NONLOCAL PROBLEMS WITH SINGULAR NONLINEARITY

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Abstract. We investigate existence and uniqueness of solutions for a class of nonlinear nonlocal problems involving the fractional p-Laplacian operator and singular nonlinearities.

Contents

1. Introduction

Let $p \in (1, \infty)$, $s \in (0, 1)$ and $\gamma > 0$. Let $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain with $N > sp$. We shall consider the following nonlocal quasilinear singular problem

$$
\begin{cases} (-\Delta)_p^s\, u = \frac{f(x)}{u^\gamma} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{1.1}
$$

where $(-\Delta)_p^s$ is the fractional p-Laplacian operator, formally defined by

$$
(-\Delta)_p^s\, u(x) := 2 \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{|u(x) - u(y)|^{p-2}\, (u(x) - u(y))}{|x - y|^{N+s\,p}}\, dy, \qquad x \in \mathbb{R}^N.
$$

The aim of the paper is to prove the existence and the uniqueness of the solution to (1.1).

Let us first discuss the semilinear local case, $s = 1$, $p = 2$. In this setting the study of singular elliptic equations goes back to the pioneering work [12]. Avoiding to disclose the discussion, we only mention here the contributions in [3, 7–9, 17, 19, 21–23, 28], that settled up the issue in the semi-linear local case. The reader could be interested in observing that, in this case, by a simple change of variables it follows that the problem is also related to problems involving a first order term of the type $\frac{|\nabla u|^2}{u}$. We refer the readers to [1, 4, 18] for related results in this setting. To deal with singular problems, we have to face the fact that solutions are not in general in classical Sobolev spaces, because of the lack of regularity near the boundary. If already we consider the semi-linear local case it has been shown in [23] that the solution cannot belong to $H_0^1(\Omega)$ if $\gamma \geq 3$. Let us now state our result. Note that, due to the lack of regularity of the solutions near the boundary, the notion of solution has to be understood in the weak distributional meaning, for test functions compactly supported in the domain. Furthermore, the nonlocal nature of the operator has to be

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taken into account. Having this remarks in mind, the basic definition of solution can be formulated in the following:

Definition 1.1. A positive function $u \in W^{s,p}_{\mathrm{loc}}(\Omega) \cap L^{p-1}(\Omega)$ is a weak solution to problem (1.1) if

$$u^{\max\{\frac{\gamma+p-1}{p},1\}} \in W^{s,p}_0(\Omega), \qquad \frac{f(x)}{u^\gamma} \in L^1_{\mathrm{loc}}(\Omega),$$

and we have

$$\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}\,(u(x)-u(y))\,(\varphi(x)-\varphi(y))}{|x-y|^{N+s\,p}}\,dx\,dy = \int_\Omega \frac{f(x)}{u^\gamma}\,\varphi\,dx, \tag{1.2}$$

for every $\varphi \in C^\infty_c(\Omega)$.

According to such definition we have:

Theorem 1.2 (Existence)**.** *Let $0<\gamma\le 1$ and assume that*

$$f \in L^m(\Omega), \quad m := \frac{Np}{N(p-1)+sp+\gamma(N-sp)}.$$

Then (1.1) has a weak solution $u \in W^{s,p}_0(\Omega)$ with $\operatorname{essinf}_K u > 0$ for any compact $K \Subset \Omega$.
If $\gamma > 1$ and $f \in L^1(\Omega)$, then (1.1) has a weak a solution $u \in W^{s,p}_{\mathrm{loc}}(\Omega) \cap L^{p-1}(\Omega)$ such that $u^{(\gamma+p-1)/p} \in W^{s,p}_0(\Omega)$ and $\operatorname{essinf}_K u > 0$ for any compact $K \Subset \Omega$.

Actually the proof of Theorem 1.2 will be carried out considering first the simplest case $0<\gamma\le 1$ (see Theorem 3.2) and then the case $\gamma>1$ (see Theorem 3.6). The proof relies on the well established technique introduce in [3]. Actually, via Shauder fixed point theorem, we find a solution to a regularized problem and then we perform a-priori uniform estimates to pass to the limit. Such a procedure has been investigated in the nonlocal case for $p=2$ in [2], where a slightly different very weak notion of solution is considered that is allowed by the fact that the operator is linear and admits a double integration by parts. Since this is not the case for $p\neq 2$, we need a different approach which is rather technical and will be clear to the reader while reading the paper.
Let us now turn to the uniqueness of the solution. Since the way of understanding the boundary condition is not unambiguous, we start with the following:

Definition 1.3. Let u be such that $u=0$ in $\mathbb{R}^N\setminus\Omega$. We say that $u\le 0$ on $\partial\Omega$ if, for every $\varepsilon>0$, it follows that

$$(u-\varepsilon)^+ \in W^{s,p}_0(\Omega)\,.$$

We will say that $u=0$ on $\partial\Omega$ if u is non-negative and $u\le 0$ on $\partial\Omega$.

Adopting such definition, we will prove the following uniqueness result:

Theorem 1.4 (Uniqueness)**.** *Let $\gamma>0$ and let $f\in L^1(\Omega)$ be non-negative. Then, under zero Dirichlet boundary conditions in the sense of Definition 1.3, the solution to (1.1) is unique.*

It is worth emphasizing that the proof of Theorem 1.4 will follow via a more general comparison principle.

Having in mind Theorem 1.2, one may say that u has zero Dirichlet boundary datum if

$$u^{\max\{\frac{\gamma+p-1}{p},1\}} \in W^{s,p}_0(\Omega). \tag{1.3}$$

Our result applies in this case too since we have the following

Proposition 1.5. *Let $\gamma>0$ and let u be non-negative with $u^{\max\{\frac{\gamma+p-1}{p},1\}} \in W^{s,p}_0(\Omega)$. Then u fulfills zero Dirichlet boundary conditions in the sense of Definition 1.3.*

Proof. We only need to prove the result in the case $\gamma > 1$. For $\varepsilon > 0$ let us set

$$\mathcal{S}_\varepsilon := \operatorname{supp}(u-\varepsilon)^+ \qquad \mathcal{Q}_\varepsilon := \mathbb{R}^{2N} \setminus \left(\mathcal{S}_\varepsilon^c \times \mathcal{S}_\varepsilon^c\right).$$

By Lemma 3.5 with $q = \frac{\gamma+p-1}{p}$, we have that

$$\left|u^{\frac{\gamma+p-1}{p}}(x) - u^{\frac{\gamma+p-1}{p}}(y)\right|^p \geq \varepsilon^{\gamma-1}|u(x)-u(y)|^p \qquad \text{in } \mathcal{Q}_\varepsilon,$$

since either $u(x) \geq \varepsilon$ in $\mathcal{Q}_\varepsilon$ or $u(y) \geq \varepsilon$ in $\mathcal{Q}_\varepsilon$. From this we easily infer that

$$\begin{aligned}\int_{\mathbb{R}^{2N}} \frac{|(u-\varepsilon)^+(x)-(u-\varepsilon)^+(y)|^p}{|x-y|^{N+s\,p}}\,dx\,dy &\leq \int_{\mathcal{Q}_\varepsilon} \frac{|u(x)-u(y)|^p}{|x-y|^{N+s\,p}}\,dx\,dy \\ &\leq \varepsilon^{1-\gamma}\int_{\mathbb{R}^{2N}} \frac{|u^{\frac{\gamma+p-1}{p}}(x)-u^{\frac{\gamma+p-1}{p}}(y)|^p}{|x-y|^{N+s\,p}}\,dx\,dy,\end{aligned}$$

which concludes the proof. □

By Theorem 1.4, thanks to Proposition 1.5, we deduce in fact a more general uniqueness result:

Theorem 1.6 (Uniqueness). *Let $\gamma > 0$ and let $f \in L^1(\Omega)$ be non-negative and let u, v be weak solutions to (1.1). Assume that u and v have zero Dirichlet boundary datum either in the sense of Definition 1.3. Then $u \equiv v$.*

The technique used in the proof of the uniqueness result goes back to [8] where the uniqueness of the solution is implicitly proved in the case $f = 1$, $p = 2$ and $s = 1$. Such a technique was already improved in [10] in the local semilinear case. In this setting it is worth mentioning the recent result in [27] where singular problems with measure data are considered. The local quasilinear case $p \neq 2$ was considered with a different technique in [11]. For the nonlocal case we mention a related uniqueness result in [13], where the case $f = 1$ and $p = 2$ is considered among other problems.

We end the introduction pointing out a first simple consequence of the uniqueness result:

Theorem 1.7 (Symmetry). *Let u be the solution to (1.1) under zero Dirichlet boundary condition. Assume that the domain Ω is symmetric with respect to some hyperplane*

$$T_\lambda^\nu := \{x \cdot \nu = \lambda\}, \quad \lambda \in \mathbb{R}, \quad \nu \in S^{N-1}.$$

Then, if f is symmetric with respect to the hyperplane T_λ^ν, then u is symmetric with respect to the hyperplane T_λ^ν too. In particular, if Ω is a ball or an annulus centered at the origin and f is radially symmetric, then u is radially symmetric.

2. Approximations

We denote by $B_r(x_0)$ the $N-$dimensional open ball of radius r, centered at a point $x_0 \in \mathbb{R}^N$. The symbol $\|\cdot\|_{L^p(\Omega)}$ stands for the standard norm for the $L^p(\Omega)$ space. For a measurable function $u : \mathbb{R}^N \to \mathbb{R}$, we let

$$[u]_{D^{s,p}(\mathbb{R}^N)} := \left(\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,dxdy\right)^{1/p}$$

be its Gagliardo seminorm. We consider the space

$$W^{s,p}(\mathbb{R}^N) := \left\{u \in L^p(\mathbb{R}^N) : [u]_{D^{s,p}(\mathbb{R}^N)} < \infty\right\},$$

endowed with norm $\|\cdot\|_{L^p(\mathbb{R}^N)} + [\cdot]_{D^{s,p}(\mathbb{R}^N)}$. For $\Omega \subset \mathbb{R}^N$ open and bounded, we consider

$$W_0^{s,p}(\Omega) := \left\{u \in W^{s,p}(\mathbb{R}^N) : u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega\right\},$$

endowed with norm $[\,\cdot\,]_{D^{s,p}(\mathbb{R}^N)}$. The imbedding $W_0^{s,p}(\Omega) \hookrightarrow L^r(\Omega)$ is continuous for $1 \le r \le p_s^*$ and compact for $1 \le r < p_s^*$, where $p_s^* := N\,p/(N - s\,p)$ and $N > sp$ (as we are assuming throughout the paper). The space $W_0^{s,p}(\Omega)$ can be equivalently defined as the completion of $C_0^\infty(\Omega)$ in the norm $[\,\cdot\,]_{D^{s,p}(\mathbb{R}^N)}$, provided $\partial\Omega$ is smooth enough, see [16]. In this context by $C_0^\infty(\Omega)$ we mean the space

$$C_0^\infty(\Omega) := \{f : \mathbb{R}^N \to \mathbb{R} \,:\, f \in C^\infty(\mathbb{R}^N), \text{ support } f \text{ is compact and support } f \subseteq \Omega\}.$$

We shall denote the localized Gagliardo seminorm by

$$[u]_{W^{s,p}(\Omega)} := \left(\int_{\Omega\times\Omega} \frac{|u(x) - u(y)|^p}{|x - y|^{N+s\,p}}\, dx\, dy\right)^{1/p}.$$

Finally define the space

$$W^{s,p}_{\rm loc}(\Omega) := \big\{u \in L^p(K) : [u]_{W^{s,p}(K)} < \infty\big\}, \quad \text{for all } K \Subset \Omega.$$

We first state a lemma dealing with the existence and uniqueness of solutions to $(-\Delta)_p^s\, u = f$.

Lemma 2.1. *Let $f \in L^\infty(\Omega)$ and $f \ge 0$, $f \not\equiv 0$. Then the problem*

$$\begin{cases} (-\Delta)_p^s\, u = f & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{2.1}$$

admits a unique solution $u \in W_0^{s,p}(\Omega)$.

Proof. To prove the existence of a solution to (2.1), we minimize the functional

$$\mathcal{J}(u) = \frac{1}{p} \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}}\, dx\, dy - \int_\Omega f(x)\, u\, dx, \qquad u \in W_0^{s,p}(\Omega),$$

and then we look for a solution to (2.1) as a critical point of $\mathcal{J}(u)$. In fact, we have that

(i) $\mathcal{J}(u)$ is coercive, since by the Sobolev embedding it follows

$$\mathcal{J}(u) \ge \frac{1}{p} \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}}\, dx\, dy - C\|f\|_{L^\infty(\Omega)} \left(\int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}}\, dx\, dy\right)^{\frac{1}{p}}.$$

(ii) $\mathcal{J}(u)$ is weakly lower semi-continuous in $W_0^{s,p}(\Omega)$.

Then choosing a non-negative minimizing sequence $\{u_n\}_{n\in\mathbb{N}}$ (and since $f \ge 0$ it is not restrictive to assume that $u_n(x) \ge 0$ a.e. in $\mathbb{R}^N$, if not take $\{|u_n(x)|\}_{n\in\mathbb{N}}$), the existence of a minimum of $\mathcal{J}$ and thus of a non-negative solution u to (2.1), follows by a standard minimization procedure. That $u > 0$ follows by the strong maximum principle stated in [5, Theorem A.1]. We show now that the solution to problem (2.1) is unique. Let us suppose that $u_1, u_2 \in W_0^{s,p}(\Omega)$ are weak solutions to (2.1). Therefore, for all $\varphi \in W_0^{s,p}(\Omega)$ we have

$$\int_{\mathbb{R}^{2N}} \frac{|u_1(x) - u_1(y)|^{p-2}\,(u_1(x) - u_1(y))\,(\varphi(x) - \varphi(y))}{|x - y|^{N+s\,p}}\, dx\, dy = \int_\Omega f(x)\,\varphi\, dx,$$
$$\int_{\mathbb{R}^{2N}} \frac{|u_2(x) - u_2(y)|^{p-2}\,(u_2(x) - u_2(y))\,(\varphi(x) - \varphi(y))}{|x - y|^{N+s\,p}}\, dx\, dy = \int_\Omega f(x)\,\varphi\, dx.$$

Subtracting the two equations yields

$$\int_{\mathbb{R}^{2N}} \frac{\big(|u_1(x) - u_1(y)|^{p-2}\,(u_1(x) - u_1(y)) - |u_2(x) - u_2(y)|^{p-2}\,(u_2(x) - u_2(y))\big)\,(\varphi(x) - \varphi(y))}{|x - y|^{N+s\,p}} = 0.$$

Inserting $\varphi(x) = w(x) := u_1(x) - u_2(x)$, using elementary inequalities (cf. [24, Section 10]), yields

$$\int_{\mathbb{R}^{2N}} \frac{(|u_1(x)-u_1(y)|+|u_2(x)-u_2(y)|)^{p-2}|w(x)-w(y)|^2}{|x-y|^{N+sp}}\,dxdy \le 0, \qquad \text{if } 1<p<2,$$

as well as

$$\int_{\mathbb{R}^{2N}} \frac{|w(x)-w(y)|^p}{|x-y|^{N+sp}}\,dxdy \le 0, \qquad \text{if } p \ge 2.$$

In both cases the inequalities yield $w(x) = C$ for all $x \in \mathbb{R}^N$ and some constant $C \in \mathbb{R}$. Since $u_i = 0$ on Ω^c, we get $w = 0$ on Ω^c. Therefore $C = 0$ and the assertion follows. □

Solutions of the problem $(-\Delta)_p^s u = f(x)$ enjoy the useful L^q-estimate (cf. [26, Lemma 2.3]), that we state in the following

Lemma 2.2 (Summability lemma). *Let $f \in L^q(\Omega)$ for some $1 < q \le \infty$ and assume that $u \in W_0^{s,p}(\Omega)$ is a weak solution of the equation $(-\Delta)_p^s u = f(x)$ in Ω. Then*

$$\|u\|_r \le C\|f\|_q^{1/(p-1)},$$

where

$$r := \begin{cases} \dfrac{N\,(p-1)\,q}{N-spq}, & 1<q<\dfrac{N}{sp}, \\[2ex] \infty, & \dfrac{N}{sp}<q\le\infty, \end{cases}$$

and $C = C(N,\Omega,p,s,q) > 0$.

We consider, for a given $f \in L^1(\Omega)$, with $f \ge 0$ the truncation

$$f_n(x) := \min\{f(x), n\}, \qquad x \in \Omega.$$

Then, we consider the approximating problems

$$(\mathcal{P}_n) \qquad \begin{cases} (-\Delta)_p^s\, u_n = \frac{f_n(x)}{(u_n+1/n)^\gamma} & \text{in } \Omega, \\ u_n > 0 & \text{in } \Omega, \\ u_n = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

Proposition 2.3. *For any $n \ge 1$ there exists a weak solution $u_n \in W_0^{s,p}(\Omega) \cap L^\infty(\Omega)$ to $(\mathcal{P}_n)$.*

Proof. Given $n \in \mathbb{N}$ and a function $u \in L^p(\Omega)$, in light of Lemma 2.1 there exists a unique solution $w \in W_0^{s,p}(\Omega)$ to the problem

$$(2.2) \qquad \begin{cases} (-\Delta)_p^s\, w = \frac{f_n(x)}{(u^++1/n)^\gamma} & \text{in } \Omega, \\ w > 0 & \text{in } \Omega, \\ w = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where $u^+ := \max\{u, 0\}$. Therefore we may define the map $L^p(\Omega) \ni u \mapsto w := S(u) \in W_0^{s,p}(\Omega) \subset L^p(\Omega)$, where w is the unique solution to (2.2). Using w as test function in (2.2) we obtain

$$\int_{\mathbb{R}^{2N}} \frac{|w(x)-w(y)|^p}{|x-y|^{N+sp}}\,dx\,dy = \int_\Omega \frac{f_n(x)}{(u^++1/n)^\gamma}\,w\,dx \le n^{\gamma+1}\|w\|_{L^1(\Omega)},$$

and thus by, Sobolev imbedding, we have that

$$(2.3) \qquad \left(\int_{\mathbb{R}^{2N}} \frac{|w(x)-w(y)|^p}{|x-y|^{N+sp}}\,dx\,dy\right)^{\frac{1}{p}} \le Cn^{\frac{\gamma+1}{p-1}},$$

for some $C = C(p, s, N, \Omega)$ (independent of u), so that the ball of radius $R := Cn^{\frac{\gamma+1}{p-1}}$ in $W_0^{s,p}(\Omega)$, is invariant under the action of S. Now, in order to apply the Schauder's fixed point theorem to S and then to obtain a solution to (2.2), we have to prove the continuity and the compactness of S as an operator from $W_0^{s,p}(\Omega)$ to $W_0^{s,p}(\Omega)$.

• (*Continuity of S*). Denoting $w_k := S(u_k)$ and $w := S(u)$, then

$$\lim_{k\to\infty} \|w_k - w\|_{W_0^{s,p}(\Omega)} = 0, \qquad \text{if } \lim_{k\to\infty} \|u_k - u\|_{W_0^{s,p}(\Omega)} = 0.$$

By the strong convergence of $\{u_k\}_{k\in\mathbb{N}}$ in $W_0^{s,p}(\Omega)$, up to a subsequence, we have $u_k \to u$ in $L^{p_s^*}(\Omega)$ and $u_k \to u$ a.e. in Ω as $k \to \infty$. Considering the corresponding sequence of solutions $\{w_k\}_{k\in\mathbb{N}}$, arguing as in the proof of Lemma 2.1, setting $\bar{w}_k(x) := w_k(x) - w(x)$ we obtain

$$\begin{aligned} &\int_{\mathbb{R}^{2N}} \frac{(|w_k(x) - w_k(y)| + |w(x) - w(y)|)^{p-2}|\bar{w}_k(x) - \bar{w}_k(y)|^2}{|x - y|^{N+sp}}\, dxdy \\ &\leq \int_\Omega \left(\frac{f_n(x)}{(u_k^+ + 1/n)^\gamma} - \frac{f_n(x)}{(u^+ + 1/n)^\gamma} \right) (w_k - w)\, dx, \qquad \text{if } 1 < p < 2, \end{aligned} \tag{2.4}$$

as well as

$$\begin{aligned} &\int_{\mathbb{R}^{2N}} \frac{|\bar{w}_k(x) - \bar{w}_k(y)|^p}{|x - y|^{N+sp}}\, dxdy \\ &\leq \int_\Omega \left(\frac{f_n(x)}{(u_k^+ + 1/n)^\gamma} - \frac{f_n(x)}{(u^+ + 1/n)^\gamma} \right) (w_k - w)\, dx, \qquad \text{if } p \geq 2. \end{aligned} \tag{2.5}$$

Let us consider the right-hand side of (2.5). Using Hölder and Sobolev inequalities we infer that

$$\begin{aligned} &\left| \int_\Omega \left(\frac{f_n(x)}{(u_k^+ + 1/n)^\gamma} - \frac{f_n(x)}{(u^+ + 1/n)^\gamma} \right) (w_k - w)\, dx \right| \\ &\leq \left(\int_\Omega \left| \frac{f_n(x)}{(u_k^+ + 1/n)^\gamma} - \frac{f_n(x)}{(u^+ + 1/n)^\gamma} \right|^{(p_s^*)'} dx \right)^{\frac{1}{(p_s^*)'}} \|\bar{w}_k\|_{L^{p_s^*}(\Omega)} \\ &\leq C \left(\int_\Omega \left| \frac{f_n(x)}{(u_k^+ + 1/n)^\gamma} - \frac{f_n(x)}{(u^+ + 1/n)^\gamma} \right|^{(p_s^*)'} dx \right)^{\frac{1}{(p_s^*)'}} \|\bar{w}_k\|_{W_0^{s,p}(\Omega)}, \end{aligned}$$

where $(p_s^*)' = Np/(N(p-1) + sp)$ and $C = C(p, s, N)$ is a positive constant. From (2.5) we get

$$\begin{aligned} &\|w_k - w\|_{W_0^{s,p}(\Omega)} \\ &\leq C \left(\int_\Omega \left| \frac{f_n(x)}{(u_k^+ + 1/n)^\gamma} - \frac{f_n(x)}{(u^+ + 1/n)^\gamma} \right|^{(p_s^*)'} dx \right)^{\frac{1}{(p-1)\cdot(p_s^*)'}} \qquad \text{if } p \geq 2. \end{aligned} \tag{2.6}$$

Observing that

$$\left| \frac{f_n(x)}{(u_k^+ + 1/n)^\gamma} - \frac{f_n(x)}{(u^+ + 1/n)^\gamma} \right| \leq 2n^{\gamma+1}, \tag{2.7}$$

by the dominated convergence theorem and by the fact that $u_k(x) \to u(x)$ a.e., from (2.6) we conclude that

$$\lim_{k\to+\infty} \|w_k - w\|_{W_0^{s,p}(\Omega)} = 0,$$

showing, in the case $p \geq 2$, that the operator S is continuous from $W_0^{s,p}(\Omega)$ to $W_0^{s,p}(\Omega)$. From (2.4), a similar argument, shows the continuity of S from $W_0^{s,p}(\Omega)$ to $W_0^{s,p}(\Omega)$ for $1 < p < 2$.

• (*Compactness of S*). Let $\{u_k\}_{k\in\mathbb{N}} \subset W_0^{s,p}(\Omega)$ a bounded sequence. Denoting $w_k := S(u_k)$, we show that, up to a subsequence and for some $w \in W_0^{s,p}(\Omega)$, it holds

$$\lim_{k\to\infty} \|w_k - w\|_{W_0^{s,p}(\Omega)} = 0.$$

Let $\{u_k\}_{k\in\mathbb{N}} \subset W_0^{s,p}(\Omega)$ with $\|u_k\|_{W_0^{s,p}(\Omega)} \leq C$ for all $k \geq 1$. Then, up to a subsequence, we have $u_k \rightharpoonup u$ in $W_0^{s,p}(\Omega)$, as well as $u_k \to u$ in $L^r(\Omega)$, for $1 \leq r < p_s^*$. In view of (2.3), we have $\|S(u_k)\|_{W_0^{s,p}(\Omega)} \leq C$ for some constant C independent of k, and therefore

$$S(u_k) \rightharpoonup w, \ \text{ in } W_0^{s,p}(\Omega), \qquad S(u_k) \to w, \ \text{ in } L^r(\Omega), \ \text{ for } 1 \leq r < p_s^*,$$

for some $w \in W_0^{s,p}(\Omega)$. Then for all $\varphi \in W_0^{s,p}(\Omega)$

$$\int_{\mathbb{R}^{2N}} \frac{|w_k(x) - w_k(y)|^{p-2}\,(w_k(x) - w_k(y))\,(\varphi(x) - \varphi(y))}{|x-y|^{N+s\,p}}\,dx\,dy = \int_\Omega \frac{f_n(x)}{(u_k^+ + 1/n)^\gamma}\,\varphi\,dx. \tag{2.8}$$

We show now that, letting k to infinity, (2.8) converges to

$$\int_{\mathbb{R}^{2N}} \frac{|w(x) - w(y)|^{p-2}\,(w(x) - w(y))\,(\varphi(x) - \varphi(y))}{|x-y|^{N+s\,p}}\,dx\,dy = \int_\Omega \frac{f_n(x)}{(u^+ + 1/n)^\gamma}\,\varphi\,dx. \tag{2.9}$$

By the dominated convergence theorem, it is readily seen that

$$\lim_{k\to\infty} \int_\Omega \frac{f_n(x)}{(u_k^+ + 1/n)^\gamma}\,\varphi\,dx = \int_\Omega \frac{f_n(x)}{(u^+ + 1/n)^\gamma}\,\varphi\,dx.$$

Furthermore, since the sequence

$$\left\{ \frac{|w_k(x) - w_k(y)|^{p-2}\,(w_k(x) - w_k(y))}{|x-y|^{\frac{N+s\,p}{p'}}} \right\}_{k\in\mathbb{N}} \qquad \text{is bounded in } L^{p'}(\mathbb{R}^{2N}),$$

and by the pointwise convergence of $w_k(x)$ to $w(x)$

$$\frac{|w_k(x) - w_k(y)|^{p-2}\,(w_k(x) - w_k(y))}{|x-y|^{\frac{N+s\,p}{p'}}} \to \frac{|w(x) - w(y)|^{p-2}\,(w(x) - w(y))}{|x-y|^{\frac{N+s\,p}{p'}}} \quad \text{a.e. in } \mathbb{R}^{2N},$$

it follows by standard results that, up to a subsequence,

$$\frac{|w_k(x) - w_k(y)|^{p-2}\,(w_k(x) - w_k(y))}{|x-y|^{\frac{N+s\,p}{p'}}} \rightharpoonup \frac{|w(x) - w(y)|^{p-2}\,(w(x) - w(y))}{|x-y|^{\frac{N+s\,p}{p'}}} \quad \text{weakly in } L^{p'}(\mathbb{R}^{2N}).$$

Then, since

$$\frac{\varphi(x) - \varphi(y)}{|x-y|^{\frac{N+s\,p}{p}}} \in L^p(\mathbb{R}^{2N}),$$

we conclude that the l.h.s. of (2.8) converges to the l.h.s. of (2.9). Whence, (2.9) holds, that is, in particular, $w = S(u)$. Arguing as for (2.4) and (2.5) setting $w_k(x) = S(u_k)$ and $\bar{w}_k(x) := w_k(x) - w(x)$, we infer that

$$\begin{aligned}
&\int_{\mathbb{R}^{2N}} \frac{(|w_k(x) - w_k(y)| + |w(x) - w(y)|)^{p-2}|\bar{w}_k(x) - \bar{w}_k(y)|^2}{|x-y|^{N+s\,p}}\,dxdy \\
&\leq \|w_k - w\|_{L^p(\Omega)} \left(\int_\Omega \left| \frac{f_n(x)}{(u_k^+ + 1/n)^\gamma} - \frac{f_n(x)}{(u^+ + 1/n)^\gamma} \right|^{p'} dx \right)^{\frac{1}{p'}}, \qquad \text{if } 1 < p < 2,
\end{aligned}$$

as well as

$$\int_{\mathbb{R}^{2N}} \frac{|\bar{w}_k(x) - \bar{w}_k(y)|^p}{|x-y|^{N+sp}}\, dxdy$$
$$\leq \|w_k - w\|_{L^p(\Omega)} \left(\int_\Omega \left| \frac{f_n(x)}{(u_k^+ + 1/n)^\gamma} - \frac{f_n(x)}{(u^+ + 1/n)^\gamma} \right|^{p'} dx \right)^{\frac{1}{p'}}, \quad \text{if } p \geq 2,$$

where $p' = p/(p-1)$. Using (2.7), the last two equations imply that

$$\lim_{k\to+\infty} \|S(u_k) - S(u)\|_{W_0^{s,p}(\Omega)} = 0,$$

that is the compactness of S from $W_0^{s,p}(\Omega)$ to $W_0^{s,p}(\Omega)$. Schauder's fixed point theorem provides that existence of $u_n \in W_0^{s,p}(\Omega)$ such that $u_n = S(u_n)$, that is a weak solution to

$$\begin{cases} (-\Delta)_p^s\, u_n = \frac{f_n(x)}{(u_n+1/n)^\gamma} & \text{in } \Omega, \\ u_n > 0 & \text{in } \Omega, \\ u_n = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{2.10}$$

Since the r.h.s. of (2.10) belongs to $L^\infty(\Omega)$, by virtue of Lemma 2.2 we have $u_n \in L^\infty(\Omega)$. □

Lemma 2.4 (Monotonicity). *The sequence $\{u_n\}_{n\in\mathbb{N}}$ found in the previous lemma satisfies*

$$u_n(x) \leq u_{n+1}(x), \quad \textit{for a.e. } x \in \Omega,$$

and

$$u_n(x) \geq \sigma > 0, \quad \textit{for a.e. } x \in \omega \Subset \Omega,$$

for some positive constant $\sigma = \sigma(\omega)$.

Proof. We have, for any $n \in \mathbb{N}$, that for all $\varphi \in W_0^{s,p}(\Omega)$

$$\int_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^{p-2}\,(u_n(x) - u_n(y))\,(\varphi(x) - \varphi(y))}{|x-y|^{N+sp}}\, dx\, dy = \int_\Omega \frac{f_n(x)}{(u_n + 1/n)^\gamma}\, \varphi\, dx.$$

as well as for all $\varphi \in W_0^{s,p}(\Omega)$

$$\int_{\mathbb{R}^{2N}} \frac{|u_{n+1}(x) - u_{n+1}(y)|^{p-2}\,(u_{n+1}(x) - u_{n+1}(y))\,(\varphi(x) - \varphi(y))}{|x-y|^{N+sp}}\, dx\, dy = \int_\Omega \frac{f_{n+1}(x)}{(u_{n+1} + 1/(n+1))^\gamma}\, \varphi\, dx.$$

By taking $\varphi = w = (u_n - u_{n+1})^+ \in W_0^{s,p}(\Omega)$ as test function in the formula above and subtracting the second from the first, concerning the r.h.s. (and recalling that $f_n \leq f_{n+1}$ a.e.) we get

$$\int_\Omega \frac{f_n(x)}{(u_n + 1/n)^\gamma}\,(u_n - u_{n+1})^+\, dx - \int_\Omega \frac{f_{n+1}(x)}{(u_{n+1} + 1/(n+1))^\gamma}(u_n - u_{n+1})^+\, dx$$
$$\leq \int_\Omega f_{n+1}(u_n - u_{n+1})^+ \frac{(u_{n+1} + 1/(n+1))^\gamma - (u_n + 1/n)^\gamma}{(u_n + 1/n)^\gamma (u_{n+1} + 1/(n+1))^\gamma} dx \leq 0.$$

Then, if $I_p(s) := |s|^{p-2}s$, we conclude that

$$\int_{\mathbb{R}^{2N}} \frac{\big(I_p(u_n(x) - u_n(y)) - I_p(u_{n+1}(x) - u_{n+1}(y))\big)\,(w(x) - w(y))}{|x-y|^{N+sp}} \leq 0. \tag{2.11}$$

Now, arguing exactly as in the proof of [25, Lemma 9], we get

$$\big(I_p(u_n(x) - u_n(y)) - I_p(u_{n+1}(x) - u_{n+1}(y))\big)\,(w(x) - w(y)) \geq 0, \quad \text{for a.e. } (x,y) \in \mathbb{R}^{2N},$$

with the *strict* inequality, unless it holds

$$(u_n(x) - u_{n+1}(x))^+ = (u_n(y) - u_{n+1}(y))^+, \quad \text{for a.e. } (x,y) \in \mathbb{R}^{2N}. \tag{2.12}$$

On the other hand, by (2.11), we have

$$\big(I_p(u_n(x)-u_n(y))-I_p(u_{n+1}(x)-u_{n+1}(y))\big)\,(w(x)-w(y))=0,\quad \text{for a.e. } (x,y)\in\mathbb{R}^{2N}.$$

Therefore, (2.12) holds true, namely

$$(u_n(x)-u_{n+1}(x))^+=C,\quad \text{for a.e. } x\in\mathbb{R}^N,$$

for some constant C. Since $u_n=u_{n+1}=0$ on $\mathbb{R}^N\setminus\Omega$ it follows that $C=0$, which implies in turn that $u_n(x)\le u_{n+1}(x)$, for a.e. $x\in\Omega$. This concludes the proof of the first assertion. Concerning the second assertion, we observe that we know that $u_1\in L^\infty(\Omega)$, yielding

$$(-\Delta)^s_p u_1=\frac{f_1(x)}{(u_1+1)^\gamma}\in L^\infty(\Omega).$$

Then, by [20, Theorem 1.1] we deduce that $u_1\in C^{0,\alpha}(\bar\Omega)$ for some $\alpha\in(0,1)$. In particular by the strong maximum principle,

$$u_1(x)\ge\sigma>0,\quad \text{for a.e. } x\in\omega\subset\subset\Omega$$

and $\sigma=\sigma(\omega)$. The second assertion then follows by monotonicity. □

3. Existence of solutions

To prove the existence of a solution to (1.1) we use the sequence of solutions $\{u_n\}_{n\in\mathbb{N}}$ of problem ($\mathcal{P}_n$) (see Proposition 2.3) and then, using some a-priori estimates, we pass to the limit.

3.1. **Existence in the case $0<\gamma\le 1$.** First of all, we prove the following

Lemma 3.1. *Let $\{u_n\}_{n\in\mathbb{N}}\subset W^{s,p}_0(\Omega)\cap L^\infty(\Omega)$ be the sequence of solution to problem ($\mathcal{P}_n$) provided by Proposition 2.3. Assume that*

$$0<\gamma\le 1,\qquad f\ge 0,\quad f\in L^m(\Omega),\quad m:=\frac{Np}{N(p-1)+sp+\gamma(N-sp)}. \tag{3.1}$$

Then $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $W^{s,p}_0(\Omega)$.

Proof. In the case $0<\gamma<1$, taking u_n as test function in ($\mathcal{P}_n$), as $f_n\le f$ we get

$$\begin{aligned}\int_{\mathbb{R}^{2N}}\frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+ps}}\,dx\,dy&\le\int_\Omega\frac{f_n(x)}{(u_n+1/n)^\gamma}\,u_n\,dx\le\int_\Omega f_n(x)\,u_n^{1-\gamma}\,dx\\&\le\|f\|_{L^m(\Omega)}\left(\int_\Omega u_n^{(1-\gamma)m'}\,dx\right)^{\frac{1}{m'}},\end{aligned}\tag{3.2}$$

where $m'=m/(m-1)$. Since $(1-\gamma)m'=p_s^*$, by the Sobolev embedding, we obtain

$$\left(\int_\Omega u_n^{(1-\gamma)m'}\,dx\right)^{\frac{1}{m'}}\le C\left(\int_{\mathbb{R}^{2N}}\frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+ps}}\,dx\,dy\right)^{\frac{p_s^*}{pm'}},$$

for some constant $C=C(p,s,N)>0$. Finally, since $p_s^*/(pm')<1$, from (3.2) we get

$$\sup_{n\in\mathbb{N}}\|u_n\|_{W^{s,p}_0(\Omega)}\le C(f,p,s,\gamma,N).$$

If instead $\gamma=1$, then arguing as for (3.2), we get

$$\int_{\mathbb{R}^{2N}}\frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+ps}}\,dx\,dy\le\int_\Omega\frac{f_n(x)}{u_n+1/n}\,u_n\,dx\le\int_\Omega f(x)dx,$$

which yields again the desired boundedness. □

Theorem 3.2 (Existence, $0 < \gamma \le 1$). *Assume that* (3.1) *holds. Then problem* (1.1) *admits a weak solution* $u \in W^{s,p}_0(\Omega)$.

Proof. By virtue of Lemma 3.1, the sequence of solutions $\{u_n\}_{n\in\mathbb{N}} \subset W^{s,p}_0(\Omega) \cap L^\infty(\Omega)$ of problem ($\mathcal{P}_n$) provided by Proposition 2.3 is bounded in $W^{s,p}_0(\Omega)$. Then, up to a subsequence, we have $u_n \rightharpoonup u$ in $W^{s,p}_0(\Omega)$, $u_n \to u$ in $L^r(\Omega)$ for $1 \le r < p^*_s$ and $u_n \to u$ a.e. in Ω and, furthermore, by Lemma 2.4, we have

$$\text{for all } K \Subset \Omega \text{ there exists } \sigma_K > 0 \text{ such that } u(x) \ge \sigma_K > 0, \quad \text{for a.e. } x \in K.$$

We have

$$\int_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^{p-2}\,(u_n(x) - u_n(y))\,(\varphi(x) - \varphi(y))}{|x-y|^{N+s\,p}}\,dx\,dy = \int_\Omega \frac{f_n(x)}{(u_n + 1/n)^\gamma}\,\varphi\,dx, \tag{3.3}$$

for all $\varphi \in C^\infty_c(\Omega)$. Since the sequence

$$\left\{ \frac{|u_n(x) - u_n(y)|^{p-2}\,(u_n(x) - u_n(y))}{|x-y|^{\frac{N+s\,p}{p'}}} \right\}_{n\in\mathbb{N}} \quad \text{is bounded in } L^{p'}(\mathbb{R}^{2N}),$$

and by the point-wise convergence of u_n to u

$$\frac{|u_n(x) - u_n(y)|^{p-2}\,(u_n(x) - u_n(y))}{|x-y|^{\frac{N+s\,p}{p'}}} \to \frac{|u(x) - u(y)|^{p-2}\,(u(x) - u(y))}{|x-y|^{\frac{N+s\,p}{p'}}} \quad \text{a.e. in } \mathbb{R}^{2N},$$

it follows by standard results that

$$\frac{|w_k(x) - w_k(y)|^{p-2}\,(w_k(x) - w_k(y))}{|x-y|^{\frac{N+s\,p}{p'}}} \rightharpoonup \frac{|w(x) - w(y)|^{p-2}\,(w(x) - w(y))}{|x-y|^{\frac{N+s\,p}{p'}}} \quad \text{weakly in } L^{p'}(\mathbb{R}^{2N}).$$

Then, since for $\varphi \in C^\infty_c(\Omega)$ we have

$$\frac{\varphi(x) - \varphi(y)}{|x-y|^{\frac{N+s\,p}{p}}} \in L^p(\mathbb{R}^{2N}),$$

we conclude that

$$\lim_{n\to+\infty} \int_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^{p-2}\,(u_n(x) - u_n(y))\,(\varphi(x) - \varphi(y))}{|x-y|^{N+s\,p}}\,dx\,dy =$$
$$\int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{p-2}\,(u(x) - u(y))\,(\varphi(x) - \varphi(y))}{|x-y|^{N+s\,p}}\,dx\,dy,$$

for all $\varphi \in C^\infty_c(\Omega)$. Concerning the right-hand side of formula (3.3), recalling Lemma 2.4, for any $\varphi \in C^\infty_c(\Omega)$ with $\mathrm{supp}(\varphi) = K$, there exists $\sigma_K > 0$ independent of n such that

$$\left| \frac{f_n(x)\,\varphi}{(u_n + 1/n)^\gamma} \right| \le \sigma_K^\gamma |f(x)\varphi(x)| \in L^1(\Omega).$$

By the dominated convergence theorem we conclude that

$$\lim_{n\to\infty} \int_\Omega \frac{f_n(x)\,\varphi}{(u_n + 1/n)^\gamma}\,\varphi\,dx = \int_\Omega \frac{f(x)}{u^\gamma}\,\varphi\,dx.$$

Finally, passing to the limit in (3.3), we conclude that

$$\int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{p-2}\,(u(x) - u(y))\,(\varphi(x) - \varphi(y))}{|x-y|^{N+s\,p}}\,dx\,dy = \int_\Omega \frac{f(x)}{u^\gamma}\,\varphi\,dx,$$

for all $\varphi \in C^\infty_c(\Omega)$, namely u is a solution to (1.1). $\square$

3.2. **Existence in the case $\gamma>1$.** First, we recall the following result from [6, Lemma 3.3].

Proposition 3.3. *Let $F\in L^q(\Omega)$ with $q>N/(sp)$ and let $u\in W^{s,p}_0(\Omega)\cap L^\infty(\Omega)$ be such that*

$$\int_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^{p-2}\,(u(x)-u(y))\,(\varphi(x)-\varphi(y))}{|x-y|^{N+s\,p}}\,dx\,dy=\int_\Omega F\varphi\,dx,$$

for all $\varphi\in W^{s,p}_0(\Omega)$. Then, for every convex C^1 function $\Phi:\mathbb{R}\to\mathbb{R}$, we have

$$\int_{\mathbb{R}^{2N}}\frac{|\Phi(u)(x)-\Phi(u)(y)|^{p-2}\,(\Phi(u)(x)-\Phi(u)(y))\,(\varphi(x)-\varphi(y))}{|x-y|^{N+s\,p}}\,dx\,dy\le\int_\Omega F|\Phi'(u)|^{p-2}\Phi'(u)\varphi\,dx,$$

for every non-negative function $\varphi\in W^{s,p}_0(\Omega)$.

Lemma 3.4. *Let $\{u_n\}_{n\in\mathbb{N}}\subset W^{s,p}_0(\Omega)\cap L^\infty(\Omega)$ be the sequence of solution to ($\mathcal{P}_n$) provided by Proposition 2.3. Let $\gamma>1$ and $f\in L^1(\Omega)$. Then $\{u_n^{(\gamma+p-1)/p}\}_{n\in\mathbb{N}}$ is bounded in $W^{s,p}_0(\Omega)$.*

Proof. We can apply Proposition 3.3 to each $u_n\ge 0$ by choosing

$$F(x):=\frac{f_n(x)}{(u_n+1/n)^\gamma}\in L^\infty(\Omega),\qquad \Phi(s):=s^{(\gamma+p-1)/p},\ \ s\ge 0,$$

by noticing that Φ is C^1 and convex on $\mathbb{R}^+$ since $\gamma>1$. Then

$$\begin{aligned}&\int_{\mathbb{R}^{2N}}\frac{|\Phi(u_n)(x)-\Phi(u_n)(y)|^{p-2}\,(\Phi(u_n)(x)-\Phi(u_n)(y))\,(\varphi(x)-\varphi(y))}{|x-y|^{N+s\,p}}\,dx\,dy\\&\le\int_\Omega\frac{f_n(x)}{(u_n+1/n)^\gamma}|\Phi'(u_n)|^{p-2}\Phi'(u_n)\varphi\,dx,\end{aligned}$$

for every n and all non-negative function $\varphi\in W^{s,p}_0(\Omega)$. By choosing $\varphi:=\Phi(u_n)$ as test function (which belongs to $W^{s,p}_0(\Omega)$, since Φ is Lipschitz on bounded intervals), we infer

$$\int_{\mathbb{R}^{2N}}\frac{|\Phi(u_n)(x)-\Phi(u_n)(y)|^{p}}{|x-y|^{N+s\,p}}\,dx\,dy\le\int_\Omega\frac{f_n(x)}{(u_n+1/n)^\gamma}|\Phi'(u_n)|^{p-1}\Phi(u_n)\,dx.\tag{3.4}$$

Note that

$$|\Phi'(u_n)|^{p-1}\Phi(u_n)\le Cu_n^\gamma,\qquad\text{for any } n\in\mathbb{N}\text{ and some } C=C(\gamma,p)>0.$$

In turn,

$$\int_\Omega\frac{f_n(x)}{(u_n+1/n)^\gamma}|\Phi'(u_n)|^{p-1}\Phi(u_n)\,dx\le C\int_\Omega|f_n|\,dx\le C\int_\Omega|f|\,dx,$$

since $f_n\le f$. From inequality (3.4) it follows that $\{\Phi(u_n)\}_{n\in\mathbb{N}}$ is bounded in $W^{s,p}_0(\Omega)$. □

Lemma 3.5. *Let $q>1$ and $\varepsilon>0$. In the plane $\mathbb{R}^2$ with the notation $p=(x,y)$, let us set*

$$S^x_\varepsilon:=\{x\ge\varepsilon\}\cap\{y\ge 0\}\qquad S^y_\varepsilon:=\{y\ge\varepsilon\}\cap\{x\ge 0\}\,.$$

Then, we have that

$$|x^q-y^q|\ge\varepsilon^{q-1}|x-y|\qquad\text{in}\quad S^x_\varepsilon\cup S^y_\varepsilon\,.\tag{3.5}$$

Proof. With no loss of generality, we may assume that $x\ge y$. Let us first note that

$$x^q-y^q=q\lambda^{q-1}(x-y),\qquad\text{for some }\lambda\in(y,x)\,.$$

Whence (3.5) holds true, since $q>1$, if $(x,y)\in S^x_\varepsilon\cap S^y_\varepsilon$, namely if $y\ge\varepsilon$ in the case that we are considering. Then, let us deal with the case $0\le y<\varepsilon\le x$. Since $t\mapsto t^q$ is (strictly) convex for $q>1$, then we have

$$\frac{x^q-y^q}{x-y}\ge\frac{x^q}{x}=x^{q-1}\ge\varepsilon^{q-1}\,.$$

Thus, inequality (3.5) is proved. □

Next we turn to the existence result for $\gamma > 1$.

Theorem 3.6 (Existence for $\gamma > 1$)**.** *Let $f \geq 0$, $f \in L^1(\Omega)$ and $\gamma > 1$. Then problem (1.1) admits a weak a solution $u \in W^{s,p}_{\rm loc}(\Omega)$ with $u^{(\gamma+p-1)/p} \in W^{s,p}_0(\Omega)$.*

Proof. In light of Lemma 3.4, the sequence $\{u_n\}_{n\in\mathbb{N}}$ of solution to ($\mathcal{P}_n$) of Proposition 2.3 satisfies

$$\sup_{n\in\mathbb{N}} \left[u_n^{(\gamma+p-1)/p}\right]_{W^{s,p}(\mathbb{R}^N)} \leq C. \tag{3.6}$$

Since $\{u_n\}_{n\in\mathbb{N}}$ is increasing it admits pointwise limit u as $n \to \infty$. In particular, by Fatou's lemma

$$\left[u^{(\gamma+p-1)/p}\right]_{W^{s,p}(\mathbb{R}^N)} \leq \liminf_n \left[u_n^{(\gamma+p-1)/p}\right]_{W^{s,p}(\mathbb{R}^N)} \leq C. \tag{3.7}$$

Then $u^{(\gamma+p-1)/p} \in W^{s,p}_0(\Omega)$ and therefore $u \in L^p(\Omega)$ since $\gamma > 1$. Notice also that, by virtue of Lemma 2.4, for all $K \Subset \Omega$ there exists $\sigma_K > 0$ such that $u(x) \geq \sigma_K > 0$ for a.e. $x \in K$. Therefore, in light of Lemma 3.5, we have

$$\frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} \leq \sigma_K^{1-\gamma} \frac{|u^{\frac{\gamma+p-1}{p}}(x) - u^{\frac{\gamma+p-1}{p}}(y)|^p}{|x-y|^{N+sp}}, \qquad x, y \in K, \quad K \Subset \mathbb{R}^N.$$

This yields

$$u \in W^{s,p}_{\rm loc}(\Omega).$$

We have, for any $n \in \mathbb{N}$

$$\int_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^{p-2}\,(u_n(x)-u_n(y))\,(\varphi(x)-\varphi(y))}{|x-y|^{N+sp}}\,dx\,dy = \int_\Omega \frac{f_n(x)}{(u_n+1/n)^\gamma}\,\varphi\,dx, \tag{3.8}$$

for all $\varphi \in C^\infty_c(\Omega)$. In order to pass to the limit in (3.8), we observe the following. By the elementary inequality (see e.g. [14, 15]) $||\xi|^{p-2}\xi - |\xi'|^{p-2}\xi'| \leq C(|\xi| + |\xi'|)^{p-2}|\xi - \xi'|$ for $\xi, \xi' \in \mathbb{R}$ with $|\xi| + |\xi'| > 0$, we get

$$\begin{aligned} &\left| \int_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^{p-2}\,(u_n(x)-u_n(y))\,(\varphi(x)-\varphi(y))}{|x-y|^{N+sp}}\,dx\,dy - \right. \\ &\left. \int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}\,(u(x)-u(y))\,(\varphi(x)-\varphi(y))}{|x-y|^{N+sp}}\,dx\,dy \right| \\ &\leq \int_{\mathbb{R}^{2N}} \frac{(|u_n(x)-u_n(y)| + |u(x)-u(y)|)^{p-2}|\bar{u}_n(x)-\bar{u}_n(y)||\varphi(x)-\varphi(y)|}{|x-y|^{N+sp}}\,dx\,dy, \end{aligned} \tag{3.9}$$

where $\bar{u}_n(x) := u_n(x) - u(x)$. Let us fix $\varepsilon > 0$.

We *claim* that there exist a compact $\mathcal{K} \subset \mathbb{R}^{2N}$ such that

$$\int_{\mathbb{R}^{2N}\setminus\mathcal{K}} \frac{(|u_n(x)-u_n(y)| + |u(x)-u(y)|)^{p-2}|\bar{u}_n(x)-\bar{u}_n(y)||\varphi(x)-\varphi(y)|}{|x-y|^{N+sp}}\,dx\,dy \leq \frac{\varepsilon}{2}, \tag{3.10}$$

for all $n \in \mathbb{N}$. Let us set

$$\mathcal{S}_\varphi := \operatorname{supp}\varphi \qquad \mathcal{Q}_\varphi := \mathbb{R}^{2N} \setminus \left(\mathcal{S}^c_\varphi \times \mathcal{S}^c_\varphi\right).$$

By triangular and Hölder inequalities we get

(3.11)
$$
\begin{aligned}
&\int_{\mathbb{R}^{2N}\setminus\mathcal{K}} \frac{(|u_n(x)-u_n(y)|+|u(x)-u(y)|)^{p-2}|\bar{u}_n(x)-\bar{u}_n(y)||\varphi(x)-\varphi(y)|}{|x-y|^{N+s\,p}}\,dx\,dy\\
&=\int_{\mathcal{Q}_\varphi\setminus\mathcal{K}} \frac{(|u_n(x)-u_n(y)|+|u(x)-u(y)|)^{p-1}|\varphi(x)-\varphi(y)|}{|x-y|^{N+s\,p}}\,dx\,dy\\
&\leq\left(\int_{\mathcal{Q}_\varphi\setminus\mathcal{K}} \frac{(|u_n(x)-u_n(y)|+|u(x)-u(y)|)^{p}}{|x-y|^{N+s\,p}}\,dx\,dy\right)^{\frac{p-1}{p}}\left(\int_{\mathcal{Q}_\varphi\setminus\mathcal{K}} \frac{|\varphi(x)-\varphi(y)|^p}{|x-y|^{N+s\,p}}\,dx\,dy\right)^{\frac{1}{p}}\\
&=\left(\int_{\mathcal{Q}_\varphi\setminus\mathcal{K}} \frac{(|u_n(x)-u_n(y)|+|u(x)-u(y)|)^{p}}{|x-y|^{N+s\,p}}\,dx\,dy\right)^{\frac{p-1}{p}}\left(\int_{\mathbb{R}^{2N}\setminus\mathcal{K}} \frac{|\varphi(x)-\varphi(y)|^p}{|x-y|^{N+s\,p}}\,dx\,dy\right)^{\frac{1}{p}}.
\end{aligned}
$$

By Lemma 2.4, there exists $\sigma_{\mathcal{S}_\varphi}>0$ independent of n such that $u(x)\geq\sigma_{\mathcal{S}_\varphi}$ for a.e. $x\in\mathcal{S}_\varphi$. Moreover, by using Lemma 3.5 with $q=(\gamma+p-1)/p$, we have

$$
\begin{aligned}
&\frac{(|u_n(x)-u_n(y)|+|u(x)-u(y)|)^{p}}{|x-y|^{N+s\,p}}\\
&\leq C(p)\frac{|u_n(x)-u_n(y)|^p+|u(x)-u(y)|^p}{|x-y|^{N+s\,p}} \qquad (3.12)\\
&\leq C(p)\sigma_{\mathcal{S}_\varphi}^{1-\gamma}\frac{|u_n^{\frac{\gamma+p-1}{p}}(x)-u_n^{\frac{\gamma+p-1}{p}}(y)|^p+|u^{\frac{\gamma+p-1}{p}}(x)-u^{\frac{\gamma+p-1}{p}}(y)|^p}{|x-y|^{N+s\,p}},\qquad \text{a.e. } (x,y)\in\mathcal{Q}_\varphi.
\end{aligned}
$$

Then from (3.11) and (3.12) we infer that

$$
\begin{aligned}
&\int_{\mathbb{R}^{2N}\setminus\mathcal{K}} \frac{(|u_n(x)-u_n(y)|+|u(x)-u(y)|)^{p-2}|\bar{u}_n(x)-\bar{u}_n(y)||\varphi(x)-\varphi(y)|}{|x-y|^{N+s\,p}}\,dx\,dy\\
&\leq C\left(\int_{\mathbb{R}^{2N}} \frac{|u_n^{\frac{\gamma+p-1}{p}}(x)-u_n^{\frac{\gamma+p-1}{p}}(y)|^p}{|x-y|^{N+s\,p}}\,dx\,dy\right)^{\frac{p-1}{p}}\left(\int_{\mathbb{R}^{2N}\setminus\mathcal{K}} \frac{(|\varphi(x)-\varphi(y)|)^p}{|x-y|^{N+s\,p}}\,dx\,dy\right)^{\frac{1}{p}}\\
&+C\left(\int_{\mathbb{R}^{2N}} \frac{|u^{\frac{\gamma+p-1}{p}}(x)-u^{\frac{\gamma+p-1}{p}}(y)|^p}{|x-y|^{N+s\,p}}\,dx\,dy\right)^{\frac{p-1}{p}}\left(\int_{\mathbb{R}^{2N}\setminus\mathcal{K}} \frac{(|\varphi(x)-\varphi(y)|)^p}{|x-y|^{N+s\,p}}\,dx\,dy\right)^{\frac{1}{p}}\\
&\leq C\left(\int_{\mathbb{R}^{2N}\setminus\mathcal{K}} \frac{|\varphi(x)-\varphi(y)|^p}{|x-y|^{N+s\,p}}\,dx\,dy\right)^{\frac{1}{p}},
\end{aligned}
$$

with $C=C(p,\gamma,\sigma_{\mathcal{S}_\varphi})$ and where we used (3.6) and (3.7). Then, since $\varphi\in C_c^\infty(\Omega)$, there exists $\mathcal{K}=\mathcal{K}(\varepsilon)$ such that (3.10) holds, proving the *claim*.

On the other hand, consider now an arbitrary measurable subset $E\subset\mathcal{K}$. Arguing as in (3.11) and (3.12), we reach the inequality

$$
\begin{aligned}
&\int_E \frac{(|u_n(x)-u_n(y)|+|u(x)-u(y)|)^{p-2}|\bar{u}_n(x)-\bar{u}_n(y)||\varphi(x)-\varphi(y)|}{|x-y|^{N+s\,p}}\,dx\,dy\\
&\leq C\left(\int_E \frac{|\varphi(x)-\varphi(y)|^p}{|x-y|^{N+s\,p}}\,dx\,dy\right)^{\frac{1}{p}},
\end{aligned}
$$

that is

$$\int_E \frac{(|u_n(x)-u_n(y)|+|u(x)-u(y)|)^{p-2}|\bar u_n(x)-\bar u_n(y)||\varphi(x)-\varphi(y)|}{|x-y|^{N+s\,p}}\,dx\,dy \to 0,$$

uniformly on n, if the Lebesgue measure of E goes to zero. Moreover

$$\frac{(|u_n(x)-u_n(y)|+|u(x)-u(y)|)^{p-2}|\bar u_n(x)-\bar u_n(y)||\varphi(x)-\varphi(y)|}{|x-y|^{N+s\,p}} \to 0 \quad \text{a.e. in } \mathbb{R}^{2N}.$$

Vitali's Theorem now implies that, given $\varepsilon>0$, there exists $\bar n>0$ such that, if $n\geq \bar n$, it follows

$$\int_{\mathcal{K}} \frac{(|u_n(x)-u_n(y)|+|u(x)-u(y)|)^{p-2}|\bar u_n(x)-\bar u_n(y)||\varphi(x)-\varphi(y)|}{|x-y|^{N+s\,p}}\,dx\,dy \leq \frac{\varepsilon}{2}. \tag{3.13}$$

From (3.9), using (3.10) and (3.13), we are able to pass to the limit in the left-hand side of $(\mathcal{P}_n)$, that is

$$\lim_{n\to+\infty}\int_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^{p-2}\,(u_n(x)-u_n(y))\,(\varphi(x)-\varphi(y))}{|x-y|^{N+s\,p}}\,dx\,dy =$$
$$\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}\,(u(x)-u(y))\,(\varphi(x)-\varphi(y))}{|x-y|^{N+s\,p}}\,dx\,dy,$$

for all $\varphi\in C^\infty_c(\Omega)$. Finally, arguing for the right-hand side as in the proof of Theorem 3.2, we pass to the limit in (3.3), concluding that

$$\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}\,(u(x)-u(y))\,(\varphi(x)-\varphi(y))}{|x-y|^{N+s\,p}}\,dx\,dy = \int_\Omega \frac{f(x)}{u^\gamma}\,\varphi\,dx,$$

for all $\varphi\in C^\infty_c(\Omega)$, namely u is a solution to (1.1). □

Remark 3.7. In the previous results, if furthermore

$$f\in L^{\frac{Np}{N(p-1)+sp}}(\Omega),$$

then (1.2) is satisfied for all $\varphi\in W^{s,p}_0(\Omega)$ such that $\mathrm{supp}(\varphi)\Subset\Omega$.

Remark 3.8. With reference to the proof of Theorem 3.6, we observe that

$$\text{the sequence } \{u_n\}_{n\in\mathbb{N}} \text{ is bounded in } W_0^{\frac{sp}{\gamma+p-1},\gamma+p-1}(\Omega).$$

In fact, since $(\gamma+p-1)/p>1$, by the Hölderianity of the map $t\mapsto t^{p/(\gamma+p-1)}$, we get

$$\frac{|u_n(x)-u_n(y)|^{\gamma+p-1}}{|x-y|^{N+sp}} \leq \frac{|u_n^{(\gamma+p-1)/p}(x)-u_n^{(\gamma+p-1)/p}(y)|^p}{|x-y|^{N+sp}}, \quad x,y\in\mathbb{R}^N.$$

Therefore, the weak solution u of Theorem 3.6 also belongs to $u\in W_0^{\frac{sp}{\gamma+p-1},\gamma+p-1}(\Omega)$.

4. Proof of the uniqueness results

Let us start defining the real valued function g_k by

$$g_k(s) := \begin{cases} \min\{s^{-\gamma}\,,\,k\} & \text{if } s>0,\\ k & \text{if } s\leq 0.\end{cases} \tag{4.1}$$

Then we consider the real valued function Φ_k defined to be the primitive of g_k that is equal to zero for $s=1$. Let us consequently consider the functional $J_k : W^{s,p}_0(\Omega)\to[-\infty\,,+\infty]$ defined by

$$J_k(\varphi) := \frac{1}{p}\int_{\mathbb{R}^{2N}} \frac{|\varphi(x)-\varphi(y)|^p}{|x-y|^{N+ps}}\,dx\,dy - \int_{\mathbb{R}^N} f(x)\Phi_k(\varphi)\,dx \qquad \varphi\in W^{s,p}_0(\Omega)\,.$$

Let us now recall that, given $z \in W^{s,p}_{\mathrm{loc}}(\Omega)\cap L^{p-1}(\Omega)$ with $z \geq 0$, we say that z is a weak supersolution (subsolution) to (1.1), if

$$\int_{\mathbb{R}^{2N}} \frac{|z(x)-z(y)|^{p-2}\,(z(x)-z(y))\,(\varphi(x)-\varphi(y))}{|x-y|^{N+s\,p}}\,dx\,dy \underset{(\leq)}{\geq} \int_\Omega \frac{f(x)}{z^\gamma}\,\varphi\,dx \qquad \forall \varphi \in C^\infty_c(\Omega)\,,\ \varphi \geq 0\,.$$

For a fixed supersolution v, we consider w defined as the minimum of J_k on the convex set

$$\mathcal{K} := \{\varphi \in W^{s,p}_0(\Omega)\,:\, 0 \leq \varphi \leq v \text{ a.e. in } \Omega\}\,.$$

By direct computation, we deduce that

$$\begin{aligned}&\int_{\mathbb{R}^{2N}} \frac{|w(x)-w(y)|^{p-2}\,(w(x)-w(y))\,(\psi(x)-w(x)-(\psi(y)-w(y)))}{|x-y|^{N+s\,p}}\,dx\,dy\\ &\geq \int_\Omega f(x)\Phi_k'(w)(\psi-w) \qquad \text{for } \psi \in w + (W^{s,p}_0(\Omega)\cap L^\infty_c(\Omega)) \text{ and } 0\leq\psi\leq v\,,\end{aligned} \tag{4.2}$$

where by $L^\infty_c(\Omega)$ we denote the space of L^∞-functions, with compact support in Ω. With such notation, we have the following

Lemma 4.1. *For all $\psi \in C^\infty_c(\Omega)$ with $\psi \geq 0$ we have*

$$\int_{\mathbb{R}^{2N}} \frac{|w(x)-w(y)|^{p-2}\,(w(x)-w(y))\,(\psi(x)-\psi(y))}{|x-y|^{N+s\,p}}\,dx\,dy \geq \int_\Omega f(x)\Phi_k'(w)\psi dx. \tag{4.3}$$

Proof. Let us consider a real valued function $g \in C^\infty_c(\mathbb{R})$ with $0\leq g(t)\leq 1$, $g(t)=1$ for $t\in[-1,1]$ and $g(t)=0$ for $t\in(-\infty,-2]\cup[2,\infty)$. Then, for any non-negative $\varphi \in C^\infty_c(\Omega)$, we set $\varphi_h := g(\frac{w}{h})\,\varphi$ and $\varphi_{h,t} := \min\{w+t\varphi_h\,,\, v\}$ with $h\geq 1$ and $t>0$. We have that $\varphi_{h,t} \in w+(W^{s,p}_0(\Omega)\cap L^\infty_c(\Omega))$ and $0\leq \varphi_{h,t}\leq v$, so that, by (4.2), we deduce that

$$\begin{aligned}&\int_{\mathbb{R}^{2N}} \frac{|w(x)-w(y)|^{p-2}\,(w(x)-w(y))\,(\varphi_{h,t}(x)-w(x)-(\varphi_{h,t}(y)-w(y)))}{|x-y|^{N+s\,p}}\,dx\,dy\\ &\geq \int_\Omega f(x)\Phi_k'(w)(\varphi_{h,t}-w)\,dx.\end{aligned}$$

By standard manipulations, by the inequality (see e.g. [14,15])

$$(|\xi|^{p-2}\xi-|\xi'|^{p-2}\xi')(\xi-\xi') \geq C(|\xi|+|\xi'|)^{p-2}|\xi-\xi'|^2, \tag{4.4}$$

for $\xi,\xi'\in\mathbb{R}$ with $|\xi|+|\xi'|>0$, and by (4.2), we deduce that

$$\begin{aligned}\mathbb{I}_1 &:= c\int_{\mathbb{R}^{2N}} \frac{(|\varphi_{h,t}(x)-\varphi_{h,t}(y)|+|w(x)-w(y)|)^{p-2}\,(\varphi_{h,t}(x)-w(x)-(\varphi_{h,t}(y)-w(y)))^2}{|x-y|^{N+s\,p}}\,dx\,dy\\ &\leq \int_{\mathbb{R}^{2N}} \frac{|\varphi_{h,t}(x)-\varphi_{h,t}(y)|^{p-2}\,(\varphi_{h,t}(x)-\varphi_{h,t}(y))\,(\varphi_{h,t}(x)-w(x)-(\varphi_{h,t}(y)-w(y)))}{|x-y|^{N+s\,p}}\,dx\,dy\\ &-\int_{\mathbb{R}^{2N}} \frac{|w(x)-w(y)|^{p-2}\,(w(x)-w(y))\,(\varphi_{h,t}(x)-w(x)-(\varphi_{h,t}(y)-w(y)))}{|x-y|^{N+s\,p}}\,dx\,dy\\ &\leq \int_{\mathbb{R}^{2N}} \frac{|\varphi_{h,t}(x)-\varphi_{h,t}(y)|^{p-2}\,(\varphi_{h,t}(x)-\varphi_{h,t}(y))\,(\varphi_{h,t}(x)-w(x)-(\varphi_{h,t}(y)-w(y)))}{|x-y|^{N+s\,p}}\,dx\,dy\\ &-\int_\Omega f(x)\Phi_k'(w)(\varphi_{h,t}-w)\end{aligned}$$

We rewrite this as

$$
\begin{aligned}
&\mathbb{I}_1 - \int_\Omega f(x)(\Phi_k'(\varphi_{h,t}) - \Phi_k'(w))(\varphi_{h,t} - w)\,dx \\
&\le \int_{\mathbb{R}^{2N}} \frac{|\varphi_{h,t}(x) - \varphi_{h,t}(y)|^{p-2}\,(\varphi_{h,t}(x) - \varphi_{h,t}(y))\,(\varphi_{h,t}(x) - w(x) - (\varphi_{h,t}(y) - w(y)))}{|x-y|^{N+sp}}\,dx\,dy \\
&- \int_\Omega f(x)\cdot\Phi_k'(\varphi_{h,t})(\varphi_{h,t} - w) \\
&= \int_{\mathbb{R}^{2N}} \mathcal{G}(x,y)\,dxdy - \int_\Omega f(x)\cdot\Phi_k'(\varphi_{h,t})(\varphi_{h,t} - w - t\varphi_h) \\
&+ t\int_{\mathbb{R}^{2N}} \frac{|\varphi_{h,t}(x) - \varphi_{h,t}(y)|^{p-2}\,(\varphi_{h,t}(x) - \varphi_{h,t}(y))\,(\varphi_h(x) - \varphi_h(y))}{|x-y|^{N+sp}}\,dx\,dy \\
&- t\int_\Omega f(x)\Phi_k'(\varphi_{h,t})\varphi_h
\end{aligned}
\tag{4.5}
$$

where, if $I_p(t) := |t|^{p-2}t$, we have set

$$
\mathcal{G}(x,y) := \frac{I_p(\varphi_{h,t}(x) - \varphi_{h,t}(y))\,(\varphi_{h,t}(x) - w(x) - t\varphi_h(x) - (\varphi_{h,t}(y) - w(y) - t\varphi_h(y)))}{|x-y|^{N+sp}}.
$$

For future use, let us also set

$$
\mathcal{G}_v(x,y) := \frac{I_p(v(x) - v(y))\,(\varphi_{h,t}(x) - w(x) - t\varphi_h(x) - (\varphi_{h,t}(y) - w(y) - t\varphi_h(y)))}{|x-y|^{N+sp}}.
$$

Now we set $S_v := \{\varphi_{h,t} = v\}$ and note that, actually, $S_v = \{v \le w + t\varphi_h\}$. We use the decomposition

$$
\mathbb{R}^{2N} = (S_v \cup S_v^c) \times (S_v \cup S_v^c).
$$

Taking into account that

$$
\mathcal{G}(\cdot,\cdot) = 0 \text{ in } S_v^c \times S_v^c,
$$

we deduce that

$$
\begin{aligned}
&\int_{\mathbb{R}^{2N}} \mathcal{G}(x,y)\,dx\,dy \\
&= \int_{S_v}\int_{S_v} \mathcal{G}(x,y)\,dx\,dy + \int_{S_v^c}\int_{S_v} \mathcal{G}(x,y)\,dx\,dy + \int_{S_v}\int_{S_v^c} \mathcal{G}(x,y)\,dx\,dy \\
&\le \int_{S_v}\int_{S_v} \mathcal{G}_v(x,y)\,dx\,dy + \int_{S_v^c}\int_{S_v} \mathcal{G}_v(x,y)\,dx\,dy + \int_{S_v}\int_{S_v^c} \mathcal{G}_v(x,y)\,dx\,dy \\
&= \int_{\mathbb{R}^{2N}} \mathcal{G}_v(x,y)\,dx\,dy,
\end{aligned}
$$

where the inequality follows since $\mathcal{G} \le \mathcal{G}_v$. In particular, to see this, write down explicitly the expression of $\mathcal{G}$ and exploit the monotonicity of the real valued function $|t - t_0|^{p-2}(t - t_0)$ together with the definition of $\varphi_{h,t}$. Thence we go back to (4.5) and get that

$$\begin{aligned}
&\mathbb{I}_1 - \int_\Omega f(x)\cdot(\Phi_k'(\varphi_{h,t}) - \Phi_k'(w))(\varphi_{h,t}-w)\,dx \\
&\le \int_{\mathbb{R}^{2N}} \mathcal{G}_v(x,y)\,dx\,dy - \int_\Omega f(x)\cdot\Phi_k'(\varphi_{h,t})(\varphi_{h,t}-w-t\varphi_h) \\
&+ t\int_{\mathbb{R}^{2N}} \frac{|\varphi_{h,t}(x)-\varphi_{h,t}(y)|^{p-2}\,(\varphi_{h,t}(x)-\varphi_{h,t}(y))\,(\varphi_h(x)-\varphi_h(y))}{|x-y|^{N+s\,p}}\,dx\,dy \\
&- t\int_\Omega f(x)\cdot\Phi_k'(\varphi_{h,t})\varphi_h\,.
\end{aligned}$$

By the definition of Φ_k, it follows that v is a supersolution to the equation $(-\Delta)^s_p z = \Phi_k'(z)$ too. Therefore, recalling that $\varphi_{h,t} - w - t\varphi_h \le 0$, we deduce that

$$\begin{aligned}
&\mathbb{I}_1 - \int_\Omega f(x)\cdot(\Phi_k'(\varphi_{h,t}) - \Phi_k'(w))(\varphi_{h,t}-w)\,dx \\
&\le t\int_{\mathbb{R}^{2N}} \frac{|\varphi_{h,t}(x)-\varphi_{h,t}(y)|^{p-2}\,(\varphi_{h,t}(x)-\varphi_{h,t}(y))\,(\varphi_h(x)-\varphi_h(y))}{|x-y|^{N+s\,p}}\,dx\,dy - t\int_\Omega f(x)\cdot\Phi_k'(\varphi_{h,t})\varphi_h\,.
\end{aligned}$$

Exploiting the fact $\mathbb{I}_1 \ge 0$ and again that $\varphi_{h,t} - w \le t\varphi_h$, we deduce that

$$\begin{aligned}
&\int_{\mathbb{R}^{2N}} \frac{|\varphi_{h,t}(x)-\varphi_{h,t}(y)|^{p-2}\,(\varphi_{h,t}(x)-\varphi_{h,t}(y))\,(\varphi_h(x)-\varphi_h(y))}{|x-y|^{N+s\,p}}\,dx\,dy - \int_\Omega f(x)\cdot\Phi_k'(\varphi_{h,t})\varphi_h \\
&\ge -\int_\Omega f(x)\cdot|\Phi_k'(\varphi_{h,t}) - \Phi_k'(w)||\varphi_h|\,dx\,.
\end{aligned}$$

Recalling the defintion of $\varphi_{h,t}$ and that $w \in W^{s,p}_0(\Omega)$, we can now pass to the limit for $t \to 0$ exploiting the Lebesgue Theorem obtaining

$$\int_{\mathbb{R}^{2N}} \frac{|w(x)-w(y)|^{p-2}\,(w(x)-w(y))\,(\varphi_h(x)-\varphi_h(y))}{|x-y|^{N+s\,p}}\,dx\,dy - \int_\Omega f(x)\cdot\Phi_k'(w)\varphi_h \ge 0.$$

The claim, namely the proof of (4.3), follows letting $h \to \infty$. □

Now we are in position to prove our *weak comparison principle*, namely we have the following:

Theorem 4.2. *Let $\gamma > 0$ and let $f \in L^1(\Omega)$ be non-negative. Let u be a subsolution to (1.1) such that $u \le 0$ on $\partial\Omega$ and let v be a supersolution to (1.1). Then, $u \le v$ a.e. in Ω.*

Proof. For $\varepsilon > 0$ and w as in Lemma 4.1, it follows that

$$(u-w-\varepsilon)^+ \in W^{s,p}_0(\Omega)\,.$$

This can be easily deduced by the fact that $w \in W^{s,p}_0(\Omega)$ and $w \ge 0$ a.e. in Ω, so that the support of $(u-w-\varepsilon)^+$ is contained in the support of $(u-\varepsilon)^+$. Therefore, by (4.3) and by standard density arguments, it follows

$$\begin{aligned}
&\int_{\mathbb{R}^{2N}} \frac{|w(x)-w(y)|^{p-2}\,(w(x)-w(y))\,(T_\tau\left((u-w-\varepsilon)^+\right)(x) - T_\tau\left((u-w-\varepsilon)^+\right)(y))}{|x-y|^{N+s\,p}}\,dx\,dy \\
&\ge \int_\Omega f(x)\cdot\Phi_k'(w)T_\tau\left((u-w-\varepsilon)^+\right)
\end{aligned} \tag{4.6}$$

for $T_\tau(s) := \min\{s,\tau\}$ for $s \ge 0$ and $T_\tau(-s) := -T_\tau(s)$ for $s < 0$. Let now $\varphi_n \in C^\infty_c(\Omega)$ such that $\varphi_n \to (u-w-\varepsilon)^+$ in $W^{s,p}_0(\Omega)$ and set

$$\tilde\varphi_{\tau,n} := T_\tau(\min\{(u-w-\varepsilon)^+, \varphi_n^+\})\,.$$

It follows that $\tilde{\varphi}_{\tau,n} \in W_0^{s,p}(\Omega) \cap L_c^\infty(\Omega)$ so that, by a density argument

$$\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}\,(u(x)-u(y))\,(\tilde{\varphi}_{\tau,n}(x)-\tilde{\varphi}_{\tau,n}(y))}{|x-y|^{N+s\,p}}\,dx\,dy \leq \int_\Omega \frac{f(x)}{u^\gamma}\,\tilde{\varphi}_{\tau,n}\,dx\,.$$

Passing to the limit as n tends to infinity, it is easy to deduce that

$$\begin{aligned}&\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}\,(u(x)-u(y))\,(T_\tau\left((u-w-\varepsilon)^+(x)\right)-T_\tau\left((u-w-\varepsilon)^+(y)\right))}{|x-y|^{N+s\,p}}\,dx\,dy\\ &\leq \int_\Omega \frac{f(x)}{u^\gamma} T_\tau\left((u-w-\varepsilon)^+\right)\,dx\,.\end{aligned} \tag{4.7}$$

It is convenient now to set

$$g(t) \;:=\; T_\tau((t-\varepsilon)^+) = \min\{\tau\,,\,\max\{t-\varepsilon\,,\,0\}\}\,.$$

With such a notation we have

$$\begin{aligned}&|u(x)-u(y)|^{p-2}\,(u(x)-u(y))\,(T_\tau\left((u-w-\varepsilon)^+(x)\right)-T_\tau\left((u-w-\varepsilon)^+(y)\right)) =\\ &= |u(x)-u(y)|^{p-2}\,(u(x)-u(y))(u(x)-w(x)-(u(y)-w(y))\,H(x,y)\end{aligned}$$

with

$$H(x,y) := \frac{g(u(x)-w(x))-g(u(y)-w(y))}{(u(x)-w(x)-(u(y)-w(y))},$$

where $(u(x)-w(x)-(u(y)-w(y)) \neq 0$. In the same way we deduce that

$$\begin{aligned}&|w(x)-w(y)|^{p-2}\,(w(x)-w(y))\,(T_\tau\left((u-w-\varepsilon)^+(x)\right)-T_\tau\left((u-w-\varepsilon)^+(y)\right)) =\\ &= |w(x)-w(y)|^{p-2}\,(w(x)-w(y))(u(x)-w(x)-(u(y)-w(y))\,H(x,y)\,.\end{aligned}$$

Then, we subtract (4.6) to (4.7) using (4.4) and the fact that $H(x,y)$ is non-negative by the definition of g (see (4.1)) that is nondecreasing. Therefore, choosing $\varepsilon > 0$ such that $\varepsilon^{-\gamma} < k$, we deduce that

$$\begin{aligned}&c\int_{\mathbb{R}^{2N}} \frac{(|u(x)-u(y)|+|w(x)-w(y)|)^{p-2}\,(u(x)-w(x)-(u(y)-w(y)))^2}{|x-y|^{N+s\,p}} H(x,y)\,dx\,dy\\ &\leq \int_\Omega f(x)\cdot\left(\frac{1}{u^\gamma}-\Phi_k'(w)\right) T_\tau\left((u-w-\varepsilon)^+\right)\,dx\\ &= \int_\Omega f(x)\cdot\left(\Phi_k'(u)-\Phi_k'(w)\right) T_\tau\left((u-w-\varepsilon)^+\right)\,dx \leq 0\,,\end{aligned}$$

where the equality in the last line follows using the definition (4.1) and the fact that we are working in a region where $T_\tau(\cdot) \neq 0$. Thus

$$\int_{\mathbb{R}^{2N}} \frac{(|u(x)-u(y)|+|w(x)-w(y)|)^{p-2}\,(T_\tau\left((u-w-\varepsilon)^+(x)\right)-T_\tau\left((u-w-\varepsilon)^+(y)\right))^2}{|x-y|^{N+s\,p}}\,dx\,dy \leq 0$$

and letting $\tau \to +\infty$, by Fatou's Lemma

$$\int_{\mathbb{R}^{2N}} \frac{(|u(x)-u(y)|+|w(x)-w(y)|)^{p-2}\,((u-w-\varepsilon)^+(x)-(u-w-\varepsilon)^+(y))^2}{|x-y|^{N+s\,p}}\,dx\,dy \leq 0.$$

Being the integrand non-negative, it is standard to obtain

$$u \leq w+\varepsilon \leq v+\varepsilon \qquad \text{a.e. in} \quad \Omega$$

and the thesis follows letting $\varepsilon \to 0$. $\square$

In light of Theorem 4.2 we are now in position to conclude the proof of our main results.

Proof of Theorem 1.4. If u and v are two solutions to (1.1) with zero Dirichlet boundary condition, then we have that $u \leq v$ by Theorem 4.2. In the same way it follows that $v \leq u$. □

We now deduce a symmetry result from the uniqueness of the solution. We have the following:

Proof of Theorem 1.7. By rotation and translation invariance, we may and we will assume that Ω is symmetric in the x_1-direction and $f(x_1, x') = f(-x_1, x')$ (with $x' \in \mathbb{R}^{N-1}$). Setting $v(x_1, x') := u(-x_1, x')$ it follows that v is a solution to (1.1) with zero Dirichlet boundary condition. By uniqueness, namely applying Theorem 1.4, it follows that $u = v$, that is $u(x_1, x') = u(-x_1, x')$ a.e., ending the proof. □

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