)$ and g is an upper gradient of u we can infer that

$$|u_0(x) - u_0(\gamma(t_0))| = |u(x) - u(\gamma(t_0))| \le \int_{\gamma|_{[a,t_0]}} g_0\, ds.$$

If $\gamma(b) \notin G$, then from the above we have that $|u_0(x) - u_0(y)| = |u_0(x) - u_0(\gamma(t_0))| \le \int_\gamma g_0\, ds$. It now follows that g_0 is a p-weak upper gradient of u_0 and so $u_0 \in N^{1,\infty}(X)$. □

We next introduce the notion of p-Poincaré inequalities, which play a main role in this paper.

Definition 2.9 *Given $1 \le p < \infty$, we say that a metric measure space X supports a p-Poincaré inequality if there are positive constants C, λ such that whenever $B = B(x, r)$ is a ball in X and g is an upper gradient of u,*

$$\fint_B |u - u_B|\, d\mu \le C\, r \left(\fint_{\lambda B} g^p\, d\mu \right)^{1/p}.$$

Here $u_B := \mu(B)^{-1} \int_B u\, d\mu =: \fint_B u\, d\mu$ is the average of u on the ball B, and $\lambda B := B(x, \lambda r)$. We say that X supports an ∞-Poincaré inequality if there are positive constants C, λ such that whenever $B = B(x, r)$ is a ball in X and g is an upper gradient of u,

$$\fint_B |u - u_B|\, d\mu \le C\, r\, \|g\|_{L^\infty(\lambda B)}.$$

By Hölder's inequality, we know that every metric measure space supporting a p-Poincaré inequality for some $1 \le p < \infty$ must necessarily support an ∞-Poincaré inequality. The converse need not hold true, as demonstrated in [18].

The following geometric characterization of ∞-Poincaré inequality was established in [15].

Theorem 2.10 ([15, Theorem 3.1]) *Let (X, d, μ) be a complete metric measure space with μ be doubling. Then the following are equivalent:*

(1) *X supports an ∞-Poincaré inequality.*

(2) *There exists a constant $C \geq 1$ such that if $u \in N^{1,\infty}(X)$ with an ∞-weak upper gradient $g \in L^\infty(X)$, then u is $C\|g\|_{L^\infty(X)}$-Lipschitz continuous on X.*

(3) *There is a constant $C \geq 1$ such that whenever $N \subset X$ with $\mu(N) = 0$ and $x, y \in X$ with $x \neq y$, then there is a rectifiable curve γ with end points x, y such that $\ell(\gamma) \leq Cd(x, y)$ and $\mathcal{H}^1(\gamma^{-1}(N)) = 0$*

Criteria (1) and (3) above immediately imply that X is connected. The following argument shows that Criterion (2) also implies the connectivity of X. Indeed, if X is not connected, then there are two non-empty open subsets $U, V \subset X$ such that $U \cup V = X$ and $U \cap V$ is empty. We set $u = \chi_U$, and note that then $u \in N^{1,\infty}(X)$ with $g_u \equiv 0$. Observe that u is not 0-Lipschitz continuous as u is not constant on X, violating Criterion (2) above. Note that in Criterion (2), if we remove the requirement that u is $C\|g\|_{L^\infty(X)}$-Lipschitz, then we need to add the requirement that X is connected.

We end this section with a technical lemma that will be needed in Section 3, showing a locality property of minimal ∞-weak upper gradients of functions in $N^{1,\infty}(X)$.

Lemma 2.11 *Let X be a metric measure space and E a measurable subset of X. Suppose that $u, v \in N^{1,\infty}(X)$ are such that $u = v$ on E. Then $g_u = g_v$ almost everywhere on E.*

Proof. By Lemma 2.3, the minimal gradients g_u and g_v are assumed to be upper gradients of u and v respectively. It can be shown as in [6, Theorem 2.18] that $g_1 = g_u + g_v \cdot \chi_E$ and $g_2 = g_u \cdot \chi_{X \setminus E} + g_v$ are upper gradients of u. Then by [29, Lemma 4.1] we have that the pointwise minimum $g = \inf\{g_1, g_2\}$ is an ∞-weak upper gradient of u, and it is clear that $g = g_u \cdot \chi_{X \setminus E} + g_v \cdot \chi_E$. By Lemma 2.3 we can modify g in a set of zero measure to obtain an upper gradient of u. In the same way, a modification of $g_v \cdot \chi_{X \setminus E} + g_u \cdot \chi_E$ is an upper gradient of v. By minimality, we obtain that $g_u = g_v$ a.e. on E. $\square$

3 Existence of ∞-harmonic functions

In this section we show the existence of an ∞-harmonic function on a domain $\Omega \subset X$ with prescribed Lipschitz boundary data. To do so, we solve a variational (minimization) problem corresponding to each exponent $p > 1$ and then let $p \to \infty$ to obtain the solution. A similar technique was employed in [24] where the variational problem was to minimize the L^p-energy and obtain a p-harmonic function for each finite p; however, without a p-Poincaré inequality for some finite value of p, we have no control over the behavior of p-harmonic functions, and hence the variational problem we consider is different.

Standing Assumptions: Throughout this section we assume that (X, d, μ) is a complete metric measure space with μ doubling and supporting an ∞-Poincaré inequality. We fix a bounded domain $\Omega \subset X$ and we assume that $\mu(X \setminus \Omega) > 0$ in order to avoid trivial statements.

As a tool for the proof of the main theorem of this section, Theorem 3.3, we need the following notions.

Definition 3.1 *Given $L > 0$, let $N^{1,\infty}_L(X)$ be the collection of all functions u in $N^{1,\infty}(X)$ that have an upper gradient g with $\|g\|_{L^\infty(X)} \le L$. For $u \in N^{1,\infty}_L(X)$ we set $D_L(u)$ to be the collection of all upper gradients g of u such that $\|g\|_{L^\infty(X)} \le L$.*

Now consider a function $f : X \to \mathbb{R}$ with $f \in N^{1,\infty}(X)$. Our goal in this section is to establish the existence of an ∞-harmonic function u in Ω with boundary data f. Note that, since X is complete and supports an ∞-Poincaré inequality, by Theorem 2.10 we know that every function in $N^{1,\infty}(X)$ is in fact Lipschitz continuous on X, so f is Lipschitz. Let $L > 0$ such that $D_L(f)$ is non-empty.

Definition 3.2 *Fix $1 < p < \infty$. For $u \in N^{1,\infty}_L(X)$ we set*

$$I^p_L(u) := \int_\Omega g_u^p \, d\mu = \inf \left\{ \int_\Omega g^p \, d\mu \, : \, g \in D_L(u) \right\},$$

and let

$$J^p_f = \inf \left\{ I^p_L(u) \, : \, u \in N^{1,\infty}_L(X);\, u = f \text{ on } X \setminus \Omega \right\}. \tag{3}$$

Theorem 3.3 *Let (X, d, μ) be a complete metric measure space with μ doubling and supporting an ∞-Poincaré inequality, and let $\Omega \subset X$ be a bounded domain such that $X \setminus \Omega$ has positive measure. Given a Lipschitz function $f : X \to \mathbb{R}$, there is a Lipschitz function $\varphi : \overline{\Omega} \to \mathbb{R}$ such that $\varphi = f$ on $\partial\Omega$ and φ is ∞-harmonic in Ω.*

If $f : \partial\Omega \to \mathbb{R}$ is an L-Lipschitz function, then using the McShane extension theorem [28] (see Section 1 of this paper), we can extend f to a bounded Lipschitz function defined on X. Hence in the above theorem it suffices to prescribe f only on $\partial\Omega$. The remainder of this section is devoted to the proof of this theorem. The proof will be divided into different steps:

Step 1. Fix $L > 0$ such that f is L-Lipschitz on X, and note that the constant function $g = L$ is an upper gradient of f.

For every $1 < p < \infty$, we will show that there is a Lipschitz function u_p on X, which is a solution of the variational problem J^p_f defined in (3), and such that $u_p = f$ on $X \setminus \Omega$. Recall that, by Theorem 2.10, there is a constant $C > 0$ such that every function in $N_L^{1,\infty}(X)$ is in fact CL-Lipschitz on X.

Note that $J_f^p \le I_L^p(f) \le L^p\mu(\Omega) < \infty$, and hence we can find a sequence $\{u_k\}_k \subset N_L^{1,\infty}(X)$ such that $u_k = f$ on $X \setminus \Omega$ and $\lim_k I_L^p(u_k) = J_f^p$. Since each u_k is CL-Lipschitz, the family $\{u_k\}_k$ is equicontinuous on X, and since $u_k = f$ on $X \setminus \Omega$ with Ω bounded, it follows that the family is also equibounded on X. Thus an invocation of the Arzela-Ascoli theorem leads us to conclude that, passing to a subsequence if necessary, there is a CL-Lipschitz function u_p on X such that $\{u_k\}_k \to u_p$ uniformly on X.

Lemma 3.4 *For each $1 < p < \infty$ we have that $u_p \in N_L^{1,\infty}(X)$, $u_p = f$ on $X \setminus \Omega$, and*

$$J_f^p = I_L^p(u_p) = \int_\Omega (g_{u_p})^p \, d\mu. \tag{4}$$

Proof. Since $\{u_k\}_k \to u_p$ uniformly on X, we only need to consider upper gradients of u_p now. By passing to a subsequence if needed, for each k we can find an upper gradient g_k of u_k such that $g_k \le L$ almost everywhere on X and

$$\int_\Omega g_k^p \, d\mu \le J_f^p + 1/k.$$

Fix a bounded domain Ω_0 in X such that $\Omega \Subset \Omega_0$. Thus $\{g_k\}_k$ is a bounded sequence in $L^p(\Omega_0)$. By the reflexivity of $L^p(\Omega_0)$, taking a further subsequence we may assume that $\{g_k\}_k$ is weakly convergent in $L^p(\Omega_0)$ to a non-negative Borel function $g_p \in L^p(\Omega_0)$. By Mazur's lemma, there is a convex combination subsequence $\{h_k\}_k$ (with $h_k = \sum_{j=k}^{N(k)} \lambda_{k,j} g_k$) such that $\{h_k\}_k \to g_p$ both in $L^p(\Omega_0)$ and pointwise outside a set $E \subset \Omega_0$ with $\mu(E) = 0$. From the results of [25] we know that g_p is a p-weak upper gradient of u_p on Ω_0. Note that g_p is defined only on Ω_0. On the other hand, since $u_k = f$ on $\Omega_0 \setminus \Omega$, the extension of each u_k by f to $X \setminus \Omega_0$ is also in $N^{1,\infty}(X)$ with the extension of g_p by L to $X \setminus \Omega_0$ a p-weak upper gradient of u_p on X. See Lemma 2.8. Because each $g_{u_k} \le L$ almost everywhere in X, we have that $g_p \le L$ on $X \setminus (E \cup \bigcup_k E_k)$, where each $E_k = \{g_k > L\}$; and note that by assumption on g_k, we have $\mu(E_k) = 0$. However, we do not know that g_p is an *upper gradient* of u_p. Thus we need to modify g_p suitably as follows.

Setting $F = E \cup \bigcup_k E_k$, we have $\mu(F) = 0$. Let Γ_F^+ denote the collection of all non-constant rectifiable (arc-length parametrized) curves γ in X such that $\mathcal{H}^1(\gamma^{-1}(F)) > 0$. Then, by considering $\rho = \infty \cdot \chi_F$ in Lemma 2.2, we obtain that $\mathrm{Mod}_\infty(\Gamma_F^+) = 0$. For rectifiable non-constant curves γ in X that do *not* belong to Γ_F^+ we know that $\{h_k \circ \gamma\}_k \to g_p \circ \gamma$ $\mathcal{H}^1$-a.e. on the domain of γ, and that almost everywhere there we also have each $h_k \le L$ and $g_p \le L$. Therefore by the Lebesgue dominated convergence theorem,

$$\lim_k \int_\gamma h_k \, ds = \int_\gamma g_p \, ds.$$

Denoting the endpoints of γ by x and y, and noting that the convex combination sequence $v_k = \sum_{j=k}^{N(k)} \lambda_{k,j} u_k$, with $N(k), \lambda_{k,j}$ as in the choice of h_k, converges uniformly to u_p as well on X, we have that

$$|u_p(x) - u_p(y)| = \lim_k |v_k(x) - v_k(y)| \le \lim_k \int_\gamma h_k \, ds = \int_\gamma g_p \, ds.$$

Therefore g_p is an ∞-weak upper gradient of u_p (this is stronger than saying that g_p is a p-weak upper gradient of u_p), with the upper gradient inequality being satisfied for all non-constant rectifiable curves in X that do not belong to Γ_F^+. Therefore the function $\widehat{g_p} := g_p + \infty\chi_F$ is an upper gradient of u_p on X such that $\widehat{g_p} \le L$ on $X \setminus F$. This allows us to conclude that $u_p \in N_L^{1,\infty}(X)$, and by construction we also have that $u_p = f$ on $X \setminus \Omega$. This also means that $I_L^p(u_p) \ge J_f^p$.

Finally, since $h_k \to \widehat{g_p}$ in $L^p(\Omega)$ we have that

$$\lim_k \int_\Omega h_k^p \, d\mu = \int_\Omega \widehat{g_p}^p \, d\mu.$$

By the lower continuity of L^p-norms, we deduce that

$$J_f^p \le I_L^p(u_p) \le \int_\Omega \widehat{g_p}^p \, d\mu \le \lim_k \int_\Omega g_k^p \, d\mu \le J_f^p.$$

Suppose now that $g \in D_L(u)$. Then by the lattice property of ∞-weak upper gradients (see [29, Lemma 4.1]) we have that $\min\{g_p, g\}$ is an ∞-weak upper gradient of u_p. Hence by the minimality of $I_L^p(u_p)$ we must have $g_p \le g$ almost everywhere in Ω, that is, $g_p = g_{u_p}$. Since you removed the old Lemma 3.2 and modified the statement of this lemma, the above argument is essential to put in here.

This completes the proof of the lemma. $\square$

Now let U be a subdomain of Ω, and consider the analogous variational problem on U with boundary data u_p. For $u \in N_L^{1,\infty}(X)$ we set

$$I_{L,U}^p(u) := \int_U g_u^p \, d\mu = \inf\left\{ \int_U g^p \, d\mu \, : \, g \in D_L(u) \right\},$$

and for functions $w \in N^{1,\infty}(X)$, we set

$$J_{w,U}^p := \inf\left\{ \int_U g_u^p \, d\mu \, : \, u \in N_L^{1,\infty}(X);\ u = w \text{ on } \partial U \right\}. \tag{5}$$

The next Lemma shows that the function u_p obtained above solves the minimization problem (5).

Lemma 3.5 *Let $1 < p < \infty$. Let U be a subdomain of Ω and let $v \in N_L^{1,\infty}(X)$ such that $v = u_p$ on ∂U. Then*

$$\int_U (g_{u_p})^p \, d\mu \leq \int_U g_v^p \, d\mu.$$

Proof. Consider the Lipschitz function $w = v \cdot \chi_U + u_p \cdot \chi_{X \setminus U}$. By Lemma 2.11, we have that $g_w = g_v$ almost everywhere on U and $g_w = g_{u_p}$ almost everywhere on $X \setminus U$. In particular $w \in N_L^{1,\infty}(X)$, and since $w = f$ on $X \setminus \Omega$, we obtain that:

$$\int_\Omega (g_{u_p})^p \, d\mu = J_f^p \leq I_L^p(w) \leq \int_\Omega g_w^p \, d\mu = \int_U g_v^p \, d\mu + \int_{\Omega \setminus U} (g_{u_p})^p \, d\mu.$$

Then the conclusion follows. □

Step 2: In this step we show that the function u_p obtained in Step 1 is unique and satisfies the comparison property. We start with the following lemma, which shows a strong locality property for functions in $N^{1,\infty}(X)$.

Lemma 3.6 *Let $u \in N^{1,\infty}(X)$ and suppose that $g \in L^\infty(X)$ is an upper gradient of u such that $g = 0$ almost everywhere in Ω. Then u is constant on Ω.*

Proof. Suppose that $g = 0$ almost everywhere in Ω, and let $E = \{x \in \Omega : g(x) > 0\}$. Then $\mu(E) = 0$. Since X is complete and supports an ∞-Poincaré inequality, by Theorem 2.10 we know that there is a constant $C \geq 1$ such that, for each $x \in \Omega$, with $r > 0$ such that $B(x, 2Cr) \subset \Omega$, and for each $y \in B(x,r)$ there is a C-quasiconvex curve γ connecting x to y such that $\mathcal{H}^1(\gamma^{-1}(E)) = 0$. Since g is an upper gradient of u, it follows that for all $y \in B(x,r)$,

$$|u(x) - u(y)| \leq \int_\gamma g \, ds = 0,$$

that is, $u(y) = u(x)$. Thus we conclude that u is locally constant on Ω, and hence by the connectivity of Ω we also have that u is constant on Ω. □

Next we show uniqueness.

Lemma 3.7 *Let $1 < p < \infty$. If v_p is another minimizer of J_f^p, then $v_p = u_p$.*

Proof. The proof of this follows exactly as in [10, Theorem 7.14] (see [6, Theorem 7.2] for a more detailed proof, considering the obstacle $\psi = -\infty$ there), upon noticing that $D_L(u)$ is a convex subset of $L^p(X)$ (since Ω is bounded, we may without loss of generality assume that $\mu(X) < \infty$), and by the proof of Lemma 3.4, $D_L(u)$ is closed in $L^p(X)$ as well. Strictly speaking, the proofs referred to above show that $v_p - u_p$ has an upper gradient that is zero almost everywhere in Ω. Now invoking Lemma 3.6 we obtain the desired result. $\square$

The next lemma yields the desired comparison theorem for functions u_p.

Lemma 3.8 *Let $1 < p < \infty$. Let f, F be two bounded functions in $N^{1,\infty}_L(X)$ such that $f \le F$ on $X \setminus \Omega$, and let u_p, U_p be the two respective minimizers of J^p_f and J^p_F. Then $u_p \le U_p$ on Ω.*

Proof. Since both u_p and U_p are Lipschitz continuous on X, and since $u_p = f \le F = U_p$ on $X \setminus \Omega$, it follows that $W := \{x \in X : u_p(x) > U_p(x)\}$ is an *open* subset of Ω with $u_p = U_p$ on ∂W. Suppose that W is non-empty (if W is empty, then the claim of the lemma follows). Then $u_p = U_p$ on ∂W and hence has a common L-Lipschitz extension Ψ to $X \setminus W$. It follows from the local nature of the L^p-norm that both u_p and U_p solve the minimization problem J^p_Ψ on W, and hence by Lemma 3.7 we must have $u_p = U_p$ in W, which contradicts the choice of W. Thus W must be empty. This concludes the proof of the lemma. $\square$

Step 3. In this step we fix a monotone increasing sequence $\{p_k\}_k$ with $1 < p_k < \infty$ and $\{p_k\}_k \to \infty$, and for each k we consider the function u_{p_k} constructed in Step 1. Note that $\{u_{p_k}\}$ is an equicontinuous and equibounded sequence of CL-Lipschitz functions on X. So, by passing to a subsequence if necessary, and noting that each $u_{p_k} = f$ in $X \setminus \Omega$ with $\overline{\Omega}$ compact, by the Arzela-Ascoli theorem we can assume that $\{u_{p_k}\}$ converges uniformly on X to a Lipschitz function φ on X, and it is clear that $\varphi = f$ on $X \setminus \Omega$. We next see that this limit function φ is ∞-harmonic in Ω.

Lemma 3.9 *The function φ is ∞-harmonic in Ω.*

Proof. For each $k \in \mathbb{N}$, we will denote for simplicity by g_k the minimal ∞-weak upper gradient $g_{u_{p_k}}$ of u_{p_k}. Now for each fixed $k_0 \in \mathbb{N}$ we have that

$$\int_\Omega g_k^{p_{k_0}} \, d\mu \le L^{p_{k_0}} \mu(\Omega),$$

and so $\{g_k\}_{k \ge k_0}$ forms a bounded sequence in $L^{p_{k_0}}(\Omega)$. An appeal to reflexivity of $L^{p_{k_0}}(\Omega)$ and to Mazur's lemma gives us a convex combination subsequence of the sequence $\{g_k\}_{k \ge k_0}$ that converges both in $L^{p_{k_0}}(\Omega)$ and pointwise almost everywhere in Ω (and hence in X) to some non-negative Borel function ρ_{k_0}. Since each $g_k \le L$ almost everywhere in X, by a repeat of the proof of Lemma 3.4 we see that a modification of ρ_{k_0} on a set of measure zero gives an upper gradient of φ and that $\rho_{k_0} \le L$ almost everywhere in X.

In order to check that φ is ∞-harmonic on Ω, consider $v \in N^{1,\infty}(\Omega)$ such that $v = f$ on $X \setminus \Omega$ and let g_v be its minimal ∞-weak upper gradient. If $\|g_v\|_{L^\infty(\Omega)} > L$ then as $\|\rho_{k_0}\|_{L^\infty(\Omega)} \le L$, we have the desired comparison (2). Therefore, without loss of generality, we assume that $g_v \le L$ almost everywhere in Ω. Since $v = f$ on $X \setminus \Omega$, we have by the pasting lemma [6, Theorem 2.18] together with the lattice property that the extension of g_v by L to $X \setminus \Omega$ is an ∞-weak upper gradient of v. Thus we have $g_v \le L$ almost everywhere in X. That is, $g_v \in D_L(v)$.

For each $k \in \mathbb{N}$ we know from Lemma 3.4 that

$$I_L^{p_k}(u_{p_k}) = \int_\Omega g_k^{p_k}\, d\mu \le \int_\Omega g_v^{p_k}\, d\mu.$$

Therefore, using Hölder's inequality, for each $k_0 \in \mathbb{N}$ and each $k \ge k_0$, we have that

$$\left(\fint_\Omega g_k^{p_{k_0}}\, d\mu\right)^{1/p_{k_0}} \le \left(\fint_\Omega g_k^{p_k}\, d\mu\right)^{1/p_k} \le \left(\fint_\Omega g_v^{p_k}\, d\mu\right)^{1/p_k} \le \|g_v\|_{L^\infty(\Omega)}.$$

As pointed out above, $\rho_{k_0} \le L$ almost everywhere in X. An argument analogous to the one given in the proof of Lemma 3.4 also tells us that ρ_{k_0} is an ∞-weak upper gradient of φ. Therefore $g_\varphi \le \rho_{k_0}$ almost everywhere in X. Since ρ_{k_0} is a weak limit of $\{g_{p_k}\}_{k \ge k_0}$ in $L^{p_{k_0}}(\Omega)$, it follows by letting $k \to \infty$ that

$$\left(\fint_\Omega g_\varphi^{p_{k_0}}\, d\mu\right)^{1/p_{k_0}} \le \left(\fint_\Omega \rho_{k_0}^{p_{k_0}}\, d\mu\right)^{1/p_{k_0}} \le \|g_v\|_{L^\infty(\Omega)}.$$

Now letting $k_0 \to \infty$ we obtain

$$\|g_\varphi\|_{L^\infty(\Omega)} \le \|g_v\|_{L^\infty(\Omega)}. \tag{6}$$

We now need to prove the above inequality for every open subset $V \subset \Omega$ rather than just Ω, and for every $v \in N^{1,\infty}(V)$ such that $v = \varphi$ on $X \setminus V$. To do so, consider first a connected component U of V. Note that, because of the quasiconvexity of X, each connected component of V is an open set. Furthermore, since Ω is connected and $U \subset \Omega$, it follows that ∂U is non-empty and we have $v = \varphi$ on ∂U. Thus the extension of v by φ to $X \setminus U$ is a test function for checking ∞-harmonicity of φ in U. Now for each $k \in \mathbb{N}$ consider the problem of minimizing the functional $I_{L,U}^{p_k}(\cdot)$ considered in (5) over all $u \in N_L^{1,\infty}(X)$ for which $u = \varphi$ on ∂U. As in Lemma 3.4, for each $k \in \mathbb{N}$ we obtain a minimizing function $w_{p_k} \in N_L^{1,\infty}(X)$ such that $J_{\varphi,U}^{p_k} = I_{L,U}^{p_k}(w_{p_k})$. See (5) for the definition of $J_{\varphi,U}^{p_k}$. As before, $\{w_{p_k}\}_k$ is an equicontinuous and equibounded sequence of Lipschitz functions on X. Then, passing to a subsequence we may assume that $\{w_{p_k}\}_k$ converges uniformly on X to some Lipschitz function ψ (in the same manner that we have obtained φ). Then as in (6) we have that, for every $u \in N^{1,\infty}(U)$ such that $u = \varphi$ on $X \setminus U$,

$$\|g_\psi\|_{L^\infty(U)} \leq \|g_u\|_{L^\infty(U)}.$$

In particular,

$$\|g_\psi\|_{L^\infty(U)} \leq \|g_v\|_{L^\infty(U)}.$$

Since $\{u_{p_k}\}_k$ converges uniformly to φ in X, for each $\varepsilon > 0$ there is some $k_\varepsilon \in \mathbb{N}$ such that whenever $k \in \mathbb{N}$ with $k \geq k_\varepsilon$,

$$w_{p_k} - \varepsilon = \varphi - \varepsilon < u_{p_k} < \varphi + \varepsilon = w_{p_k} + \varepsilon \text{ on } X \setminus U.$$

From Lemma 3.5 we know that u_{p_k} is a minimizer $J^{p_k}_{u_{p_k},U}$. Now by Lemma 3.8, applied to the pair of functions $w_{p_k} - \varepsilon$ and u_{p_k} on U, and again to the pair of functions u_{p_k} and $w_{p_k} + \varepsilon$ on U, we get that

$$w_{p_k} - \varepsilon \leq u_{p_k} \leq w_{p_k} + \varepsilon \text{ on } U.$$

Thus, letting $k \to \infty$, we obtain that $\psi - \varepsilon \leq \varphi \leq \psi + \varepsilon$ on V whenever $\varepsilon > 0$, that is, $\psi = \varphi$ on U. Thus from Lemma 2.11 we have that $g_\psi = g_\varphi$ almost everywhere on U. Then

$$\|g_\varphi\|_{L^\infty(U)} = \|g_\psi\|_{L^\infty(U)} \leq \|g_v\|_{L^\infty(U)} \leq \|g_v\|_{L^\infty(V)}.$$

To complete the proof, note that, since X is complete and μ doubling, we have that X is a *proper* metric space, that is, every closed ball in X is compact (see, e.g. pg. 102 in [21]). In particular X is separable, and the open set V has at most a countable number of connected components. Then we obtain that

$$\|g_\varphi\|_{L^\infty(V)} \leq \|g_v\|_{L^\infty(V)},$$

as required. □

The above three steps together complete the proof of Theorem 3.3. In the next section we consider the relationships between ∞-harmonic functions and AMLEs.

4 Coincidence of ∞-harmonicity and AMLEs under the assumption of ∞-weak Fubini property

Recall the notion of AMLEs from Definition 2.7. In this section we compare the notion of ∞-harmonicity and the notion of AMLE. We show that if X supports an ∞-weak Fubini property, then the two notions coincide.

In [24] it was shown that if the metric measure space supports a p-Poincaré inequality for some finite $p \geq 1$ and satisfies a notion of weak Fubini property associated with the index p, then a function is an AMLE if and only if it is ∞-harmonic. In our paper we only require X to support an ∞-weak Fubini property (see below). Note that ∞-weak Fubini property implies that X supports an ∞-Poincaré inequality. However, the support of a weak Fubini property as in [24] *does not* imply the support of a p-Poincaré inequality, but *does* imply the support of ∞-weak Fubini property, which in turn implies the support of an ∞-Poincaré inequality. As described in Section 2, there are metric measure spaces equipped with a doubling measure and supporting an ∞-Poincaré inequality, but supporting no p-Poincaré inequality, $1 \leq p < \infty$.

Definition 4.1 We say that the metric measure space (X, d, μ) satisfies an ∞-*weak Fubini property* if there exist constants $C > 0$ and $\tau_0 > 0$ such that, for every $0 < \tau < \tau_0$ and for every pair of balls B_1, B_2 in X with $\operatorname{dist}(B_1, B_2) > \tau \cdot \max\{\operatorname{diam}(B_1), \operatorname{diam}(B_2)\}$, we have that

$$\operatorname{Mod}_\infty(\Gamma(B_1, B_2, \tau)) > 0,$$

where $\Gamma(B_1, B_2, \tau)$ denotes the family of all paths γ in X from B_1 to B_2, with length $\ell(\gamma) \leq \operatorname{dist}(B_1, B_2) + C\tau$.

The next characterization of ∞-weak Fubini property will be useful to us. First we introduce the following notion. Given a subset N of a metric measure space X, we say that a curve γ is *transversal to* N if $\mathcal{H}^1(\gamma^{-1}(N)) = 0$. The terminology of transversality is from [8] and [9].

Proposition 4.2 *Let (X, d, μ) be a complete metric measure space with μ be doubling. Then X satisfies an ∞-weak Fubini property if and only if for every set $N \subset X$ with $\mu(N) = 0$ and every $\varepsilon > 0$, for each pair of distinct points $x, y \in X$, there is a rectifiable curve γ transversal to N, with end points x, y and such that $\ell(\gamma) \leq d(x, y) + \varepsilon$. Moreover, if X satisfies an ∞-weak Fubini property, then X supports an ∞-Poincaré inequality.*

Proof. Note first that the support of ∞-Poincaré inequality is a consequence of ∞-weak Fubini property, and this can be seen by following the same proof of $(b) \Rightarrow (f)$ given in [15, Theorem 3.1.].

Suppose first that for every set $N \subset X$ with $\mu(N) = 0$ and every $\varepsilon > 0$, for each pair of distinct points $x, y \in X$, there is a transversal rectifiable curve γ with end points x, y such that

$$\ell(\gamma) \leq d(x, y) + \varepsilon.$$

Let B_1, B_2 satisfy the hypotheses in the definition of ∞-weak Fubini property with $\tau = \varepsilon$. If, with the choice of $C = 2$, we have $\operatorname{Mod}_\infty \Gamma(B_1, B_2, \varepsilon) = 0$, then there is a non-negative

Borel measurable function ρ such that $\rho = 0$ μ-a.e. in X and for all $\gamma \in \Gamma(B_1, B_2, \varepsilon)$ we have $\int_\gamma \rho\, ds = \infty$ (see 2.2). Let $N = \{x \in X : \rho(x) > 0\}$. We choose $x_1 \in B_1$ and $x_2 \in B_2$ such that

$$d(x_1, x_2) \leq \operatorname{dist}(B_1, B_2) + \varepsilon.$$

Then by assumption of ρ we have $\mu(N) = 0$ and so there is a transversal curve γ_0 connecting x_1 and x_2 such that $\ell(\gamma_0) \leq d(x_1, x_2) + \varepsilon$. But then we have $\int_{\gamma_0} \rho\, ds = 0 < \infty$, and $\ell(\gamma_0) \leq \operatorname{dist}(B_1, B_2) + 2\varepsilon$, which means that $\gamma_0 \in \Gamma(B_1, B_2, \varepsilon)$, contradicting the choice of ρ. Thus we must have $\operatorname{Mod}_\infty(\Gamma(B_1, B_2, \varepsilon)) > 0$, that is, an ∞-weak Fubini property holds.

Conversely, suppose X satisfies an ∞-weak Fubini property. Let $N \subset X$ with $\mu(N) = 0$, $\varepsilon > 0$, and $x, y \in X$ be two distinct points. Choose $\varepsilon > 0$ such that $\tau < \min\{\varepsilon, \tau_0, d(x, y)\}/(10C)$. Let B_1 be the ball centered at x with radius τ and B_2 be the ball centered at y with radius τ. Then B_1, B_2 satisfy the hypotheses in the definition of ∞-weak Fubini property, and so $\operatorname{Mod}_\infty \Gamma(B_1, B_2, \tau) > 0$. Thus we can find $x_\tau \in B_1$, $y_\tau \in B_2$ and a transversal rectifiable curve γ_τ with end points x_τ, y_τ such that

$$\ell(\gamma_\tau) \leq \operatorname{dist}(B_1, B_2) + C\tau.$$

By choosing τ to be small enough, we can ensure that $\ell(\gamma_\tau) \leq d(x, y) + \frac{\varepsilon}{2}$. Note that $d(x, x_\tau) < \tau$ and $d(y, y_\tau) < \tau$, and so by the ∞-Poincaré inequality (a consequence of the ∞-weak Fubini property as noted above), there exist curves β_τ connecting x to x_τ and α_τ connecting y to y_τ such that $\ell(\beta_\tau) < C\tau$ and $\ell(\alpha_\tau) < C\tau$, with $\mathcal{H}^1(\beta_\tau^{-1}(N) \cup \alpha_\tau^{-1}(N)) = 0$. The concatenation $\gamma = \alpha_\tau * \gamma_\tau * \beta_\tau$ is a transversal rectifiable curve connecting x to y with

$$\ell(\gamma) \leq d(x, y) + \tfrac{\varepsilon}{2} + 2C\tau.$$

By choosing τ small enough so that we also have $2C\tau < \varepsilon/2$, we obtain the desired result. $\square$

Now, we define a geodesic distance on the metric measure space by using the notion of transversality for a given null set. This distance has been used in [8] and [9] for example.

Definition 4.3 Let X be a metric measure space. For each null set N in X we define

$$\widehat{d}_N(x, y) = \inf\{\ell(\gamma) : \gamma \text{ is a curve transversal to } N \text{ and connecting } x \text{ to } y\}.$$

It is easily seen that for null sets $N \subset X$, $\widehat{d}_N$ is an *extended* metric on X, in the sense that $\widehat{d}_N$ can possibly take infinite values (since the infimum of the empty set is ∞). Furthermore, if X supports an ∞-Poincaré inequality, then by Theorem 2.10 there exists $C \geq 1$ such that for each null set $N \subset X$,

$$d(x, y) \leq \widehat{d}_N(x, y) \leq C d(x, y). \tag{7}$$

The next result shows that, if a metric measure space X satisfies an ∞-Poincaré inequality, then there is a bi-Lipschitz equivalent length metric on X that makes X satisfy the ∞-weak Fubini property.

For $x, y \in X$ we set

$$\widehat{d}(x,y) = \sup\{\widehat{d}_N(x,y) \,:\, N \text{ null set in } X\}. \tag{8}$$

Proposition 4.4 *Let (X, d, μ) be a complete metric measure space with μ doubling.*

(a) *If $\widehat{d}$ is finite on X, then it is a length metric on X and $(X, \widehat{d}, \mu)$ satisfies an ∞-weak Fubini property.*

(b) *If X supports an ∞-Poincaré inequality, then*

 (i) *There exists $C \geq 1$ such that*

$$d(x,y) \leq \widehat{d}(x,y) \leq C d(x,y)$$

 whenever $x, y \in X$. Furthermore, $(X, \widehat{d}, \mu)$ satisfies an ∞-weak Fubini property.

 (ii) *For every domain Ω in X, a function u on Ω is ∞-harmonic in Ω with respect to $\widehat{d}$ and μ if and only if it is ∞-harmonic with respect to d and μ.*

(c) *(X, d, μ) satisfies an ∞-weak Fubini property if and only if $d = \widehat{d}$.*

Proof. To prove (a), suppose that $\widehat{d}$ is finite-valued on $X \times X$. Denoting by ℓ and $\widehat{\ell}$ the corresponding length functionals associated respectively to d and $\widehat{d}$, for every curve γ in X we have that

$$\ell(\gamma) = \widehat{\ell}(\gamma).$$

Indeed, since $d \leq \widehat{d}$ it is clear that $\ell(\gamma) \leq \widehat{\ell}(\gamma)$. On the other hand, given $\varepsilon > 0$, choose first a subdivision $P = \{t_0 < t_1 < \cdots < t_n\}$ of the interval where γ is defined so that

$$\widehat{\ell}(\gamma) - \varepsilon \leq \sum_{i=1}^{n} \widehat{d}(\gamma(t_{i-1}), \gamma(t_i)).$$

For each $i = 1, 2, \cdots, n$ choose a null set M_i of X such that

$$\widehat{d}(\gamma(t_{i-1}), \gamma(t_i)) \leq \widehat{d}_{M_i}(\gamma(t_{i-1}), \gamma(t_i)) + \frac{\varepsilon}{n}.$$

Letting $M = \bigcup_{i=1}^{n} M_i$, we have that $\widehat{d}_{M_i} \leq \widehat{d}_M$ for every $i = 1, 2, \cdots, n$. Then

$$\widehat{\ell}(\gamma) - \varepsilon \leq \sum_{i=1}^{n} \widehat{d}_M(\gamma(t_{i-1}), \gamma(t_i)) + \varepsilon \leq \sum_{i=1}^{n} \ell(\gamma_|[t_{i-1}, t_i]) + \varepsilon = \ell(\gamma) + \varepsilon.$$

As a consequence, we obtain that $\widehat{\ell}(\gamma) \le \ell(\gamma)$. Thus $\widehat{d}$ is a length metric on X. Now, by Proposition 4.2, given $x, y \in X$, a null set $N \subset X$ and $\varepsilon > 0$, from the definition of metric $\widehat{d}_N$ we obtain a curve γ transversal to N, joining x and y, and such that

$$\widehat{\ell}(\gamma) = \ell(\gamma) \le \widehat{d}_N(x, y) + \varepsilon \le \widehat{d}(x, y) + \varepsilon,$$

and the rest of the claim of (a) follows.

Claim (b)(i) follows from the discussion preceding (7). We next show (b)(ii). First note that, by the remark given at the beginning of this proof, the arc-length parametrization of every curve γ in X coincides for (X, d) and $(X, \widehat{d})$. Thus, if $\rho : X \to [0, \infty]$ is a Borel function, the path integral $\int_\gamma \rho\, ds$ coincides for (X, d, μ) and $(X, \widehat{d}, \mu)$. This means that given a function u on X, a non-negative Borel measurable function g on X is an upper gradient of u with respect to the metric d if and only if it is an upper gradient of u with respect to the metric $\widehat{d}$. Analogous statements hold for ∞-weak upper gradients of u. In particular, the corresponding Newton-Sobolev spaces coincide: $N^{1,\infty}(X, d, \mu) = N^{1,\infty}(X, \widehat{d}, \mu)$. Now the result follows directly from the definition of ∞-harmonicity. Finally, using Proposition 4.2 again we obtain (c). $\square$

The following example shows that Part (b)(i) of the above proposition is no longer true without the hypothesis of ∞-Poincaré inequality.

Example 4.5 Without ∞-Poincaré inequality $\widehat{d}$ may possibly take infinite values, and in particular it may not be equivalent to d, as shown by the example of the Sierpinski carpet endowed with the corresponding Hausdorff measure. The Sierpinski carpet does not support an ∞-Poincaré inequality and hence cannot satisfy any ∞-weak Fubini property. Since the length metric on this carpet is bi-Lipschitz equivalent to the Euclidean metric, it follows that the above statement holds also when the carpet X is equipped with the length metric. To see that $\widehat{d}$ is not equivalent to d in this case, we consider the set N constructed in Example 2.6. Observe that $\widehat{d}_N(x, y) = \infty$, and so $\widehat{d}$ is not equivalent to d in the carpet.

Lemma 4.6 *Suppose that (X, d, μ) is a complete metric measure space with μ doubling and supporting an ∞-weak Fubini property. Then for each $u \in \mathrm{LIP}^\infty(X) = N^{1,\infty}(X)$,*

$$\mathrm{LIP}(u, X) = \sup_{x \in X} \operatorname{Lip} u(x) = \|\operatorname{Lip} u\|_{L^\infty(X)} = \|g_u\|_{L^\infty(X)}.$$

Furthermore, if $V \subset X$ is a non-empty open set, then for each $u \in N^{1,\infty}(V)$ (noting that such functions are necessarily locally Lipschitz continuous in V),

$$\sup_{x \in V} \operatorname{Lip} u(x) = \|\operatorname{Lip} u\|_{L^\infty(V)} = \|g_u\|_{L^\infty(V)}. \tag{9}$$

Proof. Note that as $\operatorname{Lip} u$ is an upper gradient of u and g_u is the minimal ∞-weak upper gradient of u, we have that $g_u \leq \operatorname{Lip} u$ almost everywhere in X.

Let $u \in \mathrm{LIP}^\infty(X)$, and define $N = \{x \in X : \operatorname{Lip} u(x) > \|\operatorname{Lip} u\|_{L^\infty(X)}\}$. Now, fix $x, y \in X$. Given $\varepsilon > 0$ take γ in X connecting x and y that is transversal to N, parametrized by the arc-length, such that $\ell(\gamma) \leq d(x,y) + \varepsilon$. Then

$$|u(x) - u(y)| \leq \int_0^{\ell(\gamma)} \operatorname{Lip} u(\gamma(t))dt \leq \|\operatorname{Lip} u\|_{L^\infty(X)}\ell(\gamma) \leq \|\operatorname{Lip} u\|_{L^\infty(X)}[d(x,y) + \varepsilon].$$

Now, first let $\varepsilon \to 0$ and then take the supremum over $x, y \in X$ in the above to obtain $\mathrm{LIP}(u, X) \leq \|\operatorname{Lip} u\|_{L^\infty(X)}$.

Replacing the role of $\operatorname{Lip} u$ in the above with g_u and noting that the collection Γ of curves for which the function-upper gradient inequality does not hold has ∞-modulus zero, there must be a set $N \subset X$ with $\mu(N) = 0$ such that for each $\gamma \in \Gamma$ we must have $\mathcal{H}^1(\gamma^{-1}(N)) > 0$, which gives the last equality in the first claim.

Let $V \subset X$ be open and non-empty set, and $u \in N^{1,\infty}(V)$ with $B(x, 2r) \subset V$ Fix $r > 0$ such that $B(x, 2r) \subset V$, and $0 < \varepsilon < r/2$. Let $x \in V$ and $N = \{y \in B(x,r) : g_u(y) > \|g_u\|_{L^\infty(B(x,r))}\}$. Then $\mu(N) = 0$. Note in the above inequality that for each $y \in B(x, r/2)$ there is a rectifiable curve γ with end points x, y such that $\ell(\gamma) \leq d(x,y) + \varepsilon$, γ is transversal to N, and

$$\frac{|u(x) - u(y)|}{d(x,y)} \leq \frac{\ell(\gamma)}{d(x,y)} \fint_{[0,\ell(\gamma)]} g_u \circ \gamma \, ds \leq \frac{\ell(\gamma)}{d(x,y)} \|g_u\|_{L^\infty(B(x,r))}.$$

Note that by the choice of r and ε, $\gamma \subset V$. It follows that (by letting $\varepsilon \to 0$ and then $y \to x$)

$$\operatorname{Lip} u(x) \leq \lim_{r \to 0^+} \|g_u\|_{L^\infty(B(x,r))}.$$

From the previous inequality we also have that whenever $V \subset X$ is a non-empty open set, then

$$\|g_u\|_{L^\infty(V)} \leq \|\operatorname{Lip} u\|_{L^\infty(V)}.$$

On the other hand, for each $\varepsilon > 0$ there exists $z_0 \in V$ such that

$$\|\operatorname{Lip} u\|_{L^\infty(V)} - \varepsilon \leq \operatorname{Lip} u(z_0) \leq \lim_{r \to 0^+} \|g_u\|_{L^\infty(B(z_0,r))} \leq \|g_u\|_{L^\infty(V)}.$$

Therefore it follows that

$$\|g_u\|_{L^\infty(V)} = \|\operatorname{Lip} u\|_{L^\infty(V)},$$

for any non-empty open set $V \subset X$. $\square$

Remark 4.7 A converse of the above lemma also holds. Let (X, d, μ) be a complete metric measure space with μ doubling. Suppose that for each $u \in N^{1,\infty}(X)$ we have

$$\mathrm{LIP}(u, X) = \sup_{x \in X} \mathrm{Lip}\, u(x) = \| \mathrm{Lip}\, u\|_{L^\infty(X)} = \|g_u\|_{L^\infty(X)}. \tag{10}$$

Then X satisfies an ∞-weak Fubini property. To see this, fix a set $N \subset X$ with $\mu(N) = 0$. We consider the metric $\widehat{d}_N$ as in Definition 4.3. By the above hypothesis, it follows from Theorem 2.10 that X supports an ∞-Poincaré inequality, and so again by Theorem 2.10 it follows that there is a constant $C \geq 1$ with $\widehat{d}_N(z, w) \leq C\, d(z, w)$ whenever $z, w \in X$. We fix $y \in X$, $R > 1$, and consider the function

$$u(x) = \min \left\{ R, \inf_\gamma \int_\gamma [1 + \infty \cdot \chi_N]\, ds \right\},$$

where the infimum is over all rectifiable curves γ in X connecting x to y. Note that $u(x) \leq \widehat{d}_N(x, y)$ for each $x \in X$, and so $u \in N^{1,\infty}(X)$. Furthermore, $g = 1 + \infty\chi_N \in L^\infty(X)$ is an upper gradient of u, and so by the hypothesis we have

$$\mathrm{LIP}(u, X) = \|g_u\|_{L^\infty(X)} \leq \|g\|_{L^\infty(X)} = 1,$$

that is, u is 1-Lipschitz continuous on X. It follows that for each $x \in X$ and $\varepsilon > 0$ we can find a curve γ connecting x to y that is transversal to N and satisfying $\ell(\gamma) \leq d(x, y) + \varepsilon$. Therefore, from Proposition 4.2, X satisfies an ∞-weak Fubini property.

Under the ∞-weak Fubini property (which implies the support of ∞-Poincaré inequality), we know that $\mathrm{LIP}^\infty(X) = N^{1,\infty}(X)$, see Theorem 2.10. Hence the property of every $u \in N^{1,\infty}(X)$ satisfying (10) characterizes proper metric measure spaces that support ∞-weak Fubini property. The property (10) is crucial in understanding the connections between AMLEs and ∞-harmonic functions, see for example [24] and [12].

The following lemma is from [24], but we provide a proof here for the reader's convenience. Recall that a metric space is said to be *proper* if every closed ball in that space is compact.

Lemma 4.8 *If X is a proper length space, then whenever $V \subset X$ is a non-empty open set, we have*

$$\mathrm{LIP}(u, V) = \max \left\{ \mathrm{LIP}(u, \partial V), \sup_{z \in V} \mathrm{Lip}\, u(z) \right\}. \tag{11}$$

Proof. Since X is a proper length space, it follows from an application of the Arzela-Ascoli theorem that for each $x, y \in X$ there is a rectifiable curve, called a *geodesic curve*, with end points x, y such that $\ell(\gamma) = d(x, y)$. The fact that

$$\mathrm{LIP}(u, V) \geq \max \left\{ \mathrm{LIP}(u, \partial V), \sup_{z \in V} \mathrm{Lip}\, u(z) \right\}$$

is immediate. So it only remains to prove the reverse inequality. For $x, y \in V$ let γ be a geodesic in X connecting x to y. If γ lies within V, then

$$\frac{|u(x) - u(y)|}{d(x,y)} \le \frac{1}{d(x,y)} \int_\gamma \operatorname{Lip} u \, ds \le \sup_{z \in V} \operatorname{Lip} u(z).$$

If γ intersects ∂V, let $x_0 \in \partial V$ be the first time γ intersects ∂V, and $y_0 \in \partial V$ be the last time γ intersects ∂V; that is, there are $t_s, t_l \in [0, d(x,y)]$ such that $\gamma([0,t_s)) \subset V$, $\gamma(t_s) = x_0$, and $\gamma((t_l, d(x,y)]) \subset V$, $\gamma(t_l) = y_0$. Then

$$\begin{aligned}
|u(x) - u(y)| &\le |u(x) - u(x_0)| + |u(x_0) - u(y_0)| + |u(y_0) - u(y)| \\
&\le [\sup_{z \in V} \operatorname{Lip} u(z)] d(x, x_0) + \operatorname{LIP}(u, \partial V)\, d(x_0, y_0) + [\sup_{z \in V} \operatorname{Lip} u(z)] d(y, y_0) \\
&\le \max\left\{ \operatorname{LIP}(u, \partial V),\, \sup_{z \in V} \operatorname{Lip} u(z) \right\} [d(x, x_0) + d(x_0, y_0) + d(y_0, y)] \\
&\le \max\left\{ \operatorname{LIP}(u, \partial V),\, \sup_{z \in V} \operatorname{Lip} u(z) \right\} d(x, y).
\end{aligned}$$

Combining this with the above inequality yields that for each $x, y \in V$ with $x \neq y$,

$$\frac{|u(x) - u(y)|}{d(x,y)} \le \max\left\{ \operatorname{LIP}(u, \partial V),\, \sup_{z \in V} \operatorname{Lip} u(z) \right\}.$$

Taking the supremum over $x, y \in V$ of the above yields (11). $\square$

The following is the main theorem of this section.

Theorem 4.9 *Let (X, d, μ) be a complete metric measure space with μ a doubling measure supporting an ∞-Poincaré inequality. Let $\Omega \subset X$ be a bounded domain such that $X \setminus \Omega$ has positive measure. Let $f : \partial\Omega \to \mathbb{R}$ be a Lipschitz function, there is exactly one function $u : \overline{\Omega} \to \mathbb{R}$ such that $u = f$ on $\partial\Omega$ and u is ∞-harmonic in Ω. Furthermore, u is an AMLE of f to Ω when X is equipped with the metric $\widehat{d}$ as defined in* (8)*.*

Proof. Recall that the notion of ∞-harmonicity yields the same class of functions under each of the metrics d and $\widehat{d}$, see Proposition 4.4 (b) (ii). By Proposition 4.4 (a) we have that $(X, \widehat{d})$ is a length space, $(X, \widehat{d}, \mu)$ satisfies an ∞-weak Fubini property, and the function $u := \varphi$ given by Theorem 3.3 is ∞-harmonic in Ω for $(X, \widehat{d}, \mu)$. Also, since $(X, \widehat{d})$ is complete and μ doubling, we have that $(X, \widehat{d})$ is a proper metric space.

By Lemma 4.6 and by Lemma 4.8, if $V \subset \Omega$ is a non-empty open set and if $v : \overline{V} \to \mathbb{R}$ is such that $v = u$ on ∂V, then

$$\begin{aligned}
\mathrm{LIP}(u,V) = \max\left\{ \mathrm{LIP}(u,\partial V),\ \sup_{z\in V} \mathrm{Lip}\, u(z)\right\} &= \max\left\{ \mathrm{LIP}(v,\partial V),\ \|g_u\|_{L^\infty(V)}\right\} \\
&\le \max\left\{ \mathrm{LIP}(v,\partial V),\ \|g_v\|_{L^\infty(V)}\right\} \\
&= \max\left\{ \mathrm{LIP}(v,\partial V),\ \sup_{z\in V} \mathrm{Lip}\, v(z)\right\} = \mathrm{LIP}(v,V).
\end{aligned}$$

It follows that u is AMLE in Ω for $(X, \widehat{d})$.

Finally, by [31, Theorem 1.4] AMLEs are unique; hence the uniqueness of u. This completes the proof of the theorem. $\square$

The proof of Theorem 4.9 also shows that, under the ∞-weak Fubini property, every ∞-harmonic function is an AMLE. The converse is also true, as the following shows.

Theorem 4.10 *Let X be a complete metric measure space with the measure μ be a doubling measure satisfying an ∞-weak Fubini property. Let Ω be a bounded domain in X with $\partial\Omega$ non-empty. If $u : \overline{\Omega} \to \mathbb{R}$ is an AMLE in Ω, then u is ∞-harmonic in Ω.*

The proof of this theorem uses a notion called comparison with cones property. We say that a continuous function $u : \overline{\Omega} \to \mathbb{R}$ satisfies the *comparison with cones* in Ω if for all domains $V \subset \Omega$ and all $a \ge 0$, all $b \in \mathbb{R}$ and all $w_0 \in X \setminus V$ both the following two conditions are satisfied:

1. $u(x) \le b + ad(w_0, x)$ for all $x \in V$ whenever $u(x) \le b + ad(w_0, x)$ for all $x \in \partial V$,
2. $u(x) \ge b - ad(w_0, x)$ for all $x \in V$ whenever $u(x) \ge b - ad(w_0, x)$ for all $x \in \partial V$

Proof. In [24, Proposition 4.1] it was shown that if X is a proper length space, then AMLEs on a domain $\Omega \subset X$ satisfy a comparison with cones in Ω. Since X is complete, μ is doubling and X satisfies an ∞-weak Fubini property, we know that X is a proper length space. Therefore we know that the AMLE $u : \overline{\Omega} \to \mathbb{R}$ satisfies the comparison with cones property.

Next, it was shown in [24, Proposition 5.8] that a function that satisfies the comparison with cones property on Ω must be of strong-AMLE class in Ω provided X is a proper length space satisfying a weak Fubini property. The notion of strong-AMLE of [24] agrees with our notion of ∞-harmonicity under our hypotheses on X, see Lemma 4.6 above (more specifically, equation (9)). The proof of [24, Proposition 5.8] given there would work even if their notion of weak Fubini property is replaced with our weaker notion of ∞-weak Fubini property.

Combining the above two paragraphs, we conclude that the AMLE u must be ∞-harmonic in Ω. $\square$

Combining the above two theorems we have a proof of Theorem 1.2.

Remark 4.11 If (X, d, μ) is a complete metric measure space with μ a doubling measure and $\widehat{d} < \infty$ one can also guarantee the existence of an ∞-harmonic function. Indeed, by Proposition 4.4(a), $(X, \widehat{d})$ is a length space and $(X, \widehat{d}, \mu)$ satisfies an ∞-weak Fubini property. By [23] we can always find an AMLE in $(X, \widehat{d})$ and by Theorem 4.10 they are ∞-harmonic with respect to $\widehat{d}$ and therefore with respect to d.

Example 4.12 Example 4.5 also gives a situation where an ∞-harmonic function is not necessarily an AMLE. To construct such a function, consider the Sierpinski Carpet X endowed with its length metric and the corresponding Hausdorff measure. Set $g := \chi_N$, where N is the set given in Example 2.6. We fix the set $E := \{(0, x_2) : (0, x_2) \in X\} = \{0\} \times [0, 1]$ in X, and for points $x = (x_1, x_2)$ in X we define

$$f(x) := \inf_{\gamma} \int_{\gamma} g\, ds,$$

where the infimum is over all rectifiable curves γ in the carpet with one end point at x and the other at E. Note that $f(E) = \{0\}$, but for $x \notin E$ we have $f(x) \geq |x_1| > 0$. Hence f is non-constant. We claim that f is Lipschitz on X. To see this, note that if x, y are two points in X, then

$$|f(x) - f(y)| \leq \int_{\beta} g\, ds \leq \ell(\beta)$$

for every rectifiable curve β in X with end points x, y. Since we are considering the length metric on X, we obtain that f is 1-Lipschitz continuous. On the other hand, g is an upper gradient for f (see [14, Lemma 3.5.]). Its minimal ∞-weak upper gradient g_f then satisfies $g_f \leq g$ almost everywhere on X, and so by the fact that $\mu(N) = 0$, we have $\|g_f\|_{L^\infty(X)} \leq \|g\|_{L^\infty(X)} = 0$, and therefore f is automatically ∞-harmonic in X. However, $\mathrm{LIP}(f, X) > 0$ because f is non-constant. Consider the domain $\Omega = X \setminus E$. Then f is not an AMLE in Ω, since $f = 0$ on $\partial\Omega = E$ and the only AMLE extension is the zero extension.

References

[1] L. Ambrosio, V. Caselles, S. Masnou, J.-M. Morel: Connected components of sets of finite perimeter and applications to image processing. *J. Eur. Math. Soc. (JEMS)* **3** (2001), no. 1, 39–92.

[2] G. Aronsson: Extension of functions satisfying Lipschitz conditions, *Ark. Mat.* **6** (1967), 551–561.

[3] G. Aronsson, M. G. Crandall, P. Juutinen: A tour of the theory of absolutely minimizing functions, *Bull. Amer. Math. Soc. (N.S.)* **41** (2004), 439–505.

[4] F. H. Beatrous, T. Bieske, J. Manfredi: The maximum principle for vector fields. The p-harmonic equation and recent advances in analysis, *Contemp. Math.* **370.** Amer. Math. Soc., Providence, RI (2005) 1–9.

[5] M. Bertalmío, V. Caselles, G. Haro, G. Sapiro: PDE-based image and surface inpainting. *Handbook of mathematical models in computer vision,* (2006) 33–61, Springer, New York.

[6] A. Björn, J. Björn: Nonlinear potential theory on metric spaces, *EMS Tracts Mathematics*, vol. 17. European Math. Soc., Zurich, November 2011.

[7] M. Bourdon, H. Pajot: Quasi-conformal geometry and hyperbolic geometry. (English summary) Rigidity in dynamics and geometry (Cambridge, 2000), 1–17, Springer, Berlin, 2002.

[8] G. De Cecco, G. Palmieri: Distanza intrinseca su una variettá riemanniana di Lipschitz. *Rend. Sere. Mat. Torino*, vol. 46, 2, (1988), 157–170.

[9] G. De Cecco, G. Palmieri: Integral distance on a Lipschitz Riemannian manifold. *Math. Z.*, vol. 207, (1991), 223–243.

[10] J. Cheeger: Differentiability of Lipschitz Functions on metric measure spaces. *Geom. Funct. Anal.* **9** (1999), 428–517.

[11] M. G. Crandall, L. C. Evans, and R. F. Gariepy: Optimal Lipschitz extensions and the infinity Laplacian. *Calc. Var. Partial Differential Equations* **13** (2001), 123–139.

[12] F. Dragoni, J. Manfredi, D. Vittone: Weak Fubini property and infinity harmonic functions in Riemannian and sub-Riemannian manifolds. *Trans. Amer. Math. Soc.* **365** (2013), no. 2, 837–859.

[13] E. Durand-Cartagena, J. A. Jaramillo: Pointwise Lipschitz functions on metric spaces. *J. Math. Anal. Appl.* **363** (2010), 525–548.

[14] E. Durand-Cartagena, J. A. Jaramillo, N. Shanmugalingam: The ∞-Poincaré inequality in metric measure spaces. *Michigan Math. Journal.* **60** (2011), 63–85.

[15] E. Durand-Cartagena, J. A. Jaramillo, N. Shanmugalingam: Geometric characterizations of p-Poincaré inequalities, *Pub. Mat.* **60** (2016), 81–111.

[16] E. Durand-Cartagena, X. Li: *Preservation of bounded geometry under sphericalization and flattening: quasiconvexity and ∞-Poincaré inequality,* preprint in http://cvgmt.sns.it/paper/2571/.

[17] E. Durand-Cartagena, X. Li: *Preservation of p-Poincaré inequality for large p under sphericalization and flattening,* preprint in http://cvgmt.sns.it/paper/2791/.

[18] E. Durand-Cartagena, N. Shanmugalingam, A. Williams: p-Poincaré inequality vs. ∞-Poincaré inequality; some counter-examples. *Math. Z.* **271** (2012) 447–467.

[19] J. Heinonen: Lectures on Analysis on Metric Spaces. Springer (2001).

[20] J. Heinonen, P. Koskela: Quasiconformal maps in metric spaces with controlled geometry. *Acta Math.* **181** (1998), 1–61.

[21] J. Heinonen, P. Koskela, N. Shanmugalingam, J. Tyson: *Sobolev Spaces on Metric Measure Spaces: An Approach Based on Upper Gradients.* Cambridge University Press, New Mathematical Monographs series: February 2015.

[22] R. Jensen: Uniqueness of Lipschitz extensions: minimizing the sup norm of the gradient. *Arch. Rational Mech. Anal.* **123** (1993), no. 1, 51–74.

[23] P. Juutinen: Absolutely minimizing Lipschitz extensions on a metric space. *Ann. Acad. Sci. Fenn. Math.* 27 (2002), no. 1, 57–67.

[24] P. Juutinen, N. Shanmugalingam: Equivalence of AMLE, strong AMLE, and comparison with cones in metric measure spaces. *Math. Nachr.* **279** (2006), no. 9-10, 1083–1098.

[25] S. Kallunki, N. Shanmugalingam: Modulus and continuous capacity. *Ann. Acad. Sci. Fenn. Math.* **26** (2001), 455–464.

[26] C. Le Guyader, L. Guillot: Extrapolation of vector fields using the infinity Laplacian and with applications to image segmentation. *Commun. Math. Sci.* **7** (2009), no. 2, 423–452.

[27] X. Li, N. Shanmugalingam: Preservation of bounded geometry under sphericalization and flattening. *Indiana Univ. Math. J.* **64** (2015), no. 5, 1303–1341.

[28] E. J. McShane: Extension of range of functions. *Bull. Amer. Math. Soc.* **40** (1934), no. 12, 837–842.

[29] L. Malý: Minimal weak upper gradients in Newtonian spaces based on quasi-Banach function lattices. *Ann. Acad. Sci. Fenn. Math.* **38** (2013), no. 2, 727–745.

[30] F. Mèmoli, G. Sapiro, P. Thompson: Brain and surface warping via minimizing Lipschitz extensions. *IMA Preprint Series* **2092** (2006), Institute of Mathematics and its Applications, MN. https://www.ima.umn.edu/sites/default/files/2092.pdf

[31] Y. Peres, O. Schramm, S. Sheffield, D. B. Wilson: Tug-of-war and the infinity Laplacian. *J. Amer. Math. Soc.* **22** (2009), no. 1, 167–210.

[32] N. Shanmugalingam: Newtonian Spaces: An extension of Sobolev spaces to Metric Measure Spaces. *Rev. Mat. Iberoamericana* **16** (2000), 243–279.

Address:

E.D-C.: Departamento de Matemática Aplicada, ETSI Industriales, Universidad Nacional de Educación a Distancia (UNED), 28040 Madrid, Spain.
E-mail: edurand@ind.uned.es

J.J.: Instituto de Matemática Interdisciplinar and Departamento de Análisis Matemático, Facultad de CC. Matemáticas, Universidad Complutense de Madrid, 28040-Madrid, Spain.
E-mail: jaramil@mat.ucm.es

N.S.: Department of Mathematical Sciences, P.O.Box 210025, University of Cincinnati, Cincinnati, OH 45221–0025, U.S.A.
E-mail: shanmun@uc.edu