CAPILLARITY PROBLEMS WITH NONLOCAL SURFACE TENSION ENERGIES

FRANCESCO MAGGI AND ENRICO VALDINOCI

Abstract. We explore the possibility of modifying the classical Gauss free energy functional used in capillarity theory by considering surface tension energies of nonlocal type. The corresponding variational principles lead to new equilibrium conditions which are compared to the mean curvature equation and Young's law found in classical capillarity theory. As a special case of this family of problems we recover a nonlocal relative isoperimetric problem of geometric interest.

1. Introduction

1.1. **Overview.** Classical capillarity theory is based on the study of volume-constrained critical points and local/global minimizers of the Gauss free energy of a liquid droplet occupying a region E inside a container $\Omega \subset \mathbb{R}^n$, $n \geqslant 2$. If $\mathcal{H}^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure in $\mathbb{R}^n$, then the Gauss free energy of E is

$$\mathcal{H}^{n-1}(\Omega \cap \partial E) + \sigma\, \mathcal{H}^{n-1}(\partial\Omega \cap \partial E) + \int_E g(x)\, dx \tag{1.1}$$

where $\mathcal{H}^{n-1}(\Omega \cap \partial E)$ accounts for the surface tension energy of the interior liquid/air interface, $\sigma\, \mathcal{H}^{n-1}(\partial\Omega \cap \partial E)$ for the surface tension energy due to the liquid/solid interface (measured relatively to the liquid/air tension, so that the *relative adhesion coefficient* σ is assumed to satisfy $-1 < \sigma < 1$), and where $g(x)$ stands for the potential energy density acting on the droplet. It is well-known that when E is a volume-constrained critical point of the Gauss free energy having sufficiently smooth boundary, then the equilibrium conditions (Euler-Lagrange equations) for E take the form

$$\mathbf{H}_{\partial E}(x) + g(x) = c\,, \qquad \text{for every } x \in \Omega \cap \partial E\,, \tag{1.2}$$

$$\nu_E(x) \cdot \nu_\Omega(x) = \sigma\,, \qquad \text{for every } x \in \overline{\Omega \cap \partial E} \cap \partial\Omega\,, \tag{1.3}$$

where ν_E is the outer unit normal to E, $\mathbf{H}_{\partial E}$ is the mean curvature of ∂E (computed with respect to ν_E) and $c \in \mathbb{R}$ is a Lagrange multiplier.

In this paper we introduce and investigate a family of capillarity-type energies where the effect of surface tension is measured through nonlocal interaction energies, rather then through surface area. Given $s \in (0,1)$ and $\varepsilon \in (0,\infty]$ we denote by

$$I_s^\varepsilon(E,F) = \int_E dx \int_F \frac{1_{(0,\varepsilon)}(|x-y|)\, dy}{|x-y|^{n+s}}$$

the *fractional interaction energy of order s truncated at scale ε* between two disjoint sets E and F contained in $\mathbb{R}^n$. We then work with the following "fractional Gauss free energy"

$$I_s^\varepsilon(E, \Omega \cap E^c) + \sigma\, I_s^\varepsilon(E, \Omega^c) + \int_E g(x)\, dx\,, \tag{1.4}$$

Points in E interact with points in $\Omega \cap E^c$ and with points in Ω^c; the second type of interaction is weighted by a constant σ having the same role of the relative adhesion coefficient in the classical model, and interactions are truncated at distance ε. Since the kernel $|z|^{-n-s}$ is not locally integrable, the function $x \in E \mapsto \int_{E^c} |x-y|^{-n-s}\, dy$ explodes like $\mathrm{dist}(x,\partial E)^{-s}$ as

$x \in E$ approaches the boundary of E. Now for every $y \in \partial E$ the function $t > 0 \mapsto \mathrm{dist}(y - t\nu_E(y), \partial E)^{-s} = t^{-s}$ is integrable as $t \to 0^+$, and thus we understand a term like the integral over $x \in E$ of $x \in E \mapsto \int_{E^c} |x-y|^{-n-s}\, dy$, or more generally $I_s^\varepsilon(E, E^c)$ with $\varepsilon < \infty$, as a nonlocal measurement of the surface area of ∂E. This intuition is confirmed by the fact that, in the limit $s \to 1^-$ corresponding to highly concentrated kernels, and after scaling by the factor $(1-s)$, the nonlocal capillarity energy (1.4) converges to its local counterpart (1.1),

$$\lim_{s\to 1^-} \frac{(1-s)}{\kappa_n} \Big(I_s^\varepsilon(E, \Omega \cap E^c) + \sigma\, I_s^\varepsilon(E, \Omega^c) \Big) = \mathcal{H}^{n-1}(\Omega \cap \partial E) + \sigma\, \mathcal{H}^{n-1}(\partial\Omega \cap \partial E)$$

see Proposition 1.2 below. The latter property indicates that for s close to 1 the nonlocal model is quite close to the classical one. There are however some qualitative differences of possible interest, and the goal of this paper is starting their study.

Clearly, in order to understand these differences, the first step is deriving and discussing the Euler-Lagrange equations for the nonlocal capillarity energy (1.4). Both the interior equilibrium condition (1.2) and Young's law (the contact angle condition (1.3)) are affected by the nonlocality of the model.

A first remarkable difference is that the interior equilibrium condition feels the effect of the relative adhesion coefficient σ at interior points whose distance from $\partial\Omega$ is within the range of the interaction kernel. (This is in striking difference with the classical model, where the corresponding interior equilibrium condition, namely (1.2), is completely unaffected by the mismatch in surface tension even at points in the boundary of the droplet lying at arbitrarily small distance from the container walls.) Indeed, as proved in Theorem 1.3 below, the interior equilibrium condition in the fractional setting takes the form

$$\mathbf{H}^{s,\varepsilon}_{\partial E}(x) - (1-\sigma) \int_{\Omega^c} \frac{1_{(0,\varepsilon)}(|x-y|)}{|x-y|^{n+s}}\, dy + g(x) = c \qquad \text{for every } x \in \Omega \cap \partial E\,, \tag{1.5}$$

where $\mathbf{H}^{s,\varepsilon}_{\partial E}(x)$ is the *fractional mean curvature of* ∂E *at* x (of fractional order s and with truncation at scale ε), defined as

$$\mathbf{H}^{s,\varepsilon}_{\partial E}(x) = \mathrm{p.v.} \int_{\mathbb{R}^n} \Big(1_{E^c}(y) - 1_E(y) \Big) \frac{1_{(0,\varepsilon)}(|x-y|)}{|x-y|^{n+s}}\, dy \qquad \forall x \in \partial E\,.$$

This last integral has to be defined in the principal value sense and only for $x \in \partial E$, because in order for the integral to converge it is essential that, in a ball of radius $r > 0$ centered at x, 1_{E^c} and -1_E cancel out the presence of the non-integrable kernel on outside of a region of volume $\mathrm{o}(r^n)$. With this caveat in mind, it holds that, as $s \to 1^-$, $(1-s)\,\mathbf{H}^{s,\varepsilon}_{\partial E}(x) \to \mathbf{H}_{\partial E}(x)$ for every $x \in \partial E$ such that ∂E is an hypersurface of class C^2 around x. The novel feature of the fractional model is contained in the second term on the left-hand side of (1.5), namely

$$-(1-\sigma) \int_{\Omega^c} \frac{1_{(0,\varepsilon)}(|x-y|)}{|x-y|^{n+s}}\, dy\,.$$

Because of this term, the mismatch $1-\sigma$ in the surface tension between the liquid/air and liquid/solid interface is felt *also at point* $x \in \Omega \cap \partial E$ *lying at a distance at most* ε *from the boundary wall* $\partial\Omega$. Notice that this nonlocal term, multiplied by $(1-s)$, converges to 0 as $s \to 1^-$ for every $x \in \Omega$.

Coming to the contact angle condition, as proved in Theorem 1.4 below, when working with the fractional model one finds a different contact angle than the one predicted in the classical Young's law (1.3). Independently from the considered value of ε and on the ambient space dimension n, the fractional Young's law takes the form

$$\nu_E(x) \cdot \nu_\Omega(x) = \cos(\pi - \theta(s,\sigma))\,, \qquad \text{for every } x \in \overline{\Omega \cap \partial E} \cap \partial\Omega\,, \tag{1.6}$$

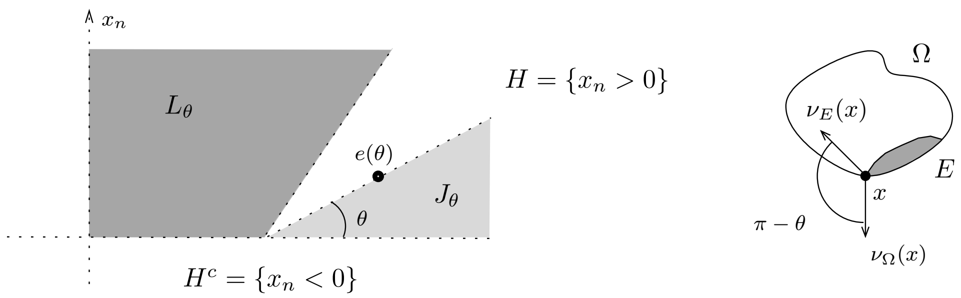

FIGURE 1.1. The contact angle for the fractional Young's law of order s is computed by balancing the volume of the cone L_θ and the volume of H^c multiplied by σ. In both cases "volume" is computed with respect to the singular density $|z-e(\theta)|^{-n-s}dz$, where both integral converge as the non-integrable singularity $e(\theta)$ is at positive distance from both L_θ and H^c. Notice that L_θ is defined by considering the reflection J_θ^* of J_θ with respect to $H \cap \partial J_\theta$, and then by setting $L_\theta = J_\theta^* \cap H$.

where $\theta = \theta(s,\sigma) \in (0,\pi)$ is uniquely defined in terms of s and σ by the identity

$$\int_{\mathbb{R}^n} \frac{(1_{J_\theta^c \cap H} + \sigma\, 1_{H^c} - 1_{J_\theta})(z)}{|e(\theta)-z|^{n+s}}\, dz \;=\; 0, \tag{1.7}$$

$$\text{where} \quad J_\theta = \Big\{x \in \mathbb{R}^n : x_n > 0 \text{ and } \cos\alpha\, x_n = \sin\alpha\, x_1 \text{ for some } \alpha \in (0,\theta)\Big\},$$

$$H = \{x \in \mathbb{R}^n : x_n > 0\}$$

$$\text{and} \quad e(\theta) = \cos\theta\, e_1 + \sin\theta\, e_n\,,$$

whose geometric significance is illustrated in Figure 1.1. (Notice that the independence of θ from n is not apparent from (1.7).) One has $\sigma \in (-1,1) \mapsto \theta(s,\sigma)$ is strictly increasing with

$$\theta(s,0) = \frac{\pi}{2}\,, \qquad \lim_{\sigma\to(-1)^+} \theta(s,\sigma) = 0\,, \qquad \lim_{\sigma\to 1^-} \theta(s,\sigma) = \pi$$

and, quite importantly,

$$\lim_{s\to 1^-} \cos(\pi - \theta(s,\sigma)) = \sigma\,,$$

so that the fractional Young's law (1.6) converges to its local counterpart (1.3) as $s \to 1^-$. The fact of obtaining a different contact angle than the classical one may be reconciliated with physical observation as the angle predicted by the classical Young's law may be actually observed in the nonlocal context *at a characteristic distance from the boundary of the container*. In other words, *the nonlocal model may predict different microscopic and macroscopic contact angles, the latter in accordance with* (1.3). We plan to address this issue in a subsequent paper, by focusing on the fractional sessile droplet problem.

Let us now comment on the mathematical background of our work. The use of fractional Sobolev norms in the analysis of partial differential equations is of course a well established area of research with a vast literature and a huge range of applications. The study of nonlocal *geometric* variational problems has attracted a large attention since the seminal work [CRS10], where *nonlocal minimal surfaces* have been introduced motivated by the study of the mean curvature flow as the limit of a process based on long range correlation. The boundary of a set E is nonlocal area minimizing in an open set Ω if the quantity

$$\int_{E\cap\Omega}\int_{E^c\cap\Omega} \frac{dxdy}{|x-y|^{n+s}} + \text{additional "lower order" interaction terms}$$

is minimized by E among all sets F such that $F\setminus\Omega = E\setminus\Omega$. The main result in [CRS10] is partial $C^{1,\alpha}$-regularity theorem outside a closed singular set of dimension $n-2$. Higher order regularity and improved dimensional estimates for the singular set have been obtained in [SV13, BFV14, FV16], examples of singular minimizing cones have been obtained in [DdPW13, DdPW14], while boundaries with constant fractional mean curvature have been studied in [CFSW16, CFW16, CFMN16, DdPDV16]. The present paper is also a contribution to the developing theory of nonlocal geometric variational problems. Indeed the minimization of (1.4) in the case $\sigma = 0$, $g = 0$, and $\varepsilon = +\infty$ leads to study a family of *relative isoperimetric problems for fractional perimeters* in the open set Ω. Relative isoperimetric problems are of course a classical subject in the calculus of variations, especially because of their importance in determining (or in bounding) sharp constants in Poincaré-type inequalities; see [Maz11]. This kind of application uses the possibility of writing Dirichlet energies as perimeter integrals over super-level sets by the coarea formula. This is possible also in the nonlocal case, where an appropriate version of the coarea formula can be found, for example, in [Vis91].

1.2. **Interaction kernels.** The study of nonlocal geometric variational problems is mainly concerned with the nonlocal perimeters defined through the infinite-range isotropic singular kernels or, briefly, fractional kernels. Given $s \in (0,1)$, the *fractional kernel* of order s is defined as

$$K_s(\zeta) = \frac{1}{|\zeta|^{n+s}}\,, \qquad \zeta \in \mathbb{R}^n \setminus \{0\}\,. \tag{1.8}$$

It also seems interesting to consider finite range interactions. We thus introduce the *truncated fractional kernel* of order s,

$$K_s^\varepsilon(\zeta) = \frac{1_{(0,\varepsilon)}(|\zeta|)}{|\zeta|^{n+s}}\,, \qquad \zeta \in \mathbb{R}^n \setminus \{0\}\,, \varepsilon \in (0,\infty]\,. \tag{1.9}$$

Given K_s and K_s^ε as the prototype kernels in our theory, we may finally want to consider possibly anisotropic interactions. We are thus led to introduce the following family of kernels.

Given $n \geqslant 2$, $s \in (0,1)$, $\lambda \geqslant 1$ and $\varepsilon \in [0,\infty]$ we consider the family of *interaction kernels*

$$\mathbf{K}(n,s,\lambda,\varepsilon) \qquad (\text{and set } \mathbf{K}(n,s,\lambda) = \mathbf{K}(n,s,\lambda,0))$$

consisting of those *even* functions $K : \mathbb{R}^n \setminus \{0\} \to [0,+\infty)$ satisfying

$$\frac{1_{B_\varepsilon}(\zeta)}{\lambda\,|\zeta|^{n+s}} \leqslant K(\zeta) \leqslant \frac{\lambda}{|\zeta|^{n+s}} \qquad \forall \zeta \in \mathbb{R}^n \setminus \{0\}\,. \tag{1.10}$$

(Here, $B_\varepsilon(x)$ is the ball of center x and radius ε, and we simply set $B_\varepsilon = B_\varepsilon(0)$.) In particular, we assume that K is bounded from above by a homogeneous kernel with polynomial decay of degree $-(n+s)$ and that is bounded from below by the same type of homogeneous kernels up to distance ε from the origin. Notice that $\mathbf{K}(n,s,1,\infty)$ contains only the fractional kernel K_s defined in (1.8). Given any $K \in \mathbf{K}(n,s,\lambda,\varepsilon)$ we set

$$K^*(\zeta) = \lim_{r\to 0^+} r^{n+s}\,K(r\,\zeta) \qquad \zeta \neq 0\,, \tag{1.11}$$

provided the limit exists. Notice that K^* is automatically $-(n+s)$-homogeneous and bounded from above by $\lambda\,|\zeta|^{-n-s}$, and that in the case of truncated fractional kernels we have

$$(K_s^\varepsilon)^* = K_s \qquad \forall s \in (0,1)\,,\ \varepsilon > 0\,.$$

Occasionally we shall need to work with smoother interaction kernels: given $h \in \mathbb{N}$ we thus introduce the class

$$\mathbf{K}^h(n,s,\lambda,\varepsilon) \qquad (\text{and set } \mathbf{K}^h(n,s,\lambda) = \mathbf{K}^h(n,s,\lambda,0))$$

consisting of those $K \in \mathbf{K}(n,s,\lambda,\varepsilon) \cap C^h(\mathbb{R}^n \setminus \{0\})$, with

$$|D^j K(\zeta)| \leqslant \frac{\lambda}{|\zeta|^{n+s+j}} \qquad \forall \zeta \in \mathbb{R}^n \setminus \{0\}\,, 1 \leqslant j \leqslant h\,. \tag{1.12}$$

Each kernel K defines an *interaction functional* between disjoint subsets of $\mathbb{R}^n$,

$$I(E,F)=\int_E\int_F K(x-y)\,dx\,dy\in[0,\infty]\,,\qquad E,F\subset\mathbb{R}^n\,,E\cap F=\varnothing\,.$$

The *nonlocal perimeter associated to* K is defined as the interaction of a set with its complement

$$P(E)=I(E,E^c)\,,\qquad E^c=\mathbb{R}^n\setminus E\,.$$

In the important cases of the fractional kernel $K=K_s$ and of truncated fractional kernel $K=K_s^\varepsilon$ we write I_s and I_s^ε in place of I, and P_s and P_s^ε in place of P, so that

$$I_s(E,F)=\int_E\int_F\frac{dx\,dy}{|x-y|^{n+s}}\,,\qquad P_s(E)=I_s(E,E^c)\,,$$

$$I_s^\varepsilon(E,F)=\int_E\int_F\frac{1_{B_\varepsilon}(x-y)\,dx\,dy}{|x-y|^{n+s}}\,,\qquad P_s^\varepsilon(E)=I_s^\varepsilon(E,E^c)\,.$$

As shown in [Dáv02] (see also [BBM01])

$$\lim_{s\to1^-}(1-s)\,P_s^\varepsilon(E)=\kappa_n\,\mathcal{H}^{n-1}(\partial^*E)\qquad \kappa_n:=\frac{1}{2}\int_{S^n}|e\cdot\omega|d\mathcal{H}^{n-1}_\omega\qquad e\in S^{n-1}\,,$$

whenever E is a set of finite perimeter in $\mathbb{R}^n$ and ∂^*E denotes the reduced boundary of E (for example, if E is a bounded open set with Lipschitz boundary, then E is a set of finite perimeter and $\partial^*E=\partial E$).

1.3. **Nonlocal capillarity energy.** Given $K\in\mathbf{K}(n,s,\lambda,\varepsilon)$, an open set $\Omega\subset\mathbb{R}^n$, and $\sigma\in(-1,1)$ we define the nonlocal capillarity energy of $E\subset\Omega$ as

$$\mathcal{E}(E)=I(E,E^c\Omega)+\sigma\,I(E,\Omega^c)\,. \tag{1.13}$$

Here and in the following we adopt the following unusual convention in order to simplify formulas involving the interaction functional: precisely, when a set intersection $F\cap G$ will appear as an argument of I, we shall write FG in place of $F\cap G$. For example,

$$I(EF,GH)\text{ stands for }I(E\cap F,G\cap H)\,. \tag{1.14}$$

Looking at (1.13), the term $I(E,E^c\Omega)$ accounts for interactions between liquid and air particles, while the term $I(E,\Omega^c)$ accounts for interactions between E and the solid walls of the container. From the physical point of view, we expect short range interactions to matter the most. When working with the fractional kernels I_s^ε, this can be taken into account either by requiring the truncation parameter ε to be small, or by taking s close to 1. As already noticed, the latter option corresponds to highly concentrated kernels whose fractional perimeter are increasingly close to the classical perimeter.

The basic variational problem we are interested in is then

$$\gamma=\inf\Big\{\mathcal{E}(E)+\int_E g(x)\,dx:\ \ E\subset\Omega\,,\ |E|=m\Big\} \tag{1.15}$$

where $m\in(0,|\Omega|)$ and $g:\mathbb{R}^n\to\mathbb{R}$ are given. As already noticed, when $\sigma=0$ and $g=0$, (1.15) is a nonlocal relative isoperimetric problem of geometric and functional interest. The minimization problem in (1.15) is indeed well-posed, according to the following simple result:

Proposition 1.1 (Existence of minimizers)**.** *If $K\in\mathbf{K}(n,s,\lambda)$, Ω is an open bounded set with $P(\Omega)<\infty$, and $g\in L^\infty(\Omega)$, then there exist minimizers in* (1.15)*. Moreover, $I(E,E^c\Omega)<\infty$ for every minimizer E.*

We have already mentioned the fact that, as $s\to1^-$, fractional perimeters converge to classical perimeters. This is true also for our nonlocal capillarity energy.

Proposition 1.2 (Convergence to the classical energy)**.** *If Ω and E are open sets with Lipschitz boundary and $E\subset\Omega$, then*

$$\begin{aligned}\lim_{s\to 1^-}(1-s)\,I_s(E,E^c\Omega) &= \kappa_n\,\mathcal{H}^{n-1}(\Omega\cap\partial E)\\ \lim_{s\to 1^-}(1-s)\,I_s(E,\Omega^c) &= \kappa_n\,\mathcal{H}^{n-1}(\partial E\cap\partial\Omega)\,.\end{aligned}$$

In particular,

$$\lim_{s\to 1^-}\frac{(1-s)}{\kappa_n}\,\mathcal{E}(E)=\mathcal{H}^{n-1}(\Omega\cap\partial E)+\sigma\,\mathcal{H}^{n-1}(\partial\Omega\cap\partial E)\,.$$

1.4. **Euler-Lagrange equations.** We now address the form taken by the equilibrium conditions (Euler-Lagrange equations) at boundary points of minimizers in the nonlocal capillarity problem. Notice that a minimizer E in (1.15) could be in principle quite irregular, and actually the property of being a minimizer is invariant under modifications of E on and by a set of volume zero. It is thus convenient to work with a robust notion of boundary of E and set

$$\partial E=\Big\{x\in\overline{\Omega}:0<|E\cap B_r(x)|<\omega_n\,r^n\quad\forall r>0\Big\}\,.$$

We shall then define the *regular part* Reg_E and the *singular part* Σ_E of ∂E by setting

$$\mathrm{Reg}_E=\left\{x\in\overline{\Omega\cap\partial E}:\begin{array}{l}\text{there exists }\varrho>0\text{ and }\alpha\in(s,1)\text{ s.t. }B_\varrho(x)\cap\partial E\text{ is a }C^{1,\alpha}\text{-manifold}\\ \text{with boundary, whose boundary points are in }\partial\Omega\end{array}\right\}$$

and $\Sigma_E=\partial E\setminus\mathrm{Reg}_E$, respectively. We expect the Euler-Lagrange equations to hold in weak form at every point $x\in\partial E$ and in a stronger, pointwise form at every $x\in\mathrm{Reg}_E$; see (1.22) and (1.23) below. Since our primary goal here is understanding the qualitative features of the proposed nonlocal capillarity model, and thus its possible physical interest, we shall not be concerned with the regularity problem, which would consists in showing the smallness of Σ_E. Let us recall that, in the local case, when $n=3$ the singular set is empty [Tay77,Luc87,DPM15].

In order to introduce the Euler-Lagrange equations for the nonlocal capillarity energy $\mathcal{E}$, it is convenient to recall the form taken by the equilibrium conditions for local minimizers of nonlocal perimeters. Given two sets E and F which are equal outside of a bounded open set A we formally have

$$P(E)-P(F)=P(E,A)-P(F,A)$$

where we have set

$$P(E;A)=I(EA,E^cA)+I(EA,E^cA^c)+I(E^cA,EA^c)\,,$$

and where the identity $P(E)-P(F)$ holds in general only in a formal sense as it involves the cancellation of the possibly infinite interaction terms $I(EA^c,E^cA^c)=I(FA^c,F^cA^c)$ (as $E\cap A^c=F\cap A^c$ by assumption). We thus say that $E\subset\mathbb{R}^n$ is a critical point of P in a bounded open set A if

$$\frac{d}{dt}\Big|_{t=0}P(f_t(E),A)=0\,,$$

for every family of diffeomorphisms $\{f_t\}_{|t|<\delta}$ such that

$$f_0=\mathrm{Id}\qquad\mathrm{spt}(f_t-\mathrm{Id}\,)\subset\subset A\qquad\forall|t|<\delta\,.\tag{1.16}$$

If $K\in\mathbf{K}^1(n,s,\lambda)$, then being a critical point is equivalent to the condition

$$\int_E\int_{E^c}\mathrm{div}_{\,(x,y)}\big(K(x-y)\,(T(x),T(y))\big)\,dx\,dy=0\qquad\forall T\in C^1_c(A;\mathbb{R}^n)\,.\tag{1.17}$$

where we have set

$$\mathrm{div}_{\,(x,y)}\big(K(x-y)\,(T(x),T(y))\big)=\mathrm{div}_{\,x}\big(K(x-y)\,T(x)\big)+\mathrm{div}_{\,y}\big(K(x-y)\,T(y)\big)\,.$$

We refer to (1.17) as to the *weak form* of the Euler-Lagrange equation of P in A. Notice that (1.17) "holds at every $x \in \partial E$" in the sense that it is satisfied by every measurable set E if restricted to vector fields T with $\operatorname{spt} T \cap \partial E = \varnothing$. If $K \in \mathbf{K}^2(n, s, \lambda)$, then (1.17) implies that

$$\mathbf{H}^K_{\partial E}(x) = 0 \qquad \forall x \in A \cap \mathrm{Reg_E} \tag{1.18}$$

where $\mathbf{H}^K_{\partial E}(x)$ is the *nonlocal mean curvature of* ∂E *at* x (with respect to the kernel K), and is defined as

$$\mathbf{H}^K_{\partial E}(x) := \mathrm{p.v.} \int_{\mathbb{R}^n} \Big(1_{E^c}(y) - 1_E(y)\Big)\, K(x-y)\, dy \qquad x \in \partial E\,. \tag{1.19}$$

This integral converges in the principal value sense as soon as E is the epigraph of a $C^{1,\alpha}$-function with $\alpha > s$ in a neighborhood of x, and actually $\mathbf{H}^K_{\partial E}$ is a continuous function on Reg_E. Equation (1.18) is the *strong form* of (1.17), and in the limit $s \to 1^-$ of highly concentrated fractional kernels we have

$$\lim_{s\to 1^-} (1-s)\, \mathbf{H}^{K_s^\varepsilon}_{\partial E}(x) = \mathbf{H}_{\partial E}(x)$$

provided ∂E is of class C^2 in a neighborhood of x.

Coming back to the capillarity problem, we say that $E \subset \Omega$ is a (volume-constrained) *critical point of* $\mathcal{E} + \int g$ if

$$\frac{d}{dt}\bigg|_{t=0} \mathcal{E}(f_t(E)) + \int_{f_t(E)} g = 0\,, \tag{1.20}$$

for every family of diffeomorphisms $\{f_t\}_{|t|<\delta}$ such that, for every $|t| < \delta$,

$$f_0 = \mathrm{Id}\,, \quad \mathrm{spt}(f_t - \mathrm{Id}\,) \subset\subset \mathbb{R}^n\,, \quad f_t(\Omega) = \Omega\,, \quad |f_t(E)| = |E|\,. \tag{1.21}$$

Global minimizers in (1.15) are of course critical sets. At regular points of a critical set of $\mathcal{E} + \int g$ the Euler-Lagrange equations take the following form.

Theorem 1.3 (Euler-Lagrange equation). *Let Ω be a bounded open set with C^1-boundary, $g \in C^1(\mathbb{R}^n)$, and E be a critical point of $\mathcal{E} + \int g$. If $K \in \mathbf{K}^1(n, s, \sigma)$, then there exists a constant $c \in \mathbb{R}$ such that*

$$\begin{aligned} &\iint_{E\times(E^c\cap\Omega)} \mathrm{div}\,_{(x,y)}\big(K(x-y)\,(T(x), T(y))\big)\, dxdy \\ &+ \sigma \iint_{E\times\Omega^c} \mathrm{div}\,_{(x,y)}\big(K(x-y)\,(T(x), T(y))\big)\, dxdy + \int_E \mathrm{div}\,(g\,T) = c \int_E \mathrm{div}\; T \end{aligned} \tag{1.22}$$

for every $T \in C^\infty_c(\mathbb{R}^n;\mathbb{R}^n)$ with

$$T \cdot \nu_\Omega = 0 \qquad \text{on } \partial\Omega\,.$$

Moreover, if $K \in \mathbf{K}^2(n, s, \sigma)$, then

$$\mathbf{H}^K_{\partial E}(x) - (1-\sigma) \int_{\Omega^c} K(x-y)\, dy + g(x) = c\,, \qquad \forall x \in \Omega \cap \mathrm{Reg}_E\,. \tag{1.23}$$

We next investigate the contact angle condition, or Young's law, in the nonlocal setting. Let us recall that in the local setting Young's law can be derived through integration by parts starting from the weak form of (1.2), that is

$$\int_{\partial E} \mathrm{div}\,^{\partial E} T + \int_{\partial E} g\,(T\cdot\nu_E) = c \int_{\partial E} T\cdot\nu_E$$

for every $T \in C^1_c(\mathbb{R}^n;\mathbb{R}^n)$ with $T \cdot \nu_\Omega = 0$ on $\partial\Omega$; see, e.g., [Mag12, Theorem 19.8], and compare with (1.22). In the nonlocal case we need to use a different approach, avoiding integration by parts. More precisely, the nonlocal Young's law will be obtained by taking blow-ups of (1.23)

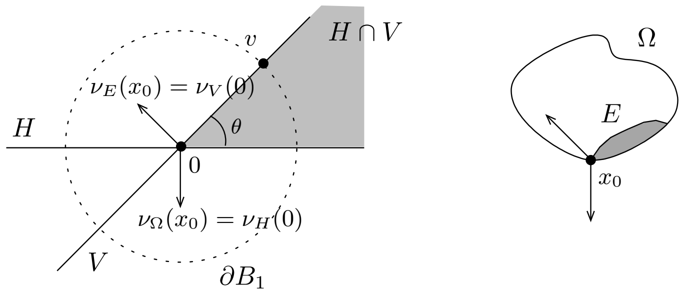

FIGURE 1.2. The nonlocal Young's law is computed at points $x_0 \in \partial\Omega$ where E blow-ups a cone of the form $V \cap H$ where V is an half-space, and H is the half-space blow-up of Ω at x_0. This law determines the angle between V and H via the identity (1.24).

along sequences of regular interior points converging to $\partial\Omega \cap \mathrm{Reg}_E$. Here and in the following we shall use the notation

$$A^{x_0,r} = \frac{A - x_0}{r}$$

for the blow-up of $A \subset \mathbb{R}^n$ at scale $r > 0$ around $x_0 \in \mathbb{R}^n$.

Theorem 1.4 (Nonlocal Young's law). *Let $K \in \mathbf{K}^2(n, s, \lambda)$ be such that the homogeneous kernel K^* is well-defined accordingly to (1.11), and let $g \in C^0(\mathbb{R}^n)$. Let Ω be a bounded open set with C^1-boundary and E be a volume-constrained critical set of $\mathcal{E} + \int g$. Given $x_0 \in \mathrm{Reg}_E \cap \partial\Omega$, let H and V be the half-spaces such that*

$$\Omega^{x_0,r} \to H \text{ and } E^{x_0,r} \to H \cap V \text{ in } L^1_{\mathrm{loc}}(\mathbb{R}^n) \text{ as } r \to 0^+$$

and set $\nu_E(x_0) := \nu_V(0)$. Then the angle between H and V must satisfy the identity

$$\mathbf{H}^{K^*}_{\partial(H\cap V)}(v) - (1 - \sigma) \int_{H^c} K^*(v - z)\, dz = 0\,, \qquad \forall v \in H \cap \partial V\,, \tag{1.24}$$

see Figure 1.2. In the special case when $K = K_s^\varepsilon$, and thus $K^* = K_s$, (1.24) uniquely identifies the angle between H and V. More precisely, for every $s \in (0, 1)$ and $\sigma \in (-1, 1)$ there exists a unique $\theta = \theta(s, \sigma) \in (0, \pi)$ such that

$$\nu_E(x_0) \cdot \nu_\Omega(x_0) = \nu_V(0) \cdot \nu_H(0) = \cos\big(\pi - \theta(s, \sigma)\big)\,. \tag{1.25}$$

The function $\sigma \in (-1, 1) \mapsto \theta(s, \sigma)$ is strictly increasing with

$$\theta(s, 0) = \frac{\pi}{2}\,, \qquad \lim_{\sigma\to(-1)^+} \theta(s, \sigma) = 0\,, \qquad \lim_{\sigma\to 1^-} \theta(s, \sigma) = \pi$$

and

$$\lim_{s\to 1^-} \cos\big(\pi - \theta(s, \sigma)\big) = \sigma\,.$$

In particular, the fractional Young's law (1.25) converges to the classical Young's law in the limit $s \to 1^-$ of highly concentrated interaction kernels.

Theorem 1.4 shows that the nonlocal Young's law may take different forms depending on the considered kernels. Even in the class of isotropic fractional kernels K_s, the contact angle will depend on s (in addition to its dependency on σ), although it will converge to the angle predicted by the classical Young's law in the limit $s \to 1^-$. The contact angle predicted by the classical Young's law may be actually observed in the nonlocal context at a characteristic distance from the boundary of the container. We plan to further investigate this issue in a subsequent paper, focusing on the sessile droplet problem.

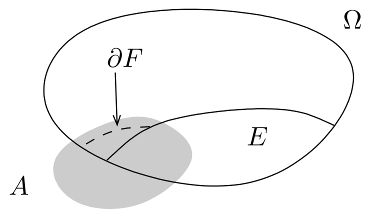

FIGURE 1.3. Since in Definition 1.5 we consider variations in an open set A which is not contained in Ω, we are in practice imposing on E a Dirichlet condition along $\Omega\cap\partial A$ and a Neumann condition along $A\cap\partial\Omega$.

We also remark that in the case $\sigma=0$ with isotropic kernel $K=K_s^\varepsilon$, the nonlocal Young's law always boils down to

$$\nu_E(x_0)\cdot\nu_\Omega(x_0)=0\qquad\forall x_0\in\partial\Omega\cap\mathrm{Reg}_E\,.$$

This is interesting as the corresponding variational problem

$$\inf\Big\{I_s^\varepsilon(E,E^c\Omega):E\subset\Omega\,,|E|=m\Big\}$$

is a natural fractional variant of the classical *relative isoperimetric problem in* Ω. Thus critical points in the relative isoperimetric problem and in all of its fractional variants share the same orthogonality condition at the boundary of Ω, independently from ε and s. At the same time, the equilibrium interior condition $\mathbf{H}_{\partial E}^{K_s^\varepsilon}=$ constant valid on $\Omega\cap\mathrm{Reg}_E$ depends on the specific values of s and ε.

1.5. **Interior regularity and other regularity properties.** In the last part of our paper we address some regularity properties of local (almost) minimizers of the nonlocal capillarity energy $\mathcal{E}$. In order to introduce the minimality condition that we shall consider, let us notice that if E and F are equal outside of an open set A (not necessarily contained in Ω, see Figure 1.3), that is, if $F\cap A^c=E\cap A^c$, then one can formally compute (with the convention (1.14) in force)

$$\begin{aligned}\mathcal{E}(E)-\mathcal{E}(F)&=I(E,E^c\Omega)+\sigma\,I(E,\Omega^c)-I(F,F^c\Omega)-\sigma\,I(F,\Omega^c)\\&=I(EA,E^c\Omega)+I(EA^c,E^c\Omega A)+\sigma\,I(EA,\Omega^c)\\&\quad-I(FA,F^c\Omega)-I(FA^c,F^c\Omega A)-\sigma\,I(FA,\Omega^c)\,.\end{aligned}$$

We are thus led to consider the following kind of local (almost) minimality inequality.

Definition 1.5 (Almost minimizers). Let $K\in\mathbf{K}(n,s,\lambda)$, Ω and A be open (possibly unbounded) sets in $\mathbb{R}^n$ such that

$$I(\Omega A,\Omega^c)<\infty\,,\tag{1.26}$$

and let $\Lambda\in[0,\infty)$, $r_0\in(0,\infty]$ and $\sigma\in(-1,1)$. Given $E\subset\Omega$, one says that E is a (Λ,r_0,σ,K)-*minimizer in* (A,Ω) if

$$\begin{aligned}&I(EA,E^c\Omega)+I(EA^c,E^c\Omega A)+\sigma\,I(EA,\Omega^c)\\\leqslant\;&I(FA,F^c\Omega)+I(FA^c,F^c\Omega A)+\sigma\,I(FA,\Omega^c)+\Lambda\,|E\Delta F|\,,\end{aligned}\tag{1.27}$$

for every $F\subset\Omega$ with $\mathrm{diam}(F\Delta E)<2\,r_0$ and $F\cap A^c=E\cap A^c$. Notice that (1.26) guarantees that $I(FA,\Omega^c)<\infty$ whenever $F\subset\Omega$, so that, even when $\sigma<0$, the quantity

$$I(FA,F^c\Omega)+I(FA^c,F^c\Omega A)+\sigma\,I(FA,\Omega^c)\,,$$

appearing on the right-hand side of (1.27) is well-defined in $(-\infty,\infty]$.

As proved in Corollary 5.5 below, if E is a minimizer in (1.15), then there exist $\Lambda \geqslant 0$ and $r_0 > 0$ (depending on E and $\|g\|_{L^\infty(\Omega)}$) such that E is a $(\Lambda, r_0, \sigma, K)$-minimizer in $(\mathbb{R}^n, \Omega)$. The same is true for local minimizers of course, and the lower order term $\Lambda\,|E\Delta F|$ in the minimality inequality (1.27) actually allows to reabsorb various type of constraints (see [Alm76, Tam84] for more examples of this idea).

We are thus interested in understanding the regularity of $(\Lambda, r_0, \sigma, K)$-minimizers. Since at present an interior regularity theory for nonlocal variational problems has only been developed in the isotropic case of the fractional kernel K_s (see [CRS10, CG10]) we shall mainly focus on this case. The first important remark is that on variations supported away from the boundary of Ω, the minimality inequality (1.27) implies the type of almost–minimality condition considered in [CRS10, CG10]. Thus, interior regularity is readily established.

Theorem 1.6 (Interior regularity)**.** *If E is a $(\Lambda, r_0, \sigma, K_s)$-minimizer in (A, Ω), then $A \cap \Omega \cap \mathrm{Reg}_E$ is a $C^{1,\alpha}$-hypersurface for some universal $\alpha \in (0,1)$ and $A \cap \Omega \cap \Sigma_E$ is a closed set with Hausdorff dimension less than $n-3$.*

The regularity problem near points on $\partial\Omega$ is more complex than its interior counterpart because it involves the study of a free boundary. Here we just address what is usually the first step in the analysis of a regularity problem, namely, we obtain perimeter and volume density estimates which hold *uniformly* up to the boundary of Ω. This problem, in the case $\sigma < 0$, presents some additional difficulties with respect to the interior case. These difficulties are addressed by exploiting some geometric inequalities for fractional perimeters.

Theorem 1.7 (Density estimates)**.** *Let $n \geqslant 2$, $s \in (0,1)$, $\sigma \in (-1,1)$, $\Lambda \geqslant 0$, and $K = K_s^\varepsilon$ for some $\varepsilon > 0$. If Ω is either a bounded open set with C^1 boundary or an half-space, then there exist positive constants C_0 (depending on n, s, σ, and Λ), c_* (depending on n and s) and κ (depending on n, s, σ and Ω) such that if E is a $(\Lambda, r_0, \sigma, K_s^\varepsilon)$-minimizer in (A, Ω), then*

$$I_s^\varepsilon(EB_r(x), (EB_r(x))^c) \leqslant C_0\, r^{n-s}\,, \tag{1.28}$$

whenever $B_r(x) \subset A$ and $r < \min\{r_0, c_*\,\kappa, c_*\,\varepsilon\}$. Moreover,

$$\frac{1}{C_0} \leqslant \frac{|E \cap B_r(x)|}{r^n} \leqslant 1 - \frac{1}{C_0} \tag{1.29}$$

whenever $B_r(x) \subset A$, $r < \min\{r_0, c_*\,\kappa, c_*\,\varepsilon\}$, and $x \in \overline{\Omega \cap \partial E}$.

Remark 1.8. Theorem 1.7 holds for a much larger class of "uniformly-C^1" open sets, of which bounded open set with C^1-boundary and half-spaces are particular cases. The dependence of κ from Ω can actually be expressed quite precisely in terms of this uniform C^1-property as explained in the course of the proof of Theorem 1.7.

1.6. **Organization of the paper.** In section 2 we address the existence of minimizers in the nonlocal capillarity problem, and the convergence of the fractional capillarity energy to the classical Gauss free-energy in the limit $s \to 1^-$. In section 3 and section 4 we discuss, respectively, the deduction of the Euler-Lagrange equations in weak and in strong form, and of the nonlocal Young's law. In section 5 we explain how to quickly deduce interior regularity, while section 6 is devoted to the proof of Theorem 1.7. Finally, in appendix A we obtain a quite natural closure result for sequences of almost-minimizers which shall be useful in future investigations.

Acknowledgment: FM was supported by NSF-DMS Grant 1265910 and NSF-DMS FRG Grant 1361122. EV was supported by ERC grant 277749 E.P.S.I.L.O.N. Elliptic PDEs and Symmetry of Interfaces and Layers for Odd Nonlinearities and PRIN grant 201274FYK7 Critical Point Theory and Perturbative Methods for Nonlinear Differential Equations.

2. Existence of minimizers and convergence to the classical energy

We start by proving the existence of minimizers in the variational problem (1.15), namely

$$\gamma = \inf\left\{\mathcal{E}(E) + \int_E g(x)\,dx : \ E \subset \Omega\,, \ |E| = m\right\} \tag{2.1}$$

under the assumptions that $K \in \mathbf{K}(n, s, \lambda, \varepsilon)$, Ω is an open bounded set with $P(\Omega) < \infty$, and $g \in L^\infty(\Omega)$, and where

$$\mathcal{E}(E) = I(E, E^c\Omega) + \sigma\, I(E, \Omega^c)\,;$$

see Proposition 1.1. The proof is based on a semicontinuity argument and on a direct minimization procedure. We premise the following lower semicontinuity lemma.

Lemma 2.1 (Lower semicontinuity)**.** *If $P(\Omega) < \infty$, $E_j \subset \Omega$, and $E_j \to E$ in $L^1(\Omega)$, then*

$$\liminf_{j\to\infty} \mathcal{E}(E_j) \geqslant \mathcal{E}(E)\,.$$

Proof. This is immediate by Fatou's lemma if $\sigma \geqslant 0$. If $\sigma \in (-1, 0)$, then we exploit the identity

$$\begin{aligned}\mathcal{E}(E) &= -P(\Omega) + I(E, E^c\Omega) + P(\Omega) - |\sigma|\, I(E, \Omega^c)\\ &= -P(\Omega) + I(E, E^c\Omega) + (1 - |\sigma|)\, I(E, \Omega^c) + I(E^c\Omega, \Omega^c)\,,\end{aligned}$$

and, again, Fatou's lemma, to complete the proof. □

Proof of Proposition 1.1. We first remark that since $K \in \mathbf{K}(n, s, \lambda, \varepsilon)$, then and any $p \in \mathbb{R}^n$,

$$P(F) \geqslant \frac{1}{\lambda}\, I_s(F\, B_{\varepsilon/2}(p), F^c B_{\varepsilon/2}(p))\,, \qquad \forall F \subset \mathbb{R}^n\,. \tag{2.2}$$

Indeed, if $x, y \in B_{\varepsilon/2}(p)$, then $|x - y| \leqslant |x - p| + |p - y| < \varepsilon$ and so, by (1.10),

$$P(F) = I(F, F^c) \geqslant \int_{F\cap B_{\varepsilon/2}(p)}\int_{F^c\cap B_{\varepsilon/2}(p)} K(x - y)\,dx\,dy \geqslant \frac{1}{\lambda}\int_{F\cap B_{\varepsilon/2}(p)}\int_{F^c\cap B_{\varepsilon/2}(p)} \frac{dx\,dy}{|x - y|^{n+s}},$$

that proves (2.2). Now, if H is a half-space such that $|H \cap \Omega| = m$ and $R > 0$ is such that $\Omega \subset B_R$, then

$$\mathcal{E}(H \cap \Omega) = I(H\Omega, (H\Omega)^c\Omega) + \sigma\, I(H\Omega, \Omega^c) \leqslant I(H\, B_R, H^c\, B_R) + P(\Omega) < \infty\,,$$

since $I(H\, B_R, H^c\, B_R) \leqslant C(n, s)\, R^{n-s}$ thanks to (1.10). As a consequence, we find that $\gamma < \infty$. Let $E_j \subset \Omega$ be such that $\mathcal{E}(E_j) + \int_{E_j} g \to \gamma$, then for j large enough

$$\gamma + 1 + \int_\Omega |g| \geqslant I(E_j, E_j^c\Omega) + \sigma\, I(E_j, \Omega^c) \geqslant I(E_j, E_j^c\Omega) - P(\Omega)\,,$$

and thus

$$P(E_j) = I(E_j, E_j^c\Omega) + I(E_j, E_j^c\Omega^c) \leqslant \gamma + 1 + \int_\Omega |g| + 2\, P(\Omega)\,.$$

Since $E_j \subset B_R$, using this and (2.2), we find that, up to extracting subsequences, $E_j \to E$ in $L^1_{\rm loc}(\mathbb{R}^n)$ for some $E \subset \Omega$ with $|E| = m$. By Lemma 2.1, we conclude that E is a minimizer. Now we remark that

$$I(E, \Omega^c) \leqslant I(\Omega, \Omega^c) = P(\Omega) < +\infty, \tag{2.3}$$

and so the fact that $\mathcal{E}(E) < \infty$ also implies that

$$I(E, E^c\Omega) < +\infty\,, \tag{2.4}$$

as claimed. □

We now turn to the convergence of the fractional capillarity energy to the Gauss free energy in the limit $s \to 1^-$, that is, we prove Proposition 1.2. Recalling that, by definition,

$$\kappa_n = \int_{S^n} |e\cdot\omega| d\mathcal{H}^{n-1}_\omega$$

we shall actually prove a stronger result, valid for every set of finite perimeter contained in Ω. Here $\partial^* E$ denotes the reduced boundary of the set of finite perimeter E, see [Mag12].

Proposition 2.2. *If Ω is an open set with Lipschitz boundary and $E \subset \Omega$ is a set of finite perimeter with $I_s(E, E^c) < \infty$, then*

$$\lim_{s\to 1^-} (1-s)\, I_s(E, E^c\Omega) = \frac{\kappa_n}{2}\, \mathcal{H}^{n-1}(\Omega\cap\partial^* E) \tag{2.5}$$

$$\lim_{s\to 1^-} (1-s)\, I_s(E, \Omega^c) = \frac{\kappa_n}{2}\, \mathcal{H}^{n-1}(\partial^* E\cap\partial\Omega) \tag{2.6}$$

Proof. Given $V \subset \mathbb{R}^n$ we define a Radon measure μ_s^V on $\mathbb{R}^n$ by setting

$$\begin{aligned}\mu_s^V(A) &= (1-s)\Big(I_s(EA, E^cV) + I_s(EV, E^cA)\Big)\\ &= (1-s)\int_A dx \int_V |1_E(x) - 1_E(y)|\, K_s(x-y)\, dy\,.\end{aligned}$$

Notice that $\mu_s^V(\mathbb{R}^n)$ is finite as $\mu_s^V(\mathbb{R}^n) \leqslant 2\,(1-s) I_s(E,E^c)$. By [Dáv02, Lemma 2] we have that

$$\mu_s^V \overset{*}{\rightharpoonup} \kappa_n \mathcal{H}^{n-1} \llcorner (V\cap\partial^* E) \quad \text{weakly-* as Radon measures in } V \tag{2.7}$$

whenever V is an open set, with

$$\kappa_n\, \mathcal{H}^{n-1}(V\cap\partial^* E) = \lim_{s\to 1^-} \mu_s^V(V) = 2 \lim_{s\to 1^-} (1-s)\, I_s(EV, E^cV) \tag{2.8}$$

provided V is open, bounded, with Lipschitz boundary. By applying (2.8) with $V = \Omega$ we find that

$$\kappa_n\, \mathcal{H}^{n-1}(\Omega\cap\partial^* E) = 2 \lim_{s\to 1^-} (1-s)\, I_s(E, E^c\Omega)\,,$$

that is (2.5). We now set

$$N_r(A) = \Big\{x\in\mathbb{R}^n : \mathrm{dist}(x,A) < r\Big\} \qquad r>0$$

and apply (2.8) with $V = N_r(\Omega^c)$, to find

$$\kappa_n\, \mathcal{H}^{n-1}(N_r(\Omega^c)\cap\partial^* E) = 2 \lim_{s\to 1^-} (1-s)\, I_s(EN_r(\Omega^c), E^cN_r(\Omega^c))$$

and thus

$$\kappa_n\, \mathcal{H}^{n-1}(\partial\Omega\cap\partial^* E) = 2 \lim_{r\to 0^+}\lim_{s\to 1^-} (1-s)\, I_s(EN_r(\Omega^c), E^cN_r(\Omega^c))\,. \tag{2.9}$$

We have

$$\begin{aligned} I_s(EN_r(\Omega^c), E^cN_r(\Omega^c)) &= I_s(E, E^cN_r(\Omega^c)) - I_s(E\setminus N_r(\Omega^c), E^cN_r(\Omega^c))\\ &= I_s(E,\Omega^c) - I_s(E, E^cN_r(\Omega^c)\Omega) - I_s(E\setminus N_r(\Omega^c), E^cN_r(\Omega^c))\end{aligned} \tag{2.10}$$

where in the last step we have use the fact that $\Omega^c \subset E^c\cap N_r(\Omega^c)$. We now want to estimate the two negative terms on the right-hand side of (2.10). First, since $E\subset\Omega$,

$$(1-s)\, I_s(E, E^cN_r(\Omega^c)\Omega) \leqslant \mu_s^\Omega(N_r(\Omega^c)\cap\Omega)$$

and since for a.e. $r>0$ we have $\mathcal{H}^{n-1}(N_r(\Omega^c)\cap\partial^* E) = 0$ we find

$$\limsup_{s\to 1^-} (1-s)\, I_s(E, E^cN_r(\Omega^c)\Omega) \leqslant \kappa_n\, \mathcal{H}^{n-1}(N_r(\Omega^c)\cap\Omega\cap\partial^* E) \qquad \text{for a.e. } r>0\,,$$

where $\mathcal{H}^{n-1}(N_r(\Omega^c)\cap\Omega\cap\partial^*E)\to 0$ as $r\to 0^+$ thanks to $\Omega\cap\partial\Omega=\varnothing$ and $\mathcal{H}^{n-1}(\partial^*E)<\infty$; summarizing,

$$\lim_{r\to 0^+}\lim_{s\to 1^-}(1-s)\,I_s(E,E^cN_r(\Omega^c)\Omega)=0\,. \tag{2.11}$$

Coming now to the second term on the right-hand side of (2.10), we have

$$\begin{aligned} I_s(E\setminus N_r(\Omega^c),E^cN_r(\Omega^c)) &= I_s(E\setminus N_r(\Omega^c),E^cN_r(\Omega^c)\Omega)+I_s(E\setminus N_r(\Omega^c),E^cN_r(\Omega^c)\Omega^c)\\ &\leqslant I_s(E,E^cN_r(\Omega^c)\Omega)+I_s(E\setminus N_r(\Omega^c),\Omega^c)\end{aligned}$$

where the first term has been addressed in (2.11), while the second satisfies

$$\begin{aligned} I_s(E\setminus N_r(\Omega^c),\Omega^c) &= \int_{E\setminus N_r(\Omega^c)}dx\int_{\Omega^c}\frac{dy}{|x-y|^{n+s}}\leqslant\int_{E\setminus N_r(\Omega^c)}dx\int_{B_r(x)^c}\frac{dy}{|x-y|^{n+s}}\\ &\leqslant C(n)\,|E|\int_r^\infty\frac{dt}{t^{1+s}}=C(n)\frac{|E|}{s\,r^s}\,,\end{aligned}$$

so that

$$\lim_{s\to 1^-}(1-s)\,I_s(E\setminus N_r(\Omega^c),\Omega^c)=0\qquad\forall r>0\,. \tag{2.12}$$

By combining (2.9), (2.10), (2.11) and (2.12) we deduce (2.6). □

3. The Euler-Lagrange equation

In this section we characterize the Euler-Lagrange equation for the nonlocal capillarity energy $\mathcal{E}$, see Theorem 1.3.

Lemma 3.1 (Weak form of the Euler-Lagrange equation)**.** *Let $K\in\mathbf{K}^1(n,\sigma,\lambda)$. If Ω is a bounded open set with C^1-boundary, $g\in C^1(\mathbb{R}^n)$, and E is a critical point of $\mathcal{E}+\int g$, then there exists a constant $c\in\mathbb{R}$ such that*

$$\begin{aligned}&\iint_{E\times(E^c\cap\Omega)}\operatorname{div}_{(x,y)}\big(K(x-y)\,(T(x),T(y))\big)\,dxdy\\ &+\sigma\iint_{E\times\Omega^c}\operatorname{div}_{(x,y)}\big(K(x-y)\,(T(x),T(y))\big)\,dxdy+\int_E\operatorname{div}(g\,T)=c\int_E\operatorname{div}T\end{aligned} \tag{3.1}$$

for every $T\in C^\infty_c(\mathbb{R}^n;\mathbb{R}^n)$ with

$$T\cdot\nu_\Omega=0\qquad\text{on }\partial\Omega\,.$$

Proof. Step one: Given $T\in C^\infty_c(\mathbb{R}^n;\mathbb{R}^n)$ satisfying

$$T\cdot\nu_\Omega=0\quad\text{on }\partial\Omega\qquad\int_E\operatorname{div}T=0 \tag{3.2}$$

the flux

$$\begin{cases}\partial_t h_t(x)=T(h_t(x))\\ h_0(x)=x\end{cases}\qquad\forall|t|<\varepsilon\,,$$

generated by T satisfies $h_t(\Omega)=\Omega$ for every $|t|<\varepsilon$ and $|h_t(E)|=|E|+\mathrm{O}(t^2)$. By picking any vector field $S\in C^\infty_c(\Omega;\mathbb{R}^n)$ with support a positive distance from the support of T and such that

$$\int_E\operatorname{div}S>0$$

and by exploiting a classical argument based on the implicit function theorem (see [Mag12, Theorem 19.8, Step one]) we can find $s\in C^\infty((-\varepsilon,\varepsilon))$ with $s(0)=s'(0)=0$ such that the family of diffeomorphisms

$$f_t(x)=x+t\,T(x)+s(t)\,S(x)\qquad(x,t)\in\mathbb{R}^n\times(-\varepsilon,\varepsilon) \tag{3.3}$$

satisfies $f_t(\Omega)=\Omega$ and $|f_t(E)|=|E|$, that is (1.21). In particular, by assumption,

$$\frac{d}{dt}\Big|_{t=0}\mathcal{E}(f_t(E))+\int_{f_t(E)}g=0\,. \tag{3.4}$$

We notice that by (3.3) (see, e.g. [Mag12, Lemma 17.4])

$$\nabla f_t=\mathrm{Id}\,+t\,\nabla T+\mathrm{O}(t^2)\,,\qquad Jf_t=\det(\nabla f_t)=1+t\,\mathrm{div}\,T+\mathrm{O}(t^2)\,, \tag{3.5}$$

uniformly on $\mathbb{R}^n$ as $t\to 0$, as well as

$$|f_t(x)-f_t(y)|\leqslant C|x-y|\qquad\forall x,y\in\mathbb{R}^n\,. \tag{3.6}$$

for some $C>0$. Moreover, if F is an arbitrary Borel set and $h\in C^1(\mathbb{R}^n)$ then

$$\frac{d}{dt}\Big|_{t=0}\int_{f_t(F)}h=\int_F\mathrm{div}\,(h\,T) \tag{3.7}$$

while if F is of locally finite perimeter in an open neighborhood of $\mathrm{spt}\,T$ and $h\in C^0(\mathbb{R}^n)$, then

$$\frac{d}{dt}\Big|_{t=0}\int_{f_t(F)}h=\int_{\partial^*F}h\,(T\cdot\nu_F)\,d\mathcal{H}^{n-1}\,, \tag{3.8}$$

see for example [Mag12, Proposition 17.8].

Step two: We assume that $K\in C^2_c(\mathbb{R}^n)$ and prove that

$$\begin{aligned}\frac{d}{dt}\Big|_{t=0}I(f_t(E),f_t(E)^c\Omega)&=\iint_{E\times(E^c\cap\Omega)}\Big[\mathrm{div}\,_x\big(K(x-y)T(x)\big)+\mathrm{div}\,_y\big(K(x-y)T(y)\big)\Big]\,dx\,dy\\ \frac{d}{dt}\Big|_{t=0}I(f_t(E),\Omega^c)&=\iint_{E\times\Omega^c}\Big[\mathrm{div}\,_x\big(K(x-y)T(x)\big)+\mathrm{div}\,_y\big(K(x-y)T(y)\big)\Big]\,dx\,dy\,.\end{aligned} \tag{3.9}$$

By (3.3) and since $s'(0)=0$ we have

$$|(f_t(x)-f_t(y))-(x-y)|\leqslant C\,t\,|x-y|\qquad\forall x,y\in\mathbb{R}^n\,,|t|<\varepsilon\,,$$

so that if , then

$$|\zeta|\geqslant\frac{|x-y|}{2}, \tag{3.10}$$

whenever $x,y\in\mathbb{R}^n$, $|t|<\varepsilon$, and ζ is a point lying on the segment joining $x-y$ and $f_t(x)-f_t(y)$. From (1.12) and (3.10), $|D^2K(\zeta)|\leqslant C\,|x-y|^{-n-s-2}$, and thus

$$|D^2K(\zeta)|\,|f_t(x)-f_t(y)-(x-y)|^2\leqslant\frac{Ct^2\min\{1,|x-y|^2\}}{|x-y|^{n+s+2}} \tag{3.11}$$

for x, y, t and ζ as in (3.10). Also, since (3.3) and $s'(0)=0$ give

$$\Big|f_t(x)-f_t(y)-(x-y)-t(T(x)-T(y))\Big|\leqslant C\,t^2|x-y|\,,\qquad\forall x,y\in\mathbb{R}^n\,,|t|<\varepsilon\,,$$

by using again (1.12) we find

$$\begin{aligned}&\Big|\nabla K(x-y)\cdot\big(f_t(x)-f_t(y)-(x-y)\big)-t\nabla K(x-y)\cdot\big(T(x)-T(y)\big)\Big|\\ \leqslant\;&\frac{C}{|x-y|^{n+s+1}}\,\Big|f_t(x)-f_t(y)-(x-y)-t(T(x)-T(y))\Big|\\ \leqslant\;&\frac{Ct^2\min\{1,|x-y|\}}{|x-y|^{n+s+1}}\end{aligned}$$

for every $x,y\in\mathbb{R}^n$, $|t|<\varepsilon$. From this and (3.11),

$$K\big(f_t(x)-f_t(y)\big)=K(x-y)+t\nabla K(x-y)\cdot\big(T(x)-T(y)\big)+t^2\Upsilon(x,y), \tag{3.12}$$

where here and in the rest of this proof, Υ denotes a generic function (which may change from line to line) such that

$$|\Upsilon(x,y)| \leqslant \frac{C\min\{1,|x-y|\}}{|x-y|^{n+s+1}}. \tag{3.13}$$

By combining (3.5) and (3.12) we find

$$\begin{aligned} &K\big(f_t(x)-f_t(y)\big)\, Jf_t(x)\, Jf_t(y)\\ &= \Big[K(x-y)+t\nabla K(x-y)\cdot\big(T(x)-T(y)\big)+t^2\Upsilon(x,y)\Big]\\ &\quad\cdot\Big[1+t\mathrm{div}\,T(x)+O(t^2)\Big]\,\Big[1+t\mathrm{div}\,T(y)+O(t^2)\Big]\\ &= K(x-y)+t\nabla K(x-y)\cdot\big(T(x)-T(y)\big)+tK(x,y)\big(\mathrm{div}\,T(x)+\mathrm{div}\,T(y)\big)+t^2\Upsilon(x,y)\,. \end{aligned} \tag{3.14}$$

Now we observe that

$$\mathrm{div}_x\big(K(x-y)T(x)\big)=\nabla K(x-y)\cdot T(x)+K(x-y)\mathrm{div}\,T(x).$$

Then, since K is even,

$$\mathrm{div}_y\big(K(x-y)T(y)\big)=\nabla K(x-y)\cdot T(y)+K(x-y)\mathrm{div}\,T(y)$$

and therefore

$$\begin{aligned} &\mathrm{div}_x\big(K(x-y)T(x)\big)+\mathrm{div}_y\big(K(x-y)T(y)\big)\\ &\qquad=\nabla K(x-y)\cdot\big(T(x)+T(y)\big)+K(x-y)\big(\mathrm{div}\,T(x)+\mathrm{div}\,T(x)\big). \end{aligned}$$

Comparing this with (3.14), we conclude that

$$\begin{aligned} &K\big(f_t(x)-f_t(y)\big)\, Jf_t(x)\, Jf_t(y)\\ &= K(x-y)+t\Big[\mathrm{div}_x\big(K(x-y)T(x)\big)+\mathrm{div}_y\big(K(x-y)T(y)\big)\Big]+t^2\Upsilon(x,y). \end{aligned}$$

Consequently, by the area formula,

$$\begin{aligned} &I(f_t(E),f_t(E)^c\cap\Omega)=I(E,E^c\cap\Omega)\\ &\qquad+t\iint_{E\times(E^c\cap\Omega)}\Big[\mathrm{div}_x\big(K(x-y)T(x)\big)+\mathrm{div}_y\big(K(x-y)T(y)\big)\Big]\,dx\,dy\\ &\qquad+t^2\iint_{E\times(E^c\cap\Omega)}\Upsilon(x,y)\,dx\,dy\\ &I(f_t(E),\Omega^c)=I(E,\Omega^c)\\ &\qquad+t\iint_{E\times\Omega^c}\Big[\mathrm{div}_x\big(K(x-y)T(x)\big)+\mathrm{div}_y\big(K(x-y)T(y)\big)\Big]\,dx\,dy\\ &\qquad+t^2\iint_{E\times\Omega^c}\Upsilon(x,y)\,dx\,dy. \end{aligned} \tag{3.15}$$

By (1.10), (3.13) and (2.3) it follows that

$$+\infty>I(E,\Omega^c)\geqslant\iint_{\substack{E\times\Omega^c\\|x-y|\leqslant\varepsilon}}\frac{dx\,dy}{\lambda\,|x-y|^{n+s}}\geqslant\frac{1}{C\,\lambda}\iint_{\substack{E\times\Omega^c\\|x-y|\leqslant\varepsilon}}\Upsilon(x,y)\,dx\,dy$$

and thus

$$\iint_{E\times\Omega^c}\Upsilon(x,y)\,dx\,dy<+\infty.$$

Similarly (using (2.4) in lieu of (2.3)), we obtain that

$$\iint_{E\times(E^c\cap\Omega)}\Upsilon(x,y)\,dx\,dy<+\infty.$$

Accordingly, we find from (3.15) that

$$
\begin{aligned}
I(f_t(E), f_t(E)^c \cap \Omega) &= I(E, E^c \cap \Omega)\\
&\quad + t \iint_{E\times(E^c\cap\Omega)} \Big[\mathrm{div}_x\big(K(x-y)T(x)\big) + \mathrm{div}_y\big(K(x-y)T(y)\big)\Big]\, dx\, dy + O(t^2)\\
I(f_t(E), \Omega^c) &= I(E, \Omega^c)\\
&\quad + t \iint_{E\times\Omega^c} \Big[\mathrm{div}_x\big(K(x-y)T(x)\big) + \mathrm{div}_y\big(K(x-y)T(y)\big)\Big]\, dx\, dy + O(t^2)\,.
\end{aligned}
$$

This completes the proof of (3.9), thus of step two.

Step three: We now claim that (3.9) holds with $K \in \mathbf{K}_*(n,\sigma,\lambda,\varepsilon)$ in place of a generic $K \in C^2_c(\mathbb{R}^n)$. For each $\delta \in (0,1/2)$, let $\eta_\delta \in C^\infty([0,+\infty))$ be such that $\eta_\delta = 1$ in $[0,\delta]\cup[1/\delta,+\infty)$, $\eta_\delta = 0$ in $[2\delta, 1/(2\delta)]$, $|\eta'_\delta| \leqslant 4/\delta$, and $\eta_\delta \to 0$ monotonically as $\delta \to 0$, and set

$$
K_\delta = (1-\eta_\delta)\, K\,. \tag{3.16}
$$

If we let

$$
\phi_\delta(t) := \mathcal{E}_\delta(f_t(E)) \qquad \phi(t) := \mathcal{E}(f_t(E))
$$

then by monotone convergence, $\phi_\delta(t) \to \phi(t)$ as $\delta \to 0^+$ for every $|t| < \varepsilon$, where ϕ_δ and ϕ are smooth functions by the area formula (and since $I(E,E^c\Omega), I(E,\Omega^c) < \infty$). On noticing that

$$
\frac{\partial f_t}{\partial t}(x) = T(x) + s'(t)\, S(x) =: T_t(x)
$$

by (3.9) we have

$$
\phi'_\delta(t) = \bigg(\iint_{E\times(E^c\cap\Omega)} + \sigma \iint_{E\times\Omega^c}\bigg)\Big[\mathrm{div}_x\big(K_\delta(x-y)T_t(x)\big) + \mathrm{div}_y\big(K_\delta(x-y)T_t(y)\big)\Big]\, dxdy\,. \tag{3.17}
$$

We now claim that

$$
\phi'_\delta(t) \to \bigg(\iint_{E\times(E^c\cap\Omega)} + \sigma \iint_{E\times\Omega^c}\bigg)\Big[\mathrm{div}_x\big(K(x-y)T_t(x)\big) + \mathrm{div}_y\big(K(x-y)T_t(y)\big)\Big]\, dxdy \tag{3.18}
$$

uniformly on $|t| < \varepsilon$ as $\delta \to 0^+$. By applying the mean value theorem to ϕ_δ and since $\phi_\delta \to \phi$ as $\delta \to 0^+$ pointwise, this will imply that

$$
\phi'(0) = \bigg(\iint_{E\times(E^c\cap\Omega)} + \sigma \iint_{E\times\Omega^c}\bigg)\Big[\mathrm{div}_x\big(K(x-y)T(x)\big) + \mathrm{div}_y\big(K(x-y)T(y)\big)\Big]\, dxdy
$$

as required. To prove (3.18) we just notice that

$$
\begin{aligned}
&\mathrm{div}_x\big(K_\delta(x-y)T_t(x)\big) + \mathrm{div}_y\big(K_\delta(x-y)T_t(y)\big)\\
&= K_\delta(x-y)\,\big(\mathrm{div}\, T_t(x) + \mathrm{div}\, T_t(y)\big) + \nabla K_\delta(x-y)\cdot(T_t(x) - T_t(y))
\end{aligned}
$$

where $|T_t(x) - T_t(y)| \leqslant C\,|x-y|$ for every $x,y \in \mathbb{R}^n$ and $|t| < \varepsilon$, so that (1.12) gives

$$
\Big|\mathrm{div}_x\big(K_\delta(x-y)T_t(x)\big) + \mathrm{div}_y\big(K_\delta(x-y)T_t(y)\big)\Big| \leqslant \frac{C}{|x-y|^{n+s}} \leqslant C\,K(x-y)\,,
$$

and, in conclusion, (3.18) holds by dominated convergence and thanks to $I(E,E^c\Omega), I(E,\Omega^c) < \infty$ (recall (2.3) and (2.4)).

Step four: Let us consider the linear functional on $T \in C^\infty_c(\mathbb{R}^n;\mathbb{R}^n)$ defined by

$$
\Lambda(T) = \bigg(\iint_{E\times(E^c\cap\Omega)} + \sigma \iint_{E\times\Omega^c}\bigg)\mathrm{div}_{(x,y)}\Big(K(x-y)\,\big(T(x),T(y)\big)\Big)\, dxdy + \int_E \mathrm{div}\,(g\,T)\,.
$$

By combining (3.4), (3.7) and step three we find that $\Lambda(T) = 0$ whenever T satisfies (3.2). If $T_1, T_2 \in C^\infty(\mathbb{R}^n;\mathbb{R}^n)$ have disjoint supports and are such that

$$T_1 \cdot \nu_\Omega = T_2 \cdot \nu_\Omega = 0 \quad \text{on } \partial\Omega \qquad \int_E \operatorname{div} T_2 \neq 0\,,$$

then

$$T = T_1 - \frac{\int_E \operatorname{div}\, T_1}{\int_E \operatorname{div}\, T_2}\, T_2$$

is admissible in (3.2), and thus satisfy $\Lambda(T) = 0$. Thus $\Lambda(T_1)/\int_E \operatorname{div} T_1 = \Lambda(T_2)/\int_E \operatorname{div} T_2$, and the proof is completed by the arbitrariness of T_1 and T_2. □

In passing from Lemma 3.1 to Theorem 1.3 we shall need the following proposition.

Proposition 3.2. *If $\sigma \in (-1,1)$ and $K \in \mathbf{K}^1(n,s,\lambda)$, then for every $E \subset \Omega$ the function*

$$\mathbf{H}^{K,\sigma,\Omega}_{\partial E}(x) := \text{p.v.} \int_{\mathbb{R}^n} K(x-y)\,\big(1_{E^c\cap\Omega}(y) + \sigma\, 1_{\Omega^c}(y) - 1_E(y)\big)\, dy \qquad x \in \partial E$$

is continuous on $\Omega \cap \mathrm{Reg}_E$ with

$$\mathbf{H}^{K_\delta,\sigma,\Omega}_{\partial E} \to \mathbf{H}^{K,\sigma,\Omega}_{\partial E} \qquad \text{as } \delta \to 0^+ \tag{3.19}$$

uniformly on compact subsets of $\Omega \cap \mathrm{Reg}_E$. Here, K_δ is defined as in (3.16).

Proof. Since $K_\delta \in C^1_c(\mathbb{R}^n)$ we definitely have

$$\mathbf{H}^{K_\delta,\sigma,\Omega}_{\partial E}(x) = \mathbf{H}^{K_\delta}_{\partial E}(x) - (1-\sigma)\int_{\Omega^c} K_\delta(x-y)\, dy \qquad \forall x \in \partial E\,, \tag{3.20}$$

see (1.19) for the definition of $\mathbf{H}^{K_\delta}_{\partial E}$. It is shown in [FFM+15, Proposition 6.3] that the continuous functions $\{\mathbf{H}^{K_\delta}_{\partial E}\}_\delta$ converge uniformly on compact subsets of $\Omega \cap \mathrm{Reg}_E$ to $\mathbf{H}^K_{\partial E}$. An identical argument leads to obtain (3.19), proves the continuity of $\mathbf{H}^{K,\sigma,\Omega}_{\partial E}$ on $\mathrm{Reg}_E \cap \Omega$. □

Proof of Theorem 1.3. Let $K_\delta \in C^2_c(\mathbb{R}^n)$ be defined as in (3.16). As soon as E has finite perimeter, one has (by [Mag12, Formula (15.11)])

$$\int_E \operatorname{div}_x\big(K_\delta(x-y)T(x)\big)\, dx = \int_{\partial^* E} K_\delta(x-y)T(x)\cdot \nu_E\, d\mathcal{H}^{n-1}_x$$

where $\partial^* E$ denotes the reduced boundary of ∂E and ν_E its measure-theoretic outer unit normal. In particular, for any set F that does not intersect E we find

$$\int_{E\times F} \operatorname{div}_x\big(K_\delta(x-y)T(x)\big)\, dx\, dy = \int_F \left(\int_{\partial^* E} K_\delta(x-y)T(x)\cdot \nu_E\, d\mathcal{H}^{n-1}_x\right) dy. \tag{3.21}$$

Similarly, for any set F that does not intersect E,

$$\int_F \operatorname{div}_y\big(K_\delta(x-y)T(y)\big)\, dy = \int_{\partial^* F} K_\delta(x-y)T(y)\cdot \nu_F\, d\mathcal{H}^{n-1}_y$$

and therefore, integrating in E and changing the names of the variables,

$$\begin{aligned}\iint_{E\times F} \operatorname{div}_y\big(K_\delta(x-y)T(y)\big)\, dx\, dy &= \int_E \left(\int_{\partial^* F} K_\delta(x-y)T(y)\cdot \nu_F\, d\mathcal{H}^{n-1}_y\right) dx \\ &= \int_E \left(\int_{\partial^* F} K_\delta(x-y)T(x)\cdot \nu_F\, d\mathcal{H}^{n-1}_x\right) dy.\end{aligned}$$

Using this formula and (3.21) with $F=E^c\cap\Omega$, we obtain that

$$\begin{aligned}&\iint_{E\times(E^c\cap\Omega)}\operatorname{div}_{(x,y)}\big(K_\delta(x-y)(T(x),T(y))\big)\,dx\,dy\\&=\int_{E^c\cap\Omega}\left(\int_{\partial^*E}K_\delta(x-y)T(x)\cdot\nu_E\,d\mathcal{H}^{n-1}_x\right)dy\\&\qquad+\int_E\left(\int_{\partial^*(E^c\cap\Omega)}K_\delta(x-y)T(x)\cdot\nu_{E^c\cap\Omega}\,d\mathcal{H}^{n-1}_x\right)dy\\&=\int_{\Omega\cap\partial^*E}T(x)\cdot\nu_E\left(\int_{\mathbb{R}^n}K_\delta(x-y)\left(1_{E^c\cap\Omega}(y)-1_E(y)\right)dy\right)d\mathcal{H}^{n-1}_x\,,\end{aligned}\tag{3.22}$$

and analogously

$$\begin{aligned}&\iint_{E\times\Omega^c}\operatorname{div}_{(x,y)}\big(K_\delta(x-y)(T(x),T(y))\big)\,dx\,dy\\&=\int_{\Omega\cap\partial^*E}T(x)\cdot\nu_E\left(\int_{\mathbb{R}^n}K_\delta(x-y)\,1_{\Omega^c}(y)\,dy\right)d\mathcal{H}^{n-1}_x.\end{aligned}\tag{3.23}$$

In particular,

$$\begin{aligned}&\left(\iint_{E\times(E^c\cap\Omega)}+\sigma\iint_{E\times\Omega^c}\right)\operatorname{div}_{(x,y)}\big(K_\delta(x-y)(T(x),T(y))\big)\,dx\,dy\\&=\int_{\Omega\cap\partial^*E}(T\cdot\nu_E)\,\mathbf{H}^{K_\delta,\sigma,\Omega}_{\partial E}\,d\mathcal{H}^{n-1}\,.\end{aligned}\tag{3.24}$$

Let us now fix $x\in\Omega\cap\mathrm{Reg}_E$ and $T\in C^1_c(B_\varrho(x)\cap\Omega)$ with $\varrho>0$ such that $B_{2\,\varrho}(x)\cap\partial E\subset\Omega\cap\mathrm{Reg}_E$. In this way, $\mathbf{H}^{K_\delta,\sigma,\Omega}_{\partial E}$ converges uniformly to $\mathbf{H}^{K,\sigma,\Omega}_{\partial E}$ on $\operatorname{spt}T$ and thus the right-hand side of (3.24) converges to $\int_{B_\varrho(x)\cap\partial E}(T\cdot\nu_E)\,\mathbf{H}^{K,\sigma,\Omega}_{\partial E}$. Since we have already shown in the proof of Lemma 3.1 that in the limit $\delta\to0^+$ we can take replace K_δ by K on the left-hand side of (3.24), we conclude that

$$\begin{aligned}&\left(\iint_{E\times(E^c\cap\Omega)}+\sigma\iint_{E\times\Omega^c}\right)\operatorname{div}_{(x,y)}\big(K(x-y)(T(x),T(y))\big)\,dx\,dy\\&=\int_{B_\varrho(x)\cap\partial E}(T\cdot\nu_E)\,\mathbf{H}^{K,\sigma,\Omega}_{\partial E}\,d\mathcal{H}^{n-1}\,.\end{aligned}$$

for every $x\in\Omega\cap\mathrm{Reg}_E$ and $T\in C^1_c(B_\varrho(x)\cap\Omega)$, for $\varrho>0$ depending on x. By combining this identity with (3.1), $\int_E\operatorname{div}(T\,g)=\int_{B_\varrho(x)\cap\partial E}g\,(T\cdot\nu_E)$, and the arbitrariness of T, we finally deduce (1.23). □

4. Nonlocal Young's law

This section addresses the proof of Theorem 1.4. We premise a simple technical lemma. Here, we decompose $x\in\mathbb{R}^n$ as $x=(x',x_n)\in\mathbb{R}^{n-1}\times\mathbb{R}$ and set

$$\mathbf{C}=\{x\in\mathbb{R}^n:|x'|<1\,,|x_n|<1\}\qquad\text{and}\qquad\mathbf{D}=\{z\in\mathbb{R}^{n-1}:|z|<1\}\,.$$

Lemma 4.1. *Let $\lambda\geqslant1$, $s\in(0,1)$ and $\alpha\in(s,1)$. If $\{F_k\}_{k\in\mathbb{N}}$ is a sequence of Borel sets in $\mathbb{R}^n$ with $0\in\partial F_k$,*

$$F_k\to F\ \textit{in}\ L^1_{\mathrm{loc}}(\mathbb{R}^n)\ \textit{for some}\ F\subset\mathbb{R}^n\,,$$

and, for some functions $u_k,u\in C^{1,\alpha}(\mathbb{R}^{n-1})$,

$$\mathbf{C}\cap F_k=\Big\{x\in\mathbf{C}:x_n\leqslant u_k(x')\Big\}\qquad\textit{and}\qquad\lim_{k\to\infty}\|u_k-u\|_{C^{1,\alpha}(\mathbf{D})}=0\,,$$

then

$$\lim_{k\to\infty}\mathbf{H}^{K_k}_{\partial F_k}(0)=\mathbf{H}^{K}_{\partial F}(0)$$

whenever $\{K_k\}_{k\in\mathbb{N}}$ *and* K *are kernels in* $\mathbf{K}(n,s,\lambda,0)$ *with* $K_k\to K$ *pointwise in* $\mathbb{R}^n\setminus\{0\}$*.*

Proof. Up to rigid motions we may assume without loss of generality that $0\in\partial F$ (so that $u(0)=u_k(0)=0$) and that $\nabla u_k(0)=\nabla u(0)=0$. Since $u\in C^{1,\alpha}(\mathbf{D})$ and $u_k\to u$ in $C^{1,\alpha}(\mathbf{D})$ we can find $\gamma>0$ such that

$$\max\{|u_k(z)|,|u(z)|\}\leqslant\gamma|z|^{1+\alpha}\qquad\forall z\in\mathbf{D}\,,k\in\mathbb{N}\,.$$

If we let

$$P_{\varepsilon,\gamma}=\big\{x\in B_\varepsilon:|x_n|<\gamma\,|x'|^{1+\alpha}\big\}\qquad\varepsilon\in(0,1)\,,$$

then $|z|^{-n-s}\in L^1(P_{\varepsilon,\gamma}\cup(B_\varepsilon)^c)$ and thus

$$\mathbf{H}^{K_k}_{\partial F_k}(0)=\int_{(B_\varepsilon)^c\cup P_{\varepsilon,\gamma}}(1^c_{F_k}-1_{F_k})\,K_k\qquad\mathbf{H}^{K}_{\partial F}(0)=\int_{(B_\varepsilon)^c\cup P_{\varepsilon,\gamma}}(1^c_{F}-1_{F})\,K\,.$$

Since $(1^c_{F_k}-1_{F_k})\to(1^c_F-1_F)$ a.e. on $\mathbb{R}^n$ we conclude by dominated convergence that $\mathbf{H}^{K_k}_{\partial F_k}(0)\to\mathbf{H}^{K}_{\partial F}(0)$. □

Proof of Theorem 1.4. Step one: We start proving the validity of (1.24). Let us fix $x_0\in\partial\Omega\cap\mathrm{Reg}_E$ so that x_0 is a boundary point of the manifold with boundary $B_\varrho(x_0)\cap\partial E$. Consider a sequence $\{x_k\}_{k\in\mathbb{N}}\subset\Omega\cap\mathrm{Reg}_E$ such that $x_k\to x_0$, and set

$$r_k=|x_k-x_0|\qquad v_k=\frac{x_k-x_0}{r_k}\qquad E^{x_0,r_k}=\frac{E-x_0}{r_k}\qquad\Omega^{x_0,r_k}=\frac{\Omega-x_0}{r_k}\,.$$

We recall that, by (1.23),

$$\mathbf{H}^K_{\partial E}(x_k)-(1-\sigma)\int_{\Omega^c}K(x_k-y)\,dy+g(x_k)=c\tag{4.1}$$

for a constant c independent of k. We have that

$$\Omega^{x_0,r_k}\to H\text{ and }E^{x_0,r_k}\to V\cap H\text{ in }L^1_{\mathrm{loc}}(\mathbb{R}^n)$$

where H and V are suitable half-spaces in $\mathbb{R}^n$ so that

$$\nu_\Omega(x_0)=\nu_H(0)\qquad\nu_V(0)=\lim_{k\to\infty}\nu_E(x_k)=:\nu_V(0)\,.$$

Up to extracting subsequences, we have that $v_k\to v$ for some $v\in S^{n-1}$. We can use the change of variables $y=x_0+r_k\,z$ to find

$$\mathbf{H}^K_{\partial E}(x_k)\;=\;\int_{\mathbb{R}^n}K(x_k-y)\big(1_{E^c}(y)-1_E(y)\big)\,dy\tag{4.2}$$

$$=\;r_k^{-s}\int_{\mathbb{R}^n}r_k^{n+s}\,K(x_k-x_0-r_k\,z)\big(1_{(E^{x_0,r_k})^c}(z)-1_{E^{x_0,r_k}}(z)\big)\,dz\tag{4.3}$$

Now, since $\{x_k\}_{k\in\mathbb{N}}\subset\Omega\cap\mathrm{Reg}_E$, we can find rigid motions $Q_k:\mathbb{R}^n\to\mathbb{R}^n$ and functions $u_k\in C^{1,\alpha}(\mathbb{R}^{n-1})$ such that if we set

$$F_k=Q_k(E^{x_0,r_k}-v_k)$$

then $0\in\partial F_k$ and

$$\mathbf{C}\cap F_k=\Big\{x\in\mathbf{C}:x_n\leqslant u_k(x')\Big\}\,.$$

Notice that

$$F_k\to F=H\cap V\text{ in }L^1_{\mathrm{loc}}(\mathbb{R}^n)$$

with $u_k\to u$ in $C^{1,\alpha}(\mathbf{D})$ for a linear function $u:\mathbb{R}^{n-1}\to\mathbb{R}$. If we set

$$K_k(\zeta)=r_k^{n-s}\,K(r_k\,\zeta)\qquad\zeta\in\mathbb{R}^n\setminus\{0\}\,,$$

then by (4.2) we get

$$\mathbf{H}^K_{\partial E}(x_k) = r_k^{-s}\,\mathbf{H}^{K_k}_{\partial F_k}(0)\,.$$

Since $K_k \to K^*$ pointwise in $\mathbb{R}^n \setminus \{0\}$, by Lemma 4.1 we find

$$\lim_{k\to\infty} r_k^s\,\mathbf{H}^K_{\partial E}(x_k) = \mathbf{H}^{K^*}_{\partial(H\cap V)}(v)\,,$$

and since $r_k^s\, g(x_k) \to 0$ (indeed $x_k \to x_0$ and g is locally bounded), (4.1) implies

$$\mathbf{H}^{K^*}_{\partial(H\cap V)}(v) - (1-\sigma)\lim_{k\to\infty} r_k^s \int_{\Omega^c} K(x_k - y)\,dy = 0\,.$$

By the change of variable $y = x_0 + r_k\, z$,

$$\int_{\Omega^c} K(x_k - y)\,dy = r_k^{-s}\int_{(\Omega^{x_0,r_k})^c} r_k^{n+s}\,K\big(r_k(v_k - z)\big)\,dz$$

where

$$\lim_{k\to\infty}\int_{(\Omega^{x_0,r_k})^c} r_k^{n+s}\,K\big(r_k(v_k - z)\big)\,dz = \int_{H^c} K^*(v-z)\,dz\,.$$

We have thus proved that

$$\mathbf{H}^{K^*}_{\partial(H\cap V)}(v) - (1-\sigma)\int_{H^c} K^*(v-z)\,dz = 0\,, \qquad \forall v \in H\cap\partial V\,, \tag{4.4}$$

that is (1.24).

Step two: We now assume that $K = K_s^\varepsilon$ for some $\varepsilon > 0$, so that $K^* = K_s$. Up to a rigid motion we can assume that H and V satisfy

$$\begin{aligned} H &= \{x\in\mathbb{R}^n : x_n > 0\} \\ H\cap V &= \Big\{x\in\mathbb{R}^n : x_n>0 \text{ and } \cos\alpha\, x_n = \sin\alpha\, x_1 \text{ for some } \alpha\in(0,\theta)\Big\} =: J_\theta\,, \end{aligned}$$

for some $\theta\in(0,\pi)$. Since (4.4) is $-s$ homogeneous in $|v|$, we find that (4.4) is equivalent to

$$\int_{\mathbb{R}^n} \frac{(1_{J_\theta^c\cap H} + \sigma\,1_{H^c} - 1_{J_\theta})(z)}{|e(\theta)-z|^{n+s}}\,dz = 0 \tag{4.5}$$

where

$$e(\theta) = \cos\theta\, e_1 + \sin\theta\, e_n\,.$$

In this step we show that there exists a unique $\theta = \theta(n,s,\sigma)\in(0,\pi)$ such that (4.5) holds – so that, correspondingly,

$$\nu_E(x_0)\cdot\nu_\Omega(x_0) = \nu_V(0)\cdot\nu_H(0) = \cos\big(\pi - \theta(n,s,\sigma)\big)$$

and (1.25) holds – and that the function $\sigma\in(-1,1)\mapsto\theta(n,s,\sigma)$ is strictly increasing with

$$\theta(n,s,0) = \frac{\pi}{2}\,, \qquad \lim_{\sigma\to(-1)^+}\theta(n,s,\sigma) = 0\,, \qquad \lim_{\sigma\to1^-}\theta(n,s,\sigma) = \pi\,. \tag{4.6}$$

We first notice that we do not need to specify the integral in (4.5) in the principal value sense as there always is a ball centered at $e(\theta)$ with one half of it contained in J_θ, the other half contained in $J_\theta^c\cap H$. It is also geometrically evident (see Figure 4.1) that the choice $\sigma = 0$, $\theta = \pi/2$ solves (4.5) and that if a pair (σ,θ) satisfies (4.5) then (i) $\theta\in(0,\pi/2)$ if and only if $\sigma\in(-1,0)$; (ii) $\theta\in(\pi/2,\pi)$ if and only if $\sigma\in(0,1)$; (iii) if $\theta\in[\pi/2,\pi)$, then $(-\sigma,\pi-\theta)$ also solves (4.5).

We are thus left to show that $\sigma\in(-1,0)$ there exists a unique $\theta\in(0,\pi/2)$ (also depending on n and s) such that (4.5) holds, and that the correspondence $\sigma\in(-1,0)\mapsto\theta(n,s,\sigma)$ is strictly increasing and satisfies $\theta(n,s,(-1)^+) = 0$. To prove this, let us notice that having restricted $\sigma\in(-1,0)$, we can directly consider (4.5) with $\theta\in(0,\pi/2)$. Since in this case the reflection

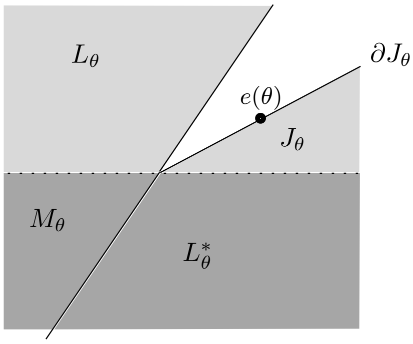

FIGURE 4.1. The cones J_θ, L_θ and M_θ when $\theta \in (0,\pi/2)$.

of J_θ with respect to the hyperplane containing $H\cap\partial J_\theta$ is entirely contained in $J_\theta^c \cap H$, (4.5) turns out to the be equivalent to

$$\int_{\mathbb{R}^n} \frac{(1_{L_\theta}+\sigma\,1_{H^c})(z)}{|e(\theta)-z|^{n+s}}\,dz = 0 \tag{4.7}$$

where L_θ is equal to H minus the union of J_θ with its reflection with respect to the hyperplane containing $H\cap\partial J_\theta$. With Figure 4.1 in mind, now let L_θ^* be the reflection of L_θ with respect to the hyperplane containing $H\cap\partial J_\theta$, so that L_θ^* is contained in H^c, and let $M_\theta = H^c\cap(L_\theta^*)^c$. As $H^c = L_\theta^*\cup M_\theta$ with $L_\theta^*\cap M_\theta=\varnothing$ and since L_θ^* is mapped into L_θ by an isometry keeping the distance from $e(\theta)$ invariant, we get

$$\int_{\mathbb{R}^n} \frac{(1_{L_\theta}+\sigma\,1_{H^c})(z)}{|e(\theta)-z|^{n+s}}\,dz = (1+\sigma)\int_{L_\theta}\frac{dz}{|e(\theta)-z|^{n+s}} + \sigma\int_{M_\theta}\frac{dz}{|e(\theta)-z|^{n+s}} \tag{4.8}$$

and thus, by (4.7), we conclude that (4.5) holds for some $\theta\in(0,\pi/2)$ if and only if

$$\int_{M_\theta}\frac{dz}{|e(\theta)-z|^{n+s}} = -\Big(1+\frac{1}{\sigma}\Big)\int_{L_\theta}\frac{dz}{|e(\theta)-z|^{n+s}}\,. \tag{4.9}$$

Let us set

$$a(\theta)=\int_{M_\theta}\frac{dz}{|e(\theta)-z|^{n+s}} \qquad b(\theta)=\int_{L_\theta}\frac{dz}{|e(\theta)-z|^{n+s}}\,.$$

Clearly $a(\theta)$ is strictly increasing on $(0,\pi/2)$, with $a(0)=0$ and $a(\pi/2)<\infty$: indeed

$$a(\theta)=\int_{U_\theta}\frac{dz}{|z-e_n|^{n+s}} \qquad U_\theta=\Big\{x\in\mathbb{R}^n : x_n<0\,,|x_1|<|x_n|\,\tan\theta\Big\}\,,$$

where the latter function is trivially increasing as $|U_{\theta_2}\setminus U_{\theta_1}|>0$ whenever $0<\theta_1<\theta_2<\pi/2$. At the same time $b(\theta)$ is strictly decreasing with $b(0^+)=+\infty$ and $b((\pi/2)^-)=0^+$. This is seen as while θ increases from 0 to $\pi/2$, the region L_θ is strictly decreasing from H to the empty set, while the distance between the singularity $e(\theta)$ and L_θ is strictly increasing. In conclusion

$$\theta\in\Big(0,\frac{\pi}{2}\Big)\mapsto\frac{a(\theta)}{b(\theta)}$$

is a strictly increasing function on $(0,\pi/2)$ with limit 0 as $\theta\to0^+$ and limit $+\infty$ as $\theta\to(\pi/2)^-$. Moreover,

$$\sigma\in(-1,0)\mapsto-\Big(1+\frac{1}{\sigma}\Big)$$

is a strictly increasing function on $(-1,0)$ with limit 0 as $\sigma\to(-1)^+$ and limit $+\infty$ as $\sigma\to0^-$. In conclusion, for every $\sigma\in(-1,0)$ there exists a unique $\theta=\theta(n,s,\sigma)\in(0,\pi/2)$ such that (4.7) holds. The resulting map $\sigma\in(-1,0)\mapsto\theta(n,s,\sigma)$ is strictly increasing and satisfies the first two properties in (4.6). This completes the proof of step two.

Step three: We conclude the proof of the theorem by showing that $\theta(n,s,\sigma)=\theta(s,\sigma)$ with

$$\lim_{s\to 1^-}\cos(\pi-\theta(s,\sigma))=\sigma\,,\qquad \forall\sigma\in(-1,1)\,.$$

To this end, let us first go back to (4.7), and notice that

$$(1_{L_\theta}+\sigma\,1_{H^c})(z)=f(z_1,z_n)$$

so that if $n\geqslant 3$, then (4.7) takes the form

$$\int_{\mathbb{R}}dz_1\int_{\mathbb{R}}f(z_1,z_n)\,dz_n\int_{\mathbb{R}^{n-2}}\frac{dw}{(\ell^2+|w|^2)^{(n+s)/2}}=0 \tag{4.10}$$

where we have set

$$\ell(z_1,z_n)=\sqrt{(z_1-\cos\theta)^2+(z_n-\sin\theta)^2}\,.$$

Now, in polar coordinates,

$$\int_{\mathbb{R}^{n-2}}\frac{dw}{(\ell^2+|w|^2)^{(n+s)/2}}=(n-2)\,\omega_{n-2}\int_0^\infty\frac{r^{n-3}\,dr}{(\ell^2+r^2)^{(n+s)/2}}$$

where, by scaling,

$$\int_0^\infty\frac{r^{n-3}\,dr}{(\ell^2+r^2)^{(n+s)/2}}=\frac{C(n,s)}{\ell^{2+s}}\,.$$

By taking (4.10) into account, the definition (4.7) of θ boils down to

$$\int_{\mathbb{R}}dz_1\int_{\mathbb{R}}\frac{f(z_1,z_n)}{\ell^{2+s}}\,dz_n=0\,,$$

which is actually equivalent to (4.7) in the case $n=2$. This proves that $\theta(n,s,\sigma)=\theta(2,s,\sigma)$ for every $n\geqslant 3$. We thus plainly set $\theta=\theta(s,\sigma)$ and then turn to the proof of $\cos(\pi-\theta(s,\sigma))\to\sigma$ as $s\to 1^-$.

By exploiting the symmetries of $\theta(s,\sigma)$ in σ, it suffices to consider the case when $\sigma\in(-1,0)$ (and thus $\theta\in(0,\pi/2)$). It is then convenient to rewrite (4.8) by using $\int_{L_\theta}=\int_{L_\theta\cup M_\theta}-\int_{M_\theta}$, to find that

$$1+\sigma=\frac{\int_{M_\theta}|z-e(\theta)|^{-(2+s)}dz}{\int_{L_\theta\cup M_\theta}|z-e(\theta)|^{-(2+s)}dz}\,.$$

Notice that $L_\theta\cup M_\theta$ is an half-plane lying at distance $\sin\theta$ from $e(\theta)$. Hence,

$$\int_{L_\theta\cup M_\theta}\frac{dz}{|z-e(\theta)|^{2+s}}=\int_{\{y_2<0\}}\frac{dy}{|y-\sin\theta e_2|^{2+s}}=\frac{1}{(\sin\theta)^s}\int_{\{x_2<0\}}\frac{dx}{|x-e_2|^{2+s}}\,.$$

At the same time, by a counter-clockwise rotation around the origin of angle $(\pi/2)-\theta$, which thus maps $e(\theta)=\cos\theta e_1+\sin\theta\,e_2$ into e_2, we find

$$\int_{M_\theta}\frac{dz}{|z-e(\theta)|^{2+s}}=\int_{\Gamma_\theta}\frac{dx}{|x-e_2|^{2+s}}$$

where we have set

$$\Gamma_\theta=\Big\{w\in\mathbb{R}^2: x_2<0\,,-\theta<\arctan\Big(-\frac{x_1}{x_2}\Big)<\theta\Big\}\,,$$

see Figure 4.2. Putting everything together we find that $\theta=\theta(s,\sigma)$ satisfies

$$\frac{(1+\sigma)}{(\sin\theta)^s}\int_{\Gamma_{\pi/2}}\frac{dx}{|x-e_2|^{2+s}}=\int_{\Gamma_\theta}\frac{dx}{|x-e_2|^{2+s}}\,, \tag{4.11}$$

(indeed $\Gamma_{\pi/2}=\{x_2<0\}$). We now consider the function $u:\{x_2<0\}\to(-\pi/2,\pi/2)$ defined by

$$u(x)=\arctan\Big(-\frac{x_1}{x_2}\Big)$$

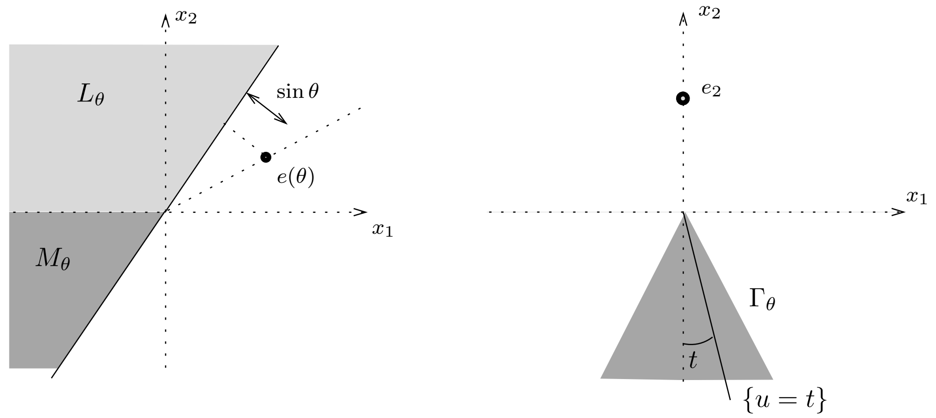

FIGURE 4.2. Notation used in computing the limit of $\theta(s,\sigma)$ as $s\to 1^-$.

and notice that $u(x)$ is a locally Lipschitz on $\{x_2<0\}$ with $\Gamma_\theta=\{-\theta<u<\theta\}$ and

$$|\nabla u|=\frac{1}{|x|}\,.$$

By the Coarea formula for every Borel function $g:\{x_2<0\}\to[0,\infty]$ we have

$$\int_{\{x_2<0\}} g(x)\,|\nabla u(x)|\,dx=\int_{-\pi/2}^{\pi/2}dt\int_{\{u=t\}}g(x)\,d\mathcal{H}^{n-1}(x)$$

so that, by choosing

$$g(x)=\frac{1_{\Gamma_\theta}(x)}{|\nabla u(x)|\,|x-e_2|^{2+s}}$$

we get

$$\int_{\Gamma_\theta}\frac{dx}{|x-e_2|^{2+s}}=\int_{-\theta}^{\theta}dt\int_{\{u=t\}}\frac{|x|}{|x-e_2|^{2+s}}\,d\mathcal{H}^1_x=2\int_0^\theta dt\int_{\{u=t\}}\frac{|x|}{|x-e_2|^{2+s}}\,d\mathcal{H}^1_x\,.$$

Now, if $t\in(0,\pi/2)$, then $\{u=t\}$ is the half-line $\{x\in\mathbb{R}^2:x_1>0\,,x_2=-(\tan t)\,x_1\}$ so that

$$x_1=|x|\,\sin t\qquad x_2=-|x|\cos t\qquad \forall x\in\{u=t\}\,.$$

Hence, setting $|x|=r$ we find

$$\int_{\{u=t\}}\frac{|x|}{|x-e_2|^{2+s}}\,d\mathcal{H}^1_x=\int_0^\infty\frac{r\,dr}{(r^2+2r\cos t+1)^{(2+s)/2}}\,.$$

By dominated convergence

$$\begin{aligned}\lim_{s\to1^-}\int_0^\theta dt\int_0^\infty\frac{r\,dr}{(r^2+2r\cos(t)+1)^{(2+s)/2}}&=\int_0^\theta dt\int_0^\infty\frac{r\,dr}{(r^2+2r\cos t+1)^{3/2}}\\&=\int_0^\theta\frac{dt}{1+\cos t}=\frac{\sin\theta}{1+\cos\theta}\,,\end{aligned}$$

where we have used

$$\int\frac{r\,dr}{(r^2+2r\cos t+1)^{3/2}}=-\frac{1}{\sin^2 t}\frac{1+r\,\cos t}{\sqrt{1+2r\cos t+r^2}}+\text{const.}$$

In summary, by taking the limit as $s\to1^-$ in (4.11) we find

$$\frac{1+\sigma}{\sin(\theta(1,\sigma))}=\frac{\sin(\theta(1,\sigma))}{1+\cos(\theta(1,\sigma))}\,,$$

which gives $\sigma = -\cos(\theta(1,\sigma)) = \cos(\pi - \theta(1,\sigma))$. This completes the proof of Theorem 1.4. $\square$

5. Almost-minimality and interior regularity

In this section we gather some simple basic properties of the almost-minimizers introduced in Definition 1.5, show that minimizers in (1.15) are almost-minimizers, and then check the interior regularity theory from [CG10] applies in our case. Let us recall that given $K \in \mathbf{K}(n,s,\lambda)$, open sets Ω and A with

$$I(\Omega A, \Omega^c) < \infty\,, \tag{5.1}$$

and $\Lambda \in [0,\infty)$, $r_0 \in (0,\infty]$ and $\sigma \in (-1,1)$, we say that $E \subset \Omega$ is $(\Lambda, r_0, \sigma, K)$-*minimizer in* (A,Ω) if

$$\begin{aligned} & I(EA, E^c\Omega) + I(EA^c, E^c\Omega A) + \sigma\, I(EA, \Omega^c) \\ \leqslant\ & I(FA, F^c\Omega) + I(FA^c, F^c\Omega A) + \sigma\, I(FA, \Omega^c) + \Lambda\, |E\Delta F|\,, \end{aligned} \tag{5.2}$$

whenever $F \subset \Omega$, $\mathrm{diam}(F\Delta E) < 2\,r_0$ and $F \cap A^c = E \cap A^c$. Thanks to (1.26), $I(FA, \Omega^c) < \infty$ whenever $F \subset \Omega$, and in particular the right hand side of (5.2) is always well definite in $(-\infty, \infty]$. We begin with two simple remarks.

Remark 5.1 (Almost-minimality and blow-ups)**.** Let us recall our notation $A^{x,r} = (A - x)/r$ for the blow-up of $A \subset \mathbb{R}^n$ near $x \in \mathbb{R}^n$ at scale $r > 0$. It is easily seen that for every $x \in \mathbb{R}^n$ and $r > 0$ one has that E is a $(\Lambda, r_0, \sigma, K)$-minimizer in (A,Ω) if and only if

$$E^{x,r} \text{ is a } (r^s\Lambda, r_0/r, \sigma, r^{n+s}\, K(r\,\cdot))\text{-minimizer in } (A^{x,r}, \Omega^{x,r})\,.$$

In particular, should $E^{x,r}$ converge to a limit set E^* as $r \to 0^+$ (for some $x \in A$ fixed), then one expects E^* to be a $(0,\infty,\sigma,K^*)$-minimizer in (B_R, H) for every $R > 0$, with $H = \mathbb{R}^n$ if $x \in A \cap \Omega$, and with $H = \{z : z \cdot \nu_\Omega(x) < 0\}$ if $x \in A \cap \partial\Omega$ and Ω is an open set of class C^1. Here K^* is defined as in (1.11).

Remark 5.2 (Almost-minimality and complement)**.** One notices that E is a $(\Lambda, r_0, \sigma, K)$-minimizer in (A,Ω) if and only if

$$\Omega \cap E^c \text{ is a } (\Lambda, r_0, -\sigma, K)\text{-minimizer in } (A,\Omega).$$

This can be easily checked by noticing that, for any set $E \subset \Omega$,

$$\begin{aligned} I(\Omega E^c A, (\Omega E^c)^c\Omega) &= I(\Omega E^c A, E) \\ &= I(EA, E^c\Omega A) + I(EA^c, E^c\Omega A)\,, \\ I(\Omega E^c A^c, (\Omega E^c)^c\Omega A) &= I(EA, E^c\Omega A^c)\,, \\ \sigma\, I(\Omega E^c A, \Omega^c) &= -\sigma\, I(EA, \Omega^c) + \sigma\, I(\Omega A, \Omega^c)\,. \end{aligned}$$

Let us recall the definition of (nonlocal) relative perimeter of E in an open set A,

$$P(E;A) = I(EA, E^cA) + I(EA, E^cA^c) + I(EA^c, E^cA)\,.$$

Proposition 5.3. *If $K \in \mathbf{K}(n,s,\lambda)$ and E is a $(\Lambda, r_0, \sigma, K)$-minimizer in (A,Ω) and x_0 and ϱ_0 are such that $B_{2\,\varrho_0}(x_0) \subset\subset \Omega \cap A$ with $\varrho_0 \leqslant r_0$, then*

$$P(E; B_{\varrho_0}(x_0)) \leqslant P(F; B_{\varrho_0}(x_0)) + C\,\frac{|E\Delta F|}{\varrho_0^s} \tag{5.3}$$

for every set F such that $E\Delta F \subset\subset B_{\varrho_0}(x_0)$, where C depends on Λ, λ, n and s.

Proof. Since $\varrho_0 \leqslant r_0$ we can plug any F such that $E\Delta F \subset\subset B_{\varrho_0}(x_0)$ into (5.2), and then deduce

$$\begin{aligned} I(EA, E^c\Omega) + I(EA^c, E^c\Omega A) &\leqslant I(FA, F^c\Omega) + I(FA^c, F^c\Omega A) \\ &\quad + |I(FA, \Omega^c) - I(EA, \Omega^c)| + \Lambda\, |E\Delta F|\,, \end{aligned}$$

where $K \in \mathbf{K}(n,s,\lambda)$ gives

$$
\begin{aligned}
|I(FA,\Omega^c)-I(EA,\Omega^c)| &\leqslant \lambda\int_{\Omega^c} dy \int_{(E\Delta F)\cap B_{\varrho_0}(x_0)} \frac{dx}{|x-y|^{n+s}}\\
&\leqslant \lambda\,|E\Delta F| \int_{B_{2\varrho_0}(x_0)^c} \frac{dy}{\mathrm{dist}(y,B_{\varrho_0}(x_0))^{n+s}}\\
&\leqslant \frac{\lambda}{\varrho_0^s}\, n\omega_n \int_2^\infty \frac{t^{n-1}\,dt}{(t-1)^{n+s}}|E\Delta F| \leqslant C\,\frac{|E\Delta F|}{\varrho_0^s}\,.
\end{aligned}
$$

We thus have

$$
I(EA,E^c\Omega)+I(EA^c,E^c\Omega A) \leqslant I(FA,F^c\Omega)+I(FA^c,F^c\Omega A)+C\,\frac{|E\Delta F|}{\varrho_0^s}\,. \tag{5.4}
$$

Let us now set $W = B_{\varrho_0}(x_0)$ for the sake of brevity. Since $W \subset\subset \Omega\cap A$ we have

$$
\begin{aligned}
I(EA,E^c\Omega)+I(EA^c,E^c\Omega A) &= I(EW,E^cW)+I(EW,E^cW^c\Omega)\\
&\quad +I(EAW^c,E^cW)+I(EAW^c,E^cW^c\Omega)\\
&\quad +I(EA^c,E^cW)+I(EA^c,E^c\Omega AW^c)
\end{aligned}
$$

where $E\Delta F \subset\subset W \subset\subset A$ implies that by replacing E with F we leave unchanged both the fourth and sixth interaction terms. We denote by κ their sum, so that $\kappa(E)=\kappa(F)$, and rewrite the above identity as

$$
\begin{aligned}
I(EA,E^c\Omega)+I(EA^c,E^c\Omega A) &= I(EW,E^cW)+I(EW,E^cW^c\Omega)\\
&\quad +I(EAW^c,E^cW)+I(EA^c,E^cW)+\kappa\\
&= I(EW,E^cW)+I(EW,E^cW^c\Omega)+I(EW^c,E^cW)+\kappa\\
&= P(E;W)-I(EW,E^c\Omega^c)+\kappa\,.
\end{aligned}
$$

Hence (5.4) is equivalent to

$$
P(E;W) \leqslant P(F;W)+I(EW,E^c\Omega^c)-I(FW,F^c\Omega^c)+C\,\frac{|E\Delta F|}{\varrho_0^s}\,. \tag{5.5}
$$

But since $E^c\cap\Omega^c = F^c\cap\Omega^c$, by arguing as before we find

$$
|I(EW,E^c\Omega^c)-I(FW,F^c\Omega^c)| \leqslant \int_{\Omega^c} dy \int_{(E\Delta F)\cap W} K(x-y)\,dx \leqslant C\,\frac{|E\Delta F|}{\varrho_0^s}\,,
$$

and (5.3) is proved. □

Corollary 5.4. *If E is a (Λ,r_0,σ,K_s)-minimizer in (A,Ω), then there exists a relatively closed subset Σ of $\Omega\cap\partial E$ such that $\Omega\cap\partial E\setminus\Sigma$ is a $C^{1,\alpha}$-hypersurface for some $\alpha\in(0,1)$ and Σ has Hausdorff dimension at most $n-3$. In particular, Σ is empty if $n=2$.*

Proof. The validity of (5.3) allows one to apply the main result of [CG10] and the deduce the above assertion with the Hausdorff dimension of Σ bounded by $n-2$. The improvement on the dimensional bound for Σ is obtained by exploiting [SV13]. □

We now show that minimizers in (1.15) are almost-minimizers.

Proposition 5.5. *If E is a minimizer in* (1.15)*, then E is a (Λ,r_0,σ,K)-minimizer in $(\mathbb{R}^n,\Omega)$ for values of r_0 and Λ depending on E and $\|g\|_{L^\infty(\Omega)}$ only.*

Proof. Let us fix two points $x_0 \neq y_0 \in \Omega\cap\partial E$ so that for some $\varrho_0>0$ we have

$$
|E\cap B_{\varrho_0}(x_0)|>0\,,\qquad |E\cap B_{\varrho_0}(y_0)|>0\,,\qquad |x_0-y_0|>4\,\varrho_0\,,\qquad B_{\varrho_0}(x_0)\cup B_{\varrho_0}(y_0)\subset\subset\Omega\,.
$$

Then there exists $T\in C^\infty_c(B_{\varrho_0}(x_0);\mathbb{R}^n)$ and $S\in C^\infty_c(B_{\varrho_0}(y_0);\mathbb{R}^n)$ such that

$$\int_E \mathrm{div}\, T=\int_E \mathrm{div}\, S=1\,,$$

see, e.g. [CM16, Lemma 3.5]. Let us now pick $F\subset\Omega$ with $\mathrm{diam}(F\Delta E)<2\,r_0$. If r_0 is small enough with respect to ϱ_0, then we either have $\mathrm{dist}(F,B_{\varrho_0}(x_0))>0$ or $\mathrm{dist}(F,B_{\varrho_0}(y_0))>0$. Without loss of generality, we may assume to be in the first case. Now let $f_t(x)=x+t\,T(x)$ and define

$$F_t=\Big(f_t(E)\cap B_{\varrho_0}(x_0)\Big)\cup\Big(F\setminus B_{\varrho_0}(x_0)\Big)=f_t(F)$$

for $|t|<\varepsilon_0$ and ε_0 small enough to ensure that $\{f_t\}_{|t|<\varepsilon_0}$ is a family of smooth diffeomorphisms with $\mathrm{spt}(f_t-\mathrm{Id}\,)\subset\subset B_{\varrho_0}(x_0)$ for every $|t|<\varepsilon_0$. If we set $\varphi(t)=|F_t|$, then

$$\varphi'(0)=\int_E \mathrm{div}\, T=1\,,$$

so that, up to decreasing the value of ε_0, φ is strictly increasing on $(-\varepsilon_0,\varepsilon_0)$, with range $(-v_0,v_0)$ for some $v_0>0$. Notice that the size of v_0 only depends on E through the choice of x_0 and of the vector field T. Thus, up to decreasing the value of r_0 depending on E, we find that $||F|-|E||<\omega_n\,r_0^n<v_0$, and thus that there exists $t_*=t_*(F)$ such that

$$|F_{t_*}|=|E|\qquad |t_*|\leqslant C\,\big||F|-|E|\big|$$

for a constant $C=C(E)$. By minimality of E we have

$$I(E,E^c\Omega)+\sigma\,I(E,\Omega^c)+\int_E g\leqslant I(F_{t_*},F_{t_*}^c\Omega)+\sigma\,I(F_{t_*},\Omega^c)+\int_{F_{t_*}} g\,.$$

Now, since for some $C=C(E)$ we have $|Jf_t(x)-1|\leqslant C\,|t|$ and $|\nabla f_t|\leqslant C$ on $\mathbb{R}^n$ for every $|t|<\varepsilon_0$, by the area formula we find

$$\begin{aligned}
&\big|I(F_t,F_t^c\Omega)-I(F,F^c\Omega)\big|\leqslant C\,|t|\,I(F,F^c\Omega)\,,\\
&\big|I(F_t,\Omega^c)-I(F,\Omega^c)\big|\leqslant C\,|t|\,I(F,\Omega^c)\,,\\
&\Big|\int_{F_t} g-\int_E g\Big|\leqslant\Big|\int_{F_t} g-\int_F g\Big|+\|g\|_{L^\infty(\Omega)}\,|E\Delta F|\leqslant C\,|t|+\|g\|_{L^\infty(\Omega)}\,|E\Delta F|\,,
\end{aligned}$$

whenever $|t|<\varepsilon_0$. By exploiting these facts with $t=t_*$ and taking into account $|t_*|\leqslant C\,\big||F|-|E|\big|$, we conclude that

$$I(E,E^c\Omega)+\sigma\,I(E,\Omega^c)+\leqslant I(F,F^c\Omega)+\sigma\,I(F,\Omega^c)+C\,\big|E\Delta F\big|\,,$$

where $\Lambda=\Lambda(E,\|g\|_{L^\infty(\Omega)})$. □

6. Density estimates at the boundary

We now discuss the proof of Theorem 1.7. We shall actually prove a more general result, involving the following notion of uniformly C^1 domain.

Definition 6.1. If $\eta>0$, A is an open set, Ω is an open set in $\mathbb{R}^n$ with boundary of class C^1 in A, and H_p denotes the affine tangent half-space to Ω at $p\in\partial\Omega$, then we define

$$\varrho_A(\eta,\Omega)$$

as the supremum of all $\varrho > 0$ such that for every $p \in A \cap \partial\Omega$ there exists a C^1-diffeomorphisms $T_p : \mathbb{R}^n \to \mathbb{R}^n$ with

$$T_p(B_\varrho(p)) = B_\varrho(p)\,, \tag{6.1}$$

$$T(B_\varrho(p) \cap \Omega) = B_\varrho(p) \cap H_p\,, \tag{6.2}$$

$$\|T_p - \mathrm{Id}\,\|_{C^0(\mathbb{R}^n)} + \|T_p^{-1} - \mathrm{Id}\,\|_{C^0(\mathbb{R}^n)} \leqslant \eta\,\varrho\,, \tag{6.3}$$

$$\|\nabla T_p - \mathrm{Id}\,\|_{C^0(\mathbb{R}^n)} + \|(\nabla T_p)^{-1} - \mathrm{Id}\,\|_{C^0(\mathbb{R}^n)} \leqslant \eta\,. \tag{6.4}$$

Remark 6.2. If Ω is a *bounded* open set with C^1-boundary, then $\varrho_{\mathbb{R}^n}(\eta, \Omega) > 0$; but, of course, one can have $\varrho_{\mathbb{R}^n}(\eta, \Omega) > 0$ even if Ω is unbounded (for example, if $\varrho_{\mathbb{R}^n}(\eta, H) = \infty$ if H is a half-space). We also notice that for every $x_0 \in \mathbb{R}^n$ and $r > 0$ one has

$$\varrho_A(\eta, \Omega) = r\,\varrho_{A^{x_0,r}}(\eta, \Omega^{x_0,r})\,. \tag{6.5}$$

Indeed, given a set of maps $\{T_p\}_{p\in\partial\Omega}$ associated to some $\varrho < \varrho_A(\eta, \Omega)$ one can use the maps $\{S_q\}_{q\in\Omega^{x_0,r}}$ defined by

$$p = x_0 + r\,q\,, \qquad S_q(y) = \frac{T_p(x_0 + r\,y) - x_0}{r}\,,$$

to show that $\varrho/r < \varrho_{A^{x_0,r}}(\eta, \Omega^{x_0,r})$. In particular, $\varrho_A(\eta, \Omega) \leqslant \varrho_{A^{x_0,r}}(\eta, \Omega^{x_0,r})$ for every $r \in (0, 1)$, that is, the positivity of $\varrho_{A^{x_0,r}}(\eta, \Omega^{x_0,r})$ is stable under blow-ups of Ω. Identity (6.5) is needed to obtain density estimates that are stable under blow-up limits.

With Definition 6.1, we can formulate the following improved version of Theorem 1.7. Notice that the assumption of Ω being a bounded open set with C^1-boundary or an half-space is replaced here by the requirement that $\varrho_A(\eta, \Omega) > 0$ for every $\eta > 0$.

Theorem 6.3 (Density estimates). *Let $n \geqslant 2$, $s \in (0, 1)$, $\sigma \in (-1, 1)$, $\Lambda \geqslant 0$ and $K = K_s^\varepsilon$ for some $\varepsilon > 0$. If A is an open set and Ω is an open set with C^1-boundary in A such that $\varrho_A(\eta, \Omega) > 0$ for every $\eta > 0$, then there exist positive constants C_0 (depending on n, s, σ and Λ), c_* (depending on n and s) and η_1 (depending on n, s and σ) with the following property: for every $(\Lambda, r_0, \sigma, K_s^\varepsilon)$-minimizer E in (A, Ω), one has*

$$I_s^\varepsilon(EB_r(x), (EB_r(x))^c) \leqslant C_0\, r^{n-s}\,, \tag{6.6}$$

whenever $B_r(x) \subset A$ and $r < \min\{r_0, c_*\,\varrho_A(\eta_1, \Omega), c_*\,\varepsilon\}$, and, moreover,

$$\frac{1}{C_0} \leqslant \frac{|E \cap B_r(x)|}{r^n} \leqslant 1 - \frac{1}{C_0}\,, \tag{6.7}$$

whenever $B_r(x) \subset A$, $r < \min\{r_0,\ c_*\varrho_A(\eta_1, \Omega),\ c_*\,\varepsilon\}$, and $x \in \overline{\Omega \cap \partial E}$.

We now turn to the proof of Theorem 6.3. A key tool is a geometric inequality, stated in Lemma 6.4 below, which can be introduced by the following considerations. A crucial role in the study of local capillarity problems is played by the geometric remark that, if Per denotes the classical (local) perimeter, then

$$\mathrm{Per}(Z; H) \geqslant \mathrm{Per}(Z; \partial H)\,, \tag{6.8}$$

whenever $Z \subset H$ is of finite perimeter and finite volume (this is a consequence of the divergence theorem; see, for example, [Mag12, Proposition 19.22]). An analogous inequality to (6.8) holds for fractional perimeters too: if H is a half-space in $\mathbb{R}^n$ and Z is a bounded subset of H, then

$$I_s(Z, Z^cH) \geqslant I_s(Z, H^c)\,. \tag{6.9}$$

Indeed, let $R > 0$ be such that $Z \subset H \cap B_R$. If we set $J = H^c$ and $Y = J \cup Z$, then J is a half-space and $Y \backslash B_R = J \backslash B_R$, so that, by [CRS10, Corollary 5.3(b)] (see also [ADPM11, Proposition 17]),

$$I_s(YB_R, Y^c) + I_s(YB_R^c, Y^cB_R) \geqslant I_s(JB_R, J^c) + I_s(JB_R^c, J^cB_R)\,. \tag{6.10}$$

Since $Y^c = Z^c \cap H$ and $Z \subset H \cap B_R$, one finds

$$I_s(YB_R, Y^c) = I_s(Z, Z^cH) + I_s(H^cB_R, Z^cH)\,,$$
$$I_s(YB_R^c, Y^cB_R) = I_s(H^cB_R^c, HZ^cB_R)\,,$$

which, combined with (6.10), gives

$$\begin{aligned} I_s(Z, Z^cH) &\geqslant I_s(H^cB_R, H) - I_s(H^cB_R, Z^cH) \\ &\quad + I_s(H^cB_R^c, HB_R) - I_s(H^cB_R^c, HZ^cB_R) \\ &= I_s(H^cB_R, Z) + I_s(H^cB_R^c, Z) = I_s(Z, H^c)\,. \end{aligned}$$

(This argument actually shows that (6.9) is equivalent to (6.10).) We now want to generalize (6.9) to the case when an open set Ω takes the place of the half-space H. The idea is that on sets of sufficiently small diameter, if the boundary Ω is regular enough to be locally close to a half-space at each of its boundary points, then an inequality like (6.9) should hold true with some error terms.

Lemma 6.4. *Given $n \geqslant 2$, $s \in (0,1)$, and $\varepsilon > 0$ there exist positive constants $C_\star$ and η_0 (depending on n and s, and with $C_\star\,\eta_0 < 1$) with the following property. If A is an open set, Ω is an open set with C^1-boundary in A, $\eta \in (0, \eta_0)$,*

$$r_\star := \min\Big\{\frac{\varrho_A(\eta,\Omega)}{4\,C_\star}, \frac{\varepsilon}{2\,C_\star}\Big\} \tag{6.11}$$

and

$$G \subset \Omega \cap B_{r_\star}(x) \qquad \text{for some } x \in \mathbb{R}^n \tag{6.12}$$

then

$$I_s^\varepsilon(G, G^c\Omega) \geqslant (1 - C_\star\,\eta)\, I_s^\varepsilon(G, \Omega^c) - \frac{C_\star}{r_\star^s}\,|G|\,, \tag{6.13}$$

Proof. Let us fix $\eta \in (0, \eta_0)$, assume without loss of generality that $\varrho_A(\eta,\Omega) > 0$, define $r_\star$ by (6.11), and directly consider the case $|G| > 0$. The idea is that when $B_{r_\star}(x)$ is sufficiently close to $\partial\Omega$, then one can first "flatten" the boundary and then exploit the local minimality of half-spaces expressed in (6.9) in order to obtain (6.13). If, instead, $B_{r_\star}(x)$ is away from $\partial\Omega$ then (6.13) follows by the isoperimetric inequality (for the fractional perimeter P_s).

Step one: We prove that if, in addition to (6.12), we have $B_{C_\star r_\star}(x) \subset \Omega$, then

$$I_s^\varepsilon(G, G^c\Omega) \geqslant I_s^\varepsilon(G, \Omega^c)\,. \tag{6.14}$$

First we notice that, trivially,

$$|z - y| \geqslant \frac{C_\star - 1}{C_\star}|z - x| \qquad \forall y \in B_{r_\star}(x)\,, z \in B_{C_\star\,r_\star}(x)^c\,. \tag{6.15}$$

(We definitely assume that $C_\star > 1$.) By assumption we have $G \subset B_{r_\star}(x)$ and $\Omega^c \subset B_{C_\star\,r_\star}(x)^c$, so that (6.15) and $K_s^\varepsilon = 1_{B_\varepsilon}\,K_s$ give us

$$\begin{aligned} I_s^\varepsilon(G, \Omega^c) &\leqslant I_s^\varepsilon(G, B_{C_\star r_\star}(x)^c) \leqslant \Big(\frac{C_\star}{C_\star - 1}\Big)^{n+s}\int_G dy \int_{B_{C_\star r_\star}(x)^c} \frac{dz}{|z-x|^{n+s}} \\ &\leqslant n\,\omega_n \Big(\frac{C_\star}{C_\star - 1}\Big)^{n+s} |G| \int_{C_\star r_\star}^{+\infty} \varrho^{-1-s}\,d\varrho = \frac{n\,\omega_n}{s}\Big(\frac{C_\star}{C_\star - 1}\Big)^{n+s} \frac{|G|}{(C_\star r)^s} \end{aligned} \tag{6.16}$$

where ω_n is the volume of the unit ball. At the same time, by the fractional isoperimetric inequality (see [FLS08, CV11]) we have that

$$P_s(G) \geqslant \frac{P_s(B_1)}{\omega_n^{(n-s)/n}}\,|G|^{(n-s)/n}\,. \tag{6.17}$$

Since $(C_\star+1)\,r_\star\leqslant 2\,C_\star r_\star<\varepsilon$ and $G\subset B_{r_\star}(x)$ we have that

$$|y-z|\leqslant\varepsilon\qquad\forall y\in G\,,z\in B_{C_\star r_\star}(x)\,,$$

so that by $K_s^\varepsilon=1_{B_\varepsilon}K_s$ and by $B_{C_\star r_\star}(x)\subset\Omega$ we find

$$I_s^\varepsilon(G,G^c\Omega)\geqslant I_s^\varepsilon(G,G^cB_{C_\star r_\star}(x))\geqslant I_s(G,G^cB_{C_\star r_\star}(x))=\Big(P_s(G)-I_s(G;B_{C_\star r_\star}(x)^c)\Big)\,.$$

Hence, by (6.17) and (6.16), we have

$$\begin{aligned}\frac{I_s^\varepsilon(G,G^c\Omega)}{I_s^\varepsilon(G,\Omega^c)}&\geqslant\frac{\frac{P_s(B_1)}{\omega_n^{(n-s)/n}}\,|G|^{(n-s)/n}-\frac{n\,\omega_n}{s}\left(\frac{C_\star}{C_\star-1}\right)^{n+s}\frac{|G|}{(C_\star r)^s}}{\frac{n\,\omega_n}{s}\left(\frac{C_\star}{C_\star-1}\right)^{n+s}\frac{|G|}{(C_\star r)^s}}\\&=\frac{s}{n}\,P_s(B_1)\Big(\frac{C_\star-1}{C_\star}\Big)^{n+s}C_\star^s\Big(\frac{|B_{r_\star}(x)|}{|G|}\Big)^{s/n}-1\\&\geqslant\frac{s}{n}\,P_s(B_1)\Big(\frac{C_\star-1}{C_\star}\Big)^{n+s}C_\star^s-1\geqslant 1\,,\end{aligned}$$

where the last inequality holds provided $C_\star$ is large enough depending on n and s.

Step two: We now complete the proof of the lemma. We first notice that

$$|K_s(\zeta_1)-K_s(\zeta_1)|\leqslant C(n,s)\,\frac{K_s(\zeta_1)}{|\zeta_1|}\,|\zeta_1-\zeta_2|\tag{6.18}$$

whenever $|\zeta_1-\zeta_2|\leqslant|\zeta_1|/2$. Indeed, if $t\in[0,1]$, then

$$|t\zeta_2+(1-t)\zeta_1|\geqslant|\zeta_1|-|\zeta_2-\zeta_1|\geqslant\frac{|\zeta_1|}{2}.$$

and thus

$$\begin{aligned}|K_s(\zeta_2)-K_s(\zeta_1)|&\leqslant\sup_{t\in[0,1]}|\nabla K_s(t\zeta_2+(1-t)\zeta_1)|\,|\zeta_2-\zeta_1|\\&\leqslant\sup_{t\in(0,1)}\frac{|\zeta_2-\zeta_1|}{|t\zeta_2+(1-t)\zeta_1|^{n+s+1}}\leqslant C(n,s)\,\frac{K_s(\zeta_1)}{|\zeta_1|}\,|\zeta_2-\zeta_1|\,.\end{aligned}$$

This proves (6.18), which we are going to use now in the proof of (6.13).

Given step one, we may directly assume that there exists $p\in B_{C_\star r_\star}(x)\cap\partial\Omega$ (as well as that $B_{r_\star}(x)\cap\Omega\neq\varnothing$, otherwise (6.12) would give $G=\varnothing$). The existence of p gives

$$B_{2\,r_\star}(x)\subset B_\varrho(p)\,,\qquad\text{for some }\varrho<\varrho(\eta,\Omega)\,.\tag{6.19}$$

Indeed, if $q\in B_{2\,r_\star}(x)$ and we pick $C_\star>2$, then we find

$$|q-p|\leqslant|q-x|+|x-p|\leqslant 2\,r_\star+C_\star r_\star<2C_\star r_\star<\varrho(\eta,\Omega)$$

by definition of $r_\star$. By definition of $\varrho_A(\eta,\Omega)$ there exists a C^1-diffeomorphisms $T_p:\mathbb{R}^n\to\mathbb{R}^n$ such that (6.1)–(6.4) hold with ϱ as in (6.19). In particular (6.4) gives that

$$\Big|\big(T_p(z)-T_p(y)\big)-(z-y)\Big|\leqslant\eta\,|z-y|\,,\qquad\forall z,\ y\in\mathbb{R}^n\,,$$

so that, provided $\eta_0<1/2$, in view of (6.18),

$$\Big|K_s\big(T_p(z)-T_p(y)\big)-K_s(z-y)\Big|\leqslant C(n,s)\,K_s(z-y)\,\eta\,,\qquad\text{if }|z-y|<\varepsilon\,.$$

At the same time (6.4) implies

$$\Big|JT_p(z)\,JT_p(y)-1\Big|\leqslant C(n)\,\eta\qquad\forall z,y\in\mathbb{R}^n\,,$$

so that in conclusion, for every $z,y\in\mathbb{R}$ we have

$$\Big|JT_p(z)\,JT_p(y)\,K_s\big(T_p(z)-T_p(y)\big)-K_s(z-y)\Big|\leqslant C_\star\,\eta\,K_s(z-y)\,.$$

By the area formula, for every pair of disjoint sets $A_1, A_2 \subset \mathbb{R}^n$ one has that $I_s(A_1, A_2)$ if finite if and only if $I_s(T_p(A_1), T_p(A_2))$ is finite, with

$$(1 - C_\star\,\eta) I_s(A_1, A_2) \leqslant I_s(T_p(A_1), T_p(A_2)) \leqslant (1 + C_\star\,\eta)\, I_s(A_1, A_2)\,. \tag{6.20}$$

We are now in the position to conclude our argument. By $G \subset B_{r_\star}(x) \cap \Omega$ and (6.19) we find $T_p(G) \subset H \cap B_\varrho(p)$, where $H = H_p$ is the affine tangent half-space to Ω at p (see Definition 6.1). Thus we can apply (6.9) to $Z = T_p(G)$ and find

$$I_s(T_p(G), T_p(G)^c H) \geqslant I_s(T_p(G), H^c)\,, \tag{6.21}$$

which is equivalently written (by using $T_p(G)^c \cap B_\varrho(p)^c = B_\varrho(p)^c$) as

$$\begin{aligned} &I_s(T_p(G), T_p(G)^c H B_\varrho(p)) + I_s(T_p(G), H B_\varrho(p)^c)\\ &\geqslant I_s(T_p(G), H^c B_\varrho(p)) + I_s(T_p(G), H^c B_\varrho(p)^c)\,. \end{aligned}$$

Since $H \cap B_\varrho(p) = T_p(\Omega \cap B_\varrho(p))$ and $H^c \cap B_\varrho(p) = T_p(\Omega^c \cap B_\varrho(p))$, by exploiting (6.20) we obtain

$$\begin{aligned} &(1 + C_\star\,\eta)\, I_s(G, G^c \Omega B_\varrho(p)) + I_s(T_p(G), H B_\varrho(p)^c) \\ \geqslant\ &(1 - C_\star\,\eta)\, I_s(G, \Omega^c B_\varrho(p)) + I_s(T_p(G), H^c B_\varrho(p)^c)\,. \end{aligned} \tag{6.22}$$

Again by (6.20) one finds

$$\begin{aligned} &|I_s(T_p(G), H B_\varrho(p)^c) - I_s(G, \Omega B_\varrho(p)^c)| \\ \leqslant\ &|I_s(T_p(G), T_p(\Omega B_\varrho(p)^c)) - I_s(G, \Omega B_\varrho(p)^c)| \\ &+|I_s(T_p(G), T_p(\Omega B_\varrho(p)^c)) - I_s(T_p(G), H B_\varrho(p)^c)| \\ \leqslant\ &C_\star \eta I_s(G, \Omega B_\varrho(p)^c) + I_s(T_p(G), B_\varrho(p)^c) \\ \leqslant\ &C_\star \eta I_s(G, \Omega B_\varrho(p)^c) + (1 + C_\star\,\eta)\, I_s(G, B_\varrho(p)^c)\,, \end{aligned}$$

so that (6.22) gives

$$\begin{aligned} &(1 + C_\star\,\eta) \big(I_s(G, G^c \Omega) + I_s(G, B_\varrho(p)^c)\big) \\ \geqslant\ &(1 - C_\star\,\eta)\, I_s(G, \Omega^c B_\varrho(p)) + I_s(T_p(G), H^c B_\varrho(p)^c)\,. \end{aligned} \tag{6.23}$$

Similarly, by using (6.20) one more time,

$$\begin{aligned} &|I_s(T_p(G), H^c B_\varrho(p)^c) - I_s(G, \Omega^c B_\varrho(p)^c)| \\ \leqslant\ &|I_s(T_p(G), T_p(\Omega^c B_\varrho(p)^c)) - I_s(G, \Omega^c B_\varrho(p)^c)| \\ &+|I_s(T_p(G), T_p(\Omega^c B_\varrho(p)^c)) - I_s(T_p(G), H^c B_\varrho(p)^c)| \\ \leqslant\ &C_\star\,\eta I_s(G, \Omega^c B_\varrho(p)^c) + I_s(T_p(G), B_\varrho(p)^c) \\ \leqslant\ &C_\star\,\eta I_s(G, \Omega^c B_\varrho(p)^c) + (1 + C_\star\,\eta)\, I_s(G, B_\varrho(p)^c)\,, \end{aligned}$$

which plugged into (6.23) gives, as $C_\star \eta_0 < 1$,

$$(1 + C_\star\,\eta)\, I_s(G, G^c \Omega) \geqslant (1 - C_\star\,\eta)\, I_s(G, \Omega^c) - 4\, I_s(G, B_\varrho(p)^c)\,. \tag{6.24}$$

Since $G \subset B_{r_\star}(x)$ with $B_{2\,r_\star}(x) \subset B_\varrho(p)$, recalling (1.10) we have

$$I_s(G, B_\varrho(p)^c) \leqslant \int_G dz \int_{B_{2\,r_\star}(x)^c} \frac{dy}{|z - y|^{n+s}} = \frac{n\omega_n}{s\,2^s\,r_\star^s}\,|G|\,,$$

and thus (6.24) implies

$$\begin{aligned} (1 + C_\star\,\eta)\, I_s(G, G^c \Omega) \ &\geqslant\ (1 - C_\star\,\eta)\, I_s^\varepsilon(G, \Omega^c) - 4\, I_s(G, B_\varrho(p)^c) && (6.25)\\ &\geqslant\ (1 - C_\star\,\eta)\, I_s^\varepsilon(G, \Omega^c) - C(n, s)\frac{|G|}{r_\star^s}\,. && (6.26) \end{aligned}$$

Now

$$I_s(G,G^c\Omega)=I_s^\varepsilon(G,G^c\Omega)+\int_G dz\int_{B_\varepsilon(z)\cap G^c\cap\Omega}\frac{dy}{|z-y|^{n+s}}\leqslant I_s^\varepsilon(G,G^c\Omega)+C(n,s)\frac{|G|}{\varepsilon^s}$$

and so we deduce (6.13) from (6.25) and $r_\star\leqslant\varepsilon/C(n,s)$. □

We are now ready for the proof of Theorem 6.3.

Proof of Theorem 6.3. Let E be a $(\Lambda,r_0,\sigma,K_\sigma^\varepsilon)$-minimizer in (A,Ω), so that

$$\begin{aligned}&I_s^\varepsilon(EA,E^c\Omega)+I_s^\varepsilon(EA^c,E^c\Omega A)+\sigma\,I_s^\varepsilon(EA,\Omega^c)\\ \leqslant\;&I_s^\varepsilon(FA,F^c\Omega)+I_s^\varepsilon(FA^c,F^c\Omega A)+\sigma\,I_s^\varepsilon(FA,\Omega^c)+\Lambda\,|E\Delta F|\,,\end{aligned}\tag{6.27}$$

whenever $F\subset\Omega$ with $\mathrm{diam}(F\Delta E)<2\,r_0$ and $F\cap A^c=E\cap A^c$.

Let us fix $B_r(x)\subset A$ with $r<r_0$ and test (6.27) with $F=E\cap B_r(x)^c$. Since $F\cap A=E\cap B_r(x)^c\cap A$ and $F^c=E^c\cup(E\cap B_r(x))$ one has

$$I_s^\varepsilon(FA,F^c\Omega)-I_s^\varepsilon(EA,E^c\Omega)=-I_s^\varepsilon(EB_r(x),E^c\Omega)+I_s^\varepsilon(EB_r(x)^cA,EB_r(x))\,.$$

Similarly, by $F\cap A^c=E\cap A^c$,

$$\begin{aligned}I_s^\varepsilon(FA^c,F^c\Omega A)-I_s^\varepsilon(EA^c,E^c\Omega A)&=I_s^\varepsilon(EA^c,EB_r(x))\,,\\ I_s^\varepsilon(FA,\Omega^c)-I_s^\varepsilon(EA,\Omega^c)&=-I_s^\varepsilon(EB_r(x),\Omega^c)\,,\end{aligned}$$

so that (6.27) gives

$$\begin{aligned}&I_s^\varepsilon(EB_r(x),E^c\Omega)+\sigma\,I_s^\varepsilon(EB_r(x),\Omega^c)\\ &\quad\leqslant I_s^\varepsilon(EB_r(x)^cA,EB_r(x))+I_s^\varepsilon(EA^c,EB_r(x))+\Lambda\,u(r)\,,\end{aligned}\tag{6.28}$$

provided $u(r)=|E\cap B_r(x)|$. By $A^c\subset B_r(x)^c$ one finds

$$I_s^\varepsilon(EB_r(x)^cA,EB_r(x))+I_s^\varepsilon(EA^c,EB_r(x))\leqslant 2\,I_s^\varepsilon(EB_r(x),EB_r(x)^c)\,,$$

so that, by adding $I_s^\varepsilon(EB_r(x),EB_r(x)^c)$ to both sides of (6.28), one gets

$$\begin{aligned}&I_s^\varepsilon(EB_r(x),(E^c\Omega)\cup(EB_r(x)^c))+\sigma\,I_s^\varepsilon(EB_r(x),\Omega^c)\\ \leqslant\;&3\,I_s^\varepsilon(EB_r(x),EB_r(x)^c)+\Lambda\,u(r)\,.\end{aligned}\tag{6.29}$$

Now let $C_\star$ and η_0 be as in Lemma 6.4 and fix $\eta_1\in(0,\eta_0)$ depending on n, s, and σ so that

$$(1-C_\star\,\eta_1)^2-|\sigma|\geqslant\eta_1\,.$$

We are going to apply Lemma 6.4 with $\eta=\eta_1$, so that (6.11) gives

$$r_\star=\min\Big\{\frac{\varrho_A(\eta_1,\Omega)}{4\,C_\star},\frac{\varepsilon}{2\,C_\star}\Big\}\leqslant c_*\,\min\big\{\varrho_A(\eta_1,\Omega),\varepsilon\big\}$$

for a constant c_* depending on n and s. If we set $G=E\cap B_r(x)$, then $G\subset\Omega\cap B_{r_\star}(x)$ provided $r<r_\star$. In particular, by (6.13) we find

$$I_s^\varepsilon(G,G^c\Omega)\geqslant(1-C_\star\,\eta_1)\,I_s^\varepsilon(G,\Omega^c)-\frac{C_\star}{r_\star^s}\,|G|\,.\tag{6.30}$$

Moreover,

$$G^c\cap\Omega=(E^c\cap\Omega)\cup(B_r(x)^c\cap\Omega)=(E^c\cap\Omega)\cup(E\cap B_r(x)^c)\,,$$

so that (6.29) gives

$$3\,I_s^\varepsilon(EB_r(x),EB_r(x)^c)+\Lambda\,u(r)\geqslant I_s^\varepsilon(G,G^c\Omega)+\sigma\,I_s^\varepsilon(G,\Omega^c).$$

By (6.13),

$$
\begin{aligned}
& 3\, I_s^\varepsilon(EB_r(x), EB_r(x)^c) + \Lambda\, u(r) \\
\geqslant\ & C_\star\, \eta_1\ I_s^\varepsilon(G, G^c\Omega) + (1 - C_\star\, \eta_1) I_s^\varepsilon(G, G^c\Omega) - |\sigma|\, I_s^\varepsilon(G, \Omega^c) \\
\geqslant\ & C_\star\, \eta_1\ I_s^\varepsilon(G, G^c\Omega) + [(1 - C_\star\, \eta_1)^2 - |\sigma|]\, I_s^\varepsilon(G, \Omega^c) - (1 - C_\star\, \eta_1)\, \frac{C_\star}{r_\star^s}\, u(r) \\
\geqslant\ & \eta_1 \Big(I_s^\varepsilon(G, G^c\Omega) + I_s^\varepsilon(G, \Omega^c) \Big) - \frac{C_\star}{r_\star^s}\, u(r) = \eta_1\, P_s^\varepsilon(G) - \frac{C_\star}{r_\star^s}\, u(r)\,.
\end{aligned}
$$

Summarizing, if $B_r(x) \subset A$ with $r < \min\{r_0, r_\star\}$, then

$$
3\, I_s^\varepsilon(EB_r(x), EB_r(x)^c) + \Big(\Lambda + \frac{C_\star}{r_\star^s}\Big)\, u(r) \geqslant \eta_1\, P_s^\varepsilon(G)\,, \qquad G = E \cap B_r(x)\,. \tag{6.31}
$$

Since $I_s^\varepsilon(EB_r(x), EB_r(x)^c) \leqslant P_s^\varepsilon(B_r(x)) \leqslant C(n,s) r^{n-s}$ and $u(r) \leqslant \omega_n r^n$, we see that (6.31) immediately implies (6.6). Next, we apply the fractional isoperimetric inequality (6.17) to bound from below $P_s^\varepsilon(G)$ in (6.31). More precisely, we notice that

$$
P_s(G) = P_s^\varepsilon(G) + \int_G dz \int_{G^c \cap B_\varepsilon(z)^c} \frac{dy}{|z-y|^{n+s}} \leqslant P_s^\varepsilon(G) + C(n,s)\, \frac{|G|}{\varepsilon^s} \leqslant P_s^\varepsilon(G) + C(n,s)\, \frac{|G|}{r_\star^s}
$$

so that, up to increasing the value of $C_\star$, (6.17) gives

$$
3\, I_s^\varepsilon(EB_r(x), EB_r(x)^c) + \Big(\Lambda + \frac{C_\star}{r_\star^s}\Big)\, u(r) \geqslant \frac{P(B_1)}{\omega_n^{(n-s)/n}}\, \eta_1\, u(r)^{(n-s)/n}\,. \tag{6.32}
$$

By exploiting $u(r) \leqslant (\omega_n\, r^n)^{s/n} u(r)^{(n-s)/n}$ we find that if

$$
\Big(\Lambda + \frac{C_\star}{r_\star^s}\Big) (\omega_n\, r^n)^{s/n} \leqslant \frac{P(B_1)\, \eta_1}{2\, \omega_n^{(n-s)/n}}\,, \tag{6.33}
$$

then (6.32) implies

$$
u(r)^{(n-s)/n} \leqslant C(n,s,\Lambda,\sigma)\, I_s^\varepsilon(EB_r(x), EB_r(x)^c)\,. \tag{6.34}
$$

We notice that (6.33) is equivalent to

$$
\Big(\frac{r}{r_\star}\Big)^s \leqslant \frac{P(B_1)\, \eta_1}{2\omega_n (r_\star^s \Lambda + C_\star)}
$$

which, by $r_\star^s \Lambda \leqslant \Lambda$, is in turn implied by

$$
r \leqslant c(n,s)\, r_\star
$$

and thus by $r \leqslant c_*(n,s) \min\{\varrho_A(\eta_1, \Omega), \varepsilon\}$. We have thus proved the validity of (6.34) provided $r \leqslant r_0$ and $r \leqslant c_*(n,s) \min\{\varrho_A(\eta_1, \Omega), \varepsilon\}$. Arguing as in [CRS10, Lemma 4.2], we conclude that if $B_r(x) \subset A$, $x \in \Omega \cap \partial E$, and r satisfies the above constraints, then $u(r) \geqslant c_0\, r^n$ for some $c_0 = c_0(n,s,\sigma,\Lambda)$. By Remark 5.2, $\Omega \cap E^c$ is a $(\Lambda, r_0, -\sigma)$-minimizer in (A, Ω), and since $\Omega \cap \partial(\Omega \cap E^c) = \Omega \cap \partial E$ one can repeat the above argument with $\Omega \cap E^c$ in place of E to find the upper volume density estimate in (6.7). □

Appendix A. Closure theorem for almost-minimizers and blow-up limits

In this appendix we prove a closure theorem for sequences of $(\Lambda, r_0, \sigma, K)$-minimizers (Theorem A.1). As an application, we then show that blow-up limits exists and are in turn minimizers (Theorem A.2). In the following, given an interaction kernel $K \in \mathbf{K}(n, s\lambda)$, we set

$$
w_F(x) := 1_F(x) \int_{F^c} K(x-y)\, dy \qquad F \subset \mathbb{R}^n \tag{A.1}
$$

so that

$$
w_F \text{ belongs to } L^1(A) \text{ if (and only if) } I(AF, F^c) < +\infty. \tag{A.2}
$$

Theorem A.1. *Let $n \geqslant 2$, $s \in (0,1)$, $\sigma \in (-1,1)$, $\lambda \geqslant 1$, $\Lambda \geqslant 0$, $r_0 > 0$, $K \in \mathbf{K}(n,s,\lambda)$ and A be an open set. Consider a sequence $\{E_j\}_{j\in\mathbb{N}}$ of $(\Lambda, r_0, \sigma, K)$-minimizers in (A, Ω_j), where $\{\Omega_j\}_{j\in\mathbb{N}}$ is a sequence of open sets. If there exists an open set Ω with $P(\Omega) < \infty$ such that*

$$E_j \to E \text{ and } \Omega_j \to \Omega \text{ in } L^1_{\rm loc}(A) \tag{A.3}$$

and

$$w_{\Omega_j} \text{ converges to } w_\Omega \text{ weakly in } L^1_{\rm loc}(A) \tag{A.4}$$

then E is a $(\Lambda, r_0, \sigma, K)$-minimizer in (A, Ω).

Moreover, in the case when $K = K_s^\varepsilon$ and $\Omega_j = \Omega$ is an open set with C^1-boundary such that $\varrho(\eta, \Omega) > 0$ for every $\eta > 0$, one has that:

(i) if $x_j \in A \cap \overline{\Omega \cap \partial E_j}$ and $x_j \to x$ for some $x \in A$, then $x \in \partial E$;

(ii) if $x \in \overline{\Omega \cap \partial E}$ then there exists $x_j \in \partial E_j$ such that $x_j \to x$.

Proof. Step one: We want to prove that (5.2) holds whenever $F \subset \Omega$, $\mathrm{diam}(F\Delta E) < 2r_0$ and $F \cap A^c = E \cap A^c$. Of course, without loss of generality, we may assume that

$$I(FA, F^c\Omega) + I(FA^c, F^c\Omega A) + \sigma\, I(FA, \Omega^c) < \infty\,.$$

Since $I(FA, \Omega^c) \leqslant I(\Omega A, \Omega^c) < \infty$, this implies

$$I(FA, F^c\Omega) + I(FA^c, F^c\Omega A) < \infty \tag{A.5}$$

and hence

$$\int_A w_F = I(AF, F^c) = I(AF, F^c\Omega) + I(AF, \Omega^c) \leqslant I(AF, F^c\Omega) + P(\Omega) < \infty\,.$$

Similarly,

$$\begin{aligned}\int_A w_{F^c} &= I(AF^c, F) = I(AF^c, AF) + I(AF^c, A^cF)\\ &\leqslant I(AF^c, AF) + I(AF^c\Omega^c, A^cF) + I(AF^c\Omega, A^cF)\end{aligned}$$

where $I(AF^c, AF) \leqslant \int_A w_F < \infty$, $I(AF^c\Omega^c, A^cF) \leqslant P(\Omega) < \infty$, and $I(AF^c\Omega, A^cF) < \infty$ by (A.5). We have thus proved that in showing (5.2) for a given $F \subset \Omega$ with $\mathrm{diam}(F\Delta E) < 2r_0$ and $F \cap A^c = E \cap A^c$, we can directly assume that

$$w_F\,, w_{F^c} \in L^1(A)\,. \tag{A.6}$$

Now we fix a bounded set W with $w_W, w_{W^c} \in L^1(\mathbb{R}^n)$ such that $F\Delta E \subset\subset W \subset\subset A$ and $\mathrm{diam}(W) < 2r_0$ (we can achieve this by taking W in the form of a finite union of balls, say). Our goal is thus proving that

$$\begin{aligned}&I(EW, E^c\Omega W) + I(EW, E^c\Omega W^c) + I(EW^c, E^c\Omega W) + \sigma\, I(EW, \Omega^c) \\ \leqslant\ &I(FW, F^c\Omega W) + I(FW, E^c\Omega W^c) + I(EW^c, F^c\Omega W) + \sigma\, I(FW, \Omega^c) + \Lambda\, |E\Delta F|\,.\end{aligned} \tag{A.7}$$

To this end we set $F_j = (F \cap \Omega_j \cap W) \cup (E_j \cap W^c)$, and test the minimality inequality of E_j (see (1.27)) on F_j. In this way we find

$$\begin{aligned}&I(E_jW, E_j^c\Omega_jW) + I(E_jW, E_j^c\Omega_jW^c) + I(E_jW^c, E_j^c\Omega_jW) + \sigma\, I(E_jW, \Omega_j^c) \\ \leqslant\ &I(F\Omega_jW, F^c\Omega_jW) + I(F\Omega_jW, E_j^c\Omega_jW^c) + I(E_jW^c, F^c\Omega_jW) + \sigma\, I(F\Omega_jW, \Omega_j^c) \\ &+\Lambda\, |E_j\Delta F_j|.\end{aligned} \tag{A.8}$$

We claim that in the limit $j \to \infty$, (A.8) implies (A.7). By Fatou's lemma and (A.3), the inferior limit as $j \to \infty$ of the sum of first three terms on the left-hand side of (A.8) is bounded from below by the corresponding sum on the left-hand side of (A.7). We thus have to address the behavior of the two σ-terms in (A.8), and of the first three terms appearing on its right-hand side. We start with the first of these terms, and find by (1.10) and (A.1) that

$$\begin{aligned}|I(F\Omega_jW,F^c\Omega_jW)-I(F\Omega W,F^c\Omega W)| &\leqslant \big|I(F\Omega_jW,F^c\Omega_jW)-I(F\Omega_jW,F^c\Omega W)\big| \\ &\quad+\big|I(F\Omega_jW,F^c\Omega W)-I(F\Omega W,F^c\Omega W)\big| \\ &\leqslant \int_{(\Omega_j\Delta\Omega)\cap W} w_{F^c}+\int_{(\Omega_j\Delta\Omega)\cap W} w_F\,.\end{aligned}$$

By (A.3) and (A.6) we thus find

$$\lim_{j\to+\infty} I(F\Omega_jW,F^c\Omega_jW)=I(F\Omega W,F^c\Omega W)\,. \tag{A.9}$$

Now we claim that if $G_j\subset\Omega_j$ and $G_j\to G$ in $L^1_{\rm loc}(A)$, then

$$\lim_{j\to+\infty} I(G_jW,\Omega_j^c)=I(GW,\Omega^c). \tag{A.10}$$

To this end, we recall that (A.4) implies

$$\lim_{j\to\infty}\int_{U_j} w_{\Omega_j}=0 \qquad \text{whenever} \qquad \lim_{j\to\infty}|U_j|=0\,,$$

see [AFP00, Theorem 1.38]. Since, by definition, (A.4) gives us $\int_{\mathbb{R}^n} u\,w_{\Omega_j}\to\int_{\mathbb{R}^n} u\,w_\Omega$ for every $u\in L^\infty_{\rm loc}(A)$, we conclude that

$$\begin{aligned}|I(G_jW,\Omega_j^c)-I(G_jW,\Omega^c)| &= \left|\int_{\mathbb{R}^n} 1_{G_j\cap W}(x)\,\big(w_{\Omega_j}(x)-w_\Omega(x)\big)\,dx\right| \\ \leqslant \left|\int_{\mathbb{R}^n} 1_{GW}(x)\,\big(w_{\Omega_j}(x)-w_\Omega(x)\big)\,dx\right| &+\int_{(G_j\Delta G)\cap W}\big(w_{\Omega_j}(x)+w_\Omega(x)\big)\,dx \;\longrightarrow 0\end{aligned}$$

as $j\to+\infty$. From this last fact and thanks to $w_\Omega\in L^1(\mathbb{R}^n)$ we have

$$\begin{aligned}&|I(G_jW,\Omega_j^c)-I(GW,\Omega^c)| \\ \leqslant\;& |I(G_jW,\Omega_j^c)-I(G_jW,\Omega^c)|+|I(G_jW,\Omega^c)-I(GW,\Omega^c)| \\ \leqslant\;& |I(G_jW,\Omega_j^c)-I(G_jW,\Omega^c)|+\int_{(G_j\Delta G)W} w_\Omega(x)\,dx \;\longrightarrow 0\end{aligned}$$

as $j\to+\infty$, which proves (A.10). We now exploit (A.10) with $G_j=E_j$ and with $G_j=F\cap\Omega_j$ to take care of the σ-terms in (A.8) and find

$$\lim_{j\to+\infty} I(E_jW,\Omega_j^c)=I(EW,\Omega^c) \qquad \lim_{j\to+\infty} I(F\Omega_jW,\Omega_j^c)=I(F\Omega W,\Omega^c)\,. \tag{A.11}$$

We are left to take care of the second and third terms on the right-hand side of (A.8). To this end we notice that since $w_W,w_{W^c}\in L^1(\mathbb{R}^n)$, if G_j, $L_j\subset\Omega_j$ with $G_j\to G$ and $L_j\to L$ in $L^1_{\rm loc}(A)$, then

$$\begin{aligned}&|I(G_jW,L_jW^c)-I(GW,LW^c)| \\ \leqslant\;& |I(G_jW,L_jW^c)-I(G_jW,LW^c)|+|I(G_jW,LW^c)-I(GW,LW^c)| \\ \leqslant\;& \int_{L_j\Delta L} w_{W^c}(x)\,dx+\int_{G_j\Delta G} w_W(x)\,dx \;\longrightarrow 0\end{aligned}$$

as $j\to+\infty$. By using this observation first with $G_j:=F\cap\Omega_j$ and $L_j:=E_j^c\Omega_j$, and then with $G_j:=F^c\Omega_j$ and $L_j:=E_j$, we finally obtain that

$$\begin{aligned}&\lim_{j\to+\infty} I(F\Omega_jW,E_j^c\Omega_jW^c)=I(F\Omega W,E^c\Omega W^c) \\ &\lim_{j\to+\infty} I(E_j\Omega_jW^c,F^c\Omega_jW)=I(E\Omega_jW^c,F^c\Omega W).\end{aligned} \tag{A.12}$$

We have thus completed the proof of (A.7).

Step two: Let us now assume that $K = K_s^\varepsilon$ and that $\Omega_j = \Omega$ for an open set Ω with C^1-boundary such that $\varrho(\eta, \Omega) > 0$ for every $\eta > 0$, so that the density estimates of Theorem 6.3 hold. Let us pick $x_j \in A \cap \overline{\Omega \cap \partial E_j}$ and $x_j \to x$ for some $x \in A$, then $x \in \partial E$. By (6.7)

$$\frac{1}{C_0} \leqslant \frac{|E_j \cap B_r(x_j)|}{r^n} \leqslant 1 - \frac{1}{C_0}\,,$$

for every $r < \min\{\mathrm{dist}(x_j, \partial A), r_0,\, c_*\varrho(\eta_1, \Omega),\, c_*\,\varepsilon\}$ where $C_0 = C_0(n, s, \sigma, \Lambda)$ and $c_* = c_*(n, s)$. As $j \to \infty$ we find

$$\frac{1}{C_0} \leqslant \frac{|E \cap B_r(x)|}{r^n} \leqslant 1 - \frac{1}{C_0}\,,$$

for every $r < \min\{\mathrm{dist}(x, \partial A), r_0,\, c_*\varrho(\eta_1, \Omega),\, c_*\,\varepsilon\}$, that is $x \in \partial E$. Now let us consider $x \in \overline{\Omega \cap \partial E}$, and assume that for some $\tau > 0$ and for infinitely many values of j we have $B_\tau(x) \cap \partial E_j = \varnothing$. Without loss of generality we may assume that either $|E_j \cap B_\tau(x)| = 0$ or $|E_j^c \cap B_\tau(x)| = 0$ for infinitely many j. In this way, by Fatou's lemma,

$$0 = \lim_{j\to\infty} I_s^\varepsilon(E_j B_\tau(x), E_j^c B_\tau(x)) \geqslant I_s^\varepsilon(E B_\tau(x), E^c B_\tau(x))$$

so that either $|E \cap B_\tau(x)| = 0$ or $|E^c \cap B_\tau(x)| = 0$, against the fact that $x \in \overline{\Omega \cap \partial E}$ and thus the density estimates (6.7) hold for E at x, being E a $(\Lambda, r_0, \sigma, K_s^\varepsilon)$-minimizer. □

As an application of Theorem A.1 we obtain the following compactness result for blow-ups. For the sake of simplicity, we limit our analysis to the case $K = K_s$.

Theorem A.2. *Let $n \geqslant 2$, $s \in (0,1)$, $\sigma \in (-1,1)$, $\Lambda \geqslant 0$, $r_0 > 0$, and let A be an open set. Let E be a $(\Lambda, r_0, \sigma, K_s)$-minimizer in (A, Ω) where Ω is an open set with C^1-boundary in A and with $\varrho_A(\eta, \Omega) > 0$ for every $\eta > 0$, let $x_0 \in A \cap \overline{\Omega \cap \partial E}$, and given a positive vanishing sequence $\{r_j\}_{j\in\mathbb{N}}$, set*

$$E_j = E^{x_0, r_j} = \frac{E - x_0}{r_j} \qquad \Omega_j = \Omega^{x_0, r_j}\,.$$

Then there exists an half-space H with $0 \in \partial H$ and a set $E \subset H$ with $0 \in \partial E$ such that, up to extracting a subsequence, $E_j \to E$ and $\Omega_j \to \Omega$ in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ as $j \to \infty$ and E is a $(0, \infty, \sigma, K_s)$-minimizer in $(\mathbb{R}^n, H)$.

We shall need the following simple lemma.

Lemma A.3. *If $f : \mathbb{R}^n \to \mathbb{R}^n$ iw a bi-Lipschitz diffeomorphism and $\Omega \subset \mathbb{R}^n$, then*

$$w_{f(\Omega)}(x) \leqslant C\, w_\Omega(f^{-1}(x)) \qquad \forall x \in \mathbb{R}^n\,,$$

with a constant C depending only on the Lipschitz constants of f and f^{-1}, and converging to 1 when these Lipschitz constants converge to 1.

Proof of Lemma A.3. Setting $\tilde{y} = f^{-1}(y)$ and $\tilde{x} = f^{-1}(x)$ the area formula gives

$$w_{f(\Omega)}(x) = 1_{f(\Omega)}(x) \int_{f(\Omega^c)} \frac{dy}{|x-y|^{n+s}} = 1_\Omega(\tilde{x}) \int_{\Omega^c} \frac{Jf(\tilde{y})\, d\tilde{y}}{|f(\tilde{x}) - f(\tilde{y})|^{n+s}}\,.$$

We conclude as $\|Jf\|_{L^\infty(\mathbb{R}^n)} \leqslant 1 + C(n)\,|\mathrm{Lip}(f) - 1|$ and $|f(\tilde{x}) - f(\tilde{y})| \geqslant \mathrm{Lip}(f^{-1})\,|\tilde{x} - \tilde{y}|$. □

Proof of Theorem A.2. By regularity of $\partial\Omega$ we have that $\Omega_j \to H$ in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$, while if we set $A_j = (A - x_0)/r_j$, then $A_j \to \mathbb{R}^n$ in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ as $x_0 \in A$. By Remark 5.1, E_j is a $(\Lambda\, r_j^s, r_0/r_j, \sigma, K_s)$-minimizer in (A_j, Ω_j). Let us fix $\tau > 0$ and $R > 0$ and notice that for j

large enough we certainly have that $E_j \cap B_{2R}$ is a $(\tau, \tau^{-1}, \sigma, K_s)$-minimizer in $(B_R, \Omega_j \cap B_{2R})$. Let us also notice that by (6.2) we can certainly assume that

$$\begin{aligned} \varrho_{B_R}(\eta, \Omega_j \cap B_{2R}) &= \varrho_{(B_{Rr_j}(x_0))^{x_0,r_j}}\Big(\eta, (\Omega \cap B_{2\,Rr_j}(x_0))^{x_0,r_j}\Big) \\ &= \frac{1}{r_j}\,\varrho_{B_{Rr_j}(x_0)}\Big(\eta, \Omega \cap B_{2\,Rr_j}(x_0)\Big) \geqslant \frac{\varrho_A(\Omega,\eta)}{r_j} \end{aligned}$$

In particular, for $\eta_1 = \eta_1(n,s,\sigma)$ as in Theorem 6.3 we have

$$\theta := \inf_{j\in\mathbb{N}} \varrho_{B_R}(\eta, \Omega_j \cap B_{2R}) > 0\,.$$

Now we show that, up to extracting a subsequence, $E_j \to E$ in $L^1_{\rm loc}(B_{2R})$ for some set $E \subset H$. Indeed, by (6.6) we have

$$I_s(E_j B_r(x), (E_j B_r(x))^c) \leqslant C_0\, r^{n-s}\,,$$

whenever $B_r(x) \subset B_R$ and $r < \min\{\tau^{-1}, c_*\,\theta\}$ where $C_0 = C_0(n,s,\sigma,\Lambda)$, $c_* = c_*(n,s)$ and $\theta > 0$ is as above. By a covering argument we see that

$$\sup_{j\in\mathbb{N}} I_s(E_j W, E_j^c) < \infty$$

for every $W \subset\subset B_{2R}$. In particular, $E_j \to E$ in $L^1_{\rm loc}(B_{2R})$ for some set E, and the fact that $E \subset H$ follows immediately from $E_j \subset \Omega_j$ and $\Omega_j \to H$ in $L^1_{\rm loc}(\mathbb{R}^n)$.

We claim that $w_{\Omega_j \cap B_{2R}}$ converges weakly in $L^1(B_R)$ to $w_{H\cap B_{2R}}$. Indeed, there exists a bi-Lipschitz family of diffeomorphisms $f_j : \mathbb{R}^n \to \mathbb{R}^n$ such that $f_j(0) = 0$, $f_j(\Omega_j \cap B_{2R}) = H \cap B_{2R}$ and

$$(1-\delta_j)\,|x-y| \leqslant |f_j(x) - f_j(y)| \leqslant (1+\delta_j)\,|x-y| \qquad \forall x,y \in \mathbb{R}^n$$

where $\delta_j \to 0$. In particular, by Lemma A.3,

$$(1 - C(n)\,\delta_j)\, w_{H\cap B_{2R}} \leqslant w_{\Omega_j\cap B_{2R}} \leqslant (1 + C(n)\,\delta_j)\, w_{H\cap B_{2R}} \qquad \text{on } \mathbb{R}^n\,.$$

This proves our claim.

Since $P_s(H \cap B_{2R}) < \infty$ we can apply Theorem A.1 to conclude that, for every $\tau > 0$, E is $(\tau, \tau^{-1}, \sigma, K_s)$-minimizer in $(B_R, H \cap B_{2R})$. By the arbitrariness of τ, E is $(0, \infty, \sigma, K_s)$-minimizer in $(B_R, H \cap B_{2R})$. By the arbitrariness of R, E is $(0, \infty, \sigma, K_s)$-minimizer in $(\mathbb{R}^n, H)$. Again by Theorem A.1, since $0 \in \overline{\Omega_j \cap \partial E_j}$ for every j, it follows that $0 \in \partial E$. □

References

[ADPM11] Luigi Ambrosio, Guido De Philippis, and Luca Martinazzi. Gamma-convergence of nonlocal perimeter functionals. *Manuscripta Math.*, 134(3-4):377–403, 2011.

[AFP00] L. Ambrosio, N. Fusco, and D. Pallara. *Functions of bounded variation and free discontinuity problems.* Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 2000. xviii+434 pp pp.

[Alm76] F. J. Jr. Almgren. Existence and regularity almost everywhere of solutions to elliptic variational problems with constraints. *Mem. Amer. Math. Soc.*, 4(165):viii+199 pp, 1976.

[BBM01] Jean Bourgain, Haim Brezis, and Petru Mironescu. Another look at sobolev spaces. In *in Optimal Control and Partial Differential Equations*, pages 439–455, 2001.

[BFV14] Begoña Barrios, Alessio Figalli, and Enrico Valdinoci. Bootstrap regularity for integro-differential operators and its application to nonlocal minimal surfaces. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 13(3):609–639, 2014.

[CFMN16] Giulio Ciraolo, Alessio Figalli, Francesco Maggi, and Matteo Novaga. Rigidity and sharp stability estimates for hypersurfaces with constant and almost-constant nonlocal mean curvature. *To appear on J. Reine Angew. Math.*, 2016. Preprint arXiv:1503.00653.

[CFSW16] Xavier Cabré, Mouhamed Moustapha Fall, Joan Solà-Morales, and Tobias Weth. Curves and surfaces with constant nonlocal mean curvature: meeting Alexandrov and Delaunay. *To appear on J. Reine Angew. Math.*, 2016. Preprint arXiv:1503.00469.

[CFW16] Xavier Cabre, Mouhamed Moustapha Fall, and Tobias Weth. Delaunay hypersurfaces with constant nonlocal mean curvature. 2016. Preprint arXiv:1602.02623.

[CG10] M. C. Caputo and N. Guillen. Regularity for non-local almost minimal boundaries and applications. 2010. Preprint arXiv:1003.2470.

[CM16] Maria Colombo and Francesco Maggi. Existence and almost everywhere regularity of isoperimetric clusters for fractional perimeters, 2016. Preprint arXiv:1605.05641.

[CRS10] L. Caffarelli, J.-M. Roquejoffre, and O. Savin. Nonlocal minimal surfaces. *Comm. Pure Appl. Math.*, 63(9):1111–1144, 2010.

[CV11] Luis Caffarelli and Enrico Valdinoci. Uniform estimates and limiting arguments for nonlocal minimal surfaces. *Calc. Var. Partial Differential Equations*, 41(1-2):203–240, 2011.

[Dáv02] J. Dávila. On an open question about functions of bounded variation. *Calc. Var. Partial Differential Equations*, 15(4):519–527, 2002.

[DdPDV16] Juan Davila, Manuel del Pino, Serena Dipierro, and Enrico Valdinoci. Nonlocal Delaunay surfaces. *Nonlinear Anal.*, 137:357–380, 2016.

[DdPW13] Juan Davila, Manuel del Pino, and Juncheng Wei. Nonlocal minimal Lawson cones, 2013. Preprint arXiv:1303.0593.

[DdPW14] Juan Dvila, Manuel del Pino, and Juncheng Wei. Nonlocal s-minimal surfaces and Lawson cones, 2014. Preprint arXiv:1402.4173.

[DPM15] G. De Philippis and F. Maggi. Regularity of free boundaries in anisotropic capillarity problems and the validity of Young's law. *Arch. Ration. Mech. Anal.*, 216(2):473–568, 2015.

[FFM$^+$15] A. Figalli, N. Fusco, F. Maggi, V. Millot, and M. Morini. Isoperimetry and stability properties of balls with respect to nonlocal energies. *Comm. Math. Phys.*, 336(1):441–507, 2015.

[FLS08] Rupert L. Frank, Elliott H. Lieb, and Robert Seiringer. Hardy-Lieb-Thirring inequalities for fractional Schrödinger operators. *J. Amer. Math. Soc.*, 21(4):925–950, 2008.

[FV16] Alessio Figalli and Enrico Valdinoci. Regularity and Bernstein-type results for nonlocal minimal surfaces. *To appear on J. Reine Angew. Math.*, 2016. Preprint arXiv:1307.0234.

[Luc87] S. Luckhaus. Partial boundary regularity for the parametric capillarity problem in arbitrary space dimension. 1987. Sonderforschungsbereich 123, Universität Heidelberg.

[Mag12] F. Maggi. *Sets of finite perimeter and geometric variational problems: an introduction to Geometric Measure Theory*, volume 135 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, 2012.

[Maz11] Vladimir Maz'ya. *Sobolev spaces with applications to elliptic partial differential equations*, volume 342 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer, Heidelberg, augmented edition, 2011. ISBN 978-3-642-15563-5. xxviii+866 pp.

[SV13] Ovidiu Savin and Enrico Valdinoci. Regularity of nonlocal minimal cones in dimension 2. *Calc. Var. Partial Differential Equations*, 48(1-2):33–39, 2013.

[Tam84] I. Tamanini. *Regularity results for almost minimal oriented hypersurfaces in* $\mathbb{R}^N$. 1984. available for download at http://cvgmt.sns.it/paper/1807/.

[Tay77] J. E. Taylor. Boundary regularity for solutions to various capillarity and free boundary problems. *Comm. Partial Differential Equations*, 2(4):323–357, 1977.

[Vis91] Augusto Visintin. Generalized coarea formula and fractal sets. *Japan J. Indust. Appl. Math.*, 8(2): 175–201, 1991.

(Francesco Maggi) Abdus Salam International Center for Theoretical Physics, Strada Costiera 11, I-34151, Trieste, Italy. On leave from the University of Texas at Austin

E-mail address: fmaggi@ictp.it

(Enrico Valdinoci) School of Mathematics and Statistics, University of Melbourne, 813 Swanston Street, Parkville VIC 3010, Australia, and Weierstrass Institut für Angewandte Analysis und Stochastik, Mohrenstrasse 39, 10117 Berlin, Germany, and Dipartimento di Matematica, Università degli studi di Milano, Via Saldini 50, 20133 Milan, Italy

E-mail address: enrico@mat.uniroma3.it