FROM VISCO TO PERFECT PLASTICITY IN THERMOVISCOELASTIC MATERIALS

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Abstract. We consider a thermodynamically consistent model for thermoviscoplasticity. For the related PDE system, coupling the heat equation for the absolute temperature, the momentum balance with viscosity and inertia for the displacement variable, and the flow rule for the plastic strain, we propose two weak solvability concepts, 'entropic' and 'weak energy' solutions, where the highly nonlinear heat equation is suitably formulated. Accordingly, we prove two existence results by passing to the limit in a carefully devised time discretization scheme.

Furthermore, we study the asymptotic behavior of weak energy solutions as the rate of the external data becomes slower and slower, which amounts to taking the vanishing viscosity and inertia limit of the system. We prove their convergence to a global energetic solution to the Prandtl-Reuss model for perfect plasticity, whose evolution is 'energetically' coupled to that of the (spatially constant) limiting temperature.

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1. Introduction

Over the last decade, the mathematical study of rate-independent systems has received strong impulse. This is undoubtedly due to their ubiquity in several branches of continuum mechanics, see [Mie05, MR15], but also to the manifold challenges posed by the analysis of rate-independent evolution. In particular, its intrinsically nonsmooth (in time) character makes it necessary to resort to suitable weak solvability notions: First and foremost, the concept of *(global) energetic solution*, developed in [MT99, MT04, Mie05], cf. also the notion of *quasistatic evolution*, first introduced for models of crack propagation, cf. e.g. [DMT02, DMFT05a].

Alternative solution concepts for rate-independent models have been subsequently proposed, on the grounds that the *global stability* condition prescribed by the energetic notion fails to accurately describe the behavior of the system at jump times, as soon as the driving energy is nonconvex. Among the various selection criteria of mechanically feasible concepts, let us mention here the *vanishing viscosity* approach, pioneered in [EM06] and subsequently developed in the realm of *abstract* rate-independent systems in [MRS09, MRS12, MRS] and, in parallel, in the context of specific models in crack propagation and damage, cf. e.g. [TZ09, KMZ08, LT11, KRZ13], as well as in plasticity, see e.g. [DDS11, DDS12, BFM12, FS13]. In all of these applications, the evolution of the displacement variable is governed by the elastic equilibrium equation (with no viscosity or inertial terms), which is coupled to the rate-independent flow rule for the internal parameter describing the mechanical phenomenon under consideration. In the 'standard' vanishing-viscosity approach, the viscous term, regularizing the temporal evolution and then sent to zero, is added only to the flow rule.

Recent papers have started to extend the vanishing-viscosity analysis to *coupled* systems, typically for the displacement and the internal variable. In the context of the rate-dependent model, *both* the displacements and the internal variable are subject to viscous dissipation (and possibly to inertia in the momentum balance), and the vanishing-viscosity limit is taken *both* in the momentum balance, and in the flow rule. The very first paper initiating this analysis is [DMS14], obtaining a *perfect plasticity* (rate-independent) system in the limit of dynamic processes. We also quote [Sca16], where this kind of approach was developed in the realm of a model for delamination, as well as [MRS16], tackling the analysis of *abstract, finite dimensional* systems where the viscous terms vanish with different rates.

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The model for small-strain associative elastoplasticity with the Prandtl-Reuss flow rule (without hardening) for the plastic strain, chosen in [DMS14] to pioneer the 'full vanishing-viscosity' approach, has been extensively studied. In fact, the existence theory for perfect plasticity is by now classical, dating back to [Joh76, Suq81, KT83], cf. also [Tem83]. It was revisited in [DMDM06] in the framework of the aforementioned concept of (global) energetic solution to rate-independent system, with the existence result established by passing to the limit in time-incremental minimization problems; a fine study of the flow rule for the plastic strain was also carried out. This variational approach has apparently given new impulse to the analysis of perfect plasticity, extended to the case of heterogeneous materials in [Sol09, FG12, Sol14, Sol15]; we also quote [BMR12] on the vanishing-hardening approximation of the Prandtl-Reuss model.

In [DMS14], first of all an existence result for a dynamic viscoelastoplastic system approximating the perfectly plastic one, featuring viscosity and inertia in the momentum balance, and viscosity in the flow rule for the plastic tensor, has been obtained. Secondly, the authors have analyzed its behavior as the rate of the external data becomes slower and slower: with a suitable rescaling, this amounts to taking the vanishing-viscosity and inertia limit of the system. They have shown that the (unique) solutions to the viscoplastic system converge, up to a subsequence, to a (global) energetic solution of the perfectly plastic system.

In this paper, we aim to **use the model for perfect plasticity as a *case study*** for the vanishing-viscosity analysis of rate-dependent systems also *encompassing thermal effects.*

Indeed, the analysis of systems with a *mixed* rate-dependent/rate-independent character, coupling the evolution of the (absolute) temperature and of the displacement/deformation variables with the *rate-independent* flow rule of an internal variable, has been initiated in [Rou10], and subsequently particularized to various mechanical models, among which we mention perfect plasticity [Rou13] and damage [LRTT14]. In the latter paper, a vanishing-viscosity analysis (as the rate of the external loads and heat sourced tends to zero) for the *mixed* rate-dependent/independent model, has also been performed.

Let us stress that, instead, here the (approximating) thermoviscoplastic system will feature a *rate-dependent* flow rule for the plastic strain, and thus will be fully rate-dependent. First of all, we will focus on its analysis. Exploiting the techniques from [RR15], we will obtain two existence results, which might be interesting in their own right, for two notions of solutions, referred to as 'entropic' and 'weak energy', by passing to the limit in a carefully tailored time discretization scheme. Secondly, in the case of 'weak energy' solutions we will perform the vanishing-viscosity asymptotics, obtaining a system where the evolution of the displacement and of the elastic and plastic strains, in the sense of (global) energetic solutions, is coupled to that of the (spatially constant) temperature variable.

Let us now get further insight into our analysis, first in the visco-, and then in the perfectly plastic cases.

1.1. **The thermoviscoplastic system.** The reference configuration is a bounded, open, Lipschitz domain $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$, and we consider the evolution of the system in a time interval $(0,T)$. Within the small-strain approximation, the momentum balance features the linearized strain tensor $E(u) = \frac{1}{2}\left(\nabla u + \nabla u^{\perp}\right)$, decomposed as

$$E(u) = e + p \qquad \text{in } \Omega \times (0,T), \tag{1.1}$$

with $e \in \mathbb{M}^{d\times d}_{\mathrm{sym}}$ (the space of symmetric $(d\times d)$-matrices) and $p \in \mathbb{M}^{d\times d}_{\mathrm{D}}$ (the space of symmetric $(d\times d)$-matrices with null trace) the elastic and plastic strains, respectively. In according to Kelvin-Voigt rheology for materials subject to thermal expansion, the stress is given by

$$\sigma = \mathbb{D}\dot{e} + \mathbb{C}(e - \mathbb{E}\vartheta), \tag{1.2}$$

with ϑ the absolute temperature, and the elasticity, viscosity, and thermal expansion tensors $\mathbb{C}$, $\mathbb{D}$, $\mathbb{E}$ depending on the space variable x (which shall be overlooked in this Introduction for simplicity of exposition), symmetric, $\mathbb{C}$ and $\mathbb{D}$ positive definite. Then, we consider the following PDE system:

$$\dot{\vartheta} - \mathrm{div}(\kappa(\vartheta)\nabla\vartheta) = H + \mathrm{R}(\vartheta, \dot{p}) + \dot{p} : \dot{p} + \mathbb{D}\dot{e} : \dot{e} - \vartheta\mathbb{C}\mathbb{E} : \dot{e} \qquad \text{in } \Omega \times (0,T), \tag{1.3a}$$

$$\rho\ddot{u} - \mathrm{div}\sigma = F \qquad \text{in } \Omega \times (0,T), \tag{1.3b}$$

$$\partial_{\dot{p}}\mathrm{R}(\vartheta, \dot{p}) + \dot{p} \ni \sigma_{\mathrm{D}} \qquad \text{in } \Omega \times (0,T). \tag{1.3c}$$

The heat equation (1.3a) features as heat conductivity coefficient a nonlinear function $\kappa \in \mathrm{C}^0(\mathbb{R}^+)$, which shall be supposed to have a suitable growth. In the momentum balance (1.3b), $\rho > 0$ is the (constant, for simplicity) mass density. The evolution of the plastic strain p is given by the flow rule (1.3c), where σ_{D} is the deviatoric part of the stress σ, and the dissipation potential $\mathrm{R} : \mathbb{R}^+ \times \mathbb{M}^{d\times d}_{\mathrm{D}} \to [0,+\infty)$ is lower semicontinuous, and associated with a multifunction $K : \mathbb{R}^+ \rightrightarrows \mathbb{M}^{d\times d}_{\mathrm{D}}$, with values in the compact and convex subsets of $\mathbb{M}^{d\times d}_{\mathrm{D}}$, via the relation

$$R(\vartheta,\dot p) = \sup_{\pi\in K(\vartheta)} \pi{:}\dot p \qquad \text{for all } (\vartheta,\dot p) \in \mathbb{R}^+ \times \mathbb{M}^{d\times d}_{\mathrm{D}}$$

(the dependence of K and R on $x \in \Omega$ is overlooked within this section). Namely, for every $\vartheta \in \mathbb{R}^+$ $R(\vartheta,\cdot)$ is the support function of the convex and compact set $K(\vartheta)$, which can be interpreted as the domain of viscoelasticity, allowed to depend on $x \in \Omega$ as well as on the temperature variable. In fact, $R(\vartheta,\cdot)$ is the Fenchel-Moreau conjugate of the indicator function $I_{K(\vartheta)}$, and thus (1.3c) (where $\partial_{\dot p}$ denotes the subdifferential in the sense of convex analysis w.r.t. the variable $\dot p$) rephrases as

$$\dot p \in \partial I_{K(\vartheta)}(\sigma_{\mathrm{D}}-\dot p) \;\Leftrightarrow\; \dot p = \sigma_{\mathrm{D}} - \mathrm{P}_{K(\vartheta)}(\sigma_{\mathrm{D}}) \tag{1.4}$$

in $\Omega \times (0,T)$, with $\mathrm{P}_{K(\vartheta)}$ the projection operator onto $K(\vartheta)$. The PDE system (1.3) is supplemented by the boundary conditions

$$\sigma\nu = g \qquad \text{on } \Gamma_{\mathrm{Neu}} \times (0,T), \tag{1.5a}$$

$$u = w \qquad \text{on } \Gamma_{\mathrm{Dir}} \times (0,T), \tag{1.5b}$$

$$\kappa(\vartheta)\nabla\vartheta\nu = h \qquad \text{on } \partial\Omega \times (0,T), \tag{1.5c}$$

where ν is the external unit normal to $\partial\Omega$, with Γ_{Neu} and Γ_{Dir} its Neumann and Dirichlet parts. The body is subject to the volume force F, to the applied traction g on Γ_{Neu}, and solicited by a displacement field w applied on Γ_{Dir}, while H and h are bulk and surface (positive) heat sources, respectively.

A PDE system with the same structure as (1.3, 1.5) was proposed in [Rou13] to model the thermodynamics of perfect plasticity: i.e., a heat equation akin to (1.3a) and the momentum balance (1.3b) were coupled to the *rate-independent version* of the flow rule (1.13b), cf. (1.13b) below. While the mixed rate-dependent/independent system in [Rou13] calls for a completely different analysis from our own, the modeling discussion developed in [Rou13, Sec. 2] can be easily adapted to (1.3, 1.5) to show its compliance with the first and second principle of thermodynamics. In particular, let us stress that, due to the presence of the *quadratic* terms $\dot p : \dot p$, $\mathbb{D}\dot e : \dot e$, and $\vartheta\mathbb{C}\mathbb{E} : \dot e$ on the right-hand side of (1.3a), system (1.3, 1.5) is <u>thermodynamically consistent</u>.

The analysis of (the Cauchy problem associated with) system (1.3, 1.5) poses some significant mathematical difficulties:

(1): First and foremost, its nonlinear character, and in particular the quadratic terms on the r.h.s. of (1.3a), which is thus only estimated in $L^1((0,T)\times\Omega)$ as soon as $\dot p$ and $\dot e$ are estimated in $L^2((0,T)\times\Omega;\mathbb{M}^{d\times d}_{\mathrm{D}})$ and $L^2((0,T)\times\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})$, respectively. Because of this, on the one hand obtaining suitable estimates of the temperature variable turns out to be challenging. On the other hand, suitable weak formulations of (1.3a) are called for.

In the one-dimensional case, existence results have been obtained for thermodynamically consistent (visco)-plasticity models with hysteresis in [KS97, KSS02, KSS03]. In higher dimensions, suitable adjustments of the toolbox by Boccardo & Gallouët [BG89] to handle the heat equation with L^1/measure data have been devised in a series of recent papers on thermoviscoelasticity with rate-dependent/independent plasticity. In particular, we quote [BR08], dealing with a (rate-dependent) thermoviscoplastic model, where thermal expansion effects are neglected, as well as [BR11], addressing rate-independent plasticity with hardening coupled with thermal effects, with the stress tensor given by $\sigma = \mathbb{D}E(u_t) + \mathbb{C}e - \mathbb{C}\mathbb{E}\vartheta$, and finally [Rou13], handling the thermodynamics of perfect plasticity. Let us point out that, in the estimates developed in [BR11, Rou13], a crucial role is played by a sort of 'compatibility condition' between the growth exponents of the (ϑ-dependent) heat capacity coefficient multiplying ϑ_t, and of the heat conduction coefficient $\kappa(\vartheta)$. This allows for Boccardo-Gallouët type estimates, drawn from [Rou10].

Here we instead suppose that the heat capacity coefficient is constant (cf. also Remark 2.6 ahead), and develop different arguments to derive estimates on the temperature variable. Along the footsteps of [FPR09, RR15], analyzing thermodynamically consistent models for phase transitions and with damage, respectively, we will rely on a growth condition for the heat conduction, only. Namely, we shall suppose that

$$\kappa(\vartheta) \sim \vartheta^{\mu} \qquad \text{with } \mu > 1. \tag{1.6}$$

We shall exploit (1.6) upon testing (1.3a) by a suitable negative power of ϑ (all calculations can be rendered rigorously on the level of a time discretization scheme). In this way, we will deduce a crucial estimate for ϑ in $L^2(0,T;H^1(\Omega))$. Under (1.6) we will address the weak solvability of (1.3, 1.5) in terms of the 'entropic' notion of solution, proposed in the framework of models for heat conduction in fluids, cf. e.g. [Fei07, BFM09], and later used to weakly formulate models for phase change [FPR09] and, among other applications, for damage in thermoviscoelastic materials [RR15]. In the framework of our plasticity system, this solution concept features the momentum balance (1.3b) and the flow rule (1.3c), stated a.e. in $\Omega\times(0,T)$, and coupled with

- the *entropy inequality*

$$
\begin{aligned}
&\int_s^t\int_\Omega \log(\vartheta)\dot\varphi\,\mathrm{d}x\,\mathrm{d}r-\int_s^t\int_\Omega\Big(\kappa(\vartheta)\nabla\log(\vartheta)\nabla\varphi-\kappa(\vartheta)\frac{\varphi}{\vartheta}\nabla\log(\vartheta)\nabla\vartheta\Big)\,\mathrm{d}x\,\mathrm{d}r\\
&\leq\int_\Omega\log(\vartheta(t))\varphi(t)\,\mathrm{d}x-\int_\Omega\log(\vartheta(s))\varphi(s)\,\mathrm{d}x\\
&\quad-\int_s^t\int_\Omega\left(H+\mathrm{R}(\vartheta,\dot p)+|\dot p|^2+\mathbb{D}\dot e:\dot e-\vartheta\mathbb{B}:\dot e\right)\frac{\varphi}{\vartheta}\,\mathrm{d}x\,\mathrm{d}r-\int_s^t\int_{\partial\Omega}h\frac{\varphi}{\vartheta}\,\mathrm{d}x\,\mathrm{d}r
\end{aligned}
\tag{1.7}
$$

  with φ a sufficiently regular, *positive* test function,
- the *total energy inequality*

$$
\begin{aligned}
&\frac{\rho}{2}\int_\Omega|\dot u(t)|^2\,\mathrm{d}x+\mathcal{E}(\vartheta(t),e(t))\\
&=\frac{\rho}{2}\int_\Omega|\dot u(s)|^2\,\mathrm{d}x+\mathcal{E}(\vartheta(s),e(s))+\int_s^t\langle\mathcal{L},\dot u-\dot w\rangle_{H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^d)}+\int_s^t\int_\Omega H\,\mathrm{d}x\,\mathrm{d}r+\int_s^t\int_{\partial\Omega}h\,\mathrm{d}S\,\mathrm{d}r\\
&\quad+\rho\left(\int_\Omega\dot u(t)\dot w(t)\,\mathrm{d}x-\int_\Omega\dot u_0\dot w(0)\,\mathrm{d}x-\int_s^t\int_\Omega\dot u\ddot w\,\mathrm{d}x\,\mathrm{d}r\right)+\int_s^t\int_\Omega\sigma:E(\dot w)\,\mathrm{d}x\,\mathrm{d}r
\end{aligned}
\tag{1.8}
$$

  involving the total load $\mathcal{L}$ associated with the external forces F and g, and the energy functional $\mathcal{E}(\vartheta,e):=\int_\Omega\vartheta\,\mathrm{d}x+\int_\Omega\frac12\mathbb{C}e{:}e\,\mathrm{d}x\,.$

Both (1.7) and (1.8) are required to hold for almost all $t\in(0,T]$ and almost all $s\in(0,t)$, and for $s=0$.

While referring to [FPR09, RR15] for more details and to Sec. 2.2 for a formal derivation of (1.7)–(1.8), let us point out here that this solution concept reflects the thermodynamic consistency of the model, since it corresponds to the requirement that the system should satisfy the second and first principle of Thermodynamics. From an analytical viewpoint, observe that the entropy inequality (1.7) has the advantage that all the quadratic terms on the right-hand side of (1.3a) feature as multiplied by a negative test function. This allows for upper semicontinuity arguments in the limit passage in a suitable approximation of (1.7)–(1.8). Furthermore, despite its weak character, *weak-strong uniqueness* results can be obtained for the entropic formulation, cf. e.g. [FN12] in the context of the Navier-Stokes-Fourier system modeling heat conduction in fluids.

(2): An additional analytical challenge is related to handling a non-zero applied traction g on the Neumann part of the boundary Γ_{Neu}. This results in the term $\int_0^T\langle\mathcal{L},\dot u\rangle_{H^1(\Omega;\mathbb{R}^d)}\,\mathrm{d}t$ on the r.h.s. of (1.8), whose time discrete version is, in fact, the starting point in the derivation of all of the a priori estimates. The estimate of this term is delicate, since it would in principle involve the $H^1(\Omega;\mathbb{R}^d)$-norm of $\dot u$, which is not controlled by the left-hand side of (1.8). A by-part integration in time shifts the problem to estimating the $H^1(\Omega;\mathbb{R}^d)$-norm of u, but the l.h.s. of (1.8) only controls the $L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})$-norm of e. Observe that this is ultimately due to the form (1.2) of the stress σ.

To overcome this problem, we will impose that the data F and g comply with a suitable *safe load* condition, see also Remark 4.4.

Finally,

(3): the presence of adiabatic effects in the momentum balance, accounted for by the thermal expansion term coupling it with the heat equation, leads to yet another technical problem. In fact, the estimate of the term $\int_0^T\int_\Omega\vartheta\mathbb{C}\mathbb{E}{:}E(\dot w)\,\mathrm{d}x\,\mathrm{d}t$ contributing to the integral $\int_0^T\int_\Omega\sigma:E(\dot w)\,\mathrm{d}x\,\mathrm{d}t$ on the r.h.s. of (1.8) calls for suitable assumptions on the Dirichlet loading w, since the l.h.s. of (1.8) only controls the $L^1(\Omega)$-norm of ϑ, cf. again Remark 4.4.

As already mentioned, we will tackle the existence analysis for the entropic formulation of system (1.3, 1.5) by approximation via time discretization. In particular, along the footsteps of [RR15], we will carefully devise our time-discretization scheme in such a way that the approximate solutions obtained by interpolation of the discrete ones fulfill discrete versions of the entropy and total energy inequalities, in addition to the discrete momentum balance and flow rule. We will then obtain a series of a priori estimates allowing us to deduce suitable compactness information on the approximate solutions, and thus to pass to the limit.

In this way, under the basic growth condition (1.6) on κ and under appropriate assumptions on the data, also tailored to the technical problems **(2)**&**(3)**, we will prove our first main result, **Theorem 1**, stating the existence of entropic solutions to the Cauchy problem for system (1.3, 1.5).

Under a more stringent growth condition on κ, we will prove in **Theorem 2** an existence result for an enhanced notion of solution, featuring

- a 'conventional' weak formulation of the heat equation (1.3a), namely

$$\langle \dot\vartheta, \varphi\rangle + \int_\Omega \kappa(\vartheta)\nabla\vartheta\nabla\varphi\,\mathrm{d}x = \int_\Omega \left(H + \mathrm{R}(\vartheta,\dot p) + \dot p : \dot p + \mathbb{D}\dot e : e - \vartheta\mathbb{C}\mathbb{E} : \dot e\right)\varphi\,\mathrm{d}x + \int_{\partial\Omega} h\varphi\,\mathrm{d}S \qquad (1.9)$$

  for all test functions $\varphi \in W^{1,p}(\Omega)$, with $\vartheta \in W^{1,1}(0,T;W^{1,p}(\Omega)^*)$ and $p>1$ sufficiently big, and
- the *total energy balance*, i.e. (1.8) as an equality.

In view of the latter feature, we will refer to these improved solutions as 'weak energy'.

1.2. **The perfectly plastic system.** In investigating the vanishing viscosity and inertia limit of system (1.3, 1.5), we shall confine the discussion to the asymptotic behavior of a family of *weak energy solutions*. In fact, the case of entropic solutions could be also encompassed in our approach, but the final result would be less meaningful due to the weak character of the formulation of the heat equation. Furthermore, we will extend the analysis developed in [DMS14] to the *spatially heterogeneous case*, i.e. with the tensors $\mathbb{C}$, $\mathbb{D}$, $\mathbb{E}$, and the elastic domain K, depending on $x\in\Omega$. However, we will drop the dependence of K on the (spatially discontinuous) temperature variable ϑ.

Mimicking [DMS14], we will supplement the thermoviscoplastic system with rescaled data F^ε g^ε, w^ε, H^ε, h^ε, with $\mathfrak{f}^\varepsilon(t) = \mathfrak{f}(\varepsilon t)$, for $t\in[0,T/\varepsilon]$ and for $\mathfrak{f}\in\{F,\, g,\, w,\, H,\, h\}$. Correspondingly, we will consider a family $(\vartheta^\varepsilon, u^\varepsilon, e^\varepsilon, p^\varepsilon)_\varepsilon$ of weak energy solutions to (the Cauchy problem for) system (1.3, 1.5), defined on $[0,T/\varepsilon]$, We will further rescale them in such a way that they are defined on $[0,T]$, by setting $\vartheta_\varepsilon(t) = \vartheta^\varepsilon(t/\varepsilon)$, and defining analogously u_ε, e_ε, p_ε and the data F_ε, g_ε, w_ε, H_ε, h_ε. Hence, the functions $(\vartheta_\varepsilon, u_\varepsilon, e_\varepsilon, p_\varepsilon)$ are *weak energy* solutions of the rescaled system

$$\varepsilon\dot\vartheta - \mathrm{div}(\kappa(\vartheta)\nabla\vartheta) = H + \varepsilon\mathrm{R}(\vartheta,\dot p) + \varepsilon^2\dot p : \dot p + \varepsilon^2\mathbb{D}\dot e : \dot e - \vartheta\mathbb{C}\mathbb{E}_\varepsilon : \dot e \qquad \text{in } \Omega\times(0,T), \qquad (1.10a)$$

$$\rho\varepsilon^2\ddot u - \mathrm{div}\left(\varepsilon\mathbb{D}\dot e + \mathbb{C}(e - \mathbb{E}_\varepsilon\vartheta)\right) = F \qquad \text{in } \Omega\times(0,T), \qquad (1.10b)$$

$$\partial_{\dot p}\mathrm{R}(\vartheta,\dot p) + \varepsilon\dot p \ni \left(\varepsilon\mathbb{D}\dot e + \mathbb{C}(e - \mathbb{E}_\varepsilon\vartheta)\right)_{\mathrm{D}} \qquad \text{in } \Omega\times(0,T), \qquad (1.10c)$$

supplemented with the boundary conditions (1.5) featuring the rescaled data g_ε, w_ε, h_ε. Observe that we will let the thermal expansion tensors vary with ε.

For technical reasons expounded at length in Section 6, we will address the asymptotic analysis of system (1.10, 1.5) only under the assumption that the tensors $\mathbb{E}_\varepsilon$ scale in a suitable way with ε, namely

$$\mathbb{E}_\varepsilon = \varepsilon^\alpha\mathbb{E} \quad \text{with a given } \mathbb{E}\in\mathbb{M}^{d\times d\times d\times d}_{\mathrm{sym}} \text{ and } \alpha > \frac{1}{2}. \qquad (1.11)$$

Under (1.11), the *formal* limit of system (1.10, 1.5) then consists of

- the stationary heat equation

$$-\,\mathrm{div}(\kappa(\vartheta)\nabla\vartheta) = H \qquad \text{in } \Omega\times(0,T), \qquad (1.12)$$

  supplemented with the Neumann condition (1.5c);
- the system for perfect plasticity

$$-\,\mathrm{div}\sigma = F \qquad \text{in } \Omega\times(0,T), \qquad (1.13a)$$

$$\partial_{\dot p}\mathrm{R}(\Theta,\dot p) \ni \sigma_{\mathrm{D}} \qquad \text{in } \Omega\times(0,T), \qquad (1.13b)$$

with the boundary conditions (1.5a) and (1.5b), complemented by the kinematic admissibility condition and Hooke's law

$$E(u) = e + p \qquad \text{in } \Omega \times (0,T), \tag{1.13c}$$

$$\sigma = \mathbb{C}e \qquad \text{in } \Omega \times (0,T). \tag{1.13d}$$

In fact, system (1.13) has to be weakly formulated in function spaces reflecting the fact that the plastic strain p is only a Radon measure on Ω, and so is $E(u)$ (so that the displacement variable u is only a function of bounded deformation), and that, in principle, we only have BV-regularity for $t \mapsto p(t)$.

Our asymptotic result, **Theorem 3**, states that, under suitable conditions on the data $(F_\varepsilon\, g_\varepsilon,\, w_\varepsilon,\, H_\varepsilon,\, h_\varepsilon)_\varepsilon$, up to a subsequence the functions $(\vartheta_\varepsilon, u_\varepsilon, e_\varepsilon, p_\varepsilon)_\varepsilon$ converge as $\varepsilon \downarrow 0$ to a quadruple (Θ, u, e, p) such that

(1) Θ is constant in space,
(2) (u, e, p) comply with the *(global) energetic formulation* of system (1.13), consisting of a global stability condition and of an energy balance;
(3) there additionally holds a balance between the energy dissipated through changes of the plastic strain and the thermal energy on almost every sub-interval of $(0, T)$, i.e.

$$\mathrm{Var}(p; [s,t]) = \int_\Omega \Theta(s)\,\mathrm{d}x - \int_\Omega \Theta(t)\,\mathrm{d}x \qquad \text{for almost all } s,t \in (0,T) \text{ with } s<t. \tag{1.14}$$

Observe that (1.14) couples the evolution of the temperature to that of p, and thus of the solution triple (u, e, p).

Finally, based on the arguments from [DMS14], in Theorem 3 we will also obtain that (u, e, p) are, ultimately, *absolutely continuous* as functions of time. This is a special feature of the perfectly plastic system, already observed in [DMDM06]. It is in accordance with the time regularity results proved in [MT04] for energetic solutions to rate-independent systems driven by uniformly convex energy functionals. It is in fact because of the 'convex character' of the problem that we retrieve *(global) energetic* solutions, upon taking the vanishing-viscosity and inertia limit, cf. also [MRS09, Prop. 7]. Also in view of the similar vanishing-viscosity analysis developed in [LRTT14] in the context of a thermodynamically consistent model for damage, we expect to obtain a different kind of solution when performing the same analysis for thermomechanical systems driven by nonconvex (mechanical) energies. We plan to address these studies in the future.

Plan of the paper. In Section 2 we establish all the assumptions on the thermoviscoplastic system (1.3, 1.5) and its data, introduce the two solvability concepts we will address, and state our two existence results, Theorems 1 & 2. Section 3 is devoted to the analysis of the time discretization scheme for (1.3, 1.5). In Section 4 we pass to the time-continuous limit and conclude the proofs of Thms. 1 & 2. In Section 5 we set up the limiting perfectly plastic system and give its (global) energetic formulation. The vanishing viscosity and inertia analysis is carried out in Section 6 with Theorem 3, whose proof also relies on some Young measure tools recapitulated in the Appendix.

Notation 1.1 (General notation)**.** In what follows, $\mathbb{R}^+$ shall stand for $(0, +\infty)$. We will denote by $\mathbb{M}^{d\times d}$ ($\mathbb{M}^{d\times d\times d\times d}$) the space of $d{\times}d$ ($d{\times}d{\times}d{\times}d$, respectively) matrices. We consider $\mathbb{M}^{d\times d}$ endowed with the Frobenius inner product $\eta : \xi := \sum_{ij} \eta_{ij}\xi_{ij}$ for two matrices $\eta = (\eta_{ij})$ and $\xi = (\xi_{ij})$, which induces the matrix norm $|\cdot|$. $\mathbb{M}^{d\times d}_{\mathrm{sym}}$ stands for the subspace of symmetric matrices, and $\mathbb{M}^{d\times d}_{\mathrm{D}}$ for the subspace of symmetric matrices with null trace. In fact, $\mathbb{M}^{d\times d}_{\mathrm{sym}} = \mathbb{M}^{d\times d}_{\mathrm{D}} \oplus \mathbb{R}I$ (I denoting the identity matrix), since every $\eta \in \mathbb{M}^{d\times d}_{\mathrm{sym}}$ can be written as

$$\eta = \eta_{\mathrm{D}} + \frac{\mathrm{tr}(\eta)}{d} I$$

with η_{D} the orthogonal projection of η into $\mathbb{M}^{d\times d}_{\mathrm{D}}$. We will refer to η_{D} as the deviatoric part of η.

With the symbol $\odot$ we will denote the symmetrized tensor product of two vectors a, $b \in \mathbb{R}^d$, defined as the symmetric matrix with entries $\frac{a_i b_j + a_j b_i}{2}$. Note that the trace $\mathrm{tr}(a \odot b)$ coincides with the scalar product $a \cdot b$.

Given a Banach space X we shall use the symbol $\langle \cdot, \cdot \rangle_X$ for the duality pairing between X^* and X; if X is a Hilbert space, $(\cdot, \cdot)_X$ will stand for its inner product. To avoid overburdening notation, we shall often write $\|\cdot\|_X$ both for the norm on X, and on the product space $X \times \ldots \times X$. With the symbol $\overline{B}_{1,X}(0)$ we will denotes the closed unitary ball in X. We shall denote by the symbols

$$\text{(i) } \mathrm{B}([0,T];X), \qquad \text{(ii) } \mathrm{C}^0_{\mathrm{weak}}([0,T];X), \qquad \text{(iii) } \mathrm{BV}([0,T];X)$$

the spaces of functions from $[0,T]$ with values in X that are defined at *every* $t \in [0,T]$ and (i) are measurable; (ii) are *weakly* continuous on $[0,T]$; (iii) have bounded variation on $[0,T]$.

Finally, we shall use the symbols c, c', C, C', etc., whose meaning may vary even within the same line, to denote various positive constants depending only on known quantities. Furthermore, the symbols I_i, $i = 0, 1, ...$, will be used as place-holders for several integral terms (or sums of integral terms) popping in the various estimates: we warn the reader that we will not be self-consistent with the numbering, so that, for instance, the symbol I_1 will occur several times with different meanings.

2. **Main results for the thermoviscoplastic system**

First, in Section 2.1, for the thermoviscoplastic system (1.3, 1.5) we establish all the basic assumptions on the reference configuration Ω, on the tensors $\mathbb{C}$, $\mathbb{D}$, $\mathbb{E}$, on the set of admissible stresses K (and, consequently, on the dissipation potential R), on the external data g, h, f, ℓ, and w, and on the initial data (ϑ_0, u_0, $\dot{u}_0$, e_0, p_0). In Section 5.1 later on, we will revisit and strengthen some of these conditions in order to deal with the limiting perfectly plastic system. In view of this, to distinguish the two sets of assumptions, we will label them by indicating the number of the section (i.e., 2 for the thermoviscoplastic, and 5 for the perfectly plastic, system).

Second, in Sec. 2.2 we introduce the weak solvabilty concepts for the (Cauchy problem associated with the) viscoplastic system (1.3, 1.5), and state our existence results in Sec. 2.3.

2.1. **Setup.**

The reference configuration. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain, with Lipschitz boundary; we set $Q := (0,T)\times\Omega$. The boundary $\partial\Omega$ is given by

$$\begin{gathered}\partial\Omega = \Gamma_{\mathrm{Dir}} \cup \Gamma_{\mathrm{Neu}} \cup \partial\Gamma \quad \text{with } \Gamma_{\mathrm{Dir}},\, \Gamma_{\mathrm{Neu}},\, \partial\Gamma \text{ pairwise disjoint,}\\ \Gamma_{\mathrm{Dir}} \text{ and } \Gamma_{\mathrm{Neu}} \text{ relatively open in } \partial\Omega, \text{ and } \partial\Gamma \text{ their relative boundary in } \partial\Omega,\\ \text{with Hausdorff measure } \mathcal{H}^{d-1}(\partial\Gamma) = 0.\end{gathered} \tag{2.Ω}$$

We will denote by $|\Omega|$ the Lebesgue measure of Ω. On the Dirichlet part Γ_{Dir}, assumed with $\mathcal{H}^{d-1}(\Gamma_{\mathrm{Dir}}) > 0$, we shall prescribe the displacement, while on Γ_{Neu} we will impose a Neumann condition. The trace of a function v on Γ_{Dir} or Γ_{Neu} shall be still denoted by the symbol v. The symbol $H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^d)$ shall indicate the space of functions with null trace on Γ_{Dir}. The symbol $W^{1,p}_{\mathrm{Dir}}(\Omega;\mathbb{R}^d)$, $p > 1$, shall denote the analogous $W^{1,p}$-space. Finally, we will use the notation

$$W^{1,p}_+(\Omega) := \left\{\zeta \in W^{1,p}(\Omega) \,:\, \zeta(x) \geq 0 \quad \text{for a.a. } x \in \Omega\right\}, \qquad \text{and analogously for } W^{1,p}_-(\Omega). \tag{2.1}$$

In what follows, we shall extensively use Korn's inequality (cf. [GS86]): for every $1 < p < \infty$ there exists a constant $C_K = C_K(\Omega, p) > 0$ such that there holds

$$\|u\|_{W^{1,p}(\Omega;\mathbb{R}^d)} \leq C_K \|E(u)\|_{L^p(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})} \qquad \text{for all } u \in W^{1,p}_{\mathrm{Dir}}(\Omega;\mathbb{R}^d)\,. \tag{2.2}$$

Kinematic admissibility and stress. First of all, let us formalize the decomposition of the linearized strain $E(u)$ as the sum of the elastic and the plastic strain. Given a function $w \in H^1(\Omega;\mathbb{R}^d)$, we say that a triple (u, e, p) is *kinematically admissible with boundary datum* w, and write $(u, e, p) \in \mathcal{A}(w)$, if

$$u \in H^1(\Omega;\mathbb{R}^d), \quad e \in L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}}), \quad p \in L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{D}}), \tag{2.3a}$$

$$E(u) = e + p \quad \text{a.e. in } \Omega, \tag{2.3b}$$

$$u = w \quad \text{on } \Gamma_{\mathrm{Dir}}. \tag{2.3c}$$

The elasticity, viscosity, and thermal expansion tensors are symmetric and fulfill

$$\begin{gathered}\mathbb{C},\, \mathbb{D},\, \mathbb{E} \in L^\infty(\Omega;\mathrm{Lin}(\mathbb{M}^{d\times d}_{\mathrm{sym}}))\,, \text{ and}\\ \exists\, C^1_{\mathbb{C}},\, C^2_{\mathbb{C}},\, C^1_{\mathbb{D}},\, C^2_{\mathbb{D}} > 0 \text{ for a.a. } x \in \Omega \ \ \forall\, A \in \mathbb{M}^{d\times d}_{\mathrm{sym}} \,:\, \left\{\begin{array}{l} C^1_{\mathbb{C}}|A|^2 \leq \mathbb{C}(x)A : A \leq C^2_{\mathbb{C}}|A|^2,\\ C^1_{\mathbb{D}}|A|^2 \leq \mathbb{D}(x)A : A \leq C^2_{\mathbb{D}}|A|^2,\end{array}\right.\end{gathered} \tag{2.T}$$

where $\mathrm{Lin}(\mathbb{M}^{d\times d}_{\mathrm{sym}})$ denotes the space of linear operators from $\mathbb{M}^{d\times d}_{\mathrm{sym}}$ to $\mathbb{M}^{d\times d}_{\mathrm{sym}}$. Observe that with (2.T) we also encompass in our analysis the case of an anisotropic and inhomogeneous material. Throughout the paper, we will use the short-hand notation

$$\mathbb{B} := \mathbb{C}\mathbb{E} \tag{2.4}$$

Remark 2.1. In [DMS14] the viscosity tensor $\mathbb{D}$ was assumed (constant in space and) positive semidefinite, only: In particular, the case $\mathbb{D} \equiv 0$ was encompassed in the existence and vanishing-viscosity analysis. We are not able to extend our own analysis in this direction, though. In fact, the coercivity condition required on $\mathbb{D}$ (joint with $E(\dot{u}) = \dot{e} + \dot{p}$, following from kinematic admissibility), will play a crucial role in estimating the term $\iint \vartheta \mathbb{B}{:}E(\dot{u})\,\mathrm{d}x\,\mathrm{d}t$, which arises from the mechanical energy balance (2.19) ahead.

External heat sources. For the volume and boundary heat sources H and h we require

$$H \in L^1(0,T;L^1(\Omega)) \cap L^2(0,T;H^1(\Omega)^*), \qquad H \geq 0 \quad \text{a.e. in } Q\,, \tag{2.H$_1$}$$

$$h \in L^1(0,T;L^2(\partial\Omega)), \qquad h \geq 0 \quad \text{a.e. in } (0,T)\times\partial\Omega\,. \tag{2.H$_2$}$$

Indeed, the positivity of H and h is necessary for obtaining the strict positivity of the temperature ϑ.

Body force and traction. Our basic conditions on the volume force F and the assigned traction g are

$$F \in L^2(0,T;H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^d)^*), \qquad g \in L^2(0,T;H^{1/2}_{00,\Gamma_{\mathrm{Dir}}}(\Gamma_{\mathrm{Neu}};\mathbb{R}^d)^*), \tag{2.L$_1$}$$

recalling that $H^{1/2}_{00,\Gamma_{\mathrm{Dir}}}(\Gamma_{\mathrm{Neu}};\mathbb{R}^d)$ is the space of functions $\gamma \in H^{1/2}(\Gamma_{\mathrm{Neu}};\mathbb{R}^d)$ such that there exists $\tilde{\gamma} \in H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^d)$ with $\tilde{\gamma} = \gamma$ in Γ_{Neu}.

Furthermore, for technical reasons that will be expounded in Remark 4.4 ahead (cf. also the text preceding the proof of Proposition 4.3), in order to allow for a non-zero traction g, also for the viscoplastic system we will need to require a *uniform safe load* type condition, which usually occurs in the analysis of perfectly plastic systems, cf. Sec. 5 later on. Namely, we impose that there exists a function $\varrho : [0,T] \to L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})$ solving for almost all $t \in (0,T)$ the following elliptic problem

$$\begin{cases} -\mathrm{div}(\varrho(t)) = F(t) & \text{in } \Omega, \\ \varrho(t)\nu = g(t) & \text{on } \Gamma_{\mathrm{Neu}} \end{cases}$$

such that

$$\varrho \in W^{1,1}(0,T;L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})) \qquad \text{and} \qquad \varrho_{\mathrm{D}} \in L^1(0,T;L^\infty(\Omega;\mathbb{M}^{d\times d}_{\mathrm{D}}))\,. \tag{2.L$_2$}$$

Indeed, condition (2.L$_2$) will enter into play only starting from the derivation of a priori estimates on the approximate solutions to the viscoplastic system, uniform with respect to the time discretization parameter τ. When not explicitly using (2.L$_2$), to shorten notation we will incorporate the volume force F and the traction g into the total load induced by them, namely the function $\mathcal{L} : (0,T) \to H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^d)^*$ given at $t \in (0,T)$ by

$$\langle \mathcal{L}(t), u\rangle_{H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^d)} := \langle F(t), u\rangle_{H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^d)} + \langle g(t), u\rangle_{H^{1/2}_{00,\Gamma_{\mathrm{Dir}}}(\Gamma_{\mathrm{Neu}};\mathbb{R}^d)} \qquad \text{for all } u \in H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^d), \tag{2.5}$$

which fulfills $\mathcal{L} \in L^2(0,T;H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^d)^*)$ in view of (2.L$_1$).

Dirichlet loading. Finally, we will suppose that the hard device w to which the body is subject on Γ_{Dir} is the trace on Γ_{Dir} of a function, denoted by the same symbol, fulfilling

$$w \in L^1(0,T;W^{1,\infty}(\Omega;\mathbb{R}^d)) \cap W^{2,1}(0,T;H^1(\Omega;\mathbb{R}^d)) \cap H^2(0,T;L^2(\Omega;\mathbb{R}^d))\,. \tag{2.W}$$

We postpone to Remark 4.4 some explanations on the use of, and need for, conditions (2.W). Let us only mention here that the requirement $w \in L^1(0,T;W^{1,\infty}(\Omega;\mathbb{R}^d))$ could be replaced by asking for $\mathbb{B}{:}E(w) = 0$ a.e. in Q, as imposed, e.g., in [Rou13].

The weak formulation of the momentum balance. The variational formulation of (1.3b), supplemented with the boundary conditions (1.5a) and (1.5b), reads

$$\begin{aligned} \rho\int_\Omega \ddot{u}(t)v\,\mathrm{d}x + \int_\Omega (\mathbb{D}\dot{e}(t) + \mathbb{C}e(t) - \vartheta(t)\mathbb{B}) : E(v)\,\mathrm{d}x = \langle \mathcal{L}(t), v\rangle_{H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^d)} & \\ \text{for all } v \in H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^d), \text{ for a.a. } t \in (0,T)\,. & \end{aligned} \tag{2.6}$$

We will often use the short-hand notation $-\mathrm{div}_{\mathrm{Dir}}$ for the elliptic operator defined by

$$\langle -\mathrm{div}_{\mathrm{Dir}}(\sigma), v\rangle_{H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^d)} := \int_\Omega \sigma : E(v)\,\mathrm{d}x \qquad \text{for all } v \in H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^d)\,. \tag{2.7}$$

The plastic dissipation. Prior to stating our precise assumptions on the multifunction $K : \Omega \times \mathbb{R}^+ \rightrightarrows \mathbb{M}^{d\times d}_{\mathrm{D}}$, following [CV77] let us recall the notions of measurability, lower semicontinuity, and upper semicontinuity, for a general multifunction $\mathsf{F} : X \rightrightarrows Y$. Although the definitions and results given in [CV77] cover much more general situations, for simplicity here we shall confine the discussion to the case of a topological measurable space $(X, \mathscr{M})$, and a (separable) Hilbert space Y. For a set $B \subset Y$, we set

$$\mathsf{F}^{-1}(B) := \{x \in X \,:\, \mathsf{F}(x) \cap B \neq \emptyset\}.$$

We say that

$$\mathsf{F} \text{ is measurable if for every open subset } U \subset Y,\ \mathsf{F}^{-1}(U) \in \mathscr{M}; \tag{2.8a}$$

$$\mathsf{F} \text{ is lower semicontinuous if for every open set } U \subset Y, \text{ the set } \mathsf{F}^{-1}(U) \text{ is open;} \tag{2.8b}$$

$$\mathsf{F} \text{ is upper semicontinuous if for every open set } U \subset Y, \text{ the set } \{x \in X \,:\, \mathsf{F}(x) \subset U\} \text{ is open.} \tag{2.8c}$$

Finally, F is continuous if it is both lower and upper semicontinuous.

Let us now turn back to the multifunction $K : \Omega \times \mathbb{R}^+ \rightrightarrows \mathbb{M}^{d\times d}_{\mathrm{D}}$. We suppose that

$$\begin{aligned} &K : \Omega \times \mathbb{R}^+ \rightrightarrows \mathbb{M}^{d\times d}_{\mathrm{D}} && \text{is measurable w.r.t. the variables } (x,\vartheta),\\ &K(x,\cdot) : \mathbb{R}^+ \rightrightarrows \mathbb{M}^{d\times d}_{\mathrm{D}} && \text{is continuous} \quad \text{for almost all } x \in \Omega. \end{aligned} \tag{2.K$_1$}$$

Furthermore, we require that

$$\begin{aligned} &K(x,\vartheta) \text{ is a convex and compact set in } \mathbb{M}^{d\times d}_{\mathrm{D}} \qquad \text{for all } \vartheta \in \mathbb{R}^+, \text{ for almost all } x \in \Omega,\\ &\exists\, 0 < c_r < C_R \quad \text{for a.a. } x \in \Omega,\ \forall\, \vartheta \in \mathbb{R}^+ : \quad B_{c_r}(0) \subset K(x,\vartheta) \subset B_{C_R}(0). \end{aligned} \tag{2.K$_2$}$$

Therefore, the support function associated with the multifunction K, i.e.

$$\mathrm{R} : \Omega \times \mathbb{R}^+ \times \mathbb{M}^{d\times d}_{\mathrm{D}} \to [0,+\infty) \quad \text{defined by } \mathrm{R}(x,\vartheta,\dot p) := \sup_{\pi \in K(x,\vartheta)} \pi : \dot p \tag{2.9}$$

is positive, with $\mathrm{R}(x,\vartheta,\cdot) : \mathbb{M}^{d\times d}_{\mathrm{D}} \to [0,+\infty)$ convex and 1-positively homogeneous for almost all $x \in \Omega$ and for all $\vartheta \in \mathbb{R}^+$. By the first of (2.K$_1$), the function $\mathrm{R} : \Omega \times \mathbb{R}^+ \times \mathbb{M}^{d\times d}_{\mathrm{D}} \to [0,+\infty)$ is measurable. Moreover, by the second of (2.K$_1$), in view of [CV77, Thms. II.20, II.21] (cf. also [Sol09, Prop. 2.4]) the function

$$\mathrm{R}(x,\cdot,\cdot) : \mathbb{R}^+ \times \mathbb{M}^{d\times d}_{\mathrm{D}} \to [0,+\infty) \text{ is (jointly) lower semicontinuous,} \tag{2.10a}$$

for almost all $x \in \Omega$, i.e. R is a *normal integrand*, and

$$\mathrm{R}(x,\cdot,\dot p) : \mathbb{R}^+ \to \mathbb{R}^+ \text{ is continuous for every } \dot p \in \mathbb{M}^{d\times d}_{\mathrm{D}}. \tag{2.10b}$$

Finally, it follows from the second of (2.K$_2$) that

$$c_r|\dot p| \le \mathrm{R}(x,\theta,\dot p) \le C_R|\dot p| \qquad \text{for all } (\vartheta,\dot p) \in \mathbb{R}^+ \times \mathbb{M}^{d\times d}_{\mathrm{D}} \text{ for almost all } x \in \Omega\,, \tag{2.11a}$$

and that

$$\partial_{\dot p}\mathrm{R}(\vartheta,\dot p) \subset \partial_{\dot p}\mathrm{R}(\vartheta,0) = K(x,\vartheta) \subset B_{C_R}(0) \qquad \text{for all } (\vartheta,\dot p) \in \mathbb{R}^+ \times \mathbb{M}^{d\times d}_{\mathrm{D}} \quad \text{for almost all } x \in \Omega. \tag{2.11b}$$

Finally, we also introduce the *plastic dissipation potential* $\mathcal{R} : L^1(\Omega;\mathbb{R}^+) \times L^1(\Omega;\mathbb{M}^{d\times d}_{\mathrm{D}})$ given by

$$\mathcal{R}(\vartheta,\dot p) := \int_\Omega \mathrm{R}(x,\vartheta(x),\dot p(x))\,\mathrm{d}x\,. \tag{2.12}$$

The plastic flow rule. Taking into account the 1-positive homogeneity of $\mathrm{R}(x,\vartheta,\cdot)$, yielding the following characterization of $\partial_{\dot p}\mathrm{R}(x,\vartheta,\dot p) : \mathbb{M}^{d\times d}_{\mathrm{D}} \rightrightarrows \mathbb{M}^{d\times d}_{\mathrm{D}}$:

$$\zeta \in \partial_{\dot p}\mathrm{R}(x,\vartheta,\dot p) \quad \Leftrightarrow \quad \left\{ \begin{array}{ll} \zeta : \eta \le \mathrm{R}(x,\vartheta,\eta) & \text{for all } \eta \in \mathbb{M}^{d\times d}_{\mathrm{D}} \\ \zeta : \dot p = \mathrm{R}(x,\vartheta,\dot p), & \end{array} \right. \quad \Leftrightarrow \quad \left\{ \begin{array}{l} \zeta \in \partial_{\dot p}\mathrm{R}(x,\vartheta,0) = K(x,\vartheta),\\ \zeta : \dot p \ge \mathrm{R}(x,\vartheta,\dot p), \end{array} \right. \tag{2.13}$$

the plastic flow rule

$$\partial_{\dot p}\mathrm{R}(x,\vartheta(t,x),\dot p(t,x)) + \dot p(t,x) \ni \sigma_{\mathrm{D}}(t,x) \qquad \text{for a.a. } (t,x) \in Q, \tag{2.14}$$

reformulates as

$$\begin{cases} (\sigma_{\mathrm{D}}(t,x) - \dot p(t,x)) : \eta \le \mathrm{R}(x,\vartheta(t,x),\eta) \quad \text{for all } \eta \in \mathbb{M}^{d\times d}_{\mathrm{D}} \\ (\sigma_{\mathrm{D}}(t,x) - \dot p(t,x)) : \dot p(t,x) \ge \mathrm{R}(x,\vartheta(t,x),\dot p(t,x)) \end{cases} \qquad \text{for a.a. } (t,x) \in Q\,. \tag{2.15}$$

Cauchy data. We will supplement the thermoviscoplastic system with initial data

$$\begin{aligned} &\vartheta_0 \in L^1(\Omega), \text{ fulfilling the strict positivity condition } \exists\, \vartheta_* > 0 : \inf_{x\in\Omega} \vartheta_0(x) \geq \vartheta_*, \\ &\qquad\qquad \text{and such that } \log(\vartheta_0) \in L^1(\Omega), \end{aligned} \tag{2.16a}$$

$$u_0 \in H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^d), \ \dot{u}_0 \in L^2(\Omega;\mathbb{R}^d), \tag{2.16b}$$

$$e_0 \in L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}}), \quad p_0 \in L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{D}}) \quad \text{such that } (u_0,e_0,p_0) \in \mathcal{A}(w(0))\,. \tag{2.16c}$$

2.2. **Weak solvability concepts for the thermoviscoplastic system.** Throughout this section, we shall suppose that the functions $\mathbb{C},\ldots,\mathrm{R}$, the data $g,\ldots,w$, and the initial data $(\vartheta_0,\, u_0, \dot{u}_0, e_0, p_0)$ fulfill the conditions stated in Section 2.1. We now motivate the weak solvabilty concepts for the (Cauchy problem associated with the) viscoplastic system (1.3, 1.5) with some heuristic calculations.

Heuristics for entropic and weak solutions to system (1.3, 1.5). As already mentioned in the Introduction, we shall formulate the heat equation (1.3a) by means of an entropy inequality and a total energy inequality, featuring the stored energy of the system. The latter is given by the sum of the elastic and of the internal energies, i.e.

$$\mathcal{E}(\vartheta,u,e,p) = \mathcal{E}(\vartheta,e) := \mathcal{F}(\vartheta) + \mathcal{Q}(e) \quad \text{with} \quad \begin{cases} \mathcal{F}(\vartheta) := \int_\Omega \vartheta \,\mathrm{d}x, \\ \mathcal{Q}(e) := \frac{1}{2}\int_\Omega \mathbb{C}e : e \,\mathrm{d}x\,. \end{cases} \tag{2.17}$$

Let us formally derive (in particular, without specifying the needed regularity on the solution quadruple (ϑ,u,e,p)) the total energy inequality (indeed, we will formally obtain a total energy *balance*), starting from the energy estimate associated with system (1.3, 1.5). The latter consists in testing the momentum balance by $\dot{u}-\dot{w}$, the heat equation by 1, and the plastic flow rule by $\dot{p}$, adding the resulting relations and integrating in space and over a generic interval $(s,t)\subset(0,T)$. More in detail, the test of (1.3b) and of (1.3c) yields, after some elementary calculations,

$$\begin{aligned} &\frac{\rho}{2}\int_\Omega |\dot{u}(t)|^2 \,\mathrm{d}x + \int_s^t \int_\Omega (\mathbb{D}\dot{e} + \mathbb{C}e - \vartheta\mathbb{B}) : E(\dot{u}) \,\mathrm{d}x\,\mathrm{d}r + \int_s^t \int_\Omega \left(|\dot{p}|^2 + \mathrm{R}(\vartheta,\dot{p})\right) \mathrm{d}x\,\mathrm{d}r \\ &= \frac{\rho}{2}\int_\Omega |\dot{u}(s))|^2 \,\mathrm{d}x + \int_s^t \langle \mathcal{L}, \dot{u}-\dot{w}\rangle_{H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^d)} \,\mathrm{d}r + \int_s^t \int_\Omega (\mathbb{D}\dot{e} + \mathbb{C}e - \vartheta\mathbb{B}) : E(\dot{w}) \,\mathrm{d}x\,\mathrm{d}r \\ &\quad + \rho\left(\int_\Omega \dot{u}(t)\dot{w}(t)\,\mathrm{d}x - \int_\Omega \dot{u}_0 \dot{w}(0)\,\mathrm{d}x - \int_s^t \int_\Omega \dot{u}\ddot{w}\,\mathrm{d}x\,\mathrm{d}r \right) + \int_s^t \int_\Omega \sigma_{\mathrm{D}} : \dot{p} \,\mathrm{d}x\,\mathrm{d}r\,. \end{aligned} \tag{2.18}$$

Now, taking into account that $E(\dot{u}) = \dot{e}+\dot{p}$ by the kinematical admissibility condition, rearranging some terms one has that

$$\begin{aligned} \int_s^t \int_\Omega (\mathbb{D}\dot{e} + \mathbb{C}e - \vartheta\mathbb{B}) : E(\dot{u}) \,\mathrm{d}x\,\mathrm{d}r = \int_s^t \int_\Omega (\mathbb{D}\dot{e} : \dot{e} + \mathbb{C}\dot{e} : e) \,\mathrm{d}x - \int_s^t \int_\Omega \vartheta\mathbb{B} : \dot{e} \,\mathrm{d}x\,\mathrm{d} \\ + \int_s^t \int_\Omega (\mathbb{D}\dot{e} + \mathbb{C}e - \vartheta\mathbb{B}) : \dot{p} \,\mathrm{d}x\,\mathrm{d}r\,. \end{aligned}$$

Substituting this in (2.18) and noting that $\int_s^t \int_\Omega (\mathbb{D}\dot{e} + \mathbb{C}e - \vartheta\mathbb{B}) : \dot{p} \,\mathrm{d}x\,\mathrm{d}r = \int_s^t \int_\Omega \sigma_{\mathrm{D}} : \dot{p} \,\mathrm{d}x\,\mathrm{d}r$, so that the last term on the right-hand side of (2.18) cancels out, we get the *mechanical energy balance*, featuring the kinetic and dissipated energies

$$\begin{aligned} &\underbrace{\frac{\rho}{2}\int_\Omega |\dot{u}(t)|^2 \,\mathrm{d}x}_{\text{kinetic}} + \underbrace{\int_s^t \int_\Omega \left(\mathbb{D}\dot{e} : \dot{e} + |\dot{p}|^2\right) \mathrm{d}x\,\mathrm{d}r + \int_s^t \mathcal{R}(\vartheta,\dot{p})\,\mathrm{d}r}_{\text{dissipated}} + \mathcal{Q}(e(t)) \\ &= \frac{\rho}{2}\int_\Omega |\dot{u}(s)|^2 \,\mathrm{d}x + \mathcal{Q}(e(s)) + \int_s^t \langle \mathcal{L}, \dot{u}-\dot{w}\rangle_{H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^d)} \,\mathrm{d}r + \int_s^t \int_\Omega \vartheta\mathbb{B} : \dot{e}\,\mathrm{d}x\,\mathrm{d}r \\ &\quad + \rho\left(\int_\Omega \dot{u}(t)\dot{w}(t)\,\mathrm{d}x - \int_\Omega \dot{u}(s)\dot{w}(s)\,\mathrm{d}x - \int_s^t \int_\Omega \dot{u}\ddot{w}\,\mathrm{d}x\,\mathrm{d}r \right) + \int_s^t \int_\Omega (\mathbb{D}\dot{e} + \mathbb{C}e - \vartheta\mathbb{B}) : E(\dot{w}) \,\mathrm{d}x\,\mathrm{d}r, \end{aligned} \tag{2.19}$$

which will also have a significant role for our analysis.

Summing this with the heat equation tested by 1 and integrated in time and space gives, after cancelation of some terms, the *total energy balance*

$$
\begin{aligned}
&\frac{\rho}{2}\int_\Omega |\dot u(t)|^2\,\mathrm{d}x + \mathcal{E}(\vartheta(t),e(t))\\
&=\frac{\rho}{2}\int_\Omega |\dot u(s)|^2\,\mathrm{d}x + \mathcal{E}(\vartheta(s),e(s)) + \int_s^t \langle \mathcal{L}, \dot u - \dot w\rangle_{H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^d)} + \int_s^t\int_\Omega H\,\mathrm{d}x\,\mathrm{d}r + \int_s^t\int_{\partial\Omega} h\,\mathrm{d}S\,\mathrm{d}r\\
&\quad + \rho\left(\int_\Omega \dot u(t)\dot w(t)\,\mathrm{d}x - \int_\Omega \dot u_0 \dot w(0)\,\mathrm{d}x - \int_s^t\int_\Omega \dot u\ddot w\,\mathrm{d}x\,\mathrm{d}r\right) + \int_s^t\int_\Omega \sigma : E(\dot w)\,\mathrm{d}x\,\mathrm{d}r\,.
\end{aligned}
\tag{2.20}
$$

As for the *entropy inequality*, let us only mention that it can be formally obtained by multiplying the heat equation (1.3a) by φ/ϑ, with φ a smooth and *positive* test function. Integrating in space and over a generic interval $(s,t) \subset (0,T)$ leads to the identity

$$
\begin{aligned}
&\int_s^t\int_\Omega \partial_t \log(\vartheta)\varphi\,\mathrm{d}x\,\mathrm{d}r + \int_s^t\int_\Omega \Big(\kappa(\vartheta)\nabla\log(\vartheta)\nabla\varphi - \kappa(\vartheta)\frac{\varphi}{\vartheta}\nabla\log(\vartheta)\nabla\vartheta\Big)\,\mathrm{d}x\,\mathrm{d}r\\
&= \int_s^t\int_\Omega \big(H + \mathrm{R}(\vartheta,\dot p) + |\dot p|^2 + \mathbb{D}\dot e : \dot e - \vartheta\mathbb{B} : \dot e\big)\,\frac{\varphi}{\vartheta}\,\mathrm{d}x\,\mathrm{d}r + \int_s^t\int_{\partial\Omega} h\frac{\varphi}{\vartheta}\,\mathrm{d}x\,\mathrm{d}r\,.
\end{aligned}
\tag{2.21}
$$

The entropic solution concept given in Definition 2.2 below will feature the inequality version of (2.21), where the first term on the left-hand side is integrated by parts in time, as well as the inequality version of (2.20).

Definition 2.2 (Entropic solutions to the thermoviscoplastic system)**.** *Given initial data* $(\vartheta_0, u_0, \dot u_0, e_0, p_0)$ *fulfilling* (2.16)*, we call a quadruple* (ϑ, u, e, p) *an* entropic solution *to the Cauchy problem for system (1.3, 1.5), if*

$$\vartheta \in L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;L^1(\Omega)), \tag{2.22a}$$
$$\log(\vartheta) \in L^2(0,T;H^1(\Omega)), \tag{2.22b}$$
$$u \in H^1(0,T;H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^d)) \cap W^{1,\infty}(0,T;L^2(\Omega;\mathbb{R}^d)) \cap H^2(0,T;H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^d)^*), \tag{2.22c}$$
$$e \in H^1(0,T;L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})), \tag{2.22d}$$
$$p \in H^1(0,T;L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{D}})), \tag{2.22e}$$

(u,e,p) *comply with the initial conditions*

$$u(0,x) = u_0(x), \quad \dot u(0,x) = \dot u_0(x) \qquad \text{for a.a. } x\in\Omega, \tag{2.23a}$$
$$e(0,x) = e_0(x) \qquad \text{for a.a. } x\in\Omega, \tag{2.23b}$$
$$p(0,x) = p_0(x) \qquad \text{for a.a. } x\in\Omega, \tag{2.23c}$$

(while the initial condition for ϑ *is implicitly formulated in* (2.25) *and* (2.26) *below), and with*

- *the* strict positivity *of* ϑ*:*

$$\exists\,\bar\vartheta > 0 \text{ for a.a. } (t,x)\in Q: \quad \vartheta(t,x) > \bar\vartheta; \tag{2.24}$$

- *the* entropy inequality*, to hold for almost all* $t\in(0,T]$ *and almost all* $s\in(0,t)$*, and for* $s=0$ *(where* $\log(\vartheta(0))$ *is to be understood as* $\log(\vartheta_0)$*),*

$$
\begin{aligned}
&\int_s^t\int_\Omega \log(\vartheta)\dot\varphi\,\mathrm{d}x\,\mathrm{d}r - \int_s^t\int_\Omega \Big(\kappa(\vartheta)\nabla\log(\vartheta)\nabla\varphi - \kappa(\vartheta)\frac{\varphi}{\vartheta}\nabla\log(\vartheta)\nabla\vartheta\Big)\,\mathrm{d}x\,\mathrm{d}r\\
&\le \int_\Omega \log(\vartheta(t))\varphi(t)\,\mathrm{d}x - \int_\Omega \log(\vartheta(s))\varphi(s)\,\mathrm{d}x\\
&\quad - \int_s^t\int_\Omega \big(H + \mathrm{R}(\vartheta,\dot p) + |\dot p|^2 + \mathbb{D}\dot e : \dot e - \vartheta\mathbb{B} : \dot e\big)\,\frac{\varphi}{\vartheta}\,\mathrm{d}x\,\mathrm{d}r - \int_s^t\int_{\partial\Omega} h\frac{\varphi}{\vartheta}\,\mathrm{d}x\,\mathrm{d}r
\end{aligned}
\tag{2.25}
$$

for all φ *in* $L^\infty([0,T];W^{1,\infty}(\Omega)) \cap H^1(0,T;L^{6/5}(\Omega))$*, with* $\varphi \ge 0$*;*

- *the* total energy inequality*, to hold for almost all $t \in (0,T]$ and almost all $s \in (0,t)$, and for $s=0$:*

$$
\begin{aligned}
&\frac{\rho}{2}\int_\Omega |\dot u(t)|^2\,\mathrm{d}x + \mathcal{E}(\vartheta(t),e(t))\\
&= \frac{\rho}{2}\int_\Omega |\dot u(s)|^2\,\mathrm{d}x + \mathcal{E}(\vartheta(s),e(s)) + \int_s^t \langle \mathcal{L}, \dot u - \dot w\rangle_{H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^d)} + \int_s^t\int_\Omega H\,\mathrm{d}x\,\mathrm{d}r + \int_s^t\int_{\partial\Omega} h\,\mathrm{d}S\,\mathrm{d}r \\
&\quad +\rho\left(\int_\Omega \dot u(t)\dot w(t)\,\mathrm{d}x - \int_\Omega \dot u_0 \dot w(0)\,\mathrm{d}x - \int_s^t\int_\Omega \dot u \ddot w\,\mathrm{d}x\,\mathrm{d}r\right) + \int_s^t\int_\Omega \sigma : E(\dot w)\,\mathrm{d}x\,\mathrm{d}r,
\end{aligned}
\tag{2.26}
$$

where for $s=0$ we read $\vartheta(0)=\vartheta_0$, with the stress σ given by the constitutive equation

$$
\sigma = \mathbb{D}\dot e + \mathbb{C}e - \vartheta\mathbb{B} \qquad \text{a.e. in } Q; \tag{2.27}
$$

- *the* kinematic admissibility *condition*

$$
(u(t,x),e(t,x),p(t,x)) \in \mathcal{A}(w(t,x)) \qquad \text{for a.a. } (t,x)\in Q; \tag{2.28}
$$

- *the weak formulation* (2.6) *of the* momentum balance*;*
- *the* plastic flow rule (2.14).

Remark 2.3. Observe that with the entropy inequality (2.25) we are tacitly claiming that, in addition to (2.22a) and (2.22b), the temperature variable has the following summability properties

$$
\kappa(\vartheta)|\nabla\log(\vartheta)|^2\varphi \in L^1(Q), \qquad \kappa(\vartheta)\nabla\log(\vartheta)\in L^1(Q)
$$

for every positive admissible test function φ. In fact, we shall retrieve the above properties (and improve the second one, cf. (2.34)), within the proof of Theorem 1. Furthermore, note that the integral $\int_\Omega \log(\vartheta(t))\varphi(t)\,\mathrm{d}x$ makes sense for almost all $t\in(0,T)$, since the estimate

$$
|\log(\vartheta(t,x))| \le \vartheta(t,x) + \frac{1}{\vartheta(t,x)} \le \vartheta(t,x) + \frac{1}{\bar\vartheta} \qquad \text{for a.a. } (t,x)\in Q,
$$

(with the second inequality due to (2.24)), and the fact $\vartheta \in L^\infty(0,T;L^1(\Omega))$, guarantee that $\log(\vartheta) \in L^\infty(0,T;L^1(\Omega))$ itself. Finally, the requirement that $\varphi\in H^1(0,T;L^{6/5}(\Omega))$ ensures that $\int_s^t\int_\Omega \log(\vartheta)\dot\varphi\,\mathrm{d}x\,\mathrm{d}r$ is a well defined integral, since $\log(\vartheta)\in L^2(0,T;L^6(\Omega))$ by (2.22b).

We refer to [RR15, Rmk 2.6] for a thorough discussion on the consistency between the entropic and the classical formulation of the heat equation (1.3a).

In our second solvability concept for the initial-boundary value problem associated with system (1.3), the temperature has the enhanced time regularity (2.29) below, which allows us to give an improved variational formulation of the heat equation (1.3a). Observe that, in [RR15] this solution notion was referred to as *weak*. In this paper we will instead prefer the term *weak energy solution*, in order to highlight the validity of the total energy *balance* on *every* interval $[s,t]\subset[0,T]$, cf. Corollary 2.5 below.

Definition 2.4 (Weak energy solutions to the thermoviscoplastic system)**.** *Given initial data $(\vartheta_0,u_0,\dot u_0,e_0,p_0)$ fulfilling* (2.16)*, we call a quadruple (ϑ,u,e,p) a* weak energy solution *to the Cauchy problem for system (1.3, 1.5), if*

- *in addition to the regularity and summability properties* (2.22)*, there holds*

$$
\vartheta \in W^{1,1}(0,T;W^{1,\infty}(\Omega)^*), \tag{2.29}
$$

- *in addition to the initial conditions* (2.23)*, ϑ complies with*

$$
\vartheta(0)=\vartheta_0 \qquad \text{in } W^{1,\infty}(\Omega)^*. \tag{2.30}
$$

- *in addition to the strict positivity* (2.24)*, the kinematic admissibility* (2.28)*, the weak mometum balance* (2.6)*, and the flow rule* (2.14)*, (ϑ,u,e,p) comply for almost all $t\in(0,T)$ with the following weak formulation of the heat equation*

$$
\begin{aligned}
&\langle \dot\vartheta,\varphi\rangle_{W^{1,\infty}(\Omega)} + \int_\Omega \kappa(\vartheta)\nabla\vartheta\nabla\varphi\,\mathrm{d}x\\
&= \int_\Omega \left(H + \mathrm{R}(\vartheta,\dot p) + |\dot p|^2 + \mathbb{D}\dot e : \dot e - \vartheta\mathbb{B}:\dot e\right)\varphi\,\mathrm{d}x + \int_{\partial\Omega} h\varphi\,\mathrm{d}S \quad \text{for all } \varphi\in W^{1,\infty}(\Omega).
\end{aligned}
\tag{2.31}
$$

Along the lines of Remark 2.3, we may observe that, underlying the weak formulation (2.31) is the property $\kappa(\vartheta)\nabla\vartheta \in L^1(Q;\mathbb{R}^d)$, which shall be in fact (slightly) improved in Theorem 2.

We conclude the section with the following result, under the (tacitly assumed) conditions from Sec. 2.1.

Lemma 2.5. *(1) Let (ϑ, u, e, p) be either an* entropic *or a* weak energy solution *to (the Cauchy problem for) system (1.3, 1.5). Then, the functions (ϑ, u, e, p) comply with the mechanical energy balance* (2.19) *for every $0 \le s \le t \le T$.*

(2) Let (ϑ, u, e, p) be a weak energy solution *to (the Cauchy problem for) system (1.3, 1.5). Then, the total energy* balance

$$\begin{aligned}&\frac{\rho}{2}\int_\Omega |\dot u(t)|^2\,\mathrm{d}x + \langle \vartheta(t), 1\rangle_{W^{1,\infty}(\Omega)} + \mathcal{Q}(e(t))\\&= \frac{\rho}{2}\int_\Omega |\dot u(s)|^2\,\mathrm{d}x + \langle \vartheta(s), 1\rangle_{W^{1,\infty}(\Omega)} + \mathcal{Q}(e(s)) + \int_s^t \langle \mathcal{L}, \dot u - \dot w\rangle_{H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^d)} + \int_s^t\int_\Omega H\,\mathrm{d}x\,\mathrm{d}r + \int_s^t\int_{\partial\Omega} h\,\mathrm{d}S\,\mathrm{d}r\\&\quad + \rho\left(\int_\Omega \dot u(t)\dot w(t)\,\mathrm{d}x - \int_\Omega \dot u_0 \dot w(0)\,\mathrm{d}x - \int_s^t\int_\Omega \dot u \ddot w\,\mathrm{d}x\,\mathrm{d}r\right) + \int_s^t\int_\Omega \sigma : E(\dot w)\,\mathrm{d}x\,\mathrm{d}r\end{aligned} \tag{2.32}$$

holds for all $0 \le s \le t \le T$.

Observe that, since $\vartheta \in L^\infty(0,T;L^1(\Omega))$, there holds $\langle \vartheta(t), 1\rangle_{W^{2,d+\epsilon}(\Omega)} = \int_\Omega \vartheta(t)\,\mathrm{d}x = \mathcal{F}(\vartheta(t))$ for almost all $t \in (0,T)$ and for $t = 0$.

Proof. The energy balance (2.19) follows from testing the momentum balance (2.6) by $\dot u - \dot w$, the plastic flow rule by $\dot p$, adding the resulting relations, and integrating in time.

As for (2.32), it is sufficient to test the weak formulation (2.31) of the heat equation by $\varphi = 1$, integrate in time taking into account that $\vartheta \in W^{1,1}(0,T;W^{1,\infty}(\Omega)^*)$, and add the resulting identity to (2.19). □

2.3. **Existence results for the thermoviscoplastic system.** Our first result states the existence of entropic solutions, under a mild growth condition on the thermal conductivity κ. For shorter notation, in the statement below we shall write (2.H) in place of (2.H$_1$), (2.H$_2$), and analogously (2.L), (2.K).

Theorem 1. *Assume* (2.Ω), (2.T), (2.H), (2.L), (2.W), *and* (2.K). *In addition, suppose that*

$$\begin{aligned}&\text{the function } \kappa : \mathbb{R}^+ \to \mathbb{R}^+ \text{ is continuous and}\\&\exists\, c_0, c_1 > 0 \quad \mu > 1 \quad \forall \vartheta \in \mathbb{R}^+ : \quad c_0(1+\vartheta^\mu) \le \kappa(\vartheta) \le c_1(1+\vartheta^\mu)\,.\end{aligned} \tag{2.κ_1}$$

Then, for every $(\vartheta_0, u_0, \dot u_0, e_0, p_0)$ satisfying (2.16) *there exists an entropic solution (ϑ, u, e, p) such that, in addition, ϑ complies with the positivity property*

$$\vartheta(t,x) \ge \bar\vartheta := \left(\bar c T + \frac{1}{\vartheta_*}\right)^{-1} \quad \text{for almost all } (t,x) \in Q, \tag{2.33}$$

where $\vartheta_* > 0$ is from (2.16a) *and $\bar c := \frac{|\mathbb{B}|^2}{2C^1_{\mathbb{D}}}$, with $C^1_{\mathbb{D}} > 0$ from* (2.T). *Finally, there holds*

$$\begin{aligned}&\kappa(\vartheta)\nabla\log(\vartheta) \in L^{1+\bar\delta}(Q;\mathbb{R}^d) \text{ with } \bar\delta = \frac{\alpha}{\mu} \text{ and } \alpha \in [2-\mu, 1), \text{ and}\\&\kappa(\vartheta)\nabla\log(\vartheta) \in L^1(0,T;X) \quad \text{with } X = \begin{cases} L^{2-\eta}(\Omega;\mathbb{R}^d) & \text{for all } \eta \in (0,1] \quad \text{if } d = 2,\\ L^{3/2-\eta}(\Omega;\mathbb{R}^d) & \text{for all } \eta \in (0,1/2] \quad \text{if } d = 3,\end{cases}\end{aligned} \tag{2.34}$$

so that the entropy inequality (2.25) *in fact holds for all positive test functions $\varphi \in L^\infty([0,T];W^{1,d+\epsilon}(\Omega)) \cap H^1(0,T;L^{6/5}(\Omega))$, for every $\epsilon > 0$.*

Remark 2.6. In [LRTT14] we proved an existence result for a PDE system modeling *rate-independent* damage in thermoviscoelastic materials, featuring a temperature equation with the same structure as (1.3a). Also in that context we obtained a strict positivity property with the same constant as in (2.33). Moreover, we showed that, if the heat source function H and the initial temperature ϑ_0 fulfill

$$H(t,x) \ge H_* > 0 \text{ for a.a. } (t,x) \in Q \quad \text{and} \quad \vartheta_0(x) \ge \sqrt{H_*/\bar c} \text{ for a.a. } x \in \Omega,$$

with $\bar c > 0$ from (2.33), then the enhanced positivity property

$$\vartheta(t,x) \ge \max\{\bar\vartheta, \sqrt{H_*/\bar c}\} \quad \text{for a.a. } (t,x) \in Q \tag{2.35}$$

holds. In the setting of the thermoviscoplastic system (1.3), too, it would be possible to prove (2.35). Observe that, choosing suitable data for the heat equation, the threshold $\max\{\bar\vartheta, \sqrt{H_*/\bar{c}}\}$, and thus the temperature, may be tuned to stay above a given constant. Choosing such a constant as the so-called *Debye temperature* (cf., e.g., [Wed97, Sec. 4.2, p. 761]), according to the Debye model one can thus justify the assumption that the heat capacity is constant.

Under a more stringent growth condition on κ, we obtain the existence of weak energy solutions.

Theorem 2. *Assume* (2.Ω), (2.T), (2.H), (2.L), (2.W), (2.K), *and* (2.κ_1). *In addition, suppose that the exponent μ in* (2.κ_1) *fulfills*

$$\begin{cases} \mu \in (1,2) & \text{if } d=2, \\ \mu \in \left(1,\frac{5}{3}\right) & \text{if } d=3. \end{cases} \tag{2.κ_2}$$

Then, for every $(\vartheta_0, u_0, \dot{u}_0, e_0, p_0)$ satisfying (2.16) *there exists a weak energy solution (ϑ, u, e, p) to the Cauchy problem for system (1.3, 1.5) satisfying* (2.33)–(2.34), *as well as*

$$\nabla(\hat\kappa(\vartheta)) \in W^{1+\tilde\delta}(Q) \text{ for some } \tilde\delta > 0, \tag{2.36}$$

with $\hat\kappa$ a primitive of κ. Therefore, (2.31) *in fact holds for all test functions $\varphi \in W^{1,1+1/\tilde\delta}(\Omega)$ and, ultimately, ϑ has the enhanced regularity $\vartheta \in W^{1,1}(0,T;W^{1,1+1/\tilde\delta}(\Omega)^*)$.*

As it will be clear from the proof of Thm. 2, in the case $d=3$ the exponent $\tilde\delta$ is in fact given by $\tilde\delta = \frac{2-3\mu+3\alpha}{3(\mu-\alpha+2)}$ for all $\alpha \in (\bar\alpha, 1)$ with $\bar\alpha := \max\{\mu-\frac{2}{3}, 2-\mu\}$: The condition $\mu < \frac{5}{3}$ for $d=3$ in fact ensures that it is possible to choose $\alpha < 1$ with $\alpha > \mu - \frac{2}{3}$. Also, note that for every α in the prescribed range we have that $\tilde\delta < 2$, so that $1 + 1/\tilde\delta > 3$. This yields

$$W^{1,1+1/\tilde\delta}(\Omega) \subset L^\infty(\Omega) \qquad \text{for } d \in \{2,3\}, \tag{2.37}$$

so that every $\varphi \in W^{1,1+1/\tilde\delta}(\Omega)$ can multiply the L^1-r.h.s. of (2.31) and, moreover, has trace in $L^2(\partial\Omega)$. Therefore, $W^{1,1+1/\tilde\delta}(\Omega)$ is an admissible space of test functions for (2.31). Clearly, in the case $d=2$ as well one can explicitly compute $\tilde\delta$, exploiting the condition $\mu < 2$, leading to a better range of indexes.

The proofs of Theorems 1 and 2, developed in Section 4, shall result from passing to the limit in a carefully tailored time discretization scheme of the thermoviscoplastic system (1.3, 1.5), analyzed in detail in Section 3.

3. Analysis of the thermoviscoplastic system: time discretization

The analysis of the time-discrete scheme for system (1.3, 1.5) shall often follow the lines of that developed for the phase transition/damage system analyzed in [RR15] (cf. also the proof of [LRTT14, Thm. 2.7]). Therefore, to avoid overburdening the exposition we will not fully develop all the arguments, but frequently refer to [RR15, LRTT14] for all details.

In the statements of all of the results of this section we will always tacitly assume the conditions on the problem data from Section 2.1.

Given an equidistant partition of $[0,T]$, with time-step $\tau > 0$ and nodes $t_\tau^k := k\tau$, $k = 0, \ldots, K_\tau$, we approximate the data F, g, w, H, and h by local means as follows

$$\begin{gathered} F_\tau^k := \frac{1}{\tau}\int_{t_\tau^{k-1}}^{t_\tau^k} F(s)\,\mathrm{d}s\,, \qquad g_\tau^k := \frac{1}{\tau}\int_{t_\tau^{k-1}}^{t_\tau^k} g(s)\,\mathrm{d}s\,, \qquad w_\tau^k := \frac{1}{\tau}\int_{t_\tau^{k-1}}^{t_\tau^k} w(s)\,\mathrm{d}s\,, \\ H_\tau^k := \frac{1}{\tau}\int_{t_\tau^{k-1}}^{t_\tau^k} H(s)\,\mathrm{d}s\,, \qquad h_\tau^k := \frac{1}{\tau}\int_{t_\tau^{k-1}}^{t_\tau^k} h(s)\,\mathrm{d}s \end{gathered} \tag{3.1}$$

for all $k = 1, \ldots, K_\tau$. From the terms F_τ^k and H_τ^k one then defines the elements $\mathcal{L}_\tau^k$, which are the local-mean approximations of $\mathcal{L}$. Hereafter, given elements $(v_\tau^k)_{k=1,\ldots,K_\tau}$ in a Banach space B, we will use the notation

$$\mathrm{D}_{k,\tau}(v) := \frac{v_\tau^k - v_\tau^{k-1}}{\tau}, \qquad \mathrm{D}^2_{k,\tau}(v) := \frac{v_\tau^k - 2v_\tau^{k-1} + v_\tau^{k-2}}{\tau^2}.$$

We construct discrete solutions to system (1.3, 1.5) by recursively solving an elliptic system, cf. the forthcoming Problem 3.1, where the weak formulation of the discrete heat equation features the function space

$$X := \{\theta \in H^1(\Omega) \,:\, \kappa(\theta)\nabla\theta\nabla v \in L^1(\Omega) \text{ for all } v \in H^1(\Omega)\}, \tag{3.2}$$

and, for $k \in \{1, \ldots, K_\tau\}$, the elliptic operator

$$A^k : X \to H^1(\Omega)^* \text{ defined by } \langle A^k(\theta), v\rangle_{H^1(\Omega)} := \int_\Omega \kappa(\theta)\nabla\theta\nabla v \,\mathrm{d}x - \int_{\partial\Omega} h_\tau^k v \,\mathrm{d}S\,. \tag{3.3}$$

We also mention in advance that, for technical reasons connected both with the proof of existence of discrete solutions to Problem 3.1 (cf. the upcoming Lemma 3.4), and with the rigorous derivation of a priori estimates on them (cf. Remark 3.2 below), it will be necessary to add the regularizing term $-\tau\mathrm{div}(|e_\tau^k|^{\gamma-2}e_\tau^k)$ to the discrete momentum equation, as well as the term $\tau|p_\tau^k|^{\gamma-2}p_\tau^k$ to the discrete plastic flow rule, with $\gamma > 4$. That is why, we will seek for discrete solutions with $e_\tau^k \in L^\gamma(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})$ and $p_\tau^k \in L^\gamma(\Omega;\mathbb{M}^{d\times d}_{\mathrm{D}})$, giving $E(u_\tau^k) \in L^\gamma(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})$ by the kinematic admissibility condition and thus, via Korn's inequality (2.2), $u_\tau^k \in W^{1,\gamma}_{\mathrm{Dir}}(\Omega;\mathbb{R}^d)$. Because of these regularizations, it will be necessary to supplement the discrete system with approximate initial data

$$\begin{aligned} &(e_\tau^0)_\tau \subset L^\gamma(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}}) \quad \text{such that } \sup_{\tau>0}\tau^{1/\gamma}\|e_\tau^0\|_{L^\gamma(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})} \le C \text{ and } e_\tau^0 \to e_0 \text{ in } L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}}),\\ &(p_\tau^0)_\tau \subset L^\gamma(\Omega;\mathbb{M}^{d\times d}_{\mathrm{D}}) \quad \text{such that } \sup_{\tau>0}\tau^{1/\gamma}\|p_\tau^0\|_{L^\gamma(\Omega;\mathbb{M}^{d\times d}_{\mathrm{D}})} \le C \text{ and } p_\tau^0 \to p_0 \text{ in } L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{D}}). \end{aligned} \tag{3.4}$$

Problem 3.1. *Let $\gamma > 4$. Starting from*

$$\vartheta_\tau^0 := \vartheta_0, \qquad u_\tau^0 := u_0, \qquad u_\tau^{-1} := u_0 - \tau\dot u_0, \qquad e_\tau^0 := e_\tau^0, \qquad p_\tau^0 := p_\tau^0, \tag{3.5}$$

find $\{(\vartheta_\tau^k, u_\tau^k, e_\tau^k, p_\tau^k)\}_{k=1}^{K_\tau} \subset X \times W^{1,\gamma}_{\mathrm{Dir}}(\Omega;\mathbb{R}^d) \times L^\gamma(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}}) \times L^\gamma(\Omega;\mathbb{M}^{d\times d}_{\mathrm{D}})$ *fulfilling for all* $k = 1, \ldots, K_\tau$

- *the discrete heat equation*

$$\begin{aligned} &\mathrm{D}_{k,\tau}(\vartheta) + A^k(\vartheta_\tau^k)\\ &= H_\tau^k + \mathrm{R}\left(\vartheta_\tau^{k-1}, \mathrm{D}_{k,\tau}(p)\right) + |\mathrm{D}_{k,\tau}(p)|^2 + \mathbb{D}\mathrm{D}_{k,\tau}(e) : \mathrm{D}_{k,\tau}(e) - \vartheta_\tau^k\mathbb{B} : \mathrm{D}_{k,\tau}(e) \quad \textit{in } H^1(\Omega)^*; \end{aligned} \tag{3.6a}$$

- *the kinematic admissibility* $(u_\tau^k, e_\tau^k, p_\tau^k) \in \mathcal{A}(w_\tau^k)$ *(in the sense of* (2.3)*);*
- *the discrete momentum balance*

$$\rho\int_\Omega \mathrm{D}^2_{k,\tau}(u)v\,\mathrm{d}x + \int_\Omega \sigma_\tau^k : E(v)\,\mathrm{d}x = \langle \mathcal{L}_\tau^k, v\rangle_{H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^d)} \quad \textit{for all } v \in W^{1,\gamma}_{\mathrm{Dir}}(\Omega;\mathbb{R}^d); \tag{3.6b}$$

- *the discrete plastic flow rule*

$$\zeta_\tau^k + \mathrm{D}_{k,\tau}(p) + \tau|p_\tau^k|^{\gamma-2}p_\tau^k \ni (\sigma_\tau^k)_{\mathrm{D}}, \quad \textit{with } \zeta_\tau^k \in \partial_{\dot p}\mathrm{R}\left(\vartheta_\tau^{k-1}, \mathrm{D}_{k,\tau}(p)\right), \qquad \textit{a.e. in } \Omega, \tag{3.6c}$$

where we have used the place-holder $\sigma_\tau^k := \mathbb{D}\mathrm{D}_{k,\tau}(e) + \mathbb{C}e_\tau^k + \tau|e_\tau^k|^{\gamma-2}e_\tau^k - \vartheta_\tau^k\mathbb{B}$.

Remark 3.2 (Main features of the time-discretization scheme)**.** Observe that the discrete heat equation (3.6a) is coupled with the momentum balance (3.6b) through the implicit term ϑ_τ^k, which therefore contributes to the stress σ_τ^k in (3.6b) and in (3.6c). This makes the time discretization scheme (3.6) fully implicit, as it is not possible to decouple any of the equations from the others. In turn, the 'implicit coupling' between the heat equation and the momentum balance is crucial for the argument leading to the (strict) positivity of the discrete temperatures: we refer to the proof of [RR15, Lemma 4.4] for all details. In fact, the time discretization schemes in [BR11, Rou13] are fully implicit as well, again in view of the positivity of the temperature (though the arguments there are different, based on the approach via the enthalpy transformation in the heat equation).

The role of the terms $-\tau\mathrm{div}(|e_\tau^k|^{\gamma-2}e_\tau^k)$ and $\tau|p_\tau^k|^{\gamma-2}p_\tau^k$, added to the discrete momentum equation and plastic flow rule, respectively, is to 'compensate' the quadratic terms on the right-hand side of (3.6a). More precisely, they ensure that the pseudomonotone operator by means of which we will reformulate our approximation of system (3.6), i.e. (3.14) ahead, is coercive, in particular w.r.t. the $H^1(\Omega)$-norm in the variable ϑ. This will allow us to apply a result from the theory of pseudomonotone operators in order to obtain the existence of solutions to (3.14) and, a fortiori, to (3.6).

Proposition 3.3 (Existence of discrete solutions)**.** *Under the growth condition* (2.κ_1)*, Problem 3.1 admits a solution* $\{(\vartheta_\tau^k, u_\tau^k, e_\tau^k, p_\tau^k)\}_{k=1}^{K_\tau}$*. Furthermore, any solution to Problem 3.1 fulfills*

$$\vartheta_\tau^k \ge \bar\vartheta > 0 \qquad \textit{for all } k = 1, \ldots, K_\tau, \tag{3.7}$$

with $\bar\vartheta$ *from* (2.33)*.*

Along the lines of [RR15, LRTT14], we will prove the existence of a solution to Problem 3.1 by

(1) constructing an approximate problem where the thermal conductivity coefficient κ is truncated and, accordingly, so are the occurrences of ϑ in the thermal expansion terms coupling the discrete heat and momentum equations, cf. system (3.12) below;
(2) proving the existence of a solution to the approximate discrete problem by resorting to a general existence result from [Rou05] for elliptic systems featuring pseudomonotone operators;
(3) passing to the limit with the truncation parameter.

As the statement of Proposition 3.3 suggests, the positivity property (3.7) can be proved for *all* discrete solutions to Problem 3.1 (i.e. not only for those deriving from the aforementioned approximation procedure). Since its proof can be carried out by repeating the arguments for positivity in [RR15, LRTT14], we choose to omit it and refer to these papers for all details. We shall instead focus on the existence argument, dwelling with some detail on the parts which differ from [RR15, LRTT14].

The proof of Proposition 3.3 will be split some steps:
Step 1: existence for the approximate discrete system. We introduce the truncation operator

$$\mathcal{T}_M : \mathbb{R} \to \mathbb{R}, \qquad \mathcal{T}_M(r) := \begin{cases} -M & \text{if } r < -M, \\ r & \text{if } |r| \leq M, \\ M & \text{if } r > M, \end{cases} \tag{3.8}$$

and define

$$\kappa_M(r) := \kappa(\mathcal{T}_M(r)) := \begin{cases} \kappa(-M) & \text{if } r < -M, \\ \kappa(r) & \text{if } |r| \leq M, \\ \kappa(M) & \text{if } r > M, \end{cases} \tag{3.9}$$

$$A_M^k : H^1(\Omega) \to H^1(\Omega)^* \text{ by } \langle A_M^k(\theta), v \rangle_{H^1(\Omega)} := \int_\Omega \kappa_M(\theta) \nabla\theta \nabla v \,\mathrm{d}x - \int_{\partial\Omega} h_\tau^k v \,\mathrm{d}S. \tag{3.10}$$

For later use, we observe that, thanks to (2.κ_1) there still holds $\kappa_M(r) \geq c_0$ for all $r \in \mathbb{R}$, and therefore

$$\forall\, \delta > 0 \ \ \exists\, C_\delta > 0 \ \ \forall\, \theta \in H^1(\Omega) \ : \qquad \langle \mathcal{A}_M^k(\theta), \theta \rangle_{H^1(\Omega)} \geq c_0 \int_\Omega |\nabla\theta|^2 \,\mathrm{d}x - \delta \|\theta\|_{L^2(\partial\Omega)}^2 - C_\delta \|h_\tau^k\|_{L^2(\partial\Omega)}^2 . \tag{3.11}$$

The approximate version of system (3.6) reads (to avoid overburdening notation, for the time being we will not highlight the dependence of the solution quadruple on the truncation parameter M):

$$\begin{aligned} &\mathrm{D}_{k,\tau}(\vartheta) + A_M^k(\vartheta_\tau^k) \\ &= H_\tau^k + \mathrm{R}\left(\vartheta_\tau^{k-1}, \mathrm{D}_{k,\tau}(p)\right) + |\mathrm{D}_{k,\tau}(p)|^2 + \mathbb{D}\mathrm{D}_{k,\tau}(e) : \mathrm{D}_{k,\tau}(e) - \mathcal{T}_M(\vartheta_\tau^k)\mathbb{B} : \mathrm{D}_{k,\tau}(e) \quad \text{in } H^1(\Omega)^*; \end{aligned} \tag{3.12a}$$

$$\rho \int_\Omega \mathrm{D}_{k,\tau}^2(u) v \,\mathrm{d}x + \int_\Omega \sigma_\tau^k : E(v) \,\mathrm{d}x = \langle \mathcal{L}_\tau^k, v \rangle_{H_{\mathrm{Dir}}^1(\Omega;\mathbb{R}^d)} \quad \text{ for all } v \in W_{\mathrm{Dir}}^{1,\gamma}(\Omega;\mathbb{R}^d); \tag{3.12b}$$

$$\zeta_\tau^k + \mathrm{D}_{k,\tau}(p) + \tau |p_\tau^k|^{\gamma-2} p_\tau^k \ni (\sigma_{M,\tau}^k)_{\mathrm{D}}, \quad \text{with } \zeta_\tau^k \in \partial_{\dot{p}} \mathrm{R}\left(\vartheta_\tau^{k-1}, \mathrm{D}_{k,\tau}(p)\right), \qquad \text{a.e. in } \Omega, \tag{3.12c}$$

coupled with the kinematic admissibility

$$(u_\tau^k, e_\tau^k, p_\tau^k) \in \mathcal{A}(w_\tau^k), \tag{3.13}$$

where now

$$\sigma_\tau^k := \mathbb{D}\mathrm{D}_{k,\tau}(e) + \mathbb{C} e_\tau^k + \tau |e_\tau^k|^{\gamma-2} e_\tau^k - \mathcal{T}_M(\vartheta_\tau^k)\mathbb{B} .$$

The following result states the existence of solutions to system (3.12) for $k \in \{1, \ldots, K_\tau\}$ fixed: in its proof, we make use of the higher order terms added to the discrete momentum equation and plastic flow rule.

Lemma 3.4. *Under the growth condition* (2.κ_1)*, there exists $\bar\tau > 0$ such that for $0 < \tau < \bar\tau$ and for every $k = 1, \ldots, K_\tau$ there exists a solution $(\vartheta_\tau^k, u_\tau^k, e_\tau^k, p_\tau^k) \in H^1(\Omega) \times W_{\mathrm{Dir}}^{1,\gamma}(\Omega;\mathbb{R}^d) \times L^\gamma(\Omega;\mathbb{M}_{\mathrm{sym}}^{d\times d}) \times L^\gamma(\Omega;\mathbb{M}_{\mathrm{D}}^{d\times d})$ to system* (3.12)*, such that ϑ_τ^k complies with the positivity property* (3.7)*.*

Proof. The positivity (3.7) follows from the same argument developed in the proof of [RR15, Lemma 4.4]. As for existence: For fixed $k \in \{1, \ldots, K_\tau\}$, we reformulate system (3.12), coupled with (3.13), as

$$\partial\Psi_k(\vartheta_\tau^k, u_\tau^k - w_\tau^k, p_\tau^k) + \mathscr{A}_k(\vartheta_\tau^k, u_\tau^k - w_\tau^k, p_\tau^k) \ni \mathscr{B}_k, \tag{3.14}$$

where the elliptic operator $\mathscr{A}_k : \mathbf{B} \to \mathbf{B}^*$, with $\mathbf{B} := H^1(\Omega) \times W^{1,\gamma}_{\mathrm{Dir}}(\Omega;\mathbb{R}^d) \times L^\gamma(\Omega;\mathbb{M}^{d\times d}_{\mathrm{D}})$, is given componentwise by

$$\begin{aligned}\mathscr{A}_k^1(\vartheta,\tilde u,p) :=&\vartheta + A_M^k(\vartheta) - \mathrm{R}(\vartheta_\tau^{k-1}, p - p_\tau^{k-1}) - \frac{1}{\tau}|p|^2 - \frac{2}{\tau} p : p_\tau^{k-1}\\ &- \frac{1}{\tau}\mathbb{D}\left(E(\tilde u + w_\tau^k) - p\right) : \left(E(\tilde u + w_\tau^k) - p\right) - \frac{2}{\tau}\mathbb{D}\left(E(\tilde u + w_\tau^k) - p\right) : e_\tau^{k-1}\\ &+ \mathfrak{T}_M(\vartheta)\mathbb{B}\left(E(\tilde u + w_\tau^k) - p - e_\tau^{k-1}\right),\end{aligned} \tag{3.15a}$$

$$\begin{aligned}\mathscr{A}_k^2(\vartheta,\tilde u,p) := \rho(\tilde u - w_\tau^k) - \mathrm{div}_{\mathrm{Dir}}\Big(&\tau\mathbb{D}\left(E(\tilde u + w_\tau^k) - p\right) + \tau^2\mathbb{C}\left(E(\tilde u + w_\tau^k) - p\right)\\ &+ \tau^3\left|E(\tilde u + w_\tau^k) - p\right|^{\gamma-2}\left(E(\tilde u + w_\tau^k) - p\right) - \tau^2\mathfrak{T}_M(\vartheta)\mathbb{B}\Big),\end{aligned} \tag{3.15b}$$

$$\begin{aligned}\mathscr{A}_k^3(\vartheta,\tilde u,p) := p + \tau^2|p|^{\gamma-2}p - \Big(&\mathbb{D}\left(E(\tilde u + w_\tau^k) - p\right) + \tau\mathbb{C}\left(E(\tilde u + w_\tau^k) - p\right)\\ &+ \tau^2\left|E(\tilde u + w_\tau^k) - p\right|^{\gamma-2}\left(E(\tilde u + w_\tau^k) - p\right) - \tau\mathfrak{T}_M(\vartheta)\mathbb{B}\Big)_{\mathrm{D}},\end{aligned} \tag{3.15c}$$

with $-\mathrm{div}_{\mathrm{Dir}}$ defined by (2.7), while the vector $\mathscr{B}_k \in \mathbf{B}^*$ on the right-hand side of (3.14) has components

$$\mathscr{B}_k^1 := H_\tau^k + \frac{1}{\tau}|p_\tau^{k-1}|^2 + \frac{1}{\tau}\mathbb{D}e_\tau^{k-1} : e_\tau^{k-1}, \tag{3.16a}$$

$$\mathscr{B}_k^2 := \mathcal{L}_\tau^k + 2\rho u_\tau^{k-1} - \rho u_\tau^{k-1} - \mathrm{div}_{\mathrm{Dir}}(\tau\mathbb{D}e_\tau^{k-1}), \tag{3.16b}$$

$$\mathscr{B}_k^3 := p_\tau^{k-1} - (\mathbb{D}e_\tau^{k-1})_{\mathrm{D}}, \tag{3.16c}$$

and $\partial\Psi_k : \mathbf{B} \rightrightarrows \mathbf{B}^*$ is the subdifferential of the lower semicontinuous and convex potential $\Psi_k(\vartheta,\tilde u,p) := \mathrm{R}(\vartheta_\tau^{k-1}, p - p_\tau^{k-1})$. We shall therefore prove the existence of a solution to the abstract subdifferential inclusion (3.14) by applying the existence result [Rou05, Thm. 5.15], which amounts to verifying that $\mathscr{A}_k : \mathbf{B} \to \mathbf{B}^*$ is pseudomonotone and coercive.

To check coercivity, we compute

$$\begin{aligned}\langle\mathscr{A}_k(\vartheta,\tilde u,p),(\vartheta,\tilde u,p)\rangle_{\mathbf{B}} =&\ \langle\mathscr{A}_k^1(\vartheta,\tilde u,p),\vartheta\rangle_{H^1(\Omega)} + \langle\mathscr{A}_k^2(\vartheta,\tilde u,p),\tilde u\rangle_{W^{1,\gamma}_{\mathrm{Dir}}(\Omega;\mathbb{R}^d)} + \int_\Omega \mathscr{A}_k^3(\vartheta,\tilde u,p) : p\,\mathrm{d}x\\ \overset{(1)}{\geq}&\ \|\vartheta\|^2_{L^2(\Omega)} + c_0\|\nabla\vartheta\|^2_{L^2(\Omega)} + \rho\|\tilde u\|^2_{L^2(\Omega)} + \left(\tau C_{\mathbb{D}}^1 + \tau^2 C_{\mathbb{C}}^1\right)\|E(\tilde u) + E(w_\tau^k) - p\|^2_{L^2(\Omega)}\\ &+ \tau^3\|E(\tilde u) + E(w_\tau^k) - p\|^\gamma_{L^\gamma(\Omega)} + \|p\|^2_{L^2(\Omega)} + \tau^2\|p\|^\gamma_{L^\gamma(\Omega)} - I_1 - I_2 - I_3\end{aligned} \tag{3.17}$$

where (1) follows from (2.T) and (3.11), with $\delta > 0$ to be specified. Taking into account (2.11a), again (2.T), and the fact that $|\mathfrak{T}_M(\vartheta)| \leq M$ a.e. in Ω, we have

$$\begin{aligned}I_1 =& -\delta\|\vartheta\|^2_{L^2(\partial\Omega)} - C_\delta\|h_\tau^k\|^2_{L^2(\partial\Omega)}\\ &- C_R\|p - p_\tau^{k-1}\|_{L^2(\Omega)}\|\vartheta\|_{L^2(\Omega)} - \frac{1}{\tau}\|p\|^2_{L^4(\Omega)}\|\vartheta\|_{L^2(\Omega)} - \frac{2}{\tau}\|p_\tau^{k-1}\|_{L^4(\Omega)}\|p\|_{L^4(\Omega)}\|\vartheta\|_{L^2(\Omega)}\\ &- \frac{C_{\mathbb{D}}^2}{\tau}\|E(\tilde u) + E(w_\tau^k) - p\|^2_{L^4(\Omega)}\|\vartheta\|_{L^2(\Omega)} - C\|E(\tilde u) + E(w_\tau^k) - p\|_{L^4(\Omega)}\|e_\tau^{k-1}\|_{L^4(\Omega)}\|\vartheta\|_{L^2(\Omega)}\\ &- C\|E(\tilde u) + E(w_\tau^k) - p\|^2_{L^2(\Omega)} - C\|e_\tau^{k-1}\|^2_{L^2(\Omega)},\end{aligned} \tag{3.18a}$$

as well as

$$\begin{aligned}I_2 =& -\rho\|\tilde u\|_{L^2(\Omega)}\|w_\tau^k\|_{L^2(\Omega)} - C\|E(\tilde u) + E(w_\tau^k) - p\|_{L^2(\Omega)}\|E(w_\tau^k) - p\|_{L^2(\Omega)}\\ &- \tau^3\int_\Omega |E(\tilde u) + E(w_\tau^k) - p|^{\gamma-1}|E(w_\tau^k) - p|\,\mathrm{d}x - C\int_\Omega |E(\tilde u)|\,\mathrm{d}x,\end{aligned} \tag{3.18b}$$

and

$$I_3 = -C\|E(\tilde u) + E(w_\tau^k) - p\|_{L^2(\Omega)}\|p\|_{L^2(\Omega)} - \tau^2\int_\Omega |E(\tilde u) + E(w_\tau^k) - p|^{\gamma-1}|p|\,\mathrm{d}x - C\int_\Omega |p|\,\mathrm{d}x\,. \tag{3.18c}$$

Now, with straightforward calculations it is possible to absorb the negative terms I_1, I_2, I_3 into the positive terms on the right-hand side of (3.17): without entering into details, let us only observe that, for example, the sixth term on the right-hand side of (3.18a) can be estimated by means of Young's inequality as

$$
\begin{aligned}
-\frac{C_{\mathbb{D}}^2}{\tau}\|E(\tilde u)+E(w_\tau^k)-p\|_{L^4(\Omega)}^2\|\vartheta\|_{L^2(\Omega)} &\geq -\delta\|\vartheta\|_{L^2(\Omega)}^2 - C\|E(\tilde u)+E(w_\tau^k)-p\|_{L^4(\Omega)}^4\\
&\geq -\delta\|\vartheta\|_{L^2(\Omega)}^2 - \frac{\tau^3}{2}\|E(\tilde u)+E(w_\tau^k)-p\|_{L^\gamma(\Omega)}^\gamma - C,
\end{aligned}
$$

using that $\gamma>4$. The fourth term can be dealt with in the same way, so that one of the resulting terms is absorbed into $\tau^2\|p\|_{L^\gamma(\Omega)}^\gamma$. The other terms contributing to I_1, I_2, and I_3 can be handled analogously. Let us now observe that the positive terms on the right-hand side of (3.17) bound the desired norms of ϑ, $\tilde u$, p. Indeed, also taking into account that, again by Young's inequality

$$
\|E(\tilde u)+E(w_\tau^k)-p\|_{L^2(\Omega)}^2 \geq c\|E(\tilde u)\|_{L^2(\Omega)}^2 - C\|E(w_\tau^k)\|_{L^2(\Omega)}^2 - \frac{\tau^2}{4}\|p\|_{L^\gamma(\Omega)}^\gamma - C,
$$

and repeatedly using the well-known estimate $(a+b)^\gamma \leq 2^{\gamma-1}(a^\gamma+b^\gamma)$ for all $a,\,b\in[0,+\infty)$, which gives

$$
\begin{aligned}
\tau^3\|E(\tilde u)+E(w_\tau^k)-p\|_{L^\gamma(\Omega)}^\gamma + \frac{\tau^2}{4}\|p\|_{L^\gamma(\Omega)}^\gamma &\geq \frac{\tau^3}{2^{\gamma-1}}\|E(\tilde u)+E(w_\tau^k)\|_{L^\gamma(\Omega)}^\gamma + \left(\frac{\tau^2}{4}-\tau^3\right)\|p\|_{L^\gamma(\Omega)}^\gamma\\
&\geq \frac{\tau^3}{2^{2\gamma-2}}\|E(\tilde u)\|_{L^\gamma(\Omega)}^\gamma + \frac{\tau^2}{8}\|p\|_{L^\gamma(\Omega)}^\gamma - \frac{\tau^3}{2^{\gamma-1}}\|E(w_\tau^k)\|_{L^\gamma(\Omega)}^\gamma
\end{aligned}
$$

(where we have also used that, for $\tau<\bar\tau:=1/8$, there holds $\tau^2/8\geq\tau^3$), we end up with

$$
\langle \mathscr{A}_k(\vartheta,\tilde u,p),(\vartheta,\tilde u,p)\rangle_{\mathbf{B}} \geq c\left(\|\vartheta\|_{H^1(\Omega)}^2 + \|\tilde u\|_{L^2(\Omega)}^2 + \|E(\tilde u)\|_{L^2(\Omega)}^2 + \|E(\tilde u)\|_{L^\gamma(\Omega)}^\gamma + \|p\|_{L^2(\Omega)}^2 + \|p\|_{L^\gamma(\Omega)}^\gamma\right) - C
$$

for two positive constants c and C, depending on τ, on M, and on w. Thanks to Korn's inequality (2.2), this shows the coercivity of $\mathscr{A}_k$. Its pseudomonotonicity ensues from standard arguments. We thus conclude the existence of solutions to system (3.12). $\square$

Step 2: a priori estimates on the solutions of the approximate discrete system. Let now

$$
(\vartheta_{M,\tau}^k, u_{M,\tau}^k, e_{M,\tau}^k, p_{M,\tau}^k)_M
$$

be a family of solutions to system (3.12). The following result collects a series of a priori estimates uniform w.r.t. the parameter M (but not w.r.t. τ): a crucial ingredient to derive them will be a discrete version of the total energy inequality (2.26), cf. (3.20) below, featuring the discrete total energy

$$
\mathcal{E}_\tau(\vartheta,e,p) := \int_\Omega \vartheta\,\mathrm{d}x + \frac12\int_\Omega \mathbb{C}e : e\,\mathrm{d}x + \frac{\tau}{\gamma}\int_\Omega (|e|^\gamma+|p|^\gamma)\,\mathrm{d}x\,. \tag{3.19}
$$

Lemma 3.5. *Let $k\in\{1,\ldots,K_\tau\}$ and $\tau\in(0,\bar\tau)$ be fixed. Under the growth condition (2.κ_1), the solution quadruple $(\vartheta_{M,\tau}^k, u_{M,\tau}^k, e_{M,\tau}^k, p_{M,\tau}^k)$ to (3.12) satisfies*

$$
\begin{aligned}
&\frac{\rho}{2}\int_\Omega\left|\frac{u_{M,\tau}^k-u_\tau^{k-1}}{\tau}\right|^2\mathrm{d}x + \mathcal{E}_\tau(\vartheta_{M,\tau}^k, e_{M,\tau}^k, p_{M,\tau}^k)\\
&\leq \frac{\rho}{2}\int_\Omega\left|\frac{u_\tau^{k-1}-u_\tau^{k-2}}{\tau}\right|^2\mathrm{d}x + \mathcal{E}_\tau(\vartheta_\tau^{k-1}, e_\tau^{k-1}, p_\tau^{k-1}) + \tau\int_\Omega H_\tau^k\,\mathrm{d}x + \tau\int_{\partial\Omega} h_\tau^k\,\mathrm{d}x\\
&\quad + \tau\,\langle \mathcal{L}_\tau^k, \frac{u_{M,\tau}^k-u_\tau^{k-1}}{\tau} - \mathrm{D}_{k,\tau}(w)\rangle_{H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^d)} + \tau\int_\Omega \sigma_{M,\tau}^k : E(\mathrm{D}_{k,\tau}(w))\\
&\quad + \rho\int_\Omega\left(\frac{u_{M,\tau}^k-u_\tau^{k-1}}{\tau} - \mathrm{D}_{k-1,\tau}(u)\right)\mathrm{D}_{k,\tau}(w)\,\mathrm{d}x\,.
\end{aligned} \tag{3.20}
$$

Moreover, there exists a constant $C>0$ such that for all $M>0$

$$
\|\vartheta_{M,\tau}^k\|_{L^1(\Omega)} + \|u_{M,\tau}^k\|_{L^2(\Omega;\mathbb{R}^d)} + \|e_{M,\tau}^k\|_{L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})} \leq C, \tag{3.21a}
$$

$$
\tau^{1/\gamma}\|u_{M,\tau}^k\|_{W^{1,\gamma}(\Omega;\mathbb{R}^d)} + \tau^{1/\gamma}\|e_{M,\tau}^k\|_{L^\gamma(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})} + \tau^{1/\gamma}\|p_{M,\tau}^k\|_{L^\gamma(\Omega;\mathbb{M}^{d\times d}_{\mathrm{D}})} \leq C, \tag{3.21b}
$$

$$
\|\vartheta_{M,\tau}^k\|_{H^1(\Omega)} + \|\vartheta_{M,\tau}^k\|_{L^{3\mu+6}(\Omega)} \leq C \qquad \textit{with } \mu>1 \textit{ from } (2.\kappa_1), \tag{3.21c}
$$

$$
\|\zeta_\tau^k\|_{L^\infty(\Omega;\mathbb{M}^{d\times d}_{\mathrm{D}})} \leq C, \tag{3.21d}
$$

where $\zeta_\tau^k\in\partial_{\dot p}\mathrm{R}(\vartheta_\tau^{k-1},(p_{M,\tau}^k-p_\tau^{k-1})/\tau)$ fulfills (3.12c).

Proof. Inequality (3.20) follows by multiplying (3.12a) by τ, testing (3.12b) by $u^k_{M,\tau} - w^k_\tau - (u^{k-1}_\tau - w^{k-1}_\tau)$, and (3.12c) by $p^k_{M,\tau} - p^{k-1}_{M,\tau}$. We integrate in space the resulting relations and develop the following estimates

$$\begin{aligned} &\frac{\rho}{\tau^2}\int_\Omega \left(u^k_{M,\tau}-u^{k-1}_\tau-(u^{k-1}_\tau-u^{k-2}_\tau)\right)(u^k_{M,\tau}-u^{k-1}_\tau)\,\mathrm{d}x \\ &\geq \frac{\rho}{2}\left\|\frac{u^k_{M,\tau}-u^{k-1}_\tau}{\tau}\right\|^2_{L^2(\Omega)} - \frac{\rho}{2}\left\|\frac{u^{k-1}_\tau-u^{k-2}_\tau}{\tau}\right\|^2_{L^2(\Omega)}, \end{aligned} \tag{3.22a}$$

$$\begin{aligned} &\int_\Omega \mathbb{D}\left(\frac{e^k_{M,\tau}-e^{k-1}_\tau}{\tau}\right):E(u^k_{M,\tau}-u^{k-1}_\tau)\,\mathrm{d}x \\ &= \tau\int_\Omega \mathbb{D}\frac{e^k_{M,\tau}-e^{k-1}_\tau}{\tau}:\frac{e^k_{M,\tau}-e^{k-1}_\tau}{\tau}\,\mathrm{d}x + \int_\Omega \mathbb{D}\frac{e^k_{M,\tau}-e^{k-1}_\tau}{\tau}:(p^k_{M,\tau}-p^{k-1}_\tau)\,\mathrm{d}x, \end{aligned} \tag{3.22b}$$

$$\begin{aligned} \int_\Omega \mathbb{C}e^k_{M,\tau}:E(u^k_{M,\tau}-u^{k-1}_\tau)\,\mathrm{d}x &= \int_\Omega \mathbb{C}e^k_{M,\tau}:(e^k_{M,\tau}-e^{k-1}_\tau) + \mathbb{C}e^k_{M,\tau}:(p^k_{M,\tau}-p^{k-1}_\tau)\,\mathrm{d}x \\ &\geq \int_\Omega \left(\tfrac12\mathbb{C}e^k_{M,\tau}:e^k_{M,\tau} - \tfrac12\mathbb{C}e^{k-1}_\tau:e^{k-1}_\tau + \mathbb{C}e^k_{M,\tau}:(p^k_{M,\tau}-p^{k-1}_\tau)\right)\mathrm{d}x, \end{aligned} \tag{3.22c}$$

$$\begin{aligned} &\int_\Omega |e^k_{M,\tau}|^{\gamma-2}e^k_{M,\tau}:E(u^k_{M,\tau}-u^{k-1}_\tau)\,\mathrm{d}x \\ &= \int_\Omega |e^k_{M,\tau}|^{\gamma-2}e^k_{M,\tau}:(e^k_{M,\tau}-e^{k-1}_\tau)\,\mathrm{d}x + \int_\Omega |e^k_{M,\tau}|^{\gamma-2}e^k_{M,\tau}:(p^k_{M,\tau}-p^{k-1}_\tau)\,\mathrm{d}x \\ &\geq \int_\Omega \left(\tfrac1\gamma|e^k_{M,\tau}|^\gamma - \frac1\gamma|e^{k-1}_\tau|^\gamma + |e^k_{M,\tau}|^{\gamma-2}e^k_{M,\tau}:(p^k_{M,\tau}-p^{k-1}_\tau)\right)\mathrm{d}x. \end{aligned} \tag{3.22d}$$

Observe that (3.22b)–(3.22d) mimic the calculations on the time-continuous level leading to (2.20) and in fact rely on the kinematic admissibility condition. The terms on the right-hand side of (3.22b) cancel with the fourth term on the r.h.s. of (3.12a), multiplied by τ, and with the analogous term deriving from (3.12c), tested by $p^k_{M,\tau}-p^{k-1}_\tau$. In the same way, the last terms on the r.h.s. of (3.22c) and (3.22d) cancel with the ones coming from (3.12c). In fact, it can be easily checked that, with the exception of τH^k_τ, all the terms on the r.h.s. of (3.12a) cancel out: for instance, $\tau\int_\Omega \mathrm{R}(\vartheta^{k-1}_\tau, p^k_{M,\tau}-p^{k-1}_\tau)\,\mathrm{d}x$ cancels with the term $\int_\Omega \zeta^k_{\tau,M}:(p^k_{M,\tau}-p^{k-1}_\tau)\,\mathrm{d}x$ in view of (2.13). In this way, we conclude (3.20).

In order to derive estimates (3.21a)–(3.21b), we observe that that the first four terms on the right-hand side of (3.20) are bounded, depending on the quantities $\|u^{k-1}_\tau\|_{L^2(\Omega;\mathbb{R}^d)}$, $\mathcal{E}_\tau(\vartheta^{k-1}_\tau, e^{k-1}_\tau, p^{k-1}_\tau)$, $\|H^k_\tau\|_{L^1(\Omega)}$, $\|h^k_\tau\|_{L^2(\partial\Omega)}$, whereas the remaining ones can be controlled by the ones on the left-hand side. In fact, we have

$$\begin{aligned} \left|\tau\left\langle \mathcal{L}^k_\tau, \frac{u^k_{M,\tau}-u^{k-1}_\tau}{\tau} - \mathrm{D}_{k,\tau}(w)\right\rangle_{H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^d)}\right| &\overset{(1)}{\leq} \delta\|u^k_{M,\tau}-w^k_\tau\|^2_{H^1(\Omega;\mathbb{R}^d)} + \delta\|u^{k-1}_\tau-w^{k-1}_\tau\|^2_{H^1(\Omega;\mathbb{R}^d)} + C_\delta\|\mathcal{L}^k_\tau\|^2_{H^1(\Omega;\mathbb{R}^d)^*} \\ &\overset{(2)}{\leq} \delta C_K\|E(u^k_{M,\tau})\|^2_{L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})} + C \\ &\overset{(3)}{\leq} 2\delta C_K^2\|e^k_{M,\tau}\|^2_{L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})} + 2\delta C_K\|p^k_{M,\tau}\|^2_{L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{D}})} + C, \end{aligned}$$

$$\begin{aligned} &\left|\tau\int_\Omega \sigma^k_{M,\tau}:E(\mathrm{D}_{k,\tau}(w))\,\mathrm{d}x\right| \\ &\overset{(4)}{\leq} \frac{\delta}{\tau}\|e^k_{M,\tau}-e^{k-1}_\tau\|^2_{L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})} + \delta\tau\|e^k_{M,\tau}\|^2_{L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})} + C_\delta\|E(\mathrm{D}_{k,\tau}(w))\|^2_{L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})} \\ &\quad + C\|E(\mathrm{D}_{k,\tau}(w))\|_{L^\infty(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})}\left(\int_\Omega |\vartheta^k_{M,\tau}| + |e^k_{M,\tau}|^{\gamma-1}\,\mathrm{d}x\right) \end{aligned}$$

$$\begin{aligned} \left|\rho\int_\Omega \left(\frac{u^k_{M,\tau}-u^{k-1}_\tau}{\tau} - \mathrm{D}_{k-1,\tau}(u)\right)\mathrm{D}_{k,\tau}(w)\,\mathrm{d}x\right| &\leq \frac{\rho}{4}\int_\Omega \left|\frac{u^k_{M,\tau}-u^{k-1}_\tau}{\tau}\right|^2\mathrm{d}x + \frac{\rho}{4}\|\mathrm{D}_{k-1,\tau}(u)\|^2_{L^2(\Omega;\mathbb{R}^d)} \\ &\quad + \rho\|\mathrm{D}_{k,\tau}(w)\|^2_{L^2(\Omega;\mathbb{R}^d)}, \end{aligned}$$

where $\delta > 0$ in (1) and in the other estimates is an arbitrary positive constant, to be specified later, while (2) ensues from Korn's inequality (2.2) and from the bounds on the quantities $\|\mathcal{L}^k_\tau\|_{H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^d)^*}$, $\|w^k_\tau\|_{H^1(\Omega;\mathbb{R}^d)}$, $\|w^{k-1}_\tau\|_{H^1(\Omega;\mathbb{R}^d)}$, $\|u^{k-1}_\tau\|_{H^1(\Omega;\mathbb{R}^d)}$, and (3) from the kinematic admissibility condition. For (4) we have used that

$\sigma_{M,\tau}^k := \mathbb{D}\mathrm{D}_{k,\tau}(e)+\mathbb{C}e_\tau^k+\tau|e_\tau^k|^{\gamma-2}e_\tau^k-\mathcal{T}_M(\vartheta_\tau^k)\mathbb{B}$, as well as the fact that $\|\mathcal{T}_M(\vartheta_{M,\tau}^k)\|_{L^1(\Omega)} \leq \|\vartheta_{M,\tau}^k\|_{L^1(\Omega)}$. It is now immediate to check that the terms on the right-hand sides of the above estimates are either bounded, due to our assumptions, or can be absorbed into the left-hand side of (3.20), suitably tuning the positive constant δ. All in all, we conclude that

$$\int_\Omega \left| \frac{u_{M,\tau}^k - u_\tau^{k-1}}{\tau} \right|^2 \mathrm{d}x + \mathcal{E}_\tau(\vartheta_{M,\tau}^k, e_{M,\tau}^k, p_{M,\tau}^k) \leq C$$

for a constant independent of M. Estimates (3.21a) and (3.21b) then ensue, also taking into account Korn's inequality.

Estimate (3.21c) is proved in two steps, by testing (3.12a) first by $\mathcal{T}_M(\vartheta_{M,\tau}^k)$, and secondly by $\vartheta_{M,\tau}^k$. We refer to the proof of [RR15, Lemma 4.4] for all the calculations.

Estimate (3.21d) follows from the fact that $\zeta_{\tau,M}^k \in \partial_{\dot{p}}\mathrm{R}(\vartheta_\tau^{k-1}, (p_{M,\tau}^k - p_\tau^{k-1})/\tau)$ and from (2.11b). □

Step 3: limit passage in the approximate discrete system. With the following result we conclude the proof of Proposition 3.3. From now on, we suppose that $M \in \mathbb{N} \setminus \{0\}$.

Lemma 3.6. *Let $k \in \{1, \ldots, K_\tau\}$ and $\tau \in (0, \bar{\tau})$ be fixed. Under the growth condition $(2.\kappa_1)$, there exist a (not relabeled) subsequence of $(\vartheta_{M,\tau}^k, u_{M,\tau}^k, e_{M,\tau}^k, p_{M,\tau}^k)_M$ and of $(\zeta_{\tau,M}^k)_M$, and a quadruple $(\vartheta_\tau^k, u_\tau^k, e_\tau^k, p_\tau^k) \in H^1(\Omega) \times W_{\mathrm{Dir}}^{1,\gamma}(\Omega;\mathbb{R}^d) \times L^\gamma(\Omega;\mathbb{M}_{\mathrm{sym}}^{d\times d}) \times L^\gamma(\Omega;\mathbb{M}_{\mathrm{D}}^{d\times d})$ and $\zeta_\tau^k \in L^\infty(\Omega;\mathbb{M}_{\mathrm{D}}^{d\times d})$, such that the following convergences hold as $M \to \infty$*

$$\vartheta_{M,\tau}^k \rightharpoonup \vartheta_\tau^k \qquad \text{in } H^1(\Omega) \cap L^{3\mu+6}(\Omega), \tag{3.23a}$$

$$u_{M,\tau}^k \to u_\tau^k \qquad \text{in } W_{\mathrm{Dir}}^{1,\gamma}(\Omega;\mathbb{R}^d), \tag{3.23b}$$

$$e_{M,\tau}^k \to e_\tau^k \qquad \text{in } L^\gamma(\Omega;\mathbb{M}_{\mathrm{sym}}^{d\times d}), \tag{3.23c}$$

$$p_{M,\tau}^k \to p_\tau^k \qquad \text{in } L^\gamma(\Omega;\mathbb{M}_{\mathrm{D}}^{d\times d}), \tag{3.23d}$$

$$\zeta_{\tau,M}^k \overset{*}{\rightharpoonup} \zeta_\tau^k \qquad \text{in } L^\infty(\Omega;\mathbb{M}_{\mathrm{D}}^{d\times d}), \tag{3.23e}$$

and the quintuple $(\vartheta_\tau^k, u_\tau^k, e_\tau^k, p_\tau^k, \zeta_\tau^k)$ fulfill system (3.6).

Proof. It follows from estimates (3.21) that convergences (3.23a), (3.23e), and the weak versions of (3.23b)–(3.23d) hold as $M \to \infty$, along a suitable subsequence. Moreover, there exist $\varepsilon_\tau^k \in L^{\gamma/(\gamma-1}(\Omega;\mathbb{M}_{\mathrm{sym}}^{d\times d})$ and $\pi_\tau^k \in L^{\gamma/(\gamma-1}(\Omega;\mathbb{M}_{\mathrm{D}}^{d\times d})$ such that

$$|e_{M,\tau}^k|^{\gamma-2}e_{M,\tau}^k \rightharpoonup \varepsilon_\tau^k \quad \text{in } L^{\gamma/(\gamma-1}(\Omega;\mathbb{M}_{\mathrm{sym}}^{d\times d}), \qquad |p_{M,\tau}^k|^{\gamma-2}p_{M,\tau}^k \rightharpoonup \pi_\tau^k \quad \text{in } L^{\gamma/(\gamma-1}(\Omega;\mathbb{M}_{\mathrm{D}}^{d\times d}).$$

Furthermore, from (3.23a) one deduces that $\vartheta_{M,\tau}^k \to \vartheta_\tau^k$ strongly in $L^{3\mu+6-\rho}(\Omega)$ for all $\rho \in (0, 3\mu+5]$. Hence, it is not difficult to conclude that

$$\mathcal{T}_M(\vartheta_{M,\tau}^k) \to \vartheta_\tau^k \quad \text{in } L^{3\mu+6-\rho}(\Omega) \text{ for all } \rho \in (0, 3\mu+5]. \tag{3.24}$$

With these convergences at hand, it is possible to pass to the limit in (3.12b)–(3.12c) and prove that the functions $(\vartheta_\tau^k, u_\tau^k, e_\tau^k, p_\tau^k, \zeta_\tau^k, \varepsilon_\tau^k, \pi_\tau^k)$ fulfill

$$\begin{aligned} &\rho \mathrm{D}_{k,\tau}^2(u) - \mathrm{div}_{\mathrm{Dir}}(\bar{\sigma}_\tau^k) = \mathcal{L}_\tau^k && \text{in } H_{\mathrm{Dir}}^1(\Omega)^*, \\ &\zeta_\tau^k + \mathrm{D}_{k,\tau}(p) + \pi_\tau^k = (\bar{\sigma}_\tau^k)_{\mathrm{D}} && \text{a.e. in } \Omega, \end{aligned} \tag{3.25}$$

with $\bar{\sigma}_\tau^k = \mathbb{D}\mathrm{D}_{k,\tau}(e) + \mathbb{C}e_\tau^k + \varepsilon_\tau^k - \vartheta_\tau^k\mathbb{B}$. In order to conclude the discrete momentum equation and plastic flow rule, it thus remains to show that

$$\varepsilon_\tau^k = |e_\tau^k|^{\gamma-2}e_\tau^k, \qquad \pi_\tau^k = |p_\tau^k|^{\gamma-2}p_\tau^k, \qquad \zeta_\tau^k \in \partial_{\dot{p}}\mathrm{R}(\vartheta_\tau^{k-1}, p_\tau^k - p_\tau^{k-1}). \tag{3.26}$$

With this aim, on the one hand we observe that

$$
\begin{aligned}
&\limsup_{M\to\infty}\left(\int_\Omega \zeta_{\tau,M}^k{:}p_{M,\tau}^k\,\mathrm{d}x+\tau\int_\Omega |p_{M,\tau}^k|^\gamma\,\mathrm{d}x+\tau\int_\Omega |e_{M,\tau}^k|^\gamma\,\mathrm{d}x\right)\\
&\overset{(1)}{\le}\limsup_{M\to\infty}\left(-\int_\Omega \frac{p_{M,\tau}^k-p_\tau^{k-1}}{\tau}:p_{M,\tau}^k\,\mathrm{d}x+\int_\Omega(\sigma_{M,\tau}^k)_{\mathrm{D}}:p_{M,\tau}^k\,\mathrm{d}x+\tau\int_\Omega|e_{M,\tau}^k|^\gamma\,\mathrm{d}x\right)\\
&\overset{(2)}{\le}-\int_\Omega\frac{p_\tau^k-p_\tau^{k-1}}{\tau}:p_\tau^k\,\mathrm{d}x+\limsup_{M\to\infty}\int_\Omega \underbrace{\sigma_{M,\tau}^k:E(u_{M,\tau}^k)}_{=\sigma_{M,\tau}^k\,:\,E(u_{M,\tau}^k-w_\tau^k)\,+\,\sigma_{M,\tau}^k:E(w_\tau^k)}-\sigma_{M,\tau}^k:e_{M,\tau}^k+\tau|e_{M,\tau}^k|^\gamma\,\mathrm{d}x\\
&\overset{(3)}{=}-\int_\Omega\frac{p_\tau^k-p_\tau^{k-1}}{\tau}:p_\tau^k\,\mathrm{d}x+\limsup_{M\to\infty}\Big(-\int_\Omega\rho\frac{u_{M,\tau}^k-2u_\tau^{k-1}+u_\tau^{k-2}}{\tau^2}(u_{M,\tau}^k-w_\tau^k)\,\mathrm{d}x\\
&\qquad\qquad+\ \langle\mathcal{L}_\tau^k,u_{M,\tau}^k-w_\tau^k\rangle_{H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^d)}+\int_\Omega\sigma_{M,\tau}^k{:}E(w_\tau^k)\,\mathrm{d}x\\
&\qquad\qquad-\int_\Omega\left(\mathbb{D}\frac{e_{M,\tau}^k-e_\tau^{k-1}}{\tau}+\mathbb{C}e_{M,\tau}^k-\mathcal{T}_M(\vartheta_{M,\tau}^k)\mathbb{B}\right){:}e_{M,\tau}^k\,\mathrm{d}x\Big)\\
&\overset{(4)}{\le}-\int_\Omega\frac{p_\tau^k-p_\tau^{k-1}}{\tau}:p_\tau^k\,\mathrm{d}x-\rho\int_\Omega\mathrm{D}^2_{k,\tau}(u)(u_\tau^k-w_\tau^k)\,\mathrm{d}x+\ \langle\mathcal{L}_\tau^k,u_\tau^k-w_\tau^k\rangle_{H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^d)}+\int_\Omega\sigma_\tau^k{:}E(w_\tau^k)\,\mathrm{d}x\\
&\qquad-\int_\Omega\left(\mathbb{D}\frac{e_\tau^k-e_\tau^{k-1}}{\tau}+\mathbb{C}e_\tau^k-\vartheta_\tau^k\mathbb{B}\right){:}e_\tau^k\,\mathrm{d}x\\
&\overset{(5)}{=}\int_\Omega\zeta_\tau^k{:}p_\tau^k\,\mathrm{d}x+\int_\Omega|\pi_\tau^k|^\gamma\,\mathrm{d}x+\int_\Omega|\varepsilon_\tau^k|^\gamma\,\mathrm{d}x.
\end{aligned}
\tag{3.27}
$$

In (3.27), (1) follows from testing (3.12c) by $p_{M,\tau}^k$, (2) from the weak convergence $p_{M,\tau}^k\to p_\tau^k$ in $L^2(\Omega;\mathbb{M}_{\mathrm{D}}^{d\times d})$ and the discrete admissibility condition, (3) from rewriting the term $\int_\Omega\sigma_{M,\tau}^k : E(u_{M,\tau}^k-w_\tau^k)\,\mathrm{d}x$ in terms of (3.12b) tested by $u_{M,\tau}^k-w_\tau^k$, and from using the explicit expression of $\sigma_{M,\tau}^k$ (which leads to the cancelation of the term $\int_\Omega\tau|e_{M,\tau}^k|^\gamma\,\mathrm{d}x$), (4) from the previously proved convergences via lower semicontinuity arguments, and (5) from repeating the above calculations in the frame of system (3.25), fulfilled by the limiting seventuple $(\vartheta_\tau^k,u_\tau^k,e_\tau^k,p_\tau^k,\zeta_\tau^k,\varepsilon_\tau^k,\pi_\tau^k)$. On the other hand, we have that

$$
\liminf_{M\to\infty}\int_\Omega\zeta_{\tau,M}^k{:}p_{M,\tau}^k\,\mathrm{d}x\ge\int_\Omega\zeta_\tau^k{:}p_\tau^k\,\mathrm{d}x,\qquad
\begin{aligned}
&\liminf_{M\to\infty}\int_\Omega|p_{M,\tau}^k|^\gamma\,\mathrm{d}x\ge\int_\Omega|\pi_\tau^k|^\gamma\,\mathrm{d}x,\\
&\liminf_{M\to\infty}\int_\Omega|e_{M,\tau}^k|^\gamma\,\mathrm{d}x\ge\int_\Omega|\varepsilon_\tau^k|^\gamma\,\mathrm{d}x\,,
\end{aligned}
\tag{3.28}
$$

where the second and the third inequalities follow from the weak convergence of $(p_{M,\tau}^k)_M$ and $(e_{M,\tau}^k)_M$ to π_τ^k and ε_τ^k, whereas the first inequality ensues from

$$
\begin{aligned}
\liminf_{M\to\infty}\left(\int_\Omega\zeta_{\tau,M}^k{:}p_{M,\tau}^k-\zeta_\tau^k{:}p_\tau^k\right)\mathrm{d}x&\ge\liminf_{M\to\infty}\int_\Omega\zeta_{\tau,M}^k{:}(p_{M,\tau}^k-p_\tau^k)\,\mathrm{d}x+\liminf_{M\to\infty}\int_\Omega(\zeta_{\tau,M}^k-\zeta_\tau^k){:}p_\tau^k\,\mathrm{d}x\\
&\overset{(1)}{\ge}\liminf_{M\to\infty}\int_\Omega\big(\mathrm{R}(\vartheta_\tau^{k-1},p_{M,\tau}^k-p_\tau^{k-1})-\mathrm{R}(\vartheta_\tau^{k-1},p_\tau^k-p_\tau^{k-1})\big)\,\mathrm{d}x\overset{(2)}{\ge}0
\end{aligned}
$$

with (1) due to the fact that $\zeta_{\tau,M}^k\in\partial_{\dot p}\mathrm{R}(\vartheta_\tau^{k-1},p_{M,\tau}^k-p_\tau^{k-1})$ and from $\zeta_{\tau,M}^k\overset{*}{\rightharpoonup}\zeta_\tau^k$ in $L^\infty(\Omega;\mathbb{M}_{\mathrm{D}}^{d\times d})$ as $M\to\infty$, and (2) following from the lower semicontinuity w.r.t. to the weak $L^2(\Omega;\mathbb{M}_{\mathrm{D}}^{d\times d})$-convergence of the integral functional $p\mapsto\int_\Omega\mathrm{R}(\vartheta_\tau^{k-1},p-p_\tau^{k-1})\,\mathrm{d}x$. Combining (3.27) and (3.28) we obtain that

$$
\begin{cases}
\lim_{M\to\infty}\int_\Omega\zeta_{\tau,M}^k{:}p_{M,\tau}^k\,\mathrm{d}x=\int_\Omega\zeta_\tau^k{:}p_\tau^k\,\mathrm{d}x & \overset{(1)}{\Rightarrow}\quad \zeta_\tau^k\in\partial_{\dot p}\mathrm{R}(\vartheta_\tau^{k-1},p_\tau^k-p_\tau^{k-1}),\\
\lim_{M\to\infty}\int_\Omega|p_{M,\tau}^k|^\gamma\,\mathrm{d}x=\int_\Omega|\pi_\tau^k|^\gamma\,\mathrm{d}x & \Rightarrow\quad p_{M,\tau}^k\to\pi_\tau^k\quad\text{ in }L^\gamma(\Omega;\mathbb{M}_{\mathrm{sym}}^{d\times d}),\\
\lim_{M\to\infty}\int_\Omega|e_{M,\tau}^k|^\gamma\,\mathrm{d}x=\int_\Omega|\varepsilon_\tau^k|^\gamma\,\mathrm{d}x & \Rightarrow\quad e_{M,\tau}^k\to\varepsilon_\tau^k\quad\text{ in }L^\gamma(\Omega;\mathbb{M}_{\mathrm{D}}^{d\times d})\,,
\end{cases}
$$

with (1) due to Minty's trick. Hence we conclude convergences (3.23c)–(3.23d) (and (3.23b), via the kinematic admissibility $E(u_{M,\tau}^k)=e_{M,\tau}^k+p_{M,\tau}^k$ and Korn's inequality), as well as (3.26). Therefore $(\vartheta_\tau^k,u_\tau^k,e_\tau^k,p_\tau^k,\zeta_\tau^k)$ fulfill the discrete momentum balance (3.6b) and flow rule (3.6c).

Exploiting convergences (3.23a) we pass to the limit as $M\to\infty$ on the right-hand side of (3.12a). In order to take the limit of the elliptic operator on the left-hand side, we repeat the argument from the proof of [RR15, Lemma 4.4]. Namely, we observe that, due to convergence (3.24), $\kappa_M(\vartheta_{M,\tau}^k) = \kappa(\mathcal{T}_M(\vartheta_{M,\tau}^k)) \to \kappa(\vartheta_\tau^k)$ in $L^q(\Omega)$ for all $1 \le q < 3 + \frac{6}{\mu}$, and combine this with the fact that $\nabla\vartheta_{M,\tau}^k \rightharpoonup \nabla\vartheta$ in $L^2(\Omega)$, and with the fact that, by comparison in (3.12a), $(\mathcal{A}_M^k(\vartheta_{M,\tau}^k))_M$ is bounded in $H^1(\Omega)^*$. All in all, we conclude that $\mathcal{A}_M^k(\vartheta_{M,\tau}^k) \rightharpoonup \mathcal{A}^k(\vartheta_\tau^k)$ in $H^1(\Omega)^*$ as $M \to \infty$, yielding the discrete heat equation (3.6a). □

4. Proof of Theorems 1 and 2

In the statements of all of the results of this section, leading to the proofs of Thms. 1 & 2, we will always tacitly assume the conditions on the problem data from Section 2.1.

We start by fixing some notation for the approximate solutions.

Notation 4.1 (Interpolants)**.** For a given Banach space B and a K_τ-tuple $(\mathfrak{h}_\tau^k)_{k=0}^{K_\tau} \subset B$, we introduce the left-continuous and right-continuous piecewise constant, and the piecewise linear interpolants of the values $\{\mathfrak{h}_\tau^k\}_{k=0}^{K_\tau}$, i.e.

$$\left.\begin{array}{lll} \overline{\mathfrak{h}}_\tau : (0,T] \to B & \text{defined by} & \overline{\mathfrak{h}}_\tau(t) := \mathfrak{h}_\tau^k, \\ \underline{\mathfrak{h}}_\tau : (0,T] \to B & \text{defined by} & \underline{\mathfrak{h}}_\tau(t) := \mathfrak{h}_\tau^{k-1}, \\ \mathfrak{h}_\tau : (0,T] \to B & \text{defined by} & \mathfrak{h}_\tau(t) := \frac{t-t_\tau^{k-1}}{\tau}\mathfrak{h}_\tau^k + \frac{t_\tau^k-t}{\tau}\mathfrak{h}_\tau^{k-1} \end{array}\right\} \qquad \text{for } t \in (t_\tau^{k-1}, t_\tau^k],$$

setting $\overline{\mathfrak{h}}_\tau(0) = \underline{\mathfrak{h}}_\tau(0) = \mathfrak{h}_\tau(0) := h_\tau^0$. We also introduce the piecewise linear interpolant of the values $\{\mathrm{D}_{k,\tau}(\mathfrak{h}) = \frac{\mathfrak{h}_\tau^k - \mathfrak{h}_\tau^{k-1}}{\tau}\}_{k=1}^{K_\tau}$ (which are the values taken by the piecewise constant function $\dot{\mathfrak{h}}_\tau$), viz.

$$\widehat{\mathfrak{h}}_\tau : (0,T) \to B \ \text{ defined by } \ \widehat{\mathfrak{h}}_\tau(t) := \frac{(t - t_\tau^{k-1})}{\tau}\mathrm{D}_{k,\tau}(\mathfrak{h}) + \frac{(t_\tau^k - t)}{\tau}\mathrm{D}_{k-1,\tau}(\mathfrak{h}) \qquad \text{for } t \in (t_\tau^{k-1}, t_\tau^k].$$

Note that $\partial_t\widehat{\mathfrak{h}}_\tau(t) = \mathrm{D}_{k,\tau}^2(\mathfrak{h})$ for $t \in (t_\tau^{k-1}, t_\tau^k]$.

Furthermore, we denote by $\bar{\mathfrak{t}}_\tau$ and by $\underline{\mathfrak{t}}_\tau$ the left-continuous and right-continuous piecewise constant interpolants associated with the partition, i.e. $\bar{\mathfrak{t}}_\tau(t) := t_\tau^k$ if $t_\tau^{k-1} < t \le t_\tau^k$ and $\underline{\mathfrak{t}}_\tau(t) := t_\tau^{k-1}$ if $t_\tau^{k-1} \le t < t_\tau^k$. Clearly, for every $t \in [0,T]$ we have $\bar{\mathfrak{t}}_\tau(t) \downarrow t$ and $\underline{\mathfrak{t}}_\tau(t) \uparrow t$ as $\tau \to 0$.

In view of (2.H$_1$), (2.H$_2$), and (2.L$_1$) it is easy to check that the piecewise constant interpolants $(\overline{H}_\tau)_\tau$, $(\overline{h}_\tau)_\tau$, and $(\overline{\mathcal{L}}_\tau)_\tau$ of the values H_τ^k, h_τ^k, and $\mathcal{L}_\tau^k$, cf. (3.1), fulfill as $\tau \downarrow 0$

$$\overline{H}_\tau \to H \text{ in } L^1(0,T;L^1(\Omega)) \cap L^2(0,T;H^1(\Omega)*). \tag{4.1a}$$

$$\overline{h}_\tau \to h \text{ in } L^1(0,T;L^2(\partial\Omega)), \tag{4.1b}$$

$$\overline{\mathcal{L}}_\tau \to \mathcal{L} \text{ in } L^2(0,T;H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^d)^*). \tag{4.1c}$$

Furthermore, it follows from (2.W) that

$$\begin{gathered} \overline{w}_\tau \to w \quad \text{in } L^1(0,T;W^{1,\infty}(\Omega;\mathbb{R}^d)), \qquad w_\tau \to w \quad \text{in } W^{1,p}(0,T;H^1(\Omega;\mathbb{R}^d)) \text{ for all } 1 \le p < \infty, \\ \widehat{w}_\tau \to w \quad \text{in } W^{1,1}(0,T;H^1(\Omega;\mathbb{R}^d) \cap H^1(0,T;L^2(\Omega;\mathbb{R}^d)). \end{gathered} \tag{4.1d}$$

We now reformulate the discrete system (3.6) in terms of the approximate solutions constructed interpolating the discrete solutions $(\vartheta_\tau^k, u_\tau^k, e_\tau^k, p_\tau^k)_{k=1}^{K_\tau}$. Therefore, we have

$$\begin{aligned} &\partial_t\vartheta_\tau(t) + \mathcal{A}^{\frac{\bar{\mathfrak{t}}_\tau(t)}{\tau}}(\overline{\vartheta}_\tau(t)) \\ &\quad = \overline{H}_\tau(t) + \mathrm{R}(\underline{\vartheta}_\tau(t), \dot{p}_\tau(t)) + |\dot{p}_\tau(t)|^2 + \mathbb{D}\dot{e}_\tau(t){:}\dot{e}_\tau(t) - \overline{\vartheta}_\tau(t)\mathbb{B} : \dot{e}_\tau(t), \quad \text{in } H^1(\Omega)^*, \end{aligned} \tag{4.2a}$$

$$\rho\int_\Omega \partial_t\widehat{u}_\tau(t)v\,\mathrm{d}x + \int_\Omega \overline{\sigma}_\tau(t){:}E(v)\,\mathrm{d}x = \langle\overline{\mathcal{L}}_\tau(t), v\rangle_{H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^d)} \quad \text{for all } v \in W^{1,\gamma}_{\mathrm{Dir}}(\Omega;\mathbb{R}^d), \tag{4.2b}$$

$$\overline{\zeta}_\tau(t) + \dot{p}_\tau(t) + \tau|\overline{p}_\tau(t)|^{\gamma-2}\overline{p}_\tau(t) \ni (\overline{\sigma}_\tau(t))_{\mathrm{D}} \quad \text{a.e. in } \Omega \tag{4.2c}$$

for almost all $t \in (0,T)$, with $\overline{\zeta}_\tau \in \partial_{\dot{p}}\mathrm{R}(\underline{\vartheta}_\tau, \dot{p}_\tau)$ a.e. in Q, and where we have used the notation

$$\overline{\sigma}_\tau := \mathbb{D}\dot{e}_\tau + \mathbb{C}\overline{e}_\tau + \tau|\overline{e}_\tau|^{\gamma-2}\overline{e}_\tau - \overline{\vartheta}_\tau\mathbb{B}. \tag{4.2d}$$

We now show that the approximate solutions fulfill the approximate versions of the entropy inequality (2.25), of the total energy inequality (2.26), and of the mechanical energy (in)equality (2.19). These discrete inequalities will have a pivotal role in the derivation of a priori estimates on the approximate solutions. Moreover, we will take their limit in order to obtain the entropy and total energy inequalities prescribed by the notion of *entropic solution*, cf. Definition 2.2.

For stating the discrete entropy inequality (4.5) below, we need to introduce *discrete* test functions. For technical reasons, we will need to pass to the limit with test functions enjoying a slightly stronger time regularity than that required by Def. 2.2. Namely, we fix a positive test function φ, with $\varphi \in \mathrm{C}^0([0,T];W^{1,\infty}(\Omega)) \cap H^1(0,T;L^{6/5}(\Omega))$. We set

$$
\varphi_\tau^k := \frac{1}{\tau}\int_{t_\tau^{k-1}}^{t_\tau^k} \varphi(s)\,\mathrm{d}s \qquad \text{for } k=1,\ldots,K_\tau, \tag{4.3}
$$

and consider the piecewise constant and linear interpolants $\overline{\varphi}_\tau$ and φ_τ of the values $(\varphi_\tau^k)_{k=1}^{K_\tau}$. It can be shown that the following convergences hold as $\tau\to 0$

$$
\overline{\varphi}_\tau \to \varphi \quad \text{ in } L^\infty(0,T;W^{1,\infty}(\Omega)) \text{ and } \quad \partial_t\varphi_\tau \to \partial_t\varphi \quad \text{ in } L^2(0,T;L^{6/5}(\Omega)). \tag{4.4}
$$

Observe that the first convergence property easily follows from the fact that the map $\varphi : [0,T]\to W^{1,\infty}(\Omega)$ is uniformly continuous.

Lemma 4.2 (Discrete entropy, mechanical, and total energy inequalities)**.** *The interpolants of the discrete solutions* $(\vartheta_\tau^k, u_\tau^k, e_\tau^k, p_\tau^k)_{k=1}^{K_\tau}$ *to Problem 3.1 fulfill*

- *the* discrete *entropy inequality*

$$
\begin{aligned}
&\int_{\overline{\mathrm{t}}_\tau(s)}^{\overline{\mathrm{t}}_\tau(t)}\int_\Omega \log(\underline{\vartheta}_\tau(r))\dot{\varphi}_\tau(r)\,\mathrm{d}x\,\mathrm{d}r - \int_{\overline{\mathrm{t}}_\tau(s)}^{\overline{\mathrm{t}}_\tau(t)}\int_\Omega \kappa(\overline{\vartheta}_\tau(r))\nabla\log(\overline{\vartheta}_\tau(r))\nabla\overline{\varphi}_\tau(r)\,\mathrm{d}x\,\mathrm{d}r \\
&\le \int_\Omega \log(\overline{\vartheta}_\tau(t))\overline{\varphi}_\tau(t)\,\mathrm{d}x - \int_\Omega \log(\overline{\vartheta}_\tau(s))\overline{\varphi}_\tau(s)\,\mathrm{d}x - \int_{\overline{\mathrm{t}}_\tau(s)}^{\overline{\mathrm{t}}_\tau(t)}\int_\Omega \kappa(\overline{\vartheta}_\tau(r))\frac{\overline{\varphi}_\tau(r)}{\overline{\vartheta}_\tau(r)}\nabla\log(\overline{\vartheta}_\tau(r))\nabla\overline{\vartheta}_\tau(r)\,\mathrm{d}x\,\mathrm{d}r \\
&\quad - \int_{\overline{\mathrm{t}}_\tau(s)}^{\overline{\mathrm{t}}_\tau(t)}\int_\Omega \left(\overline{H}_\tau(r) + \mathrm{R}(\underline{\vartheta}_\tau(r),\dot{p}_\tau(r)) + |\dot{p}_\tau(r)|^2 + \mathbb{D}\dot{e}_\tau(r){:}\dot{e}_\tau(r) - \overline{\vartheta}_\tau(r)\mathbb{B}{:}\dot{e}_\tau(r)\right)\frac{\overline{\varphi}_\tau(r)}{\overline{\vartheta}_\tau(r)}\,\mathrm{d}x\,\mathrm{d}r \\
&\quad - \int_{\overline{\mathrm{t}}_\tau(s)}^{\overline{\mathrm{t}}_\tau(t)}\int_{\partial\Omega} \overline{h}_\tau(r)\frac{\overline{\varphi}_\tau(r)}{\overline{\vartheta}_\tau(r)}\,\mathrm{d}S\,\mathrm{d}r
\end{aligned} \tag{4.5}
$$

for all $0\le s\le t\le T$ *and for all* $\varphi \in \mathrm{C}^0([0,T];W^{1,\infty}(\Omega)) \cap H^1(0,T;L^{6/5}(\Omega))$ *with* $\varphi\ge 0$*;*
- *the* discrete *total energy inequality for all* $0\le s\le t\le T$*, viz.*

$$
\begin{aligned}
&\frac{\rho}{2}\int_\Omega |\dot{u}_\tau(t)|^2\,\mathrm{d}x + \mathcal{E}_\tau(\overline{\vartheta}_\tau(t),\overline{e}_\tau(t)) \\
&\le \frac{\rho}{2}\int_\Omega |\dot{u}_\tau(s)|^2\,\mathrm{d}x + \mathcal{E}_\tau(\overline{\vartheta}_\tau(s),\overline{e}_\tau(s)) + \int_{\overline{\mathrm{t}}_\tau(s)}^{\overline{\mathrm{t}}_\tau(t)} \langle\overline{\mathcal{L}}_\tau(r),\dot{u}_\tau(r)-\dot{w}_\tau(r)\rangle_{H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^d)}\,\mathrm{d}r \\
&\quad + \int_{\overline{\mathrm{t}}_\tau(s)}^{\overline{\mathrm{t}}_\tau(t)}\left(\int_\Omega \overline{H}_\tau\,\mathrm{d}x + \int_{\partial\Omega}\overline{h}_\tau\,\mathrm{d}S\right)\mathrm{d}r \\
&\quad + \rho\int_\Omega \dot{u}_\tau(t)\dot{w}_\tau(t)\,\mathrm{d}x - \rho\int_\Omega \dot{u}_\tau(s)\dot{w}_\tau(s)\,\mathrm{d}x - \rho\int_{\overline{\mathrm{t}}_\tau(s)}^{\overline{\mathrm{t}}_\tau(t)} \dot{u}_\tau(r-\tau)\partial_t\widehat{\dot{w}}_\tau(r)\,\mathrm{d}x\,\mathrm{d}r \\
&\quad + \int_{\overline{\mathrm{t}}_\tau(s)}^{\overline{\mathrm{t}}_\tau(t)}\int_\Omega \overline{\sigma}_\tau(r){:}E(\dot{w}_\tau(r))\,\mathrm{d}x\,\mathrm{d}r
\end{aligned} \tag{4.6}
$$

with the discrete total energy functional $\mathcal{E}_\tau$ *from* (3.19)*;*

- *the* discrete *mechanical energy inequality for all* $0 \le s \le t \le T$, *viz.*

$$
\begin{aligned}
&\frac{\rho}{2}\int_\Omega |\dot{u}_\tau(t)|^2\,\mathrm{d}x + \int_{\bar{\mathfrak{t}}_\tau(s)}^{\bar{\mathfrak{t}}_\tau(t)} \int_\Omega \left(\mathbb{D}\dot{e}_\tau(r):\dot{e}_\tau(r) + \mathrm{R}(\underline{\vartheta}_\tau(r),\dot{p}_\tau(r)) + |\dot{p}_\tau(r)|^2\right)\mathrm{d}x\,\mathrm{d}r + \frac{1}{2}\int_\Omega \mathbb{C}\overline{e}_\tau(t):\overline{e}_\tau(t)\,\mathrm{d}x \\
&\qquad + \frac{\tau}{\gamma}\int_\Omega \left(|\overline{e}_\tau(t)|^\gamma + |\overline{p}_\tau(t)|^\gamma\right)\mathrm{d}x \\
&\le \frac{\rho}{2}\int_\Omega |\dot{u}_\tau(s)|^2\,\mathrm{d}x + \int_{\bar{\mathfrak{t}}_\tau(s)}^{\bar{\mathfrak{t}}_\tau(t)} \langle \overline{\mathcal{L}}_\tau(r), \dot{u}_\tau(r) - \dot{w}_\tau(r)\rangle_{H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^d)}\,\mathrm{d}r + \int_{\bar{\mathfrak{t}}_\tau(s)}^{\bar{\mathfrak{t}}_\tau(t)} \int_\Omega \overline{\vartheta}_\tau(r)\mathbb{B}:\dot{e}_\tau\,\mathrm{d}x\,\mathrm{d}r \\
&\quad + \rho\int_\Omega \dot{u}_\tau(t)\dot{w}_\tau(t)\,\mathrm{d}x - \rho\int_\Omega \dot{u}_\tau(s)\dot{w}_\tau(s)\,\mathrm{d}x - \rho\int_{\bar{\mathfrak{t}}_\tau(s)}^{\bar{\mathfrak{t}}_\tau(t)} \dot{u}_\tau(r-\tau)\partial_t\widehat{w}_\tau(r)\,\mathrm{d}x\,\mathrm{d}r \\
&\quad + \int_{\bar{\mathfrak{t}}_\tau(s)}^{\bar{\mathfrak{t}}_\tau(t)} \int_\Omega \overline{\sigma}_\tau(r):E(\dot{w}_\tau(r))\,\mathrm{d}x\,\mathrm{d}r\,.
\end{aligned}
\tag{4.7}
$$

Below we will only outline the argument for Lemma 4.2, referring to the proof of the analogous [RR15, Prop. 4.8] for most of the details. Let us only mention in advance that we will make use of the *discrete by-part integration* formula, holding for all K_τ-uples $\{\mathfrak{h}_\tau^k\}_{k=0}^{K_\tau} \subset B$, $\{v_\tau^k\}_{k=0}^{K_\tau} \subset B^*$ in a given Banach space B:

$$
\sum_{k=1}^{K_\tau} \tau\,\langle v_\tau^k, \mathrm{D}_{k,\tau}(\mathfrak{h})\rangle_B = \langle v_\tau^{K_\tau}, \mathfrak{h}_\tau^{K_\tau}\rangle_B - \langle v_\tau^0, \mathfrak{h}_\tau^0\rangle_B - \sum_{k=1}^{K_\tau} \tau\,\langle \mathrm{D}_{k,\tau}(v), \mathfrak{h}_\tau^{k-1}\rangle_B\,, \tag{4.8}
$$

as well as of the following inequality, satisfied by any <u>concave</u> (differentiable) function $\psi : \mathrm{dom}(\psi) \to \mathbb{R}$:

$$
\psi(x) - \psi(y) \le \psi'(y)(x-y) \qquad \text{for all } x,\, y \in \mathrm{dom}(\psi). \tag{4.9}
$$

Sketch of the proof of Lemma 4.2. The entropy inequality (4.5) follows from testing (3.6a) by $\frac{\varphi_\tau^k}{\vartheta_\tau^k}$, for $k \in \{1,\ldots,K_\tau\}$ fixed, with $\varphi \in \mathrm{C}^0([0,T];W^{1,\infty}(\Omega)) \cap H^1(0,T;L^{6/5}(\Omega))$ an arbitrary positive test function. Observe that $\frac{\varphi_\tau^k}{\vartheta_\tau^k} \in H^1(\Omega)$, as $\vartheta_\tau^k \in H^1(\Omega)$ is bounded away from zero by a strictly positive constant, cf. (3.7). We then obtain

$$
\begin{aligned}
&\int_\Omega \left(H_\tau^k + \mathrm{R}\left(\vartheta_\tau^{k-1}, \mathrm{D}_{k,\tau}(p)\right) + |\mathrm{D}_{k,\tau}(p)|^2 + \mathbb{D}\mathrm{D}_{k,\tau}(e) : \mathrm{D}_{k,\tau}(e) - \vartheta_\tau^k\mathbb{B} : \mathrm{D}_{k,\tau}(e)\right)\frac{\varphi_\tau^k}{\vartheta_\tau^k}\,\mathrm{d}x + \int_{\partial\Omega} h_\tau^k \frac{\varphi_\tau^k}{\vartheta_\tau^k}\,\mathrm{d}S \\
&= \int_\Omega \frac{\vartheta_\tau^k - \vartheta_\tau^{k-1}}{\tau}\frac{\varphi_\tau^k}{\vartheta_\tau^k}\,\mathrm{d}x + \int_\Omega \kappa(\vartheta_\tau^k)\nabla\vartheta_\tau^k\nabla\left(\frac{\varphi_\tau^k}{\vartheta_\tau^k}\right)\mathrm{d}x \\
&\overset{(1)}{\le} \int_\Omega \frac{\log(\vartheta_\tau^k) - \log(\vartheta_\tau^{k-1})}{\tau}\varphi_\tau^k\,\mathrm{d}x + \int_\Omega \left(\frac{\kappa(\vartheta_\tau^k)}{\vartheta_\tau^k}\nabla\vartheta_\tau^k\nabla\varphi_\tau^k - \frac{\kappa(\vartheta_\tau^k)}{|\vartheta_\tau^k|^2}|\nabla\vartheta_\tau^k|^2\varphi_\tau^k\right)\mathrm{d}x
\end{aligned}
$$

where (1) follows from (4.9) with $\psi = \log$. Then, one sums the above inequality, multiplied by τ, over $k = m,\ldots,j$, for amy couple of indices $m,\, j \in \{1,\ldots,K_\tau\}$, and uses the discrete by-part integration formula (4.8) to deal with the term $\sum_{k=m}^{j} \frac{\log(\vartheta_\tau^k) - \log(\vartheta_\tau^{k-1})}{\tau}\varphi_\tau^k$. This leads to (4.5).

As for the discrete total energy inequality, with the very same calculations developed in the proof of Lemma 3.5, one shows that the solution quadruple $(\vartheta_\tau^k, u_\tau^k, e_\tau^k, p_\tau^k)$ to system (3.6) fulfills the energy inequality (3.20). Note that the two inequalities, i.e. the one for system (3.6) and (3.20) for the truncated version (3.12) of (3.6), in fact coincide since they neither involve the elliptic operator in the discrete heat equation, nor the thermal expansion terms coupling the heat equation and the momentum balance, which are the terms affected by the truncation procedure. Then, (4.6) ensues by adding (3.20) over the index $k = m,\ldots,j$, for any couples of indices $m,\, j \in \{1,\ldots,K_\tau\}$.

The mechanical energy inequality (4.7) is derived by subtracting from (4.6) the discrete heat equation (3.6a) multiplied by τ and integrated over Ω. □

4.1. **A priori estimates.** The following result collects the a priori estimates on the approximate solutions of system (4.2). Let us mention in advance that, along the footsteps of [RR15], we shall derive from the discrete entropy inequality (4.5) a *weak version* of the estimate on the total variation of $\log(\overline{\vartheta}_\tau)$, cf. (4.10m) and (4.34) below, which will play a crucial role in the compactness arguments for the approximate temperatures $(\overline{\vartheta}_\tau)_\tau$.

Proposition 4.3. *Assume* $(2.\kappa_1)$. *Then, there exists a constant* $S>0$ *such that for all* $\tau>0$ *the following estimates hold*

$$\|\overline{u}_\tau\|_{L^\infty(0,T;H^1(\Omega;\mathbb{R}^d))} \leq S, \tag{4.10a}$$

$$\|u_\tau\|_{H^1(0,T;H^1(\Omega;\mathbb{R}^d))\cap W^{1,\infty}(0,T;L^2(\Omega;\mathbb{R}^d))} \leq S, \tag{4.10b}$$

$$\|\widehat{u}_\tau\|_{L^2(0,T;H^1(\Omega;\mathbb{R}^d))\cap L^\infty(0,T;L^2(\Omega;\mathbb{R}^d))\cap W^{1,\gamma/(\gamma-1)}(0,T;W^{1,\gamma}(\Omega;\mathbb{R}^d)^*)} \leq S, \tag{4.10c}$$

$$\|\overline{e}_\tau\|_{L^\infty(0,T;L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}}))} \leq S, \tag{4.10d}$$

$$\|e_\tau\|_{H^1(0,T;L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}}))} \leq S, \tag{4.10e}$$

$$\tau^{1/\gamma}\|\overline{e}_\tau\|_{L^\infty(0,T;L^\gamma(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}}))} \leq S, \tag{4.10f}$$

$$\|\overline{p}_\tau\|_{L^\infty(0,T;L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{D}}))} \leq S, \tag{4.10g}$$

$$\|p_\tau\|_{H^1(0,T;L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{D}}))} \leq S, \tag{4.10h}$$

$$\tau^{1/\gamma}\|\overline{p}_\tau\|_{L^\infty(0,T;L^\gamma(\Omega;\mathbb{M}^{d\times d}_{\mathrm{D}}))} \leq S, \tag{4.10i}$$

$$\|\log(\overline{\vartheta}_\tau)\|_{L^2(0,T;H^1(\Omega))} \leq S, \tag{4.10j}$$

$$\|\overline{\vartheta}_\tau\|_{L^2(0,T;H^1(\Omega))\cap L^\infty(0,T;L^1(\Omega))} \leq S, \tag{4.10k}$$

$$\|(\overline{\vartheta}_\tau)^{(\mu+\alpha)/2}\|_{L^2(0,T;H^1(\Omega))},\ \|(\overline{\vartheta}_\tau)^{(\mu-\alpha)/2}\|_{L^2(0,T;H^1(\Omega))} \leq C \quad \textit{for all } \alpha\in[2-\mu,1) \tag{4.10l}$$

$$\sup_{\varphi\in W^{1,d+\epsilon}(\Omega),\ \|\varphi\|_{W^{1,d+\epsilon}(\Omega)}\leq 1} \mathrm{Var}(\ \langle\log(\overline{\vartheta}_\tau),\varphi\rangle_{W^{1,d+\epsilon}(\Omega)};[0,T]) \leq S\,. \tag{4.10m}$$

Furthermore, if κ *fulfills* $(2.\kappa_2)$, *there holds in addition*

$$\sup_{\tau>0}\|\vartheta_\tau\|_{\mathrm{BV}([0,T];W^{1,\infty}(\Omega)^*)} \leq S. \tag{4.10n}$$

The starting point in the proof is the discrete total energy inequality (4.6), giving rise to the second of (4.10b), the first of (4.10c) (4.10d), (4.10f), (4.10i), and the second of (4.10k): we will detail the related calculations, in particular showing how the terms arising from the external forces F and g, and those involving the Dirichlet loading w can be handled. Let us also refer to the upcoming Remark 4.4 for more comments.

The *dissipative* estimates, i.e. the first of (4.10b), (4.10e), and (4.10h), then follow from the discrete mechanical energy inequality (4.7). The remaining estimates on the approximate temperature can be performed with the very same arguments as in the proof of [RR15, Prop. 4.10], to which we shall refer for all details.

Proof. **First a priori estimate:** We write the total energy inequality (4.6) for $s=0$ and estimate the terms on its right-hand side:

$$\frac{\rho}{2}\int_\Omega |\dot{u}_\tau(t)|^2\,\mathrm{d}x + \mathcal{E}_\tau(\overline{\vartheta}_\tau(t),\overline{e}_\tau(t),\overline{p}_\tau(t)) \leq I_1+I_2+I_3+I_4+I_5+I_6, \tag{4.11}$$

with

$$I_1 = \frac{\rho}{2}\int_\Omega |\dot{u}_\tau(0)|^2\,\mathrm{d}x + \mathcal{E}_\tau(\overline{\vartheta}_\tau(0),\overline{e}_\tau(0),\overline{p}_\tau(0)) \leq C$$

thanks to (2.16a), (2.16b), (3.4), and (3.5). To estimate I_2 we use the safe load condition $(2.\mathrm{L}_2)$, namely

$$\begin{aligned} I_2 &= \int_0^{\bar{\mathfrak{t}}_\tau(t)} \langle\overline{\mathcal{L}}_\tau(r),\dot{u}_\tau(r){-}\dot{w}_\tau(r)\rangle_{H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^d)}\,\mathrm{d}r \\ &= \int_0^{\bar{\mathfrak{t}}_\tau(t)} \langle\overline{F}_\tau(r),\dot{u}_\tau(r){-}\dot{w}_\tau(r)\rangle_{H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^d)}\,\mathrm{d}r + \int_0^{\bar{\mathfrak{t}}_\tau(t)} \langle\overline{g}_\tau(r),\dot{u}_\tau(r){-}\dot{w}_\tau(r)\rangle_{H^{1/2}_{00,\Gamma_{\mathrm{Dir}}}(\Gamma_{\mathrm{Neu}};\mathbb{R}^d)}\,\mathrm{d}r \\ &= \int_0^{\bar{\mathfrak{t}}_\tau(t)}\int_\Omega \overline{\varrho}_\tau(r) : (E(\dot{u}_\tau(r)){-}E(\dot{w}_\tau(r)))\,\mathrm{d}x\,\mathrm{d}r \\ &\overset{(1)}{=} \int_0^{\bar{\mathfrak{t}}_\tau(t)}\int_\Omega \overline{\varrho}_\tau(r):\dot{e}_\tau(r)\,\mathrm{d}x\,\mathrm{d}r + \int_0^{\bar{\mathfrak{t}}_\tau(t)}\int_\Omega \overline{\varrho}_\tau(r):\dot{p}_\tau(r)\,\mathrm{d}x\,\mathrm{d}r - \int_0^{\bar{\mathfrak{t}}_\tau(t)}\int_\Omega \overline{\varrho}_\tau(r):E(\dot{w}_\tau(r))\,\mathrm{d}x\,\mathrm{d}r \\ &\doteq I_{2,1}+I_{2,2}+I_{2,3} \end{aligned} \tag{4.12}$$

where $\overline{F}_\tau$, $\overline{g}_\tau$, $\overline{\varrho}_\tau$, ϱ_τ denote the approximations of F, g, ϱ. Equality (1) follows from the kinematic admissibility condition $E(\dot{u}_\tau)=\dot{e}_\tau+\dot{p}_\tau$. Observe that, thanks to $(2.\mathrm{L}_2)$, there holds

$$\|\overline{\varrho}_\tau\|_{L^\infty(0,T;L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}}))} + \|\varrho_\tau\|_{W^{1,1}(0,T;L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}}))} + \|(\overline{\varrho}_\tau)_{\mathrm{D}}\|_{L^1(0,T;L^\infty(\Omega;\mathbb{M}^{d\times d}_{\mathrm{D}}))} \leq C\,. \tag{4.13}$$

Now, using the discrete by-part integration formula (4.8) we see that

$$
\begin{aligned}
I_{2,1} &= -\int_0^{\bar{t}_\tau(t)} \int_\Omega \dot{\varrho}_\tau(r) : \underline{e}_\tau(r)\,\mathrm{d}x\,\mathrm{d}r + \int_\Omega \overline{\varrho}_\tau(t) : \overline{e}_\tau(t)\,\mathrm{d}x - \int_\Omega \overline{\varrho}_\tau(0) : \overline{e}_\tau(0)\,\mathrm{d}x \\
&\stackrel{(2)}{\leq} \int_0^{\bar{t}_\tau(t)} \|\dot{\varrho}_\tau(r)\|_{L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})} \|\underline{e}_\tau(r)\|_{L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})}\,\mathrm{d}r + \frac{C^1_{\mathbb{C}}}{16}\|\overline{e}_\tau(t)\|^2_{L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})} + C\|\overline{\varrho}_\tau(t)\|^2_{L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})} + C
\end{aligned}
$$

where estimate (2) follows from Young's inequality. The choice of the coefficient $\frac{C^1_{\mathbb{C}}}{16}$ will allow us to absorb the second term into the left-hand side of (4.11), taking into account the coercivity property (2.T) of $\mathbb{C}$, which ensures that $\mathcal{E}_\tau(\overline{\vartheta}_\tau(t), \overline{e}_\tau(t), \overline{p}_\tau(t))$ on the left-hand side of (4.11) bounds $\|\overline{e}_\tau(t)\|^2_{L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})}$. As for I_{22}, using the discrete flow rule (4.2c) and taking into the expression of $(\overline{\sigma}_\tau)_{\mathrm{D}}$ we gather

$$
\begin{aligned}
I_{2,2} &= \int_0^{\bar{t}_\tau(t)} \int_\Omega (\overline{\varrho}_\tau(r))_{\mathrm{D}} \left(\mathbb{D}\dot{e}_\tau(r) + \mathbb{C}\overline{e}_\tau(r) + \tau|\overline{e}_\tau(r)|^{\gamma-2}\overline{e}_\tau(r) - \overline{\vartheta}_\tau(r)\mathbb{B} - \overline{\zeta}_\tau(r) - \tau|\overline{p}_\tau(r)|^{\gamma-2}\overline{p}_\tau(r)\right)\mathrm{d}x\,\mathrm{d}r \\
&\doteq I_{2,2,1} + I_{2,2,2} + I_{2,2,3} + I_{2,2,4} + I_{2,2,5} + I_{2,2,6}
\end{aligned}
$$

and we estimate the above terms as follows. First, for $I_{2,2,1}$ we resort to the by-parts integration formula (4.8) with the very same calculations as in the estimate of the integral term $I_{2,1}$. Second, we estimate

$$
I_{2,2,2} \leq \frac{C^1_{\mathbb{C}}}{16}\|\overline{e}_\tau(t)\|\|^2_{L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})} + C\|\overline{\varrho}_\tau(t)\|\|^2_{L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})}\,.
$$

In the estimate of $I_{2,2,3}$ we use Hölder's inequality

$$
I_{2,2,3} \leq \frac{\tau\gamma}{2}\int_0^{\bar{t}_\tau(t)} \|(\overline{\varrho}_\tau(r))_{\mathrm{D}}\|^\gamma_{L^\gamma(\Omega;\mathbb{M}^{d\times d}_{\mathrm{D}})}\,\mathrm{d}r + \frac{\tau}{2\gamma}\int_0^{\bar{t}_\tau(t)} \|\overline{e}_\tau(r)\|^\gamma_{L^\gamma(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})}\,\mathrm{d}r\,.
$$

For $I_{2,2,4}$ we resort to estimate (4.13) for $(\overline{\varrho}_\tau)_{\mathrm{D}}$ in $L^1(0,T;L^\infty(\Omega;\mathbb{M}^{d\times d}_{\mathrm{D}}))$, so that

$$
I_{2,2,4} \leq C\int_0^{\bar{t}_\tau(t)} \|(\overline{\varrho}_\tau(r))_{\mathrm{D}}\|_{L^\infty(\Omega;\mathbb{M}^{d\times d}_{\mathrm{D}})}\|\overline{\vartheta}_\tau(r)\|_{L^1(\Omega)}\,\mathrm{d}r;
$$

again, this term will be estimated via Gronwall's inequality, taking into account that $\mathcal{E}_\tau(\overline{\vartheta}_\tau(t), \overline{e}_\tau(t), \overline{p}_\tau(t))$ on the left-hand side of (4.11) bounds $\|\overline{\vartheta}_\tau(t)\|_{L^1(\Omega)}$. Finally, since $\|\overline{\zeta}_\tau(t)\|_{L^\infty(\Omega;\mathbb{M}^{d\times d}_{\mathrm{D}})} \leq C_R$ thanks to (2.11b), we find that $I_{2,2,5} \leq C_R \int_0^{\bar{t}_\tau(t)} \|\overline{\varrho}_\tau(r)\|_{L^1(\Omega;\mathbb{M}^{d\times d}_{\mathrm{D}})}\,\mathrm{d}r \leq C$ by (4.13), while with Hölder's inequality we have

$$
I_{2,2,6} \leq \frac{\tau\gamma}{2}\int_0^{\bar{t}_\tau(t)} \|(\overline{\varrho}_\tau(r))_{\mathrm{D}}\|^\gamma_{L^\gamma(\Omega;\mathbb{M}^{d\times d}_{\mathrm{D}})}\,\mathrm{d}r + \frac{\tau}{2\gamma}\int_0^{\bar{t}_\tau(t)} \|\overline{p}_\tau(r)\|^\gamma_{L^\gamma(\Omega;\mathbb{M}^{d\times d}_{\mathrm{D}})}\,\mathrm{d}r\,.
$$

This concludes the estimation of $I_{2,2}$. Finally, we have

$$
I_{2,3} \leq \int_0^{\bar{t}_\tau(t)} \|\overline{\varrho}_\tau(r)\|_{L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})}\|E(\dot{w}_\tau(r))\|_{L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})}\,\mathrm{d}r \leq C
$$

in view of (4.13) and (4.1d), which provides a bound for w_τ, and we have thus handled all the terms contributing to I_2. We also have

$$
I_3 = \int_0^{\bar{t}_\tau(t)} \left(\int_\Omega \overline{H}_\tau\,\mathrm{d}x + \int_{\partial\Omega} \overline{h}_\tau\,\mathrm{d}S\right)\mathrm{d}r \leq \|\overline{H}_\tau\|_{L^1(0,T;L^1(\Omega))} + \|\overline{h}_\tau\|_{L^1(0,T;L^2(\partial\Omega))} \leq C,
$$

due to (4.1);

$$
\begin{aligned}
I_4 &= \rho\int_\Omega \dot{u}_\tau(t)\dot{w}_\tau(t)\,\mathrm{d}x - \rho\int_\Omega \dot{u}_0\dot{w}_\tau(0)\,\mathrm{d}x - \rho\int_0^{\bar{t}_\tau(t)} \dot{u}_\tau(r-\tau)\partial_t\widehat{w}_\tau(r)\,\mathrm{d}x\,\mathrm{d}r \\
&\stackrel{(1)}{\leq} C + \frac{\rho}{8}\int_\Omega |\dot{u}_\tau(t)|^2\,\mathrm{d}x + 2\rho\int_\Omega |\dot{w}_\tau(t)|^2\,\mathrm{d}x + \rho\int_0^{\bar{t}_\tau(t)-\tau} \|\dot{u}_\tau(s)\|_{L^2(\Omega;\mathbb{R}^d)}\|\partial_t\widehat{w}_\tau(s+\tau)\|_{L^2(\Omega;\mathbb{R}^d)}\,\mathrm{d}s,
\end{aligned}
$$

where (1) follows from (2.16b), (3.5), and (4.1d), and we are tacitly assuming that $\dot{u}_\tau$ extends identically to zero on the interval $(-\tau, 0)$. Moreover,

$$
\begin{aligned}
I_5 &= \int_0^{\bar{\mathsf{t}}_\tau(t)} \int_\Omega \left(\mathbb{D}\dot{e}_\tau(r) + \mathbb{C}\overline{e}_\tau(r) - \overline{\vartheta}_\tau(r)\mathbb{B}\right) : E(\dot{w}_\tau(r))\,\mathrm{d}x\,\mathrm{d}r \\
&\overset{(2)}{\le} \int_\Omega \mathbb{D}\overline{e}_\tau(t) : E(\dot{w}_\tau(t))\,\mathrm{d}x - \int_\Omega \mathbb{D}\overline{e}_\tau(0) : E(\dot{w}_\tau(0))\,\mathrm{d}x - \int_0^{\bar{\mathsf{t}}_\tau(t)} \int_\Omega \mathbb{D}\overline{e}_\tau(r-\tau) : E(\partial_t \widehat{w}_\tau(r))\,\mathrm{d}x\,\mathrm{d}r \\
&\quad + C_{\mathbb{C}}^2 \int_0^{\bar{\mathsf{t}}_\tau(t)} \|\overline{e}_\tau(r)\|_{L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})} \|E(\dot{w}_\tau(r))\|_{L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})}\,\mathrm{d}r + C\int_0^{\bar{\mathsf{t}}_\tau(t)} \|\overline{\vartheta}_\tau(r)\|_{L^1(\Omega)} \|E(\dot{w}_\tau(r))\|_{L^\infty(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})}\,\mathrm{d}r \\
&\le C + \frac{C_{\mathbb{C}}^1}{8} \int_\Omega |\overline{e}_\tau(t)|^2\,\mathrm{d}x + \int_0^{\bar{\mathsf{t}}_\tau(t)} \left(\|E(\partial_t \widehat{w}_\tau(r))\|_{L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})} + \|E(\dot{w}_\tau(r))\|_{L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})} \right) \|\overline{e}_\tau(r)\|_{L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})}\,\mathrm{d}r \\
&\quad + C\int_0^{\bar{\mathsf{t}}_\tau(t)} \|\overline{\vartheta}_\tau(r)\|_{L^1(\Omega)} \|E(\dot{w}_\tau(r))\|_{L^\infty(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})}\,\mathrm{d}r
\end{aligned}
\tag{4.14}
$$

where (2) follows from integrating by parts the term $\iint \mathbb{D}\dot{e}_\tau{:}E(\dot{w}_\tau)$ (again, setting $\overline{e}_\tau \equiv 0$ on $(-\tau, 0)$). Collecting all of the above estimates and taking into account the coercivity properties of $\mathcal{E}_\tau$, as well as the bounds provided by (4.13) and (4.1d), we get

$$
\begin{aligned}
&\frac{3}{8}\rho \int_\Omega |\dot{u}_\tau(t)|^2\,\mathrm{d}x + \|\overline{\vartheta}_\tau(t)\|_{L^1(\Omega)} + \frac{1}{4} C_{\mathbb{C}}^1 \|\overline{e}_\tau(t)\|^2_{L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})} + \frac{\tau}{2\gamma} \|\overline{e}_\tau(t)\|^\gamma_{L^\gamma(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})} + \frac{\tau}{2\gamma} \|\overline{p}_\tau(t)\|^\gamma_{L^\gamma(\Omega;\mathbb{M}^{d\times d}_{\mathrm{D}})} \\
&\le C + \int_0^{\bar{\mathsf{t}}_\tau(t)} \|\dot{\varrho}_\tau(r)\|_{L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})} \|\underline{e}_\tau(r)\|_{L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})}\,\mathrm{d}r + C\int_0^{\bar{\mathsf{t}}_\tau(t)} \|(\overline{\varrho}_\tau(r))_{\mathrm{D}}\|_{L^\infty(\Omega;\mathbb{M}^{d\times d}_{\mathrm{D}})} \|\overline{\vartheta}_\tau(r)\|_{L^1(\Omega)}\,\mathrm{d}r \\
&\quad + \rho \int_0^{\bar{\mathsf{t}}_\tau(t)-\tau} \|\partial_t \widehat{w}_\tau(s+\tau)\|_{L^2(\Omega;\mathbb{R}^d)} \|\dot{u}_\tau(s)\|_{L^2(\Omega;\mathbb{R}^d)}\,\mathrm{d}s \\
&\quad + \int_0^{\bar{\mathsf{t}}_\tau(t)} \left(\|E(\partial_t \widehat{w}_\tau(r))\|_{L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})} + \|E(\dot{w}_\tau(r))\|_{L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})} \right) \|\overline{e}_\tau(r)\|_{L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})}\,\mathrm{d}r \\
&\quad + C\int_0^{\bar{\mathsf{t}}_\tau(t)} \|E(\dot{w}_\tau(r))\|_{L^\infty(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})} \|\overline{\vartheta}_\tau(r)\|_{L^1(\Omega)}\,\mathrm{d}r\,.
\end{aligned}
$$

Applying a suitable version of Gronwall's Lemma (cf., e.g., [Dra03, Thm. 21]), we conclude that

$$
\|\dot{u}_\tau\|_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^d)} + \mathcal{E}_\tau(\overline{\vartheta}_\tau(t), \overline{e}_\tau(t), \overline{p}_\tau(t)) \le C,
$$

whence the second of (4.10b), the first of (4.10c), (4.10d), (4.10f), (4.10i), and the second of (4.10k).

Remark 4.4. The safe load condition ($2.\mathrm{L}_2$) is crucial handling $\int_0^{\bar{\mathsf{t}}_\tau(t)} \langle \overline{\mathcal{L}}_\tau, \dot{u}_\tau - \dot{w}_\tau \rangle_{H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^d)}\,\mathrm{d}r$ on the r.h.s. of (4.11), cf. (4.12). In fact, this term involves the *dissipative* variable $\dot{u}_\tau$, whose $L^2(\Omega;\mathbb{R}^d)$-norm, *only*, is estimated by the r.h.s. of (4.11). Condition ($2.\mathrm{L}_2$) then allows us to rewrite the above integral in terms of the functions $\overline{\varrho}_\tau$ and $\dot{e}_\tau$, p_τ, and the resulting integrals are then treated via integration by parts, leading to quantities that can be controlled by the l.h.s. of (4.11).

Without ($2.\mathrm{L}_2$), the term $\int_0^{\bar{\mathsf{t}}_\tau(t)} \langle \overline{\mathcal{L}}_\tau, \dot{u}_\tau - \dot{w}_\tau \rangle_{H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^d)}\,\mathrm{d}r$ could be treated only by supposing that $g \equiv 0$, and that $F \in L^2(Q;\mathbb{R}^d)$.

The estimates for the term $\int_0^{\bar{\mathsf{t}}_\tau(t)} \int_\Omega \overline{\sigma}_\tau{:}E(\dot{w}_\tau)\,\mathrm{d}x\,\mathrm{d}r$, cf. (4.14), unveil the role of the condition $w \in L^1(0,T;W^{1,\infty}(\Omega;\mathbb{R}^d))$, which allows us to control the term $\int_0^{\bar{\mathsf{t}}_\tau(t)} \int_\Omega \overline{\vartheta}_\tau \mathbb{B}{:}E(\dot{w}_\tau)\,\mathrm{d}x\,\mathrm{d}r$ exploiting the $L^1(\Omega)$-bound provided by the l.h.s. of (4.11). Alternatively, one could impose some sort of 'compatibility' between the thermal expansion tensor $\mathbb{B} = \mathbb{C}\mathbb{E}$ and the Dirichlet loading w, by requiring that $\mathbb{B}{:}E(\dot{w}) \equiv 0$, cf. [Rou13]. Analogously, the condition $w \in W^{2,1}(0,T;H^1(\Omega;\mathbb{R}^d))$ has been used in the estimation of the term I_5, cf. (4.14).

Second a priori estimate: we test (3.6a) by $(\vartheta_\tau^k)^{\alpha-1}$, with $\alpha \in (0,1)$, thus obtaining

$$
\begin{aligned}
&\int_\Omega \left(H_\tau^k + \mathrm{R}\left(\vartheta_\tau^{k-1}, \mathrm{D}_{k,\tau}(p)\right) + |\mathrm{D}_{k,\tau}(p)|^2 + \mathbb{D}\mathrm{D}_{k,\tau}(e) : \mathrm{D}_{k,\tau}(e) \right) (\vartheta_\tau^k)^{\alpha-1} \,\mathrm{d}x \\
&\quad - \int_\Omega \kappa(\vartheta_\tau^k) \nabla \vartheta_\tau^k \nabla (\vartheta_\tau^k)^{\alpha-1} \,\mathrm{d}x + \int_{\partial\Omega} h_\tau^k (\vartheta_\tau^k)^{\alpha-1} \,\mathrm{d}S \\
&\leq \int_\Omega \left(\frac{1}{\alpha} \frac{(\vartheta_\tau^k)^\alpha - (\vartheta_\tau^{k-1})^\alpha}{\tau} + \vartheta_\tau^k \mathbb{B} : \mathrm{D}_{k,\tau}(e) (\vartheta_\tau^k)^{\alpha-1} \right) \mathrm{d}x
\end{aligned}
\tag{4.15}
$$

where we have applied the concavity inequality (4.9), with the choice $\psi(\vartheta) = \frac{1}{\alpha}\vartheta^\alpha$, to estimate the term $\frac{1}{\tau}\int_\Omega (\vartheta_\tau^k - \vartheta_\tau^{k-1})(\vartheta_\tau^k)^{\alpha-1} \,\mathrm{d}x$. Therefore, multiplying by τ, summing over the index k and neglecting some positive terms on the left-hand side of (4.15), we obtain for all $t \in (0,T]$

$$
\begin{aligned}
&\frac{4(1-\alpha)}{\alpha^2} \int_0^{\bar{\mathsf{t}}_\tau(t)} \int_\Omega \kappa(\overline{\vartheta}_\tau) |\nabla((\overline{\vartheta}_\tau)^{\alpha/2})|^2 \,\mathrm{d}x\,\mathrm{d}s + \int_0^{\bar{\mathsf{t}}_\tau(t)} \int_\Omega C_{\mathbb{D}}^1 |\dot{e}_\tau|^2 (\overline{\vartheta}_\tau)^{\alpha-1} \,\mathrm{d}x\,\mathrm{d}s \\
&\leq I_1 + I_2 + I_3,
\end{aligned}
\tag{4.16}
$$

with

$$
I_1 = \frac{1}{\alpha} \int_\Omega (\overline{\vartheta}_\tau(t))^\alpha \,\mathrm{d}x \leq \frac{1}{\alpha} \|\overline{\vartheta}_\tau\|_{L^\infty(0,T;L^1(\Omega))} + C \leq C \tag{4.17}
$$

via Young's inequality (using that $\alpha \in (0,1)$) and the second of (4.10k); similarly $I_2 = -\frac{1}{\alpha}\int_\Omega (\vartheta_0)^\alpha \,\mathrm{d}x \leq \frac{1}{\alpha}\|\vartheta_0\|_{L^1(\Omega)} + C$, whereas

$$
I_3 = \int_0^{\bar{\mathsf{t}}_\tau(t)} \int_\Omega \overline{\vartheta}_\tau(t) \mathbb{B} : \dot{e}_\tau(t) (\overline{\vartheta}_\tau(t))^{\alpha-1} \,\mathrm{d}x \leq \frac{C_{\mathbb{D}}^1}{4} \int_0^{\bar{\mathsf{t}}_\tau(t)} \int_\Omega |\dot{e}_\tau|^2 (\overline{\vartheta}_\tau)^{\alpha-1} \,\mathrm{d}x\,\mathrm{d}s + C \int_0^{\bar{\mathsf{t}}_\tau(t)} \int_\Omega (\overline{\vartheta}_\tau)^{\alpha+1} \,\mathrm{d}x\,\mathrm{d}s \,. \tag{4.18}
$$

All in all, absorbing the first term on the right-hand side of (4.18) into the left-hand side of (4.16) and taking into account the growth condition (2.κ_1) on κ, which yields with easy calculations that

$$
\int_0^{\bar{\mathsf{t}}_\tau(t)} \int_\Omega \kappa(\overline{\vartheta}_\tau) |\nabla((\overline{\vartheta}_\tau)^{\alpha/2})|^2 \,\mathrm{d}x\,\mathrm{d}s \stackrel{(1)}{\geq} c \int_0^{\bar{\mathsf{t}}_\tau(t)} \int_\Omega |\overline{\vartheta}_\tau|^{\mu+\alpha-2} |\nabla \overline{\vartheta}_\tau|^2 \,\mathrm{d}x\,\mathrm{d}s = c \int_0^{\bar{\mathsf{t}}_\tau(t)} \int_\Omega |\nabla(\overline{\vartheta}_\tau)^{(\mu+\alpha)/2}|^2 \,\mathrm{d}x\,\mathrm{d}s, \tag{4.19}
$$

we conclude from (4.16) that

$$
c \int_0^{\bar{\mathsf{t}}_\tau(t)} \int_\Omega |\nabla(\overline{\vartheta}_\tau)^{(\mu+\alpha)/2}|^2 \,\mathrm{d}x\,\mathrm{d}s \leq C + C \int_0^{\bar{\mathsf{t}}_\tau(t)} \int_\Omega (\overline{\vartheta}_\tau)^{\alpha+1} \,\mathrm{d}x\,\mathrm{d}s \,. \tag{4.20}
$$

From now on, the calculations follow exactly the same lines as those developed in [RR15, (3.8)–(3.12)] for the analogous estimate, in turn based on the ideas from [FPR09]. While referring to [RR15] for all details, let us just give the highlights. Setting $\overline{\xi}_\tau := (\overline{\vartheta}_\tau \vee 1)^{(\mu+\alpha)/2}$, we deduce from (4.20) the following inequality

$$
\int_0^{\bar{\mathsf{t}}_\tau(t)} \int_\Omega |\nabla \overline{\xi}_\tau|^2 \,\mathrm{d}x\,\mathrm{d}s \leq C + C \int_0^{\bar{\mathsf{t}}_\tau(t)} \|\overline{\xi}_\tau\|_{L^q(\Omega)}^q \,\mathrm{d}s \leq C + C \int_0^{\bar{\mathsf{t}}_\tau(t)} \|\overline{\xi}_\tau\|_{L^r(\Omega)}^s \,\mathrm{d}s + C \int_0^{\bar{\mathsf{t}}_\tau(t)} \|\overline{\xi}_\tau\|_{L^r(\Omega)}^q \,\mathrm{d}s, \tag{4.21}
$$

with $q \in [1,6)$ satisfying $\frac{\mu+\alpha}{2} \geq \frac{\alpha+1}{q}$. The very last estimate ensues from the Gagliardo-Nirenberg inequality, which in fact yields

$$
\|\overline{\xi}_\tau\|_{L^q(\Omega)} \leq C_{\mathrm{GN}} \|\nabla \overline{\xi}_\tau\|_{L^2(\Omega;\mathbb{R}^d)}^\theta \|\overline{\xi}_\tau\|_{L^r(\Omega)}^{1-\theta} + C\|\overline{\xi}_\tau\|_{L^r(\Omega)} \qquad \text{for } \theta \in (0,1) \text{ s.t. } \frac{1}{q} = \frac{\theta}{6} + \frac{1-\theta}{r} \tag{4.22}
$$

with $r \in [1,q]$. Then, s in (4.21) is a third exponent, related to q and r via (4.22). In [RR15] it is shown that the exponents q and r can be chosen in such a way as to have $\|\overline{\xi}_\tau\|_{L^\infty(0,T;L^r(\Omega))} \leq C\|\overline{\vartheta}_\tau\|_{L^\infty(0,T;L^1(\Omega))} + C \leq C$ thanks to second of (4.10k). Inserting this into (4.21) one concludes that $\|\nabla \overline{\xi}_\tau\|_{L^2(0,T;L^2(\Omega;\mathbb{R}^d))} \leq C$. All in all, this argument yields a bound for $\overline{\xi}_\tau$ in $L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;L^r(\Omega))$. Since $\overline{\xi}_\tau = (\overline{\vartheta}_\tau \vee 1)^{(\mu+\alpha)/2}$, we ultimately conclude that

$$
\|(\overline{\vartheta}_\tau)^{(\mu+\alpha)/2}\|_{L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;L^r(\Omega))} \leq C. \tag{4.23}
$$

Then, from inequality (1) in (4.19) we deduce that $\int_0^T \int_\Omega |\nabla \overline{\vartheta}_\tau|^2 \,\mathrm{d}x\,\mathrm{d}s \leq C$ provided that

$$
\mu + \alpha - 2 \geq 0 \;\Leftrightarrow\; \alpha \geq 2 - \mu. \tag{4.24}
$$

Thus, we infer the first of (4.10k) via the Poincaré's inequality. Interpolating between the two estimates in (4.10k) via the Gagliardo-Niremberg inequality gives

$$
\|\overline{\vartheta}_\tau\|_{L^h(Q)} \leq C \quad \text{with } h = \frac{8}{3} \text{ if } d = 3 \text{ and } h = 3 \text{ if } d = 2. \tag{4.25}
$$

Estimate (4.10j) follows from taking into account that

$$\|\log(\overline{\vartheta}_\tau)\|_{L^2(0,T;H^1(\Omega))} \leq C\left(1+\|\overline{\vartheta}_\tau\|_{L^2(0,T;H^1(\Omega))}\right)$$

thanks to the strict positivity (3.7). For later use, let us also observe that

$$\int_\Omega |\nabla(\overline{\vartheta}_\tau)^{(\mu-\alpha)/2}|^2 \,\mathrm{d}x = C\int_\Omega (\overline{\vartheta}_\tau)^{\mu-\alpha-2}|\nabla\overline{\vartheta}_\tau|^2 \,\mathrm{d}x \leq \frac{C}{\bar{\vartheta}^{2\alpha}}\int_\Omega (\overline{\vartheta}_\tau)^{\mu+\alpha-2}|\nabla\overline{\vartheta}_\tau|^2 \,\mathrm{d}x \leq C$$

thanks to (4.23). Then, taking into account that $(\overline{\vartheta}_\tau)^{(\mu-\alpha)/2} \leq (\overline{\vartheta}_\tau)^{(\mu+\alpha)/2}+1$ a.e. in Q, and the previously proved (4.23), we conclude estimate (4.10l).

Third a priori estimate: We consider the mechanical energy inequality (4.7) written for $s=0$. We estimate the terms on its right-hand side by the very same calculations developed in the *First a priori estimate* for the right-hand side terms of (4.6). We also use that

$$\int_0^{\bar{\mathrm{t}}_\tau(t)}\int_\Omega \overline{\vartheta}_\tau(r)\mathbb{B}{:}\dot{e}_\tau \,\mathrm{d}x\,\mathrm{d}r \leq \delta\int_0^{\bar{\mathrm{t}}_\tau(t)}\int_\Omega |\dot{e}_\tau|^2 \,\mathrm{d}x\,\mathrm{d}r + C_\delta\|\overline{\vartheta}_\tau\|^2_{L^2(0,T;L^2(\Omega))}$$

via Young's inequality, with the constant $\delta>0$ chosen in such a way as to absorb the term $\iint|\dot{e}_\tau|^2$ into the left-hand side of (4.7). Since $\|\overline{\vartheta}_\tau\|^2_{L^2(0,T;L^2(\Omega))} \leq C$ by the previously proved (4.10k) , we ultimately conclude that the terms on the right-hand side of (4.7) are all bounded, uniformly w.r.t. τ. This leads to (4.10e) and (4.10h), whence (4.10g), as well as (4.10a) and the first of (4.10b) by kinematic admissibility.

Fourth a priori estimate: It follows from estimates (4.10d), (4.10e), (4.10f), and (4.10k) that the stresses $(\overline{\sigma}_\tau)_\tau$ are uniformly bounded in $L^{\gamma/(\gamma-1)}(Q;\mathbb{M}^{d\times d}_{\mathrm{sym}})$. Therefore, also taking into account (4.1c), a comparison argument in the discrete momentum balance (4.2b) yields that the derivatives $(\partial_t\widehat{u}_\tau)_\tau$ are bounded in $L^{\gamma/(\gamma-1)}(0,T;W^{1,\gamma}(\Omega;\mathbb{R}^d)^*)$, whence the second of (4.10c).

Fifth a priori estimate: We will now sketch the argument for (4.10m), referring to the proof of [RR15, Prop. 4.10] for all details. Indeed, let us fix a partition $0=\sigma_0<\sigma_1<\ldots<\sigma_J=T$ of the interval $[0,T]$. From the discrete entropy inequality (4.5) written on the interval $[\sigma_{i-1},\sigma_i]$ and for a *constant-in-time* test function we deduce that

$$\begin{aligned}&\int_\Omega (\log(\overline{\vartheta}_\tau(\sigma_i))-\log(\overline{\vartheta}_\tau(\sigma_{i-1})))\varphi\,\mathrm{d}x+\Lambda_{i,\tau}(\varphi)\geq 0 \quad \text{for all } \varphi\in W^{1,d+\epsilon}_+(\Omega),\\ &\int_\Omega (\log(\overline{\vartheta}_\tau(\sigma_{i-1}))-\log(\overline{\vartheta}_\tau(\sigma_i)))\varphi\,\mathrm{d}x-\Lambda_{i,\tau}(\varphi)\geq 0 \quad \text{for all } \varphi\in W^{1,d+\epsilon}_-(\Omega),\end{aligned} \tag{4.26}$$

where we have used the place-holder

$$\begin{aligned}\Lambda_{i,\tau}(\varphi) = &\int_{\bar{\mathrm{t}}_\tau(\sigma_{i-1})}^{\bar{\mathrm{t}}_\tau(\sigma_i)}\int_\Omega \kappa(\overline{\vartheta}_\tau)\nabla\log(\overline{\vartheta}_\tau)\nabla\varphi\,\mathrm{d}x\,\mathrm{d}r+\int_{\bar{\mathrm{t}}_\tau(\sigma_{i-1})}^{\bar{\mathrm{t}}_\tau(\sigma_i)}\int_\Omega \mathbb{B}{:}\dot{e}_\tau\varphi\,\mathrm{d}x\,\mathrm{d}r\\ &-\int_{\bar{\mathrm{t}}_\tau(\sigma_{i-1})}^{\bar{\mathrm{t}}_\tau(\sigma_i)}\int_\Omega \kappa(\overline{\vartheta}_\tau)\frac{\varphi}{\overline{\vartheta}_\tau}\nabla(\log(\overline{\vartheta}_\tau))\nabla\overline{\vartheta}_\tau\,\mathrm{d}x\,\mathrm{d}r-\int_{\bar{\mathrm{t}}_\tau(\sigma_{i-1})}^{\bar{\mathrm{t}}_\tau(\sigma_i)}\int_{\partial\Omega}\overline{h}_\tau\frac{\varphi}{\overline{\vartheta}_\tau}\,\mathrm{d}S\,\mathrm{d}r\\ &-\int_{\bar{\mathrm{t}}_\tau(\sigma_{i-1})}^{\bar{\mathrm{t}}_\tau(\sigma_i)}\int_\Omega\left(\overline{H}_\tau+\mathrm{R}(\underline{\vartheta}_\tau,\dot{p}_\tau)+|\dot{p}_\tau|^2+\mathbb{D}\dot{e}_\tau{:}\dot{e}_\tau\right)\frac{\varphi}{\overline{\vartheta}_\tau}\,\mathrm{d}x\,\mathrm{d}r.\end{aligned} \tag{4.27}$$

Arguing as in the proof of [RR15, Prop. 4.10], from (4.26) we deduce that

$$\begin{aligned}&\sum_{i=1}^J\left|\langle\log(\overline{\vartheta}_\tau(\sigma_i))-\log(\overline{\vartheta}_\tau(\sigma_{i-1})),\varphi\rangle_{W^{1,d+\epsilon}(\Omega)}\right|\\ &\leq\sum_{i=1}^J\int_\Omega(\log(\overline{\vartheta}_\tau(\sigma_i))-\log(\overline{\vartheta}_\tau(\sigma_{i-1})))|\varphi|\,\mathrm{d}x+\Lambda_{i,\tau}(|\varphi|)+|\Lambda_{i,\tau}(\varphi^+)|+|\Lambda_{i,\tau}(\varphi^-)|\end{aligned} \tag{4.28}$$

for all $\varphi\in W^{1,d+\epsilon}(\Omega)$. Then, we infer the bound (4.10m) by estimating the terms on the right-hand side of (4.28), uniformly w.r.t. φ. In particular, to handle the second, fourth, and fifth integral terms arising from $\Lambda_{i,\tau}(\varphi)$ (cf. (4.27)), we use the previously proved estimates (4.10e), (4.10h), as well as the bounds provided by (4.1a) and (4.1b) on $\overline{H}_\tau$ and $\overline{h}_\tau$, cf. [RR15] for all details. Let us only comment on the estimates for the first and third integral terms on the r.h.s. of (4.27). We remark that for every $\varphi \in W^{1,d+\epsilon}(\Omega)$ we have

$$
\begin{aligned}
& \left| \int_{\bar{\mathsf{t}}_\tau(\sigma_{i-1})}^{\bar{\mathsf{t}}_\tau(\sigma_i)} \int_\Omega \kappa(\overline{\vartheta}_\tau) \nabla \log(\overline{\vartheta}_\tau) \nabla \varphi \,\mathrm{d}x \,\mathrm{d}r \right| \\
& \stackrel{(1)}{\leq} C \int_{\bar{\mathsf{t}}_\tau(\sigma_{i-1})}^{\bar{\mathsf{t}}_\tau(\sigma_i)} \int_\Omega \left(|\overline{\vartheta}_\tau|^{\mu-1} |\nabla \overline{\vartheta}_\tau| + \frac{1}{\overline{\vartheta}_\tau} |\nabla \overline{\vartheta}_\tau| \right) |\nabla \varphi| \,\mathrm{d}x \,\mathrm{d}r \\
& \stackrel{(2)}{\leq} C \int_{\bar{\mathsf{t}}_\tau(\sigma_{i-1})}^{\bar{\mathsf{t}}_\tau(\sigma_i)} \int_\Omega | \left(\overline{\vartheta}_\tau\right)^{(\mu+\alpha-2)/2} \nabla \overline{\vartheta}_\tau| \left(\overline{\vartheta}_\tau\right)^{(\mu-\alpha)/2} |\nabla \varphi| + \frac{1}{\bar{\vartheta}} |\nabla \overline{\vartheta}_\tau| |\nabla \varphi| \,\mathrm{d}x \,\mathrm{d}r \\
& \stackrel{(3)}{\leq} C \int_{\bar{\mathsf{t}}_\tau(\sigma_{i-1})}^{\bar{\mathsf{t}}_\tau(\sigma_i)} \| \left(\overline{\vartheta}_\tau\right)^{(\mu+\alpha-2)/2} \nabla \overline{\vartheta}_\tau \|_{L^2(\Omega;\mathbb{R}^d)} \| \left(\overline{\vartheta}_\tau\right)^{(\mu-\alpha)/2} \|_{L^{d^\star}(\Omega)} \|\nabla \varphi\|_{L^{d+\epsilon}(\Omega;\mathbb{R}^d)} \,\mathrm{d}r \\
& \quad + C \int_{\bar{\mathsf{t}}_\tau(\sigma_{i-1})}^{\bar{\mathsf{t}}_\tau(\sigma_i)} \|\nabla \overline{\vartheta}_\tau\|_{L^2(\Omega;\mathbb{R}^d)} \|\nabla \varphi\|_{L^2(\Omega;\mathbb{R}^d)} \,\mathrm{d}r
\end{aligned}
\tag{4.29}
$$

where (1) follows from the growth condition (2.κ_1) on κ, (2) from the discrete positivity property (3.7), and (3) from Hölder's inequality, in view of the continuous embedding

$$
H^1(\Omega) \subset L^{d^\star}(\Omega) \quad \text{with } d^\star \begin{cases} \in [1,\infty) & \text{if } d = 2, \\ = 6 & \text{if } d = 3. \end{cases}
\tag{4.30}
$$

Therefore, observe that only $\varphi \in W^{1,d+\epsilon}(\Omega)$, with $\epsilon > 0$, is needed. Then, we use estimates (4.10k) and (4.10l) to bound the terms on the r.h.s. of (4.29). As for the third term on the r.h.s. of (4.27), we use that

$$
\begin{aligned}
& \left| \int_{\bar{\mathsf{t}}_\tau(\sigma_{i-1})}^{\bar{\mathsf{t}}_\tau(\sigma_i)} \int_\Omega \kappa(\overline{\vartheta}_\tau) \frac{\varphi}{\overline{\vartheta}_\tau} \nabla (\log(\overline{\vartheta}_\tau)) \nabla \overline{\vartheta}_\tau \,\mathrm{d}x \,\mathrm{d}r \right| \\
& \stackrel{(4)}{\leq} C \int_{\bar{\mathsf{t}}_\tau(\sigma_{i-1})}^{\bar{\mathsf{t}}_\tau(\sigma_i)} \int_\Omega \left(|\overline{\vartheta}_\tau|^{\mu-2} |\nabla \overline{\vartheta}_\tau|^2 + \frac{1}{\bar{\vartheta}^2} |\nabla \overline{\vartheta}_\tau|^2 \right) |\varphi| \,\mathrm{d}x \,\mathrm{d}r \\
& \stackrel{(5)}{\leq} C \|\varphi\|_{L^\infty(\Omega)} \int_{\bar{\mathsf{t}}_\tau(\sigma_{i-1})}^{\bar{\mathsf{t}}_\tau(\sigma_i)} \int_\Omega |\overline{\vartheta}_\tau|^{\mu+\alpha-2} |\nabla \overline{\vartheta}_\tau|^2 + |\nabla \overline{\vartheta}_\tau|^2 \,\mathrm{d}x \,\mathrm{d}r,
\end{aligned}
$$

with (4) due to (2.κ_1) and the positivity property (3.7), and (5) following from the estimate $|\overline{\vartheta}_\tau|^{\mu-2} \leq |\overline{\vartheta}_\tau|^{\mu+\alpha-2} + 1$, combined with the fact that $\varphi \in W^{1,d+\epsilon}(\Omega) \subset L^\infty(\Omega)$. Again, we conclude via the bounds (4.10k) and (4.10l).

Sixth a priori estimate: Under the stronger condition (2.κ_2), we multiply the discrete heat equation (3.6a) by a function $\varphi \in W^{1,\infty}(\Omega)$. Integrating in space we thus obtain for almost all $t \in (0,T)$

$$
\left| \int_\Omega \dot{\vartheta}_\tau(t) \varphi \,\mathrm{d}x \right| \leq \left| \int_\Omega \kappa(\overline{\vartheta}_\tau(t)) \nabla \overline{\vartheta}_\tau(t) \nabla \varphi \,\mathrm{d}x \right| + \left| \int_\Omega \overline{J}_\tau(t) \varphi \,\mathrm{d}x \right| + \left| \int_{\partial\Omega} \overline{h}_\tau(t) \varphi \,\mathrm{d}S \right| \doteq I_1 + I_2 + I_3 \,,
\tag{4.31}
$$

where we have used the place-holder $\overline{J}_\tau(t) := \overline{H}_\tau(t) + \mathrm{R}\left(\underline{\vartheta}_\tau(t), \dot{p}_\tau(t)\right) + |\dot{p}_\tau(t)|^2 + \mathbb{D}\dot{e}_\tau(t) : \dot{e}_\tau(t) - \overline{\vartheta}_\tau(t)\mathbb{B} : \dot{e}_\tau(t)$. Now, in view of (4.1a) for $\overline{H}_\tau$ and of estimates (4.10e), (4.10h), and (4.10k), it is clear that

$$
I_2 \leq \mathcal{J}_\tau(t) \|\varphi\|_{L^\infty(\Omega)} \quad \text{with } \mathcal{J}_\tau(t) := \|\overline{J}_\tau(t)\|_{L^1(\Omega)}.
$$

Observe that the family $(\mathcal{J}_\tau)_\tau$ is uniformly bounded in $L^1(0,T)$. The third term on the r.h.s. of (4.31) is analogously bounded thanks to (4.1b). As for the first one, we use that

$$
I_1 \leq C \|(\overline{\vartheta}_\tau)^{(\mu-\alpha+2)/2}\|_{L^2(\Omega)} \|(\overline{\vartheta}_\tau)^{(\mu+\alpha-2)/2} \nabla \overline{\vartheta}_\tau\|_{L^2(\Omega;\mathbb{R}^d)} \|\nabla \varphi\|_{L^\infty(\Omega;\mathbb{R}^d)} + C \|\nabla \overline{\vartheta}_\tau\|_{L^2(\Omega;\mathbb{R}^d)} \|\nabla \varphi\|_{L^2(\Omega;\mathbb{R}^d)},
$$

based on the growth condition (2.κ_1) on κ. By (2.κ_2) we have $\mu < 5/3$ if $d = 3$, and $\mu < 2$ if $d = 2$. Since α can be chosen arbitrarily close to 1, from (4.25) we gather that $(\overline{\vartheta}_\tau)^{(\mu-\alpha+2)/2}$ is bounded in $L^2(Q)$. Therefore, also taking into account (4.23) and (4.10k) we infer that $I_1 \leq \mathcal{K}_\tau(t) \|\varphi\|_{L^\infty(\Omega)}$ with $(\mathcal{K}_\tau)_\tau$ bounded in $L^1(0,T)$. Hence, estimate (4.10n) follows. □

4.2. **Passage to the limit.** In this section, we conclude the proof of Theorems 1 & 2. First of all, from the a priori estimates obtained in Proposition 4.3 we deduce the convergence (along a subsequence, in suitable topologies) of the approximate solutions, to a quadruple (ϑ, u, e, p). In the proofs of Thm. 1 (2, respectively), we then proceed to show that (ϑ, u, e, p) is an *entropic* (a *weak energy*, respectively) solution to (the Cauchy problem for) system (1.3, 1.5), by passing to the limit in the approximate system (4.2), and in the discrete entropy and total energy inequalities. Let us mention that, in order to recover the kinematic admissibility, the weak momentum balance, and the plastic flow rule, we will follow an approach different from that developed in [DMS14]. The latter paper exploited a reformulation of the (discrete) momentum balance and flow rule in terms of a mechanical energy balance, and a variational inequality, based on the results from [DMDM06]. Let us point out that it would be possible to repeat this argument in the present setting as well. Nonetheless, the limit passage procedure that we will develop in Step 2 of the proof of Thm. 1 will lead us to conclude, via careful lim sup-arguments, additional strong convergences that will allow us to take the limit of the quadratic terms on the r.h.s. of the heat equation (1.3a).

Prior to our compactness statement for the sequence of approximate solutions, we recall here a compactness result, akin to the Helly Theorem and tailored to the bounded variation type estimate (4.10m), which will have a pivotal role in establishing the convergence properties for (a subsequence of) the approximate temperatures. Theorem 4.5 below was proved in [RR15], cf. Thm. A.5 therein, with the exception of convergence (4.37). We will give its proof in the Appendix, and in doing so we will shortly recapitulate the argument for [RR15, Thm. A.5]. Since in the proof we shall resort to a compactness result from the theory of Young measures, also invoked in the proof of Thm. 3, we shall recall such result, together with some basics of the theory, in the Appendix.

Theorem 4.5. *Let $\mathbf{V}$ and $\mathbf{Y}$ be two (separable) reflexive Banach spaces such that $\mathbf{V} \subset \mathbf{Y}^*$ continuously. Let $(\ell_k)_k \subset L^p(0,T;\mathbf{V}) \cap \mathrm{B}([0,T];\mathbf{Y}^*)$ be bounded in $L^p(0,T;\mathbf{V})$ and suppose in addition that*

$$(\ell_k(0))_k \subset \mathbf{Y}^* \text{ is bounded,} \tag{4.32}$$

$$\exists\, C>0 \quad \forall\, \varphi \in \overline{B}_{1,\mathbf{Y}}(0) \quad \forall\, k \in \mathbb{N}: \qquad \mathrm{Var}(\,\langle \ell_k, \varphi \rangle_{\mathbf{Y}}; [0,T]) \leq C, \tag{4.33}$$

where, for given $\ell \in \mathrm{B}([0,T];\mathbf{Y}^*)$ and $\varphi \in \mathbf{Y}$ we set

$$\mathrm{Var}(\,\langle \ell, \varphi \rangle_{\mathbf{Y}}; [0,T]) := \sup\Big\{ \sum_{i=1}^{J} |\, \langle \ell(\sigma_i), \varphi \rangle_{\mathbf{Y}} - \langle \ell(\sigma_{i-1}), \varphi \rangle_{\mathbf{Y}}| \;:\; 0=\sigma_0 < \sigma_1 < \ldots < \sigma_J = T \Big\}\,. \tag{4.34}$$

Then, there exist a (not relabeled) subsequence $(\ell_k)_k$ and a function $\ell \in L^p(0,T;\mathbf{V}) \cap L^\infty(0,T;\mathbf{Y}^*)$ such that as $k \to \infty$

$$\ell_k \overset{*}{\rightharpoonup} \ell \quad \text{ in } L^p(0,T;\mathbf{V}) \cap L^\infty(0,T;\mathbf{Y}^*), \tag{4.35}$$

$$\ell_k(t) \rightharpoonup \ell(t) \quad \text{ in } \mathbf{V} \quad \text{for a.a. } t \in (0,T). \tag{4.36}$$

Furthermore, for almost all $t \in (0,T)$ and any sequence $(t_k)_k \subset [0,T]$ with $t_k \to t$ there holds

$$\ell_k(t_k) \rightharpoonup \ell(t) \qquad \text{ in } \mathbf{Y}^*. \tag{4.37}$$

We are now in the position to prove the following compactness result where, in particular, we show that, along a subsequence, the sequences $(\overline{\vartheta}_\tau)_\tau$ and $(\underline{\vartheta}_\tau)_\tau$ converge, in suitable topologies, to the *same* limit ϑ. This is not a trivial consequence of the obtained a priori estimates, as no bound on the total variation of the functions $\overline{\vartheta}_\tau$ is available. In fact, this fact stems from the 'generalized BV' estimate (4.10m), via the convergence property (4.37) from Theorem 4.5.

Lemma 4.6 (Compactness)**.** *Assume* $(2.\kappa_1)$*. Then, for any sequence* $\tau_k \downarrow 0$ *there exist a (not relabeled) subsequence and a quintuple* $(\vartheta, u, e, p, \zeta)$ *such that the following convergences hold*

$$u_{\tau_k} \overset{*}{\rightharpoonup} u \quad \text{in } H^1(0,T;H^1(\Omega;\mathbb{R}^d)) \cap W^{1,\infty}(0,T;L^2(\Omega;\mathbb{R}^d)), \tag{4.38a}$$

$$\overline{u}_{\tau_k}, \underline{u}_{\tau_k} \to u \quad \text{in } L^\infty(0,T;H^{1-\epsilon}(\Omega;\mathbb{R}^d)) \text{ for all } \epsilon \in (0,1], \tag{4.38b}$$

$$u_{\tau_k} \to u \quad \text{in } \mathrm{C}^0([0,T];H^{1-\epsilon}(\Omega;\mathbb{R}^d)) \text{ for all } \epsilon \in (0,1], \tag{4.38c}$$

$$\widehat{u}_{\tau_k} \to \dot{u} \quad \text{in } \mathrm{C}^0_{\mathrm{weak}}([0,T];L^2(\Omega;\mathbb{R}^d)) \cap L^2(0,T;H^{1-\epsilon}(\Omega;\mathbb{R}^d)) \text{ for all } \epsilon \in (0,1], \tag{4.38d}$$

$$\partial_t \widehat{u}_{\tau_k} \rightharpoonup \ddot{u} \quad \text{in } L^{\gamma/(\gamma-1)}(0,T;W^{1,\gamma}(\Omega;\mathbb{R}^d)^*), \tag{4.38e}$$

$$\overline{e}_{\tau_k} \overset{*}{\rightharpoonup} e \quad \text{in } L^\infty(0,T;L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})), \tag{4.38f}$$

$$e_{\tau_k} \rightharpoonup e \quad \text{in } H^1(0,T;L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})), \tag{4.38g}$$

$$e_{\tau_k} \to e \quad \text{in } \mathrm{C}^0_{\mathrm{weak}}([0,T];L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})), \tag{4.38h}$$

$$\tau|\overline{e}_{\tau_k}|^{\gamma-2}\overline{e}_{\tau_k} \to 0 \quad \text{in } L^\infty(0,T;L^{\gamma/(\gamma-1)}(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})), \tag{4.38i}$$

$$\overline{p}_{\tau_k} \overset{*}{\rightharpoonup} p \quad \text{in } L^\infty(0,T;L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})), \tag{4.38j}$$

$$p_{\tau_k} \rightharpoonup p \quad \text{in } H^1(0,T;L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})), \tag{4.38k}$$

$$p_{\tau_k} \to p \quad \text{in } \mathrm{C}^0_{\mathrm{weak}}([0,T];L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{D}})), \tag{4.38l}$$

$$\tau|\overline{p}_{\tau_k}|^{\gamma-2}\overline{p}_{\tau_k} \to 0 \quad \text{in } L^\infty(0,T;L^{\gamma/(\gamma-1)}(\Omega;\mathbb{M}^{d\times d}_{\mathrm{D}})), \tag{4.38m}$$

$$\overline{\vartheta}_{\tau_k} \rightharpoonup \vartheta \quad \text{in } L^2(0,T;H^1(\Omega)), \tag{4.38n}$$

$$\log(\overline{\vartheta}_{\tau_k}) \overset{*}{\rightharpoonup} \log(\vartheta) \quad \text{in } L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;W^{1,d+\epsilon}(\Omega)^*) \quad \text{for every } \epsilon > 0, \tag{4.38o}$$

$$\log(\overline{\vartheta}_{\tau_k}(t)) \rightharpoonup \log(\vartheta(t)) \quad \text{in } H^1(\Omega) \text{ for almost all } t \in (0,T), \tag{4.38p}$$

$$\log(\underline{\vartheta}_{\tau_k}(t)) \rightharpoonup \log(\vartheta(t)) \quad \text{in } H^1(\Omega) \text{ for almost all } t \in (0,T), \tag{4.38q}$$

$$\overline{\vartheta}_{\tau_k} \to \vartheta \quad \text{in } L^h(Q) \text{ for all } h \in [1,8/3) \text{ for } d=3 \text{ and all } h \in [1,3) \text{ if } d=2, \tag{4.38r}$$

$$\underline{\vartheta}_{\tau_k} \to \vartheta \quad \text{in } L^h(Q) \text{ for all } h \in [1,8/3) \text{ for } d=3 \text{ and all } h \in [1,3) \text{ if } d=2, \tag{4.38s}$$

$$(\overline{\vartheta}_{\tau_k})^{(\mu+\alpha)/2} \rightharpoonup \vartheta^{(\mu+\alpha)/2} \quad \text{in } L^2(0,T;H^1(\Omega)) \text{ for every } \alpha \in [2-\mu,1), \tag{4.38t}$$

$$(\overline{\vartheta}_{\tau_k})^{(\mu-\alpha)/2} \rightharpoonup \vartheta^{(\mu-\alpha)/2} \quad \text{in } L^2(0,T;H^1(\Omega)) \text{ for every } \alpha \in [2-\mu,1), \tag{4.38u}$$

$$\overline{\zeta}_{\tau_k} \overset{*}{\rightharpoonup} \zeta \quad \text{in } L^\infty(Q;\mathbb{M}^{d\times d}_{\mathrm{D}}). \tag{4.38v}$$

The triple (u,e,p) *complies with the kinematic admissibility condition* (2.28)*, while* ϑ *also fulfills*

$$\vartheta \in L^\infty(0,T;L^1(\Omega)) \text{ and } \vartheta \geq \bar{\vartheta} \text{ a.e. in } Q \tag{4.38w}$$

with $\bar{\vartheta}$ *from* (3.7)*.*

Furthermore, under condition $(2.\kappa_2)$ *we also have* $\vartheta \in \mathrm{BV}([0,T];W^{1,\infty}(\Omega)^*)$*, and*

$$\overline{\vartheta}_{\tau_k} \to \vartheta \quad \text{in } L^2(0,T;Y) \text{ for all } Y \text{ such that } H^1(\Omega) \Subset Y \subset W^{1,\infty}(\Omega)^*, \tag{4.38x}$$

$$\overline{\vartheta}_{\tau_k}(t) \overset{*}{\rightharpoonup} \vartheta(t) \quad \text{in } W^{1,\infty}(\Omega)^* \text{ for all } t \in [0,T]. \tag{4.38y}$$

Let us mention beforehand that, in the proof of Thm. 1 we will obtain further convergence properties for the sequences of approximate solutions, cf. also Remark 4.7 ahead.

Sketch of the proof. Convergences (4.38a)–(4.38c), (4.38f)–(4.38h), (4.38j)–(4.38l), and (4.38v) follow from the a priori estimates in Proposition 4.3 via well known weak and strong compactness results (cf. e.g. [Sim87]), also taking into account that

$$\|\overline{e}_{\tau_k} - e_{\tau_k}\|_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^d))} \leq C\tau_k^{1/2} \to 0 \qquad \text{as } k \to \infty, \tag{4.39}$$

and the analogous relations involving $\overline{p}_{\tau_k}$, p_{τ_k}, etc. Passing to the limit as $k \to \infty$ in the discrete kinematic admissibility condition $(\overline{u}_{\tau_k}(t), \overline{e}_{\tau_k}(t), \overline{p}_{\tau_k}(t)) \in \mathcal{A}(\overline{w}_{\tau_k}(t))$ for a.a. $t \in (0,T)$, also in view of convergence (4.1d) for $\overline{w}_{\tau_k}$, we conclude that the triple (u,e,p) is admissible. In view of estimate (4.10c) for $(\widehat{u}_{\tau_k})_k$, again by the Aubin-Lions type compactness results from [Sim87] we conclude that there exists v such that $\widehat{u}_{\tau_k} \to v$ in

$L^2(0,T;H^{1-\epsilon}(\Omega;\mathbb{R}^d)) \cap \mathrm{C}^0_{\mathrm{weak}}([0,T];L^2(\Omega;\mathbb{R}^d))$ for every $\epsilon \in (0,1]$. Taking into account that

$$\|\widehat{u}_{\tau_k} - \dot{u}_{\tau_k}\|_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^d))} \leq \tau_k^{1/2} \|\partial_t \widehat{u}_{\tau_k}\|_{L^2(0,T;L^2(\Omega;\mathbb{R}^d))} \leq S{\tau_k}^{1/2}, \tag{4.40}$$

we conclude that $v = \dot{u}$, whence (4.38d). It then follows from (4.40) that

$$\dot{u}_{\tau_k}(t) \rightharpoonup \dot{u}(t) \qquad \text{in } L^2(\Omega;\mathbb{R}^d) \quad \text{for every } t \in [0,T]. \tag{4.41}$$

Moreover, thanks to (4.40) we identify the weak limit of $\partial_t \widehat{u}_{\tau_k}$ in $L^{\gamma/(\gamma-1)}(0,T;W^{1,\gamma}(\Omega;\mathbb{R}^d)^*)$ with $\ddot{u}$, and (4.38e) ensues. In order to prove (4.38i) (an analogous argument yields (4.38m)), it is sufficient to observe that

$$\|\tau|\overline{e}_{\tau_k}|^{\gamma-2}\overline{e}_{\tau_k}\|_{L^\infty(0,T;L^{\gamma/(\gamma-1)}(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}}))} = \tau^{1/\gamma}\left(\tau^{1/\gamma}\|\overline{e}_{\tau_k}\|_{L^\infty(0,T;L^\gamma(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}}))}\right)^{\gamma-1} \to 0$$

thanks to estimate (4.10f).

For the convergences of the functions $(\overline{\vartheta}_{\tau_k})_k$, we briefly recap the arguments from the proof of [RR15, Lemma 5.1]. On account of estimates (4.10j) and (4.10m) we can apply the compactness theorem (4.5) to the functions $\ell_k = \log(\overline{\vartheta}_{\tau_k})$, in the setting of the spaces $\mathbf{V} = H^1(\Omega)$, $\mathbf{Y} = W^{1,d+\epsilon}(\Omega)$, and with $p = 2$. Hence we conclude that, up to a subsequence the functions $\log(\overline{\vartheta}_{\tau_k})$ weakly* converge to some λ in $L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;W^{1,d+\epsilon}(\Omega)^*)$ for all $\epsilon > 0$, i.e. (4.38o), and that $\log(\overline{\vartheta}_{\tau_k}(t)) \rightharpoonup \lambda(t)$ in $H^1(\Omega)$ for almost all $t \in (0,T)$, i.e. (4.38p). Therefore, up to a further subsequence we have $\log(\overline{\vartheta}_{\tau_k}) \to \lambda$ almost everywhere in Q. Thus, $\overline{\vartheta}_{\tau_k} \to \vartheta := e^\lambda$ almost everywhere in Q. Convergences (4.38n) and (4.38r) then follow from estimates (4.10k) and (4.25), respectively. An immediate lower semicontinuity argument combined with estimate (4.10k) allows us to conclude (4.38w); the strict positivity of ϑ follows from (3.7). Concerning convergence (4.38t), we use (4.38r) to deduce that $(\overline{\vartheta}_{\tau_k})^{(\mu+\alpha)/2} \to \vartheta^{(\mu+\alpha)/2}$ in $L^{2h/(\mu+\alpha)}(Q)$ for h as in (4.38r). Since $(\overline{\vartheta}_{\tau_k})^{(\mu+\alpha)/2}$ is itself bounded in $L^2(0,T;H^1(\Omega)$ by estimate (4.10l), (4.38t) ensues, and so does (4.38u) by a completely analogous argument.

Let us now address convergences (4.38q) and (4.38s) for the sequence $(\underline{\vartheta}_{\tau_k})_k$. On the one hand, observe that estimates (4.10j)–(4.10m) also hold for $(\underline{\vartheta}_{\tau_k})_k$. Therefore, we may apply Thm. 4.5 to the functions $\log(\underline{\vartheta}_{\tau_k})$ and conclude that there exists $\underline{\lambda}$ such that $\log(\underline{\vartheta}_{\tau_k}) \overset{*}{\rightharpoonup} \underline{\lambda}$ in $L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;W^{1,d+\epsilon}(\Omega)^*)$ for all $\epsilon > 0$, as well as $\log(\underline{\vartheta}_{\tau_k}(t)) \rightharpoonup \underline{\lambda}(t)$ in $H^1(\Omega)$ for almost all $t \in (0,T)$. On the other hand, since $\underline{\vartheta}_{\tau_k}(t) = \overline{\vartheta}_{\tau_k}(t-\tau_k)$ for almost all $t \in (0,T)$, from (4.37) we conclude that $\log(\underline{\vartheta}_{\tau_k}(t)) \rightharpoonup \log(\vartheta(t))$ in $W^{1,d+\epsilon}(\Omega)^*$ for almost all $t \in (0,T)$. Hence we identify $\underline{\lambda}(t) = \log(\vartheta(t))$ for almost all $t \in (0,T)$. Then, convergences (4.38q) and (4.38s) ensue from the very same arguments as for the sequence $(\overline{\vartheta}_{\tau_k})_k$ (in fact, the analogue of (4.38o) also holds for $\log(\underline{\vartheta}_{\tau_k})_k$).

Finally, under condition (2.κ_2), we can also count on the BV-estimate (4.10n) for $(\overline{\vartheta}_\tau)_\tau$. We may then apply [DMDM06, Lemma 7.2], which generalizes the classical Helly Theorem to functions with values in the dual of a separable Banach space, and conclude the pointwise convergence (4.38y). Convergence (4.38x) follows from estimate (4.10n) combined with (4.10k), via an Aubin-Lions type compactness result for BV-functions (see, e.g., [Rou05, Chap. 7, Cor. 4.9]). □

We are now in the position to develop the **proof of Theorem 1**. Let (τ_k) be a null sequence of time steps, and let

$$(\overline{\vartheta}_{\tau_k}, \underline{\vartheta}_{\tau_k}, \vartheta_{\tau_k}, \overline{u}_{\tau_k}, u_{\tau_k}, \widehat{u}_{\tau_k}, \overline{e}_{\tau_k}, e_{\tau_k}, \overline{p}_{\tau_k}, p_{\tau_k}, \overline{\zeta}_{\tau_k})_k,$$

be a sequence of solutions to the approximate PDE system (4.2) for which the convergences stated in Lemma 4.6 hold to a quintuple $(\vartheta, u, e, p, \zeta)$. We will pass to the limit in the time-discrete versions of the momentum balance and of the plastic flow rule, in the discrete entropy inequality and in the discrete total energy inequality, to conclude that (ϑ, u, e, p) is an entropic solution to the thermoviscoplastic system in the sense of Def. 2.2.

Step 0*: ad the initial conditions* (2.23) *and the kinematic admissibility* (2.28). It was shown in Lemma 4.6 that the limit triple (u, e, p) is kinematically admissible. Passing to the limit in the initial conditions (3.5), on account of (3.4) and of the pointwise convergences (4.38c), (4.38h), (4.38l), and (4.41), we conclude that the triple (u, e, p) comply with initial conditions (2.23).

Step 1*: ad the momentum balance* (2.6). Thanks to convergences (4.38f)–(4.38i) and (4.38n) we have that

$$\overline{\sigma}_{\tau_k} = \mathbb{D}\dot{e}_{\tau_k} + \mathbb{C}\overline{e}_{\tau_k} + \tau|\overline{e}_{\tau_k}|^{\gamma-2}\overline{e}_{\tau_k} - \overline{\vartheta}_{\tau_k}\mathbb{B} \rightharpoonup \sigma = \mathbb{D}\dot{e} + \mathbb{C}e - \vartheta\mathbb{B} \qquad \text{in } L^{\gamma/(\gamma-1)}(Q;\mathbb{M}^{d\times d}_{\mathrm{sym}}). \tag{4.42}$$

Combining this with convergence (4.38e) and with (4.1c) for $(\overline{\mathcal{L}}_{\tau_k})_k$, we pass to the limit in the discrete momentum balance (4.2b) and conclude that (ϑ, u, e) fulfill (2.6) with test functions in $W^{1,\gamma}_{\mathrm{Dir}}(\Omega;\mathbb{R}^d)$. By comparison in (2.6) we conclude that $\ddot{u} \in L^2(0,T;H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^d)^*)$, whence (2.22c). Moreover, a density argument yields that (2.6) holds with test functions in $H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^d)$. This concludes the proof of the momentum balance.

Step 2: ad the plastic flow rule (2.14). Convergences (4.38k)–(4.38m), (4.38v), and (4.42) ensure that the functions (ϑ, e, p, ζ) fulfill

$$\zeta + \dot{p} \ni \sigma_{\mathrm{D}} \qquad \text{a.e. in } Q. \tag{4.43}$$

In order to conclude (2.14) it remains to show that $\zeta \in \partial_{\dot{p}}\mathrm{R}(\vartheta,\dot{p})$ a.e. in Q, which can be reformulated via (2.13). In turn, the latter relations are equivalent to

$$\begin{cases} \iint_Q \zeta{:}\eta \,\mathrm{d}x\,\mathrm{d}t \le \int_0^T \mathcal{R}(\vartheta(t),\eta(t))\,\mathrm{d}t & \text{for all } \eta \in L^2(Q;\mathbb{M}^{d\times d}_{\mathrm{D}}), \\ \iint_Q \zeta{:}\dot{p}\,\mathrm{d}x\,\mathrm{d}t \ge \int_0^T \mathcal{R}(\vartheta(t),\dot{p}(t))\,\mathrm{d}t. \end{cases} \tag{4.44}$$

To obtain (4.44) we will pass to the limit in the analogous relations satisfied at level k, namely

$$\begin{cases} \iint_Q \overline{\zeta}_{\tau_k}{:}\eta \,\mathrm{d}x\,\mathrm{d}t \le \int_0^T \mathcal{R}(\underline{\vartheta}_{\tau_k}(t),\eta(t))\,\mathrm{d}t & \text{for all } \eta \in L^2(Q;\mathbb{M}^{d\times d}_{\mathrm{D}}), \\ \iint_Q \overline{\zeta}_{\tau_k}{:}\dot{p}_{\tau_k}\,\mathrm{d}x\,\mathrm{d}t \ge \int_0^T \mathcal{R}(\underline{\vartheta}_{\tau_k}(t),\dot{p}_{\tau_k}(t))\,\mathrm{d}t. \end{cases} \tag{4.45}$$

With this aim, we use conditions (2.10) on the dissipation metric R. In order to pass to the limit in the first of (4.45) for a fixed $\eta \in L^2(Q;\mathbb{M}^{d\times d}_{\mathrm{D}})$, we use convergence (4.38v) for $(\overline{\zeta}_{\tau_k})_k$, and the fact that

$$\lim_{k\to\infty} \iint_Q \mathrm{R}(\underline{\vartheta}_{\tau_k},\eta)\,\mathrm{d}x\,\mathrm{d}t = \iint_Q \mathrm{R}(\vartheta,\eta)\,\mathrm{d}x\,\mathrm{d}t.$$

The latter limit passage follows from convergence (4.38s) for $\underline{\vartheta}_{\tau_k}$ which, combined with the continuity property (2.10b), gives that $\mathrm{R}(\underline{\vartheta}_{\tau_k},\eta) \to \mathrm{R}(\vartheta,\eta)$ almost everywhere in Q. Then we use the dominated convergence theorem, taking into account that for every $k\in\mathbb{N}$ we have $\mathrm{R}(\underline{\vartheta}_{\tau_k},\eta) \le C_R|\eta|$ a.e. in Q thanks to (2.11a).

As for the second inequality in (4.45), we use (2.10a) and the convexity of the map $\dot{p} \mapsto \mathrm{R}(\vartheta,\dot{p})$, combined with convergences (4.38k) and (4.38s), to conclude via the Ioffe theorem [Iof77] that

$$\liminf_{k\to\infty} \int_0^T \mathcal{R}(\underline{\vartheta}_{\tau_k}(t),\dot{p}_{\tau_k}(t))\,\mathrm{d}t \ge \int_0^T \mathcal{R}(\vartheta(t),\dot{p}(t))\,\mathrm{d}t\,. \tag{4.46}$$

Secondly, we show that

$$\limsup_{k\to\infty} \iint_Q \overline{\zeta}_{\tau_k}{:}\dot{p}_{\tau_k}\,\mathrm{d}x\,\mathrm{d}t \le \iint_Q \zeta{:}\dot{p}\,\mathrm{d}x\,\mathrm{d}t. \tag{4.47}$$

For (4.47) we repeat the same argument developed to obtain (3.26) in the proof of Lemma 3.6, and observe that

$$\begin{aligned}
&\limsup_{k\to\infty}\left(\iint_Q \overline{\zeta}_{\tau_k}{:}\dot{p}_{\tau_k}\,\mathrm{d}x\,\mathrm{d}t + \iint_Q |\dot{p}_{\tau_k}|^2\,\mathrm{d}x\,\mathrm{d}t + \iint_Q \mathbb{D}\dot{e}_{\tau_k}{:}\dot{e}_{\tau_k}\,\mathrm{d}x\,\mathrm{d}t\right) \\
&\stackrel{(1)}{=} \limsup_{k\to\infty}\left(\iint_Q \left(\mathbb{D}\dot{e}_{\tau_k} + \mathbb{C}\overline{e}_{\tau_k} + \tau|\overline{e}_{\tau_k}|^{\gamma-2}\overline{e}_{\tau_k} - \overline{\vartheta}_{\tau_k}\mathbb{B}\right){:}\dot{p}_{\tau_k}\,\mathrm{d}x\,\mathrm{d}t + \iint_Q \mathbb{D}\dot{e}_{\tau_k}{:}\dot{e}_{\tau_k}\,\mathrm{d}x\,\mathrm{d}t\right) + \underbrace{\lim_{k\to\infty}\tau_k\iint_Q |\dot{p}_{\tau_k}|^\gamma\,\mathrm{d}x\,\mathrm{d}t}_{=0} \\
&\stackrel{(2)}{\le} \limsup_{k\to\infty} \underbrace{\iint_Q \left(\mathbb{D}\dot{e}_{\tau_k} + \mathbb{C}\overline{e}_{\tau_k} + \tau|\overline{e}_{\tau_k}|^{\gamma-2}\overline{e}_{\tau_k} - \overline{\vartheta}_{\tau_k}\mathbb{B}\right){:}E(\dot{u}_{\tau_k}-\dot{w}_{\tau_k})\,\mathrm{d}x\,\mathrm{d}t}_{=\int_0^T \langle\overline{\mathcal{L}}_{\tau_k},\dot{u}_{\tau_k}-\dot{w}_{\tau_k}\rangle_{H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^d)^*}\,\mathrm{d}t - \rho\iint_Q \partial_t\widehat{u}_{\tau_k}(\dot{u}_{\tau_k}-\dot{w}_{\tau_k}))\,\mathrm{d}x\,\mathrm{d}t} + \limsup_{k\to\infty}\iint_Q \overline{\sigma}_{\tau_k}{:}E(\dot{w}_{\tau_k})\,\mathrm{d}x\,\mathrm{d}t \\
&\quad - \liminf_{k\to\infty}\iint_Q \left(\mathbb{C}\overline{e}_{\tau_k} + \tau|\overline{e}_{\tau_k}|^{\gamma-2}\overline{e}_{\tau_k} - \overline{\vartheta}_{\tau_k}\mathbb{B}\right){:}\dot{e}_{\tau_k}\,\mathrm{d}x\,\mathrm{d}t \\
&\stackrel{(3)}{\le} \int_0^T \langle\mathcal{L},\dot{u}-\dot{w}\rangle_{H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^d)^*}\,\mathrm{d}t - \iint_Q \left(\rho\ddot{u}(\dot{u}-\dot{w}) + \sigma{:}E(\dot{w}) + \mathbb{C}e{:}\dot{e} - \vartheta\mathbb{B}{:}\dot{e}\right)\mathrm{d}x\,\mathrm{d}t \\
&\stackrel{(4)}{=} \iint_Q \zeta{:}\dot{p}\,\mathrm{d}x\,\mathrm{d}t + \iint_Q |\dot{p}|^2\,\mathrm{d}x\,\mathrm{d}t + \iint_Q \mathbb{D}\dot{e}{:}\dot{e}\,\mathrm{d}x\,\mathrm{d}t,
\end{aligned}$$

where (1) follows from testing the discrete flow rule (4.2c) by $\dot{p}_{\tau_k}$, (2) from the kinematic admissibility condition, yielding $\dot{p}_{\tau_k} = E(\dot{u}_{\tau_k}) - \dot{e}_{\tau_k} = E(\dot{u}_{\tau_k}-\dot{w}_{\tau_k}) - \dot{e}_{\tau_k} + E(\dot{u}_{\tau_k})$, which also leads to the cancellation of the term

$\iint_Q \mathbb{D}\dot{e}_{\tau_k}{:}\dot{e}_{\tau_k}$. The limit passage in (3) follows from convergence (4.1c) for $(\overline{\mathcal{L}}_{\tau_k})_k$, from (4.38b), (4.1d) for $(w_{\tau_k})_k$, and from (4.48) and (4.49) below. In fact we have

$$\begin{aligned}
&\limsup_{k\to\infty}\Big(-\iint_Q \rho\partial_t\widehat{u}_{\tau_k}(\dot{u}_{\tau_k}-\dot{w}_{\tau_k})\,\mathrm{d}x\,\mathrm{d}t\Big)\\
&\le -\liminf_{k\to\infty}\tfrac{\rho}{2}\int_\Omega|\dot{u}_{\tau_k}(T)|^2\,\mathrm{d}x+\tfrac{\rho}{2}\int_\Omega|\dot{u}_{\tau_k}(0)|^2\,\mathrm{d}x-\lim_{k\to\infty}\rho\iint_Q\partial_t\widehat{u}_{\tau_k}\dot{w}_{\tau_k}\,\mathrm{d}x\,\mathrm{d}t\\
&\overset{(A)}{\le} -\tfrac{\rho}{2}\int_\Omega|\dot{u}(T)|^2\,\mathrm{d}x+\rho\int_\Omega|\dot{u}_0|^2\,\mathrm{d}x-\rho\iint_Q\ddot{u}\dot{w}\,\mathrm{d}x\,\mathrm{d}t
\end{aligned}\tag{4.48}$$

with (A) due to (4.38e), (4.1d), and (4.41). Furthermore,

$$\begin{aligned}
-\liminf_{k\to\infty}\iint_Q\mathbb{C}\overline{e}_{\tau_k}{:}\dot{e}_{\tau_k}\,\mathrm{d}x\,\mathrm{d}t&\le-\liminf_{k\to\infty}\int_\Omega\tfrac12\mathbb{C}\overline{e}_{\tau_k}(T){:}\overline{e}_{\tau_k}(T)\,\mathrm{d}x+\int_\Omega\tfrac12\mathbb{C}e_0{:}e_0\,\mathrm{d}x\\
&\overset{(B)}{\le}-\int_\Omega\tfrac12\mathbb{C}e(T){:}e(T)\,\mathrm{d}x+\int_\Omega\tfrac12\mathbb{C}e_0{:}e_0,\\
-\liminf_{k\to\infty}\iint_Q\tau|\overline{e}_{\tau_k}|^{\gamma-2}\overline{e}_{\tau_k}{:}\dot{e}_{\tau_k}\,\mathrm{d}x\,\mathrm{d}t&\le-\liminf_{k\to\infty}\int_\Omega\tfrac{\tau}{\gamma}|\overline{e}_{\tau_k}(T)|^\gamma\,\mathrm{d}x+\int_\Omega\tfrac{\tau}{\gamma}|e_0|^\gamma\,\mathrm{d}x\overset{(C)}{\le}0,\\
\lim_{k\to\infty}\iint_Q\overline{\vartheta}_{\tau_k}\mathbb{B}{:}\dot{e}_{\tau_k}\,\mathrm{d}x\,\mathrm{d}t&\overset{(D)}{=}\iint_Q\vartheta\mathbb{B}{:}\dot{e}\,\mathrm{d}x\,\mathrm{d}t,
\end{aligned}\tag{4.49}$$

with (B) due to (4.38h), (C) due to (4.38i) and (3.4), and (D) due to (4.38g) and (4.38r). Finally, (4) follows from testing (2.6) by $\dot{u}-\dot{w}$, and (4.43) by $\dot{p}$. From the thus obtained lim sup-inequality, arguing in the very same way as in the proof of Lemma 3.6, we conclude that

$$\begin{aligned}
&\lim_{k\to\infty}\iint_Q\overline{\zeta}_{\tau_k}{:}\dot{p}_{\tau_k}\,\mathrm{d}x\,\mathrm{d}t=\iint_Q\zeta{:}\dot{p}\,\mathrm{d}x\,\mathrm{d}t,\\
&\dot{p}_{\tau_k}\to\dot{p}\qquad\text{in }L^2(Q;\mathbb{M}^{d\times d}_{\mathrm{D}}),\\
&\dot{e}_{\tau_k}\to\dot{e}\qquad\text{in }L^2(Q;\mathbb{M}^{d\times d}_{\mathrm{sym}}).
\end{aligned}\tag{4.50}$$

Hence, combining the first of (4.50) with (4.46), we take the limit in the second inequality in (4.45). All in all, we deduce (4.44). Hence, the functions (ϑ,e,p,ζ) fulfill the plastic flow rule (2.14).

Step 3: enhanced convergences. For later use, observe that (4.50) give

$$e_{\tau_k}\to e\quad\text{in }H^1(0,T;L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})),\qquad p_{\tau_k}\to p\quad\text{in }H^1(0,T;L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{D}})).\tag{4.51}$$

Moreover, by the kinematic admissibility condition we deduce the strong convergence of $E(\dot{u}_{\tau_k})$ in $L^2(Q;\mathbb{M}^{d\times d}_{\mathrm{sym}})$, hence, by Korn's inequality,

$$u_{\tau_k}\to u\quad\text{in }H^1(0,T;H^1(\Omega;\mathbb{R}^d)).\tag{4.52}$$

Finally, repeating the lim sup argument leading to (4.50) on a generic interval $[0,t]$, we find that

$$\dot{u}_{\tau_k}(t)\to\dot{u}(t)\quad\text{in }L^2(\Omega;\mathbb{R}^d)\quad\text{for every }t\in[0,T].\tag{4.53}$$

All in all, also on account of (4.38i) and (4.38m), we have that convergence (4.42) improves to a strong one. Therefore, from (4.2c) we deduce that

$$\overline{\zeta}_{\tau_k}=(\overline{\sigma}_{\tau_k})_{\mathrm{D}}-\dot{p}_{\tau_k}-\tau_k|\overline{p}_{\tau_k}|^{\gamma-2}\overline{p}_{\tau_k}\to\sigma_{\mathrm{D}}-\dot{p}=\zeta\qquad\text{a.e. in }Q.$$

We will use this to pass to the limit in the pointwise inequality

$$\overline{\zeta}_{\tau_k}(t,x){:}\,(\dot{p}(t,x)-\dot{p}_{\tau_k}(t,x))+\mathrm{R}(\underline{\vartheta}_{\tau_k}(t,x),\dot{p}_{\tau_k}(t,x))\le\mathrm{R}(\underline{\vartheta}_{\tau_k}(t,x),\dot{p}(t,x))\qquad\text{for a.a. }(t,x)\in Q.$$

Indeed, in view of (4.50), which gives $\lim_{k\to\infty}\overline{\zeta}_{\tau_k}{:}(\dot{p}-\dot{p}_{\tau_k})=0$ a.e. in Q, of convergence (4.38s) for $\underline{\vartheta}_{\tau_k}$, and of the continuity property (2.10b), from the above inequality we conclude that

$$\limsup_{k\to\infty}\mathrm{R}(\underline{\vartheta}_{\tau_k}(x,t),\dot{p}_{\tau_k}(x,t))\le\mathrm{R}(\vartheta(x,t),\dot{p}(x,t))\qquad\text{for a.a. }(x,t)\in Q.$$

Combining this with the lower semicontinuity inequality which derives from (2.10a), we ultimately have that $\mathrm{R}(\underline{\vartheta}_{\tau_k},\dot{p}_{\tau_k})\to\mathrm{R}(\vartheta,\dot{p})$ a.e. in Q, hence

$$\mathrm{R}(\underline{\vartheta}_{\tau_k},\dot{p}_{\tau_k})\to\mathrm{R}(\vartheta,\dot{p})\quad\text{in }L^2(Q)\tag{4.54}$$

by the dominated convergence theorem.

Step 4: ad the entropy inequality (2.25). Let us fix a positive test function $\varphi \in \mathrm{C}^0([0,T];W^{1,\infty}(\Omega)) \cap H^1(0,T;L^{6/5}(\Omega))$ for (2.25), and approximate it with the discrete test functions from (4.3): their interpolants $\overline{\varphi}_\tau$, φ_τ comply with convergences (4.4) and with the discrete entropy inequality (4.5), where we pass to the limit. We take the limit of the first integral term on the left-hand side of (4.5) based on convergence (4.38o) for $\log(\underline{\vartheta}_{\tau_k})$.

For the second integral term, we will prove that

$$\kappa(\overline{\vartheta}_{\tau_k})\nabla\log(\overline{\vartheta}_{\tau_k}) \rightharpoonup \kappa(\vartheta)\nabla\log(\vartheta) \qquad \text{in } L^{1+\bar\delta}(Q;\mathbb{R}^d) \text{ with } \bar\delta = \frac{\alpha}{\mu} \text{ and } \alpha \in [2-\mu,1), \tag{4.55}$$

cf. (4.24). First of all, let us prove that $(\kappa(\overline{\vartheta}_{\tau_k})\nabla\log(\overline{\vartheta}_{\tau_k}))_k$ is bounded in $L^{1+\bar\delta}(Q;\mathbb{R}^d)$. To this aim, we argue as in the proof of the *Fifth a priori estimate* from Prop. 4.3 and observe that

$$\left|\kappa(\overline{\vartheta}_{\tau_k})\nabla\log(\overline{\vartheta}_{\tau_k})\right| \leq C\left(|\overline{\vartheta}_{\tau_k}|^{\mu-1} + \frac{1}{\bar\vartheta}\right)|\nabla\overline{\vartheta}_{\tau_k}| \qquad \text{a.e. in } Q,$$

by the growth condition (2.κ_1) and the positivity (2.33). Let us now focus on the first term on the r.h.s.: with Hölder's inequality we have that, for a positive exponent r,

$$\begin{aligned}\iint_Q \left(|\overline{\vartheta}_{\tau_k}|^{\mu-1}|\nabla\overline{\vartheta}_{\tau_k}|\right)^r \,\mathrm{d}x\,\mathrm{d}t &\leq \|(|\overline{\vartheta}_{\tau_k}|^{(\mu-\alpha)/2})^r\|_{L^{2/(2-r)}(Q)}\|(|\overline{\vartheta}_{\tau_k}|^{(\mu+\alpha-2)/2}|\nabla\overline{\vartheta}_{\tau_k}|)^r\|_{L^{2/r}(Q;\mathbb{R}^d)} \\ &\leq C\|(|\overline{\vartheta}_{\tau_k}|^{(\mu-\alpha)/2})^r\|_{L^{2/(2-r)}(Q)},\end{aligned}$$

where the second inequality follows from the estimate for $|\overline{\vartheta}_{\tau_k}|^{(\mu+\alpha-2)/2}\nabla\overline{\vartheta}_{\tau_k}$ in $L^2(Q;\mathbb{R}^d)$ thanks to (4.10l). The latter also yields a bound for $(\overline{\vartheta}_{\tau_k})^{(\mu+\alpha)/2}$ in $L^2(Q)$, hence an estimate for $(\overline{\vartheta}_{\tau_k})^{(\mu-\alpha)/2}$ in $L^{2(\mu+\alpha)/(\mu-\alpha)}(Q)$. Therefore, for $r = (\mu+\alpha)/\mu = 1+\alpha/\mu$ we obtain that $\|(|\overline{\vartheta}_{\tau_k}|^{(\mu-\alpha)/2})^r\|_{L^{2/(2-r)}(Q)} \leq C$, and the estimate for $\kappa(\overline{\vartheta}_{\tau_k})\nabla\log(\overline{\vartheta}_{\tau_k})$ follows. For the proof of convergence (4.55), relying on convergences (4.38n)–(4.38r), we refer to [RR15, Thm. 1]. Therefore we conclude the first of (2.34).

To take the limit in the right-hand side terms in the entropy inequality (4.5), for the first two integrals we use convergence (4.38p) combined with (4.4). A lower semicontinuity argument also based on the Ioffe theorem [Iof77] and on convergences (4.4), (4.38o), and (4.38r) gives that

$$\begin{aligned}\limsup_{k\to\infty}\left(-\int_{\bar{\mathrm{t}}_{\tau_k}(s)}^{\bar{\mathrm{t}}_{\tau_k}(t)}\int_\Omega \kappa(\overline{\vartheta}_{\tau_k})\frac{\overline{\varphi}_{\tau_k}}{\overline{\vartheta}_{\tau_k}}\nabla\log(\overline{\vartheta}_{\tau_k})\nabla\overline{\vartheta}_{\tau_k}\,\mathrm{d}x\,\mathrm{d}r\right) &= -\liminf_{k\to\infty}\int_{\bar{\mathrm{t}}_{\tau_k}(s)}^{\bar{\mathrm{t}}_{\tau_k}(t)}\int_\Omega \kappa(\overline{\vartheta}_{\tau_k})\overline{\varphi}_{\tau_k}|\nabla\log(\overline{\vartheta}_{\tau_k})|^2\,\mathrm{d}x\,\mathrm{d}r \\ &\leq -\int_s^t\int_\Omega \kappa(\vartheta)\varphi|\nabla\log(\vartheta)|^2\,\mathrm{d}x\,\mathrm{d}r,\end{aligned}$$

which allows us to deal with the third integral term on the r.h.s. of (4.5). Finally, we take the $\limsup_{k\to\infty}$ of the last two integral terms taking into account convergences (4.1a), (4.1b), (4.38r), which yields

$$\frac{1}{\overline{\vartheta}_{\tau_k}} \to \frac{1}{\vartheta} \qquad \text{in } L^p(Q) \quad \text{for all } 1 \leq p < \infty,$$

since $\left|\frac{1}{\overline{\vartheta}_{\tau_k}}\right| \leq \frac{1}{\bar\vartheta}$ a.e. in Q, as well as the previously established strong convergences (4.50) and (4.54).

Finally, we establish the summability property $\kappa(\vartheta)\nabla\log(\vartheta)$ in $L^1(0,T;X)$, with X from (2.34), by combining the facts that $\vartheta^{(\mu+\alpha-2)/2}\nabla\vartheta \in L^2(Q;\mathbb{R}^d)$ thanks to convergence (4.38t), with the information that $\vartheta^{(\mu-\alpha)/2} \in L^2(0,T;H^1(\Omega))$ by (4.38u), and by arguing in the very same way as in the proof of the *Fifth a priori estimate* from Prop. 4.3. In view of (2.34), the entropy inequality (2.25) in fact makes sense for all positive test functions φ in $H^1(0,T;L^{6/5}(\Omega)) \cup L^\infty(0,T;W^{1,d+\epsilon}(\Omega))$ with $\epsilon > 0$. Therefore, with a density argument we conclude it for this larger test space.

Step 5: ad the total energy inequality (2.26). It is deduced by passing to the limit in the discrete total energy inequality (4.6). For the first integral term on the left-hand side, we use that $\dot u_{\tau_k}(t) \rightharpoonup \dot u(t)$ in $L^2(\Omega;\mathbb{R}^d)$ for all $t \in [0,T]$, cf. (4.41). For the second term we observe that $\liminf_{k\to\infty}\mathcal{E}_{\tau_k}(\overline{\vartheta}_{\tau_k}(t),\overline{e}_{\tau_k}(t)) \geq \mathcal{E}(\vartheta(t),e(t))$ for *almost all* $t \in (0,T)$ by convergence (4.38r) for $\overline{\vartheta}_{\tau_k}$ and by (4.51), combined with (4.39). The limit passage on the right-hand side, for almost all $s \in (0,t)$, follows from (4.52), again (4.38r) and (4.51), from convergence (4.42) for $\overline{\sigma}_{\tau_k}$, which turns out to hold strongly in view of (4.51), and from convergences (4.1) for the interpolants $(\overline{H}_{\tau_k})_k$, $(\overline{h}_{\tau_k})_k$, $(\overline{\mathcal{L}}_{\tau_k})_k$, $(\overline{w}_{\tau_k})_k$.

This concludes the proof of Theorem 1. ∎

We now briefly sketch the **proof of Theorem 2**. The limit passage in the discrete momentum balance and in the plastic flow rule, cf. (4.2b) and (4.2c), follows from the arguments in the proof of Thm. 1.

As for the heat equation, we shall as a first step prove that the limit quadruple (ϑ, u, e, p) complies with

$$\begin{aligned} &\langle \vartheta(t), \varphi(t) \rangle_{W^{1,\infty}(\Omega)} - \int_0^t \int_\Omega \vartheta \varphi_t \,\mathrm{d}x\,\mathrm{d}s + \int_0^t \int_\Omega \kappa(\vartheta) \nabla \vartheta \nabla \varphi \,\mathrm{d}x\,\mathrm{d}s \\ &= \int_\Omega \vartheta_0 \varphi(0)\,\mathrm{d}x + \int_0^t \int_\Omega \big(H + \mathrm{R}(\vartheta, \dot p) + |\dot p|^2 + \mathbb{D}\dot e{:}\dot e - \vartheta \mathbb{B}\dot e\big)\,\varphi \,\mathrm{d}x\,\mathrm{d}s + \int_0^t \int_\Omega \int_{\partial\Omega} h\varphi \,\mathrm{d}S\,\mathrm{d}s\,. \end{aligned} \tag{4.56}$$

for all test functions $\varphi \in \mathrm{C}^0([0,T]; W^{1,\infty}(\Omega)) \cap H^1(0,T; L^{6/5}(\Omega))$ and for all $t \in (0,T]$. With this aim, we pass to the limit in the approximate temperature equation (4.2a), tested by the approximate test functions from (4.3), where we integrate by parts in time the term $\iint_Q \dot\vartheta_{\tau_k} \overline{\varphi}_{\tau_k}\,\mathrm{d}x\,\mathrm{d}t$. For this limit passage, we exploit convergences (4.4) as well as (4.38x) and (4.38y).

For the limit passage in the term $\iint_Q \kappa(\overline{\vartheta}_{\tau_k}) \nabla\overline{\vartheta}_{\tau_k} \nabla \overline{\varphi}_{\tau_k}\,\mathrm{d}x\,\mathrm{d}t$ we prove that $\kappa(\overline{\vartheta}_{\tau_k})\nabla\overline{\vartheta}_{\tau_k} \rightharpoonup \kappa(\vartheta)\nabla\vartheta$ in $L^{1+\tilde\delta}(Q;\mathbb{R}^d)$, with $\tilde\delta > 0$ given by (2.36). Let us check the bound

$$\|\kappa(\overline{\vartheta}_{\tau_k})\nabla\overline{\vartheta}_{\tau_k}\|_{L^{1+\tilde\delta}(Q;\mathbb{R}^d)} \le C, \tag{4.57}$$

by again resorting to estimates (4.10k) and (4.10l). Indeed, by (2.κ_1) we have that

$$|\kappa(\overline{\vartheta}_{\tau_k})\nabla\overline{\vartheta}_{\tau_k}| \le C|\overline{\vartheta}_{\tau_k}|^{(\mu-\alpha+2)/2} |\overline{\vartheta}_{\tau_k}|^{(\mu+\alpha-2)/2} |\nabla\overline{\vartheta}_{\tau_k}| + C|\nabla\overline{\vartheta}_{\tau_k}| \qquad \text{a.e. in } Q, \tag{4.58}$$

and we estimate the first term on the r.h.s. by observing that $|\overline{\vartheta}_{\tau_k}|^{(\mu+\alpha-2)/2}|\nabla\overline{\vartheta}_{\tau_k}|$ is bounded in $L^2(Q)$ thanks to (4.10l). On the other hand, in the case $d=3$, to which we confine the discussion, by interpolation arguments $\overline{\vartheta}_{\tau_k}$ is bounded in $L^h(Q)$ for every $1 \le h < \frac{8}{3}$. Therefore, for $\alpha > \mu - \frac{2}{3}$ (so that $\mu - \alpha + 2 < \frac{8}{3}$), the functions $(|\overline{\vartheta}_{\tau_k}|^{(\mu-\alpha+2)/2})_k$ are bounded in $L^r(Q)$ with $1 \le r < \frac{16}{3(\mu-\alpha+2)}$. Then, (4.57) follows from (4.58) via the Hölder inequality. The corresponding weak convergence can be proved arguing in the very same way as in the proof of [RR15, Thm. 2], to which we refer the reader. Therefore we conclude that $\kappa(\vartheta)\nabla\vartheta \in L^{1+\tilde\delta}(Q;\mathbb{R}^d)$. Observe that $\kappa(\vartheta)\nabla\vartheta = \nabla(\hat\kappa(\vartheta))$ thanks to [MM79]. Since $\hat\kappa(\vartheta)$ itself is a function in $L^{1+\tilde\delta}(Q)$ (for $d=3$, this follows from the fact that $\hat\kappa(\vartheta) \sim \vartheta^{\mu+1} \in L^{h/(\mu+1)}(Q)$ for every $1 \le h < \frac{8}{3}$), we conclude (2.36).

The limit passage on the r.h.s. of the discrete heat equation (4.2a) results from (4.1a), from (4.38x), the strong convergences (4.50), and (4.54).

All in all, we obtain (4.56), whence for every $\varphi \in W^{1,\infty}(\Omega)$ and every $0 \le s \le t \le T$

$$\begin{aligned} &\langle \vartheta(t) - \vartheta(s), \varphi \rangle_{W^{1,\infty}(\Omega)} \\ &= -\int_s^t \int_\Omega \kappa(\vartheta)\nabla\vartheta\nabla\varphi \,\mathrm{d}x\,\mathrm{d}r + \int_s^t \int_\Omega \big(H + \mathrm{R}(\vartheta,\dot p) + |\dot p|^2 + \mathbb{D}\dot e{:}\dot e - \vartheta\mathbb{B}\dot e\big)\,\varphi\,\mathrm{d}x\,\mathrm{d}r + \int_s^t \int_\Omega \int_{\partial\Omega} h\varphi \,\mathrm{d}S\,\mathrm{d}r\,. \end{aligned}$$

From this we easily conclude the enhanced regularity (2.29). Thanks to [DMDM06, Thm. 7.1], the absolutely continuous function $\vartheta : [0,T] \to W^{1,\infty}(\Omega)^*$ admits at almost all $t \in (0,T)$ the derivative $\dot\vartheta(t)$, which turns out to be the limit as $h \to 0$ of the incremental quotients $\frac{\vartheta(t+h)-\vartheta(t)}{h}$, w.r.t. the weak*-topology of $W^{1,\infty}(\Omega)^*$. Therefore, the enhanced weak formulation of the heat equation (2.31) follows.

Recall (cf. the comments following the statement of Thm. 2) that $\tilde\delta$ is small enough as to ensure that $W^{1,1+1/\tilde\delta}(\Omega) \subset L^\infty(\Omega)$. Therefore, the terms on the r.h.s. of (2.31) can be multiplied by test functions $\varphi \in W^{1,1+1/\tilde\delta}(\Omega)$. Thanks to (2.36), also the second term on the l.h.s. of (2.31) admits such test functions. Therefore by comparison we conclude that $\dot\vartheta \in L^1(0,T; W^{1,1+1/\tilde\delta}(\Omega)^*)$ and, with a density argument, extend (2.31) to this (slightly) larger space of test functions. This finishes the proof of Theorem 2. ■

Remark 4.7 (Energy convergences for the approximate solutions)**.** As a by-product of the proofs of Theorems 1 and 2, we improve convergences (4.38) of the approximate solutions to an entropic/weak energy solution of the thermoviscoplastic system. More specifically, it follows from (4.38r) and (4.50)–(4.53) that we have the convergence of the kinetic energies

$$\frac{\varrho}{2}\int_\Omega |\dot u_{\tau_k}(t)|^2\,\mathrm{d}x \to \frac{\varrho}{2}\int_\Omega |\dot u(t)|^2\,\mathrm{d}x \qquad \text{for all } t \in [0,T],$$

of dissipated energies

$$\int_0^T \int_\Omega \mathbb{D}\dot e_{\tau_k}{:}\dot e_{\tau_k}\,\mathrm{d}x\,\mathrm{d}t \to \int_0^T \int_\Omega \mathbb{D}\dot e{:}\dot e\,\mathrm{d}x\,\mathrm{d}t, \qquad \int_0^T \mathcal{R}(\underline{\vartheta}_{\tau_k}, \dot p_{\tau_k})\,\mathrm{d}t \to \int_0^T \mathcal{R}(\vartheta,\dot p)\,\mathrm{d}t,$$

and of the thermal and mechanical energies

$$\mathcal{F}(\overline{\vartheta}_{\tau_k}(t)) \to \mathcal{F}(\vartheta(t)) \quad \text{for a.a. } t \in (0,T), \qquad \mathcal{Q}(\overline{e}_{\tau_k}) \to \mathcal{Q}(e) \qquad \text{uniformly in } [0,T].$$

5. Setup for the perfectly plastic system

As already mentioned in the Introduction, in the vanishing-viscosity limit of the thermoviscoplastic system we will obtain the *(global) energetic formulation* for the perfectly plastic system, coupled with the stationary limit of the heat equation. Prior to performing this asymptotic analysis, in this section we gain further insight into the concept of energetic solution for perfect plasticity.

For the energetic formulation to be fully meaningful, in Sec. 5.1 we need to strengthen the assumptions, previously given in Section 2.1, on the reference configuration Ω, on the elasticity tensor $\mathbb{C}$, and on the elastic domain $x \in \Omega \rightrightarrows K(x) \subset \mathbb{M}^{d\times d}_{\mathrm{D}}$ (indeed, we will drop the dependence of K on the (spatially and temporally) nonsmooth variable ϑ). Instead, we will weaken the regularity requirements on the Dirichlet loading w.

Preliminarily, let us recall some basic facts about the space of functions with bounded deformation in Ω.

The space $\mathrm{BD}(\Omega;\mathbb{R}^d)$. It is defined by

$$\mathrm{BD}(\Omega;\mathbb{R}^d) := \{u \in L^1(\Omega;\mathbb{R}^d) \,:\, E(u) \in \mathrm{M}(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})\}, \tag{5.1}$$

with $\mathrm{M}(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})$ the space of Radon measures on Ω with values in $\mathbb{M}^{d\times d}_{\mathrm{sym}}$, with norm $\|\lambda\|_{\mathrm{M}(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})} := |\lambda|(\Omega)$ and $|\lambda|$ the variation of the measure. Recall that, by the Riesz representation theorem, $\mathrm{M}(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})$ can be identified with the dual of the space $\mathrm{C}_0(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})$ of the continuous functions $\varphi : \Omega \to \mathbb{M}^{d\times d}_{\mathrm{sym}}$ such that the sets $\{|\varphi| \geq c\}$ are compact for every $c > 0$. The space $\mathrm{BD}(\Omega;\mathbb{R}^d)$ is endowed with the graph norm

$$\|u\|_{\mathrm{BD}(\Omega;\mathbb{R}^d)} := \|u\|_{L^1(\Omega;\mathbb{R}^d)} + \|E(u)\|_{\mathrm{M}(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})},$$

which makes it a Banach space. It turns out that $\mathrm{BD}(\Omega;\mathbb{R}^d)$ is the dual of a normed space, cf. [TS80].

In addition to the strong convergence induced by $\|\cdot\|_{\mathrm{BD}(\Omega;\mathbb{R}^d)}$, this duality defines a notion of weak* convergence on $\mathrm{BD}(\Omega;\mathbb{R}^d)$: a sequence (u_k) converges weakly* to u in $\mathrm{BD}(\Omega;\mathbb{R}^d)$ if $u_k \rightharpoonup u$ in $L^1(\Omega;\mathbb{R}^d)$ and $E(u_k) \overset{*}{\rightharpoonup} E(u)$ in $\mathrm{M}(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})$. Every bounded sequence in $\mathrm{BD}(\Omega;\mathbb{R}^d)$ has a weakly* converging subsequence and, furthermore, a subsequence converging weakly in $L^{d/(d-1)}(\Omega;\mathbb{R}^d)$ and strongly in $L^p(\Omega;\mathbb{R}^d)$ for every $1 \leq p < \frac{d}{d-1}$.

Finally, we recall that for every $u \in \mathrm{BD}(\Omega;\mathbb{R}^d)$ the trace $u|_{\partial\Omega}$ is well defined as an element in $L^1(\partial\Omega;\mathbb{R}^d)$, and that (cf. [Tem83, Prop. 2.4, Rmk. 2.5]) a Poincaré-type inequality holds:

$$\exists\, C > 0 \ \ \forall\, u \in \mathrm{BD}(\Omega;\mathbb{R}^d) \,: \quad \|u\|_{L^1(\Omega;\mathbb{R}^d)} \leq C \left(\|u\|_{L^1(\Gamma_{\mathrm{Dir}};\mathbb{R}^d)} + \|E(u)\|_{\mathrm{M}(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})} \right). \tag{5.2}$$

5.1. **Setup.** Let us now detail the basic assumptions on the data of the perfectly plastic system. We postpone to the end of Section 5.2 a series of comments on the outcome of the conditions given below, as well as on the possibility of weakening some of them.

The reference configuration. For technical reasons related to the definition of the stress-strain duality, cf. Remark 5.7 later on, in addition to conditions (2.Ω) required in Sec. 2.1, we will suppose from now on that

$$\partial\Omega \text{ and } \partial\Gamma \text{ are of class } \mathrm{C}^2. \tag{5.Ω}$$

The latter requirement means that for every $x \in \partial\Gamma$ there exists a C^2-diffeomorphism defined in an open neighborhood of x that maps $\partial\Omega$ into a $(d-1)$-dimensional plane, and $\partial\Gamma$ into a $(d-2)$-dimensional plane.

Kinematic admissibility and stress. Given a function $w \in H^1(\Omega;\mathbb{R}^d)$, we say that a triple (u,e,p) is *kinematically admissible with boundary datum w for the perfectly plastic system* (*kinematically admissible*, for short), and write $(u,e,p) \in \mathcal{A}_{\mathrm{BD}}(w)$, if

$$u \in \mathrm{BD}(\Omega;\mathbb{R}^d), \quad e \in L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}}), \quad p \in \mathrm{M}(\Omega\cup\Gamma_{\mathrm{Dir}};\mathbb{M}^{d\times d}_{\mathrm{D}}), \tag{5.3a}$$

$$E(u) = e + p, \tag{5.3b}$$

$$p = (w-u) \odot \nu \mathscr{H}^{d-1} \quad \text{on } \Gamma_{\mathrm{Dir}}. \tag{5.3c}$$

Observe that (5.3a) reflects the fact that the plastic strain is now a measure that can concentrate on Lebesgue-negligible sets. Furthermore, (5.3c) relaxes the Dirichlet condition $w = u$ on Γ_{Dir} imposed by the kinematic admissibility condition (2.3) and represents a plastic slip (mathematically described by the singular part of the measure p) occurring on Γ_{Dir}. It can be checked that $\mathcal{A}(w) \subset \mathcal{A}_{\mathrm{BD}}(w)$. In the proof of Theorem 3 we will make use of the following closedness property, proved in [DMDM06, Lemma 2.1].

Lemma 5.1. *Assume* (2.Ω) *and* (5.Ω). *Let* $(w_k) \subset H^1(\Omega;\mathbb{R}^d)$ *and* $(u_k, e_k, p_k) \in \mathcal{A}_{\mathrm{BD}}(w_k)$ *for every* $k \in \mathbb{N}$. *Assume that*

$$
\begin{aligned}
&w_k \rightharpoonup w_\infty \ \text{ in } H^1(\Omega;\mathbb{R}^d), && u_k \overset{*}{\rightharpoonup} u_\infty \ \text{ in } \mathrm{BD}(\Omega;\mathbb{R}^d),\\
&e_k \rightharpoonup e_\infty \ \text{ in } L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}}), && p_k \overset{*}{\rightharpoonup} p \ \text{ in } \mathrm{M}(\Omega\cup\Gamma_{\mathrm{Dir}};\mathbb{M}^{d\times d}_{\mathrm{D}}).
\end{aligned}
\tag{5.4}
$$

Then, $(u_\infty, e_\infty, p_\infty) \in \mathcal{A}_{\mathrm{BD}}(w_\infty)$.

In the perfectly plastic system, the stress is given by $\sigma = \mathbb{C}e$, cf. (1.13d). Following [DMDM06, Sol09, FG12, Sol14] in addition to (2.T), we suppose that for almost all $x \in \Omega$ the elastic tensor $\mathbb{C}(x) \in \mathrm{Lin}(\mathbb{M}^{d\times d}_{\mathrm{sym}})$ maps the orthogonal spaces $\mathbb{M}^{d\times d}_{\mathrm{D}}$ and $\mathbb{R}I$ into themselves. Namely, there exists functions

$$
\mathbb{C}_{\mathrm{D}} \in L^\infty(\Omega;\mathrm{Lin}(\mathbb{M}^{d\times d}_{\mathrm{sym}})) \text{ and } \eta \in L^\infty(\Omega;\mathbb{R}^+) \text{ s.t. } \forall\, A \in \mathbb{M}^{d\times d}_{\mathrm{sym}} \ \ \mathbb{C}(x)A = \mathbb{C}_{\mathrm{D}}(x)A_{\mathrm{D}} + \eta(x)\mathrm{tr}(A)I, \tag{5.T}
$$

with I the identity matrix.

Body force, traction, and Dirichlet loading. Along the footsteps of [DMDM06], we enhance our conditions on F and g (cf. (2.L$_1$)), by requiring that

$$
F \in \mathrm{AC}([0,T];L^d(\Omega;\mathbb{R}^d)), \qquad g \in \mathrm{AC}([0,T];L^\infty(\Gamma_{\mathrm{Neu}};\mathbb{R}^d)). \tag{5.L$_1$}
$$

Therefore, for every $t \in [0,T]$ the element $\mathcal{L}(t)$ defined by (2.5) belongs to $\mathrm{BD}(\Omega;\mathbb{R}^d)^*$, and moreover (see [DMDM06, Rmk. 4.1]) $\dot{\mathcal{L}}(t)$ exists in $\mathrm{BD}(\Omega;\mathbb{R}^d)^*$ for almost all $t \in (0,T)$. Moreover, we shall strengthen the safe load condition from (2.L$_2$) to

$$
\varrho \in W^{1,1}(0,T;L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})) \quad \text{ and } \varrho_{\mathrm{D}} \equiv 0, \tag{5.5}
$$

cf. Remark 5.2 below.

Remark 5.2. The second of (5.5) is required only for consistency with the upcoming conditions (6.3b) on the stresses ϱ_ε associated via the safe load condition with the forces $(F_\varepsilon)_\varepsilon$ and $(g_\varepsilon)_\varepsilon$ for the thermoviscoplastic systems approximating the perfectly plastic one. The feasibility of (5.5) is completely open, though. Hence, we might as well confine the discussion to the case the body force F and the assigned traction g are null. We have chosen not to do so because (5.5) is the natural counterpart to (6.3b).

Further, we consider the body to be solicited by a hard device w on the Dirichlet boundary Γ_{Dir}, for which we suppose

$$
w \in \mathrm{AC}([0,T;H^1(\Omega;\mathbb{R}^d)), \tag{5.W}
$$

which is a weaker requirement than (2.W).

The plastic dissipation. Since the plastic strain (and, accordingly, the plastic strain rate) is a measure on $\Omega\cup\Gamma_{\mathrm{Dir}}$, from now on we will suppose that the multifunction K is defined on $\Omega\cup\Gamma_{\mathrm{Dir}}$. Furthermore, following [Sol09], we will require that

$$
K : \Omega\cup\Gamma_{\mathrm{Dir}} \rightrightarrows \mathbb{M}^{d\times d}_{\mathrm{D}} \text{ is continuous} \tag{5.K$_1$}
$$

in the sense specified by (2.8) and that

$$
\begin{aligned}
&K(x) \ \text{ is a convex and compact subset of } \mathbb{M}^{d\times d}_{\mathrm{D}} \text{ for all } x \in \Omega\cup\Gamma_{\mathrm{Dir}} \text{ and}\\
&\exists\, 0 < c_r < C_R \quad \forall\, x \in \Omega\cup\Gamma_{\mathrm{Dir}} : \quad B_{c_r}(0) \subset K(x) \subset B_{C_R}(0).
\end{aligned}
\tag{5.K$_2$}
$$

In order to state the stress constraint $\sigma_{\mathrm{D}} \in K$ a.e. in Ω in a more compact form, we also introduce the set

$$
\mathcal{K}(\Omega) := \{\zeta \in L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{D}}) : \zeta(x) \in K(x) \text{ for a.a. } x \in \Omega\}. \tag{5.6}
$$

We will denote by $\mathrm{P}_{\mathcal{K}(\Omega)} : L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{D}}) \to L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{D}})$ the projection operator onto the closed convex set $\mathcal{K}(\Omega)$, induced by the projection operators onto the sets $K(x)$, namely, for a given $\sigma \in L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{D}})$,

$$
\xi = \mathrm{P}_{\mathcal{K}(\Omega)}(\sigma) \qquad \text{if and only if} \qquad \xi(x) = \mathrm{P}_{K(x)}(\sigma(x)) \text{ for a.a. } x \in \Omega. \tag{5.7}
$$

We introduce the support function $\mathrm{R} : \Omega \times \mathbb{M}^{d\times d}_{\mathrm{D}} \to [0,+\infty)$ associated with the multifunction K. In order to define the related dissipation functional, we have to resort to the theory of convex function of measures [GS64, Res68], since the tensor p and its rate $\dot{p}$ are now Radon measures on $\Omega\cup\Gamma_{\mathrm{Dir}}$. Therefore, with every

$\dot{p} \in \mathrm{M}(\Omega\cup\Gamma_{\mathrm{Dir}};\mathbb{M}^{d\times d}_{\mathrm{sym}})$ (for convenience, we will keep to the notation $\dot{p}$ for the independent variable in the plastic dissipation potential) we associate the nonnegative Radon measure $\overline{\mathrm{R}}(\dot{p})$ defined by

$$\overline{\mathrm{R}}(\dot{p})(B) := \int_B \mathrm{R}\left(x, \frac{\dot{p}}{|\dot{p}|}(x)\right) \mathrm{d}|\dot{p}|(x) \qquad \text{for every Borel set } B \subset \Omega\cup\Gamma_{\mathrm{Dir}}, \tag{5.8}$$

with $\dot{p}/|\dot{p}|$ the Radon-Nykodým derivative of the measure $\dot{p}$ w.r.t. its variation $|\dot{p}|$. We then consider the *plastic dissipation potential*

$$\mathcal{R} : \mathrm{M}(\Omega\cup\Gamma_{\mathrm{Dir}};\mathbb{M}^{d\times d}_{\mathrm{sym}}) \to [0,+\infty) \text{ defined by } \mathcal{R}(\dot{p}) := \overline{\mathrm{R}}(\dot{p})(\Omega\cup\Gamma_{\mathrm{Dir}}). \tag{5.9}$$

Observe that the definition of the functional $\mathcal{R}$ on $\mathrm{M}(\Omega\cup\Gamma_{\mathrm{Dir}};\mathbb{M}^{d\times d}_{\mathrm{sym}})$ is consistent with that given on $L^1(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})$ in (2.12) (in the case the yield surface K does not depend on ϑ), namely

$$\mathcal{R}(\dot{p}) = \int_\Omega \mathrm{R}(x,\dot{p}(x))\,\mathrm{d}x \qquad \text{if } \dot{p} \in L^1(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}}). \tag{5.10}$$

This justifies the abuse in the notation for $\mathcal{R}$.

It follows from the lower semicontinuity of $x \rightrightarrows K(x)$ that its support function R is lower semicontinuous on $\Omega \times \mathbb{M}^{d\times d}_{\mathrm{D}}$. Since $\mathrm{R}(x,\cdot)$ is also convex and 1-homogeneous, Reshetnyak's Theorem (cf., e.g., [AFP05, Thm. 2.38]) applies to ensure that the functional $\mathcal{R}$ from (5.9) is lower semicontinuous w.r.t. the weak*-topology on $\mathrm{M}(\Omega\cup\Gamma_{\mathrm{Dir}};\mathbb{M}^{d\times d}_{\mathrm{sym}})$. Accordingly, the induced total variation functional, defined for every function $p : [0,T] \to \mathrm{M}(\Omega\cup\Gamma_{\mathrm{Dir}};\mathbb{M}^{d\times d}_{\mathrm{sym}})$ by

$$\mathrm{Var}_{\mathcal{R}}(p;[a,b]) := \sup\left\{ \sum_{i=1}^N \mathcal{R}(p(t_i)-p(t_{i-1})) \ : \ a = t_0 < t_1 < \ldots < t_{N-1} = t_N = b \right\} \tag{5.11}$$

for $[a,b] \subset [0,T]$, is lower semicontiuous w.r.t. the pointwise (in time) convergence of p in the weak* topology of $\mathrm{M}(\Omega\cup\Gamma_{\mathrm{Dir}};\mathbb{M}^{d\times d}_{\mathrm{sym}})$.

Finally, we recall [DMDM06, Thm. 7.1], stating that for every $p \in \mathrm{AC}([0,T];\mathrm{M}(\Omega\cup\Gamma_{\mathrm{Dir}};\mathbb{M}^{d\times d}_{\mathrm{sym}}))$, there exists

$$\dot{p}(t) := \text{weak}^* - \lim_{h\to 0} \frac{p(t+h)-p(t)}{h} \qquad \text{for a.a. } t \in (0,T),$$

where the limit is w.r.t. the weak*-topology of $\mathrm{M}(\Omega\cup\Gamma_{\mathrm{Dir}};\mathbb{M}^{d\times d}_{\mathrm{sym}})$. Moreover, there holds

$$\mathrm{Var}_{\mathcal{R}}(p;[a,b]) = \int_a^b \mathcal{R}(\dot{p}(t))\,\mathrm{d}t \qquad \text{for all } [a,b] \subset [0,T], \tag{5.12}$$

cf. [DMDM06, Thm. 7.1] and [Sol09, Thm. 3.6].

Cauchy data. We will supplement the perfectly plastic system with initial data

$$u_0 \in \mathrm{BD}(\Omega;\mathbb{R}^d), \tag{5.13a}$$

$$e_0 \in L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}}), \quad p_0 \in \mathrm{M}(\Omega\cup\Gamma_{\mathrm{Dir}};\mathbb{M}^{d\times d}_{\mathrm{D}}) \quad \text{such that } (u_0,e_0,p_0) \in \mathcal{A}_{\mathrm{BD}}(w(0))\,. \tag{5.13b}$$

5.2. **Energetic solutions to the perfectly plastic system.** Throughout this section we will tacitly suppose the validity of conditions (5.Ω), (5.K$_1$)–(5.K$_2$). We are now in the position to give the notion of energetic solution (or *quasistatic evolution*) for the perfectly plastic system (in the isothermal case).

Definition 5.3 (Global energetic solutions to the perfectly plastic system)**.** *Given initial data* (u_0,e_0,p_0) *fulfilling* (5.13)*, we call a triple* (u,e,p) *a* global energetic solution *to the Cauchy problem for system* (1.13)*, with boundary datum* w *on* Γ_{Dir}*, if*

$$u \in \mathrm{BV}([0,T];\mathrm{BD}(\Omega;\mathbb{R}^d)), \tag{5.14a}$$

$$e \in \mathrm{BV}([0,T];L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})), \tag{5.14b}$$

$$p \in \mathrm{BV}([0,T];\mathrm{M}(\Omega;\mathbb{M}^{d\times d}_{\mathrm{D}})), \tag{5.14c}$$

(u,e,p) *comply with the initial conditions*

$$u(0,x) = u_0(x), \qquad e(0,x) = e_0(x), \qquad p(0,x) = p_0(x) \qquad \textit{for a.a. } x \in \Omega, \tag{5.15}$$

and with the following conditions for every $t \in [0,T]$*:*

- kinematic admissibility*:* $(u(t),e(t),p(t)) \in \mathcal{A}_{\mathrm{BD}}(w(t))$*;*

- global stability*:*

$$\mathcal{Q}(e(t)) - \langle \mathcal{L}(t), u(t)\rangle_{\mathrm{BD}(\Omega;\mathbb{R}^d)} \leq \mathcal{Q}(\tilde{e}) + \mathcal{R}(\tilde{p} - p(t)) - \langle \mathcal{L}(t), \tilde{u}\rangle_{\mathrm{BD}(\Omega;\mathbb{R}^d)} \qquad \text{for all } (\tilde{u},\tilde{e},\tilde{p}) \in \mathcal{A}_{\mathrm{BD}}(w(t)) \quad \text{(S)}$$

- energy balance*:*

$$\begin{aligned}\mathcal{Q}(e(t)) + \mathrm{Var}_{\mathcal{R}}(p;[0,t]) = \mathcal{Q}(e_0) + \int_0^t \int_\Omega \sigma : E(\dot{w})\,\mathrm{d}x\,\mathrm{d}s - \int_0^t \langle \mathcal{L}, \dot{w}\rangle_{H^1(\Omega;\mathbb{R}^d)}\,\mathrm{d}s \\ + \langle \mathcal{L}(t), u(t)\rangle_{\mathrm{BD}(\Omega;\mathbb{R}^d)} - \langle \mathcal{L}(0), u_0\rangle_{\mathrm{BD}(\Omega;\mathbb{R}^d)} - \int_0^t \langle \dot{\mathcal{L}}, u\rangle_{\mathrm{BD}(\Omega;\mathbb{R}^d)}\,\mathrm{d}s,\end{aligned} \qquad \text{(E)}$$

with the stress σ given by $\sigma(t) = \mathbb{C}e(t)$ for every $t \in [0,T]$.

Remark 5.4. It follows from [DMDM06, Thm. 4.4] (cf. also [DMS14, Rmk. 5]), that (E) is equivalent to the condition that

$$\begin{aligned}&p \in \mathrm{BV}([0,T];\mathrm{M}(\Omega;\mathbb{M}^{d\times d}_{\mathrm{D}})),\\ &\mathcal{Q}(e(t)) + \mathrm{Var}_{\mathcal{R}}(p;[0,t]) - \int_\Omega \varrho(t) : (e(t) - E(w(t)))\,\mathrm{d}x \\ &= \mathcal{Q}(e_0) - \int_\Omega \varrho(0) : (e_0 - E(w(0)))\,\mathrm{d}x + \int_0^t \int_\Omega \sigma : E(\dot{w})\,\mathrm{d}x\,\mathrm{d}s - \int_0^t \int_\Omega \dot{\varrho} : (e - E(w))\,\mathrm{d}x\,\mathrm{d}s\end{aligned} \qquad (5.16)$$

for every $t \in [0,T]$.

In the above definition, for consistency with the standard concept of energetic solution, we have required only BV-time regularity for the functions (e,p) (and, accordingly, for u). On the other hand, an important feature of perfect plasticity is that, due to its *convex character*, the maps $t \mapsto u(t)$, $t \mapsto e(t)$, $t \mapsto p(t)$ are ultimately *absolutely continuous* on $[0,T]$. In fact, the following result ensures that, if a triple (u,e,p) complies with (S) and (E) *at almost all* $t \in (0,T)$, then it satisfies said conditions *for every* $t \in [0,T]$, and in addition the maps $t \mapsto u(t)$, $t \mapsto e(t)$, $t \mapsto p(t)$ are absolutely continuous. The proof of Thm. 5.5 given for [DMS14, Thm. 5] carries over to the present case of a spatially-dependent dissipation metric R.

Theorem 5.5. *Let $S \subset [0,T]$ be a set of full measure containing 0. Let $(u,e,p) : S \to \mathrm{BD}(\Omega;\mathbb{R}^d) \times L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}}) \times \mathrm{M}(\Omega;\mathbb{M}^{d\times d}_{\mathrm{D}})$ be measurable and bounded functions satisfying the Cauchy conditions* (5.15) *with a triple (u_0,e_0,p_0) as in* (5.13)*, as well as the kinematic admissibility, the global stability condition, and the energy balance* for every $t \in S$*. Suppose in addition that $p \in \mathrm{BV}([0,T];\mathrm{M}(\Omega;\mathbb{M}^{d\times d}_{\mathrm{D}}))$.*

Then, the pair (u,e) extends to an absolutely continuous function $(u,e) \in \mathrm{AC}([0,T];\mathrm{BD}(\Omega;\mathbb{R}^d)\times L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}}))$. Moreover, $p \in \mathrm{AC}([0,T];\mathrm{M}(\Omega;\mathbb{M}^{d\times d}_{\mathrm{D}}))$ and the triple (u,e,p) is a global energetic solution to the perfectly plastic system in the sense of Definition 5.3.

In the proof of the forthcoming Theorem 3, we will also make use of the following result, first proved in [DMDM06, Thm. 3.6] in the homogeneous case, and extended to a spatially-dependent yield surface in [Sol09, Thm. 3.10].

Lemma 5.6. *Let $S \subset [0,T]$ be a set of full measure containing 0. Let $(u,e,p) : S \to \mathrm{BD}(\Omega;\mathbb{R}^d)\times L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})\times \mathrm{M}(\Omega;\mathbb{M}^{d\times d}_{\mathrm{D}})$ fulfill the kinematic admissibility at $t \in S$. Then, the following conditions are equivalent at t:*

- *$(u(t),e(t),p(t))$ comply with the global stability condition* (S)*;*
- *the stress $\sigma = \mathbb{C}e$ satisfies $\sigma(t) \in \mathcal{K}(\Omega)$ and the boundary value problem*

$$\begin{cases} -\mathrm{div}_{\mathrm{Dir}}(\sigma(t)) = F(t) & \text{in } \Omega,\\ \sigma(t)\nu = g(t) & \text{on } \Gamma_{\mathrm{Neu}},\end{cases} \qquad (5.17)$$

with the operator $-\mathrm{div}_{\mathrm{Dir}}$ from (2.7)*.*

Remark 5.7. In the proof of Thm. 3, Lemma 5.6 will play a pivotal role in the argument for the global stability condition (S). In the spatially homogeneous case addressed in [DMDM06], the proof of the analogue of Lemma 5.6 relies on a careful definition of the duality between the (deviatoric part of the) stress σ, which is typically not continuous as a function of the space variable, and the strain $E(u)$, as well as the plastic strain p, which in turn are just measures. In particular, the regularity conditions (5.Ω) on $\partial\Omega$ and $\partial\Gamma$ entail the validity of a by-part integration formula (cf. [DMDM06, Prop. 2.2]), which is at the core of the proof of [DMDM06, Thm. 3.6]. Another crucial point is the validity of the inequality (between measures)

$$\overline{\mathrm{R}}(\dot{p}) \geq [\sigma_{\mathrm{D}}:\dot{p}], \tag{5.18}$$

where $\overline{\mathrm{R}}(\dot{p})$ is the Radon measure defined by (5.8), and the measure $[\sigma_{\mathrm{D}}:\dot{p}]$ (we refer to [DMDM06, Sec. 2] for its definition), 'surrogates' the duality between σ_{D} and $\dot{p}$.

In [Sol09] it was shown that, if $K : \Omega \cup \Gamma_{\mathrm{Dir}} \rightrightarrows \mathbb{M}^{d\times d}_{\mathrm{D}}$ is continuous, then (5.18) holds also in the spatially heterogeneous case and, based on that, Lemma 5.6 was derived. However, it was observed in [FG12] that the continuity of K is a quite restrictive condition. Adopting a slightly different approach to the proof of existence of 'quasistatic evolutions', the authors carried out the analysis under a much weaker, and more mechanically feasible, set of conditions on the multifunction K. Such conditions were later revisited in [Sol14] in a more abstract framework.

We believe that the asymptotic analysis developed in the upcoming Section 6 could be extended to the general setting of conditions proposed in [FG12]. We have stayed with the continuity requirement (5.K$_1$) on K, though, to keep the presentation simpler, in order to focus on the outcome of the vanishing-viscosity analysis in the case with temperature.

6. From the thermoviscoplastic to the perfectly plastic system

In this section we address the limiting behavior of *weak energy solutions* to the thermoviscoplastic system (1.3, 1.5) as the rate of the external loads F, g, w and of the heat sources H, h becomes slower and slower. Accordingly, we will rescale time by a factor $\varepsilon > 0$. Before detailing our conditions on the data F, g, w, H, and h for performing the vanishing-viscosity analysis of system (1.3, 1.5), let us specify that, already on the "viscous" level, we will confine the discussion to the case in which

the elastic domain K does not depend on ϑ but only on $x \in \Omega$, and fulfills (5.K$_1$)–(5.K$_2$).

Moreover, we will suppose that the thermal expansion tensor also depends on ε, i.e. $\mathbb{E} = \mathbb{E}_\varepsilon$, with the scaling given by (1.11). Hence, the tensors $\mathbb{B}_\varepsilon := \mathbb{C}\mathbb{E}_\varepsilon$ have the form

$$\mathbb{B}_\varepsilon = \varepsilon^\alpha \mathbb{B} \qquad \text{with } \alpha > \frac{1}{2} \text{ and } \mathbb{B} \in L^\infty(\Omega;\mathbb{R}^{d\times d}). \tag{6.1}$$

We postpone to Remark 6.3 some comments on the role of condition (6.1).

Let $(H_\varepsilon, h_\varepsilon, F_\varepsilon, g_\varepsilon, w_\varepsilon)_\varepsilon$ be a family of data for system (1.3, 1.5), and let us rescale them by the factor $\varepsilon > 0$, thus introducing

$$H^\varepsilon(t) := H_\varepsilon(\varepsilon t), \quad h^\varepsilon(t) := h_\varepsilon(\varepsilon t), \quad F^\varepsilon(t) := F_\varepsilon(\varepsilon t), \quad g^\varepsilon(t) := g_\varepsilon(\varepsilon t), \quad w^\varepsilon(t) := w_\varepsilon(\varepsilon t) \qquad t \in \left[0, \frac{T}{\varepsilon}\right].$$

Correspondingly, we denote by $(\vartheta^\varepsilon, u^\varepsilon, e^\varepsilon, p^\varepsilon)$ a weak energy solution to the thermoviscoplastic system, with the tensor $\mathbb{B}_\varepsilon$ from (6.1), starting from a family of initial data $(\vartheta^0_\varepsilon, u^0_\varepsilon, e^0_\varepsilon, p^0_\varepsilon)_\varepsilon$: under the conditions on the functions $(H_\varepsilon, h_\varepsilon, F_\varepsilon, g_\varepsilon, w_\varepsilon)_\varepsilon$ and the data $(\vartheta^0_\varepsilon, u^0_\varepsilon, e^0_\varepsilon, p^0_\varepsilon)_\varepsilon$ stated in Sec. 2.1, the existence of $(\vartheta^\varepsilon, u^\varepsilon, e^\varepsilon, p^\varepsilon)$ is ensured by Thm. 2. We further rescale the functions $(\vartheta^\varepsilon, u^\varepsilon, e^\varepsilon, p^\varepsilon)$ in such a way as to have them defined on the interval $[0, T]$, setting

$$\vartheta_\varepsilon(t) := \vartheta^\varepsilon\left(\frac{t}{\varepsilon}\right), \quad u_\varepsilon(t) := u^\varepsilon\left(\frac{t}{\varepsilon}\right), \quad e_\varepsilon(t) := e^\varepsilon\left(\frac{t}{\varepsilon}\right), \quad p_\varepsilon(t) := p^\varepsilon\left(\frac{t}{\varepsilon}\right), \qquad t \in [0, T].$$

For later reference, here we state the defining properties of weak energy solutions in terms of the rescaled quadruple $(\vartheta_\varepsilon, u_\varepsilon, e_\varepsilon, p_\varepsilon)$, taking into account the improved formulation of the heat equation provided in Theorem 2. In addition to the kinematic admissibility $E(u_\varepsilon) = e_\varepsilon + p_\varepsilon$, we have:

- strict positivity: $\vartheta_\varepsilon \geq \bar{\vartheta} > 0$ a.e. in Ω, with $\bar{\vartheta}$ given by (2.33);

- weak formulation of the heat equation, for almost all $t \in (0,T)$ and all test functions $\varphi \in W^{1,1+1/\tilde\delta}(\Omega)$, with $\tilde\delta > 0$ such that (2.36) holds:

$$\begin{aligned} &\varepsilon \langle \dot\vartheta_\varepsilon(t), \varphi\rangle_{W^{1,1+1/\tilde\delta}(\Omega)} + \int_\Omega \kappa(\vartheta_\varepsilon(t))\nabla\vartheta_\varepsilon(t)\nabla\varphi \,\mathrm{d}x \\ &= \int_\Omega \big(H_\varepsilon(t) + \varepsilon \mathrm{R}(x,\dot p_\varepsilon(t)) + \varepsilon^2 \dot p_\varepsilon(t) : \dot p_\varepsilon(t) + \varepsilon^2 \mathbb{D}\dot e_\varepsilon(t) : \dot e_\varepsilon(t) - \varepsilon\vartheta_\varepsilon(t)\mathbb{B}_\varepsilon : \dot e_\varepsilon(t)\big)\,\varphi\,\mathrm{d}x \\ &\quad + \int_{\partial\Omega} h_\varepsilon(t)\varphi\,\mathrm{d}S \end{aligned} \tag{6.2a}$$

- weak momentum balance for almost all $t \in (0,T)$ and all test functions $v \in H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^d)$

$$\rho\varepsilon^2 \int_\Omega \ddot u_\varepsilon(t) v\,\mathrm{d}x + \int_\Omega (\mathbb{D}\varepsilon\dot e_\varepsilon(t) + \mathbb{C}e_\varepsilon(t) - \vartheta_\varepsilon(t)\mathbb{B}_\varepsilon) : E(v)\,\mathrm{d}x = \langle \mathcal{L}_\varepsilon(t), v\rangle_{H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^d)} \tag{6.2b}$$

  for almost all $t \in (0,T)$, with $\mathcal{L}_\varepsilon$ defined from F_ε and g_ε as in (2.5);
- the plastic flow rule (1.10c), rewritten (cf. (1.4)) for later use as

$$\sigma_\varepsilon - \varepsilon\dot p_\varepsilon = \mathrm{P}_{\mathcal{K}(\Omega)}((\sigma_\varepsilon)_{\mathrm{D}}) \qquad \text{a.e. in } Q, \tag{6.2c}$$

  with $\sigma_\varepsilon = \varepsilon\mathbb{D}\dot e_\varepsilon + \mathbb{C}e_\varepsilon - \vartheta_\varepsilon\mathbb{B}_\varepsilon$ and the projection operator $\mathrm{P}_{\mathcal{K}(\Omega)}$ from (5.7).

We also record the

- (rescaled) mechanical energy balance

$$\begin{aligned} &\frac{\rho\varepsilon^2}{2}\int_\Omega |\dot u_\varepsilon(t)|^2\,\mathrm{d}x + \varepsilon\int_0^t\int_\Omega \mathbb{D}\dot e_\varepsilon : \dot e_\varepsilon\,\mathrm{d}x\,\mathrm{d}r + \frac{\varepsilon}{2}\int_0^t\int_\Omega |\dot p_\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}r + \frac{1}{2\varepsilon}\int_0^t d^2((\sigma_\varepsilon)_{\mathrm{D}}, \mathcal{K}(\Omega))\,\mathrm{d}t \\ &\quad + \int_0^t \mathcal{R}(\dot p_\varepsilon)\,\mathrm{d}r + \mathcal{Q}(e_\varepsilon(t)) \\ &= \frac{\rho\varepsilon^2}{2}\int_\Omega |\dot u^0_\varepsilon|^2\,\mathrm{d}x + \mathcal{Q}(e^0_\varepsilon) + \int_0^t \langle \mathcal{L}_\varepsilon, \dot u_\varepsilon - \dot w_\varepsilon\rangle_{H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^d)}\,\mathrm{d}r + \int_0^t\int_\Omega \vartheta_\varepsilon\mathbb{B}_\varepsilon : \dot e_\varepsilon\,\mathrm{d}x\,\mathrm{d}r \\ &\quad + \rho\varepsilon^2\left(\int_\Omega \dot u_\varepsilon(t)\dot w_\varepsilon(t)\,\mathrm{d}x - \int_\Omega \dot u^0_\varepsilon \dot w_\varepsilon(0)\,\mathrm{d}x - \int_0^t\int_\Omega \dot u_\varepsilon \ddot w_\varepsilon\,\mathrm{d}x\,\mathrm{d}r\right) + \int_0^t\int_\Omega \sigma_\varepsilon : E(\dot w_\varepsilon)\,\mathrm{d}x\,\mathrm{d}r \end{aligned} \tag{6.2d}$$

  for every $t \in [0,T]$, where we have used (6.2c), yielding that

$$\varepsilon|\dot p_\varepsilon|^2 = \frac{\varepsilon}{2}|\dot p_\varepsilon|^2 + \frac{1}{2\varepsilon}\underbrace{|\sigma_\varepsilon - \varepsilon\dot p_\varepsilon|^2}_{= d^2((\sigma_\varepsilon)_{\mathrm{D}}, \mathcal{K}(\Omega))} \qquad \text{a.e. in } Q,$$

  with $d(\cdot, \mathcal{K}(\Omega))$ the distance function from the closed and convex set $\mathcal{K}(\Omega)$.

Finally, adding (6.2d) with (6.2a) tested by $\frac{1}{\varepsilon}$ we obtain the

- (rescaled) total energy balance

$$\begin{aligned} &\frac{\rho\varepsilon^2}{2}\int_\Omega |\dot u_\varepsilon(t)|^2\,\mathrm{d}x + \langle \vartheta_\varepsilon(t), 1\rangle_{W^{1,\infty}} + \frac12\int_\Omega \mathbb{C}e_\varepsilon(t){:}e_\varepsilon(t)\,\mathrm{d}x \\ &= \frac{\rho\varepsilon^2}{2}\int_\Omega |\dot u^0_\varepsilon|^2\,\mathrm{d}x + \mathcal{E}(\vartheta^0_\varepsilon, e^0_\varepsilon) + \int_0^t \langle \mathcal{L}_\varepsilon, \dot u_\varepsilon - \dot w_\varepsilon\rangle_{H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^d)}\,\mathrm{d}r + \int_0^t\int_\Omega \frac{1}{\varepsilon}H_\varepsilon\,\mathrm{d}x\,\mathrm{d}r + \int_0^t\int_{\partial\Omega}\frac{1}{\varepsilon}h_\varepsilon\,\mathrm{d}S\,\mathrm{d}r \\ &\quad + \rho\varepsilon^2\left(\int_\Omega \dot u_\varepsilon(t)\dot w_\varepsilon(t)\,\mathrm{d}x - \int_\Omega \dot u^0_\varepsilon\dot w_\varepsilon(0)\,\mathrm{d}x - \int_0^t\int_\Omega \dot u_\varepsilon\ddot w_\varepsilon\,\mathrm{d}x\,\mathrm{d}r\right) + \int_0^t\int_\Omega \sigma_\varepsilon : E(\dot w_\varepsilon)\,\mathrm{d}x\,\mathrm{d}r \end{aligned} \tag{6.2e}$$

  for every $t \in [0,T]$.

Indeed, (6.2e) will be the starting point in the derivation of the priori estimates, *uniform* w.r.t. the parameter ε, on the functions $(\vartheta_\varepsilon, u_\varepsilon, e_\varepsilon, p_\varepsilon)_\varepsilon$, under the following

Hypotheses on the data $(H_\varepsilon, h_\varepsilon, F_\varepsilon, g_\varepsilon, w_\varepsilon)_\varepsilon$ *and on the initial data* $(\vartheta^0_\varepsilon, u^0_\varepsilon, \dot u^0_\varepsilon, e^0_\varepsilon, p^0_\varepsilon)_\varepsilon$. We require that there exists a constant $\overline{C} > 0$ such that for every $\varepsilon > 0$

$$\|H_\varepsilon\|_{L^1(0,T;L^1(\Omega))} \le \overline{C}\varepsilon, \qquad \|h_\varepsilon\|_{L^1(0,T;L^2(\partial\Omega))} \le \overline{C}\varepsilon. \tag{6.3a}$$

As for the body and surface forces, for every $\varepsilon > 0$ the functions F_ε and g_ε have to comply with (2.L_1) and the safe-load condition (2.L_2), with associated stresses ϱ_ε. We impose that there exist F and g as in (5.L_1)

to which $(F_\varepsilon)_\varepsilon$ and $(g_\varepsilon)_\varepsilon$ converge in topologies that we choose not to specify, and that the sequence $(\varrho_\varepsilon)_\varepsilon$ converge to the stress ϱ from (5.5) (hence, with $\varrho_{\mathrm{D}} \equiv 0$) associated with F and g, namely

$$\begin{aligned} &\varrho_\varepsilon \to \varrho \text{ in} \qquad \text{in } W^{1,1}(0,T;L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})),\\ &\|(\varrho_\varepsilon)_{\mathrm{D}}\|_{L^1(0,T;L^\infty(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}}))} \leq \overline{C}\varepsilon, \qquad \|(\dot\varrho_\varepsilon)_{\mathrm{D}}\|_{L^1(0,T;L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}}))} \leq \overline{C}\varepsilon\,. \end{aligned} \tag{6.3b}$$

For later use, let us record here that, since $\langle \mathcal{L}_\varepsilon(t), v\rangle_{H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^d)} = \int_\Omega \varrho_\varepsilon(t){:}E(v)\,\mathrm{d}x$ for every $v \in H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^d)$ by the safe load condition, and analogously for $\dot{\mathcal{L}}_\varepsilon$, it follows from (6.3b) that

$$\mathcal{L}_\varepsilon \to \mathcal{L} \quad \text{in } W^{1,1}(0,T;H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^d)^*), \tag{6.3c}$$

with $\mathcal{L}$ the total load associated with F and g.

Furthermore, we impose that the Dirichlet loadings $(w_\varepsilon)_\varepsilon \subset L^1(0,T;W^{1,\infty}(\Omega;\mathbb{R}^d)) \cap W^{2,1}(0,T;H^1(\Omega;\mathbb{R}^d)) \cap H^2(0,T;L^2(\Omega;\mathbb{R}^d))$ (cf. (2.W)), fulfill

$$\varepsilon\|\dot w_\varepsilon\|_{L^\infty(0,T;H^1(\Omega;\mathbb{R}^d))} \leq \overline{C}, \quad \varepsilon\|\ddot w_\varepsilon\|_{L^1(0,T;H^1(\Omega;\mathbb{R}^d))} \leq \overline{C}, \quad \varepsilon^{1/2}\|E(\dot w_\varepsilon)\|_{L^1(0,T;L^\infty(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}}))} \leq \overline{C}, \tag{6.3d}$$

and that there exists $w \in H^1(0,T;H^1(\Omega;\mathbb{R}^d))$ (cf. (5.W)) such that

$$w_\varepsilon \to w \quad \text{in } H^1(0,T;H^1(\Omega;\mathbb{R}^d)). \tag{6.3e}$$

Finally, for the Cauchy data $(\vartheta^0_\varepsilon, u^0_\varepsilon, \dot u^0_\varepsilon, e^0_\varepsilon, p^0_\varepsilon)_\varepsilon$ we impose the bounds and the convergences

$$\begin{aligned} &\varepsilon\|\dot u^0_\varepsilon\|_{L^2(\Omega;\mathbb{R}^d)} \to 0, \quad \vartheta^0_\varepsilon \rightharpoonup \vartheta_0 \text{ in } L^1(\Omega),\\ &u^0_\varepsilon \overset{*}{\rightharpoonup} u_0 \text{ in } \mathrm{BD}(\Omega;\mathbb{R}^d), \qquad e^0_\varepsilon \to e_0 \text{ in } L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}}), \qquad p^0_\varepsilon \overset{*}{\rightharpoonup} p_0 \text{ in } \mathrm{M}(\Omega\cup\Gamma_{\mathrm{Dir}};\mathbb{M}^{d\times d}_{\mathrm{D}}). \end{aligned} \tag{6.3f}$$

Observe that, since $(u^0_\varepsilon, \dot u^0_\varepsilon, e^0_\varepsilon, p^0_\varepsilon) \in \mathcal{A}(w_\varepsilon(0)) \subset \mathcal{A}_{\mathrm{BD}}(w_\varepsilon(0))$, convergences (6.3e) and (6.3f), combined with Lemma 5.1, ensure that the triple (u_0, e_0, p_0) is in $\mathcal{A}_{\mathrm{BD}}(w(0))$.

We are now in the position to give our asymptotic result, stating the convergence (along a sequence $\varepsilon_k \downarrow 0$) of a family of solutions to the thermoviscoplastic system, to a quadruple (Θ, u, e, p) such that (u, e, p) is an energetic solution to the plastic system, while the limit temperature Θ is constant in space, but still time-dependent. Furthermore, we find that (Θ, u, e, p) fulfill a further energy balance, cf. (6.6) ahead, from which we deduce a balance between the energy dissipated by the plastic strain, and the thermal energy.

Theorem 3. *Let the reference configuration Ω and the elasticity tensor $\mathbb{C}$ comply with* (5.Ω) *and* (5.T)*, respectively. Let $(\vartheta_\varepsilon, u_\varepsilon, e_\varepsilon, p_\varepsilon)_\varepsilon$ be a family of* weak energy solutions *to the rescaled thermoviscoplastic systems (1.10, 1.5), with heat conduction coefficient κ fulfilling* (2.κ_1) *and* (2.κ_2)*, tensors $\mathbb{B}_\varepsilon$ satisfying* (6.1)*, and with data $(H_\varepsilon, h_\varepsilon, F_\varepsilon, g_\varepsilon, w_\varepsilon)_\varepsilon$ and initial data $(\vartheta^0_\varepsilon, u^0_\varepsilon, e^0_\varepsilon, p^0_\varepsilon)_\varepsilon$ fulfilling conditions* (6.3)*.*

Then, for every sequence $\varepsilon_k \downarrow 0$ as $k \to \infty$ there exist a (not relabeled) subsequence $(\vartheta_{\varepsilon_k}, u_{\varepsilon_k}, e_{\varepsilon_k}, p_{\varepsilon_k})_k$ and functions $\Theta \in L^\infty(0,T)$, $u \in L^\infty(0,T;\mathrm{BD}(\Omega;\mathbb{R}^d))$, $e \in L^\infty(0,T;L^2(\Omega;\mathbb{R}^d))$, $p \in \mathrm{BV}([0,T];\mathrm{M}(\Omega\cup\Gamma_{\mathrm{Dir}};\mathbb{M}^{d\times d}_{\mathrm{D}}))$ such that

(1) the following convergences hold as $k \to \infty$:

$$\vartheta_{\varepsilon_k} \rightharpoonup \Theta \qquad \text{in } L^h(Q) \qquad \text{for every } h \in \begin{cases} [1,3] & \text{if } d=2,\\ [1,8/3] & \text{if } d=3,\end{cases} \tag{6.4a}$$

$$u_{\varepsilon_k}(t) \overset{*}{\rightharpoonup} u(t) \qquad \text{in } \mathrm{BD}(\Omega;\mathbb{R}^d) \qquad \text{for a.a. } t \in (0,T), \tag{6.4b}$$

$$e_{\varepsilon_k}(t) \to e(t) \qquad \text{in } L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}}) \qquad \text{for a.a. } t \in (0,T), \tag{6.4c}$$

$$p_{\varepsilon_k}(t) \overset{*}{\rightharpoonup} p(t) \qquad \text{in } \mathrm{M}(\Omega\cup\Gamma_{\mathrm{Dir}};\mathbb{M}^{d\times d}_{\mathrm{D}}) \qquad \text{for every } t \in [0,T]; \tag{6.4d}$$

(2) Θ is constant in space;

(3) (u, e, p) is a global energetic solution to the perfectly plastic system*, with initial and boundary data (u_0, e_0, p_0) and w, and the enhanced time regularity*

$$u \in \mathrm{AC}([0,T];\mathrm{BD}(\Omega;\mathbb{R}^d)), \qquad e \in \mathrm{AC}([0,T];L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})), \qquad p \in \mathrm{AC}([0,T];L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{D}})); \tag{6.5}$$

(4) the quadruple (ϑ, u, e, p) fulfills the additional energy balance

$$\begin{aligned}\mathcal{E}(\Theta(t), e(t)) = \mathcal{E}(\vartheta_0, e_0) - \int_\Omega \varrho(0) : (e_0 - E(w(0)))\,\mathrm{d}x + \int_0^t \int_\Omega \sigma : E(\dot{w})\,\mathrm{d}x\,\mathrm{d}s \\ + \int_\Omega \varrho(t) : (e(t) - E(w(t)))\,\mathrm{d}x - \int_0^t \int_\Omega \dot{\varrho} : (e - E(w))\,\mathrm{d}x\,\mathrm{d}s\end{aligned} \tag{6.6}$$

for almost all $t \in (0, T)$, and therefore there holds

$$\mathrm{Var}(p; [s,t]) = \mathcal{F}(\Theta(s)) - \mathcal{F}(\Theta(t)) = |\Omega|(\Theta(s) - \Theta(t)) \qquad \textit{for almost all } s,t \in (0,T) \qquad \textit{with } s \le t. \tag{6.7}$$

Observe that, by virtue of convergence (6.4a), the limiting temperature Θ inherits the strict positivity property $\Theta(t) \ge \bar{\vartheta} > 0$ for almost all $t \in (0, T)$.

Remark 6.1 (An alternative scaling condition on the heat conduction coefficient κ)**.** For the vanishing viscosity and inertia analysis carried out in the frame of the damage system analyzed in [LRTT14], a scaling condition on the heat conduction coefficients κ_ε, allowed to depend on ε, was exploited, in place of (6.1). Namely, it was supposed that

$$\kappa_\varepsilon(\vartheta) = \frac{1}{\varepsilon^2}\kappa(\vartheta) \qquad \text{with } \kappa \in \mathrm{C}^0(\mathbb{R}^+) \text{ satisfying } (2.\kappa_2). \tag{6.8}$$

This reflects the view that, for the limit system, if a change of heat is caused at some spot in the material, then heat must be conducted all over the body with infinite speed. In fact, (6.8) as well led us to show that the limit temperature is constant in space, like in the present case.

This scaling condition was combined with the requirement that the Dirichlet boundary Γ_{Dir} coincides with the whole $\partial\Omega$, and that the Dirichlet loading w is null, in order to deduce

(1) the convergence (along a subsequence) of the temperatures ϑ_ε to a spatially constant function Θ;
(2) the strong convergence $\varepsilon e(\dot{u}_\varepsilon) \to 0$ in $L^2(Q; \mathbb{M}^{d\times d}_{\mathrm{sym}})$, by means of a careful argument strongly relying on the homogeneous character of the Dirichlet boundary conditions.

In this way, in [LRTT14] we bypassed one of the major difficulties in the asymptotic analysis, namely the presence of the thermal expansion term $\iint \vartheta_\varepsilon \mathbb{B}{:}\dot{e}_\varepsilon \,\mathrm{d}x\,\mathrm{d}t$ on the r.h.s. of the rescaled mechanical energy balance, which in turn is the starting point for the derivation of a priori estimates uniform w.r.t. ε for the dissipative variables $\dot{e}_\varepsilon$ and $\dot{p}_\varepsilon$.

In the present context, we have decided not to develop the approach based on condition (6.8). In fact, it would have forced us to take null Dirichlet loadings for the limit perfectly plastic system, and this, in combination with the strong safe load condition 5.5, would have been too restrictive.

We will develop the proof of Theorem 3 in the ensuing Sec. 6.1.

6.1. **Proof of Theorem 3.** We start by deriving a series of a priori estimates, *uniform* w.r.t. the parameter ε, for *a distinguished class* of weak energy solutions to system (1.10, 1.5). In fact, in the derivation of these estimates we will perform the same tests as in the proof of Prop. 4.3, in particular the test of the heat equation by $\vartheta_\varepsilon^\alpha$, with $\alpha \in [2-\mu, 1)$. Since $\vartheta_\varepsilon^\alpha$ is not an admissible test function for the rescaled heat equation (6.2a) due to its insufficient spatial regularity, the calculations related to this test can be rendered rigorously only on the time discrete level, and the resulting a priori estimates in fact only hold for the weak energy solutions arising from the time discretization scheme.

Proposition 6.2 (A priori estimates uniform w.r.t. ε)**.** *Assume* (5.Ω) *and* (5.T)*. Assume conditions* $(2.\kappa_1)$ *and* $(2.\kappa_2)$ *on* κ*,* (6.1) *on the tensors* $\mathbb{B}_\varepsilon$*, and* (6.3) *on the data* $(H_\varepsilon, h_\varepsilon, F_\varepsilon, g_\varepsilon, w_\varepsilon)_\varepsilon$ *and* $(\vartheta_\varepsilon^0, u_\varepsilon^0, e_\varepsilon^0, p_\varepsilon^0)_\varepsilon$*.*

Then, there exist a constant $C > 0$ and a family $(\vartheta_\varepsilon, u_\varepsilon, e_\varepsilon, p_\varepsilon)_\varepsilon$ of weak energy solutions *to the rescaled thermoviscoplastic systems (1.10, 1.5), such that for every $\varepsilon > 0$ the following estimates hold:*

$$\|E(u_\varepsilon)\|_{L^\infty(0,T;L^1(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}}))} + \varepsilon^{1/2}\|E(\dot u_\varepsilon)\|_{L^2(Q;\mathbb{M}^{d\times d}_{\mathrm{sym}})} + \varepsilon\|\dot u_\varepsilon\|_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^d))} + \varepsilon^2\|\ddot u_\varepsilon\|_{L^2(0,T;H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^d)^*)} \leq C, \tag{6.9a}$$

$$\|e_\varepsilon\|_{L^\infty(0,T;L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}}))} + \varepsilon^{1/2}\|\dot e_\varepsilon\|_{L^2(Q;\mathbb{M}^{d\times d}_{\mathrm{sym}})} \leq C, \tag{6.9b}$$

$$\|p_\varepsilon\|_{L^\infty(0,T;L^1(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}}))} + \|\dot p_\varepsilon\|_{L^1(Q;\mathbb{M}^{d\times d}_{\mathrm{sym}})} + \varepsilon^{1/2}\|\dot p_\varepsilon\|_{L^2(Q;\mathbb{M}^{d\times d}_{\mathrm{sym}})} \leq C, \tag{6.9c}$$

$$\frac{1}{\varepsilon^{1/2}}\|d((\sigma_\varepsilon)_{\mathrm{D}}, \mathcal{K}(\Omega))\|_{L^2(0,T)} \leq C, \tag{6.9d}$$

$$\|\vartheta_\varepsilon\|_{L^\infty(0,T;L^1(\Omega))} + \|\vartheta_\varepsilon\|_{L^h(Q)} + \frac{1}{\varepsilon^{1/2}}\|\nabla\vartheta_\varepsilon\|_{L^2(Q;\mathbb{R}^d)} \leq C \qquad \text{for } h = \begin{cases} 3 & \text{if } d = 2, \\ \frac{8}{3} & \text{if } d = 3, \end{cases} \tag{6.9e}$$

with $\alpha > \frac{1}{2}$ from (6.1).

Proof. We will follow the outline of the proof of Prop. 4.3, referring to it for all details.

First a priori estimate: We start from the rescaled total energy balance (6.2e) and estimate the terms on its right-hand side. It follows from (6.3f) that $\varepsilon^2\|\dot u^0_\varepsilon\|^2_{L^2(\Omega;\mathbb{R}^d)} \leq C$ and $\mathcal{E}(\vartheta^0_\varepsilon, e^0_\varepsilon) \leq C$.

As for the third term on the r.h.s., we use the safe load condition, yielding

$$\begin{aligned}
\int_0^t \langle \mathcal{L}_\varepsilon, \dot u_\varepsilon - \dot w_\varepsilon\rangle_{H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^d)} \,\mathrm{d}r &= \int_0^t\int_\Omega \varrho_\varepsilon : (E(\dot u_\varepsilon) - E(\dot w_\varepsilon))\,\mathrm{d}x\,\mathrm{d}r \\
&\overset{(1)}{=} \int_0^t\int_\Omega \varrho_\varepsilon : \dot e_\varepsilon \,\mathrm{d}x\,\mathrm{d}r + \int_0^t\int_\Omega \varrho_\varepsilon : \dot p_\varepsilon\,\mathrm{d}x\,\mathrm{d}r - \int_0^t\int_\Omega \varrho_\varepsilon : E(\dot w_\varepsilon)\,\mathrm{d}x\,\mathrm{d}r \\
&\overset{(2)}{=} -\int_0^t\int_\Omega \dot\varrho_\varepsilon : e_\varepsilon\,\mathrm{d}x\,\mathrm{d}r + \int_\Omega \varrho_\varepsilon(t) : e_\varepsilon(t)\,\mathrm{d}x - \int_\Omega \varrho_\varepsilon(0) : e^0_\varepsilon\,\mathrm{d}x + \int_0^t\int_\Omega (\varrho_\varepsilon)_{\mathrm{D}}\dot p_\varepsilon\,\mathrm{d}x\,\mathrm{d}r \\
&\quad - \int_0^t\int_\Omega \varrho_\varepsilon : E(\dot w_\varepsilon)\,\mathrm{d}x\,\mathrm{d}r \\
&\doteq I_1 + I_2 + I_3 + I_4 + I_5
\end{aligned} \tag{6.10}$$

with (1) due to the kinematic admissibility condition, and (2) following from integration by parts, and the fact that $\dot p_\varepsilon \in \mathbb{M}^{d\times d}_{\mathrm{D}}$ a.e. in Q. We estimate

$$\begin{aligned}
|I_1| &\leq \int_0^t \|\dot\varrho_\varepsilon\|_{L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})}\|e_\varepsilon\|_{L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})}\,\mathrm{d}r, \\
|I_2| &\overset{(1)}{\leq} C\|e_\varepsilon(t)\|_{L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})} \leq \frac{C^1_{\mathbb{C}}}{16}\|e_\varepsilon(t)\|^2_{L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})} + C, \\
|I_3| &\overset{(2)}{\leq} C\|e^0_\varepsilon\|_{L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})} \leq C
\end{aligned}$$

where (1) and (2) follow from the bound provided for $\|\varrho_\varepsilon\|_{L^\infty(0,T;L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})}$ by condition (6.3b), and from (6.3f). Instead, for I_4 we use the plastic flow rule (6.2c), rewritten as $\varepsilon\dot p_\varepsilon = (\sigma_\varepsilon)_{\mathrm{D}} - \zeta_\varepsilon$, with $\zeta_\varepsilon \in \partial_{\dot p}\mathrm{R}(\cdot, \dot p_\varepsilon)$ a.e. in Q. Then,

$$I_4 = \int_0^t\int_\Omega \tfrac{1}{\varepsilon}(\varrho_\varepsilon)_{\mathrm{D}} : (\sigma_\varepsilon)_{\mathrm{D}}\,\mathrm{d}x - \int_0^t\int_\Omega \tfrac{1}{\varepsilon}(\varrho_\varepsilon)_{\mathrm{D}} : \zeta_\varepsilon\,\mathrm{d}x \doteq I_{4,1} + I_{4,2},$$

and

$$I_{4,1} = \int_0^t\int_\Omega \tfrac{1}{\varepsilon}(\varrho_\varepsilon)_{\mathrm{D}} : (\mathbb{D}\dot e_\varepsilon + \mathbb{C}e_\varepsilon - \vartheta_\varepsilon\mathbb{B}_\varepsilon)_{\mathrm{D}}\,\mathrm{d}x\,\mathrm{d}r \doteq I_{4,1,1} + I_{4,1,2} + I_{4,1,3}$$

with

$$\begin{aligned}
I_{4,1,1} &= -\int_0^t\int_\Omega \tfrac{1}{\varepsilon}(\dot\varrho_\varepsilon)_{\mathrm{D}} : \mathbb{D}e_\varepsilon\,\mathrm{d}x\,\mathrm{d}r + \int_\Omega \tfrac{1}{\varepsilon}(\varrho_\varepsilon(t))_{\mathrm{D}} : \mathbb{D}e_\varepsilon(t)\,\mathrm{d}x - \int_\Omega \tfrac{1}{\varepsilon}(\varrho_\varepsilon(0))_{\mathrm{D}} : \mathbb{D}e^0_\varepsilon\,\mathrm{d}x \\
&\leq C\int_0^t \tfrac{1}{\varepsilon}\|(\dot\varrho_\varepsilon)_{\mathrm{D}}\|_{L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{D}})}\|e_\varepsilon\|_{L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})}\,\mathrm{d}r + \tfrac{C}{\varepsilon}\|(\varrho_\varepsilon)_{\mathrm{D}}\|^2_{L^\infty(0,T;L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{D}}))} \\
&\quad + \frac{C^1_{\mathbb{C}}}{16}\|e_\varepsilon(t)\|^2_{L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})} + C\|e^0_\varepsilon\|^2_{L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})},
\end{aligned}$$

and, analogously,

$$|I_{4,1,2}| \leq C \int_0^t \tfrac{1}{\varepsilon} \|\varrho_\varepsilon\|_{L^2(\Omega;\mathbb{M}_{\mathrm{D}}^{d\times d})} \|e_\varepsilon\|_{L^2(\Omega;\mathbb{M}_{\mathrm{sym}}^{d\times d})} \,\mathrm{d}r$$

$$|I_{4,1,3}| \leq C \int_0^t \tfrac{1}{\varepsilon} \|\varrho_\varepsilon\|_{L^\infty(\Omega;\mathbb{M}_{\mathrm{D}}^{d\times d})} \|\vartheta_\varepsilon\|_{L^1(\Omega)} \,\mathrm{d}r\,.$$

Instead, for the term $I_{4,2}$ we use that $|\zeta_\varepsilon| \leq C_R$ by (2.K$_2$), so that $|I_{4,2}| \leq \frac{C_R}{\varepsilon} \|(\varrho_\varepsilon)_{\mathrm{D}}\|_{L^1(Q;\mathbb{M}_{\mathrm{D}}^{d\times d})}$. Finally, we have

$$I_5 \leq \int_0^t \|\varrho_\varepsilon\|_{L^2(\Omega;\mathbb{M}_{\mathrm{D}}^{d\times d})} \|E(\dot{w}_\varepsilon)\|_{L^2(\Omega;\mathbb{M}_{\mathrm{sym}}^{d\times d})} \,\mathrm{d}r \leq C$$

thanks to (6.3b) and (6.3e).

The fourth and the fifth terms on the r.h.s. of (6.2e) are bounded thanks to condition (6.3a). We estimate the sixth term by

$$\frac{\rho\varepsilon^2}{4} \int_\Omega |\dot{u}_\varepsilon(t)|^2 \,\mathrm{d}x + \rho\varepsilon \int_0^t \|\dot{u}_\varepsilon\|_{L^2(\Omega;\mathbb{R}^d)} \varepsilon \|\ddot{w}_\varepsilon\|_{L^2(\Omega;\mathbb{R}^d)} \,\mathrm{d}r + C + C\varepsilon^2 \|\dot{w}_\varepsilon\|^2_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^d)}$$

where we have also used (6.3f).

As for the last term on the r.h.s. of (6.2e), arguing in the very same way as in the proof of Prop. 4.3, we estimate

$$\int_0^t \int_\Omega (\varepsilon\mathbb{D}\dot{e}_\varepsilon + \mathbb{C}e_\varepsilon - \vartheta_\varepsilon\mathbb{B}_\varepsilon) : E(\dot{w}_\varepsilon) \,\mathrm{d}x \,\mathrm{d}r \doteq I_{6,1} + I_{6,2} + I_{6,3},$$

with

$$\begin{aligned} I_{6,1} &= -\int_0^t \int_\Omega \varepsilon\mathbb{D}e_\varepsilon : E(\ddot{w}_\varepsilon) \,\mathrm{d}x \,\mathrm{d}r + \int_\Omega \varepsilon\mathbb{D}e_\varepsilon(t) : E(\dot{w}_\varepsilon(t)) \,\mathrm{d}x - \int_\Omega \varepsilon\mathbb{D}e_\varepsilon^0 : E(\dot{w}_\varepsilon(0)) \,\mathrm{d}x \\ &\leq \int_0^T \|e_\varepsilon\|_{L^2(\Omega;\mathbb{M}_{\mathrm{sym}}^{d\times d})} \varepsilon \|E(\ddot{w}_\varepsilon)\|_{L^2(\Omega;\mathbb{M}_{\mathrm{sym}}^{d\times d})} + \frac{C_{\mathbb{C}}^1}{16} \|e_\varepsilon(t)\|^2_{L^2(\Omega;\mathbb{M}_{\mathrm{sym}}^{d\times d})} + C\varepsilon^2 \|E(\dot{w}_\varepsilon)\|^2_{L^\infty(0,T;L^2(\Omega;\mathbb{M}_{\mathrm{sym}}^{d\times d}))} + C, \end{aligned}$$

where we have also used (6.3f). We also have

$$|I_{6,2}| \leq C \int_0^t \|e_\varepsilon\|_{L^2(\Omega;\mathbb{M}_{\mathrm{sym}}^{d\times d})} \|E(\dot{w}_\varepsilon)\|_{L^2(\Omega;\mathbb{M}_{\mathrm{sym}}^{d\times d})} \,\mathrm{d}r,$$

$$|I_{6,3}| \leq C \int_0^t \|\vartheta_\varepsilon\|_{L^1(\Omega)} \varepsilon^{1/2} \|E(\dot{w}_\varepsilon)\|_{L^\infty(\Omega;\mathbb{M}_{\mathrm{sym}}^{d\times d})} \,\mathrm{d}r,$$

thanks to the scaling (6.1) of the tensors $\mathbb{B}_\varepsilon$.

All in all, taking into account the bounds provided by conditions (6.3), we obtain

$$\begin{aligned} &\varepsilon^2 \int_\Omega |\dot{u}_\varepsilon(t)|^2 \,\mathrm{d}x + \int_\Omega \vartheta_\varepsilon(t) \,\mathrm{d}x + \int_\Omega |e_\varepsilon(t)|^2 \,\mathrm{d}x \\ &\leq C + C \int_0^t \Big(\|\dot{\varrho}_\varepsilon\|_{L^2(\Omega;\mathbb{M}_{\mathrm{sym}}^{d\times d})} + \tfrac{1}{\varepsilon} \|(\dot{\varrho}_\varepsilon)_{\mathrm{D}}\|_{L^2(\Omega;\mathbb{M}_{\mathrm{D}}^{d\times d})} + \tfrac{1}{\varepsilon} \|\varrho_\varepsilon\|_{L^2(\Omega;\mathbb{M}_{\mathrm{sym}}^{d\times d})} + \varepsilon \|E(\ddot{w}_\varepsilon)\|_{L^2(\Omega;\mathbb{M}_{\mathrm{sym}}^{d\times d})} \Big) \|e_\varepsilon\|_{L^2(\Omega;\mathbb{M}_{\mathrm{sym}}^{d\times d})} \,\mathrm{d}r \\ &\quad + C \int_0^t \varepsilon \|\ddot{w}_\varepsilon\|_{L^2(\Omega;\mathbb{R}^d)} \varepsilon \|\dot{u}_\varepsilon\|_{L^2(\Omega;\mathbb{R}^d)} \,\mathrm{d}r + C \int_0^t \Big(\tfrac{1}{\varepsilon} \|\varrho_\varepsilon\|_{L^\infty(\Omega;\mathbb{M}_{\mathrm{sym}}^{d\times d})} + \varepsilon^{1/2} \|E(\dot{w}_\varepsilon)\|_{L^\infty(\Omega;\mathbb{M}_{\mathrm{sym}}^{d\times d})} \| \Big) \|\vartheta_\varepsilon\|_{L^1(\Omega)} \,\mathrm{d}r. \end{aligned}$$

Applying the aforementioned variant of the Gronwall Lemma from [Dra03], we obtain $\varepsilon\|\dot{u}_\varepsilon\|_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^d))} + \sup_{t\in[0,T]} \mathcal{E}(\vartheta_\varepsilon(t), e_\varepsilon(t)) \leq C$, whence the third bound in (6.9a), and the first bounds in (6.9b) and (6.9e) .

Second a priori estimate: We (formally) test the rescaled heat equation (6.2a) by $\vartheta_\varepsilon^{\alpha-1}$ and integrate on $(0,t)$, thus retrieving the (formally written) analogue of (4.16), namely

$$\begin{aligned} &c \int_0^t \int_\Omega \kappa(\vartheta_\varepsilon) |\nabla(\vartheta_\varepsilon^{\alpha/2})|^2 \,\mathrm{d}x \,\mathrm{d}r + \varepsilon^2 C_{\mathbb{D}}^2 \int_\Omega |\dot{e}_\varepsilon|^2 \vartheta_\varepsilon^{\alpha-1} \,\mathrm{d}x \,\mathrm{d}r \\ &\leq \varepsilon \int_0^t \int_\Omega \dot{\vartheta}_\varepsilon \vartheta_\varepsilon^{\alpha-1} \,\mathrm{d}x \,\mathrm{d}r + \varepsilon \int_0^t \int_\Omega \vartheta_\varepsilon \mathbb{B}_\varepsilon{:}\dot{e}_\varepsilon \vartheta_\varepsilon^{\alpha-1} \,\mathrm{d}x \,\mathrm{d}r \\ &= \frac{\varepsilon}{\alpha} \int_\Omega (\vartheta_\varepsilon(t))^\alpha \,\mathrm{d}x - \frac{\varepsilon}{\alpha} \int_\Omega (\vartheta_\varepsilon^0)^\alpha \,\mathrm{d}x + \varepsilon \int_0^t \int_\Omega \vartheta_\varepsilon \varepsilon^\alpha \mathbb{B}{:}\dot{e}_\varepsilon \vartheta_\varepsilon^{\alpha-1} \,\mathrm{d}x \,\mathrm{d}r \doteq I_1 + I_2 + I_3. \end{aligned} \tag{6.11}$$

in view of the scaling (6.1) for $\mathbb{B}_\varepsilon$. The first two integral terms on the r.h.s. can be treated in the same way as in (4.17), taking into account the previously proved bound for $\|\vartheta_\varepsilon\|_{L^\infty(0,T;L^1(\Omega))}$. We thus obtain

$$I_1 + I_2 \leq C\varepsilon\,. \tag{6.12}$$

Again, we estimate

$$I_3 \leq \frac{\varepsilon^2 C_{\mathbb{D}}^2}{4} \int_\Omega |\dot{e}_\varepsilon|^2 \vartheta_\varepsilon^{\alpha-1} \,\mathrm{d}x\,\mathrm{d}r + C\varepsilon^{2\alpha} \int_0^t \int_\Omega \vartheta_\varepsilon^{\alpha+1} \,\mathrm{d}x\,\mathrm{d}r\,.$$

While the first term in the above formula is absorbed into the l.h.s. of (6.11), the second one is handled by the very same arguments developed in the proof of Prop. 4.3. In this way, also taking into account (6.12), we obtain,

$$\|\nabla\vartheta_\varepsilon\|_{L^2(Q;\mathbb{R}^d)}^2 + \|\nabla(\vartheta_\varepsilon)^{(\mu+\alpha)/2}\|_{L^2(Q;\mathbb{R}^d))}^2 + \|\nabla(\vartheta_\varepsilon)^{(\mu-\alpha)/2}\|_{L^2(Q;\mathbb{R}^d))}^2 \leq C\varepsilon + C'\varepsilon^{2\alpha}, \tag{6.13}$$

whence, in particular, the third bound in (6.9e). The second bound follows from interpolation arguments, cf. (4.25).

Third a priori estimate: We now address the (rescaled) mechanical energy balance (6.2d). The scaling (6.1) of $\mathbb{B}_\varepsilon$ yields for the third integral term on the right-hand side

$$\left| \int_0^t \int_\Omega \vartheta_\varepsilon \mathbb{B}{:}\varepsilon^{1/2}\dot{e}_\varepsilon \,\mathrm{d}x\,\mathrm{d}r \right| \leq \int_0^t \int_\Omega |\vartheta_\varepsilon|^2 \,\mathrm{d}x\,\mathrm{d}r + \frac{\varepsilon}{4}\int_0^t \int_\Omega |\dot{e}_\varepsilon|^2 \,\mathrm{d}x\,\mathrm{d}r, \tag{6.14}$$

so that the latter term can be absorbed into the left-hand side. The remaining terms on the r.h.s. are handled by the very same calculations developed for the *First a priori estimate*. Therefore, from the bounds for the terms on the l.h.s. of (6.2d), we conclude the second of (6.9b), (6.9c) and thus, by kinematic admissibility, the first two bounds in (6.9a). We also infer (6.9d).

Fourth a priori estimate: The last bound in (6.9a) follows from a comparison argument in the rescaled momentum balance (6.2b), taking into account the previously proved estimates, as well as the uniform bound (6.3c) for $\mathcal{L}_\varepsilon$. In the same way, a comparison in the heat equation (6.2a) leads to the last estimate in (6.9e). This concludes the proof. □

Remark 6.3. Condition (6.3b), imposing that the functions $(\varrho_\varepsilon)_{\mathrm{D}}$ tend to zero (w.r.t. suitable norms) has been crucial to compensate the blowup of the bounds for $\dot{p}_\varepsilon$, in the estimate of the term $\int_0^t \int_\Omega (\varrho_\varepsilon)_{\mathrm{D}} \dot{p}_\varepsilon \,\mathrm{d}x\,\mathrm{d}r$ contributing to $\int_0^t \langle \mathcal{L}_\varepsilon, \dot{u}_\varepsilon \rangle_{H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^d)} \,\mathrm{d}r$ on the right-hand side of the total energy balance (6.2e). A close perusal at the calculations for handling $\int_0^t \langle \mathcal{L}_\varepsilon, \dot{u}_\varepsilon \rangle_{H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^d)} \,\mathrm{d}r$ reveals that, taking the tractions g_ε null would not have allowed us to avoid (6.3b), either (unlike for the thermoviscoplastic system, cf. Remark 4.4).

For estimating the term $\iint \vartheta_\varepsilon \mathbb{B}{:}\varepsilon^{1/2}\dot{e}_\varepsilon \,\mathrm{d}x\,\mathrm{d}t$ it would in fact be sufficient that the thermal expansion tensors $\mathbb{B}_\varepsilon$ scale like $\varepsilon^{1/2}$: As we will see in the proof of Theorem 3, the (slightly) stronger scaling condition from (6.1) is necessary for the limit passage as $\varepsilon \downarrow 0$ in the mechanical energy equality.

6.2. **Proof of Theorem 3.** We split the arguments in some steps.

Step 0: compactness. It follows from (6.9e) that $\nabla\vartheta_{\varepsilon_k} \to 0$ in $L^2(Q;\mathbb{R}^d)$. Therefore, also taking into account the other bounds in (6.9e), we infer that, up to a subsequence the functions $(\vartheta_{\varepsilon_k})_k$ weakly converge to a spatially constant function $\Theta \in L^h(Q)$, with h as in (6.4a). In fact, we find that $\Theta \in L^\infty(Q)$ since for every $t \in (0,T)$ and (sufficiently small) $r > 0$

$$\int_{t-r}^{t+r} \|\Theta\|_{L^1(\Omega)} \,\mathrm{d}s \leq \liminf_{k\to\infty} \int_{t-r}^{t+r} \|\vartheta_{\varepsilon_k}\|_{L^1(\Omega)} \,\mathrm{d}s \leq 2rC,$$

where the first inequality follows from $\vartheta_{\varepsilon_k} \rightharpoonup \Theta$ in $L^1(Q)$ and the second estimate from bound (6.9e). Then, it suffices to take the limit as $r \downarrow 0$ at every Lebesgue point of the function $t \mapsto \|\Theta(t)\|_{L^1(\Omega)} = |\Theta(t)||\Omega|$.

On account of the continuous embedding $L^1(\Omega;\mathbb{M}^{d\times d}_{\mathrm{D}}) \subset \mathrm{M}(\Omega\cup\Gamma_{\mathrm{Dir}};\mathbb{M}^{d\times d}_{\mathrm{D}})$ we gather from (6.9c) that the functions p_ε have uniformly bounded variation in $\mathrm{M}(\Omega\cup\Gamma_{\mathrm{Dir}};\mathbb{M}^{d\times d}_{\mathrm{D}})$. Therefore, a generalization of Helly Theorem for functions with values in the dual of a separable space, cf. [DMDM06, Lemma 7.2], yields that there exists $p \in \mathrm{BV}([0,T];\mathrm{M}(\Omega\cup\Gamma_{\mathrm{Dir}};\mathbb{M}^{d\times d}_{\mathrm{D}}))$ such that convergence (6.4d) holds and, by the lower semicontinuity of the variation functional $\mathrm{Var}_{\mathcal{R}}$, that

$$\mathrm{Var}_{\mathcal{R}}(p;[a,b]) \leq \liminf_{k\to\infty} \mathrm{Var}_{\mathcal{R}}(p_{\varepsilon_k};[a,b]) \qquad \text{for every } [a,b]\subset[0,T]. \tag{6.15}$$

For later use, we remark that, in view of estimate (6.9c) on $(\dot p_\varepsilon)_\varepsilon$,

$$\varepsilon_k \int_0^T \mathcal{R}(\dot p_{\varepsilon_k})\,\mathrm{d}t \to 0, \qquad \varepsilon_k \dot p_{\varepsilon_k} \to 0 \text{ in } L^2(Q;\mathbb{M}^{d\times d}_{\mathrm{D}}). \tag{6.16}$$

In fact, we even have that

$$\varepsilon_k^{1/2} \dot p_{\varepsilon_k} \rightharpoonup 0 \text{ in } L^2(Q;\mathbb{M}^{d\times d}_{\mathrm{D}}). \tag{6.17}$$

Indeed, by (6.9c) there exists $\varpi \in L^2(Q;\mathbb{M}^{d\times d}_{\mathrm{D}})$ such that $\varepsilon_k^{1/2}\dot p_{\varepsilon_k} \rightharpoonup \varpi$ in $L^2(Q;\mathbb{M}^{d\times d}_{\mathrm{D}})$. We now show that $\varpi \equiv 0$. With this aim, we observe that, on the one hand the weak convergence in $L^2(Q;\mathbb{M}^{d\times d}_{\mathrm{D}})$ entails that

$$\int_\Omega \xi(x)\left(\int_0^t \varepsilon_k^{1/2}\dot p_{\varepsilon_k}(s,x)\,\mathrm{d}s\right)\mathrm{d}x \to \int_\Omega \xi(x)\left(\int_0^t \varpi(s,x)\,\mathrm{d}s\right)\mathrm{d}x$$

for every $t\in(0,T)$ and $\xi \in L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{D}})$, i.e. $\int_0^t \varepsilon_k^{1/2}\dot p_{\varepsilon_k}\,\mathrm{d}s \rightharpoonup \int_0^t \varpi\,\mathrm{d}s$ in $L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{D}})$. On the other hand, we have that

$$\left\|\int_0^t \varepsilon_k^{1/2}\dot p_{\varepsilon_k}\,\mathrm{d}r\right\|_{L^1(\Omega;\mathbb{M}^{d\times d}_{\mathrm{D}})} = \left\|\varepsilon_k^{1/2} p_{\varepsilon_k}(t) - \varepsilon_k^{1/2} p^0_{\varepsilon_k}\right\|_{L^1(\Omega;\mathbb{M}^{d\times d}_{\mathrm{D}})} \to 0$$

in view of estimate (6.9c). Hence, (6.17) ensues.

Up to a further subsequence, we have

$$e_{\varepsilon_k} \overset{*}{\rightharpoonup} e \quad \text{in } L^\infty(0,T;L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})). \tag{6.18}$$

Due to (6.9b), with the same arguments as for (6.17) we have that

$$\varepsilon_k \dot e_{\varepsilon_k} \to 0 \text{ in } L^2(Q;\mathbb{M}^{d\times d}_{\mathrm{sym}}), \qquad \varepsilon_k^{1/2}\dot e_{\varepsilon_k} \rightharpoonup 0 \text{ in } L^2(Q;\mathbb{M}^{d\times d}_{\mathrm{sym}}). \tag{6.19}$$

We combine the estimate for $E(u_\varepsilon)$ in $L^\infty(0,T;L^1(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}}))$ with the fact that the trace of u_ε on Γ_{Dir} (i.e., the trace of w_ε) is bounded in $L^\infty(0,T;L^1(\Gamma_{\mathrm{Dir}};\mathbb{R}^d))$ thanks to (6.3e). Then, via the Poincaré-type inequality (5.2) we conclude that $(u_\varepsilon)_\varepsilon$ is bounded in $L^\infty(0,T;\mathrm{BD}(\Omega;\mathbb{R}^d))$, which embeds continuosly into $L^\infty(0,T;L^{d/(d-1)}(\Omega;\mathbb{R}^d))$. Therefore, up to a subsequence

$$u_{\varepsilon_k} \overset{*}{\rightharpoonup} u \quad \text{in } L^\infty(0,T;L^{d/(d-1)}(\Omega;\mathbb{R}^d)). \tag{6.20}$$

Again via inequality (5.2) combined with estimate (6.3d) on $\dot w_\varepsilon$, we deduce from the estimate for $\varepsilon^{1/2}E(\dot u_\varepsilon)$ in $L^2(Q;\mathbb{M}^{d\times d}_{\mathrm{sym}})$ that $\varepsilon^{1/2}\dot u_\varepsilon$ is bounded in $L^2(0,T;\mathrm{BD}(\Omega;\mathbb{R}^d))$, hence in $L^2(0,T;L^{d/(d-1)}(\Omega;\mathbb{R}^d))$. Therefore, taking into account (6.20), we get that

$$\varepsilon^{1/2}\dot u_{\varepsilon_k} \rightharpoonup 0 \quad \text{in } L^2(0,T;L^{d/(d-1)}(\Omega;\mathbb{R}^d)).$$

Thus, (6.9a) also yields

$$\varepsilon_k \dot u_{\varepsilon_k} \overset{*}{\rightharpoonup} 0 \text{ in } L^\infty(\Omega;L^2(\Omega;\mathbb{R}^d)), \qquad \varepsilon_k^2 \ddot u_{\varepsilon_k} \rightharpoonup 0 \text{ in } L^2(\Omega;H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^d)^*). \tag{6.21}$$

Step 1*: ad the global stability condition* (S) *for almost all* $t\in(0,T)$*.* We will exploit Lemma 5.6 and check that

(1) the stress σ belongs to the elastic domain $\mathcal{K}(\Omega)$;
(2) it complies with the boundary value problem (5.17);
(3) the triple (u,e,p) is kinematically admissible.

Ad (1): It follows from the scaling (6.1) of the tensors $\mathbb{B}_\varepsilon$ and from estimate (6.9e) on (ϑ_ε) that the term $\vartheta_\varepsilon\mathbb{B}_\varepsilon$ strongly converges to 0 in $L^2(Q;\mathbb{M}^{d\times d}_{\mathrm{sym}})$. Therefore, also taking into account convergences (6.18) and (6.19), we deduce that

$$\sigma_{\varepsilon_k} \rightharpoonup \sigma = \mathbb{C}e \qquad \text{in } L^2(Q;\mathbb{M}^{d\times d}_{\mathrm{sym}}). \tag{6.22}$$

Hence,

$$\int_0^T d^2(\sigma_{\mathrm{D}},\mathcal{K}(\Omega))\,\mathrm{d}t \le \liminf_{k\to\infty}\int_0^T d^2((\sigma_{\varepsilon_k})_{\mathrm{D}},\mathcal{K}(\Omega))\,\mathrm{d}t = 0,$$

where the last equality follows from estimate (6.9d) deduced from the (rescaled) mechanical energy balance (6.2d). Therefore, the limit stress σ complies with the admissibility condition $\sigma(t)\in\mathcal{K}(\Omega)$ for almost all $t\in(0,T)$.

Ad (2): Exploiting convergence (6.3c) for the loads $\mathcal{L}_{\varepsilon_k}$ and (6.21) for the inertial terms $\ddot{u}_{\varepsilon_k}$, we can pass to the limit in the rescaled momentum balance (6.2b) and deduce that σ complies with

$$\int_\Omega \sigma(t){:}E(v)\,\mathrm{d}x = \langle \mathcal{L}(t), v\rangle_{H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^d)} = \int_\Omega F(t)v\,\mathrm{d}x + \int_{\Gamma_{\mathrm{Neu}}} g(t)v\,\mathrm{d}S \qquad \text{for a.a. } t\in(0,T),$$

whence (5.17).

Ad (3): In order to prove that $(u(t), e(t), p(t) \in \mathcal{A}_{\mathrm{BD}}(w(t))$ we will make use of the closedness property guaranteed by Lemma 5.1, and pass to the limit in the condition $(u_\varepsilon(t), e_\varepsilon(t), p_\varepsilon(t)) \in \mathcal{A}(w_\varepsilon(t)) \subset \mathcal{A}_{\mathrm{BD}}(w_\varepsilon(t))$ for almost all $t\in(0,T)$. However, we cannot directly apply Lemma 5.1 as, at the moment, we cannot count on *pointwise-in-time* convergences for the functions $(u_{\varepsilon_k})_k$ and $(e_{\varepsilon_k})_k$. In order to extract more information from the weak convergences (6.18) and (6.20), we resort to the Young measure compactness result stated in the upcoming Theorem A.2. Indeed, up to a further extraction, with the sequence $(u_{\varepsilon_k}, e_{\varepsilon_k})_k$, bounded in $L^\infty(0,T;\mathbf{X})$ with $\mathbf{X} = L^{d/(d-1)}(\Omega;\mathbb{R}^d)\times L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})$, we can associate a limiting Young measure $\boldsymbol{\mu}\in \mathscr{Y}(0,T;\mathbf{X})$ such that for almost all $t\in(0,T)$ the probability measure μ_t is concentrated on the set $\boldsymbol{L}_t$ of the limit points of $(u_{\varepsilon_k}(t), e_{\varepsilon_k}(t))_k$ w.r.t. the weak topology of $\mathbf{X}$, and we have the following representation formulae for the limits u and e (cf. (A.3))

$$(u(t), e(t)) = \int_{\mathbf{X}} (\mathsf{u},\mathsf{e})\,\mathrm{d}\mu_t(\mathsf{u},\mathsf{e}) \qquad \text{for a.a. } t\in(0,T).$$

Furthermore, for almost all $t\in(0,T)$ let us consider the *marginals* of μ_t, namely the probability measures μ_t^1 on $L^{d/(d-1)}(\Omega;\mathbb{R}^d)$, and μ_t^2 on $L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})$, defined by taking the push-forwards of μ_t through the projection maps $\pi_1 : \mathbf{X}\to L^{d/(d-1)}(\Omega;\mathbb{R}^d)$, and $\pi_2 : \mathbf{X}\to L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})$, i.e. $\mu_t^i = (\pi_i)_\#\mu_t$ for $i=1,2$, with $(\pi_i)_\#\mu_t$ defined by $(\pi_i)_\#\mu_t(B) := \mu_t(\pi_i^{-1}(B)$ for every $B\subset L^{d/(d-1)}(\Omega;\mathbb{R}^d)$ and $B\subset L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})$, respectively. Therefore,

$$u(t) = \pi_1\left(\int_{\mathbf{X}} (\mathsf{u},\mathsf{e})\,\mathrm{d}\mu_t(\mathsf{u},\mathsf{e})\right) = \int_{\mathbf{X}} \pi_1(\mathsf{u},\mathsf{e})\,\mathrm{d}\mu_t(\mathsf{u},\mathsf{e}) = \int_{L^{d/(d-1)}(\Omega;\mathbb{R}^d)} \mathsf{u}\,\mathrm{d}\mu_t^1(\mathsf{u}), \tag{6.23a}$$

and, analogously,

$$e(t) = \int_{L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})} \mathsf{e}\,\mathrm{d}\mu_t^2(\mathsf{e}). \tag{6.23b}$$

By (A.2) in Theorem A.2, the measure μ_t^1 (μ_t^2, respectively) is concentrated on $\boldsymbol{U}_t := \pi_1(\boldsymbol{L}_t)$, the set of the weak-$L^{d/(d-1)}(\Omega;\mathbb{R}^d)$ limit points of $(u_{\varepsilon_k}(t))_k$ ($\boldsymbol{E}_t := \pi_2(\boldsymbol{L}_t)$, the set of the weak-$L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})$ limit points of $(e_{\varepsilon_k}(t))_k$ respectively). We now combine (6.23) with the following information on the sets $\boldsymbol{U}_t$ and $\boldsymbol{E}_t$. Indeed,

$$\boldsymbol{U}_t \subset \mathrm{BD}(\Omega;\mathbb{R}^d) \qquad \text{for a.a. } t\in(0,T). \tag{6.24a}$$

Indeed, pick $\mathsf{u}\in\boldsymbol{U}_t$ and a subsequence $u_{\varepsilon^t_{k_j}}(t)$, possibly depending on t, such that $u_{\varepsilon^t_{k_j}}(t) \rightharpoonup L^{d/(d-1)}(\Omega;\mathbb{R}^d)$. Since $(u_\varepsilon)_\varepsilon$ is bounded in $L^\infty(0,T;\mathrm{BD}(\Omega;\mathbb{R}^d))$, we may suppose that the sequence $(u_{\varepsilon^t_{k_j}})$ is bounded in $\mathrm{BD}(\Omega;\mathbb{R}^d)$ and, a fortiori, weakly*-converges to u in $\mathrm{BD}(\Omega;\mathbb{R}^d)$, whence (6.24a). Ultimately,

$$u(t) = \int_{L^{d/(d-1)}(\Omega;\mathbb{R}^d)} \mathsf{u}\,\mathrm{d}\mu_t^1(\mathsf{u}) = \int_{\mathrm{BD}(\Omega;\mathbb{R}^d)} \mathsf{u}\,\mathrm{d}\mu_t^1(\mathsf{u})$$

Furthermore,

$$E(\mathsf{u}) = \mathsf{e} + p(t) \qquad \text{for every } (\mathsf{u},\mathsf{e})\in \boldsymbol{U}_t\times\boldsymbol{E}_t \text{ and for a.a. } t\in(0,T). \tag{6.24b}$$

This follows from passing to the limit in the kinematic admissibility condition $E(u_{\varepsilon_k}(t)) = e_{\varepsilon_k}(t) + p_{\varepsilon_k}(t)$, taking into account the pointwise convergence (6.4d). Finally,

$$p(t) = (w(t){-}\mathsf{u})\otimes\nu\mathscr{H}^{d-1} \qquad \text{on } \Gamma_{\mathrm{Dir}} \qquad \text{for every } \mathsf{u}\in\boldsymbol{U}_t \text{ and for a.a. } t\in(0,T), \tag{6.24c}$$

which ensues from Lemma 5.1, also taking into account convergence (6.3e) for (w_ε). Then, integrating (6.24b) w.r.t. the measure μ_t, using that

$$\iint_{\mathbf{X}} E(\mathsf{u})\,\mathrm{d}\mu_t(\mathsf{u},\mathsf{e}) = E(\int_{\mathbf{X}} \mathsf{u}\,\mathrm{d}\mu_t(\mathsf{u})) = E(\int_{L^{d/(d-1)}(\Omega;\mathbb{R}^d)} \mathsf{u}\,\mathrm{d}\mu_t^1(\mathsf{u})) = E(u(t))$$

by the linearity of the operator $E(\cdot)$, and arguing analogously for the other terms in (6.24b), we conclude that $E(u(t)) = e(t) + p(t)$. The boundary condition on Γ_{Dir} follows from integrating (6.24c). This concludes the proof of the kinematic admissibility condition, and thus of (S), *for almost all* $t\in(0,T)$.

Step 2: ad the upper energy estimate in (E) *for almost all* $t \in (0,T)$. We shall now prove the inequality $\leq$ in (E). With this aim, we pass to the limit in the (rescaled) mechanical energy balance (6.2d), integrated on a generic interval $(a,b) \subset (0,T)$. Taking into account that the the first four terms on the l.h.s. are positive, we have that

$$
\begin{aligned}
\liminf_{k\to\infty} \int_a^b (\text{l.h.s. of } (6.2\text{d}))\,\mathrm{d}t &\geq \liminf_{k\to\infty} \int_a^b \int_0^t \mathcal{R}(\dot{p}_{\varepsilon_k})\,\mathrm{d}s\,\mathrm{d}t + \liminf_{k\to\infty} \int_a^b \mathcal{Q}(e_{\varepsilon_k}(t))\,\mathrm{d}t \\
&\geq \int_a^b \mathrm{Var}_{\mathcal{R}}(p;[0,t])\,\mathrm{d}t + \int_a^b \mathcal{Q}(e(t))\,\mathrm{d}t.
\end{aligned}
\tag{6.25}
$$

The first lim inf-inequality follows from the fact that

$$
\liminf_{k\to\infty} \int_0^t \mathcal{R}(\dot{p}_{\varepsilon_k})\,\mathrm{d}s \overset{(5.12)}{=} \liminf_{k\to\infty} \mathrm{Var}_{\mathcal{R}}(p_{\varepsilon_k};[0,t]) \overset{(6.15)}{\geq} \mathrm{Var}_{\mathcal{R}}(p;[0,t]) \quad \text{for every } t \in [0,T]
$$

and from the Fatou Lemma. The second one is due to the weak convergence (6.18) for the sequence $(e_{\varepsilon_k})_k$.

As for the r.h.s. of (6.2d), we have that

$$
\begin{aligned}
\lim_{k\to\infty} \int_a^b (\text{r.h.s. of } (6.2\text{d}))\,\mathrm{d}t = \int_a^b \Big(\mathcal{Q}(e_0) - \int_\Omega \varrho(0) : (e_0 - E(w(0)))\,\mathrm{d}x + \int_0^t \int_\Omega \sigma : E(\dot{w})\,\mathrm{d}x\,\mathrm{d}s \\
+ \int_\Omega \varrho(t) : (e(t) - E(w(t)))\,\mathrm{d}x - \int_0^t \int_\Omega \dot{\varrho} : (e - E(w))\,\mathrm{d}x\,\mathrm{d}s \Big)\,\mathrm{d}t\,.
\end{aligned}
\tag{6.26}
$$

In fact, the term $\frac{\rho\varepsilon_k^2}{2}\int_\Omega |\dot{u}^0_{\varepsilon_k}|^2\,\mathrm{d}x$ on the r.h.s. of (6.2d) tends to zero by (6.3f). For the term $\iint \langle \mathcal{L}_{\varepsilon_k}, \dot{u}_{\varepsilon_k} - \dot{w}_{\varepsilon_k} \rangle$ we use the safe-load condition, yielding

$$
\int_a^b \int_0^t \langle \mathcal{L}_{\varepsilon_k}, \dot{u}_{\varepsilon_k} - \dot{w}_{\varepsilon_k} \rangle_{H^1_{\mathrm{Dir}}(\Omega;\mathbb{R}^d)}\,\mathrm{d}s\,\mathrm{d}t = \int_a^b \int_0^t \int_\Omega \varrho_{\varepsilon_k} : E(\dot{u}_{\varepsilon_k})\,\mathrm{d}x\,\mathrm{d}s\,\mathrm{d}t - \int_a^b \int_0^t \int_\Omega \varrho_{\varepsilon_k} : E(\dot{w}_{\varepsilon_k})\,\mathrm{d}x\,\mathrm{d}s\,\mathrm{d}t\,.
$$

In order to pass to the limit in the first integral term, we replace $E(\dot{u}_{\varepsilon_k})$ by $\dot{e}_{\varepsilon_k} + \dot{p}_{\varepsilon_k}$ via kinematic admissibility, and integrate by parts the term featuring $\varrho_{\varepsilon_k}\dot{e}_{\varepsilon_k}$, thus obtaining the sum of four integrals, cf. equality (2) in (6.10). Referring to the notation $I_1, \ldots, I_4$ for the terms contributing to (6.10), we find that

$$
\begin{array}{lcl}
\int_a^b I_1\,\mathrm{d}t & \overset{(1)}{\to} & -\int_a^b \int_0^t \int_\Omega \dot{\varrho} : e\,\mathrm{d}x\,\mathrm{d}s\,\mathrm{d}t \\
\int_a^b I_2\,\mathrm{d}t & \overset{(2)}{\to} & \int_a^b \int_\Omega \varrho(t) : e(t)\,\mathrm{d}t \\
\int_a^b I_3\,\mathrm{d}t & \overset{(3)}{\to} & -\int_a^b \int_\Omega \varrho(0) : e(0)\,\mathrm{d}t \\
\int_a^b I_4\,\mathrm{d}t & \overset{(4)}{\to} & 0,
\end{array}
$$

as $k \to \infty$, with convergences (1) & (2) due to the first of (6.3b) combined with (6.18), while (3) follows from (6.3b) joint with (6.3f). Finally, (4) ensues from

$$
|I_4| = \left| \int_0^t \int_\Omega \varrho_{\varepsilon_k} : \dot{p}_{\varepsilon_k}\,\mathrm{d}x\,\mathrm{d}s \right| \leq \varepsilon_k^{1/2} \tfrac{1}{\varepsilon_k} \|(\varrho_{\varepsilon_k})_{\mathrm{D}}\|_{L^2(0,t;L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{D}}))} \varepsilon_k^{1/2} \|\dot{p}_{\varepsilon_k}\|_{L^2(0,t;L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{D}}))} \leq C\varepsilon_k^{1/2} \to 0
$$

where the last estimate is a consequence of (6.3b) and of estimate (6.9c) for $\dot{p}_{\varepsilon_k}$. Finally, again thanks to (6.3b) joint with (6.3e), we find that

$$
\begin{aligned}
&- \int_a^b \int_0^t \int_\Omega \varrho_{\varepsilon_k} : E(\dot{w}_{\varepsilon_k})\,\mathrm{d}x\,\mathrm{d}s\,\mathrm{d}t \\
&\to - \int_a^b \int_0^t \int_\Omega \varrho : E(\dot{w})\,\mathrm{d}x\,\mathrm{d}s\,\mathrm{d}t \\
&= \int_a^b \int_\Omega \varrho(t) : E(w(t))\,\mathrm{d}x\,\mathrm{d}t - \int_a^b \int_\Omega \varrho(0) : E(w(0))\,\mathrm{d}x\,\mathrm{d}t + \int_a^b \int_0^t \int_\Omega \dot{\varrho} : E(w)\,\mathrm{d}x\,\mathrm{d}s\,\mathrm{d}t\,,
\end{aligned}
$$

the last equality due integration by parts.

To pass to the limit in the fourth integral term on the r.h.s. of (6.2d) we use that

$$
\left| \int_0^t \int_\Omega \vartheta_{\varepsilon_k} \mathbb{B}_{\varepsilon_k} : \dot{e}_{\varepsilon_k}\,\mathrm{d}x\,\mathrm{d}s \right| \leq {\varepsilon_k}^{\alpha} \|\vartheta_{\varepsilon_k}\|_{L^2(Q)} \|\dot{e}_{\varepsilon_k}\|_{L^2(Q;\mathbb{M}^{d\times d}_{\mathrm{sym}})} \overset{(2)}{\leq} C\varepsilon_k^{\alpha-\frac{1}{2}} \to 0
$$

for all $t \in [0,T]$, with (2) following from the scaling (6.1) for $\mathbb{B}_\varepsilon$, and estimates (6.9b) and (6.9e). The fourth integral term on the r.h.s. of (6.2d) tends to zero thanks to estimate (6.9a) for $(\dot{u}_{\varepsilon_k})_k$ and to convergence (6.3e) for $(w_{\varepsilon_k})_k$. Combining (6.22) with (6.3e) we finally show that

$$\int_a^b \int_0^t \int_\Omega \sigma_{\varepsilon_k}{:}E(\dot w_{\varepsilon_k})\,\mathrm{d}x\,\mathrm{d}s\,\mathrm{d}t \to \int_a^b \int_0^t \int_\Omega \sigma{:}E(\dot w)\,\mathrm{d}x\,\mathrm{d}s\,\mathrm{d}t\,.$$

In view of all of the above convergences, (6.26) ensues.

Combining (6.25) and (6.26) we obtain for every $(a,b)\subset(0,T)$

$$\begin{aligned}\int_a^b \left(\mathcal{Q}(e(t)){+}\mathrm{Var}_{\mathcal{R}}(p;[0,t])\right)\mathrm{d}t \leq \int_a^b \Big(\mathcal{Q}(e_0) - \int_\Omega \varrho(0):(e_0{-}E(w(0)))\,\mathrm{d}x + \int_0^t\int_\Omega \sigma : E(\dot w)\,\mathrm{d}x\,\mathrm{d}s \\ + \int_\Omega \varrho(t):(e(t){-}E(w(t)))\,\mathrm{d}x - \int_0^t\int_\Omega \dot\varrho:(e{-}E(w))\,\mathrm{d}x\,\mathrm{d}s\Big)\,\mathrm{d}t\,.\end{aligned}$$

Then, by the arbitrariness of $(a,b)\subset[0,T]$, we conclude that for almost all $t\in(0,T)$ there holds

$$\begin{aligned}\mathcal{Q}(e(t)){+}\mathrm{Var}_{\mathcal{R}}(p;[0,t]) \leq \mathcal{Q}(e_0) - \int_\Omega \varrho(0):(e_0-E(w(0)))\,\mathrm{d}x + \int_0^t\int_\Omega \sigma : E(\dot w)\,\mathrm{d}x\,\mathrm{d}s \\ + \int_\Omega \varrho(t):(e(t)-E(w(t)))\,\mathrm{d}x - \int_0^t\int_\Omega \dot\varrho:(e{-}E(w))\,\mathrm{d}x\,\mathrm{d}s\end{aligned} \tag{6.27}$$

Step 3: ad the lower energy estimate in (E) *for almost all* $t\in(0,T)$. We use a by now standard argument (cf. [DMFT05b, Mie05]), combining the stability condition (S) with the previously proved momentum balance (2.6) to deduce that the converse of inequality (6.27) holds at almost all $t\in(0,T)$. We refer to the proof of [DMS14, Thm. 6] for all details.

Step 4: conclusion of the proof. It follows from Steps 1–3 that the triple (u,e,p) complies with the kinematic admissibility and the global stability conditions, as well as with the energy balance, at every $t\in S$, with $S\subset[0,T]$ a set of full measure containing 0. We are then in the position to apply Thm. 5.5 and conclude that (u,e,p) is a global energetic solution to the perfectly plastic system with the enhanced time regularity (6.5).

We also conclude enhanced convergences for the sequences (u_{ε_k}) and (e_{ε_k}) by observing that

$$\limsup_{k\to\infty}\int_a^b (\text{l.h.s. of (6.2d)})\,\mathrm{d}t \leq \limsup_{k\to\infty}\int_a^b (\text{r.h.s. of (6.2d)})\,\mathrm{d}t \overset{(1)}{=} \int_a^b (\text{r.h.s. of (E)})\,\mathrm{d}t \overset{(2)}{=} \int_a^b (\text{l.h.s. of (E)})\,\mathrm{d}t$$

where (1) follows from the limit passage arguments in Step 2 and (2) from the energy balance (E). Arguing in the very same way as in the proof of Lemma 3.6 and Thm. 1, we conclude that for almost all $t\in(0,T)$

$$\begin{aligned} &\lim_{k\to\infty}\int_a^b \frac{\rho{\varepsilon_k}^2}{2}\int_\Omega |\dot u_{\varepsilon_k}(t)|^2\,\mathrm{d}x\,\mathrm{d}t = 0 && \text{whence} \quad \varepsilon_k \dot u_{\varepsilon_k}(t)\to 0 \text{ in } L^2(\Omega;\mathbb{R}^d),\\ &\lim_{k\to\infty}\int_a^b \varepsilon_k \int_0^t\int_\Omega \mathbb{D}\dot e_{\varepsilon_k}:\dot e_{\varepsilon_k}\,\mathrm{d}x\,\mathrm{d}r\,\mathrm{d}t = 0 && \text{whence} \quad \varepsilon_k^{1/2}\dot e_{\varepsilon_k}\to 0 \text{ in } L^2(0,t;L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})),\\ &\lim_{k\to\infty}\int_a^b \varepsilon_k \int_0^t\int_\Omega |\dot p_{\varepsilon_k}|^2\,\mathrm{d}x\,\mathrm{d}r\,\mathrm{d}t = 0 && \text{whence} \quad \varepsilon_k^{1/2}\dot p_{\varepsilon_k}\to 0 \text{ in } L^2(0,t;L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{D}})),\end{aligned} \tag{6.28a}$$

as well as the convergence

$$\int_a^b \mathcal{Q}(e_{\varepsilon_k}(t))\,\mathrm{d}t \to \int_a^b \mathcal{Q}(e(t))\,\mathrm{d}t \text{ whence } e_{\varepsilon_k}(t)\to e(t) \text{ in } L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}}) \text{ for a.a. } t\in(0,T). \tag{6.28b}$$

We use (6.28) to conclude (6.4c). With the very same arguments as in the proof of [DMS14, Thm. 6] we also infer the pointwise convergence (6.4b).

Furthermore, exploiting (6.28), the weak convergence (6.4a) for $(\vartheta_{\varepsilon_k})$, and the arguments from Step 2, we pass to the limit in the (rescaled) total energy balance (6.2e), integrated on an arbitrary interval $(a,b)\subset(0,T)$. We thus have

$$\lim_{k\to\infty}\int_a^b \left(\tfrac{\rho\varepsilon_k^2}{2}\int_\Omega |\dot u_{\varepsilon_k}|^2\,\mathrm{d}x{+}\mathcal{E}(\vartheta_{\varepsilon_k}(t),e_{\varepsilon_k}(t))\right)\mathrm{d}t = \int_a^b \mathcal{E}(\vartheta(t),e(t))\,\mathrm{d}t, \tag{6.29}$$

whereas, also taking into account (6.3a) and (6.3f), arguing as in Step 2 we find that

$$\begin{aligned}\lim_{k\to\infty}\int_a^b (\text{r.h.s. of (6.2e)})\,\mathrm{d}t = \int_a^b \Big(\mathcal{E}(\vartheta_0,e_0) - \int_\Omega \varrho(0):(e_0{-}E(w(0)))\,\mathrm{d}x + \int_0^t\int_\Omega \sigma : E(\dot w)\,\mathrm{d}x\,\mathrm{d}s \\ + \int_\Omega \varrho(t):(e(t){-}E(w(t)))\,\mathrm{d}x - \int_0^t\int_\Omega \dot\varrho:(e{-}E(w))\,\mathrm{d}x\,\mathrm{d}s\Big)\,\mathrm{d}t\,.\end{aligned} \tag{6.30}$$

Combining (6.29) and (6.30) and using the arbitrariness of the interval (a, b), we conclude the energy balance (6.6). A comparison between (6.6) and (E) yields (6.7). This concludes the proof of Theorem 3. ■

Appendix A. Auxiliary compactness results

The proof of Theorem 4.5, and the argument in Step 1 of the proof of Thm. 3, hinge on a compactness argument drawn from the theory of *parameterized* (or *Young*) measures with values in an infinite-dimensional space. Hence, for the reader's convenience, we preliminarily collect here the definition of Young measure with values in a reflexive Banach space $\mathbf{X}$. We then recall the Young measure compactness result from [MRS13], extending to the frame of the weak topology classical results within Young measure theory (see e.g. [Bal84, Thm. 1], [Val90, Thm. 16]).

Preliminarily, let us fix some notation: We denote by $\mathscr{L}_{(0,T)}$ the σ-algebra of the Lebesgue measurable subsets of the interval $(0, T)$ and, given a reflexive Banach space $\mathbf{X}$, by $\mathscr{B}(\mathbf{X})$ its Borel σ-algebra.

Definition A.1 ((Time-dependent) Young measures). *A* Young measure *in the space* $\mathbf{X}$ *is a family* $\boldsymbol{\mu} := \{\mu_t\}_{t\in(0,T)}$ *of Borel probability measures on* $\mathbf{X}$ *such that the map on* $(0, T)$

$$t \mapsto \mu_t(A) \quad \text{is} \quad \mathscr{L}_{(0,T)}\text{-measurable} \quad \text{for all } A \in \mathscr{B}(\mathbf{X}). \tag{A.1}$$

We denote by $\mathscr{Y}(0, T; \mathbf{X})$ *the set of all Young measures in* $\mathbf{X}$.

The following result subsumes only part of the statements of [MRS13, Theorems A.2, A.3]. We have in fact extrapolated the crucial finding of these results for the purposes of Theorem 4.5, and also for the proof of Thm. 3. They concern the characterization of the limit points in the weak topology of $L^p(0, T; \mathbf{X})$, $p \in (1, +\infty]$, of a bounded sequence $(\ell_n)_n \subset L^p(0, T; \mathbf{X})$. Every limit point arises as the barycenter of the limiting Young measure $\boldsymbol{\mu} = (\mu_t)_{t\in(0,T)}$ associated with (a suitable subsequence $(\ell_{n_k})_k$ of) $(\ell_n)_n$. In turn, for almost all $t \in (0, T)$ the support of the measure μ_t is concentrated in the set of limit points of $(\ell_{n_k}(t))_k$ with respect to the weak topology of $\mathbf{X}$.

Theorem A.2. [MRS13, Theorems A.2, A.3] *Let* $p > 1$ *and let* $(\ell_n)_n \subset L^p(0, T; \mathbf{X})$ *be a bounded sequence. Then, there exist a subsequence* $(\ell_{n_k})_k$ *and a Young measure* $\boldsymbol{\mu} = \{\mu_t\}_{t\in(0,T)} \in \mathscr{Y}(0, T; \mathbf{X})$ *such that for a.a.* $t \in (0, T)$

$$\mu_t \text{ is concentrated on the set } \boldsymbol{L}_t := \bigcap_{p=1}^{\infty} \overline{\{\ell_{n_k}(t) \,:\, k \geq p\}}^{\text{weak-}\mathbf{X}} \tag{A.2}$$

of the limit points of the sequence $(\ell_{n_k}(t))$ *with respect to the weak topology of* $\mathbf{X}$ *and, setting*

$$\ell(t) := \int_{\mathbf{X}} l \,\mathrm{d}\mu_t(l) \qquad \text{for a.a. } t \in (0, T)\,,$$

there holds

$$\ell_{n_k} \rightharpoonup \ell \quad \text{in } L^p(0, T; \mathbf{X}) \qquad \text{as } k \to \infty \tag{A.3}$$

with $\rightharpoonup$ *replaced by* $\overset{*}{\rightharpoonup}$ *if* $p = \infty$.

Furthermore, if $\mu_t = \delta_{\ell(t)}$ *for almost all* $t \in (0, T)$*, then, up to the extraction of a further subsequence,*

$$\ell_{n_k}(t) \rightharpoonup \ell(t) \quad \text{in } \mathbf{X} \quad \text{for a.a. } t \in (0, T). \tag{A.4}$$

We are now in the position to develop the **proof of** (4.37) **in Theorem 4.5** (recall that the other items in the statement have been proved in [RR15, Thm. A.5]). Following the outline developed in [RR15] for Thm. A.5 therein, we split the argument in some steps.

Claim 1: *Let* $F \subset \overline{B}_{1,\mathbf{Y}}(0)$ *be countable and dense in* $\overline{B}_{1,\mathbf{Y}}(0)$*. There exist a subsequence* $(\ell_{n_k})_k$ *of* $(\ell_n)_n$*, a negligible set* $\bar{J} \subset (0, T)$*, and for every* $\varphi \in F$ *a function* $\mathscr{L}_\varphi : [0, T] \to \mathbb{R}$ *such that the following convergences hold as* $k \to \infty$ *for every* $\varphi \in F$*:*

$$\langle \ell_{n_k}(t), \varphi \rangle_{\mathbf{Y}} \to \mathscr{L}_\varphi(t) \quad \text{for every } t \in [0, T], \tag{A.5}$$

$$\langle \ell_{n_k}(t_k), \varphi \rangle_{\mathbf{Y}} \to \mathscr{L}_\varphi(t) \quad \text{for every } t \in [0, T] \backslash \bar{J} \text{ and for every } (t_k)_k \subset [0, T] \text{ with } t_k \to t. \tag{A.6}$$

Convergence (A.5) was already obtained in the proof of [RR15, Thm. A.5], therefore we will only focus on the proof of (A.6). With every $\varphi \in \overline{B}_{1,\mathbf{Y}}(0)$ we associate the monotone functions $\mathcal{V}_n^\varphi : [0, T] \to [0, +\infty)$ defined by $\mathcal{V}_n^\varphi(t) := \mathrm{Var}(\langle \ell_n, \varphi \rangle_{\mathbf{Y}}; [0, t])$ for every $t \in [0, T]$. Let now $F \subset \overline{B}_{1,\mathbf{Y}}(0)$ be countable and dense and let us consider the family of functions $(\mathcal{V}_n^\varphi)_{n\in\mathbb{N},\,\varphi\in F}$ and the associated distributional derivatives $(\nu_n^\varphi)_{n\in\mathbb{N},\,\varphi\in F}$, in fact

Radon measures on $[0,T]$. It follows from estimate (4.33), combined with a diagonalization procedure based on the countability of F, that there exist a sequence of indexes $(n_k)_k$ and for every $\varphi \in F$ a Radon measure ν_∞^φ, such that $\nu_{n_k}^\varphi \stackrel{*}{\rightharpoonup} \nu_\infty^\varphi$ as $k \to \infty$. Set $\mathcal{V}_\infty^\varphi(t) := \nu_\infty^\varphi([0,t]$ for every $t \in [0,T]$. Since the function $\mathcal{V}_\infty^\varphi$ is monotone, it has an at most countable jump set (i.e., the set of atoms of the measure ν_∞^φ), which we denote by J_φ. The set $\bar{J} := \cup_{\varphi\in F} J_\varphi$ is still countable.

In order to show that (A.6) holds, let us fix $\varphi \in F$. The sequence $(\langle \ell_{n_k}(t_k), \varphi\rangle_{\mathbf{Y}})_k$ is bounded for every $\varphi \in F$ and therefore it admits a subsequence (not relabeled, possibly depending on φ), converging to some $\bar{\ell}_\varphi \in \mathbb{R}$. Observe that

$$
\begin{aligned}
|\bar{\ell}_\varphi - \mathscr{L}_\varphi(t)| = \lim_{k\to\infty} | \langle \ell_{n_k}(t_k), \varphi\rangle_{\mathbf{Y}} - \langle \ell_{n_k}(t), \varphi\rangle_{\mathbf{Y}} | &\stackrel{(1)}{\leq} \limsup_{k\to\infty} \mathrm{Var}(\langle \ell_{n_k}, \varphi\rangle_{\mathbf{Y}}; [t,t_k]) \\
&= \limsup_{k\to\infty} \nu_{n_k}^\varphi([t,t_k]) \stackrel{(2)}{\leq} \nu_\infty^\varphi(\{t\}) \stackrel{(3)}{=} 0,
\end{aligned}
$$

where (1) follows from supposing (without loss of generality) that $t \leq t_k$ for k sufficiently big, (2) from the upper semicontinuity property of weak* convergence of measures, and (3) from the fact that $t \notin \bar{J}$ is not an atom for the measure ν_∞^φ. Therefore $\bar{\ell}_\varphi = \mathscr{L}_\varphi(t)$ and, a fortiori, one has convergence (A.6) along the *whole* sequence of indexes $(n_k)_k$.

Claim 2: *Let $(\ell_{n_k})_k$ be a (not relabeled) subsequence of the sequence from Claim* 1*, with which a limiting Young measure $\boldsymbol{\mu} = \{\mu_t\}_{t\in(0,T)} \in \mathscr{Y}(0,T;\mathbf{V})$ is associated according to Theorem A.2. Then, there exists a negligible set $N \subset (0,T)$ such that for every $t \in (0,T)\setminus N$ the probability measure μ_t is a Dirac mass $\delta_{\ell(t)}$, with $\ell(t) \in \mathbf{V}$ fulfilling*

$$\langle \ell(t), \varphi\rangle_{\mathbf{Y}} = \mathscr{L}_\varphi(t) \qquad \text{for every } \varphi \in F, \tag{A.7}$$

and (4.36) *holds as $k \to \infty$.*

We refer to the proof of [RR15, Thm. A.5] for this Claim.

Claim 3: *Set $J := N\cup\bar{J}$. For every $t \in [0,T]\setminus J$ and for every $(t_k)_k \subset [0,T]$ with $t_k \to t$ there holds $\ell_{n_k}(t_k) \rightharpoonup \ell(t)$ in $\mathbf{Y}^*$.*

Indeed, the sequence $(\ell_{n_k}(t_k))_k$ is bounded in $\mathbf{Y}^*$, and therefore it admits a (not relabeled) subsequence weakly converging in $\mathbf{Y}^*$ to some $\bar{\ell}$. It follows from (A.6) and (A.7) that $\langle \bar{\ell}, \varphi\rangle_{\mathbf{Y}} = \mathscr{L}_\varphi(t) = \langle \ell(t), \varphi\rangle_{\mathbf{Y}}$ for every $\varphi \in F$. Since F is dense in $\overline{B}_{1,\mathbf{Y}}(0)$, we then conclude that $\bar{\ell}$ and $\ell(t)$ coincide on all the elements in $\overline{B}_{1,\mathbf{Y}}(0)$. Hence $\bar{\ell} = \ell(t)$ in $\mathbf{Y}^*$ and the desired claim follows. This concludes the proof of (4.37). ∎

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