Optimal waiting time bounds for flux-saturated diffusion equations

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Abstract

We consider the Cauchy problem for two prototypes of flux-saturated diffusion equations. In arbitrary space dimension, we give an optimal condition on the growth of the initial datum which discriminates between occurrence or nonoccurrence of a waiting time phenomenon. We also prove optimal upper bounds on the waiting time. Our argument is based on the introduction of suitable families of subsolutions and on a comparison result for a general class of flux-saturated diffusion equations.

Keywords. waiting time phenomena, flux-saturated diffusion equations, entropy solutions, comparison principle, conservation laws

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1 Introduction

1.1 Flux-saturated diffusion equations

Flux-saturated diffusion equations are a class of second order parabolic equations of the form

$$u_t = \operatorname{div} \mathbf{a}(u, \nabla u), \tag{1.1}$$

which are characterized by a hyperbolic scaling for large values of the modulus of the gradient, in the sense that

$$\frac{1}{\psi_0(\mathbf{v})} \lim_{t\to+\infty} \mathbf{a}(z, t\mathbf{v}) \cdot \mathbf{v} =: \varphi(z) \quad \text{for all } z \geq 0, \tag{1.2}$$

where $\psi_0 : \mathbb{R}^N \mapsto [0, +\infty)$ is a positively 1-homogeneous convex function, with $\psi_0(0) = 0$ and $\psi_0 > 0$ otherwise, accounting for possible anisotropy effects. We are interested in the degenerate case; i.e. the case in which φ is a locally Lipschitz function with $\varphi(0) = 0$ and $\varphi(z) > 0$ otherwise.

To our knowledge, flux-saturated equations were first introduced in [27] in the description of inertial confinement fusion, in which case u represents the temperature. However, they find application whenever a saturation mechanism at high gradients, imposing a-priori bounds on speed

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or flux, is modeling-wise relevant for the phenomenon to be described (see for instance [32, 33, 24, 8, 10]). In addition, they emerge from a generalization of optimal transportation theory which accounts for relativistic-type cost functions (see [13]). After pioneering contributions [11, 12, 25], the mathematical interest in this class is now steadily growing, leading to a well posedness theory based on a suitable concept of entropy solution: we refer to §2 for the precise definition and to [21, 22, 14, 15, 16] for recent overviews on modeling and analytical aspects.

Our focus is on two model equations which are known to approximate the porous medium equation ([23]): the *relativistic porous medium equation*,

$$u_t = \nu \,\mathrm{div}\left(\frac{u^m \nabla u}{\sqrt{u^2 + \nu^2 c^{-2} |\nabla u|^2}}\right), \quad m \in (1, +\infty), \tag{1.3}$$

which generalizes the so-called relativistic heat equation ($m = 1$), and the *speed-limited porous medium equation*,

$$u_t = \nu \,\mathrm{div}\left(\frac{u \nabla u^{M-1}}{\sqrt{1 + \nu^2 c^{-2} |\nabla u^{M-1}|^2}}\right), \quad M \in (1, +\infty), \tag{1.4}$$

where $\nu > 0$ is a kinematic viscosity constant and $c > 0$ represents a characteristic limiting speed. The former was proposed in [33, Eq. (16)] with $m = 3/2$ and in [13, Eq. (34)] with $m = 1$, whereas the latter was proposed in [33, Eq. (19)] (see also [22]). Up to the scaling $\hat{t} = \frac{c^2}{\nu} t$, $\hat{x} = \frac{c}{\nu} x$, we will hereafter assume without losing generality that

$$\nu = c = 1.$$

Equation (1.3) and (1.4) share common general features, such as finite speed of propagation of the support ([29]) and persistence of jump discontinuities ([21]). However, they have remarkable differences, generated by the different scaling for large gradients: in one space dimension, a monotone increasing solution to (1.3), resp. (1.4), formally satisfies

$$u_t \sim (u^m)_x \;\text{ for } u_x \gg 1, \quad \text{resp.} \quad u_t \sim u_x \;\text{ for } (u^{M-1})_x \gg 1. \tag{1.5}$$

This reflects into different qualitative behavior of solutions, highlighted also by numerical simulations as in [18, 9, 19]. For instance, (1.5) suggests that (1.3) may yield to the *formation* of jump discontinuities if $m > 1$, whereas (1.4) may not, and that the speed of propagation of the support is formally given by u^{m-1} for (1.3) and by 1 for (1.4). For this reason, in the former case we conjecture that the formation of a discontinuity is not only sufficient ([21]), but also necessary for the support to expand.

1.2 Waiting-time phenomena: the main result

The aforementioned difference manifests itself also in the *waiting time phenomenon*, a positive time before which the solutions' support does not expand around a point $x_0 \in \mathbb{R}^N$. Starting from the porous medium equation (see [34] for a review), this phenomenon is well known to occur for various classes of degenerate parabolic equations and systems, also of higher order (see e.g. [26, 30, 31, 28] and references therein). Concerning (1.3) and (1.4), after numerical and formal arguments in [9, 19], rigorous sufficient conditions for a positive waiting time have been recently given in [29]: a positive constant C, depending only on N and m (resp. M), exists such that if

$$\operatorname*{ess\,sup}_{x\in\mathbb{R}^N} |x-x_0|^{-\frac{1}{m-1}} u_0(x) = L < +\infty \qquad \text{if } u \text{ solves (1.3), or} \tag{1.6}$$

$$\operatorname*{ess\,sup}_{x\in\mathbb{R}^N} |x-x_0|^{-\frac{2}{M-1}} u_0(x) = L < +\infty \qquad \text{if } u \text{ solves (1.4),} \tag{1.7}$$

then the entropy solution to the Cauchy problem for (1.3), resp. (1.4), is such that

$$u(t,x_0) = 0 \quad \text{for all} \quad t \le T_\ell := \begin{cases} CL^{1-m} & \text{if } u \text{ solves (1.3)} \\ CL^{1-M} & \text{if } u \text{ solves (1.4)} \end{cases} \tag{1.8}$$

(we refer to Section 2 for the definition of entropy solution). This result provides a *lower bound* T_ℓ on the waiting time. Based on (1.5), in [29] it is also conjectured that these growth exponents are sharp. The main result of this paper confirms this fact.

Theorem 1.1. Let $u_0 \in L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$ be nonnegative. Let u be the solution to the Cauchy problem for (1.3) (resp. (1.4)) with initial datum u_0 and let

$$t_* = \sup\left\{ t \ge 0 :\ x_0 \in \overline{\mathbb{R}^N \setminus \operatorname{supp}(u(\tau))} \quad \text{for all } \tau \in [0,t] \right\}.$$

If $v_0 \in \mathbb{S}^{N-1}$ exists such that

$$\lim_{\rho\to 0^+} \operatorname*{ess\,inf}_{x\in B(x_0+\rho v_0,\rho)} u_0(x)|x-x_0|^{-\frac{1}{m-1}} \ge L \in (0,+\infty] \quad \text{if } u \text{ solves (1.3),} \tag{1.9}$$

or

$$\lim_{\rho\to 0^+} \operatorname*{ess\,inf}_{x\in B(x_0+\rho v_0,\rho)} u_0(x)|x-x_0|^{-\frac{2}{M-1}} \ge L \in (0,+\infty] \quad \text{if } u \text{ solves (1.4),} \tag{1.10}$$

then a positive constant W, depending on m (resp. M) and N, exists such that

$$t_* \le T_u := \begin{cases} WL^{1-m} & \text{if } u \text{ solves (1.3)} \\ WL^{1-M} & \text{if } u \text{ solves (1.4).} \end{cases} \tag{1.11}$$

In particular, $t_* = 0$ if $L = +\infty$.

The growth conditions (1.9) and (1.10) imply in particular that $\operatorname{supp}(u_0)$ satisfies an interior ball property at x_0, i.e., $R > 0$ exists such that $B(x_0 + v_0R, R) \subset \operatorname{supp}(u_0)$.

The results in Theorem 1.1 are sharp. Indeed, comparing Theorem 1.1 with (1.6)-(1.7) we see that the growth exponents in (1.9)-(1.10) are optimal. Note that the growth exponent $2/(M-1)$ coincides with that of the limiting porous medium equation, whereas $1/(m-1)$ does not. In addition, comparing Theorem 1.1 with (1.8), we see that the *upper bound* T_u on the waiting time given in (1.11) is also optimal, in terms of scaling with respect to L.

The first main ingredient in our argument is a comparison result between solutions and subsolutions (see Theorem 2.6). Based on Kruzhkov doubling method, a general approach for proving uniqueness of entropy solutions to degenerate flux-saturated equations has been introduced in [2, 3] and later followed, or referred to, in quite a few subsequent papers [5, 6, 7, 4, 20, 8, 22, 17, 29]. However, no comparison result is available when subsolutions are defined the way we need in our arguments (see Definition 2.5). Therefore, in Section 2 we revisit the notion of (sub-)solution to

Eq. (1.1), providing a comparison with subsolutions for a general class of equations (see Assumption 2.1).

The second main ingredient in our argument is the introduction of suitable families of subsolutions, built such that optimal results may be obtained: their construction is outlined in the next subsection. With such subsolutions at hand, the strategy for Theorem 1.1 becomes analogous to the one used for the porous medium equation (see [34] and references therein). It is worked out in Sections 3 and 4, where the main result is proved: we argue by comparison, showing that subsolutions exist whose support is initially contained in $B(x_0 + v_0 R, R)$ and which expands up to x_0 within time T_u.

Besides comparison arguments, energy methods have also been developed in the analysis of waiting time phenomena in [26, 28, 30, 31]. These methods are potentially capable of treating equations of general form (as opposed to explicit prototypes), leading to weaker, integral-type conditions on the initial datum. It would be interesting to explore the applicability of these methods to more general classes of flux-saturated diffusion equations of the form (1.1).

1.3 Classes of subsolutions

We now give a formal overview of the construction of subsolution in the case of (1.3). As we mentioned, we expect that the support of solutions to (1.3) expands *only if* the solution has a jump discontinuity at the support's boundary. Therefore, it is natural to look for subsolutions which share the same property. Up to scaling and translation invariance, a prototype form is

$$u(t,x) = \frac{1}{A(t)} f(r(t),|x|)\chi_{B(0,r(t))}(x), \quad f(r,y) = (1 + (r^2 - y^2)^\alpha), \quad \alpha > 0,$$

which is smooth in $B(0, r(t))$ with a moving front at $|x| = r(t)$. On the jump set, the inequality

$$u_t \le \operatorname{div}\left(\frac{u^m \nabla u}{\sqrt{u^2 + |\nabla u|^2}}\right) \tag{1.12}$$

formally translates into

$$r'(t) \lim_{|x|\to r(t)^-} u(t,x) \le \frac{x}{r} \cdot \lim_{|x|\to r(t)^-} \frac{u^m \nabla u}{\sqrt{u^2 + |\nabla u|^2}}. \tag{1.13}$$

Provided that $\alpha < 1$, we have $|\nabla u| \to +\infty$ as $x \to r(t)^-$. Therefore (1.13) reduces to $r'(t) \le A^{1-m}(t)$, consistently with the Rankine-Hugoniot condition. In order to reach optimal results, we impose the equality:

$$r'(t) = A^{1-m}(t). \tag{1.14}$$

On the other hand, when $y \ll 1$, the degenerate parabolic structure dominates and (1.12) translates into

$$u_t \lesssim \operatorname{div}\left(u^{m-1}\nabla u\right),$$

that is,

$$\begin{aligned} -\frac{A'}{A^2}(1 + r^{2\alpha}) + \frac{2\alpha}{A} r^{2\alpha-1} r' &\overset{(1.14)}{=} -\frac{A'}{A^2}(1 + r^{2\alpha}) + 2\alpha r^{2\alpha-1} A^{-m} \\ &\lesssim -2\alpha N A^{-m} r^{2\alpha-2}(1 + r^{2\alpha})^{m-1}. \end{aligned} \tag{1.15}$$

In order to enforce homogeneity of (1.15) with respect to A, we choose

$$A^{m-2}(t)A'(t) = \gamma \tag{1.16}$$

for some constant $\gamma > 0$. Combining (1.14) and (1.16) we obtain

$$A(t) = ((m-1)(1+\gamma t))^{\frac{1}{m-1}} \ , \quad r(t) = r_0 + \frac{1}{\gamma(m-1)} \log(1+\gamma t) \, .$$

As opposed to the porous medium equation, however, proving that such functions are indeed subsolutions is not obvious for two reasons: first, the crossover between the parabolic scaling (for $|y| \ll 1$) and the hyperbolic scaling (for $|r(t) - y| \ll 1$); second, the nontrivial notion of subsolution (see Def. 2.5 below). On the other hand, the appropriate identification of A and r permits to obtain optimal results in terms of both growth exponent and waiting time bounds. Analogous arguments lead to a family of subsolutions for (1.4), which up to scaling and translation invariances has the form

$$u(t,x) = b^{\frac{1}{1-M}} \left(\ell - \frac{1}{1+wt} \right)^{\frac{1}{1-M}} \left(1 - \frac{|x|^2}{(1+wt)^2} \right)_+^{\frac{1}{M-1}}$$

for suitable $b > 0$, $w > 0$ and $\ell > 0$ (see Section 3).

1.4 Notation

For $a, b, \ell \in \mathbb{R}$ we let

$$\mathcal{T}^+ = \{T^\ell_{a,b} : \ 0 < a < b, \ \ell \le a\}, \quad \text{where} \quad T^\ell_{a,b}(r) = \max\{\min\{b, r\}, a\} - \ell.$$

For a given $T = T^\ell_{a,b} \in \mathcal{T}^+$, we let $T^0 := T + \ell = T^0_{a,b}$. For $f \in L^1_{loc}(\mathbb{R})$ we let

$$J_f(r) := \int_0^r f(s) \, \mathrm{d}s.$$

We use standard notations and concepts for BV functions as in [1]; in particular, for $u \in BV(\mathbb{R}^N)$, $\nabla u \mathcal{L}^N$, resp. $D^s u$, denote the the absolutely continuous, resp. singular, parts of Du with respect to the Lebesgue measure $\mathcal{L}^N$, J_u denotes its jump set and we assume that $u^+(x) > u^-(x)$ for $x \in J_u$.

2 Entropy (sub-)solutions

In this section we revisit the notion of entropy (sub-)solution to the Cauchy problem for (1.1). Consider a function $\mathbf{a}$ satisfying the following properties:

Assumption 2.1. Let $Q = (0, \infty) \times \mathbb{R}^N$. The function $\mathbf{a} : \overline{Q} \to \mathbb{R}^N$ is such that:
(i) (*Lagrangian*) there exists $f \in C(\overline{Q})$ such that $\nabla_{\mathbf{v}} f = \mathbf{a} \in C(\overline{Q})$, $f(z, \cdot)$ is convex, $f(z, 0) = 0$ for all $z \in [0, \infty)$, and

$$C_0(z)|\mathbf{v}| - D_0(z) \le f(z, \mathbf{v}) \le M_0(z)(1 + |\mathbf{v}|) \quad \text{for all } (z, \mathbf{v}) \in Q$$

for nonnegative continuous functions $M_0, C_0 \in C([0, \infty))$ and $D_0 \in C((0, \infty))$, with $C_0(z) > 0$ for $z > 0$;

(ii) (*flux*) $D_{\mathbf{v}}\mathbf{a} \in C(\overline{Q})$; $\mathbf{a}(z,0) = \mathbf{a}(0,\mathbf{v}) = 0$ and $h(z,\mathbf{v}) := \mathbf{a}(z,\mathbf{v}) \cdot \mathbf{v} = h(z,-\mathbf{v})$ for all $(z,\mathbf{v}) \in \overline{Q}$; for any $R > 0$ there exists $M_R > 0$ such that

$$|\mathbf{a}(z,\mathbf{v}) - \mathbf{a}(\hat{z},\mathbf{v})| \le M_R |z - \hat{z}| \quad \text{for all } z, \hat{z} \in [0,R] \text{ and all } \mathbf{v} \in \mathbb{R}^N; \tag{2.1}$$

(iii) (*recession functions*) the *recession functions* f^0 and h^0, defined by

$$f^0(z,\mathbf{v}) = \lim_{t\to+\infty} \frac{1}{t} f(z,t\mathbf{v}), \quad h^0(z,\mathbf{v}) = \lim_{t\to+\infty} \frac{1}{t} h(z,t\mathbf{v}),$$

exist in $\overline{Q}$; furthermore, a function $\varphi \in \mathrm{Lip}_{loc}([0,\infty))$ with $\varphi(0) = 0$ and $\varphi > 0$ in $(0,\infty)$ and a positive 1-homogeneous convex function $\psi_0 : \mathbb{R}^N \mapsto \mathbb{R}$ with $\psi_0(0) = 0$ and $\psi_0(\mathbf{v}) > 0$ for $\mathbf{v} \neq 0$ exist such that

$$f^0(z,\mathbf{v}) = h^0(z,\mathbf{v}) = \varphi(z)\psi_0(\mathbf{v}) \quad \text{for all } (z,\mathbf{v}) \in \overline{Q} \tag{2.2}$$

(cf. (1.2)), and

$$|\mathbf{a}(z,\mathbf{w}) \cdot \mathbf{v}| \le \varphi(z)\psi_0(\mathbf{v}) \text{ for all } (z,\mathbf{v}) \in Q, \mathbf{w} \in \mathbb{R}^N. \tag{2.3}$$

The convexity of f implies that

$$(\mathbf{a}(z,\mathbf{v}) - \mathbf{a}(z,\hat{\mathbf{v}})) \cdot (\mathbf{v} - \hat{\mathbf{v}}) \ge 0 \quad \text{for all } \mathbf{v}, \hat{\mathbf{v}} \in \mathbb{R}^N$$

which, combined with (2.1), also yields

$$(\mathbf{a}(z,\mathbf{v}) - \mathbf{a}(\hat{z},\hat{\mathbf{v}})) \cdot (\mathbf{v} - \hat{\mathbf{v}}) \ge -M_R |z - \hat{z}| |\mathbf{v} - \hat{\mathbf{v}}| \tag{2.4}$$

for all $z, \hat{z} \in [0,R]$ and all $\mathbf{v}, \hat{\mathbf{v}} \in \mathbb{R}^N$.

The concept of entropy solution to the Cauchy problem for (1.1) has been introduced in [3] and later extended in [5, 20, 22]. At the core of this concept is an entropy inequality (cf. (2.5) below) which follows from formally testing (1.1) by $\phi S(u)T(u)$ with $S, T \in \mathcal{T}^+$ and ϕ smooth and non-negative. In particular, when constructing a solution as limit of solutions to suitable approximating problems, one needs to argue by lower semi-continuity on terms of the form

$$S(u)a(u,\nabla u) \cdot \nabla T(u) = S(T^0(u))h(T^0(u), \nabla T^0(u))$$

(see the discussion in [3, §2.2 and 3.2]). This leads to the following entropy inequality:

$$\begin{aligned} &\int_0^{+\infty} \langle h_S(u, DT(u)) + h_T(u, DS(u)), \phi \rangle \, \mathrm{d}t \\ &\quad \le \int_0^{+\infty} \int_{\mathbb{R}^N} \left(J_{TS}(u)\phi_t - T(u)S(u)\mathbf{a}(u,\nabla u) \cdot \nabla \phi \right) \, \mathrm{d}x \, \mathrm{d}t \,, \end{aligned} \tag{2.5}$$

where $h_S(u, DT(u))$ is the Radon measure defined by

$$\begin{aligned} \langle h_S(u, DT(u)), \phi \rangle &:= \int_{\mathbb{R}^N} \phi S(T^0(u)) h(T^0(u), \nabla T^0(u)) \, \mathrm{d}x \\ &+ \int_{\mathbb{R}^N} \phi \psi_0 \left(\frac{DT^0(u)}{|DT^0(u)|} \right) \mathrm{d}|D^s J_{S\varphi}(T^0(u))| \quad \text{for all } \phi \in C_c(\mathbb{R}^N) \end{aligned} \tag{2.6}$$

and φ,ψ_0 are defined through (2.2). This motivates the following definition:

Definition 2.2. Let $\mathbf{a}$ such that Assumption 2.1 holds and let $u_0 \in L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$ nonnegative. A nonnegative function $u \in C([0,+\infty); L^1(\mathbb{R}^N)) \cap L^\infty((0,\infty)\times\mathbb{R}^N)$ is an entropy solution to the Cauchy problem for (1.1) with initial datum u_0 if $u(0) = u_0$ and:

(i) $T^a_{a,b}(u) \in L^1_{loc}((0,+\infty); BV(\mathbb{R}^N))$ for all $0 < a < b$;

(ii) $u_t = \mathrm{div}(\mathbf{a}(u,\nabla u))$ in the sense of distributions;

(iii) inequality (2.5) holds for any $S, T \in \mathcal{T}^+$ and any nonnegative $\phi \in C_c^\infty((0,+\infty)\times\mathbb{R}^N)$.

It is easily seen that Definition 2.2 implies mass conservation:

Proposition 2.3. Any solution u in the sense of Definition 2.2 is such that

$$\int_{\mathbb{R}^N} u(t,x)\,\mathrm{d}x = \int_{\mathbb{R}^N} u_0(x)\,\mathrm{d}x \quad \text{for all } t \in [0,+\infty). \tag{2.7}$$

Proof. Let $\eta_R \in \mathcal{D}(\mathbb{R}^N)$ be an increasing sequence of nonnegative functions such that $\eta_R = 1$ on $B(0,R)$, $\eta_R = 0$ in $\mathbb{R}^N \setminus B(0,R+1)$, and $|\nabla\eta_R| \le C$. Let $\psi_\varepsilon(t) := \chi_{\{[t_1,t_2]\}} * \rho_\varepsilon$, where ρ_ε is a standard mollifier and $[t_1,t_2] \subset (0,+\infty)$. We donote by C a generic positive constant independent of ε and R. Testing (ii) in Definition 2.2 with $\psi_\varepsilon(t)\eta_R(x)$ (with ε sufficiently small) and integrating by parts we obtain

$$-\int_0^{+\infty}\int_{\mathbb{R}^N} u\eta_R\psi_\varepsilon'\,\mathrm{d}x\,\mathrm{d}t = \int_0^{+\infty}\int_{\mathbb{R}^N} \psi_\varepsilon \mathbf{a}(u,\nabla u)\cdot\nabla\eta_R\,\mathrm{d}x\,\mathrm{d}t.$$

Since $u \in C([0,+\infty); L^1(\mathbb{R}^N))$, letting $\varepsilon \to 0$ we obtain

$$\left.\int_{\mathbb{R}^N} u\eta_R\,\mathrm{d}x\right|_{t=t_1}^{t=t_2} = \int_{t_1}^{t_2}\int_{\mathbb{R}^N} \mathbf{a}(u,\nabla u)\cdot\nabla\eta_R\,\mathrm{d}x\,\mathrm{d}t. \tag{2.8}$$

Since $u \in L^\infty((0,\infty)\times\mathbb{R}^N)$, (2.1) implies that $|\mathbf{a}(u,\nabla u)| \le Cu$. Therefore

$$\left|\int_{t_1}^{t_2}\int_{\mathbb{R}^N} \mathbf{a}(u,\nabla u)\cdot\nabla\eta_R\,\mathrm{d}x\,\mathrm{d}t\right| \quad\le\quad C\int_{t_1}^{t_2}\int_{B(0,R+1)\setminus B(0,R)} u\,\mathrm{d}x\,\mathrm{d}t \to 0 \quad \text{as} \quad R \to +\infty.$$

Since $u \ge 0$, the two terms on the left-hand side of (2.8) pass to the limit as $R \to +\infty$ by monotone convergence. Therefore

$$\int_{\mathbb{R}^N} u(t_2,x)\,\mathrm{d}x = \int_{\mathbb{R}^N} u(t_1,x)\,\mathrm{d}x.$$

Again since $u \in C([0,+\infty); L^1(\mathbb{R}^N))$, passing to the limit as $t_1 \to 0^+$ we obtain (2.7). □

Remark 2.4. Existence and uniqueness of entropy solutions to the Cauchy problem for equations (1.3) and (1.4) are contained in, or follow from, earlier results in [3], resp. [22]. We refer e.g. to [29] for details. In fact, [3, 22] contain existence and uniqueness results for general classes of equations (1.1) satisfying Assumption 2.1 together with slight additional hypotheses.

To our purposes, we use the notion of subsolution for equations of the form (1.1) suggested by Caselles in [20, Section 3.3]. Such notion is analogous to the one of (entropy) solution, except that (ii) in Definition 2.2 is not required.

Definition 2.5. Let $\tau > 0$ and let $\mathbf{a}$ such that Assumption 2.1 holds. A nonnegative function $u \in C([0,\tau); L^1(\mathbb{R}^N)) \cap L^\infty([0,\tau] \times \mathbb{R}^N)$ is an entropy subsolution to equation (1.1) in $(0,\tau) \times \mathbb{R}^N$ if:

(i) $T^a_{a,b}(u) \in L^1_{loc}((0,\tau); BV(\mathbb{R}^N))$ for all $0 < a < b$;

(ii) inequality (2.5) holds for any $S, T \in \mathcal{T}^+$ and any nonnegative $\phi \in C_c^\infty((0,\tau) \times \mathbb{R}^N)$.

This notion of subsolution yields the following comparison result:

Theorem 2.6. Let $\tau > 0$ and let $\mathbf{a}$ such that Assumption 2.1 holds. Let u be an entropy solution to the Cauchy problem for (1.1) with initial datum $u_0 \in L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$ and $\underline{u}$ be an entropy subsolution to equation (1.1) in $(0,\tau)$. If $\underline{u}(0) \le u_0$, then $\underline{u}(t) \le u(t)$ for all $t \in (0,\tau)$.

Remark 2.7. Theorem 2.6 applies in particular to (1.3) and (1.4) with $\varphi(s) = s^m$, resp. $\varphi(s) = s$, and $\psi_0(\mathbf{v}) = |\mathbf{v}|$. We refer to Remark 1.3 in [29] for details.

As we mentioned in the Introduction, no comparison result is available when subsolutions are defined as in Definition 2.5. Therefore, below we provide a complete and self-contained proof of Theorem 2.6. The proof follows the approach introduced in [2, 3]: however, it also clarifies and simplifies some of the arguments, such as the choice of testing functions (see the comment after (2.12)) and the estimate of I_2 (see (2.16)-(2.26)), easing the overall presentation.

Proof of Theorem 2.6. Let $b > a > 2\varepsilon > 0$, $l \ge 0$, and $T(r) = T^a_{a,b}(r)$. We denote $\mathbf{z} = \mathbf{a}(u, \nabla u)$, $\underline{\mathbf{z}} = \mathbf{a}(\underline{u}, \nabla \underline{u})$,

$$R_{\varepsilon,l}(r) := \begin{cases} T_{l-\varepsilon,l}(r) - (l-\varepsilon) & \text{if } l > 2\varepsilon, \\ T_{\varepsilon,2\varepsilon}(r) - \varepsilon & \text{if } l < 2\varepsilon, \end{cases} \tag{2.9}$$

$$S_{\varepsilon,l}(r) := \begin{cases} T_{l,l+\varepsilon}(r) - l & \text{if } l > \varepsilon, \\ T_{\varepsilon,2\varepsilon}(r) - \varepsilon & \text{if } l < \varepsilon. \end{cases} \tag{2.10}$$

We choose two different pairs of variables $(t,x) \in Q_\tau$, $(\underline{t}, \underline{x}) \in \underline{Q}_\tau := Q_\tau$, and consider u, $\mathbf{z}$ and $\underline{u}$, $\underline{\mathbf{z}}$ as functions of (t,x), resp. $(\underline{t},\underline{x})$. Let $0 \le \phi \in \mathcal{D}((0,\tau))$, ρ_m a sequence of mollifiers in $\mathbb{R}^N$, and $\tilde{\rho}_n$ a sequence of mollifiers in $\mathbb{R}$. Define

$$\eta_{m,n}(t,x,\underline{t},\underline{x}) := \rho_m(x - \underline{x})\tilde{\rho}_n(t - \underline{t})\phi\left(\frac{t + \underline{t}}{2}\right).$$

For $(\underline{t},\underline{x})$ fixed, choosing $S = R_{\varepsilon,\underline{u}}$ in (2.5) we obtain

$$\begin{aligned} -\int_{Q_\tau} J_{T,R_{\varepsilon,\underline{u}}}(u)(\eta_{m,n})_t &+ \int_{Q_\tau} \eta_{m,n}\, \mathrm{d}\left(h_T(u, D_x R_{\varepsilon,\underline{u}}(u)) + h_{R_{\varepsilon,\underline{u}}}(u, D_x T(u))\right) \\ &+ \int_{Q_\tau} \mathbf{z} \cdot \nabla_x \eta_{m,n}\, T(u)\, R_{\varepsilon,\underline{u}}(u) \le 0. \end{aligned} \tag{2.11}$$

Similarly, for (t,x) fixed, choosing $S = S_{\varepsilon,u}$ in (2.5) we obtain

$$\begin{aligned} -\int_{\underline{Q}_\tau} J_{T,S_{\varepsilon,u}}(\underline{u})(\eta_{m,n})_{\underline{t}} &+ \int_{\underline{Q}_\tau} \eta_{m,n}\, \mathrm{d}\left(h_T(\underline{u}, D_{\underline{x}} S_{\varepsilon,u}(\underline{u})) + h_{S_{\varepsilon,u}}(\underline{u}, D_{\underline{x}} T(\underline{u}))\right) \\ &+ \int_{\underline{Q}_\tau} \underline{\mathbf{z}} \cdot \nabla_{\underline{x}} \eta_{m,n}\, T(\underline{u})\, S_{\varepsilon,u}(\underline{u}) \le 0. \end{aligned} \tag{2.12}$$

It seems that one can not directly choose $R_{\varepsilon,\underline{u}} = T_{\underline{u}-\varepsilon,\underline{u}} - (\underline{u}-\varepsilon)$ as test function: indeed, $T_{\underline{u}-\varepsilon,\underline{u}} \notin \mathcal{T}^+$ when $\underline{u} < \varepsilon$ (analogous considerations hold for $S_{\varepsilon,u} = T_{u,u+\varepsilon} - u$ when $u = 0$). This motivates the definitions in (2.9) and (2.10). However, as shown in (2.17)-(2.18) below, such simpler form will be recovered after doubling variables and integrating by parts, due to the presence of the second truncating function T.

Integrating (2.11) in $\underline{Q}_\tau$, (2.12) in Q_τ, adding the two inequalities, taking into account that $\nabla_x \eta_{m,n} + \nabla_{\underline{x}} \eta_{m,n} = 0$, and noting that

$$\int_{Q_\tau \times \underline{Q}_\tau} \eta_{m,n} \,\mathrm{d}\left(h_{R_{\varepsilon,\underline{u}}}(u, D_x T(u)) + h_{S_{\varepsilon,u}}(\underline{u}, D_{\underline{x}} T(\underline{u}))\right) \geq 0,$$

we see that

$$I_1 + I_2 \leq 0, \tag{2.13}$$

where

$$\begin{aligned} I_1 &:= -\int_{Q_\tau \times \underline{Q}_\tau} \left(J_{T,R_{\varepsilon,\underline{u}}}(u)(\eta_{m,n})_t + J_{T,S_{\varepsilon,u}}(\underline{u})(\eta_{m,n})_{\underline{t}}\right) \\ I_2 &:= \int_{Q_\tau \times \underline{Q}_\tau} \eta_{m,n} \,\mathrm{d}\left(h_T(u, D_x R_{\varepsilon,\underline{u}}(u) + h_T(\underline{u}, D_{\underline{x}} S_{\varepsilon,u}(\underline{u}))\right) \\ &\quad - \int_{Q_\tau \times \underline{Q}_\tau} \underline{\mathbf{z}} \cdot \nabla_x \eta_{m,n} T(\underline{u}) S_{\varepsilon,u}(\underline{u}) - \int_{Q_\tau \times \underline{Q}_\tau} \mathbf{z} \cdot \nabla_{\underline{x}} \eta_{m,n} T(u) R_{\varepsilon,\underline{u}}(u). \end{aligned}$$

We now divide (2.13) by ε and let $\varepsilon \to 0$. Concerning I_1, we note that

$$\frac{1}{\varepsilon} J_{T,R_{\varepsilon,l}}(r) = \int_0^r T(s) \frac{R_{\varepsilon,l}(s)}{\varepsilon} \,\mathrm{d}s \to J_T^l(r) := \int_0^r T(s)\,\mathrm{sign}(s-l)_+ \,\mathrm{d}s \tag{2.14}$$

and, analogously, $J_{T,S_{\varepsilon,l}}(r) \to J_T^l(r)$, as $\varepsilon \to 0$. Therefore, by dominated convergence,

$$\frac{I_1}{\varepsilon} \stackrel{\varepsilon \to 0}{\to} -\int_{Q_\tau \times \underline{Q}_\tau} \left((\eta_{m,n})_t J_T^{\underline{u}}(u) + (\eta_{m,n})_{\underline{t}} J_T^u(\underline{u})\right). \tag{2.15}$$

Concerning I_2, after one integration by parts we obtain

$$\begin{aligned} I_2 &= \int_{Q_\tau \times \underline{Q}_\tau} \eta_{m,n} \,\mathrm{d}h_T(u, D_x R_{\varepsilon,\underline{u}}(u)) + \int_{Q_\tau \times \underline{Q}_\tau} \eta_{m,n} T(\underline{u})\, \underline{\mathbf{z}} \cdot \,\mathrm{d}D_x S_{\varepsilon,u}(\underline{u}) \\ &+ \int_{Q_\tau \times \underline{Q}_\tau} \eta_{m,n} \,\mathrm{d}h_T(\underline{u}, D_{\underline{x}} S_{\varepsilon,u}(\underline{u})) + \int_{Q_\tau \times \underline{Q}_\tau} \eta_{m,n} T(u)\, \mathbf{z} \cdot \,\mathrm{d}D_{\underline{x}} R_{\varepsilon,\underline{u}}(u). \end{aligned} \tag{2.16}$$

Due to the presence of T, the second and the fourth integrands in I_2 are nonzero only on $\{\underline{u} > a\}$, resp. $\{u > a\}$. Moreover, for $r > a$, we have

$$R_{\varepsilon,l}(r) = \left\{ \begin{array}{ll} T_{l-\varepsilon,l}(r) - (l-\varepsilon) & \text{if } l > a \\ T_{l-\varepsilon,l}(r) - (l-\varepsilon) = \varepsilon & \text{if } 2\varepsilon < l < a \\ T_{\varepsilon,2\varepsilon}(r) - \varepsilon = \varepsilon & \text{if } l < 2\varepsilon \end{array} \right\} = T_{l-\varepsilon,l}(r) - (l-\varepsilon) \quad \text{for } r > a.$$

Analogously, $S_{\varepsilon,l}(r) = T_{l,l+\varepsilon}(r) - l$ for $r > a$. Therefore, in I_2 we may equivalently consider

$$\begin{aligned} R_{\varepsilon,\underline{u}}(u) &= T_{\underline{u}-\varepsilon,\underline{u}}(u) - (\underline{u}-\varepsilon) = T_{0,\varepsilon}(u-\underline{u}+\varepsilon), && (2.17)\\ S_{\varepsilon,u}(\underline{u}) &= T_{u,u+\varepsilon}(\underline{u}) - u = T_{0,\varepsilon}(\underline{u}-u). && (2.18)\end{aligned}$$

The latter equalities in (2.17)-(2.18) show in particular that $R_{\varepsilon,\underline{u}}(u) + S_{\varepsilon,u}(\underline{u}) \equiv \varepsilon$. Therefore,

$$D_x R_{\varepsilon,\underline{u}}(u) = -D_x S_{\varepsilon,u}(\underline{u}) \quad \text{and} \quad D_{\underline{x}} S_{\varepsilon,u}(\underline{u}) = -D_{\underline{x}} R_{\varepsilon,\underline{u}}(u).$$

Furthermore, letting

$$u_\varepsilon := T_{\underline{u}-\varepsilon,\underline{u}}(u), \quad \underline{u}_\varepsilon := T_{u,u+\varepsilon}(\underline{u}), \tag{2.19}$$

it follows from the former equalities in (2.17)-(2.18) that

$$D_x R_{\varepsilon,\underline{u}}(u) = D_x u_\varepsilon \quad \text{and} \quad D_{\underline{x}} S_{\varepsilon,u}(\underline{u}) = D_{\underline{x}} \underline{u}_\varepsilon.$$

Altogether, I_2 may be rewritten as

$$\begin{aligned} I_2 &= \int_{Q_\tau\times\underline{Q}_\tau} \eta_{m,n}\, \mathrm{d}h_T(u, D_x u_\varepsilon) - \int_{Q_\tau\times\underline{Q}_\tau} \eta_{m,n} T(\underline{u})\, \underline{\mathbf{z}}\cdot \mathrm{d}D_x u_\varepsilon \\ &+ \int_{Q_\tau\times\underline{Q}_\tau} \eta_{m,n}\, \mathrm{d}h_T(\underline{u}, D_{\underline{x}} \underline{u}_\varepsilon) - \int_{Q_\tau\times\underline{Q}_\tau} \eta_{m,n} T(u)\, \mathbf{z}\cdot \mathrm{d}D_{\underline{x}} \underline{u}_\varepsilon. \end{aligned}$$

Let us write$I_2 = I_2(ac) + I_2(s)$, where $I_2(ac)$ and $I_2(s)$ contain the absolutely continuous, resp. singular, part of the measures involved in I_2. Let us first consider $I_2(ac)$. Letting

$$\chi_\varepsilon := \chi_{\{u<\underline{u}<u+\varepsilon\}} = \chi_{\{\underline{u}-\varepsilon<u<\underline{u}\}}, \tag{2.20}$$

in view of (2.19) and (2.17)-(2.18) we have $\nabla_x u_\varepsilon = \chi_\varepsilon \nabla_x u$ and $\nabla_{\underline{x}} \underline{u}_\varepsilon = \chi_\varepsilon \nabla_{\underline{x}} \underline{u}$. Therefore, recalling (2.6), $I_2(ac)$ may be rewritten as

$$\begin{aligned} I_2(ac) &= \int_{Q_\tau\times\underline{Q}_\tau} \eta_{m,n}\chi_\varepsilon (T(u)\mathbf{z} - T(\underline{u})\underline{\mathbf{z}})\cdot(\nabla_x u - \nabla_{\underline{x}}\underline{u}) \\ &= \int_{Q_\tau\times\underline{Q}_\tau} \eta_{m,n}\chi_\varepsilon (T(u) - T(\underline{u}))\mathbf{z}\cdot(\nabla_x u - \nabla_{\underline{x}}\underline{u}) \\ &\quad + \int_{Q_\tau\times\underline{Q}_\tau} \eta_{m,n}\chi_\varepsilon T(\underline{u})(\mathbf{z} - \underline{\mathbf{z}})\cdot(\nabla_x u - \nabla_{\underline{x}}\underline{u}) \\ &=: I_{2,1}(ac) + I_{2,2}(ac). \end{aligned}$$

In view of (2.1) and since $u \in L^\infty(Q)$, we have that $\|\mathbf{z}\|_\infty \le M$. In addition, (2.20) implies that

$$\chi_\varepsilon \cap \{\underline{u} > a\} \subseteq \{u > a - \varepsilon\} \ \text{ and } \quad \chi_\varepsilon \cap \{u > a\} \subseteq \{\underline{u} > a\}. \tag{2.21}$$

Therefore, since T is 1-Lipschitz,

$$\begin{aligned} \frac{1}{\varepsilon}|I_{2,1}(ac)| &\le \frac{M}{\varepsilon}\|\eta_{m,n}\|_\infty \int_{Q_\tau\times\underline{Q}_\tau} \chi_\varepsilon \chi_{\{u>a-\varepsilon\}}\chi_{\{\underline{u}>a\}}|u-\underline{u}||\nabla_x u - \nabla_{\underline{x}}\underline{u}| \\ &\overset{(2.20)}{\le} M\|\eta_{m,n}\|_\infty \int_{Q_\tau\times\underline{Q}_\tau} \chi_\varepsilon \chi_{\{u>a-\varepsilon\}}\chi_{\{\underline{u}>a\}}(|\nabla_x u| + |\nabla_{\underline{x}}\underline{u}|). && (2.22)\end{aligned}$$

Similarly,

$$\begin{aligned}\frac{1}{\varepsilon}I_{2,2}(ac) &\overset{(2.4)}{\geq} -\frac{M}{\varepsilon}\int_{Q_\tau\times\underline{Q}_\tau}\eta_{m,n}\chi_\varepsilon T(\underline{u})|u-\underline{u}|\,|\nabla_x u-\nabla_{\underline{x}}\underline{u}| \\ &\overset{(2.21)}{\geq} -M\|\eta_{m,n}\|_\infty\int_{Q_\tau\times\underline{Q}_\tau}\chi_\varepsilon\chi_{\{\underline{u}>a\}}\chi_{\{u>a-\varepsilon\}}\big(|\nabla_x u|+|\nabla_{\underline{x}}\underline{u}|\big). \quad (2.23)\end{aligned}$$

We claim that

$$\lim_{\varepsilon\to 0}\int_{Q_\tau\times\underline{Q}_\tau}\chi_{\{\underline{u}>a\}}\,\mathrm{d}|D_x u_\varepsilon| = \lim_{\varepsilon\to 0}\int_{Q_\tau\times\underline{Q}_\tau}\chi_{\{u>a-\varepsilon\}}\,\mathrm{d}|D_{\underline{x}}\underline{u}_\varepsilon| = 0. \quad (2.24)$$

Let us show the first one (the second is identical). By the coarea formula, we have for any $l>a$

$$\int_{\mathbb{R}^N}|D_x T_{l-\varepsilon,l}(u)| = \int_{l-\varepsilon}^{l} P(T_{l-\varepsilon,l}(u)>\lambda)\,\mathrm{d}\lambda = \int_{l-\varepsilon}^{l} P(T_{a/2,\ell}(u)>\lambda)\,\mathrm{d}\lambda \to 0$$

as $\varepsilon\to 0$, since $\lambda\mapsto P(T_{a/2,\ell}(u)>\lambda)$ is integrable in $\mathbb{R}$. Then (2.24) follows from dominated convergence, since

$$(t,\underline{x},\underline{t})\mapsto \chi_{\{\underline{u}>a\}}\int_{\mathbb{R}^N}\mathrm{d}|D_x u_\varepsilon| \leq \chi_{\{\underline{u}>a\}}\int_{\mathbb{R}^N}\mathrm{d}|D_x T_{a/2,\ell}(u)| \in L^1((0,\infty)^2\times\mathbb{R}^N).$$

Since $\chi_\varepsilon|\nabla_x u| = |\nabla_x u_\varepsilon|$ and $\chi_\varepsilon|\nabla_{\underline{x}}\underline{u}| = |\nabla_{\underline{x}}\underline{u}_\varepsilon|$, combining (2.22), (2.23), and (2.24) we conclude that

$$\liminf_{\varepsilon\to 0}\frac{I_2(ac)}{\varepsilon}\geq 0. \quad (2.25)$$

Recalling again (2.6), we rewrite $I_2(s)$ as

$$\begin{aligned} I_2(s) &= \int_{Q_\tau\times\underline{Q}_\tau}\eta_{m,n}\left(\psi_0\left(\frac{Du_\varepsilon}{|Du_\varepsilon|}\right)\mathrm{d}|D_x^s J_{T\varphi}(u_\varepsilon)| - T(\underline{u})\underline{\mathbf{z}}\cdot\mathrm{d}D_x^s u_\varepsilon\right) \\ &\quad + \int_{Q_\tau\times\underline{Q}_\tau}\eta_{m,n}\left(\psi_0\left(\frac{Du_\varepsilon}{|Du_\varepsilon|}\right)\mathrm{d}|D_{\underline{x}}^s J_{T\varphi}(\underline{u}_\varepsilon)| - T(u)\mathbf{z}\cdot\mathrm{d}D_{\underline{x}}^s\underline{u}_\varepsilon\right) \\ &:= I_{2,1}(s)+I_{2,2}(s)\end{aligned}$$

and we only consider $I_{2,1}(s)$ ($I_{2,2}(s)$ is treated identically). Using the homogeneity of ψ_0 and Jensen's inequality, we have

$$\begin{aligned} -\underline{\mathbf{z}}\cdot\int_{Q_\tau}\eta_{m,n}\,\mathrm{d}D_x^s u_\varepsilon &\overset{(2.3)}{\geq} -\varphi(\underline{u})\psi_0\left(\int_{Q_\tau}\eta_{m,n}\,\mathrm{d}D_x^s u_\varepsilon\right) \\ &\geq -\varphi(\underline{u})\int_{Q_\tau}\psi_0\left(\frac{D_x^s u_\varepsilon}{|D_x^s u_\varepsilon|}\right)\eta_{m,n}\,\mathrm{d}|D_x^s u_\varepsilon|\,. \end{aligned}$$

Therefore, using the fact that $\chi_{\{\underline{u}>a\}}\leq 1$ we get

$$\begin{aligned} I_{2,1}(s) &\geq \int_{Q_\tau\times\underline{Q}_\tau}\eta_{m,n}\psi_0\left(\frac{Du_\varepsilon}{|Du_\varepsilon|}\right)\chi_{\{\underline{u}>a\}}\left(\mathrm{d}|D_x^s J_{T\varphi}(u_\varepsilon)| - T(\underline{u})\varphi(\underline{u})\,\mathrm{d}|D_x^s u_\varepsilon|\right) \\ &\geq K\int_{Q_\tau\times\underline{Q}_\tau}\eta_{m,n}\chi_{\{\underline{u}>a\}}\left(\mathrm{d}|D_x^s J_{T\varphi}(u_\varepsilon)| - T(\underline{u})\varphi(\underline{u})\,\mathrm{d}|D_x^s u_\varepsilon|\right)\,, \end{aligned}$$

where K is the minimum value of ψ_0 on S^{N-1}. We split its Cantor and jump parts. Since

$$|D_x^c J_{T\varphi}(u_\varepsilon)| = |J'_{T\varphi}(u_\varepsilon)||D_x^c u_\varepsilon| = T(u_\varepsilon)\varphi(u_\varepsilon)|D_x^c u_\varepsilon|,$$

we have

$$I_{2,1}(c) \geq K \int_{Q_\tau \times \underline{Q}_\tau} \eta_{m,n}\chi_{\{\underline{u}>a\}} \left(T(u_\varepsilon)\varphi(u_\varepsilon) - T(\underline{u})\varphi(\underline{u})\right) \, \mathrm{d}|D_x^c u_\varepsilon|.$$

Since T and φ are (locally) Lipschitz continuous and $u, \underline{u}$ are bounded, a positive constant L exists such that

$$\begin{aligned} \frac{1}{\varepsilon} I_{2,1}(c) &\geq -\frac{LK}{\varepsilon} \int_{Q_\tau \times \underline{Q}_\tau} \eta_{m,n}\chi_{\{\underline{u}>a\}} |u_\varepsilon - \underline{u}| d|D_x^c u_\varepsilon| \\ &\geq -LK\|\eta_{m,n}\|_\infty \int_{Q_\tau \times \underline{Q}_\tau} \chi_{\{\underline{u}>a\}} d|D_x^c u_\varepsilon| \overset{(2.24)}{\to} 0 \quad \text{as } \varepsilon \to 0. \end{aligned}$$

Applying the same argument to $I_{2,2}(c)$, we conclude that

$$\liminf_{\varepsilon \to 0} \frac{I_2(c)}{\varepsilon} \geq 0. \tag{2.26}$$

Concerning the jump part, recalling that $\underline{u} - \varepsilon \leq u_\varepsilon \leq \underline{u}$, we have

$$\begin{aligned} \frac{1}{\varepsilon} I_{2,1}(j) \geq \frac{K}{\varepsilon} &\int_{(0,\tau)\times \underline{Q}_\tau} \int_{J_{u_\varepsilon(t)}} \eta_{m,n}\chi_{\{\underline{u}>a\}} \times \\ &\times \left(\left(\int_{u_\varepsilon^-}^{u_\varepsilon^+} T(s)\varphi(s) \, \mathrm{d}s \right) - T(\underline{u})\varphi(\underline{u})|u_\varepsilon^+ - u_\varepsilon^-| \right) \, \mathrm{d}\mathcal{H}^{N-1}(x) \\ = \ & \frac{K}{\varepsilon} \int_{(0,\tau)\times \underline{Q}_\tau} \int_{J_{u_\varepsilon(t)}} \eta_{m,n}\chi_{\{\underline{u}>a\}} \left(\int_{u_\varepsilon^-}^{u_\varepsilon^+} \left(T(s)\varphi(s) - T(\underline{u})\varphi(\underline{u})\right) \mathrm{d}s \right) \, \mathrm{d}\mathcal{H}^{N-1}(x) \\ \geq \ & -\frac{LK}{\varepsilon} \int_{(0,\tau)\times \underline{Q}_\tau} \int_{J_{u_\varepsilon(t)}} \eta_{m,n}\chi_{\{\underline{u}>a\}} \left(\int_{u_\varepsilon^-}^{u_\varepsilon^+} (\underline{u} - s) \, \mathrm{d}s \right) \, \mathrm{d}\mathcal{H}^{N-1}(x) \\ \geq \ & -LK\|\eta_{m,n}\|_\infty \int_{(0,\tau)\times \underline{Q}_\tau} \int_{J_{u_\varepsilon(t)}} \chi_{\{\underline{u}>a\}} \left(u_\varepsilon^+ - u_\varepsilon^-\right) \, \mathrm{d}\mathcal{H}^{N-1}(x) \\ = \ & -LK\|\eta_{m,n}\|_\infty \int_{Q_\tau \times \underline{Q}_\tau} \chi_{\{\underline{u}>a\}} \, \mathrm{d}|D_x^j u_\varepsilon| \overset{(2.24)}{\to} 0 \quad \text{as } \varepsilon \to 0. \end{aligned}$$

By applying the same argument on $I_{2,2}(j)$, we conclude that

$$\liminf_{\varepsilon \to 0} \frac{I_2(j)}{\varepsilon} \geq 0. \tag{2.27}$$

Collecting (2.15), (2.25), (2.26), and (2.27) into (2.13) and using (2.15), we conclude that

$$-\int_{Q_\tau \times \underline{Q}_\tau} \left((\eta_{m,n})_t J_T^{\underline{u}}(u) + (\eta_{m,n})_{\underline{t}} J_T^u(\underline{u})\right) \leq 0.$$

Since the latter expression does not contain any spatial gradient, we can divide it by $b-a$ and easily pass it to the limit as $m\to\infty$, $a\to 0$, and $b\to 0$, in this order. Noting that

$$\lim_{b\to 0}\lim_{a\to 0}\frac{1}{b-a}T(s)=\mathrm{sign}(s),$$

$$\lim_{b\to 0}\lim_{a\to 0}\frac{1}{b-a}J_T^l(r)\overset{(2.14)}{=}\int_0^r \mathrm{sign}(s-l)_+\,\mathrm{d}s=(r-l)_+,$$

we obtain

$$-\int_{\underline{(0,\tau)}\times Q_\tau}((u-\underline{u})_+(\chi_n)_t+(\underline{u}-u)_+(\chi_n)_{\underline{t}})\le 0, \tag{2.28}$$

where

$$u=u(t,x),\quad \underline{u}=\underline{u}(\underline{t},x),\quad \chi_n=\tilde{\rho}_n(t-\underline{t})\phi\left(\frac{t+\underline{t}}{2}\right),\quad (\chi_n)_t+(\chi_n)_{\underline{t}}=\tilde{\rho}_n\phi'. \tag{2.29}$$

We write

$$\begin{aligned}
-\int_{\underline{(0,\tau)}\times Q_\tau}(\underline{u}-u)_+\tilde{\rho}_n\phi' &\overset{(2.29)}{=} -\int_{\underline{(0,\tau)}\times Q_\tau}(\underline{u}-u)_+((\chi_n)_t+(\chi_n)_{\underline{t}})\\
&\overset{(2.28)}{=} -\int_{\underline{(0,\tau)}\times Q_\tau}((\underline{u}-u)_+-(u-\underline{u})_+)\,(\chi_n)_t\\
&= \int_{\underline{(0,\tau)}\times Q_\tau}(u-\underline{u})(\chi_n)_t \overset{(2.29)}{=}\int_{Q_\tau}u(\chi_n)_t=0,
\end{aligned}$$

where in the last equality we used Proposition 2.3. Letting $n\to\infty$, we obtain

$$-\int_{Q_\tau}(\underline{u}(t,x)-u(t,x))_+\,\phi'(t)\,\mathrm{d}t\,\mathrm{d}x\le 0.$$

Since this is true for all $0\le\phi\in\mathcal{D}((0,\tau))$, it implies

$$\int_{\mathbb{R}^N}(\underline{u}(t,x)-u(t,x))_+\,\mathrm{d}x\le\int_{\mathbb{R}^N}(\underline{u}(0)-u_0)_+\,\mathrm{d}x\le 0\quad\text{for all }\ t\ge 0.$$

□

3 The speed-limited porous medium equation

With Theorem 2.6 at hand, we can now focus on the analysis of the waiting time phenomenon. Here and in the next section we will prove Theorem 1.1. We begin by considering (1.4), which is slightly simpler since the subsolutions we construct are continuous: they are of the form

$$u(t,x)=b^{\frac{1}{1-M}}\left(\ell-\frac{1}{1+wt}\right)^{\frac{1}{1-M}}\left(1-\frac{|x|^2}{(1+wt)^2}\right)_+^{\frac{1}{M-1}}. \tag{3.1}$$

The form of the x-depending factor is chosen such that the subsolution's support evolves with constant speed w; the exponent $1/(M-1)$ is chosen in order to ease the calculation of ∇u^{M-1}, but we expect that this choice is unessential. The form of the first, x-independent factor is then chosen consistently, and its exponent is dictated by the homogeneity of (1.4) for $|\nabla u^{M-1}|\ll 1$ (see (3.11) below). The following holds:

Lemma 3.1. For all $b > 0$, $\ell > 1$, $K > 0$, and $w > 0$ such that

$$\frac{2N(M-1)}{b} \leq w \leq \frac{1}{\sqrt{1+\frac{b^2}{4}(\ell-1+\frac{\ell}{K})^2}}\,, \tag{3.2}$$

the function u defined in (3.1) is a subsolution to (1.4) in $\left(0,\frac{1}{wK}\right)\times\mathbb{R}^N$.

Proof. (1) Rewriting entropy inequalities. Since $u \in W^{1,1}((0,1/(wK))\times\mathbb{R}^N)$, (2.5) reduces to a single inequality:

$$\begin{aligned} &S(T^0(u))h(T^0(u),\nabla T^0(u)) + T(S^0(u))h(S^0(u),\nabla S^0(u)) \\ &\leq \quad -(J_{TS}(u))_t + \operatorname{div}(T(u)S(u)\mathbf{a}(u,\nabla u)) \end{aligned} \tag{3.3}$$

for all $S,T \in \mathcal{T}^+$. Notice that, since $\nabla S(u) = \nabla S^0(u)$ and $u = S^0(u)$ on $\operatorname{supp}(\nabla S^0(u))$, we have

$$\begin{aligned} T(u)\mathbf{a}(u,\nabla u)\cdot\nabla S(u) &= T(u)\,\mathbf{a}(u,\nabla u)\cdot\nabla S^0(u) \\ &= T(S^0(u))\,\mathbf{a}(S^0(u),\nabla S^0(u))\cdot\nabla S^0(u) \\ &= T(S^0(u))\,h(S^0(u),\nabla S^0(u)). \end{aligned} \tag{3.4}$$

Analogously,

$$S(u)a(u,\nabla u)\cdot\nabla T(u) = S(T^0(u))h(T^0(u),\nabla T^0(u)). \tag{3.5}$$

In view of (3.4) and (3.5), (3.3) translates into

$$\begin{aligned} &\frac{u\nabla u^{M-1}\cdot\nabla u}{\sqrt{1+|\nabla u^{M-1}|^2}}(S(u)T'(u)+T(u)S'(u)) \\ &\leq \quad -T(u)S(u)u_t + \operatorname{div}\left(T(u)S(u)\frac{u\nabla u^{M-1}}{\sqrt{1+|\nabla u^{M-1}|^2}}\right), \end{aligned}$$

i.e.

$$T(u)S(u)\left(u_t - \operatorname{div}\left(\frac{u\nabla u^{M-1}}{\sqrt{1+|\nabla u^{M-1}|^2}}\right)\right) \leq 0.$$

Since $T(u)S(u) \geq 0$, (3.3) is equivalent to

$$u_t \leq \operatorname{div}\left(\frac{u\nabla u^{M-1}}{\sqrt{1+|\nabla u^{M-1}|^2}}\right). \tag{3.6}$$

(2) Constructing subsolutions. We look for subsolutions of the form

$$u(t,x) = \frac{1}{A(t)}f\left(\frac{x}{B(t)}\right), \quad f(y) = (1-|y|^2)^{\frac{1}{M-1}}, \quad B(t) = 1+wt.$$

We notice that

$$u_t = -\frac{A'}{A^2}f - \frac{B'}{AB}\nabla_y f\cdot y \tag{3.7}$$

and that, since $\nabla_y f^{M-1} = -2y$,

$$\operatorname{div}\left(\frac{u\nabla u^{M-1}}{\sqrt{1+|\nabla u^{M-1}|^2}}\right) = \frac{1}{A^M B^2}\operatorname{div}_y\left(\frac{f\nabla_y f^{M-1}}{D}\right) = -\frac{2}{A^M B^2}\operatorname{div}_y\left(\frac{fy}{D}\right),$$

where

$$D := \sqrt{1+|\nabla u^{M-1}|^2} = \sqrt{1+\frac{4|y|^2}{A^{2M-2}B^2}}.$$

Therefore

$$\operatorname{div}\left(\frac{u\nabla u^{M-1}}{\sqrt{1+|\nabla u^{M-1}|^2}}\right) = -\frac{2}{A^M B^2}\left(\frac{Nf}{D} + \frac{y\cdot\nabla_y f}{D} + |y|\frac{\mathrm{d}}{\mathrm{d}|y|}\left(\frac{f}{D}\right)\right),$$

and after straightforward computations we obtain that

$$\operatorname{div}\left(\frac{u\nabla u^{M-1}}{\sqrt{1+|\nabla u^{M-1}|^2}}\right) = -\frac{2y\cdot\nabla_y f}{A^M B^2 D} - \frac{2Nf}{A^M B^2 D^3}\left(1+\frac{N-1}{N}\frac{4|y|^2}{A^{2M-2}B^2}\right). \tag{3.8}$$

Observing that $\nabla_y f(y)\cdot y \le 0$, (3.7) and (3.8) show that (3.6) is implied by the following two inequalities:

$$\frac{A'}{A^2} \ge \frac{2N}{A^M B^2}\cdot\frac{\left(1+\frac{(N-1)}{N}\frac{4|y|^2}{A^{2(M-1)}B^2}\right)}{\left(1+\frac{4|y|^2}{A^{2(M-1)}B^2}\right)^{3/2}}, \tag{3.9}$$

$$\frac{B'}{AB} \le \frac{2}{A^M B^2\left(1+\frac{4|y|^2}{A^{2(M-1)}B^2}\right)^{1/2}}\ . \tag{3.10}$$

Since the second factor on the right-hand side of (3.9) is decreasing with respect to $\frac{4|y|^2}{A^{2(M-1)}B^2}$, (3.9) is in turn implied by

$$\frac{A'}{A^2} \ge \frac{2N}{A^M B^2}\,,\quad \text{i.e.}\quad (A^{M-1})'B^2 \ge 2N(M-1). \tag{3.11}$$

Therefore, choosing

$$A(t) = b^{\frac{1}{M-1}}\left(\ell - \frac{1}{1+wt}\right)^{\frac{1}{M-1}},$$

we see that (3.11) is satisfied if

$$(A^{M-1})'B^2 = bw \ge 2N(M-1)\,. \tag{3.12}$$

On the other hand, (3.10) is implied by

$$w = B' \le \min_{t,y}\frac{1}{\sqrt{\frac{A^{2(M-1)}B^2}{4}+|y|^2}} = \frac{1}{\sqrt{1+\frac{b^2}{4}\left(\ell-1+\frac{\ell}{K}\right)^2}}, \tag{3.13}$$

where in the last step we used

$$A^{2(M-1)}B^2 = b^2(\ell-1+\ell wt)^2 \le b^2\left(\ell-1+\frac{\ell}{K}\right)^2 \quad \text{for all } t\in(0,\tfrac{1}{wK}).$$

Combining the conditions in (3.12) and (3.13), we obtain the condition in (3.2) and the proof is complete. □

By scaling, we obtain the following family of subsolutions.

Corollary 3.2. If $b > 0$, $\ell > 1$ $K > 0$, and $w > 0$ are such that (3.2) holds, then for any $s > 0$ and any $\xi \in \mathbb{R}^N$ the function

$$\underline{u}(t,x) = b^{\frac{1}{1-M}} \left(\frac{\ell}{s} - \frac{1}{s+wt}\right)^{\frac{1}{1-M}} \left(1 - \frac{|\xi - x|^2}{(s+wt)^2}\right)_+^{\frac{1}{M-1}}, \tag{3.14}$$

is a subsolution to (1.4) in $(0, \frac{s}{wK}) \times \mathbb{R}^N$.

Proof. We use the scaling invariance of (1.4) with respect to the following transformations:

$$u = U\underline{u}, \quad x = U^{M-1}\underline{x}, \quad \text{and} \quad t = U^{M-1}\underline{t}\,.$$

By Lemma 3.1, provided (3.2) holds,

$$\begin{aligned} \underline{u}(\underline{t},\underline{x}) &= \frac{1}{U} u(U^{M-1}\underline{t}, U^{M-1}\underline{x}) \\ &= \frac{b^{\frac{1}{1-M}}}{U} \left(\ell - \frac{1}{1+wU^{M-1}\underline{t}}\right)^{\frac{1}{1-M}} \left(1 - \frac{U^{2M-2}|\underline{x}|^2}{(1+wU^{M-1}\underline{t})^2}\right)_+^{\frac{1}{M-1}} \\ &= b^{\frac{1}{1-M}} \left(\frac{\ell}{U^{1-M}} - \frac{1}{U^{1-M}+w\underline{t}}\right)^{\frac{1}{1-M}} \left(1 - \frac{|\underline{x}|^2}{(U^{1-M}+w\underline{t})^2}\right)_+^{\frac{1}{M-1}} \end{aligned}$$

is a subsolution to (1.4) in $\left(0, \frac{U^{1-M}}{wK}\right) \times \mathbb{R}^N$. The result follows replacing U^{1-M} by s, $(\underline{t}, \underline{x})$ by (t,x), and using translation invariance in space. □

We are ready to prove Theorem 1.1 in the case of Equation (1.4).

Proof of Theorem 1.1: Equation (1.4). Up to a translation and a rotation, we assume without losing generality that $x_0 = 0$ and that $v_0 = (-1, 0, \dots, 0)$. We consider the case $L < +\infty$ in (1.10) (from which the case $L = +\infty$ follows immediately).

We wish to choose the parameters in Lemma 3.1 so that the function $\underline{u}$ given in (3.14) is a subsolution to (1.4) and $\underline{u}(0) \le u_0$. We fix

$$K = 2N(M-1). \tag{3.15}$$

In Corollary (3.2) we let $w = K/b$, so that condition (3.2) reduces to

$$\frac{K}{b} \le \frac{1}{\sqrt{1 + \frac{b^2}{4}(\ell - 1 + \frac{\ell}{K})^2}}\,,$$

that is,

$$\frac{4}{b^2} + \left(\ell - 1 + \frac{\ell}{K}\right)^2 \le \frac{4}{K^2}. \tag{3.16}$$

By (1.10), $R > 0$ exists such that

$$u(0,x) \ge \frac{L}{2}|x|^{\frac{2}{M-1}} \quad \text{in } B(Rv_0, R) \subseteq \operatorname{supp}(u_0)\,.$$

Let $\xi = r_1 v_0$ with

$$0 < r_1 \leq \min\{R, L^{1-M}\}.$$

Then $\underline{u}(0) \leq u_0$ if

$$b^{\frac{1}{1-M}} \left(\frac{\ell-1}{s}\right)^{\frac{1}{1-M}} \left(1 - \frac{|r_1 v_0 - x|^2}{s^2}\right)_+^{\frac{1}{M-1}} \leq \frac{L}{2}|x|^{\frac{2}{M-1}} \quad \text{in } B(Rv_0, R)\,,$$

which may be rewritten as

$$\frac{1}{sb(\ell-1)}\left(s^2 - |r_1 v_0 - x|^2\right)_+ \leq \frac{L^{M-1}}{2^{M-1}}|x|^2 \quad \text{in } B(Rv_0, R)\,. \tag{3.17}$$

We are now going to choose $\ell > 1$, $s > 0$, and $b > 0$ such that (3.16) and (3.17) hold. Let

$$b = \frac{\alpha 2^{M-1} L^{1-M}}{s(\ell-1)} > 0, \tag{3.18}$$

where $\alpha > 0$, *depending only on N and M*, will be chosen below. Then(3.17) reduces to

$$s^2 \leq |r_1 v_0 - x|^2 + \alpha |x|^2 \quad \text{in } B(Rv_0, R).$$

Since the minimum value of the right-hand side is attained at $x = r_1 v_0/(\alpha+1)$, (3.17) is in turn implied by

$$s := \frac{\alpha}{\alpha+1} r_1. \tag{3.19}$$

In view of (3.18) and (3.19), (3.16) may be rewritten as

$$\frac{4r_1^2(\ell-1)^2}{(\alpha+1)^2 4^{M-1} L^{2(1-M)}} + \left(\ell - 1 + \frac{\ell}{K}\right)^2 \leq \frac{4}{K^2}. \tag{3.20}$$

Since $r_1 \leq L^{1-M}$, (3.20) is implied by

$$\frac{4(\ell-1)^2}{(\alpha+1)^2 4^{M-1}} + \left(\ell - 1 + \frac{\ell}{K}\right)^2 \leq \frac{4}{K^2}.$$

Therefore we can choose ℓ, *depending only on N and M*, so close to 1 that (3.16) holds. Hence $\underline{u}$ given in (3.14) is a subsolution to (1.4) in $\left(0, \frac{s}{wK}\right) \times \mathbb{R}^N$ and $\underline{u}(0) \leq u_0$.

We finally estimate t_*. The time T_u at which the support of $\underline{u}$ reaches $x_0 = 0$ is given by

$$T_u \overset{(3.14)}{=} \frac{r_1 - s}{w} \overset{(3.19)}{=} \frac{r_1}{(\alpha+1)w} = \frac{s}{\alpha w}.$$

We now choose $\alpha = 2K$ (recall (3.15)), so that $T_u < \frac{s}{wK}$. Therefore $\underline{u}$ does reach $x_0 = 0$, and recalling that $w = K/b$ we obtain

$$T_u = \frac{s}{\alpha w} = \frac{sb}{\alpha K} \overset{(3.18)}{=} \frac{2^{M-1}L^{1-M}}{K(\ell-1)} =: W L^{1-M}\,,$$

with W depending only on N and M. Therefore $t_* \leq T_*$, which concludes the proof of Theorem 1.1 in the case of Equation (1.4). $\square$

4 The relativistic porous medium equation

As we observed in the introduction, in case of (1.3) it is natural to look for subsolutions with a jump discontinuity at the boundary of their support:

$$u(t,x)=\frac{1}{A(t)}\left(1+\sqrt{r(t)^2-|x|^2}\right)\chi_{Q_0}(t,x), \tag{4.1}$$

where

$$\begin{aligned} A(t) &= [(m-1)(1+\gamma t)]^{\frac{1}{m-1}}\,, && (4.2)\\ r(t) &= r_0+\frac{1}{\gamma(m-1)}\log(1+\gamma t)\,, && (4.3)\\ Q_0 &= \{(t,x):\ t\in(0,T),\ x\in B(0,r(t))\}\,. && (4.4)\end{aligned}$$

The square root in (4.1) is chosen for convenience and we expect that it can be replaced by any exponent smaller than 1 (see (1.15)). As we discussed in the introduction, the functions A and r are chosen so that $r'=A^{1-m}$ –which is dictated by a Rankine-Hugoniot condition at the jump set ∂Q_0, see (4.11) below– and that $(A^{m-1})'$ is constant –which is dictated by homogeneity, see (4.14) below.

In this section we prove:

Proposition 4.1. Let $N\geq 1$, $m>1$, $T>0$ and $r_1>0$. Then there exist a value $\gamma_0\geq 1$ such that the function u defined by (4.1)-(4.4) is a subsolution to (1.3) for any $\gamma\geq\gamma_0$ and any $r_0\in\left[\frac{r_1}{2},r_1\right]$.

Proof. (1) Splitting entropy inequalities. Let u as in (4.1). We note that (2.5) may be rewritten in form of two inequalities between measures, splitting Lebesgue and singular parts:

$$\begin{aligned} &S(T^0(u))h(T^0(u),\nabla T^0(u))+T(S^0(u))h(S^0(u),\nabla S^0(u))\\ &\quad\leq\ -(J_{TS}(u))_t+(\mathrm{div}(T(u)S(u)\mathbf{a}(u,\nabla u)))^a \end{aligned} \tag{4.5}$$

and

$$\begin{aligned} &|D^s_x J_{S\varphi}(T^0(u))|+|D^s_x J_{T\varphi}(S^0(u))|\\ &\quad\leq\ -(D_t(J_{TS}(u)))^s+(\mathrm{div}(T(u)S(u)\mathbf{a}(u,\nabla u)))^s, \end{aligned} \tag{4.6}$$

where $S,T\in\mathcal{T}^+$ and μ^a, resp. μ^s, denote the absolutely continuous, resp. singular, part of the Radon-Nikodym decomposition of a measure μ with respect to the Lebesgue measure (see [1, Theorem 1.28]). We discuss the two inequalities separately.

(2) Subsolutions on the jump set. Let us check (4.6). We note that the singular parts are concentrated on $|x|=r(t)$: hence

$$\begin{aligned} |D^s_x J_{S\varphi}(T^0(u))| &= \left(\int_{T^0(u^-)}^{T^0(u^+)} S(\sigma)\sigma^m\,\mathrm{d}\sigma\right)\mathcal{H}^{N-1}\llcorner\{|x|=r\}\\ &= \left(\int_{u^-}^{u^+} S(\sigma)T'(\sigma)\sigma^m\,\mathrm{d}\sigma\right)\mathcal{H}^{N-1}\llcorner\{|x|=r\} \end{aligned}$$

and, analogously,

$$|D_x^s J_{T\varphi}(S^0(u))| \;=\; \left(\int_{u^-}^{u^+} S'(\sigma)T(\sigma)\sigma^m \,\mathrm{d}\sigma\right)\mathcal{H}^{N-1}\llcorner\{|x|=r\}.$$

Therefore

$$\begin{aligned}&|D_x^s J_{S\varphi}(T^0(u))| + |D_x^s J_{T\varphi}(S^0(u))|\\ &= \left(\int_0^{u^+} (S(\sigma)T(\sigma))'\sigma^m \,\mathrm{d}\sigma\right)\mathcal{H}^{N-1}\llcorner\{|x|=r\}\,. \end{aligned} \tag{4.7}$$

For the first term on the right-hand side of (4.6), arguing as in [29, proof of (3.3)] we obtain

$$\begin{aligned} -(D_t(J_{TS}(u)))^s &= -r'\left(\int_0^{u^+} S(\sigma)T(\sigma)\,\mathrm{d}\sigma\right)\mathcal{H}^{N-1}\llcorner\{|x|=r\}\\ &= \;-r'\left(u^+T(u^+)S(u^+) - \int_0^{u^+}(S(\sigma)T(\sigma))'\sigma\,\mathrm{d}\sigma\right)\mathcal{H}^{N-1}\llcorner\{|x|=r\}, \end{aligned} \tag{4.8}$$

where we used one integration by parts in the last equality. Finally, for the the second term on the right-hand side of (4.6), we have

$$(\mathrm{div}(T(u)S(u)\mathbf{a}(u,\nabla u)))^s = -\lim_{|x|\to r(t)^-} T(u)S(u)\mathbf{a}(u,\nabla u)\cdot\frac{x}{r}\mathcal{H}^{N-1}\llcorner\{|x|=r\}.$$

The fact that ∇u blows up at the boundary implies that

$$\lim_{|x|\to r(t)^-}\mathbf{a}(u,\nabla u)\cdot\frac{x}{r} = -(u^+)^{-m}.$$

Therefore

$$(\mathrm{div}(T(u)S(u)\mathbf{a}(u,\nabla u)))^s = T\left(u^+\right)S\left(u^+\right)(u^+)^m\,\mathcal{H}^{N-1}\llcorner\{|x|=r\}. \tag{4.9}$$

Combining (4.7), (4.8), and (4.9), we see that (4.6) is equivalent to

$$\int_0^{u^+}(S(\sigma)T(\sigma))'\sigma(\sigma^{m-1}-r')\,\mathrm{d}\sigma \le u^+T(u^+)S(u^+)((u^+)^{m-1}-r'). \tag{4.10}$$

Since r and A have been chosen such that

$$r' = A^{1-m} = (u^+)^{m-1}, \tag{4.11}$$

the left-hand side of (4.10) is negative (since $(ST)'\ge 0$) and the right-hand side of (4.10) is zero. Hence (4.6) holds.

(3) Subsolution in the bulk. In Q_0, arguing as in Step (1) in the proof of Lemma 3.1, we obtain that (4.5) is equivalent to

$$u_t \le \mathrm{div}\left(\frac{u^m\nabla u}{\sqrt{u^2+|\nabla u|^2}}\right)\quad \text{in } Q_0. \tag{4.12}$$

We look for subsolutions of the form (4.1). For notational convenience, we let

$$\begin{aligned} \eta(t,x) &:= \sqrt{r(t)^2-|x|^2}, \\ D(t,x) &:= \sqrt{\eta^2(1+\eta)^2+|x|^2}, \\ E(t,x) &:= (1+\eta)^2+\eta(1+\eta)-1. \end{aligned}$$

Then, we compute

$$u_t = -\frac{A'}{A^2}(1+\eta)+\frac{rr'}{A\eta}$$

and, since $\nabla\eta = -x/\eta$,

$$\begin{aligned} A(t)^m \operatorname{div}\left(\frac{u^m\nabla u}{\sqrt{u^2+|\nabla u|^2}}\right) &= -\operatorname{div}\left(\frac{(1+\eta)^m x}{D}\right) \\ &= -\left(N\frac{(1+\eta)^m}{D}+|x|\left(\frac{d}{d|x|}\frac{(1+\eta)^m}{D}\right)\right) \\ &= -\left(N\frac{(1+\eta)^m}{D}+\frac{|x|^2(1+\eta)^m E}{D^3}-\frac{m|x|^2(1+\eta)^{m-1}}{\eta D}\right) \\ &=: F(|x|,r). \end{aligned}$$

Therefore u satisfies (4.12) if and only if

$$-A^{m-2}A'(1+\eta)+A^{m-1}\frac{r}{\eta}r' \overset{(4.11)}{=} -A^{m-2}A'(1+\eta)+\frac{r}{\eta} \le F(r,|x|)\,. \tag{4.13}$$

Since A has been chosen such that

$$A^{m-2}A' = \gamma, \tag{4.14}$$

it follows from (4.13) that (4.12) is satisfied if and only if

$$\gamma \ge \frac{\frac{r}{\eta}-F(r,|x|)}{1+\eta} := G(r,|x|)\,. \tag{4.15}$$

Observe that $\gamma \mapsto \frac{1}{\gamma}\log(1+\gamma t)$ is nonincreasing. Therefore, for $\gamma \ge 1$ and $t \le T$ we have

$$\begin{aligned} \frac{r_1}{2} \le r(t) &= r_0+\frac{1}{\gamma(m-1)}\log(1+\gamma t) \le r_0+\frac{1}{(m-1)}\log(1+t) \\ &\le r_1+\frac{1}{(m-1)}\log(1+T) \end{aligned}$$

Hence, by (4.15), (4.12) is satisfied if

$$\gamma \ge \gamma_0 := \sup_{(r,|x|)\in H} G(r,|x|)\,,$$

where

$$H = \left\{(r,y)\in\mathbb{R}^2_+ :\ r \in \left[\frac{r_1}{2}, r_1+\frac{1}{(m-1)}\log(1+T)\right],\ y\in[0,r)\right\}.$$

Since $\eta = \sqrt{\rho^2 - y^2} \to 0$ as $(\rho, y) \to (r, r)$, we have

$$G(\rho, y)\eta = \frac{\rho - my^2(1+\eta)^{m-1}D^{-1}}{1+\eta} + o(1) \to r(1-m) < 0 \quad \text{as } (\rho, y) \to (r, r).$$

In addition, $G(r, y)$ is continuous in H: therefore γ_0 is finite. Since γ_0 only depends on N, m, T, and r_1, the proof is complete. □

Using the invariance of (1.3) with respect to

$$\hat{u} = Uu, \quad \text{and} \quad \hat{t} = U^{m-1}t$$

and the translation invariance of (1.3) with respect to x, we immediately obtain:

Corollary 4.2. Let $N \geq 1$, $m > 1$, $T > 0$, and $r_1 > 0$. Then $\gamma_0 \geq 1$ exists such that

$$\underline{u}(t, x) = Uu(U^{m-1}t, x - \xi) \tag{4.16}$$

is a subsolution to (1.3) in $(0, U^{1-m}T) \times \mathbb{R}^N$ for any $\gamma \geq \gamma_0$, any $r_0 \in [r_1/2, r_1]$, any $U > 0$, and any $\xi \in \mathbb{R}^N$, where u is defined by (4.1)-(4.4).

We are ready to prove Theorem 1.1 for equation (1.3).

Proof of Theorem 1.1: Equation (1.3). As for (1.4), we may assume that $x_0 = 0$, $v_0 = (-1, 0, \ldots, 0)$, and $L < +\infty$ in (1.9).

Let $\xi = r_1 v_0$ and $T = 4^{m+1}L^{1-m}$. We will choose the parameters $r_1 > 0$, $r_0 \in [r_1/2, r_1]$, and $U > 0$ in Corollary 4.2 such that the function $\underline{u}(t, x)$ in (4.16) is a subsolution to (1.3) up to time $U^{1-m}T$. By (1.9), $R > 0$ exists such that

$$u(0, x) \geq \frac{L}{2}|x|^{\frac{1}{m-1}} \quad \text{in } B(Rv_0, R).$$

Hence, provided that $r_1 \leq R$, it suffices to verify that

$$\underline{u}(0, x)^{m-1} = \frac{U^{m-1}}{m-1}\left(1 + \sqrt{r_0^2 - |x - r_1 v_0|^2}\right)^{m-1} \leq \frac{L^{m-1}}{2^{m-1}}|x| \quad \text{in } B(r_1 v_0, r_0). \tag{4.17}$$

Since $|x| \geq r_1 - r_0$ in $B(r_1 v_0, r_0)$, (4.17) is implied by

$$\frac{U^{m-1}}{m-1}(1 + r_0)^{m-1} \leq \frac{L^{m-1}}{2^{m-1}}(r_1 - r_0). \tag{4.18}$$

We fix

$$r_1 \leq \min\left\{R, \frac{1}{m-1}, 1\right\}, \quad \gamma = \max(\gamma_0, 2), \quad \text{and} \quad r_0 = r_1\frac{\gamma - 1}{\gamma}$$

(note that $\frac{r_1}{2} \leq r_0 < r_1$ since $\gamma \geq 2$). Then (4.18) reduces to

$$\frac{U^{m-1}}{m-1}\left(1 + r_1\frac{\gamma - 1}{\gamma}\right)^{m-1} \leq \frac{L^{m-1}}{2^{m-1}}\,\frac{r_1}{\gamma}\,,$$

which, since $r_1 \le 1$, is implied by

$$\frac{2^{m-1}U^{m-1}}{m-1} \le \frac{L^{m-1}}{2^{m-1}}\,\frac{r_1}{\gamma},$$

which in turn holds true choosing

$$U^{m-1} = (m-1)\frac{L^{m-1}}{4^{m-1}}\,\frac{r_1}{\gamma}. \tag{4.19}$$

The time T_u at which the support of $\underline{u}$ reaches $x_0 = 0$ is implicitly defined by

$$r_1 = r_1\frac{\gamma-1}{\gamma} + \frac{1}{\gamma(m-1)}\log(1+\gamma U^{m-1}T_u),$$

or, equivalently,

$$r_1 = \frac{1}{(m-1)}\log(1+\gamma U^{m-1}T_u),$$

that is, recalling that $(m-1)r_1 \le 1$,

$$T_u = \frac{e^{(m-1)r_1}-1}{\gamma U^{m-1}} \overset{(4.19)}{=} 4^{m-1}L^{1-m}\frac{e^{(m-1)r_1}-1}{(m-1)r_1} \le 4^m L^{1-m} := WL^{1-m} < T.$$

Therefore $t_* \le WL^{1-m}$, which concludes the proof of Theorem 1.1. □

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