ON A CONJECTURE OF CHEEGER

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ABSTRACT. This note details how a recent structure theorem for normal 1-currents proved by the first and third author allows to prove a conjecture of Cheeger concerning the structure of Lipschitz differentiability spaces. More precisely, we show that the push-forward of the measure from a Lipschitz differentiability space under a chart is absolutely continuous with respect to Lebesgue measure.

KEYWORDS: Lipschitz differentiability space, Cheeger's conjecture, Alberti representation, metric measure space.

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1. INTRODUCTION

In [Che99] Cheeger proved that in every doubling metric measure space (X, ρ, μ) satisfying a Poincaré inequality, Lipschitz functions are differentiable μ-almost everywhere. More precisely, he showed the existence of a family $\{(U_i, \varphi_i)\}_{i\in\mathbb{N}}$ of Borel charts (that is, $U_i \subset X$ is a Borel set, $X = \bigcup_i U_i$ up to a μ-negligible set, and $\varphi_i \colon X \to \mathbb{R}^{d(i)}$ is Lipschitz) such that for every Lipschitz map $f\colon X \to \mathbb{R}$ at μ-almost every $x_0 \in U_i$ there exists a unique (co-)vector $df(x_0) \in \mathbb{R}^{d(i)}$ with

$$\limsup_{x\to x_0} \frac{|f(x) - f(x_0) - df(x_0)\cdot(\varphi(x) - \varphi(x_0))|}{\rho(x, x_0)} = 0.$$

This fact was later axiomatized by Keith [Kei04], leading to the notion of *Lipschitz differentiability space*, see Section 2 below.

Cheeger also conjectured that the push-forward of the reference measure μ under every chart φ_i has to be absolutely continuous with respect to the Lebesgue measure, that is,

$$(\varphi_i)_\#(\mu \llcorner U_i) \ll \mathcal{L}^{d(i)}\,,$$

see [Che99, Conjecture 4.63]. Some consequences of this fact concerning existence of bi-Lipschitz embeddings of X into some $\mathbb{R}^N$ are detailed in [Che99, Section 14], also see [CK06, CK09]

Let us assume that $(X, \rho, \mu) = (\mathbb{R}^d, \rho_{\mathcal{E}}, \nu)$ with $\rho_{\mathcal{E}}$ the Euclidean distance and ν a positive Radon measure, is a Lipschitz differentiability space when equipped with the (single) identity chart (note that it follows a-posteriori from the validity of Cheeger's conjecture that no mapping into a higher-dimensional space can be a chart in a Lipschitz differentiability structure of $\mathbb{R}^d$). In this case the validity of Cheeger's conjecture reduces to the validity of the (weak) converse of Rademacher's theorem, which states that a positive Radon measure

ν on $\mathbb{R}^d$ with the property that all Lipschitz functions are differentiable ν-almost everywhere must be absolutely continuous with respect to $\mathcal{L}^d$. Actually, it is well known to experts that this converse of Rademacher's theorem implies Cheeger's conjecture in any metric space, see for instance [Kei04, Section 2.4], [Bat15, Remark 6.11], and [Gon12].

The (strong) converse of Rademacher's theorem has been known to be true in $\mathbb{R}$ since the work of Zahorski [Zah46], where he characterized the sets $E \subset \mathbb{R}$ that are sets of non-differentiability points of some Lipschitz function. In particular, he proved that for every Lebesgue negligible set $E \subset \mathbb{R}$ there exists a Lipschitz function which is nowhere differentiable on E.

The same result for maps $f\colon \mathbb{R}^d \to \mathbb{R}^d$ has been proved by Alberti, Csörnyei & Preiss for $d = 2$ as a consequence of a deep structural result for negligible sets in the plane [ACP05, ACP10]. In 2011, Csörnyei & Jones [Jon11] announced the extension of the above result to every Euclidean space. For Lipschitz maps $f\colon \mathbb{R}^d \to \mathbb{R}^m$ with $m < d$ the situation is fundamentally different and there exists a null set such that every Lipschitz function is differentiable at at least one point from that set, see [Pre90, PS15]. We finally remark that the weak converse of Rademacher's theorem in $\mathbb{R}^2$ can also be obtained by combining the results of [Alb93] and [AM16], see [AM16, Remark 6.2 (iv)].

Recently, a result concerning the singular structure of measures satisfying a differential constraint was proved in [DR16]. When combined with the main result of [AM16], this proves the weak converse of Rademacher's theorem in any dimension, see [DR16, Theorem 1.14].

In this note we detail how the results in [AM16, DR16] in conjunction with Bate's result on the existence of a sufficient number of independent Alberti representations in a Lipschitz differentiability space [Bat15] imply Cheeger's conjecture; see Section 2 for the relevant definitions.

Theorem 1.1. *Let (X, ρ, μ) be a Lipschitz differentiability space and let (U, φ) be a d-dimensional chart. Then, $\varphi_\#(\mu \llcorner U) \ll \mathcal{L}^d$.*

Note that by the same arguments of this paper Cheeger's conjecture would also follow from the results announced in [ACP05] and [Jon11].

After we finished writing this note we learned that similar results have been proved by Kell and Mondino [KM16] and by Gigli and Pasqualetto [GP16].

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2. Setup

2.1. **Lipschitz differentiability spaces.** In the sequel, the triple (X, ρ, μ) will always denote a *metric measure space*, that is, (X, ρ) is a separable, complete metric space and $\mu \in \mathcal{M}_+(X)$ is a positive Radon measure on X.

We call a pair (U, φ) such that $U \subset X$ is a Borel set and $\varphi \colon X \to \mathbb{R}^d$ is Lipschitz, a *d-dimensional chart*, or simply a *d-chart.* A function $f \colon X \to \mathbb{R}$ is said to be *differentiable with respect to a d-chart* (U, φ) at $x_0 \in U$ if there exists a unique (co-)vector $df(x_0) \in \mathbb{R}^d$ such that

$$\limsup_{x \to x_0} \frac{|f(x) - f(x_0) - df(x_0) \cdot (\varphi(x) - \varphi(x_0))|}{\rho(x, x_0)} = 0.$$

We call a metric measure space (X, ρ, μ) a *Lipschitz differentiability space* (also called a metric measure space that admits a *measurable differentiable structure*) if there exists a countable family of $d(i)$-charts (U_i, φ_i) $(i \in \mathbb{N})$ such that $X = \bigcup_i U_i$ and any Lipschitz map $f \colon X \to \mathbb{R}$ is differentiable with respect to every (U_i, φ_i) at μ-almost every point $x_0 \in U_i$.

2.2. **Alberti representations.** We denote by $\Gamma(X)$ the set of *curves* in X, that is, the set of all Lipschitz maps $\gamma \colon \operatorname{Dom} \gamma \to X$, for which the domain $\operatorname{Dom} \gamma \subset \mathbb{R}$ is non-empty and compact. Note that we are not requiring $\operatorname{Dom} \gamma$ to be an interval and thus the set $\Gamma(X)$ is sometimes also called the set of *curve fragments* on X. We equip $\Gamma(X)$ with the Hausdorff metric $\operatorname{dist}_{\mathcal{H}}$ on graphs and we consider it as a subspace of the Polish space

$$\mathcal{K} = \left\{ K \subset \mathbb{R} \times X \ : \ K \text{ compact} \right\}, \tag{2.1}$$

endowed with the Hausdorff metric. Moreover, by arguing as in [Sch16, Lemma 2.20], it is easy to see that $\Gamma(X)$ is an F_σ-subset of $\mathcal{K}$, i.e. a countable union of closed sets.

The decomposition of a measure into a family of 1-dimensional Hausdorff measures supported on curves leads to the notion of Alberti representation. First introduced in [Alb93] for the study of the rank-one property of BV-derivatives, this decomposition has turned out to be a key tool in the study of differentiability properties of Lipschitz functions, see for instance [ACP05, ACP10, AM16, Bat15].

Definition 2.1. *Let (X, ρ, μ) be a metric measure space. An* Alberti representation *of μ on a μ-measurable set $A \subset X$ is a parametrized family $(\mu_\gamma)_{\gamma \in \Gamma(X)}$ of positive Borel measures $\mu_\gamma \in \mathcal{M}_+(X)$ with*

$$\mu_\gamma \ll \mathcal{H}^1 \llcorner \operatorname{Im} \gamma,$$

together with a Borel probability measure $\pi \in \mathcal{P}(\Gamma(X))$ such that

$$\mu(B) = \int \mu_\gamma(B) \, \mathrm{d}\pi(\gamma) \qquad \text{for all Borel sets } B \subset A. \tag{2.2}$$

Here, the measurability of the integrand is part of the requirement of being an Alberti representation

Remark 2.2. Note that this definition is slightly different from the one in [Bat15, Definition 2.2] since there the set $\Gamma(X)$ consist of *bi-Lipschitz* curves. Clearly, the existence of a representation in the sense of [Bat15] implies the existence of a representation in our sense and this will suffice for our purposes. Let us, however, point out that the converse holds true as well. Indeed, the part of γ that contributes to the integral in (2.2) can be decomposed into countably many bi-Lipschitz pieces, see [Sch16, Remark 2.17].

We will further need the notion of *independent* Alberti-representations of a measure. Let $C \subset \mathbb{R}^d$ be a closed, convex, one-sided cone, i.e. a set of the form

$$C := \left\{\, v \in \mathbb{R}^d \;\; \textbf{:} \;\; v \cdot w \geq (1-\theta)\|v\| \,\right\}$$

for some $w \in \mathbb{S}^{d-1}$ and $\theta \in (0,1)$. With a Lipschitz map $\varphi \colon X \to \mathbb{R}^d$, we say that an Alberti representation $\int \nu_\gamma \,\mathrm{d}\pi(\gamma)$ has *φ-directions in C* if

$$(\varphi \circ \gamma)'(t) \in C \setminus \{0\} \qquad \text{for } \pi\text{-a.e. curve } \gamma \text{ and } \mathcal{H}^1\text{-a.e. } t \in \operatorname{Dom}\gamma.$$

A number of m Alberti representations of μ are *φ-independent* if there are linearly independent cones $C_1, \ldots, C_m$ such that the i'th Alberti representation has φ-directions in C_i. Here, linear independence of the cones $C_1, \ldots, C_m$ means that any collection of vectors $v_i \in C_i \setminus \{0\}$ is linearly independent. In the case $X = \mathbb{R}^d$ we will always consider $\varphi = \mathrm{Id}$.

One of the main results of [Bat15] asserts that a Lipschitz differentiability space necessarily admits many independent Alberti representations, also cf. [AM16, Theorem 1.1]. Recall that according to Remark 2.2 any representation in the sense of [Bat15] is also a representation in the sense of Definition 2.1.

Theorem 2.3. *Let (X, ρ, μ) be a Lipschitz differentiability space with a d-chart (U, φ). Then, there exists a countable decomposition*

$$U = \bigcup_{k \in \mathbb{N}} U_k, \qquad U_k \subset U \text{ Borel sets,}$$

such that every $\mu \llcorner U_k$ has d φ-independent Alberti representations.

A proof of this theorem can be found in [Bat15, Theorem 6.6].

2.3. **One-dimensional currents.** In order to use the results of [DR16] we need a link between Alberti representation and 1-dimensional currents. Recall that a 1-*dimensional current* T in $\mathbb{R}^d$ is a continuous linear functional on the space of smooth and compactly supported differential 1-forms on $\mathbb{R}^d$. The *boundary* of T, ∂T is the distribution (0-current) defined via $\langle \partial T, f \rangle := \langle T, df \rangle$ for every smooth and compactly supported function $f \colon \mathbb{R}^d \to \mathbb{R}$. The *mass* of T, denoted by $\mathbf{M}(T)$, is the supremum of $\langle T, \omega \rangle$ over all 1-forms ω such that $|\omega| \leq 1$ everywhere. In particular, finite-mass currents can be naturally identified with $\mathbb{R}^d$-valued Radon measures. A current T is called *normal* if both T and ∂T have finite mass; we denote the set of normal 1-currents by $\mathbf{N}_1(\mathbb{R}^d)$.

By the Radon–Nikodým theorem, a 1-dimensional current T with finite mass can be written in the form $T = \vec{T}\|T\|$ where $\|T\|$ is a finite positive measure and $\vec{T}$ is a vector field in $\mathrm{L}^1(\mathbb{R}^d, \|T\|)$ with $|\vec{T}(x)| = 1$ for $\|T\|$-almost every $x \in \mathbb{R}^d$. In particular, the action of T on a smooth and compactly supported 1-form ω is given by

$$\langle T, \omega \rangle = \int_{\mathbb{R}^d} \langle \omega(x), \vec{T}(x) \rangle \;\mathrm{d}\|T\|(x) \; .$$

An *integer-multiplicity rectifiable* 1-*current* (in the following called simply rectifiable 1-current) $T = [\![E, \tau, m]\!]$ is a 1-current which acts on 1-forms ω as

$$\langle T, \omega\rangle = \int_E \langle \omega(x), \tau(x)\rangle \, m(x) \, \mathrm{d}\mathcal{H}^1(x) \ ,$$

where E is a 1-rectifiable set, $\tau(x)$ is a unit vector spanning the approximate tangent space $\mathrm{Tan}(E, x)$ and m is an integer-valued function such that $\int_E m \, \mathrm{d}\mathcal{H}^1 < \infty$. More information on currents can be found in [Fed69].

The relation between Alberti representations and normal 1-currents is partially encoded in the following decomposition theorem, due to Smirnov [Smi93].

Theorem 2.4. *Let $T = \vec{T}\|T\| \in \mathbf{N}_1(\mathbb{R}^d)$ be a normal 1-current with $|\vec{T}(x)| = 1$ for $\|T\|$-almost every x. Then, there exists a family of rectifiable 1-currents*

$$T_\gamma = [\![E_\gamma, \tau_\gamma, 1]\!], \qquad \gamma \in \Gamma,$$

where Γ is a measure space endowed with a finite positive Borel measure $\pi \in \mathcal{M}_+(\Gamma)$, such that the following assertions hold:

(i) T can be decomposed as

$$T = \int_\Gamma T_\gamma \, \mathrm{d}\pi(\gamma)$$

and

$$\mathbf{M}(T) = \int_\Gamma \mathbf{M}(T_\gamma) \, \mathrm{d}\pi(\gamma) = \int_\Gamma \mathcal{H}^1(E_\gamma) \, \mathrm{d}\pi(\gamma) \ ;$$

(ii) $\tau_\gamma(x) = \vec{T}(x)$ for $\mathcal{H}^1$-almost every $x \in E_\gamma$ and for π-almost every $\gamma \in \Gamma$;

(iii) $\|T\|$ can be decomposed as

$$\|T\| = \int_\Gamma \mu_\gamma \, \mathrm{d}\pi(\gamma) \ ,$$

where each μ_γ is the restriction of $\mathcal{H}^1$ to the 1-rectifiable set E_γ.

An Alberti representation of a Euclidean measure splits it into measures concentrated on "fragments" of curves. In general, these fragments cannot be glued together to obtain a 1-dimensional normal current since the boundary may have infinite mass. Nevertheless, the "holes" of every curve appearing in an Alberti representation of a measure $\nu \in \mathcal{M}_+(\mathbb{R}^d)$ can be "filled" in such a way as to produce a normal 1-current T with $\nu \ll \|T\|$. Moreover, if the representation has directions in a cone C, then the constructed normal current T has orienting vector $\vec{T}$ in $C \setminus \{0\}$ almost everywhere (with respect to $\|T\|$). Indeed, we have the following lemma, which is essentially [AM16, Corollary 6.5]; it can be interpreted as a partial converse to Theorem 2.4:

Lemma 2.5. *Let $\nu \in \mathcal{M}_+(\mathbb{R}^d)$ be a finite Radon measure. If there is an Alberti representation $\nu = \int \nu_\gamma \, \mathrm{d}\pi(\gamma)$ with directions in a cone C, then there exists a normal 1-current $T \in \mathbf{N}_1(\mathbb{R}^d)$ such that $\vec{T}(x) \in C\setminus\{0\}$ for $\|T\|$-almost every $x \in \mathbb{R}^d$ and $\nu \ll \|T\|$.*

Proof. For the purpose of illustration we sketch the proof.

Step 1. Given ν as in the statement, we claim that there exists a normal 1-current $T = \vec{T}\|T\|$ with $\mathbf{M}(T) \le 1$ and $\mathbf{M}(\partial T) \le 2$ such that $\vec{T}(x) \in C$, for $\|T\|$-almost every x and that ν is not singular with respect to $\|T\|$.

The claim follows from the proof of [AM16, Lemma 6.12]. For the sake of completeness let us present the main line of reasoning. By arguing as in Step 1 of the proof of [AM16, Lemma 6.12], to every $\gamma \in \Gamma(\mathbb{R}^d)$ with $\gamma'(t) \in C$ and a Borel measure $\nu_\gamma \ll \mathcal{H}^1 \llcorner \operatorname{Im}\gamma$, we can associate a 1-Lipschitz map $\psi_{\nu_\gamma} \colon [0,1] \to \mathbb{R}^d$ satisfying

$$\nu_\gamma(\operatorname{Im}(\psi_{\nu_\gamma})) > 0 \qquad \text{and} \qquad \psi'_{\nu_\gamma}(t) \in C \setminus \{0\} \quad \text{for } \mathcal{H}^1\text{-a.e. } t \in [0,1].$$

This map can moreover be chosen such that $\gamma \mapsto \psi_{\nu_\gamma}$ coincides with a Borel measurable map π-almost everywhere once we endow the set of curves with the topology of uniform convergence, see Step 3 in the proof of [AM16, Lemma 6.12].

Let $T_{\nu_\gamma} := [\![\operatorname{Im}\psi_{\nu_\gamma}, \tau_{\psi_{\nu_\gamma}}, 1]\!]$ be the rectifiable 1-current associated to ψ_{ν_γ} and set

$$T := \int T_{\nu_\gamma} \, \mathrm{d}\pi(\gamma) \, .$$

Since ψ_{ν_γ} is 1-Lipschitz, $\mathcal{H}^1(\operatorname{Im}\psi_{\nu_\gamma}) \le 1$ and thus $\mathbf{M}(T) \le 1$. Moreover, for all smooth compactly supported functions $f \colon \mathbb{R}^d \to \mathbb{R}$ we have

$$\langle \partial T, f \rangle = \langle T, df \rangle = \int f(\psi_{\nu_\gamma}(1)) - f(\psi_{\nu_\gamma}(0)) \, \mathrm{d}\pi(\gamma) \, ,$$

so that $\mathbf{M}(\partial T) \le 2$.

By assumption, $\vec{T}(x) \in C \setminus \{0\}$ for $\|T\|$-almost every $x \in \mathbb{R}^d$. To show that $\|T\|$ and ν are not mutually singular, for π-almost every γ set

$$\nu'_\gamma := \nu_\gamma \llcorner \operatorname{Im}\, \psi_{\nu_\gamma} \qquad \text{and} \qquad \nu' := \int \nu'_\gamma \, \mathrm{d}\pi(\gamma) \, ,$$

so that $\nu' \neq 0$ and $\nu' \le \nu$. We will now establish that $\nu' \ll \|T\|$, for which we will prove that ν and $\|T\|$ are not mutually singular. Let $E \subset \mathbb{R}^d$ be such that $\|T\|(E) = 0$. Using

$$T = \int [\![\operatorname{Im}\psi_{\nu_\gamma}, \tau_{\psi_{\nu_\gamma}}, 1]\!] \, \mathrm{d}\pi(\gamma) \qquad \text{with} \qquad \tau_{\psi_{\nu_\gamma}} = \frac{\psi'_{\nu_\gamma}}{|\psi'_{\nu_\gamma}|} \in C \, ,$$

we get

$$\mathcal{H}^1(\operatorname{Im}\psi_{\nu_\gamma} \cap E) = 0 \qquad \text{for } \pi\text{-a.e. } \gamma.$$

Since by definition $\nu_\gamma \ll \mathcal{H}^1 \llcorner \operatorname{Im}\gamma$, we have that $\nu'_\gamma \ll \mathcal{H}^1 \llcorner \operatorname{Im}\psi_{\nu_\gamma}$. Thus, $\nu'(E) = 0$.

Step 2. Let us define

$$\mathcal{T} := \left\{ T \in \mathbf{N}_1(\mathbb{R}^d) \ \mathbf{:}\ \mathbf{M}(T) \le 1, \, \mathbf{M}(\partial T) \le 2 \text{ and } \vec{T} \in C \ \|T\|\text{-a.e.} \right\}$$

and

$$\mathcal{T}_\nu := \left\{ T \in \mathcal{T} \ \mathbf{:}\ \nu \text{ and } T \text{ are not singular} \right\} .$$

Note that if $C = \{ v \in \mathbb{R}^d \,:\, v \cdot w \geq (1-\theta)\|v\| \}$ for some $w \in \mathbb{S}^{d-1}$, $\theta \in (0,1)$, then $\vec{T} \in C$ almost everywhere implies that

$$\|T\| \geq T \cdot w \geq (1-\theta)\|T\| \tag{2.3}$$

as measures (here we are identifying T with an $\mathbb{R}^d$-valued Radon measure and use the pointwise scalar product). Moreover, as a consequence of the Radon–Nikodým theorem, for every $T \in \mathcal{T}_\nu$ we may write

$$\nu = g_{\|T\|}\|T\| + \nu^s_{\|T\|} \qquad \text{with} \qquad \nu^s_{\|T\|} \perp \|T\| \,, \int g_{\|T\|} \,\mathrm{d}\|T\| > 0 \,.$$

Let us set $M := \sup_{T \in \mathcal{T}_\nu} \int g_{\|T\|} \,\mathrm{d}\|T\| > 0$ and let $T_k \in \mathcal{T}_\nu$ be a sequence with

$$\int g_{\|T_k\|} \,\mathrm{d}\|T_k\| \to M.$$

Define

$$T := \sum_k 2^{-k} T_k$$

and note that $T \in \mathcal{T}$. Moreover, by (2.3), $\|T_k\| \ll \|T\|$ for all $k \in \mathbb{N}$, so that there exist $h_k \colon \mathbb{R}^d \to \mathbb{R}$ with

$$\int_E h_k \,\mathrm{d}\|T\| = \int_E g_{\|T_k\|} \,\mathrm{d}\|T_k\| \leq \nu(E) \qquad \text{for all Borel sets } E \subset \mathbb{R}^d.$$

In particular, $T \in \mathcal{T}_\nu$ and $h_k \leq g_{\|T\|}$. Set $m_k = \max_{1 \leq j \leq k} h_j$. By the monotone convergence theorem, $m_k \to m_\infty \leq g_{\|T\|}$ in $\mathrm{L}^1(\mathbb{R}^d, \|T\|)$ and

$$M \leq \lim_{k \to \infty} \int m_k \,\mathrm{d}\|T\| = \int m_\infty \,\mathrm{d}\|T\| \leq \int g_{\|T\|} \,\mathrm{d}\|T\| \leq M.$$

Hence, M is actually a maximum and it is attained by T.

We now claim that $\nu \ll \|T\|$. Indeed, assume by contradiction that $\nu = g_{\|T\|} \,\mathrm{d}\|T\| + \nu^s_{\|T\|}$ with $\nu^s_{\|T\|} \neq 0$. Since the Alberti representation of ν induces an Alberti representation of $\nu^s_{\|T\|}$, we can apply Step 1 to find a normal 1-current

$$S \in \mathcal{T}_{\nu^s_{\|T\|}} \subset \mathcal{T}_\nu$$

such that $\nu^s_{\|T\|}$ and $\|S\|$ are not mutually singular. In particular, if $\nu = g_{\|S\|} \,\mathrm{d}\|S\| + \nu^s_{\|S\|}$, then there exists a Borel set $F \subset \mathbb{R}^d$ such that

$$\|T\|(F) = 0 \qquad \text{and} \qquad \int_F g_{\|S\|} \,\mathrm{d}\|S\| > 0. \tag{2.4}$$

Let us define $W := (T+S)/2$ and note that by (2.3) it holds that $\|T\|, \|S\| \ll \|W\|$ so that $W \in \mathcal{T}_\nu$. Moreover, there are functions $h_T\,, h_S \leq g_{\|W\|}$ such that

$$\int_E h_T \,\mathrm{d}\|W\| = \int_E g_{\|T\|} \,\mathrm{d}\|T\| \,, \qquad \int_E h_S \,\mathrm{d}\|W\| = \int_E g_{\|S\|} \,\mathrm{d}\|S\|$$

for all Borel sets E. However, for F as in (2.4) we obtain

$$M \geq \int_{\mathbb{R}^d} g_{\|W\|} \,\mathrm{d}\|W\| \geq \int_{\mathbb{R}^d} g_{\|T\|} \,\mathrm{d}\|T\| + \int_F g_{\|S\|} \,\mathrm{d}\|S\| > M,$$

a contradiction. □

3. Proof of Cheeger's conjecture

The key tool to prove Cheeger's conjecture is the following result from [DR16, Corollary 1.12]:

Theorem 3.1. *Let $T_1 = \vec{T}_1\|T_1\|, \ldots, T_d = \vec{T}_d\|T_d\| \in \mathbf{N}_1(\mathbb{R}^d)$ be 1-dimensional normal currents. Let $\nu \in \mathcal{M}_+(\mathbb{R}^d)$ be a positive Radon measure such that*

(i) $\nu \ll \|T_i\|$ for $i = 1, \ldots, d$, and

(ii) $\operatorname{span}\{\vec{T}_1(x), \ldots, \vec{T}_d(x)\} = \mathbb{R}^d$ for ν-almost every x.

Then, $\nu \ll \mathcal{L}^d$.

Combining the above result with Lemma 2.5 we immediately get the following:

Lemma 3.2. *Let $\nu \in \mathcal{M}_+(\mathbb{R}^d)$ have d independent Alberti representations. Then, $\nu \ll \mathcal{L}^d$.*

Proof. Denote by $C_1, \ldots, C_d$ independent cones such that there are d Alberti representations having directions in these cones. By Lemma 2.5 there are d normal 1-dimensional currents $T_1 = \vec{T}_1\|T_1\|, \ldots, T_d = \vec{T}_d\|T_d\| \in \mathbf{N}_1(\mathbb{R}^d)$ such that

$$\nu \ll \|T_i\| \qquad \text{for } i = 1, \ldots, d,$$

and $\vec{T}_i(x) \in C_i$ for ν-almost every $x \in \mathbb{R}^d$. By the independence of the cones,

$$\operatorname{span}\big\{\vec{T}_1(x), \ldots, \vec{T}_d(x)\big\} = \mathbb{R}^d \qquad \text{for } \nu\text{-a.e. } x \in \mathbb{R}^d.$$

This implies $\nu \ll \mathcal{L}^d$ via Theorem 3.1. □

In order to use the above result to prove Theorem 1.1 one further needs the following "push-forward lemma".

Lemma 3.3. *Let (X, ρ, μ) be a Lipschitz differentiability space with a d-chart (U, φ). If $\mu \llcorner U$ has d φ-independent Alberti representations, then also the push-forward $\varphi_\#(\mu \llcorner U) \in \mathcal{M}_+(\mathbb{R}^d)$ has d independent Alberti representations.*

Proof. It is enough to show that if there exists a representation of the form $\mu \llcorner U = \int \mu_\gamma \, d\pi(\gamma)$ with φ-directions in a cone C (i.e. such that $(\varphi \circ \gamma)'(t) \in C \setminus \{0\}$ for almost all $t \in \operatorname{Dom}\gamma$ and for π-almost every γ), then we can build an Alberti representation

$$\varphi_\#(\mu \llcorner U) = \int \nu_{\bar\gamma} \, d\bar\pi(\bar\gamma) \qquad \text{with} \qquad \bar\pi \in \mathcal{P}(\Gamma(\mathbb{R}^d)).$$

with $\bar\gamma'(t) \in C \setminus \{0\}$ for $\bar\pi$-almost every $\bar\gamma$ and almost every $t \in \operatorname{Dom}\bar\gamma$. To this end consider the map $\Phi \colon \Gamma(X) \to \Gamma(\mathbb{R}^d)$ given by $\Phi(\gamma) := \varphi \circ \gamma$ and let $\bar\pi := \Phi_\#\pi \in \mathcal{M}_+(\Gamma(\mathbb{R}^d))$. Note that, by the very definition of the push-forward measure, for $\bar\pi$-almost every $\bar\gamma$, it holds that $\bar\gamma = \varphi \circ \gamma$ for some $\gamma \in \Gamma(X)$.

By considering π as a probability measure defined on the Polish space $\mathcal{K}$ defined in (2.1), and noting that π is concentrated on $\Gamma(X)$, we can apply the disintegration theorem for measures [AGS05, Theorem 5.3.1] to show that for

$\bar\pi$-almost every $\bar\gamma$, there exists a Borel probability measure $\eta_{\bar\gamma}$ concentrated on $\Phi^{-1}(\bar\gamma)$ and such that

$$\pi(A) = \int \eta_{\bar\gamma}(A)\,\mathrm{d}\bar\pi(\bar\gamma) \qquad \text{for all Borel sets } A \subset \Gamma(X).$$

Note also that, by the disintegration theorem, the map $\bar\gamma \mapsto \eta_{\bar\gamma}$ is Borel measurable. Let us now set

$$\nu_{\bar\gamma} := \int_{\Phi^{-1}(\bar\gamma)} \varphi_\#(\mu_\gamma)\,\mathrm{d}\eta_{\bar\gamma}(\gamma).$$

Clearly, we have the representation

$$\varphi_\#(\mu \llcorner U) = \int \nu_{\bar\gamma}\,\mathrm{d}\bar\pi(\bar\gamma)$$

and $\bar\gamma'(t) = (\varphi \circ \gamma)'(t) \in C \setminus \{0\}$ for $\bar\pi$-almost every $\bar\gamma$ and almost every $t \in \operatorname{Dom}\bar\gamma$. Hence, to conclude we only have to show that

$$\nu_{\bar\gamma} \ll \mathcal{H}^1 \llcorner \operatorname{Im}\bar\gamma \qquad \text{for } \bar\pi\text{-a.e. } \bar\gamma.$$

Let E be a set with $\mathcal{H}^1(E \cap \operatorname{Im}\bar\gamma) = 0$. Since $\bar\gamma'(t) \neq 0$ for almost every $t \in \operatorname{Dom}\gamma$, the area formula implies that $\mathcal{L}^1(\bar\gamma^{-1}(E)) = 0$. If $\gamma \in \Phi^{-1}(\bar\gamma)$, say $\bar\gamma = \varphi \circ \gamma$, then

$$\mathcal{H}^1(\varphi^{-1}(E) \cap \operatorname{Im}\gamma) \le \mathcal{H}^1(\gamma(\bar\gamma^{-1}(E))) = 0 \qquad \text{for all } \gamma \in \Phi^{-1}(\bar\gamma).$$

Hence, $\mu_\gamma(\varphi^{-1}(E)) = 0$ for all $\gamma \in \Phi^{-1}(\bar\gamma)$, which immediately gives

$$\nu_{\bar\gamma}(E) = \int_{\Phi^{-1}(\bar\gamma)} \mu_\gamma(\varphi^{-1}(E))\,\mathrm{d}\eta_{\bar\gamma}(\gamma) = 0\,.$$

This concludes the proof. □

Proof of Theorem 1.1. Let (U, φ) be a d-chart. By Theorem 2.3 there are d φ-independent Alberti representations of $\mu \llcorner U_k$, where $U = \bigcup_{k\in\mathbb{N}} U_k$ is the decomposition from Bate's theorem. Then, via Lemma 3.3, the push-forward $\varphi_\#(\mu \llcorner U_k)$ also has d independent Alberti representations. Finally, Lemma 3.2 yields $\varphi_\#(\mu \llcorner U_k) \ll \mathcal{L}^d$ and this concludes the proof. □

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