THE REGULARIZATION EFFECT OF WILLMORE ENERGY

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Abstract. We consider the functional

$$E_{p,\lambda}(R) := \int_R \mathrm{dist}^p(x, \partial R)\,\mathrm{d}x + \lambda \int_{\partial R} \kappa_{\partial R}^2 \,\mathrm{d}\mathcal{H}^1_{\llcorner\partial R},$$

where $p \geq 1$, $\lambda > 0$ are given parameters, R varies among compact, convex sets of $\mathbb{R}^2$ with Hausdorff dimension equal to 2, ∂R denotes the boundary of R, $\mathrm{dist}(x, \partial R)$ is the Hausdorff distance between $\{x\}$ and ∂R, and $\kappa_{\partial R}$ denotes the (signed) curvature of ∂R. The term $\int_R \mathrm{dist}^p(x, \partial R)\,\mathrm{d}x$ quantifies the "average distance" of points (of R) to the boundary, and $\int_{\partial R} \kappa_{\partial R}^2 \,\mathrm{d}\mathcal{H}^1_{\llcorner\partial R}$ is the integrated squared curvature. We make no a priori assumptions on the regularity of the boundary ∂R, hence even existence of minimizers is unclear. The aim of this paper is to prove existence and $C^{1,1}$ regularity of minimizers of $E_{p,\lambda}$.

Keywords. average-distance problem, regularity, Willmore energy
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1. Introduction

The curvature of boundaries plays an important role in many biological models. For instance, the elasticity of cell membranes is strongly correlated to its bending, and thus to its curvature. One way to quantity the bending energy per unit area of closed lipid bilayers was proposed by Helfrich in [9], and is now commonly referred to as "Helfrich energy". A related notion, from differential geometry, is the "Willmore energy", which measures how much a surface differs from the sphere. In 2D, the Willmore energy simplifies to the integrated squared curvature.

Easy access to the boundary is also relevant: many processes such as heat dissipation, waste disposal and nutrient adsorption, are more efficient when the whole body has "easy access" to its boundary. One way to quantify the "easiness" for points of a set R to access its boundary ∂R is an energy term of the form

$$\int_R \mathrm{dist}^p(x, \partial R)\,\mathrm{d}x. \tag{1}$$

The functional (1) is formally similar to the average-distance functional

$$\Sigma \longmapsto \int_\Omega \mathrm{dist}^p(x, \Sigma)\,\mathrm{d}x,$$

where Ω is a *given* domain, and the unknown Σ varies among compact, connected sets of Ω with Hausdorff dimension equal to 1. The average distance functional is used in many modeling applications, such as urban planning and optimal pricing. For a (non-exhaustive) list of references on the average distance problem we refer to the works by Buttazzo et al. [1, 2, 3, 4, 5, 6, 7, 8]. Also related are the papers by Paolini and Stepanov [13], Santambrogio and Tilli [14], Tilli [16], Lemenant and Mainini [11], Slepčev [15], and the review paper by Lemenant [10]. . However, a crucial difference is that the domain is given in the average-distance functional, while it is a variable in the functional (1). This makes the proof of existence significantly more involved.

In this paper we consider the two-dimensional setting, with the main functional being

$$E_{p,\lambda}(R) := \int_R \operatorname{dist}^p(x, \partial R)\, \mathrm{d}x + \lambda \int_{\partial R} \kappa_{\partial R}^2 \, \mathrm{d}\mathcal{H}^1_{\llcorner \partial R}, \tag{2}$$

where $p \geq 1$, $\lambda > 0$ are given parameters, the argument R varies among compact, convex, Hausdorff two-dimensional sets of $\mathbb{R}^2$, ∂R denotes the boundary of R, and

$$\operatorname{dist}(x, \partial R) := \inf_{y \in \partial R} |x - y|.$$

The choice to work with convex sets is due to technical reasons. The term

$$\int_{\partial R} \kappa_{\partial R}^2 \, \mathrm{d}\mathcal{H}^1_{\llcorner \partial R} \tag{3}$$

is the integrated squared curvature. Since we made no a priori assumptions on the regularity of the boundary ∂R, the integrand $\kappa_{\partial R}$ may be a curvature *measure* (instead of a *function*). For future reference we will define it as follows:

$$\int_{\partial R} \kappa_{\partial R}^2 \, \mathrm{d}\mathcal{H}^1_{\llcorner \partial R} := \begin{cases} \displaystyle\int_{\partial R} \left| \frac{\mathrm{d}\kappa_{\partial R}}{\mathrm{d}\mathcal{H}^1_{\llcorner \partial R}} \right|^2 \mathrm{d}\mathcal{H}^1_{\llcorner \partial R} & \text{if } \kappa_{\partial R} \ll \mathcal{H}^1_{\llcorner \partial R}, \\ +\infty & \text{if } \kappa_{\partial R} \not\ll \mathcal{H}^1_{\llcorner \partial R}. \end{cases} \tag{4}$$

Here the notation $\frac{\mathrm{d}\kappa_{\partial R}}{\mathrm{d}\mathcal{H}^1_{\llcorner \partial R}}$ denotes the Radon-Nikodym derivative.

The choice to define it as $+\infty$ when $\kappa_{\partial R} \not\ll \mathcal{H}^1_{\llcorner \partial R}$ is due to the following argument: if $\kappa_{\partial R} \not\ll \mathcal{H}^1_{\llcorner \partial R}$, then by definition there exists a set $Q \subseteq \partial R$ such that $\mathcal{H}^1(Q) = 0$ but $|\kappa_{\partial R}(Q)| = \int_Q |\kappa_{\partial R}| \, \mathrm{d}\mathcal{H}^1_{\llcorner \partial R} > 0$. Let Q_n be a monotonically decreasing (with respect to set inclusion) sequence of sets converging to Q, that is

$$Q_1 \supseteq Q_2 \supseteq \cdots \supseteq Q_n \supseteq \cdots \supseteq \bigcap_{k=1}^{+\infty} Q_k = Q.$$

Then it follows

$$\mathcal{H}^1(Q_n) \searrow 0, \qquad |\kappa_{\partial R}(Q_n)| \searrow |\kappa_{\partial R}(Q)|,$$

and, by the monotone convergence theorem,

$$\int_Q |\kappa_{\partial R}|^2 \, \mathrm{d}\mathcal{H}^1_{\llcorner \partial R} = \lim_{n \to +\infty} \int_{Q_n} |\kappa_{\partial R}|^2 \, \mathrm{d}\mathcal{H}^1_{\llcorner \partial R} \geq \lim_{n \to +\infty} \frac{|\kappa_{\partial R}(Q_n)|^2}{\mathcal{H}^1(Q_n)} = +\infty.$$

Clearly, the functional $E_{p,\lambda}$ is invariant under rigid movements. Further details about the space of convex sets, and its topology, will be discussed in Section 2. The main result is:

Theorem 1.1. *Given $p \geq 1$, $\lambda > 0$, any minimizer R of $E_{p,\lambda}$ is $C^{1,1}$-regular with Lipschitz constant at most*

$$Y = Y(p,\lambda) := 2(2\lambda^{-1}p(D_{p,\lambda}+1)^{p-1}\pi D_{p,\lambda}^2 + K_{p,\lambda}),$$

where

$$D_{p,\lambda} := \left(24 \cdot 2^{p+2}(p+1)(p+2)(1+\pi\lambda)^2\lambda^{-1}\right)^{\frac{1}{p+1}}, \tag{5}$$

$$K_{p,\lambda} := \left(\frac{\pi/\sqrt{2}}{\lambda^{-1}+2\pi}+2\right)(\lambda^{-1}+2\pi) + \left(\frac{\pi^2}{(\lambda^{-1}+2\pi)^2}+\frac{2\sqrt{2}(\pi+1)}{\lambda^{-1}+2\pi}\right)\sqrt{\pi D_{p,\lambda}^2(\lambda^{-1}+2\pi)}, \tag{6}$$

are geometric constants independent of R. That is, the boundary ∂R admits a $C^{1,1}$-regular, arc-length parameterization $\gamma : [0, \mathcal{H}^1(\partial R)] \longrightarrow \mathbb{R}^2$ such that

$$|\gamma'(t_1) - \gamma'(t_0)| \leq Y|t_1 - t_0|$$

for any t_0, t_1.

There are essentially two main difficulties in our analysis:

1. the space of compact, convex sets of $\mathbb{R}^2$ with Hausdorff dimension equal to 2 (i.e., the space X defined in (7) below) is *not* closed with respect to the symmetric difference distance (i.e., the distance d defined in (8) below). Thus even existence of minimizers is unclear.
2. Since we made no assumption on the regularity of admissible minimizers R, we need first to prove that all potential minimizers have an uniform bound on the integrated squared curvature. Moreover, since the proof of Theorem 1.1 is done by comparing the minimizer with suitable competitors, care is required in constructing such competitors to achieve non-trivial estimates.

Issue (1) is overcome via estimates (both from above and below) on the diameter (Lemmas 2.1 and 2.3) and area (Lemma 2.2) of minimizing sequences. Issue (2) is overcome by carefully constructing competitors (for the actual construction, we refer to the proof of Theorem 1.1 in Section 3) while preserving the regularity.

2. Preliminary results

Since we are mainly considering compact, convex, Hausdorff 2-dimensional sets of $\mathbb{R}^2$, we set

$$X := \{R : R \text{ is a convex, compact, Hausdorff 2-dimensional set of } \mathbb{R}^2\}, \tag{7}$$

and endow X with the distance

$$d(R_1, R_2) := \mathcal{H}^2(R_1 \triangle R_2), \qquad \triangle := \text{ symmetric difference} \tag{8}$$

One of the key difficulties in our analysis is that X is *not* complete with respect to d, which makes unclear if $E_{p,\lambda}$ admits minimizers in X. Set

$$\bar{X} := \text{completion of } X \text{ with respect to } d.$$

We will overcome the non-completeness of X via estimates on diameter (see Lemma 2.1), perimeter (see Lemma 2.3), and area (see Lemma 2.2) of elements of minimizing sequences.

To simplify notations, for future reference, given points $x, y \in \mathbb{R}^2$, we denote by

$$[\![x, y]\!] := \{(1-s)x + sy : s \in [0,1]\}$$

the line segment between x and y. Moreover, given $r > 0$, we will denote by $B(x, r)$ the ball with center x and radius r.

Lemma 2.1. *Given $p \geq 1$, $\lambda > 0$, for any $R \in X$ it holds*

$$\operatorname{diam}(R) \geq \frac{4\pi\lambda}{E_{p,\lambda}(R)}. \tag{9}$$

Then, for any minimizing sequence $R_n \subseteq X$ (that is, $E_{p,\lambda}(R_n) \to \inf_X E_{p,\lambda}$) it holds

$$\operatorname{diam}(R_n) \geq \frac{2\pi\lambda}{1+\pi\lambda} =: \delta_\lambda \tag{10}$$

for all sufficiently large n. Finally, any minimizer (if they exist at all) satisfies estimate (10) *too.*

Proof. Consider an arbitrary $R \in X$. Choose $x, y \in \partial R$ such that $|x - y| = \operatorname{diam}(R)$. Note that $R \subseteq B(x, \operatorname{diam}(R))$, hence due to the convexity of R, it follows

$$\mathcal{H}^1(\partial R) \leq \pi \operatorname{diam}(R).$$

As ∂R is a closed convex curve with winding number equal to 1, it follows

$$\int_{\partial R} |\kappa_{\partial B}| \, \mathrm{d}\mathcal{H}^1_{\llcorner \partial R} = 2\pi,$$

and by Hölder's inequality it holds

$$E_{p,\lambda}(R) \geq \lambda \int_{\partial R} \kappa^2_{\partial R} \, \mathrm{d}\mathcal{H}^1_{\llcorner \partial R} \geq \frac{4\pi^2\lambda}{\mathcal{H}^1(\partial R)} \geq \frac{4\pi\lambda}{\operatorname{diam}(R)},$$

hence (9).

To prove (10), we show first that $\inf_X E_{p,\lambda} < +\infty$. Consider the unit ball $B := B((0,0),1)$, and note that

$$\inf_X E_{p,\lambda} \leq E_{p,\lambda}(B) = \int_B \operatorname{dist}^p(x, \partial R) \, \mathrm{d}x + \lambda \int_{\partial B} \kappa^2_{\partial B} \, \mathrm{d}\mathcal{H}^1_{\llcorner \partial B} \leq 1 + 2\pi\lambda. \tag{11}$$

Let $R_n \subseteq X$ be an arbitrary minimizing sequence. Clearly, since $E_{p,\lambda}(R_n) \to \inf_X E_{p,\lambda}$, for all sufficiently large n it holds

$$E_{p,\lambda}(R_n) \leq \inf_X E_{p,\lambda} + 1 \overset{(11)}{\leq} 2 + 2\pi\lambda,$$

and (9) gives $\operatorname{diam}(R_n) \geq \dfrac{2\pi\lambda}{1+\pi\lambda}$, hence (10). The last part follows from the continuity of the diameter with respect to the convergence in (X, d). $\square$

Lemma 2.2. *Given $p \geq 1$, $\lambda > 0$, and $R \in X$, it holds*

$$\mathcal{H}^2(R) \geq \frac{2\pi\lambda^2}{2E_{p,\lambda}(R)^2}. \tag{12}$$

Moreover, given a minimizing sequence $R_n \subseteq X$, it holds

$$\mathcal{H}^2(R_n) \geq \frac{\pi\lambda^2}{2(1+\pi\lambda)^2} =: a_\lambda \tag{13}$$

for all sufficiently large n. Finally, any minimizer (if they exist at all) satisfies estimate (13) *too.*

To simplify notations, for future reference, given a point $z \in \mathbb{R}^2$, the notations z_x (resp. z_y) will denote the x (resp. y) coordinate of z.

Proof. Consider an arbitrary $R \in X$. Choose arbitrary points $\bar{x}, \bar{y} \in \partial R$ such that $|\bar{x} - \bar{y}| = \operatorname{diam}(R)$. Endow $\mathbb{R}^2$ with a Cartesian coordinate system, with origin at the midpoint $(\bar{x}+\bar{y})/2$, such that $\bar{x} = (-\operatorname{diam}(R)/2, 0)$, $\bar{y} = (\operatorname{diam}(R)/2, 0)$. Let $\gamma : [0, \mathcal{H}^1(\partial R)] \longrightarrow \partial R$ be an arc-length parameterization, and without loss of generality, we impose $\gamma(0) = \bar{x}$. We claim

- $\gamma'(0)_x = 0$.
  Assume the opposite, i.e., $\gamma'(0)_x \neq 0$. For $|\varepsilon| \ll 1$, since γ is C^1-regular, it holds $\gamma(\varepsilon) = \bar{x} + \varepsilon\gamma'(0) + v_\varepsilon$, for some vector v_ε with $|v_\varepsilon| = o(\varepsilon)$ as $\varepsilon \to 0$. Since $\bar{y} - \bar{x}$ is parallel to the x-axis, it follows

$$\frac{\mathrm{d}}{\mathrm{d}t}|\bar{y} - \gamma(t)|\Big|_{t=0} = \lim_{\varepsilon\to 0} \frac{|\bar{y} - (\bar{x} + \varepsilon\gamma'(0))| - |\bar{y} - \bar{x}| + o(\varepsilon)}{\varepsilon} = \gamma'(0)_x \neq 0,$$

  hence $t = 0$ is not a maximum for $t \mapsto |\bar{y} - \gamma(t)|$. This contradicts

$$|\bar{y} - \bar{x}| = \operatorname{diam}(R) = \max_{x\in\partial R} |\bar{y} - x|,$$

  and the claim is proven.

Without loss of generality, we can further impose $\gamma'(0) = (0, 1)$. Consider the region $R \cap \{y \geq 0\}$. Set

$$t_0 := \inf\{t : \gamma'(t)_y = 1/2\},$$

where $\gamma(t)_y$ denotes the y-coordinate of $\gamma(t)$. By Hölder's inequality it follows

$$\frac{1}{4t_0} = \frac{\|\gamma_y'\|^2_{TV(0,t_0)}}{t_0} \leq \int_{\partial R} \kappa^2_{\partial R}\, \mathrm{d}\mathcal{H}^1_{\llcorner \partial R} \leq \frac{E_{p,\lambda}(R)}{\lambda} \Longrightarrow t_0 \geq \frac{\lambda}{4E_{p,\lambda}(R)}. \tag{14}$$

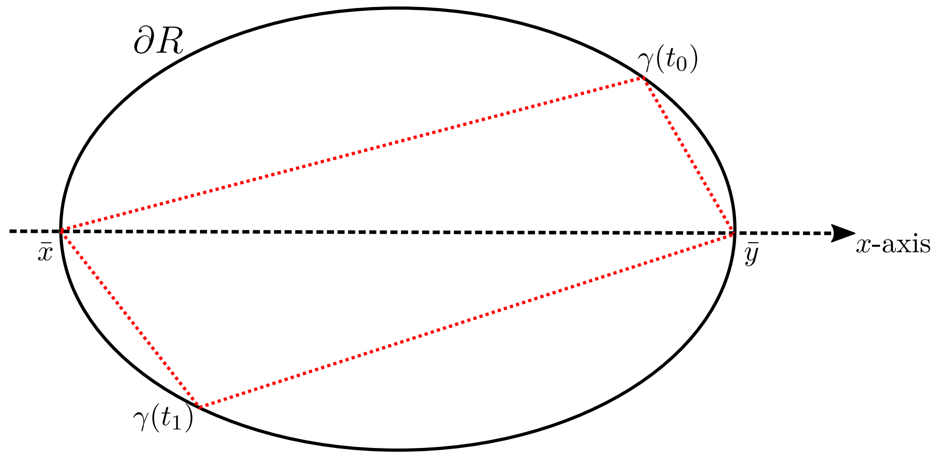

FIGURE 1. Schematic representation of the construction.

Since $1/2 \le \gamma_y'(t) \le 1$ for any $t \in [0, t_0]$, it holds $\gamma(t_0)_y \ge t_0/2$. Due to the convexity of $R \cap \{y \ge 0\}$, both line segments $[\![\gamma(t_0), \bar{x}]\!]$ and $[\![\gamma(t_0), \bar{y}]\!]$ are contained in R, hence $\triangle \bar{x}\gamma(t_0)\bar{y} \subseteq R$. By construction, the triangle $\triangle \bar{x}\gamma(t_0)\bar{y}$ has base $[\![\bar{x}, \bar{y}]\!]$ and height $[\![\gamma(t_0), (\gamma(t_0)_x, 0)]\!]$, hence

$$\mathcal{H}^2(\triangle \bar{x}\gamma(t_0)\bar{y}) = \frac{1}{2}|\bar{x} - \bar{y}| \cdot |\gamma(t_0)_y| \ge \frac{\operatorname{diam}(R) t_0}{4}. \tag{15}$$

Repeating the same construction for $R \cap \{y \le 0\}$ gives the existence of $t_1 \ge \frac{\lambda}{4E_{p,\lambda}(R)}$ such that the triangle $\triangle \bar{x}\gamma(t_1)\bar{y}$ satisfies

$$\mathcal{H}^2(\triangle \bar{x}\gamma(t_1)\bar{y}) = \frac{1}{2}|\bar{x} - \bar{y}| \cdot |\gamma(t_1)_y| \ge \frac{\operatorname{diam}(R) t_1}{4}. \tag{16}$$

Combining (15) and (16) gives

$$\mathcal{H}^2(R) \ge \frac{\operatorname{diam}(R) t_0}{4} + \frac{\operatorname{diam}(R) t_1}{4} \overset{(9),(14)}{\ge} \frac{2\pi\lambda^2}{E_{p,\lambda}(R)^2},$$

hence (12).

To prove (13), note that the above arguments give

$$\mathcal{H}^2(R_n) \overset{(12)}{\ge} \frac{2\pi\lambda^2}{E_{p,\lambda}(R_n)^2} \overset{(11)}{\ge} \frac{\pi\lambda^2}{2(1+\pi\lambda)^2},$$

for any sufficiently large n, and proof of (13) is complete. The last part follows from the continuity of the $\mathcal{H}^2$-measure with respect to the convergence in (X, d). $\square$

Lemma 2.3. *Given $p \ge 1$, $\lambda > 0$, for any $R \in X$ it holds*

$$\operatorname{diam}(R) \le \sqrt[p+1]{24 \cdot 2^p (p+1)(p+2) E_{p,\lambda}(R)^2 / \lambda}. \tag{17}$$

Moreover, for any minimizing sequence $R_n \subseteq X$ (that is, $E_{p,\lambda}(R_n) \to \inf_X E_{p,\lambda}$) it holds

$$\operatorname{diam}(R_n) \leq \sqrt[p+1]{24 \cdot 2^{p+2}(p+1)(p+2)(1+\pi\lambda)^2/\lambda} = D_{p,\lambda} \tag{18}$$

for all sufficiently large n, with $D_{p,\lambda}$ defined in (5)*. Finally, any minimizer (if they exist at all) satisfies estimate* (18) *too.*

Proof. Similarly to the proof of Lemma 2.2, consider an arbitrary $R \in X$, and choose arbitrary points $\bar{x}, \bar{y} \in \partial R$ such that $|\bar{x} - \bar{y}| = \operatorname{diam}(R)$. Endow $\mathbb{R}^2$ with a Cartesian coordinate system, with origin at the midpoint $(\bar{x} + \bar{y})/2$, such that $\bar{x} = (-\operatorname{diam}(R)/2, 0)$, $\bar{y} = (\operatorname{diam}(R)/2, 0)$.

In the proof of Lemma 2.2 we have shown the existence of a point $q \in \partial R$ (e.g., the point $\gamma(t_0)$) such that

$$\triangle \bar{x} q \bar{y} \subseteq R, \qquad |q_y| \geq \frac{\lambda}{8 E_{p,\lambda}(R)}. \tag{19}$$

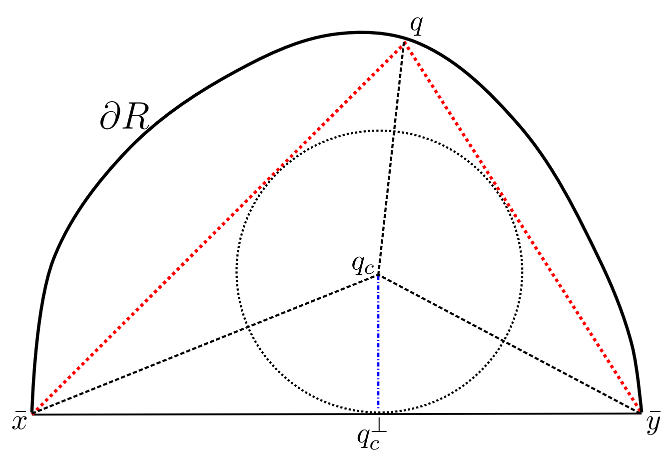

FIGURE 2. Schematic representation of the construction. Here represented only the region $R \cap \{y \geq 0\}$.

Let q_c be the incenter of $\triangle \bar{x} q \bar{y}$, and note that for any $z \in \triangle \bar{x} q_c \bar{y}$ it holds

$$\operatorname{dist}(z, \partial(\triangle \bar{x} q \bar{y})) = \operatorname{dist}(z, [\![\bar{x}, \bar{y}]\!]).$$

Denote by $q_c^\perp \in [\![\bar{x}, \bar{y}]\!]$ the projection of q_c on $[\![\bar{x}, \bar{y}]\!]$, and set

$$D_1 := |\bar{x} - q_c^\perp|, \qquad D_2 := |\bar{y} - q_c^\perp|, \qquad r := |q_c - q_c^\perp|.$$

Clearly, $D_1+D_2=\operatorname{diam}(R)$, and direct computation gives

$$\begin{aligned}\int_R \operatorname{dist}^p(z,\partial R)\,\mathrm{d}z \ge \int_{\triangle \bar{x} q_c \bar{y}} \operatorname{dist}^p(z,\partial R)\,\mathrm{d}z &\ge \int_{\triangle \bar{x} q_c^\perp \bar{y}} \operatorname{dist}^p(z,[\![\bar{x},\bar{y}]\!])\,\mathrm{d}z\\ &= \int_{\triangle \bar{x} q_c^\perp \bar{y}} z_y^p\,\mathrm{d}z = \int_0^{D_1}\int_0^{\frac{r}{D_1}x} y^p\,\mathrm{d}y\,\mathrm{d}x + \int_0^{D_2}\int_0^{\frac{r}{D_2}x} y^p\,\mathrm{d}y\,\mathrm{d}x\\ &= \frac{r(D_1^{p+1}+D_2^{p+1})}{(p+1)(p+2)} \ge \frac{r(\operatorname{diam}(R))^{p+1}}{2^p(p+1)(p+2)}. \end{aligned} \tag{20}$$

To estimate r, note that the sides $[\![\bar{x},q_c]\!]$ and $[\![\bar{y},q_c]\!]$ satisfy

$$|\bar{x}-\bar{y}| = \operatorname{diam}(R) \ge \max\{|\bar{x}-q_c|, |\bar{y}-q_c|\}.$$

Since

$$\mathcal{H}^2(\triangle \bar{x} q_c \bar{y}) = \frac{1}{2}\operatorname{diam}(R)|q_y| = \frac{1}{2}(\operatorname{diam}(R)+|\bar{x}-q_c|+|\bar{y}-q_c|)r,$$

we infer

$$r \ge \frac{|q_y|}{3} \overset{(19)}{\ge} \frac{\lambda}{24E_{p,\lambda}(R)}.$$

Plugging into (20) gives

$$\frac{\lambda}{24E_{p,\lambda}(R)}\cdot\frac{(\operatorname{diam}(R))^{p+1}}{2^p(p+1)(p+2)} \le \int_R \operatorname{dist}^p(z,\partial R)\,\mathrm{d}z \le E_{p,\lambda}(R),$$

hence (17).

To prove (18), note that for any minimizing sequence it holds

$$E_{p,\lambda}(R_n) \overset{(11)}{\le} 2(1+\pi\lambda) \Longrightarrow \operatorname{diam}(R_n) \overset{(17)}{\le} \sqrt[p+1]{24\cdot 2^{p+2}(p+1)(p+2)(1+\pi\lambda)^2/\lambda}$$

for all sufficiently large n. The last part follows from the continuity of the diameter with respect to the convergence in (X,d), concluding the proof. $\square$

Now we can prove the existence of minimizers in X (instead of just $\bar{X}$).

Lemma 2.4. *For any $p\ge 1$, $\lambda>0$, the functional $E_{p,\lambda}$ admits a minimizer in X.*

The following classic result (see for instance [15], to which we refer for the proof) will be useful.

Lemma 2.5. *Given a compact set $\Omega\subseteq\mathbb{R}^2$, and a sequence of curves $\{\gamma_k\}:[0,1]\longrightarrow\Omega$ satisfying*

$$\sup_k \|\gamma_k'\|_{BV} < +\infty, \qquad \sup_k \mathcal{H}^1(\gamma_k([0,1])) < +\infty,$$

where $\|\cdot\|_{BV}$ denotes the BV norm, then there exists a curve $\gamma:[0,1]\longrightarrow\Omega$, such that (upon subsequence) it holds:

(1) $\gamma_k\to\gamma$ in C^α for any $\alpha\in[0,1)$,
(2) $\gamma_k'\to\gamma'$ in L^p for any $p\in[1,\infty)$,
(3) $\gamma_k'' \overset{*}{\rightharpoonup} \gamma''$ in the space of signed Borel measures.

Proof. (of **Lemma 2.4**) Consider a minimizing sequence $R_n \subseteq X$. This assumption, instead of $R_n \subseteq \bar{X}$, is not restrictive since X is dense in $\bar{X}$. Since $E_{p,\lambda}$ is invariant under rigid movements, we can assume that $R_n \ni (0,0)$ for any n. In view of (11), without loss of generality, we can also impose

$$\sup_n E_{p,\lambda}(R_n) \leq 2(1+\pi\lambda).$$

Then by Lemma 2.3 we get $\sup_n \operatorname{diam}(R_n) \leq D_{p,\lambda}$, hence

$$R_n \subseteq B((0,0), D_{p,\lambda}) \quad \text{for any } n.$$

Thus R_n is a sequence of uniformly bounded, compact sets, and there exists (upon subsequence, which we do not relabel) a limit set $R \in \bar{X}$ such that $R_n \to R$ in the metric d (defined in (8)).

We claim

$$\int_R \operatorname{dist}^p(z, \partial R)\, \mathrm{d}z = \lim_{n\to+\infty} \int_{R_n} \operatorname{dist}^p(z, \partial R_n)\, \mathrm{d}z, \tag{21}$$

$$\int_{\partial R} \kappa_{\partial R}^2 \, \mathrm{d}\mathcal{H}^1_{\llcorner \partial R} \leq \liminf_{n\to+\infty} \int_{\partial R_n} \kappa_{\partial R_n}^2 \, \mathrm{d}\mathcal{H}^1_{\llcorner \partial R_n}. \tag{22}$$

Step 1. Proof of (21). This follows from the arguments from [12, Lemma 2.1]. We report the proof for completeness. We split the sums

$$\int_{R_n} \operatorname{dist}^p(z, \partial R_n)\, \mathrm{d}z = \int_{R_n \setminus R} \operatorname{dist}^p(z, \partial R_n)\, \mathrm{d}z + \int_{R_n \cap R} \operatorname{dist}^p(z, \partial R_n)\, \mathrm{d}z,$$

$$\int_{R} \operatorname{dist}^p(z, \partial R)\, \mathrm{d}z = \int_{R \setminus R_n} \operatorname{dist}^p(z, \partial R)\, \mathrm{d}z + \int_{R_n \cap R} \operatorname{dist}^p(z, \partial R)\, \mathrm{d}z,$$

and note that

$$\begin{aligned} &\left| \int_{R_n} \operatorname{dist}^p(z, \partial R_n)\, \mathrm{d}z - \int_{R} \operatorname{dist}^p(z, \partial R)\, \mathrm{d}z \right| \\ &\qquad \leq \int_{R_n \setminus R} \operatorname{dist}^p(z, \partial R_n)\, \mathrm{d}z + \int_{R \setminus R_n} \operatorname{dist}^p(z, \partial R)\, \mathrm{d}z \qquad (23) \\ &\qquad + \int_{R_n \cap R} |\operatorname{dist}^p(z, \partial R_n) - \operatorname{dist}^p(z, \partial R)|\, \mathrm{d}z. \qquad (24) \end{aligned}$$

Moreover,

$$\int_{R_n \setminus R} \operatorname{dist}^p(z, \partial R_n)\, \mathrm{d}z \leq \mathcal{H}^2(R_n \setminus R) \operatorname{diam}(R_n) \leq \mathcal{H}^2(R_n \setminus R) D_{p,\lambda} \to 0,$$

$$\int_{R \setminus R_n} \operatorname{dist}^p(z, \partial R)\, \mathrm{d}z \leq \mathcal{H}^2(R \setminus R_n) \operatorname{diam}(R) \leq \mathcal{H}^2(R \setminus R_n) D_{p,\lambda} \to 0,$$

hence

$$\lim_{n\to+\infty} \int_{R_n \setminus R} \operatorname{dist}^p(z, \partial R_n)\, \mathrm{d}z = \lim_{n\to+\infty} \int_{R \setminus R_n} \operatorname{dist}^p(z, \partial R)\, \mathrm{d}z = 0.$$

To prove

$$\lim_{n\to+\infty}\int_{R_n\cap R}|\operatorname{dist}^p(z,\partial R_n)-\operatorname{dist}^p(z,\partial R)|\,\mathrm{d}z=0,$$

denote by $d_{\mathcal{H}}$ the Hausdorff distance, and by the Mean Value theorem, it holds

$$\begin{aligned}
&\int_{R_n\cap R}|\operatorname{dist}^p(z,\partial R_n)-\operatorname{dist}^p(z,\partial R)|\,\mathrm{d}z\\
&\qquad\le\int_{R_n\cap R}|\operatorname{dist}(z,\partial R_n)-\operatorname{dist}(z,\partial R)|\cdot p\sup_{z\in R_n\cap R}\Big(\max\{\operatorname{dist}(z,\partial R_n),\operatorname{dist}(z,\partial R)\}\Big)^{p-1}\mathrm{d}z\\
&\qquad\le\mathcal{H}^2(R_n\cap R)d_{\mathcal{H}}(\partial R_n,\partial R)\cdot pD_{p,\lambda}^{p-1}\le\pi D_{p,\lambda}^2 d_{\mathcal{H}}(\partial R_n,\partial R)\cdot pD_{p,\lambda}^{p-1}\to 0.
\end{aligned}$$

Thus both terms (23) and (24) converge to zero, and (21) is proven.

Step 2. Proof of (22). Let

$$\gamma_n:[0,\mathcal{H}^1(\partial R_n)]\longrightarrow\partial R_n,\qquad\gamma_n:[0,\mathcal{H}^1(\partial R)]\longrightarrow\partial R$$

be arc-length parameterizations. Note that the hypotheses of Lemma 2.5 are satisfied, since all the γ_n are valued in $B((0,0),D_{p,\lambda})$, and, due to their convexity,

$$\sup_n\|\gamma_n'\|_{TV}=2\pi,\qquad\sup_k\mathcal{H}^1(\partial R_n)\le 2\pi D_{p,\lambda}<+\infty,$$

with $\|\cdot\|_{TV}$ denoting the total variation semi-norm. Thus $\gamma_n''\overset{*}{\rightharpoonup}\gamma''$ in the space of Borel measures. Note that $R_n\subseteq B((0,0),D_{p,\lambda})$, and by convexity, it follows that the perimeters are also uniformly bounded, i.e.,

$$L^*:=\sup_n\mathcal{H}^1(\partial R_n)<+\infty.$$

Thus we can define the curves

$$\gamma_n^*:[0,L^*]\longrightarrow\partial R_n,\qquad\gamma_n:[0,L^*]\longrightarrow\partial R$$

$$\gamma_n^*(t):=\begin{cases}\gamma_n(t) & \text{if } t\le\mathcal{H}^1(\partial R_n),\\ \gamma_n(\mathcal{H}^1(\partial R_n)) & \text{if } \mathcal{H}^1(\partial R_n)\le t\le L^*,\end{cases}$$

$$\gamma^*(t):=\begin{cases}\gamma(t) & \text{if } t\le\mathcal{H}^1(\partial R),\\ \gamma(\mathcal{H}^1(\partial R)) & \text{if } \mathcal{H}^1(\partial R)\le t\le L^*.\end{cases}$$

Note that

$$\int_{\partial R_n}\kappa_{\partial R_n}^2\,\mathrm{d}\mathcal{H}^1_{\llcorner\partial R_n}=\int_0^{L^*}|(\gamma_n^*)''|^2\,\mathrm{d}t,$$

and

$$\sup_n\int_{\partial R_n}\kappa_{\partial R_n}^2\,\mathrm{d}\mathcal{H}^1_{\llcorner\partial R_n}\le\sup_n\frac{E_{p,\lambda}(R_n)}{\lambda}<+\infty.$$

Thus the densities $(\gamma_n^*)''$ are uniformly bounded in $L^2(0,L^*;\mathbb{R})$, and converging to $(\gamma^*)''$ in the weak-* topology. Thus, by the lower-semicontinuity of norms, we infer

$$\liminf_{n\to+\infty}\int_0^{L^*}|(\gamma_n^*)''|^2\,\mathrm{d}t \geq \int_0^{L^*}|(\gamma^*)''|^2\,\mathrm{d}t = \int_{\partial R}\kappa_{\partial R}^2\,\mathrm{d}\mathcal{H}^1_{\llcorner\partial R},$$

hence (22). Combining with (21) gives

$$E_{p,\lambda}(R) \leq \liminf_{n\to+\infty} E_{p,\lambda}(R_n) = \inf_{\bar{X}} E_{p,\lambda},$$

hence R is effectively a minimizer of $E_{p,\lambda}$.

Step 3. Proof of $R\in X$. Observe that Lemma 2.2 gives $\mathcal{H}^2(R_n)\geq a_\lambda>0$ for any n, and the continuity of the $\mathcal{H}^2$-measure under convergence with respect to d gives $\mathcal{H}^2(R)\geq a_\lambda>0$. Moreover, Lemma 2.3 gives $\sup_n \operatorname{diam}(R_n)\leq D_{p,\lambda}$, and since we assumed that $R_n\ni(0,0)$ for any n, it follows also $R\ni(0,0)$. Therefore, R is bounded. Since R_n is compact and convex for any n, R is also compact and convex. Combining all the above observations gives $R\in X$. □

3. Proof of Theorem 1.1

Now we are ready to prove the main theorem. In both the proof of Theorem 1.1 and Lemmas 3.2 and 3.3 we will use the "$O(\cdot)$" notation: expressions of the form *"some quantity $X\in\mathbb{R}$ is less than or equal to $O(\varepsilon^\alpha)$* (resp. *"some quantity $X\in\mathbb{R}$ is greater than or equal to $O(\varepsilon^\alpha)$*) (for some $\alpha\geq 0$ and $\varepsilon\to 0$) will mean that there exists a constant $C\in\mathbb{R}$ (independent of ε) such that $X\leq C\varepsilon^\alpha$ (resp. $X\geq C\varepsilon^\alpha$) for any sufficiently small ε.

Proof. (of **Theorem 1.1**) Let R be a minimizer of $E_{p,\lambda}$, and let γ be an arc-length parameterization of ∂R. Assume there exist M,ε, $t_0<t_1$ such that

$$|\gamma'(t_0)-\gamma'(t_1)| = M\varepsilon, \qquad t_1-t_0=\varepsilon.$$

The goal is to find an upper bound for M.

Without loss of generality, upon rigid movements, we can assume $t_0=0$, $t_1=\varepsilon$ and $\gamma'(0)=(0,1)$. Endow $\mathbb{R}^2$ with a Cartesian coordinate system with

$$\gamma(0)\in\{x\geq 0, y=0\} \qquad \gamma(\varepsilon)\in\{y\geq 0, x=0\}, \qquad \gamma'(0)=(0,1). \tag{25}$$

The exact orientation of x-axis is not relevant. We first give an estimate on $\gamma(\varepsilon)_y$. Using Hölder's inequality, and recalling the fact that

$$\kappa_{\partial R}\ll\mathcal{H}^1_{\llcorner\partial R}, \qquad \frac{\mathrm{d}\kappa_{\partial R}}{\mathrm{d}\mathcal{H}^1_{\llcorner\partial R}}\in L^2(0,\mathcal{H}^1(\partial R);\mathbb{R}),$$

for any $t \in [0,\varepsilon]$, it holds

$$\frac{E_{p,\lambda}(R)}{\lambda} \geq \int_{\gamma([0,t])} \kappa_{\partial R}^2 \,\mathrm{d}\mathcal{H}^1_{\llcorner \partial R} = \int_0^t |\gamma''|^2 \,\mathrm{d}s \geq \frac{|\gamma'(t)-\gamma'(0)|^2}{\varepsilon} \geq \frac{|\gamma'(t)_y - 1|^2}{\varepsilon}$$
$$\Longrightarrow |\gamma'(t)_y - 1| \leq \sqrt{\varepsilon E_{p,\lambda}(R)/\lambda},$$

hence

$$\varepsilon(1-\sqrt{\varepsilon E_{p,\lambda}(R)/\lambda}) \leq \varepsilon\Big(1 - \sup_{0\leq t\leq\varepsilon} |\gamma'(t)_y|\Big)$$
$$\leq |\gamma(\varepsilon)_y - \gamma(0)_y| = |\gamma(\varepsilon)_y| \leq \varepsilon\Big(1 + \sup_{0\leq t\leq\varepsilon} |\gamma'(t)_y|\Big) \leq \varepsilon(1+\sqrt{\varepsilon E_{p,\lambda}(R)/\lambda}).$$

In particular, since we imposed $\gamma(\varepsilon)_y \geq 0$, we have $\gamma(\varepsilon)_y = \varepsilon + O(\varepsilon^{3/2})$.

Construct the competitor R_ε in the following way:

(1) denote by $t_\pm$ the two times such that $\gamma'(t_\pm) = (\pm 1, 0)$, and by $t_\perp$ the time such that $\gamma'(t_\perp) = (0,-1)$. Since we imposed $\gamma'(0) = (0,1)$, without loss of generality we can assume that the tangent direction turns *counterclockwise*, i.e.,

$$\varepsilon < t_- < t_\perp < t_+ < \mathcal{H}^1(\partial R).$$

Note that

$$\frac{\sqrt{2}}{t_\perp - t_-} = \frac{|\gamma'(t_\perp)-\gamma'(t_-)|}{t_\perp - t_-} \leq \int_{\gamma([t_-,t_\perp])} \kappa_{\partial R}^2 \,\mathrm{d}\mathcal{H}^1_{\llcorner \partial R} \leq \frac{E_{p,\lambda}(R)}{\lambda} \overset{(11)}{\leq} \lambda^{-1} + 2\pi$$
$$\Longrightarrow t_\perp - t_- \geq (\lambda^{-1}+2\pi)/\sqrt{2}.$$

Similarly, we get

$$\min\{\mathcal{H}^1(\partial R) - t_+,\ t_+ - t_\perp,\ t_-\} \geq (\lambda^{-1}+2\pi)/\sqrt{2}.$$

(2) Define the vector field $v : [t_-, t_+] \longrightarrow \mathbb{R}^2$ as

$$v(t) := \begin{cases} \left(\cos\left(\frac{\pi}{2}(1+\frac{t-t_-}{t_\perp - t_-})\right), \sin\left(\frac{\pi}{2}(1+\frac{t-t_-}{t_\perp - t_-})\right)\right), & \text{if } t \in [t_-, t_\perp], \\ \left(\cos\left(\frac{\pi}{2}(1+\frac{t_+-t}{t_+ - t_\perp})\right), -\left(\frac{t_+-t}{t_+-t_\perp}\right)^2 \sin\left(\frac{\pi}{2}(1+\frac{t_+-t}{t_+ - t_\perp})\right)\right), & \text{if } t \in [t_\perp, t_+]. \end{cases} \tag{26}$$

Note first that v is continuous (smooth outside $t_\perp$), and direct computation gives

$$v'(t) = \begin{cases} \frac{\pi/2}{t_\perp - t_-}\left(-\sin\left(\frac{\pi}{2}(1+\frac{t-t_-}{t_\perp-t_-})\right), \cos\left(\frac{\pi}{2}(1+\frac{t-t_-}{t_\perp-t_-})\right)\right), & \text{if } t \in [t_-, t_\perp), \\ \Big(\frac{\pi/2}{t_+-t_\perp}\sin\left(\frac{\pi}{2}(1+\frac{t_+-t}{t_+-t_\perp})\right), \frac{\pi/2}{t_+-t_\perp}\left(\frac{t_+-t}{t_+-t_\perp}\right)^2\cos\left(\frac{\pi}{2}(1+\frac{t_+-t}{t_+-t_\perp})\right) & \\ \qquad +2\frac{t_+-t}{t_+-t_\perp}\sin\left(\frac{\pi}{2}(1+\frac{t_+-t}{t_+-t_\perp})\right)\Big) & \text{if } t \in (t_\perp, t_+]. \end{cases}$$

In particular,

$$\lim_{t\to t_\perp^-} v'(t) = \frac{\pi/2}{t_\perp - t_-}(0,-1), \qquad \lim_{t\to t_\perp^+} v'(t) = \frac{\pi/2}{t_+ - t_\perp}(0,-1),$$

i.e., the left and right limit differ just by a multiplicative constant. This observation is crucial, since it implies that the tangent derivative of the arc-length reparameterization of v does *not* jump at $t = t_\perp$ (recall also that γ' does not jump at $t = t_\perp$, hence the tangent derivative of the arc-length reparameterization of $\gamma + cv$ does not jump at $t = t_\perp$, for any $c > 0$). We claim:

$$\|v'\|_{L^\infty} \le \max\left\{\frac{\pi/2}{t_\perp - t_-}, \frac{\pi/2}{t_+ - t_\perp} + 2\right\} \le \frac{\pi/\sqrt{2}}{\lambda^{-1} + 2\pi} + 2 =: \mu_\lambda < +\infty, \tag{27}$$

$$\|v''\|_{L^\infty} \le \frac{\pi^2}{(\lambda^{-1} + 2\pi)^2} + \frac{2\sqrt{2}(\pi+1)}{\lambda^{-1} + 2\pi} =: \nu_\lambda < +\infty. \tag{28}$$

The proofs of both claims, being quite technical, are presented in Lemma 3.1 below.

(3) Let γ_ε be the curve such that

$$\gamma_\varepsilon(t) := \begin{cases} (2\gamma(t)_x, 2\gamma(t)_y) & \text{if } t \in [0,\varepsilon], \\ \gamma(t) + (0, \gamma(\varepsilon)_y) & \text{if } t \in [\varepsilon, t_-], \\ \gamma(t) + \gamma(\varepsilon)_y v(t) & \text{if } t \in [t_-, t_+], \\ \left(\gamma(t)_x \left(1 + \frac{\gamma(0)_x}{\gamma(0)_x - \gamma(t_+)_x}\right) - \frac{\gamma(t_+)_x \gamma(0)_x}{\gamma(0)_x - \gamma(t_+)_x}, \gamma(t)_y\right) & \text{if } t \in [t_+, \mathcal{H}^1(\partial R)]. \end{cases} \tag{29}$$

Let ∂R_ε be the image of γ_ε, and R_ε be the bounded region of the plane delimited by ∂R_ε. This will be our competitor. Observe first that, as $\gamma'(t_+) = (1,0)$,

$$\lim_{t\to t_+^-} \gamma_\varepsilon'(t) = \left(1 + \gamma(\varepsilon)_y \frac{\pi/2}{t_+ - t_\perp}, 0\right), \qquad \lim_{t\to t_+^+} \gamma_\varepsilon'(t) = \left(1 + \frac{\gamma(0)_x}{\gamma(0)_x - \gamma(t_+)_x}, 0\right),$$

i.e., the left and right limit differ just by a multiplicative constant. This observation is again crucial, since it implies that the tangent derivative of the arc-length reparameterization of $\gamma + \gamma(\varepsilon)_y v$ does *not* jump at $t = t_+$.

Intuitively, for $t \in [\varepsilon, t_-]$ the competitor γ_ε is constructed from γ by:

(1) a homothety of center $(0,0)$ and ratio 2 for $t \in [0,\varepsilon]$,
(2) a translation of the vector $(0, \gamma(\varepsilon)_y)$ for $t \in [\varepsilon, t_-]$,
(3) adding the smooth vector field $\gamma(\varepsilon)_y v(t)$ for $t \in [t_-, t_+]$,
(4) a scaling of factor $1 + \frac{\gamma(0)_x}{\gamma(0)_x - \gamma(t_+)_x}$ in the x direction (with fixed line being $x = \gamma(t_+)_x$) for $t \in [t_+, \mathcal{H}^1(\partial R)]$.

It is straightforward to check compactness and convexity for R_ε. Moreover, denoting by $\tilde\gamma_\varepsilon$ the arc-length reparameterization of γ_ε, the curvature of $\tilde\gamma_\varepsilon$ is still a *function* (instead of a more generic measure), as the is γ_ε always constructed from γ via translation, scaling, or sum with

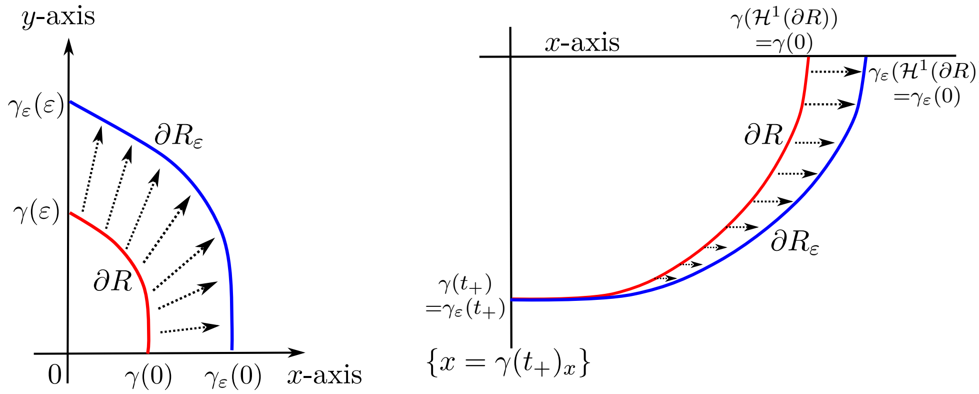

FIGURE 3. Representation of the construction of the competitor γ_ε, for $t \in [0,\varepsilon]$ (left) and $t \in [t_+, \mathcal{H}^1(\partial R)]$ (right).

smooth vector fields, and the tangent derivative $\tilde{\gamma}_\varepsilon'$ never jumps at "junction points" (i.e., for $t = \varepsilon, t_-, t_\perp, t_+, \mathcal{H}^1(\partial R)$).

Next, to estimate $E_{p,\lambda}(R_\varepsilon) - E_{p,\lambda}(R)$, we claim

$$\int_{R_\varepsilon} \operatorname{dist}^p(z, \partial R_\varepsilon)\, \mathrm{d}z - \int_R \operatorname{dist}^p(z, \partial R)\, \mathrm{d}z \le \varepsilon \cdot 2p(D_{p,\lambda}+1)^{p-1}\pi D_{p,\lambda}^2 + \varepsilon^{p+1}\pi D_{p,\lambda}, \tag{30}$$

$$\int_{\partial R_\varepsilon} \kappa_{\partial R_\varepsilon}^2 \, \mathrm{d}\mathcal{H}^1_{\llcorner \partial R_\varepsilon} - \int_{\partial R} \kappa_{\partial R}^2 \, \mathrm{d}\mathcal{H}^1_{\llcorner \partial R} \le \varepsilon\Big(K_{p,\lambda} - \frac{M}{2}\Big) + O(\varepsilon^{3/2}), \tag{31}$$

with

$$K_{p,\lambda} = \mu_\lambda(\lambda^{-1} + 2\pi) + \nu_\lambda \sqrt{\pi D_{p,\lambda}^2 (\lambda^{-1} + 2\pi)}$$

defined in (6).

Step 1. Proof of (31). Using the notation from (29), we make the following claims:

$$\int_{\gamma([\varepsilon,t_-])} \kappa_{\partial R}^2 \, \mathrm{d}\mathcal{H}^1_{\llcorner \partial R} = \int_{\gamma_\varepsilon([\varepsilon,t_-])} \kappa_{\partial R_\varepsilon}^2 \, \mathrm{d}\mathcal{H}^1_{\llcorner \partial R_\varepsilon}, \tag{32}$$

$$\int_{\gamma_\varepsilon([t_+,\mathcal{H}^1(\partial R)])} \kappa_{\partial R_\varepsilon}^2 \, \mathrm{d}\mathcal{H}^1_{\llcorner \partial R_\varepsilon} - \int_{\gamma([t_+,\mathcal{H}^1(\partial R)])} \kappa_{\partial R}^2 \, \mathrm{d}\mathcal{H}^1_{\llcorner \partial R} \le O(\varepsilon^{3/2}), \tag{33}$$

$$\int_{\gamma_\varepsilon([t_-,t_+])} \kappa_{\partial R_\varepsilon}^2 \, \mathrm{d}\mathcal{H}^1_{\llcorner \partial R_\varepsilon} - \int_{\gamma([t_-,t_+])} \kappa_{\partial R}^2 \, \mathrm{d}\mathcal{H}^1_{\llcorner \partial R} \le K_{p,\lambda}\varepsilon + O(\varepsilon^{3/2}), \tag{34}$$

$$\int_{\gamma([0,\varepsilon])} \kappa_{\partial R}^2 \, \mathrm{d}\mathcal{H}^1_{\llcorner \partial R} - \int_{\gamma_\varepsilon([0,\varepsilon])} \kappa_{\partial R_\varepsilon}^2 \, \mathrm{d}\mathcal{H}^1_{\llcorner \partial R_\varepsilon} \ge \frac{M\varepsilon}{2}. \tag{35}$$

The proof of all four assertions are quite technical, and for reader's convenience, will be done in Lemmas 3.2 and 3.3 below. Combining (32), (33), (34), and (35) gives

$$\int_{R_\varepsilon} \kappa_{\partial R_\varepsilon}^2 \, \mathrm{d}\mathcal{H}^1_{\llcorner R_\varepsilon} \le \int_R \kappa_{\partial R}^2 \, \mathrm{d}\mathcal{H}^1_{\llcorner R} + \varepsilon\Big(K_{p,\lambda} - \frac{M}{2}\Big) + O(\varepsilon^{3/2}),$$

hence (31).

Step 2. Proof of (30). Recall that, the construction of the competitor R_ε in (29) gives also

(1) for $t \in [\varepsilon, \mathcal{H}^1(\partial R)]$ the competitor γ_ε is obtained by translating γ by a vector of length at most ε,
(2) for $t \in [0, \varepsilon]$, the competitor γ_ε is is obtained by scaling γ by a factor of 2.

Hence for all t it holds $|\gamma_\varepsilon(t) - \gamma(t)| \leq 2\varepsilon$, and

$$d_{\mathcal{H}}(\partial R_\varepsilon, \partial R) \leq 2\varepsilon.$$

Thus, by the Mean Value Theorem, for each point $z \in R_\varepsilon \cap R$ it holds

$$\begin{aligned} \operatorname{dist}^p(z, \partial R_\varepsilon) - \operatorname{dist}^p(z, \partial R) &\leq (\operatorname{dist}(z, \partial R_\varepsilon) - \operatorname{dist}(z, \partial R)) \cdot \\ &\qquad \cdot p(\sup_{z \in R_\varepsilon \cap R} \max\{\operatorname{dist}(z, \partial R_\varepsilon), \operatorname{dist}(z, \partial R)\})^{p-1} \\ &\leq \varepsilon \cdot 2p(\operatorname{diam}(R) + 2\varepsilon)^{p-1} \overset{(18)}{\leq} \varepsilon \cdot 2p(D_{p,\lambda} + 1)^{p-1}, \end{aligned}$$

with $D_{p,\lambda}$ defined in (5). Thus, since clearly

$$\mathcal{H}^2(R_\varepsilon \cap R) \leq \mathcal{H}^2(R) \leq \pi(\operatorname{diam}(R)/2)^2,$$

it follows

$$\begin{aligned} \int_{R_\varepsilon \cap R} \operatorname{dist}^p(x, \partial R_\varepsilon)\, \mathrm{d}x - \int_R \operatorname{dist}^p(x, \partial R)\, \mathrm{d}x &\leq \varepsilon \cdot 2p(D_{p,\lambda} + 1)^{p-1} \mathcal{H}^2(R_\varepsilon \cap R) \\ &\leq \varepsilon \cdot p(D_{p,\lambda} + 1)^{p-1} \pi D_{p,\lambda}^2 / 2 \end{aligned} \tag{36}$$

Then note that, since by construction we have $d_{\mathcal{H}}(\partial R_\varepsilon, \partial R) \leq 2\varepsilon$, it follows

$$\int_{R_\varepsilon \setminus R} \operatorname{dist}^p(x, \partial R_\varepsilon)\, \mathrm{d}x \leq 2\varepsilon^p \mathcal{H}^2(R_\varepsilon \setminus R) \leq \varepsilon^{p+1} \mathcal{H}^1(\partial R). \tag{37}$$

Combining (36) and (37) gives

$$\begin{aligned} \int_{R_\varepsilon} \operatorname{dist}^p(x, \partial R_\varepsilon)\, \mathrm{d}x &\leq \int_R \operatorname{dist}^p(x, \partial R)\, \mathrm{d}x + \varepsilon \cdot 2p(D_{p,\lambda} + 1)^{p-1} \mathcal{H}^2(R_\varepsilon \cap R) + 2\varepsilon^{p+1} \mathcal{H}^1(\partial R) \\ &\leq \int_R \operatorname{dist}^p(x, \partial R)\, \mathrm{d}x + \varepsilon \cdot 2p(D_{p,\lambda} + 1)^{p-1} \pi D_{p,\lambda}^2 + 2\varepsilon^{p+1} \pi D_{p,\lambda}, \end{aligned} \tag{38}$$

since Lemma 2.3 gives $\mathcal{H}^2(R_\varepsilon \cap R) \leq \mathcal{H}^2(R) \leq \pi D_{p,\lambda}^2$ and $\mathcal{H}^1(\partial R) \leq \pi D_{p,\lambda}$. Thus (30) is proven.

Combining (30) and (31) we finally infer

$$\begin{aligned}
&E_{p,\lambda}(R_\varepsilon) - E_{p,\lambda}(R)\\
&\qquad = \int_{R_\varepsilon} \operatorname{dist}^p(z, \partial R_\varepsilon)\,\mathrm{d}z - \int_R \operatorname{dist}^p(z, \partial R)\,\mathrm{d}z + \lambda\Big(\int_{\partial R_\varepsilon} \kappa^2_{\partial R_\varepsilon}\,\mathrm{d}\mathcal{H}^1_{\llcorner \partial R_\varepsilon} - \int_{\partial R} \kappa^2_{\partial R}\,\mathrm{d}\mathcal{H}^1_{\llcorner \partial R}\Big)\\
&\qquad \le \varepsilon \cdot 2p(D_{p,\lambda}+1)^{p-1}\pi D^2_{p,\lambda} + 2\varepsilon^{p+1}\pi D_{p,\lambda} + \lambda\Big(\varepsilon\Big(K_{p,\lambda} - \frac{M}{2}\Big) + O(\varepsilon^{3/2})\Big).
\end{aligned}$$

Note also that the term $2\varepsilon^{p+1}\pi D_{p,\lambda}$ can be absorbed into $O(\varepsilon^{3/2})$, due to condition $p \ge 1$, hence

$$E_{p,\lambda}(R_\varepsilon) - E_{p,\lambda}(R) \le \lambda\Big(\varepsilon\Big(2\lambda^{-1}p(D_{p,\lambda}+1)^{p-1}\pi D^2_{p,\lambda} + K_{p,\lambda} - \frac{M}{2}\Big) + O(\varepsilon^{3/2})\Big).$$

The minimality assumption on R, and the arbitrariness of $\varepsilon > 0$ then imply

$$\begin{aligned}
&2\lambda^{-1}p(D_{p,\lambda}+1)^{p-1}\pi D^2_{p,\lambda} + K_{p,\lambda} - \frac{M}{2} \ge 0\\
&\qquad \Longrightarrow M \le 2(2\lambda^{-1}p(D_{p,\lambda}+1)^{p-1}\pi D^2_{p,\lambda} + K_{p,\lambda}),
\end{aligned}$$

and the proof is complete. □

Lemma 3.1. *Under the hypotheses of Theorem 1.1, assertions* (27) *and* (28) *hold.*

Proof. We use the same notations from the proof of Theorem 1.1. Since

$$v'(t) = \begin{cases} \frac{\pi/2}{t_\perp - t_-}\left(-\sin\left(\frac{\pi}{2}(1+\frac{t-t_-}{t_\perp-t_-})\right), \cos\left(\frac{\pi}{2}(1+\frac{t-t_-}{t_\perp-t_-})\right)\right), & \text{if } t \in [t_-, t_\perp),\\
\Big(\frac{\pi/2}{t_+-t_\perp}\sin\left(\frac{\pi}{2}(1+\frac{t_+-t}{t_+-t_\perp})\right), \frac{\pi/2}{t_+-t_\perp}\left(\frac{t_+-t}{t_+-t_\perp}\right)^2\cos\left(\frac{\pi}{2}(1+\frac{t_+-t}{t_+-t_\perp})\right) & \\
\qquad +2\frac{t_+-t}{t_+-t_\perp}\sin\left(\frac{\pi}{2}(1+\frac{t_+-t}{t_+-t_\perp})\right)\Big) & \text{if } t \in (t_\perp, t_+],
\end{cases}$$

it follows

$$|v'(t)| \le \frac{\pi/2}{t_\perp - t_-} \qquad \text{for any } t \in [t_-, t_\perp),$$

and

$$\begin{aligned}
|v'(t)| &= \Big[\Big(\frac{\pi/2}{t_+-t_\perp}\Big)^2 \sin^2\Big(\frac{\pi}{2}(1+\frac{t_+-t}{t_+-t_\perp})\Big) + \Big(\frac{\pi/2}{t_+-t_\perp}\Big)^2\Big(\frac{t_+-t}{t_+-t_\perp}\Big)^4\cos^2\Big(\frac{\pi}{2}(1+\frac{t_+-t}{t_+-t_\perp})\Big)\\
&\qquad + 4\Big(\frac{t_+-t}{t_+-t_\perp}\Big)^2\sin^2\Big(\frac{\pi}{2}(1+\frac{t_+-t}{t_+-t_\perp})\Big)\\
&\qquad + 4\frac{\pi/2}{t_+-t_\perp}\Big(\frac{t_+-t}{t_+-t_\perp}\Big)^3\cos\Big(\frac{\pi}{2}(1+\frac{t_+-t}{t_+-t_\perp})\Big)\sin\Big(\frac{\pi}{2}(1+\frac{t_+-t}{t_+-t_\perp})\Big)\Big]^{1/2}\\
&\le \Big[\Big(\frac{\pi/2}{t_+-t_\perp}\Big)^2 + 4 + \frac{\pi}{t_+-t_\perp}\Big]^{1/2} \le \frac{\pi/2}{t_+-t_\perp} + 2
\end{aligned}$$

for any $t \in (t_\perp, t_+]$. Thus (27) is proven.

To prove (28), note that for $t \in [t_-, t_\perp)$ it holds

$$v''(t) = -\Big|\frac{\pi/2}{t_\perp - t_-}\Big|^2 \left(\cos\Big(\frac{\pi}{2}(1+\frac{t-t_-}{t_\perp - t_-})\Big), \sin\Big(\frac{\pi}{2}(1+\frac{t-t_-}{t_\perp - t_-})\Big)\right),$$

hence $|v''(t)| \le \Big|\frac{\pi/2}{t_\perp - t_-}\Big|^2$. Similarly, for $t \in (t_\perp, t_+]$, it holds

$$\begin{aligned} v''(t) = \Big(&- \Big|\frac{\pi/2}{t_+ - t_\perp}\Big|^2 \cos\Big(\frac{\pi}{2}(1+\frac{t_+-t}{t_+-t_\perp})\Big), \\ &\frac{\pi/2}{t_+ - t_\perp}\Big[\frac{-2(t_+-t)}{t_+-t_\perp}\cos\Big(\frac{\pi}{2}(1+\frac{t_+-t}{t_+-t_\perp})\Big) + \Big(\frac{t_+-t}{t_+-t_\perp}\Big)^2 \frac{\pi/2}{t_+-t_\perp}\sin\Big(\frac{\pi}{2}(1+\frac{t_+-t}{t_+-t_\perp})\Big)\Big] \\ &- 2\frac{t_+-t}{t_+-t_\perp}\frac{\pi/2}{t_+-t_\perp}\cos\Big(\frac{\pi}{2}(1+\frac{t_+-t}{t_+-t_\perp})\Big) - \frac{2}{t_+-t_\perp}\sin\Big(\frac{\pi}{2}(1+\frac{t_+-t}{t_+-t_\perp})\Big)\Big), \end{aligned}$$

hence, using the convexity of the norm, and $t_+ - t_\perp \ge (\lambda^{-1} + 2\pi)/\sqrt{2}$ (proven in the proof of Theorem 1.1)

$$|v''(t)| \le 2\Big|\frac{\pi/2}{t_+ - t_\perp}\Big|^2 + \frac{2\pi+2}{t_+ - t_\perp} \le \frac{\pi^2}{(\lambda^{-1}+2\pi)^2} + \frac{2\sqrt{2}(\pi+1)}{\lambda^{-1}+2\pi},$$

hence (28) is proven. □

Lemma 3.2. *Under the hypotheses of Theorem 1.1, assertions* (32)*,* (33) *and* (35) *hold.*

Proof. We use the same notations from the proof of Theorem 1.1.

Proof of (32). By construction, for any $t \in [\varepsilon, t_-]$, $\gamma_\varepsilon(t)$ differ from $\gamma(t)$ by a translation, thus the curvature of these two segments are always equal, hence (32).

Proof of (33). For $t \in [t_+, \mathcal{H}^1(\partial R)]$ we have

$$\begin{aligned} \gamma_\varepsilon(t) &= \left(\gamma(t)_x\left(1+\frac{\gamma(0)_x}{\gamma(0)_x - \gamma(t_+)_x}\right) - \frac{\gamma(t_+)_x\gamma(0)_x}{\gamma(0)_x - \gamma(t_+)_x}, \gamma(t)_y\right), \\ \gamma'_\varepsilon(t) &= \left(\gamma(t)'_x\left(1+\frac{\gamma(0)_x}{\gamma(0)_x - \gamma(t_+)_x}\right), \gamma(t)'_y\right), \\ \gamma''_\varepsilon(t) &= \left(\gamma(t)''_x\left(1+\frac{\gamma(0)_x}{\gamma(0)_x - \gamma(t_+)_x}\right), \gamma(t)''_y\right). \end{aligned}$$

We claim

$$\gamma(0)_x \le \int_0^\varepsilon |\gamma'(t)_x|\,\mathrm{d}t \le \varepsilon^{3/2}\sqrt{E_{p,\lambda}(R)/\lambda}, \qquad \gamma(0)_x - \gamma(t_+)_x \ge \frac{\lambda}{2E_{p,\lambda}(R)}. \tag{39}$$

In view of (25), and noting that for any $t \in [0,\varepsilon]$ it holds

$$\frac{E_{p,\lambda}(R)}{\lambda} \geq \int_{\gamma([0,t])} \kappa_{\partial R}^2 \,\mathrm{d}\mathcal{H}^1_{\llcorner \partial R} = \int_0^t |\gamma''|^2 \,\mathrm{d}s \geq \frac{|\gamma'(t)-\gamma'(0)|^2}{\varepsilon} \geq \frac{|\gamma'(t)_x|^2}{\varepsilon}$$
$$\Longrightarrow |\gamma'(t)_x| \leq \sqrt{\varepsilon E_{p,\lambda}(R)/\lambda},$$

it follows

$$|\gamma(0)_x - \gamma(\varepsilon)_x| = |\gamma(0)_x| \leq \int_0^\varepsilon |\gamma'(t)_x| \,\mathrm{d}t \leq \varepsilon^{3/2}\sqrt{E_{p,\lambda}(R)/\lambda}.$$

Now recall that by construction $\gamma'(t_+) = (1,0)$, $\gamma'(\mathcal{H}^1(\partial R)) = \gamma'(0) = (0,1)$, $|\gamma'| \equiv 1$ for a.e. t, and let $\tau \in (t_+, \mathcal{H}^1(\partial R))$ be time for which $\gamma'(\tau) = (1/2, \sqrt{3}/2)$. Thus

$$\frac{E_{p,\lambda}(R)}{\lambda} \geq \int_{\gamma([t_+,\tau])} \kappa_{\partial R}^2 \,\mathrm{d}\mathcal{H}^1_{\llcorner \partial R} = \int_{t_+}^\tau |\gamma''|^2 \,\mathrm{d}s \geq \frac{|\gamma'(\tau)-\gamma'(t_+)|^2}{\tau - t_+} = \frac{1}{\tau - t_+}$$
$$\Longrightarrow \tau - t_+ \geq \lambda/E_{p,\lambda}(R),$$

and since $\gamma'_x \geq 0$ on $[t_+, \mathcal{H}^1(\partial R)]$, and $\gamma_x(t)' \geq 1/2$ for all $t \in [t_+,\tau]$, it follows $\gamma(0)_x - \gamma(t_+)_x \geq \frac{\lambda}{2E_{p,\lambda}(R)}$, hence (39) is proven. Consequently,

$$\left|\frac{\gamma(0)_x}{\gamma(0)_x - \gamma(t_+)_x}\right| \leq 2(\varepsilon E_{p,\lambda}(R)/\lambda)^{3/2} = O(\varepsilon^{3/2}).$$

Observe that for $t \in [t_+, \mathcal{H}^1(\partial R)]$ we have

$$|\gamma'_\varepsilon|^{-2} = \left(|\gamma'_x|^2\Big(1 + \frac{\gamma(0)_x}{\gamma(0)_x - \gamma(t_+)_x}\Big)^2 + |\gamma'_y|^2\right)^{-1}$$
$$= \left(1 + |\gamma'_x|^2 \frac{2\gamma(0)_x}{\gamma(0)_x - \gamma(t_+)_x} + |\gamma'_x|^2 \left|\frac{\gamma(0)_x}{\gamma(0)_x - \gamma(t_+)_x}\right|^2\right)^{-1} = 1 + O(\varepsilon^{3/2}),$$

and

$$\int_{t_+}^{\mathcal{H}^1(\partial R)} \frac{|\gamma''_\varepsilon|^2}{|\gamma'_\varepsilon|^2} \,\mathrm{d}t = \int_{t_+}^{\mathcal{H}^1(\partial R)} \left(|\gamma''|^2 + |\gamma''_x|^2 \frac{2\gamma(0)_x}{\gamma(0)_x - \gamma(t_+)_x}\right)(1 + O(\varepsilon^{3/2}))\,\mathrm{d}t$$
$$= \int_{t_+}^{\mathcal{H}^1(\partial R)} |\gamma''|^2 \,\mathrm{d}t + O(\varepsilon^{3/2}),$$

hence (33).

Proof of (35). In the time interval $[0,\varepsilon]$, the competitor is obtained by scaling by a factor of 2, and direct computations give that the integrated squared curvature scales by a factor of 1/2. Thus

$$\int_0^\varepsilon \left|\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{\gamma'}{|\gamma'|}\Big)\right|^2 \mathrm{d}\mathcal{H}^1_{\llcorner \partial R} - \int_0^{2\varepsilon} \left|\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{\gamma'_\varepsilon}{|\gamma'_\varepsilon|}\Big)\right|^2 \mathrm{d}\mathcal{H}^1_{\llcorner \partial R_\varepsilon}$$
$$\geq \frac{1}{2}\int_0^\varepsilon \left|\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{\gamma'}{|\gamma'|}\Big)\right|^2 \mathrm{d}\mathcal{H}^1_{\llcorner \partial R} \geq \frac{M}{2}\varepsilon, \tag{40}$$

hence (35). □

Lemma 3.3. *Under the hypotheses of Theorem 1.1, assertion* (34) *holds.*

Proof. We use the same notations from the proof of Theorem 1.1. In the time interval $[t_-, t_+]$, γ_ε is given by

$$\gamma_\varepsilon(t) = \gamma(t) + \gamma(\varepsilon)_y v(t), \qquad t \in [t_-, t_+].$$

Note first that since R is a minimizer of $E_{p,\lambda}$, it must hold

$$\int_{\gamma([t_-,t_+])} \kappa^2_{\partial R} \, \mathrm{d}\mathcal{H}^1_{\llcorner \partial R} < +\infty,$$

and recalling our definition of integrated squared curvature in (4), it follows that the Radon-Nikodym derivative $\frac{\mathrm{d}\kappa_{\partial R}}{\mathrm{d}\mathcal{H}^1_{\llcorner \partial R}}$ is square integrable. In terms of the parameterization γ_ε, this gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\gamma_\varepsilon'}{|\gamma_\varepsilon'|}\right) = \frac{\gamma_\varepsilon''}{|\gamma_\varepsilon'|} - \gamma_\varepsilon' \frac{\langle \gamma_\varepsilon'', \gamma_\varepsilon' \rangle}{|\gamma_\varepsilon'|^3} \in L^2(0, \mathcal{H}^1(\partial R); \mathbb{R}).$$

Recall that

$$\varepsilon - \varepsilon^{3/2}\sqrt{E_{p,\lambda}(R)/\lambda} \leq \gamma(\varepsilon)_y \leq \varepsilon + \varepsilon^{3/2}\sqrt{E_{p,\lambda}(R)/\lambda}.$$

As γ is parameterized by arc-length (i.e., $|\gamma'| = 1$ for a.e. t), and v was defined in (26) (in particular, $|v'|$ was uniformly bounded from above), it follows

$$|\gamma_\varepsilon'| = \sqrt{1 + 2\varepsilon\langle \gamma', v' \rangle + O(\varepsilon^{3/2})}.$$

Then, for any $\alpha \in \mathbb{R}$ and sufficiently small ε, we have

$$|\gamma_\varepsilon'|^\alpha = 1 + \alpha\varepsilon\langle \gamma', v' \rangle + O(\varepsilon^{3/2}). \tag{41}$$

Thus

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\gamma_\varepsilon'}{|\gamma_\varepsilon'|}\right) = \frac{\gamma_\varepsilon''}{|\gamma_\varepsilon'|} - \gamma_\varepsilon' \frac{\langle \gamma_\varepsilon'', \gamma_\varepsilon' \rangle}{|\gamma_\varepsilon'|^3} = \frac{\gamma'' + \varepsilon v''}{|\gamma_\varepsilon'|} - \frac{\gamma_\varepsilon'}{|\gamma_\varepsilon'|^3}\left(\langle \gamma'', \gamma' \rangle + \varepsilon^2\langle v'', v' \rangle + \varepsilon\langle \gamma'', v' \rangle + \varepsilon\langle \gamma', v'' \rangle\right).$$

Now observe:

(1) since γ is parameterized by arc-length,

$$\langle \gamma'', \gamma' \rangle = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\gamma'|^2 = 0. \tag{42}$$

(2) As both $|v'|$ and $|v''|$ are uniformly bounded from above, the term $\varepsilon^2\langle v'', v' \rangle$ is of order $O(\varepsilon^2)$.

(3) The norm of $\varepsilon\gamma_\varepsilon'\langle \gamma', v'' \rangle/|\gamma_\varepsilon'|^3$ is estimated by

$$\varepsilon\left|\frac{\gamma_\varepsilon'\langle \gamma', v'' \rangle}{|\gamma_\varepsilon'|^3}\right| \leq \varepsilon\frac{|\gamma'| \cdot |v''|}{|\gamma_\varepsilon'|^2} \leq \varepsilon\|v''\|_{L^\infty} + O(\varepsilon^{3/2}). \tag{43}$$

(4) The norm of $\varepsilon\gamma_\varepsilon'\langle \gamma'', v' \rangle/|\gamma_\varepsilon'|^3$ is estimated by

$$\varepsilon\left|\frac{\gamma_\varepsilon'\langle \gamma'', v' \rangle}{|\gamma_\varepsilon'|^3}\right| \leq \varepsilon\frac{|\gamma''| \cdot |v'|}{|\gamma_\varepsilon'|^2}. \tag{44}$$

Thus combining (42), (43) and (44) gives

$$\begin{aligned}\int_{t_-}^{t_+}\left|\gamma_\varepsilon'\frac{\langle\gamma_\varepsilon'',\gamma_\varepsilon'\rangle}{|\gamma_\varepsilon'|^3}\right|^2\mathrm{d}t&=\int_{t_-}^{t_+}\left|\frac{\gamma_\varepsilon'}{|\gamma_\varepsilon'|^3}(\langle\gamma'',\gamma'\rangle+\varepsilon^2\langle v'',v'\rangle+\varepsilon\langle\gamma'',v'\rangle+\varepsilon\langle\gamma',v''\rangle)\right|^2\mathrm{d}t\\&\le\int_{t_-}^{t_+}|\gamma_\varepsilon'|^{-4}\big|\varepsilon\langle\gamma'',v'\rangle+\varepsilon\langle\gamma',v''\rangle\big|^2\,\mathrm{d}t+O(\varepsilon^4)\\&\le 2\int_{t_-}^{t_+}|\gamma_\varepsilon'|^{-4}(|\varepsilon\langle\gamma'',v'\rangle|^2+|\varepsilon\langle\gamma',v''\rangle|^2)\,\mathrm{d}t+O(\varepsilon^4)\\&=2\varepsilon^2\int_{t_-}^{t_+}|\gamma_\varepsilon'|^{-4}|\langle\gamma'',v'\rangle|^2\,\mathrm{d}t+O(\varepsilon^{3/2})\le 2\varepsilon^2\|v'\|_{L^\infty}^2\int_{t_-}^{t_+}|\gamma_\varepsilon'|^{-4}|\gamma''|^2\,\mathrm{d}t+O(\varepsilon^{3/2}).\end{aligned}$$

In view of (41), we get

$$\int_{t_-}^{t_+}|\gamma_\varepsilon'|^{-4}|\gamma''|^2\,\mathrm{d}t\le 2\int_{t_-}^{t_+}|\gamma''|^2\,\mathrm{d}t\le 2\int_{\partial R}\kappa_{\partial R}^2\,\mathrm{d}\mathcal{H}^1_{\llcorner R}<+\infty,$$

hence

$$2\varepsilon^2\|v'\|_{L^\infty}^2\int_{t_-}^{t_+}|\gamma_\varepsilon'|^{-4}|\gamma''|^2\,\mathrm{d}t\le O(\varepsilon^2).$$

Thus

$$\int_{t_-}^{t_+}\left|\gamma_\varepsilon'\frac{\langle\gamma_\varepsilon'',\gamma_\varepsilon'\rangle}{|\gamma_\varepsilon'|^3}\right|^2\mathrm{d}t\le O(\varepsilon^{3/2}),$$

and

$$\begin{aligned}\int_{\gamma([t_-,t_+])}\kappa_{\partial R}^2\,\mathrm{d}\mathcal{H}^1_{\llcorner\partial R}&=\int_{t_-}^{t_+}\left|\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{\gamma_\varepsilon'}{|\gamma_\varepsilon'|}\Big)\right|^2\mathrm{d}t=\int_{t_-}^{t_+}\left|\frac{\gamma_\varepsilon''}{|\gamma_\varepsilon'|}-\gamma_\varepsilon'\frac{\langle\gamma_\varepsilon'',\gamma_\varepsilon'\rangle}{|\gamma_\varepsilon'|^3}\right|^2\mathrm{d}t\\&=\int_{t_-}^{t_+}\left|\frac{\gamma_\varepsilon''}{|\gamma_\varepsilon'|}\right|^2\mathrm{d}t+O(\varepsilon^{3/2}).\end{aligned}$$

Again, in view of (41), it follows

$$\begin{aligned}\int_{t_-}^{t_+}\left|\frac{\gamma_\varepsilon''}{|\gamma_\varepsilon'|}\right|^2\mathrm{d}t&=\int_{t_-}^{t_+}\left|\frac{\gamma''+\varepsilon v''}{|\gamma_\varepsilon'|}\right|^2\mathrm{d}t=\int_{t_-}^{t_+}(\langle\gamma''+\varepsilon v'',\gamma''+\varepsilon v''\rangle)(1-2\varepsilon\langle\gamma',v'\rangle+O(\varepsilon^{3/2}))\,\mathrm{d}t\\&=\int_{t_-}^{t_+}(|\gamma''|^2+2\varepsilon\langle\gamma'',v''\rangle+\varepsilon^2|v''|)(1-2\varepsilon\langle\gamma',v'\rangle+O(\varepsilon^{3/2}))\,\mathrm{d}t\\&\le(1+2\varepsilon\|v'\|_{L^\infty})\int_{t_-}^{t_+}|\gamma''|^2\,\mathrm{d}t+2\varepsilon\|v''\|_{L^\infty}\int_{t_-}^{t_+}|\gamma''|\,\mathrm{d}t+O(\varepsilon^{3/2})\\&\le(1+2\varepsilon\|v'\|_{L^\infty})\int_{t_-}^{t_+}|\gamma''|^2\,\mathrm{d}t+2\varepsilon\|v''\|_{L^\infty}\Big(\mathcal{H}^1(\partial R)\int_{t_-}^{t_+}|\gamma''|^2\,\mathrm{d}t\Big)^{1/2}+O(\varepsilon^{3/2})\\&\le\int_{t_-}^{t_+}|\gamma''|^2\,\mathrm{d}t+2\varepsilon\|v'\|_{L^\infty}E_{p,\lambda}(R)/\lambda+2\varepsilon\|v''\|_{L^\infty}\sqrt{\mathcal{H}^1(\partial R)E_{p,\lambda}(R)/\lambda}+O(\varepsilon^{3/2}).\end{aligned}\tag{45}$$

Note that

$$2\|v'\|_{L^\infty}E_{p,\lambda}(R)/\lambda + 2\|v''\|_{L^\infty}\sqrt{\mathcal{H}^1(\partial R)E_{p,\lambda}(R)/\lambda}$$
$$\le \mu_\lambda(\lambda^{-1}+2\pi) + \nu_\lambda\sqrt{\pi D^2_{p,\lambda}(\lambda^{-1}+2\pi)} = K_{p,\lambda}$$

in view of (6), (18), (27) and (28). Hence (34) follows from (45). □

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