QUASI-STATIC HYDRAULIC CRACK GROWTH DRIVEN BY DARCY'S LAW

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Abstract. In the framework of rate independent processes, we present a variational model of quasi-static crack growth in hydraulic fracture. We first introduce the energy functional and study the equilibrium conditions of an unbounded linearly elastic body subject to a remote strain $\epsilon \in \mathbb{R}$ and with a sufficiently regular crack Γ filled by a volume V of incompressible fluid. In particular, we are able to find the pressure p of the fluid inside the crack as a function of Γ, V, and ϵ. Then, we study the problem of quasi-static evolution for our model, imposing that the fluid volume V and the fluid pressure p are related by Darcy's law. We show the existence of such an evolution, and we prove that it satisfies a weak notion of the so-called Griffith's criterion.

Keywords: variational models, free-discontinuity problems, crack propagation, brittle fractures, hydraulic fractures, quasi-static evolution, energy release rate, Griffith's criterion.

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1. Introduction

Hydraulic fracture studies the process of crack growth in rocks driven by the injection of high pressure fluids. The main use of hydraulic fracturing is the extraction of natural gas or oil. In these cases, a fluid at high pressure is pumped into a pre-existing fracture through a wellbore, causing the enlargement of the crack.

In biology, a similar phenomenon has been identified in epithelial tissues. Here, an elastic body with initial cracks (a cell monolayer) is bonded and hydraulically connected to a poroelastic material, typically a hydrogel. The evolution of the fractures is due to the motion of the solvent inside the poroelastic body: when the system is under tension or compression, the fluid experiences a change of pressure and is driven towards the existing cracks at cell-cell junctions.

In [16] the authors develop a 2-dimensional model which captures the main features of the phenomenon mentioned above. The system is assumed to be unbounded and composed by a linearly elastic body $\mathbb{R}^2_+ := \{x = (x_1, x_2) \in \mathbb{R}^2 : x_2 > 0\}$ with an initial crack Γ_0 starting from the origin and lying on the x_2-axis. The elastic material is supposed to be homogeneous, isotropic, and impermeable. The fracture can only move along a straight line and is hydraulically connected to an infinite hydrogel substrate, which contains a solvent modeled as an incompressible fluid.

Since the above model is the starting point of our investigation, let us briefly describe the evolution problem considered in [16]. As usual in the study of hydraulic fracturing, the inertial forces are ignored. Given $T > 0$, the whole system is supposed to be subject to a remote time-dependent strain field with modulus $\epsilon(t) \in \mathbb{R}$, $t \in [0, T]$, which generates a pressure gradient $\nabla p(t)$ in the hydrogel. According to Darcy's law, the exchange of fluid volume $V(t)$ between the fracture and the poroelastic material should be described by the equation $\dot{V}(t) = -\nabla p(t)$, where the dot denotes the time derivative. Motivated by the

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small scale of the problem, the authors approximate the pressure gradient with the finite difference $(p_\infty(t) - p(t))/\ell$, where $p_\infty(t)$ is the fluid pressure generated by $\epsilon(t)$ far from the crack inlet, $p(t)$ is the pressure of the fluid inside the crack, and $\ell > 0$ is a length scale which, for simplicity, we will assume to be equal to 1. Using the asymptotic expansion formulas of linear elastic fracture mechanics in a half plane (see, e.g., [20]), the authors study the qualitative properties of the evolution of the crack. In particular, their analysis is based on the Griffith's criterion (see [12]): whenever the so-called *stress intensity factor* reaches a critical value (the toughness of the material), the fracture can grow. We notice that, in general, the above approach needs some regularity hypotheses on the crack evolution, such as a prescribed and sufficiently smooth crack path.

Starting from the key ideas of [16], the aim of this paper is to approach the problem of quasi-static hydraulic crack growth in the general setting of *rate independent processes* [17], adapting the variational model of brittle fracture [9] to our purposes. In particular, we look for an energetic formulation of the evolution problem based on global stability, energy balance, and, in this case, Darcy's law.

In order to be more precise on the notion of evolution we want to discuss, let us start with a brief presentation of the mathematical setting we are going to consider. The geometry of the problem is similar to the one presented in [16]: an unbounded elastic body filling $\mathbb{R}^2_+$ is adhered and hydraulically connected to an infinite hydrogel substrate. The most relevant difference, which is also one of the fundamental features of the variational model of fracture [9], is that we do not have to assume to know a priori the crack path. Indeed, the behavior of the crack set will be governed by the energy minimization procedure (global stability) described below. Therefore, in comparison with [16], we are able to enlarge the class of admissible fractures, keeping some regularity properties: every crack has to be the graph of a $C^{1,1}$-function starting from the origin and with first and second derivatives uniformly bounded by a constant η (see Definition 3.1 for further details and comments). Hence, the family $\mathcal{C}_\eta$ of admissible cracks depends on a positive parameter η which is fixed once and for all.

For every $t \in [0,T]$, we assume to know the "far" pressure $p_\infty(t)$ of the hydrogel and the remote strain field acting on the system $\epsilon(t)\mathrm{I}$, where $\epsilon(t) \in \mathbb{R}$ and I is the identity matrix of order 2. From a mathematical viewpoint, it is not necessary, as it has been done in [16], to make explicit the dependence of p_∞ on ϵ, so that with our approach we can also describe a more general situation in which an incompressible fluid is injected into a pre-existing crack with an initial pressure p_∞. For technical reasons, we suppose $p_\infty, \epsilon \in C([0,T])$, the space of continuous functions from $[0,T]$ to $\mathbb{R}$.

As in [16], we suppose that the elastic part of the system is homogeneous, isotropic, and impermeable outside of the crack, and behaves accordingly to the rules of linear elasticity, so that it is fully characterized by a constant elasticity tensor $\mathbb{C}$.

The presence of the far strain field $\epsilon(t)\mathrm{I}$ is intended in the following way: at infinity, the elastic body has to accommodate for a displacement of the form $\epsilon(t)\, id$, where id stands for the identity map in $\mathbb{R}^2$. Equivalently, the strain of the body, represented by the symmetric part of the gradient $\mathrm{E}u$ of the displacement $u\colon \mathbb{R}^2_+ \to \mathbb{R}^2$, has to be close to $\epsilon(t)\mathrm{I}$ far from the origin. In our setting, we will require $\mathrm{E}u - \epsilon(t)\mathrm{I}$ to be an L^2-function. This implies that the usual *stored elastic energy*

$$\frac{1}{2}\int_{\mathbb{R}^2_+\setminus\Gamma} \mathbb{C}\mathrm{E}u\cdot\mathrm{E}u\,\mathrm{d}x \tag{1.1}$$

associated to a displacement u and a crack $\Gamma \in \mathcal{C}_\eta$ can not be finite, unless $\epsilon = 0$ (the dot in (1.1) denotes the scalar product between matrices). Since we look for a meaningful energetic formulation of the problem of quasi-static evolution in hydraulic fracture, we have, of course, to deal with a finite energy. Therefore, the stored elastic energy (1.1) is replaced by the *renormalized stored elastic energy*

$$\mathcal{E}^{el}(u,\Gamma,\epsilon(t)) := \frac{1}{2}\int_{\mathbb{R}^2_+\setminus\Gamma} \mathbb{C}(\mathrm{E}u-\epsilon(t)\mathrm{I})\cdot(\mathrm{E}u-\epsilon(t)\mathrm{I})\,\mathrm{d}x\,. \tag{1.2}$$

We refer to Section 3 for a rigorous derivation of (1.2).

According to the pioneering work by Griffith [12] and to the mathematical model developed in [9], the fracture process is governed by the competition between the renormalized stored elastic energy (1.2) and the energy *dissipated* by the crack production, which is assumed to be of the form

$$\kappa\mathcal{H}^1(\Gamma) \qquad \text{for every } \Gamma\in\mathcal{C}_\eta\,, \tag{1.3}$$

where $\mathcal{H}^1$ denotes the 1-dimensional Hausdorff measure in $\mathbb{R}^2$ and κ is a positive constant related to the toughness of the material. Therefore, the *total energy* $\mathcal{E}$ of the system is the sum of (1.2) and (1.3), i.e.,

$$\mathcal{E}(u,\Gamma,\epsilon(t)) := \mathcal{E}^{el}(u,\Gamma,\epsilon(t)) + \kappa\mathcal{H}^1(\Gamma)\,. \tag{1.4}$$

Moreover, since we work in a quasi-static setting, which means that at every instant $t\in[0,T]$ we assume that the system is at the equilibrium, the evolution is given in terms of a *reduced* energy $\mathcal{E}_m(t,\Gamma,V)$, which is obtained from (1.4) by minimizing with respect to u in a certain class of admissible displacements. We refer to Sections 4 and 5 for the precise definition of $\mathcal{E}_m$.

In this mathematical framework, a quasi-static evolution is described by two functions defined on the interval $[0,T]$: the fracture $t\mapsto\Gamma(t)$ and the volume $t\mapsto V(t)$ of the fluid inside the crack, to which corresponds a function $t\mapsto p(t)$ standing for the fluid pressure into the fracture. The notion of evolution is based on the following properties (see Definition 5.1):

- *global stability condition:* for every $t\in[0,T]$ the crack set $\Gamma(t)$ minimizes the reduced energy $\mathcal{E}_m(t,\Gamma,V(t))$ among all admissible fractures $\Gamma\in\mathcal{C}_\eta$ containing $\Gamma(t)$;
- *energy-dissipation balance:* the rate of change of the reduced energy along the evolution (Γ,V) has to be equal to the power expended by the strain $\epsilon(t)$ and by the fluid pressure $p(t)$ acting on the fracture lips;
- *Darcy's law:* the functions $V(\cdot)$, $p_\infty(\cdot)$ and $p(\cdot)$ are related by the equation

$$\dot V(t) = p_\infty(t) - p(t) \qquad \text{for } t\in[0,T]\,, \tag{1.5}$$

  where we have set the length scale $\ell := 1$.

We stress the fact that our definition of quasi-static evolution is variational in nature, since it is based on energy minimization (global stability condition), which can be interpreted, together with the energy-dissipation balance, as a weak notion of the Griffith's criterion. The advantage of this variational approach is that it allows to study the crack growth problem in a "derivative free setting", weakening the *a priori* regularity requirements needed in [16].

We notice that a problem of quasi-static evolution in hydraulic fracture has already been considered in [2] in a 3-dimensional setting. However, the starting point of the problem studied in [2] is completely different, since the authors assumed that the crack path is known a priori and that the volume $V(\cdot)$ of fluid injected into the crack is a datum. On the contrary, in this paper the fracture set is free to choose its path and the volume function $t\mapsto V(t)$ is a result of the evolution process.

The plan of the paper is the following: in Section 2 we define the function spaces used throughout the paper. In Section 3 we make more precise the mathematical framework of the model: we introduce the class $\mathcal{C}_\eta$ of admissible cracks (Definition 3.1) and the energy which will drive the quasi-static evolution problem (Proposition 3.3). In Section 4, we analyze the auxiliary static problem of a linearly elastic body filling $\mathbb{R}^2_+$, subject to a uniform strain field $\epsilon\mathrm{I}$, $\epsilon\in\mathbb{R}$, and with a fracture $\Gamma\in\mathcal{C}_\eta$ filled by a volume $V\in[0,+\infty)$ of incompressible fluid. According to the variational principles of linear elasticity, the static problem is solved by minimizing the total energy of the system among a certain class of admissible displacements. We determine the equilibrium system satisfied by a solution u of the static problem (see Remark 4.4) and make more precise the relation between the strain fields $\mathrm{E}u$ and $\epsilon\mathrm{I}$, showing that they are L^∞-close at infinity (see Remark 4.5). Moreover, in Proposition 4.3 and Remarks 4.8 and 4.10 we determine the value of the pressure $p = p(\Gamma, V, \epsilon)$ of the fluid inside the crack.

Eventually, in Section 5 we focus on the evolution problem. The main result of this section is the existence of a *quasi-static evolution of the hydraulic crack growth problem* satisfying the global stability condition, the energy-dissipation balance, and the Darcy's law (1.5) (see Definition 5.1 and Theorem 5.2). The proof of this result relies on a time discretization procedure introduced in [9] and frequently used in the study of rate independent processes [17].

Finally, in Section 6 we discuss some properties of a quasi-static evolution under an additional assumption of regularity of the crack set $\Gamma(t)$. More precisely, in Theorem 6.8 we show that if $\Gamma(t)$ is a $C^{2,1}$-curve for every $t \in [0,T]$, then a Griffith's criterion is satisfied also in our context: the *energy release rate*, i.e., the derivative of the renormalized stored elastic energy with respect to the crack length, has to be always less than or equal to the constant κ defined in (1.3), and the equality is satisfied whenever the crack tip moves with positive velocity.

2. Preliminaries and notation

For every set E, the symbol $\mathbf{1}_E$ stands for the characteristic function of E, i.e., the function defined by $\mathbf{1}_E(x) = 1$ for $x \in E$ and $\mathbf{1}_E(x) = 0$ for $x \notin E$. For every $\delta > 0$, we set

$$\mathcal{I}_\delta(E) := \{x \in \mathbb{R}^2 : d(x, E) < \delta\}, \tag{2.1}$$

where $d(\cdot, E)$ is the usual distance function from the set E.

For every $r > 0$ and every $x \in \mathbb{R}^2$, we denote by $\mathrm{B}_r(x)$ the open ball of radius r and center x, and we set $\mathrm{B}_r^+(x) := \mathrm{B}_r(x) \cap \mathbb{R}^2_+$, where $\mathbb{R}^2_+ := \{(x_1, x_2) \in \mathbb{R}^2 : x_2 > 0\}$. When $x = (0,0)$, we use the shorter notation B_r and B_r^+. We also define $\Sigma := \partial\mathbb{R}^2_+ = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 = 0\}$, and we denote by ν_Σ the unit vector $(0,1)$ normal to Σ.

We say that an open subset Ω of $\mathbb{R}^2$ has Lipschitz boundary if for every $x \in \partial\Omega$ there exist an open neighborhood U of x, $\delta > 0$, and a Lipschitz function $h_x \colon \mathbb{R} \to \mathbb{R}$ such that, up to a change of coordinate system,

$$\Omega \cap U = \{y \in U : |y_1 - x_1| < \delta,\, y_2 < h_x(y_1)\}.$$

We say that $\partial\Omega$ has Lipschitz constant $L > 0$ if h_x has Lipschitz constant smaller than L for every $x \in \partial\Omega$.

Throughout the paper $\mathcal{H}^1$ stands for the 1-dimensional Hausdorff measure in $\mathbb{R}^2$ and $\mathcal{K}$ denotes the set of all compact subsets of $\mathbb{R}^2$.

Given $K_1, K_2 \in \mathcal{K}$, the Hausdorff distance $d_H(K_1, K_2)$ between K_1 and K_2 is defined by

$$d_H(K_1, K_2) := \max\left\{\max_{x \in K_1} d(x, K_2), \max_{x \in K_2} d(x, K_1)\right\}.$$

We say that $K_h \to K$ in the Hausdorff metric if $d_H(K_h, K) \to 0$. We refer to [19] for the main properties of the Hausdorff metric.

We say that a function $K \colon [0,T] \to \mathcal{K}$ is increasing if $K(s) \subseteq K(t)$ for every $0 \le s \le t \le T$. We recall two results concerning increasing set functions which can be found for instance in [6, Section 6].

Theorem 2.1. *Let $H \in \mathcal{K}$ and let $K \colon [0,T] \to \mathcal{K}$ be an increasing set function such that $K(t) \subseteq H$ for every $t \in [0,T]$. Let $K^- \colon (0,T] \to \mathcal{K}$ and $K^+ \colon [0,T) \to \mathcal{K}$ be the functions* *defined by*

$$K^-(t) := \overline{\bigcup_{s<t} K(s)} \quad \text{for } 0 < t \le T,$$
$$K^+(t) := \bigcap_{s<t} K(s) \quad \text{fot } 0 \le t < T.$$

Then

$$K^-(t) \subseteq K(t) \subseteq K^+(t) \quad \text{for } 0 < t < T.$$

Let Θ be the set of points $t \in (0,T)$ such that $K^+(t) = K^-(t)$. Then $[0,T] \setminus \Theta$ is at most countable and $K(t_h) \to K(t)$ in the Hausdorff metric for every $t \in \Theta$ and every sequence t_h in $[0,T]$ converging to t.

Theorem 2.2. *Let K_h be a sequence of increasing set functions from $[0,T]$ to $\mathcal{K}$. Assume that there exists $H \in \mathcal{K}$ such that $K_h(t) \subseteq H$ for every $t \in [0,T]$ and every $h \in \mathbb{N}$. Then there exist a subsequence, still denoted by K_h, and an increasing set function $K\colon [0,T] \to \mathcal{K}$ such that $K_h(t) \to K(t)$ in the Hausdorff metric for every $t \in [0,T]$.*

We denote by $\mathbb{M}^2$ the space of square matrices of order 2 with real coefficients, and by $\mathbb{M}^2_{sym}$, $\mathbb{M}^2_{skw}$ the subspaces of $\mathbb{M}^2$ of symmetric and skew-symmetric matrices, respectively. For every $\mathrm{F} \in \mathbb{M}^2$ and every $i,j = 1,2$, F_{ij} stands for the (i,j)-element of F. We denote by $cof\,\mathrm{F}$ the cofactor matrix of F. Finally, the scalar product between matrices is defined by

$$\mathrm{F} \cdot \mathrm{G} := tr(\mathrm{F}\mathrm{G}^T) \qquad \text{for every } \mathrm{F}, \mathrm{G} \in \mathbb{M}^2,$$

where the symbol tr stands for the trace of a matrix and G^T is the transpose matrix of G.

Let us introduce the main function spaces used in this paper. For every $E \subseteq \mathbb{R}^2$ and every $1 \le p < +\infty$, the space $L^p(E;\mathbb{R}^2)$ is defined as the set of functions $u\colon E \to \mathbb{R}^2$ measurable and p-integrable. For every function $u \in L^p(E;\mathbb{R}^2)$, u_i indicates the i-th component of u. As before, $L^p(E;\mathbb{M}^2)$ is the set of functions $u\colon E \to \mathbb{M}^2$ measurable and p-integrable. In both cases, we denote by $\|\cdot\|_{p,E}$ the L^p-norm on E.

For every open set $\Omega \subseteq \mathbb{R}^2$ and every $1 \le p < +\infty$, $W^{1,p}(\Omega;\mathbb{R}^2)$ is the set of functions $u \in L^p(\Omega;\mathbb{R}^2)$ whose gradient ∇u belongs to $L^p(\Omega;\mathbb{M}^2)$. The space $W^{1,p}(\Omega;\mathbb{R}^2)$ is a Banach space equipped with the norm $\|u\|_{W^{1,p}(\Omega)} := \|u\|_{p,\Omega} + \|\nabla u\|_{p,\Omega}$. In the case $p = 2$, the space $W^{1,2}(\Omega;\mathbb{R}^2)$ will be denoted by $H^1(\Omega;\mathbb{R}^2)$. In particular, $H^1(\Omega;\mathbb{R}^2)$ is a Hilbert space, and we denote its norm by $\|\cdot\|_{H^1(\Omega)}$.

We say that $u \in L^p_{loc}(\Omega;\mathbb{R}^2)$ (resp. $u \in W^{1,p}_{loc}(\Omega;\mathbb{R}^2)$) if $u \in L^p(\Omega';\mathbb{R}^2)$ (resp. $u \in W^{1,p}(\Omega';\mathbb{R}^2)$) for every $\Omega' \subset\subset \Omega$.

Following [21] and [3], we define the space

$$LD^2(\Omega;\mathbb{R}^2) := \{u \in L^2_{loc}(\Omega;\mathbb{R}^2) : \mathrm{E}u \in L^2(\Omega;\mathbb{M}^2_{sym})\}, \tag{2.2}$$

where $\mathrm{E}u$ stands for the symmetric gradient of u, namely, $\mathrm{E}u = \frac{1}{2}(\nabla u + \nabla u^T)$. For every $i,j = 1,2$, $\mathrm{E}_{ij}u$ stands for the (i,j)-component of $\mathrm{E}u$.

We now recall the relationship between the space $LD^2(\Omega;\mathbb{R}^2)$ and the space $H^1(\Omega;\mathbb{R}^2)$.

Proposition 2.3. *Let Ω be a bounded open subset of $\mathbb{R}^2$ with Lipschitz boundary. Then $LD^2(\Omega;\mathbb{R}^2) = H^1(\Omega;\mathbb{R}^2)$. In particular, there exists a constant $C = C(\Omega)$ such that for every $u \in LD^2(\Omega;\mathbb{R}^2)$*

$$\int_\Omega |\nabla u|^2 \,\mathrm{d}x \le C\Big(\int_\Omega |u|^2 \,\mathrm{d}x + \int_\Omega |\mathrm{E}u|^2 \,\mathrm{d}x\Big). \tag{2.3}$$

Moreover, if $E \subset\subset \Omega$ is open, $E \neq \emptyset$, then there exists $C' := C'(\Omega, E)$ such that

$$\int_\Omega |\nabla u|^2 \,\mathrm{d}x \le C' \int_\Omega |\mathrm{E}u|^2 \,\mathrm{d}x \tag{2.4}$$

for every $u \in LD^2(\Omega;\mathbb{R}^2)$ with

$$\int_E (\nabla u - \nabla u^T) \,\mathrm{d}x = 0.$$

Proof. See [7, Section 4] and [5, Appendix]. □

Since in the space $LD^2(\Omega;\mathbb{R}^2)$ we can control only the symmetric part of the gradient, we have that $\|\mathrm{E}u\|_{2,\Omega}$ is not a norm. Indeed, if we define

$$\mathcal{R} := \{v\colon \mathbb{R}^2 \to \mathbb{R}^2 : v(x) = Ax + b \text{ with } b \in \mathbb{R}^2,\, A \in \mathbb{M}^2_{skw}\}\,,$$

the set of rigid motion in $\mathbb{R}^2$, we have that $\mathcal{R} \subset LD^2(\Omega;\mathbb{R}^2)$ and $\|\mathrm{E}u\|_{2,\Omega} = 0$ for every $u \in \mathcal{R}$.

In Sections 3 and 6 we shall use the following subspace of $LD^2(\Omega;\mathbb{R}^2)$ on which $\|\mathrm{E}u\|_{2,\Omega}$ is a norm. Let Ω be an open subset of $\mathbb{R}^2_+$ such that $\mathcal{H}^1(\partial\Omega \cap \Sigma) > 0$. For every open set $E \subset\subset \Omega$ we define

$$LD^2_E(\Omega;\mathbb{R}^2) := \Big\{u \in LD^2(\Omega;\mathbb{R}^2) : \int_E u_1 \,\mathrm{d}x = 0 \text{ and } u_2 = 0 \text{ on } \Sigma\Big\}. \tag{2.5}$$

It is easy to see that $LD^2_E(\Omega;\mathbb{R}^2) \cap \mathcal{R} = \{0\}$.

In the following proposition, we prove that $\|\mathrm{E}u\|_{2,\Omega}$ is a norm on $LD^2_E(\Omega;\mathbb{R}^2)$.

Proposition 2.4. *Let Ω be an open bounded subset of $\mathbb{R}^2_+$ with Lipschitz boundary, and let $E \subset\subset \Omega$, $E \neq \emptyset$, be open. Assume that $\mathcal{H}^1(\partial\Omega\cap\Sigma) > 0$. Then there exists $C = C(\Omega, E)$ such that*

$$\|u\|_{H^1(\Omega)} \leq C\|\mathrm{E}u\|_{2,\Omega} \qquad \text{for every } u \in LD^2_E(\Omega;\mathbb{R}^2)\,. \tag{2.6}$$

Proof. By Proposition 2.3 we have that $LD^2_E(\Omega;\mathbb{R}^2) \subseteq H^1(\Omega;\mathbb{R}^2)$. To prove (2.6), in view of (2.3) it is enough to show that

$$\|u\|_{2,\Omega} \leq C\|\mathrm{E}u\|_{2,\Omega}$$

for some positive constant C.

Let us assume by contradiction that there exists a sequence u_n in $LD^2_E(\Omega;\mathbb{R}^2)$ such that $\|u_n\|_{2,\Omega} > n\|\mathrm{E}u_n\|_{2,\Omega}$. It is not restrictive to assume that $\|u_n\|_{2,\Omega} = 1$ for every n. From (2.3) we deduce that u_n is bounded in $H^1(\Omega;\mathbb{R}^2)$. Therefore there exists $u \in LD^2_E(\Omega;\mathbb{R}^2)$ such that, up to a subsequence, u_n converges to u weakly in $H^1(\Omega;\mathbb{R}^2)$ and strongly in $L^2(\Omega;\mathbb{R}^2)$. In particular $\|u\|_{2,\Omega} = 1$.

From the strong convergence of $\mathrm{E}u_n$ to 0 in $L^2(\Omega;\mathbb{M}^2_{sym})$, we deduce that $u \in \mathcal{R}$, and hence $u = 0$, which is a contradiction. □

Remark 2.5. Let Ω and E be as in Proposition 2.4. For every $\lambda > 0$ let us set $\Omega_\lambda := \lambda\Omega$ and $E_\lambda := \lambda E$. Then, for every $u \in LD^2_{E_\lambda}(\Omega_\lambda;\mathbb{R}^2)$ we have

$$\|u\|_{2,\Omega_\lambda} \leq C\lambda\|\mathrm{E}u\|_{2,\Omega_\lambda}\,,$$

where $C = C(\Omega; E)$ is the constant found in (2.6).

As a straightforward consequence of Proposition 2.4 we have the following corollary.

Corollary 2.6. *Let Ω be an open subset of $\mathbb{R}^2_+$ with $\mathcal{H}^1(\partial\Omega\cap\Sigma) > 0$. Let $E \subset\subset \Omega$, $E \neq \emptyset$, be open. Then the space $LD^2_E(\Omega;\mathbb{R}^2)$ is a Hilbert space equipped with the norm $\|\mathrm{E}u\|_{2,\Omega}$.*

Finally, we state a stability property of the Korn's inequality shown in Proposition 2.4.

Proposition 2.7. *Let Ω_n, Ω_∞ be bounded open subsets of $\mathbb{R}^2$ with Lipschitz boundaries. Assume that $\mathcal{H}^1(\partial\Omega_n \cap \Sigma) > 0$, $\mathcal{H}^1(\partial\Omega_\infty \cap \Sigma) > 0$, $\overline{\Omega}_n \to \overline{\Omega}_\infty$ in the Hausdorff metric and that $\partial\Omega_n$, $\partial\Omega_\infty$ have Lipschitz constant $L > 0$. Let, in addition, $E \neq \emptyset$ be an open subset of $\bigcap \Omega_n$. Then, there exists $C = C(E)$ such that, for n sufficiently large, (2.6) holds for every $u \in LD^2_E(\Omega_n;\mathbb{R}^2)$.*

Proof. The proof can be carried out following the steps of [7, Theorem 4.2] using the results of Proposition 2.3. □

For simplicity of notation, from now on we will use the shorter symbols $L^p(\Omega)$, $W^{1,p}(\Omega)$, $H^1(\Omega)$, $LD^2(\Omega)$, and $LD^2_E(\Omega)$, instead of $L^p(\Omega;\mathbb{R}^2)$, $L^p(\Omega;\mathbb{M}^2)$, $W^{1,p}(\Omega;\mathbb{R}^2)$, $H^1(\Omega;\mathbb{R}^2)$, $LD^2(\Omega;\mathbb{R}^2)$, and $LD^2_E(\Omega;\mathbb{R}^2)$. Moreover, in view of Corollary 2.6, we will always consider $LD^2_E(\Omega;\mathbb{R}^2)$ endowed with the norm $\|\mathrm{E}u\|_{2,\Omega}$.

3. Mathematical model

We describe the mathematical framework we will consider in our model inspired by [16], to which we refer for more details on the physical interpretation.

To fix the simplest possible geometry, we consider a system made of an elastic body filling the whole $\mathbb{R}^2_+$ which is adhered to a poroelastic body occupying $\mathbb{R}^2\setminus\mathbb{R}^2_+$.

As we have said in the Introduction, we assume that the incompressible fluid inside the poroelastic material is subject to a pressure p_∞ far from the crack inlet. For technical reasons, we will assume $p_\infty\in C([0,T])$.

Let us concentrate on the main features of the elastic part of the system. We assume that it presents a regular enough initial crack Γ_0. More precisely, we suppose that there exists a $C^{1,1}$-function $\gamma_0\colon[0,a_{\Gamma_0}]\to\mathbb{R}$, $a_{\Gamma_0}>0$, defined on the x_2-axis and such that $\gamma_0(0)=0$, $\gamma_0'(0)\neq 0$, and

$$\Gamma_0=\mathit{graph}(\gamma_0)=\{(\gamma_0(x_2),x_2):\,x_2\in[0,a_{\Gamma_0}]\}\,.$$

In particular, $\Gamma_0\subseteq\overline{\mathbb{R}^2_+}$, $0<\mathcal{H}^1(\Gamma_0)<+\infty$, $\Gamma_0\cap\Sigma=\{(0,0)\}$, and $|\nu_{\Gamma_0}\cdot\nu_\Sigma|\neq 1$ at the origin, where ν_{Γ_0} denotes the unit normal to Γ_0 and the dot stands for the usual scalar product in $\mathbb{R}^2$. We refer to Remark 5.3 for further comments on Γ_0.

In our model, especially in the evolution problem studied in Section 5, we do not suppose to know a priori the crack path, which will be a result of an energy minimization procedure (see Definition 5.1), but we keep a technical regularity assumption on the fracture set, which is specified in the following definition of the class of admissible cracks.

Definition 3.1. Let $\eta>0$. We define $\mathcal{C}_\eta$ to be the set of all closed curves Γ of class $C^{1,1}$ in $\overline{\mathbb{R}^2_+}$ such that the following properties hold:

(*a*) $\Gamma\supseteq\Gamma_0$ and $\Gamma\setminus\Gamma_0\subset\subset\mathbb{R}^2_+$;

(*b*) there exist $a_\Gamma>0$ and $\gamma\in C^{1,1}([0,a_\Gamma])$ such that $\|\gamma'\|_{\infty,[0,a_\Gamma]},\|\gamma''\|_{\infty,[0,a_\Gamma]}\leq\eta$ and $\Gamma=\mathit{graph}(\gamma)=\{(\gamma(x_2),x_2):\,x_2\in[0,a_\Gamma]\}$.

By definition of Γ_0, we can always find a sufficiently large η so that $\|\gamma_0'\|_{\infty,[0,a_{\Gamma_0}]}\leq\eta$ and $\|\gamma_0''\|_{\infty,[0,a_{\Gamma_0}]}\leq\eta$. Clearly, the requirements of Definition 3.1 ensure that for every $\Gamma\in\mathcal{C}_\eta$ there are no self-intersections. Moreover, for every $\Gamma\in\mathcal{C}_\eta$ it is convenient to fix an orientation and a unit normal vector ν_Γ to Γ.

We show a compactness property of the class $\mathcal{C}_\eta$ with respect to the Hausdorff convergence of sets.

Proposition 3.2. *Let Γ_k be a sequence in $\mathcal{C}_\eta$ such that $\mathcal{H}^1(\Gamma_k)$ is uniformly bounded with respect to k. Then there exists $\Gamma_\infty\in\mathcal{C}_\eta$ such that, up to a subsequence, $\Gamma_k\to\Gamma_\infty$ in the Hausdorff metric. Moreover, $\mathcal{H}^1(\Gamma_k)\to\mathcal{H}^1(\Gamma_\infty)$.*

Proof. Let $\Gamma_k\in\mathcal{C}_\eta$ be as in the statement of the proposition and let $a_{\Gamma_k}>0$ and $\gamma_k\in C^{1,1}([0,a_{\Gamma_k}])$ be as in Definition 3.1. Since $\mathcal{H}^1(\Gamma_k)$ is bounded, we have that the sequence a_{Γ_k} is bounded in $\mathbb{R}$ and

$$\sup_k\|\gamma_k\|_{W^{2,\infty}([0,a_{\Gamma_k}])}<+\infty\,. \tag{3.1}$$

Therefore, we may assume that, up to a subsequence, $a_{\Gamma_k}\to a$. Moreover, we may rescale γ_k on the interval $[0,a]$ by

$$\tilde\gamma_k(x_2):=\gamma_k\Big(\frac{x_2a_{\Gamma_k}}{a}\Big)\quad\text{for }x_2\in[0,a]\,,$$

so that

$$\Gamma_k = \Big\{ \Big(\tilde{\gamma}_k(x_2), \frac{x_2 a_{\Gamma_k}}{a} \Big) : x_2 \in [0,a] \Big\} .$$

By (3.1) we have that, up to a subsequence, $\tilde{\gamma}_k$ weakly*-converges in $W^{2,\infty}([0,a])$ to some γ. Let us set $\Gamma := \mathit{graph}(\gamma)$. It is clear from the convergence of a_{Γ_k} to a and of $\tilde{\gamma}_k$ to γ that $\Gamma \in \mathcal{C}_\eta$ and that Γ_k converges to Γ in the Hausdorff metric. Moreover, since $\tilde{\gamma}_k'$ converges to γ' uniformly in the interval $[0,a]$, we get that

$$\lim_k \mathcal{H}^1(\Gamma_k) = \lim_k \int_0^a \sqrt{\Big(\frac{a_{\Gamma_k}}{a}\Big)^2 + \tilde{\gamma}_k'^2(y)}\,\mathrm{d}y = \int_0^a \sqrt{1+\gamma'^2(y)}\,\mathrm{d}y = \mathcal{H}^1(\Gamma) ,$$

and this concludes the proof of the proposition. □

We assume that outside the crack the elastic body is isotropic, homogeneous, and impermeable. Therefore, the behavior of the elastic body is fully characterized by the elasticity tensor $\mathbb{C}\colon \mathbb{M}^2_{sym} \to \mathbb{M}^2_{sym}$ defined by

$$\mathbb{C}\mathrm{F} := \lambda tr(\mathrm{F})\mathrm{I} + 2\mu \mathrm{F} \qquad \text{for every } \mathrm{F} \in \mathbb{M}^2_{sym} , \tag{3.2}$$

λ and μ being the Lamé coefficients of the body. As usual, we assume that $\mathbb{C}\mathrm{F} = 0$ for every $\mathrm{F} \in \mathbb{M}^2_{skw}$ and that $\mathbb{C}$ is positive definite, that is, there exist two constants $0 < \alpha \le \beta < +\infty$ such that

$$\alpha|\mathrm{F}|^2 \le \mathbb{C}\mathrm{F}\cdot\mathrm{F} \le \beta|\mathrm{F}|^2 \qquad \text{for every } \mathrm{F} \in \mathbb{M}^2_{sym} . \tag{3.3}$$

Our aim is now to define the set of admissible displacements and the energy of the elastic body $\mathbb{R}^2_+$ subject to a *remote strain field* $\epsilon\mathrm{I}$, $\epsilon \in \mathbb{R}$, and with a crack $\Gamma \in \mathcal{C}_\eta$ filled by a volume $V \in [0,+\infty)$ of incompressible fluid.

Let us start with a simpler case in which we do not consider the volume of fluid inside the crack. As we have already mentioned in the Introduction, the action of the strain $\epsilon\mathrm{I}$ is intended in the following way: the displacement $u\colon \mathbb{R}^2_+ \to \mathbb{R}^2$ of the elastic body has to induce a strain field $\mathrm{E}u$ which is close to $\epsilon\mathrm{I}$ at infinity. The previous requirement is translated into the condition $u - \epsilon\, id \in LD^2(\mathbb{R}^2_+ \setminus \Gamma)$, where $LD^2(\mathbb{R}^2_+\setminus\Gamma)$ is defined in (2.2). For what follows, we notice that, for every open bounded subset of $\mathbb{R}^2_+$ with Lipschitz boundary and every $\Gamma \in \mathcal{C}_\eta$ with $\Gamma \setminus \Gamma_0 \subset\subset \Omega$, Propositions 2.3-2.7 are still valid in $LD^2(\Omega\setminus\Gamma)$.

In view of the previous comments, for every $\epsilon \in \mathbb{R}$ and every $\Gamma \in \mathcal{C}_\eta$ we introduce the set of admissible displacements (without volume constraint)

$$\mathcal{AD}(\Gamma,\epsilon) := \{u\colon \mathbb{R}^2_+ \to \mathbb{R}^2 : u - \epsilon\, id \in LD^2(\mathbb{R}^2_+\setminus\Gamma),\, u_2 = 0 \text{ on } \Sigma,\, [u]\cdot\nu_\Gamma \ge 0 \text{ on } \Gamma\}, \tag{3.4}$$

where $[u]$ stands for the jump of u through Γ, that is, $[u] := u^+ - u^-$, with u^+ and u^- denoting the traces of u on the two sides Γ^+ and Γ^- of Γ, defined according to the orientation of ν_Γ.

Let us give some comments on $\mathcal{AD}(\Gamma,\epsilon)$. The choice of the space $LD^2(\mathbb{R}^2_+\setminus\Gamma)$ has some important consequences. First of all, it says that every admissible displacement is Sobolev regular (see Proposition 2.3) outside of the curve Γ, hence the crack is actually contained in Γ. Furthermore, the fact that $\mathrm{E}u - \epsilon\mathrm{I} \in L^2(\mathbb{R}^2_+\setminus\Gamma)$ means, in a suitable weak sense, that $\mathrm{E}u$ has to coincide with the uniform strain $\epsilon\mathrm{I}$ at infinity. We refer to Remark 4.5 for further comments on the relation between $\mathrm{E}u$ and $\epsilon\mathrm{I}$. In what follows, we will assume, when needed, that $\mathrm{E}u - \epsilon\mathrm{I}$ is a function in $L^2(\mathbb{R}^2_+)$. For instance, this is true if we extend it by zero on Γ.

The boundary condition $u_2 = 0$ on Σ reflects the fact that, according to the model studied in [16], the elastic body is adhered to the poroelastic substrate. Finally, the inequality in formula (3.4), which is assumed to hold $\mathcal{H}^1$-a.e. in Γ, takes into account the non-interpenetration condition: the fracture lips can not cross each other.

Let us now define the elastic energy of the body for a displacement $u \in \mathcal{AD}(\Gamma,\epsilon)$. Due to the summability hypothesis made on $\mathrm{E}u-\epsilon\mathrm{I}$, we get that $\mathrm{E}u \notin L^2(\mathbb{R}^2_+)$ whenever $\epsilon \neq 0$. Hence, from (3.3) we deduce that the usual *stored elastic energy*

$$\frac{1}{2}\int_{\mathbb{R}^2_+\setminus\Gamma} \mathbb{C}\mathrm{E}u\cdot\mathrm{E}u\,\mathrm{d}x$$

is not finite. Therefore, in order formulate our problem in the setting of rate independent processes [17], for every displacement $u \in \mathcal{AD}(\Gamma,\epsilon)$ we have to define the *renormalized* energy

$$\mathcal{F}^{el}(u,\Gamma,\epsilon) := \frac{1}{2}\int_{\mathbb{R}^2_+\setminus\Gamma} \mathbb{C}(\mathrm{E}u-\epsilon\mathrm{I})\cdot(\mathrm{E}u-\epsilon\mathrm{I})\,\mathrm{d}x - \int_\Gamma \boldsymbol{\sigma}(\epsilon)\nu_\Gamma\cdot[u]\,\mathrm{d}\mathcal{H}^1 \tag{3.5}$$

where $\boldsymbol{\sigma}(\epsilon) := \epsilon\mathbb{C}\mathrm{I}$ is the far stress field associated to ϵ. For simplicity, we set also

$$\sigma(\epsilon) := (3\lambda+2\mu)\epsilon\,, \tag{3.6}$$

so that, by (3.2), $\boldsymbol{\sigma}(\epsilon) = \sigma(\epsilon)\mathrm{I}$ and (3.5) becomes

$$\mathcal{F}^{el}(u,\Gamma,\epsilon) = \frac{1}{2}\int_{\mathbb{R}^2_+\setminus\Gamma} \mathbb{C}(\mathrm{E}u-\epsilon\mathrm{I})\cdot(\mathrm{E}u-\epsilon\mathrm{I})\,\mathrm{d}x - \sigma(\epsilon)\int_\Gamma [u]\cdot\nu_\Gamma\,\mathrm{d}\mathcal{H}^1\,. \tag{3.7}$$

Besides $\mathcal{F}^{el}$, it is useful to introduce also the *renormalized stored elastic energy*

$$\mathcal{E}^{el}(u,\Gamma,\epsilon) := \frac{1}{2}\int_{\mathbb{R}^2_+\setminus\Gamma} \mathbb{C}(\mathrm{E}u-\epsilon\mathrm{I})\cdot(\mathrm{E}u-\epsilon\mathrm{I})\,\mathrm{d}x\,. \tag{3.8}$$

The definition of the renormalized energy given in (3.5) is also motivated by the fact that $\mathcal{F}^{el}(u,\Gamma,\epsilon)$ can be obtained as limit of the stored elastic energy on bounded domains which tend to $\mathbb{R}^2_+$, as we show below. Let us consider $R>0$ such that $\Gamma \subseteq \overline{\mathrm{B}}^+_R$ and let us set

$$\mathcal{E}^{el}_R(u,\Gamma) := \frac{1}{2}\int_{\mathrm{B}^+_R\setminus\Gamma} \mathbb{C}\mathrm{E}u\cdot\mathrm{E}u\,\mathrm{d}x$$

for every displacement $u \in \mathcal{AD}_R(\Gamma,\epsilon)$, where

$$\mathcal{AD}_R(\Gamma,\epsilon) := \{u \in H^1(\mathrm{B}^+_R\setminus\Gamma) : u = \epsilon\, id \text{ on } \partial\mathrm{B}^+_R\setminus\Sigma,\ u_2 = 0 \text{ on } \partial\mathrm{B}^+_R\cap\Sigma,\ [u]\cdot\nu_\Gamma \geq 0 \text{ on } \Gamma\}\,.$$

We notice that the Dirichlet condition $u = \epsilon\, id$ on $\partial\mathrm{B}^+_R\setminus\Sigma$ corresponds, in the bounded case, to the condition $u - \epsilon\, id \in LD^2(\mathbb{R}^2_+\setminus\Gamma)$ in (3.4). Indeed, if we extend $u \in \mathcal{AD}_R(\Gamma,\epsilon)$ by $\epsilon\, id$ in $\mathbb{R}^2_+\setminus\mathrm{B}^+_R$, it is straightforward to see that we obtain an element of $\mathcal{AD}(\Gamma,\epsilon)$. In what follows, we will denote by $\bar{u}$ this extension.

An integration by parts shows that for every $u \in \mathcal{AD}_R(\Gamma,\epsilon)$ the following equality holds:

$$\mathcal{E}^{el}_R(u,\Gamma) - \mathcal{E}^{el}_R(\epsilon\, id,\Gamma) = \mathcal{E}^{el}_R(u-\epsilon\, id,\Gamma) - \int_\Gamma \boldsymbol{\sigma}(\epsilon)\nu_\Gamma\cdot[u]\,\mathrm{d}\mathcal{H}^1 =: \mathcal{F}^{el}_R(u,\Gamma)\,. \tag{3.9}$$

The aim of the following proposition is to pass to the limit in (3.9) as $R\to+\infty$, recovering the renormalized energy defined in (3.5) and (3.7).

Proposition 3.3. *Let $\Gamma \in \mathcal{C}_\eta$ and $\epsilon \in \mathbb{R}$. Then the following facts hold:*

(*a*) *for every sequence u_R in $\mathcal{AD}_R(\Gamma,\epsilon)$ such that*

$$\sup_{R>0} \mathcal{F}^{el}_R(u_R,\Gamma) < +\infty \tag{3.10}$$

there exists $u \in \mathcal{AD}(\Gamma,\epsilon)$ such that, up to a subsequence, $\mathrm{E}\bar{u}_R - \epsilon\mathrm{I} \rightharpoonup \mathrm{E}u - \epsilon\mathrm{I}$ weakly in $L^2(\mathbb{R}^2_+)$ and

$$\mathcal{F}^{el}(u,\Gamma,\epsilon) \leq \liminf_{R\to+\infty} \mathcal{F}^{el}_R(u_R - \epsilon\, id,\Gamma)\,;$$

(*b*) *for every* $u \in \mathcal{AD}(\Gamma,\epsilon)$ *there exists a sequence* v_R *in* $\mathcal{AD}_R(\Gamma,\epsilon)$ *such that* $\mathrm{E}\bar{v}_R - \epsilon\mathrm{I} \to \mathrm{E}u - \epsilon\mathrm{I}$ *in* $L^2(\mathbb{R}^2_+)$ *and*

$$\mathcal{F}^{el}(u,\Gamma,\epsilon) = \lim_{R\to+\infty} \mathcal{F}^{el}_R(v_R,\Gamma)\,.$$

Proof. Let us prove (a). Let Γ, ϵ, and u_R be as in the statement of the proposition. It is easy to see from (3.7) and (3.9) that $\mathcal{F}^{el}_R(u_R,\Gamma) = \mathcal{F}^{el}(\bar{u}_R,\Gamma,\epsilon)$ for every $R>0$ such that $\Gamma\setminus\Gamma_0 \subset\subset \mathrm{B}^+_R$.

Let $E \subset\subset \mathbb{R}^2_+\setminus\Gamma$, $E\neq\emptyset$, be an open bounded set. For every $R>0$ such that $E\subset\subset \mathrm{B}^+_R$, there exists a horizontal translation t_R such that $\bar{u}_R - \epsilon\, id - t_R \in LD^2_E(\mathbb{R}^2_+\setminus\Gamma)$. In view of Proposition 2.4, for $r>0$ sufficiently large there exists a positive constant C_r satisfying

$$\|\bar{u}_R - \epsilon\, id - t_R\|_{H^1(\mathrm{B}^+_r\setminus\Gamma)} \leq C_r \|\mathrm{E}\bar{u}_R - \epsilon\mathrm{I}\|_{2,\mathbb{R}^2_+} \qquad \text{for every } R>0 \text{ with } E\subset\subset \mathrm{B}^+_R. \tag{3.11}$$

By (3.3) and (3.10) we have that $\mathrm{E}\bar{u}_R - \epsilon\mathrm{I}$ is bounded in $L^2(\mathbb{R}^2_+)$. Hence, by Proposition 2.4 and by inequality (3.11), there exist $v\in H^1_{loc}(\mathbb{R}^2_+\setminus\Gamma)$ and $\psi\in L^2(\mathbb{R}^2_+)$ such that, up to a subsequence, $\mathrm{E}\bar{u}_R - \epsilon\mathrm{I} \rightharpoonup \psi$ weakly in $L^2(\mathbb{R}^2_+)$ and $\bar{u}_R - \epsilon\, id - t_R \rightharpoonup v$ weakly in $H^1(\mathrm{B}^+_r\setminus\Gamma)$ for every $r>0$. Therefore, $\mathrm{E}v = \psi$ and $v\in LD^2(\mathbb{R}^2_+\setminus\Gamma)$. By continuity of the traces with respect to the weak convergence in H^1, we have that $v_2 = 0$ on Σ and

$$[u_R]\cdot\nu_\Gamma = [\bar{u}_R - \epsilon\, id - t_R]\cdot\nu_\Gamma \to [v]\cdot\nu_\Gamma \quad \text{in } L^2(\Gamma) \text{ as } R\to+\infty. \tag{3.12}$$

Let us set $u := v + \epsilon\, id$. From the previous convergences we deduce that $u\in\mathcal{AD}(\Gamma,\epsilon)$ and that, up to a subsequence, $\mathrm{E}\bar{u}_R - \epsilon\mathrm{I} \rightharpoonup \mathrm{E}u - \epsilon\mathrm{I}$ weakly in $L^2(\mathbb{R}^2_+)$. Moreover, by (3.12) we get

$$\mathcal{F}^{el}(u,\Gamma,\epsilon) \leq \liminf_{R\to+\infty} \mathcal{F}^{el}(\bar{u}_R,\Gamma,\epsilon) = \liminf_{R\to+\infty} \mathcal{F}^{el}_R(u_R,\Gamma)\,,$$

which concludes the proof of (a).

Let us now prove (b). Let $u\in\mathcal{AD}(\Gamma,\epsilon)$ and let $E\subset\subset \mathrm{B}^+_{1/2}\setminus\overline{\mathrm{B}}^+_{1/4}$, $E\neq\emptyset$, be an open set. Let $\varphi\in C^\infty_c(\mathrm{B}_{1/2})$ be a cut-off function such that $0\leq\varphi\leq1$ and $\varphi=1$ on $\overline{\mathrm{B}}_{1/4}$. Let us set $E_R := RE$ and $\varphi_R(x) := \varphi(x/R)$ for every $x\in\mathbb{R}^2$ and every $R>0$. It is clear that

$$\|\nabla\varphi_R\|_{\infty,\mathbb{R}^2} = \frac{\|\nabla\varphi\|_{\infty,\mathbb{R}^2}}{R}\,. \tag{3.13}$$

Let us restrict our attention to $R>0$ such that $\Gamma\setminus\Gamma_0\subset\subset\mathrm{B}^+_{R/4}$. Arguing as in point (a), for such R we find a horizontal translation t_R such that $u - \epsilon\, id - t_R \in LD^2_{E_R}(\mathbb{R}^2_+\setminus\Gamma)$. In particular, by Proposition 2.4 and Remark 2.5, there exists a positive constant $C = C(E)$ such that

$$\|u-\epsilon\, id - t_R\|_{2,\mathrm{B}^+_R\setminus\overline{\mathrm{B}}^+_{R/4}} \leq CR\|\mathrm{E}u - \epsilon\mathrm{I}\|_{2,\mathrm{B}^+_R\setminus\overline{\mathrm{B}}^+_{R/4}}. \tag{3.14}$$

We define $v_R := \varphi_R(u - t_R) + (1-\varphi_R)\epsilon\, id$. By construction, we have $v_R = u - t_R$ in $\mathrm{B}^+_{R/4}$ and $v_R = \epsilon\mathrm{I}$ in $\mathbb{R}^2_+\setminus\mathrm{B}^+_{R/2}$. Therefore, for every $R>0$ such that $\Gamma\subseteq\overline{\mathrm{B}}^+_{R/4}$ we have $v_R\in\mathcal{AD}_R(\Gamma,\epsilon)$ and v_R coincides with $\bar{v}_R$. Moreover,

$$\|\mathrm{E}\bar{v}_R - \mathrm{E}u\|^2_{2,\mathbb{R}^2_+} \leq \|\mathrm{E}u-\epsilon\mathrm{I}\|^2_{2,\mathbb{R}^2_+\setminus\mathrm{B}^+_{R/4}} + \int_{\mathrm{B}^+_R\setminus\mathrm{B}^+_{R/4}} |\nabla\varphi_R\odot(u-\epsilon\, id - t_R)|^2\,\mathrm{d}x\,, \tag{3.15}$$

where the symbol $\odot$ denotes the symmetric tensor product. Combining (3.13)-(3.15) we obtain

$$\|\mathrm{E}\bar{v}_R - \mathrm{E}u\|^2_{2,\mathbb{R}^2_+} \leq \|\mathrm{E}u-\epsilon\mathrm{I}\|^2_{2,\mathbb{R}^2_+\setminus\mathrm{B}^+_{R/4}} + C\|\mathrm{E}u-\epsilon\mathrm{I}\|^2_{2,\mathbb{R}^2_+\setminus\mathrm{B}^+_{R/4}}\,, \tag{3.16}$$

for some constant $C > 0$ independent of R. Passing to the limit as $R \to +\infty$ in (3.16) we deduce that $\mathrm{E}\bar{v}_R - \epsilon \mathrm{I} \to \mathrm{E}u - \epsilon \mathrm{I}$ in $L^2(\mathbb{R}^2_+)$. Finally, it is clear that

$$\mathcal{F}^{el}(u,\Gamma,\epsilon) = \lim_{R\to+\infty} \mathcal{F}^{el}_R(v_R,\Gamma) = \lim_{R\to+\infty} \mathcal{F}^{el}(\bar{v}_R,\Gamma,\epsilon)\,,$$

and the proof is thus concluded. □

We are now in a position to define the *total* energy of the system: for every $\Gamma \in \mathcal{C}_\eta$, every $\epsilon \in \mathbb{R}$, and every displacement $u \in \mathcal{AD}(\Gamma,\epsilon)$, we set

$$\mathcal{F}(u,\Gamma,\epsilon) := \mathcal{F}^{el}(u,\Gamma,\epsilon) + \kappa\mathcal{H}^1(\Gamma)\,, \tag{3.17}$$

where κ is a positive constant related to the fracture toughness. The energy $\mathcal{F}(u,\Gamma,\epsilon)$ is the sum of the renormalized elastic energy (3.5) and of a *surface energy.* The latter, which is proportional to the length of the crack Γ, in the framework of Griffith's theory [12] is interpreted as the energy dissipated by the fracture process.

We conclude this section considering the additional volume constraint in the definitions of admissible displacements (3.4) and of the total energy $\mathcal{F}$ in (3.17). Let us assume that the elastic body $\mathbb{R}^2_+$, subject to a far strain field $\epsilon\mathrm{I}$, $\epsilon \in \mathbb{R}$, has a crack $\Gamma \in \mathcal{C}_\eta$ filled by a volume $V \in [0,+\infty)$ of incompressible fluid. Since we are dealing with linearized elasticity, for the volume of the cavity determined by the crack lips we use the approximate formula

$$\int_\Gamma [u]\cdot\nu_\Gamma \,\mathrm{d}\mathcal{H}^1\,,$$

so that the class of admissible displacements becomes

$$\mathcal{A}(\Gamma,V,\epsilon) := \Big\{u \in \mathcal{AD}(\Gamma,\epsilon) : \int_\Gamma [u]\cdot\nu_\Gamma\,\mathrm{d}\mathcal{H}^1 = V\Big\}\,. \tag{3.18}$$

It is clear that a result similar to Proposition 3.3 can be stated adding the volume constraint of (3.18). Therefore, also in this case the use of the energy (3.5) is fully justified. Moreover, thanks to the volume condition we have that

$$\mathcal{F}^{el}(u,\Gamma,\epsilon) = \mathcal{E}^{el}(u,\Gamma,\epsilon) - \sigma(\epsilon)V \qquad \text{for every } u \in \mathcal{A}(\Gamma,V,\epsilon)\,.$$

Since $\sigma(\epsilon)$ and V are given constants, as total energy of the system we consider

$$\mathcal{E}(u,\Gamma,\epsilon) := \mathcal{E}^{el}(u,\Gamma,\epsilon) + \kappa\mathcal{H}^1(\Gamma)\,, \tag{3.19}$$

for every displacement $u \in \mathcal{A}(\Gamma,V,\epsilon)$. In particular, the energy (3.19) is the sum of the renormalized stored elastic energy (3.8) and of the energy dissipated by the crack production.

4. Static problem

In this section, we analyze the equilibrium condition for the elastic body $\mathbb{R}^2_+$ subject to a far strain field $\epsilon\mathrm{I}$, $\epsilon \in \mathbb{R}$, when a crack $\Gamma \in \mathcal{C}_\eta$ is filled by a prescribed volume $V \in [0,+\infty)$ of incompressible fluid.

According to the variational principles of linear elasticity, the equilibrium of the elastic body with a prescribed crack $\Gamma \in \mathcal{C}_\eta$ is achieved if the displacement u is a solution of the minimum problem

$$\min_{u\in\mathcal{A}(\Gamma,V,\epsilon)} \mathcal{E}(u,\Gamma,\epsilon)\,, \tag{4.1}$$

where the set $\mathcal{A}(\Gamma,V,\epsilon)$ of admissible displacements is defined in (3.18) and the energy $\mathcal{E}$ is given by (3.19). The existence of solutions of (4.1) follows from the direct method of the calculus of variations and Proposition 4.1 below, and is discussed in Corollary 4.2. Proposition 4.1 is stated in a more general form than the one needed here since we shall use it also in the study of the evolution problem in Section 5.

Proposition 4.1. *Let $\Gamma, \Gamma_k, \Gamma_\infty \in \mathcal{C}_\eta$ be such that $\Gamma \subseteq \Gamma_k$ and $\Gamma_k \to \Gamma_\infty$ in the Hausdorff metric. Let $V_k, V_\infty \in [0, +\infty)$ with $V_k \to V_\infty$, and let $\epsilon_k, \epsilon_\infty \in \mathbb{R}$ with $\epsilon_k \to \epsilon_\infty$. Assume that $u_k \in \mathcal{A}(\Gamma_k, V_k, \epsilon_k)$ is such that*

$$\sup_k \|\mathrm{E}u_k - \epsilon_k \mathrm{I}\|_{2,\mathbb{R}^2_+} < +\infty . \tag{4.2}$$

Then, there exists $u_\infty \in \mathcal{A}(\Gamma_\infty, V_\infty, \epsilon_\infty)$ such that, up to a subsequence, $\mathrm{E}u_k - \epsilon_k \mathrm{I}$ converges to $\mathrm{E}u_\infty - \epsilon_\infty \mathrm{I}$ weakly in $L^2(\mathbb{R}^2_+)$.

Proof. By the Hausdorff convergence of Γ_k to Γ_∞, it is easy to see that $\Gamma \subseteq \Gamma_\infty$.

Since the sequence $\mathrm{E}u_k - \epsilon_k \mathrm{I}$ is bounded in $L^2(\mathbb{R}^2_+)$, we may assume that there exists $\varphi \in L^2(\mathbb{R}^2_+)$ such that, up to a subsequence, $\mathrm{E}u_k - \epsilon_k \mathrm{I} \rightharpoonup \varphi$ weakly in $L^2(\mathbb{R}^2_+)$.

Let $r > 0$ be such that $\Gamma_\infty \setminus \Gamma \subset\subset \mathrm{B}^+_r$. Thanks to the regularity of the sets Γ_k, Γ_∞, and to the convergence of Γ_k to Γ_∞ in the Hausdorff metric, arguing as in the proof of Proposition 3.3 and applying Proposition 2.7 we have that there exist a positive constant C_r and a sequence t_k of horizontal translations such that, for k large enough, the following inequality holds:

$$\|u_k - \epsilon_k\, id - t_k\|_{H^1(\mathrm{B}^+_r \setminus \Gamma_k)} \leq C_r \|\mathrm{E}u_k - \epsilon_k \mathrm{I}\|_{2,\mathbb{R}^2_+} . \tag{4.3}$$

In view of (4.2) and (4.3), we may further assume that there exists a function $v \in H^1_{loc}(\mathbb{R}^2_+ \setminus \Gamma_\infty)$ such that for $r, \delta > 0$

$$u_k - \epsilon_k\, id - t_k \rightharpoonup v \quad \text{weakly in } H^1(\mathrm{B}^+_r \setminus \overline{\mathcal{I}_\delta}(\Gamma_\infty \setminus \Gamma)) , \tag{4.4}$$

where $\mathcal{I}_\delta(\Gamma_\infty \setminus \Gamma)$ is defined in (2.1). Clearly, $\mathrm{E}v = \varphi$ and $v \in LD^2(\mathbb{R}^2_+ \setminus \Gamma_\infty)$.

Let us show that v satisfies the non-interpenetration and the volume constraints appearing in (3.18). Let us fix Ω_k, Ω_∞ bounded open subsets of $\mathbb{R}^2_+$ with Lipschitz boundaries such that $\Gamma_k \setminus \Gamma \subset\subset \Omega_k$, $\Gamma_\infty \setminus \Gamma \subset\subset \Omega_\infty$, and $\overline{\Omega}_k \to \overline{\Omega}_\infty$ in the Hausdorff metric. By the convergence of Γ_k to Γ_∞, we may split Ω_k (resp. Ω_∞) in two open subsets $\Omega^\pm_k$ (resp. $\Omega^\pm_\infty$) with Lipschitz boundaries such that the following properties hold:

$$\Gamma_k \subseteq \partial\Omega^\pm_k \setminus \partial\Omega_k \qquad \text{and} \qquad \Gamma_\infty \subseteq \partial\Omega^\pm_\infty \setminus \partial\Omega_\infty , \tag{4.5}$$

$$\overline{\Omega}^\pm_k \to \overline{\Omega}^\pm_\infty \text{ in the Hausdorff metric} , \tag{4.6}$$

$$\nu_{\Gamma_k} \text{ points towards } \Omega^+_k \text{ and } \nu_{\Gamma_\infty} \text{ points towards } \Omega^+_\infty . \tag{4.7}$$

By (4.3), (4.6), and by a simple reflection argument, we get that

$$(u_k - \epsilon_k\, id - t_k)\, \mathbf{1}_{\Omega^\pm_k} \to v\, \mathbf{1}_{\Omega^\pm_\infty} \quad \text{strongly in } L^2(\mathbb{R}^2_+). \tag{4.8}$$

By Proposition 2.3, $u_k - \epsilon_k\, id - t_k \in H^1(\Omega_k \setminus \Gamma_k)$ and $v \in H^1(\Omega_\infty \setminus \Gamma_\infty)$. Thus, by the properties of the traces of Sobolev functions (see, e.g., [21]) and by (4.5) and (4.7), for every $k \in \mathbb{N}$ and every $\psi \in C^1(\mathbb{R}^2)$ with $supp(\psi) \cap \partial\Omega_k \setminus \Sigma = \emptyset$ we have

$$\begin{aligned} &\sum_{i=1}^2 \int_{\Omega_k} (u_k - \epsilon_k\, id - t_k)_i (\nabla\psi)_i \,\mathrm{d}x + \sum_{i=1}^2 \int_{\Omega_k} \psi(\mathrm{E}_{ii} u_k - \epsilon_k) \,\mathrm{d}x \\ &\qquad = -\int_{\Gamma_k} \psi[u_k] \cdot \nu_{\Gamma_k} \,\mathrm{d}\mathcal{H}^1 - \int_{\Sigma \cap \partial\Omega_k} \psi t_k \cdot \nu_\Sigma \,\mathrm{d}\mathcal{H}^1 = -\int_{\Gamma_k} \psi[u_k] \cdot \nu_{\Gamma_k} \,\mathrm{d}\mathcal{H}^1 , \end{aligned} \tag{4.9}$$

where, in the last equality, we have used the fact that t_k is a horizontal translation and $\nu_\Sigma = (0, 1)$ is the normal vector to Σ.

Let us consider $\psi \in C^1(\mathbb{R}^2)$ such that $supp(\psi) \cap \partial\Omega_\infty \setminus \Sigma = \emptyset$. Since $\overline{\Omega}_k \to \overline{\Omega}_\infty$ in the Hausdorff metric, for k large enough we have $supp(\psi) \cap \partial\Omega_k \setminus \Sigma = \emptyset$, so that (4.9) holds. Taking into account (4.8) and the weak convergence of $\mathrm{E}u_k - \epsilon_k \mathrm{I}$ to $\mathrm{E}v$ in $L^2(\mathbb{R}^2_+)$, passing to the limit in (4.9) as $k \to +\infty$ we obtain

$$
\begin{aligned}
&-\lim_k \left(\int_{\Gamma_k} \psi [u_k] \cdot \nu_{\Gamma_k} \, \mathrm{d}\mathcal{H}^1 + \int_{\Sigma \cap \partial\Omega_k} \psi t_k \cdot \nu_\Sigma \, \mathrm{d}\mathcal{H}^1 \right) = -\lim_k \int_{\Gamma_k} \psi [u_k] \cdot \nu_{\Gamma_k} \, \mathrm{d}\mathcal{H}^1 \\
&= \lim_k \left(\sum_{i=1}^2 \int_{\Omega_k} (u_k - \epsilon_k id - t_k)_i (\nabla \psi)_i \mathrm{d}x + \sum_{i=1}^2 \int_{\Omega_k} \psi (\mathrm{E}_{ii} u_k - \epsilon_k) \mathrm{d}x \right) \qquad (4.10) \\
&= \sum_{i=1}^2 \int_{\Omega_\infty} v_i (\nabla \psi_i) \mathrm{d}x + \sum_{i=1}^2 \int_{\Omega_\infty} \psi \mathrm{E}_{ii} v \mathrm{d}x = -\int_{\Gamma_\infty} \psi [v] \cdot \nu_{\Gamma_\infty} \, \mathrm{d}\mathcal{H}^1 - \int_{\Sigma \cap \partial\Omega_\infty} \psi v \cdot \nu_\Sigma \, \mathrm{d}\mathcal{H}^1 ,
\end{aligned}
$$

where, in the last equality, we have used again the properties of the traces of Sobolev functions.

By (4.4) we have that

$$
0 = \lim_k \int_{\Sigma \cap \partial\Omega_k} \psi t_k \cdot \nu_\Sigma \, \mathrm{d}\mathcal{H}^1 = \int_{\Sigma \cap \partial\Omega_\infty} \psi v \cdot \nu_\Sigma \, \mathrm{d}\mathcal{H}^1 ,
$$

which implies, in view of (4.10), that

$$
\lim_k \int_{\Gamma_k} \psi [u_k] \cdot \nu_{\Gamma_k} \, \mathrm{d}\mathcal{H}^1 = \int_{\Gamma_\infty} \psi [v] \cdot \nu_{\Gamma_\infty} \, \mathrm{d}\mathcal{H}^1 . \qquad (4.11)
$$

for every $\psi \in C^1(\mathbb{R}^2)$ such that $supp(\psi) \cap \partial\Omega_\infty \setminus \Sigma = \emptyset$. By the hypotheses and the arbitrariness of ψ, from (4.11) we easily get that

$$
[v] \cdot \nu_{\Gamma_\infty} \geq 0 \quad \text{on } \Gamma_\infty \qquad \text{and} \qquad \int_{\Gamma_\infty} [v] \cdot \nu_{\Gamma_\infty} \, \mathrm{d}\mathcal{H}^1 = V_\infty .
$$

In view of (4.4), we also have that $v_2 = 0$ on Σ, hence $v \in \mathcal{A}(\Gamma_\infty, V_\infty, 0)$. Thus, it is clear that $u_\infty := v + \epsilon_\infty \, id \in \mathcal{A}(\Gamma_\infty, V_\infty, \epsilon_\infty)$. Since $\mathrm{E}u_\infty = \mathrm{E}v + \epsilon_\infty \mathrm{I}$, we finally get that $\mathrm{E}u_k - \epsilon_k \mathrm{I} \rightharpoonup \mathrm{E}u_\infty - \epsilon_\infty \mathrm{I}$ weakly in $L^2(\mathbb{R}^2_+)$, and the proof is thus concluded.

□

We are now ready to discuss existence and uniqueness of solution of (4.1).

Corollary 4.2. *The minimum problem* (4.1) *admits a unique solution, up to a translation parallel to the x_1-axis.*

Proof. We apply the direct method of the calculus of variations. Let u_k be a minimizing sequence. It is clear that the sequence $\mathrm{E}u_k - \epsilon \mathrm{I}$ is bounded in $L^2(\mathbb{R}^2_+)$. Hence, by Proposition 4.1, there exists $u \in \mathcal{A}(\Gamma, V, \epsilon)$ such that, up to a subsequence, $\mathrm{E}u_k - \epsilon \mathrm{I} \rightharpoonup \mathrm{E}u - \epsilon \mathrm{I}$ weakly in $L^2(\mathbb{R}^2_+)$. Therefore,

$$
\mathcal{E}(u, \Gamma, \epsilon) \leq \liminf_k \mathcal{E}(u_k, \Gamma, \epsilon) ,
$$

and this concludes the proof of existence.

The uniqueness of solution up to a horizontal translation follows by the strict convexity of the energy, by the convexity of the constraints on the crack Γ, and by the boundary condition $u_2 = 0$ on Σ.

□

In the following propositions and remarks we study some properties of a solution u of the minimum problem (4.1).

Proposition 4.3. *Let $u \in \mathcal{A}(\Gamma, V, \epsilon)$ be a solution of* (4.1) *with $\Gamma \in \mathcal{C}_\eta$, $V \in [0, +\infty)$, and $\epsilon \in \mathbb{R}$. Then, for every $v \in LD^2(\mathbb{R}^2_+ \setminus \Gamma)$ such that $[v] \cdot \nu_\Gamma = 0$ on Γ and $v_2 = 0$ on Σ it holds*

$$
\int_{\mathbb{R}^2_+ \setminus \Gamma} \mathbb{C}(\mathrm{E}u - \epsilon \mathrm{I}) \cdot \mathrm{E}v \, \mathrm{d}x = 0 . \qquad (4.12)
$$

Moreover, there exists $q(\Gamma, V, \epsilon) \geq 0$ *such that for every* $\varphi \in C_c^1(\mathbb{R}^2)$

$$\int_{\mathbb{R}^2_+ \setminus \Gamma} \mathbb{C}(\mathrm{E}u - \epsilon \mathrm{I}) \cdot \mathrm{E}(\varphi(u - \epsilon\, id))\, \mathrm{d}x = q(\Gamma, V, \epsilon) \int_\Gamma \varphi[u] \cdot \nu_\Gamma \, \mathrm{d}\mathcal{H}^1 . \tag{4.13}$$

Before proving Proposition 4.3, we briefly discuss some consequences of formula (4.12).

Remark 4.4 (Equilibrium system). Let u be a solution of (4.1) and let us set

$$\boldsymbol{\sigma}(u) := \mathbb{C}\mathrm{E}u , \tag{4.14}$$

the stress field associated to u. Formula (4.12) means that u is a weak solution of

$$\mathrm{div}(\boldsymbol{\sigma}(u) - \boldsymbol{\sigma}(\epsilon)) = 0 \qquad \text{in } \mathbb{R}^2_+ \setminus \Gamma, \tag{4.15}$$

which reduces to

$$\mathrm{div}\, \boldsymbol{\sigma}(u) = 0 \quad \text{in } \mathbb{R}^2_+ \setminus \Gamma , \tag{4.16}$$

since $\boldsymbol{\sigma}(\epsilon)$ is a constant matrix. Equation (4.16) says that u satisfies the usual balance of forces.

Moreover, integrating by parts in (4.12), we deduce that u fulfills also the condition $\boldsymbol{\sigma}(u)_{12} = 0$ on Σ, that is, the shear stress applied on the boundary of the elastic body is zero.

Remark 4.5 (Strain field). Since a solution u to (4.1) is also a weak solution of the system (4.15), applying Proposition 2.3 and the standard regularity theory for systems with constant coefficients (see, for instance, [10, Chapter 2]), we have that for every $R > 0$ there exists a constant $C = C(R)$ satisfying the following condition: for every $x_0 \in \mathbb{R}^2_+ \setminus \Gamma$ such that $\mathrm{B}_R(x_0) \subset\subset \mathbb{R}^2_+ \setminus \Gamma$

$$\|\mathrm{E}u - \epsilon \mathrm{I}\|_{\infty, \mathrm{B}_{R/2}(x_0)} \leq C \|\mathrm{E}u - \epsilon \mathrm{I}\|_{2, \mathrm{B}_R(x_0)} .$$

This implies that

$$\lim_{|x| \to +\infty} \|\mathrm{E}u - \epsilon \mathrm{I}\|_{\infty, \mathrm{B}_{R/2}(x)} = 0 ,$$

which means that at infinity $\mathrm{E}u$ tends to coincide with the strain $\epsilon \mathrm{I}$. Therefore, the choice of the function space $LD^2(\mathbb{R}^2_+ \setminus \Gamma)$ is fully justified.

Remark 4.6. From (4.12) we deduce that for every $v, w \in LD^2(\mathbb{R}^2_+ \setminus \Gamma)$ such that $[v] \cdot \nu_\Gamma = [w] \cdot \nu_\Gamma$ on Γ and $v_2 = w_2$ on Σ the following equality holds:

$$\int_{\mathbb{R}^2_+ \setminus \Gamma} \mathbb{C}(\mathrm{E}u - \epsilon \mathrm{I}) \cdot \mathrm{E}v \, \mathrm{d}x = \int_{\mathbb{R}^2_+ \setminus \Gamma} \mathbb{C}(\mathrm{E}u - \epsilon \mathrm{I}) \cdot \mathrm{E}w \, \mathrm{d}x . \tag{4.17}$$

This property will be extensively used in the sequel.

Proof of Proposition 4.3. When $V = 0$ we have, up to a horizontal translation, $u = \epsilon\, id$, thus we can take $q(\Gamma, 0, \epsilon) = 0$.

Assume now $V > 0$. Let $v \in LD^2(\mathbb{R}^2_+ \setminus \Gamma)$ be such that $[v] \cdot \nu_\Gamma = 0$ on Γ and $v_2 = 0$ on Σ. Then, for every $\delta \in \mathbb{R}$ the function $u + \delta v$ belongs to $\mathcal{A}(\Gamma, V, \epsilon)$. Therefore,

$$\mathcal{E}(u, \Gamma, \epsilon) \leq \mathcal{E}(u + \delta v, \Gamma, \epsilon) ,$$

which implies

$$\mathcal{E}^{el}(u, \Gamma, \epsilon) \leq \mathcal{E}^{el}(u + \delta v, \Gamma, \epsilon) = \mathcal{E}^{el}(u, \Gamma, \epsilon) + \delta \int_{\mathbb{R}^2_+ \setminus \Gamma} \mathbb{C}(\mathrm{E}u - \epsilon \mathrm{I}) \cdot \mathrm{E}v \, \mathrm{d}x + \frac{\delta^2}{2} \int_{\mathbb{R}^2_+ \setminus \Gamma} \mathbb{C}\mathrm{E}v \cdot \mathrm{E}v \, \mathrm{d}x ,$$

where $\mathcal{E}^{el}$ is defined in (3.8). By the arbitrariness of δ, from the previous inequality we get (4.12).

Let us now prove (4.13). We define two linear operators L and M on $C^1_c(\mathbb{R}^2)$:

$$L(\varphi) := \int_{\mathbb{R}^2_+\setminus\Gamma} \mathbb{C}(\mathrm{E}u - \epsilon \mathrm{I}) \cdot \mathrm{E}(\varphi u)\,\mathrm{d}x\,,$$
$$M(\varphi) := \int_{\Gamma} \varphi[u]\cdot\nu_\Gamma\,\mathrm{d}\mathcal{H}^1\,.$$

For every $\varphi \in C^1_c(\mathbb{R}^2)$ with $M(\varphi) = 0$, we consider the function $(1+\delta\varphi)u$. For $|\delta|$ small enough, we have $(1+\delta\varphi)u \in \mathcal{A}(\Gamma, V, \epsilon)$. Arguing as in the previous step, we get that

$$\int_{\mathbb{R}^2_+\setminus\Gamma} \mathbb{C}(\mathrm{E}u - \epsilon \mathrm{I}) \cdot \mathrm{E}(\varphi u)\,\mathrm{d}x = 0\,. \tag{4.18}$$

Let us denote by $\mathcal{N}(L)$ and $\mathcal{N}(M)$ the kernels of the linear operators L and M, respectively. Equality (4.18), which is satisfied for every $\varphi \in \mathcal{N}(M)$, implies that $\mathcal{N}(M) \subseteq \mathcal{N}(L)$. Therefore, there exists $q = q(\Gamma, V, \epsilon) \in \mathbb{R}$ such that $L = qM$.

It is clear that for every $\varphi \in C^1_c(\mathbb{R}^2)$ we have

$$[\varphi(u - \epsilon\, id)]\cdot\nu_\Gamma = [\varphi u]\cdot\nu_\Gamma \quad \text{ on } \Gamma\,. \tag{4.19}$$

Recalling (4.12) and Remark 4.6, equality (4.19) implies that

$$\int_{\mathbb{R}^2_+\setminus\Gamma} \mathbb{C}(\mathrm{E}u - \epsilon \mathrm{I}) \cdot (\mathrm{E}\varphi(u - \epsilon\, id))\,\mathrm{d}x = \int_{\mathbb{R}^2_+\setminus\Gamma} \mathbb{C}(\mathrm{E}u - \epsilon \mathrm{I}) \cdot \mathrm{E}(\varphi u)\,\mathrm{d}x = q\int_\Gamma \varphi[u]\cdot\nu_\Gamma\,\mathrm{d}\mathcal{H}^1\,, \tag{4.20}$$

which is (4.13). Taking in (4.20) a function $\varphi \in C^1_c(\mathbb{R}^2)$ such that $\varphi = 1$ on Γ and using again Remark 4.6, we get that

$$\int_{\mathbb{R}^2_+\setminus\Gamma} \mathbb{C}(\mathrm{E}u - \epsilon \mathrm{I}) \cdot (\mathrm{E}u - \epsilon \mathrm{I})\,\mathrm{d}x = qV\,, \tag{4.21}$$

which implies that $q > 0$. This concludes the proof of the proposition. □

Remark 4.7. In the case $V > 0$, from (4.21) we get immediately an explicit formula for $q(\Gamma, V, \epsilon)$ in terms of the elastic energy and of the volume V:

$$q(\Gamma, V, \epsilon) = \frac{1}{V}\int_{\mathbb{R}^2_+\setminus\Gamma} \mathbb{C}(\mathrm{E}u - \epsilon \mathrm{I}) \cdot (\mathrm{E}u - \epsilon \mathrm{I})\,\mathrm{d}x\,. \tag{4.22}$$

Remark 4.8 (Fluid pressure). Let us consider the constant

$$p(\Gamma, V, \epsilon) := q(\Gamma, V, \epsilon) - \sigma(\epsilon)\,, \tag{4.23}$$

where $q(\Gamma, V, \epsilon)$ and $\sigma(\epsilon)$ are defined in Proposition 4.3 and in formula (3.6), respectively. We want now to explain why $p(\Gamma, V, \epsilon)$ can be interpreted as a fluid pressure. It is clear that, if u is a solution of (4.1) without the non-interpenetration condition, then $q(\Gamma, V, \epsilon)$ is a Lagrange multiplier due to the volume constraint, and hence we have

$$\int_{\mathbb{R}^2_+\setminus\Gamma} \mathbb{C}(\mathrm{E}u - \epsilon \mathrm{I}) \cdot \mathrm{E}v\,\mathrm{d}x = q(\Gamma, V, \epsilon)\int_\Gamma [v]\cdot\nu_\Gamma\,\mathrm{d}\mathcal{H}^1 \tag{4.24}$$

for every $v \in LD^2(\mathbb{R}^2_+ \setminus \Gamma)$ such that $v_2 = 0$ on Σ. Thus, with the notation introduced in (4.14), u satisfies the condition

$$\boldsymbol{\sigma}(u)\nu_\Gamma = \boldsymbol{\sigma}(\epsilon)\nu_\Gamma - q(\Gamma, V, \epsilon)\nu_\Gamma = (\sigma(\epsilon) - q(\Gamma, V, \epsilon))\nu_\Gamma = -p(\Gamma, V, \epsilon)\nu_\Gamma \quad \text{on } \Gamma\,. \tag{4.25}$$

Formula (4.25) means that the total force that the elastic body exerts on the crack Γ has modulus $-p(\Gamma, V, \epsilon)$ and is directed along ν_Γ. On the contrary, the fluid inside the crack exerts a force $p(\Gamma, V, \epsilon)\nu_\Gamma$ on the fracture lips. Therefore, we are allowed to interpret $p(\Gamma, V, \epsilon)$ as the fluid pressure. According to (4.25), the pressure $p(\Gamma, V, \epsilon)$ is acting on Γ along its normal ν_Γ in the reference configuration rather than in the deformed one. This does not affect our interpretation, since we are dealing with a linearized model.

To justify the same interpretation of $p(\Gamma, V, \epsilon)$ when the non-interpenetration condition is considered, we have to show that (4.24) holds for a sufficiently large class of functions in $LD^2(\mathbb{R}^2_+ \setminus \Gamma)$. To do this, we adapt the argument used in [2, Proposition 3.6].

Proposition 4.9. *Let u be the solution of (4.1) with $\Gamma \in \mathcal{C}_\eta$, $V \in [0, +\infty)$, and $\epsilon \in \mathbb{R}$. Then (4.24) holds for every $v \in LD^2(\mathbb{R}^2_+ \setminus \Gamma)$ such that $supp(v) \subset\subset \mathbb{R}^2_+$ and $|[v] \cdot \nu_\Gamma| \leq C[u] \cdot \nu_\Gamma$ for some $C \geq 0$.*

Proof. When $V = 0$ we have $u = \epsilon\, id$ and the statement is true with $q(\Gamma, 0, \epsilon) = 0$.

Let us assume that $V > 0$. Let v be as in the statement of the proposition, and let us fix $\overline{\varphi} \in C^1_c(\mathbb{R}^2_+)$ such that $\overline{\varphi} = 1$ on $supp(v)$ and

$$\int_\Gamma \overline{\varphi}^2 [u] \cdot \nu_\Gamma \, \mathrm{d}\mathcal{H}^1 > 0 \,. \tag{4.26}$$

If we set $\overline{u} := \overline{\varphi} u$, thanks to Proposition 2.3 we have that $\overline{u} \in H^1(\mathbb{R}^2_+ \setminus \Gamma)$. In view of (4.17), we now modify the functions $\overline{u}$ and v, keeping the same values of $[\overline{u}] \cdot \nu_\Gamma$ and $[v] \cdot \nu_\Gamma$ on Γ. Let us fix Ω a bounded open subset of $\mathbb{R}^2_+$ with smooth boundary such that $\Gamma \setminus \Gamma_0 \subset\subset \Omega$, $supp(\overline{u}) \subset\subset \Omega$, and $supp(v) \subset\subset \Omega$. We may assume that there exists an extension $\hat{\Gamma}$ of Γ in $\mathcal{C}_\eta$ such that $\nu_{\hat{\Gamma}} = \nu_\Gamma$ on Γ, $\Omega \setminus \hat{\Gamma}$ is the disjoint union of two open subsets $\Omega^\pm$ with Lipschitz boundaries and with $\nu_{\hat{\Gamma}}$ pointing towards Ω^+. We consider a scalar function $\tilde{u} \in H^1(\mathbb{R}^2_+ \setminus \Gamma)$ such that $supp(\tilde{u}) \subset\subset \Omega$, $\tilde{u} \geq 0$ on $\mathbb{R}^2_+$, $\tilde{u} = 0$ on Ω^-, and $(\tilde{u})^+ = [\overline{u}] \cdot \nu_\Gamma$ on Γ. Similarly, we can find a scalar function $\tilde{v} \in H^1(\mathbb{R}^2_+ \setminus \Gamma)$ such that $supp(\tilde{v}) \subset\subset \Omega$, $\tilde{v} = 0$ on Ω^-, $(\tilde{v})^+ = [v] \cdot \nu_\Gamma$ on Γ, and

$$|\tilde{v}| \leq C |\tilde{u}| \quad \text{a.e. on } \mathbb{R}^2_+ \,. \tag{4.27}$$

Besides $\tilde{u}$ and $\tilde{v}$, we also fix a $C^{0,1}$-extension $\tilde{\nu}_{\hat{\Gamma}}$ of the unit normal $\nu_{\hat{\Gamma}}$ to $\hat{\Gamma}$. We further assume that $\tilde{\nu}_{\hat{\Gamma}}$ has compact support in $\mathbb{R}^2$. In what follows, we will consider the functions $\tilde{u}$, $\tilde{v}$, $\tilde{u}\tilde{\nu}_{\hat{\Gamma}}$, and $\tilde{v}\tilde{\nu}_{\hat{\Gamma}}$. By construction, they belong to $H^1(\mathbb{R}^2_+ \setminus \Gamma)$ and have compact support in $\mathbb{R}^2_+$.

We now need to approximate $\tilde{u}$ and $\tilde{v}$ by truncation. Let $T_k \colon \mathbb{R} \to \mathbb{R}$ be the truncation function defined by $T_k(s) := s$ if $-k \leq s \leq k$, $T_k(s) := -k$ if $s < -k$, and $T_k(s) := k$ if $s > k$. We shall also need the function $S_k \colon \mathbb{R} \to \mathbb{R}$ defined by $S_k(s) := s - T_k(s)$.

From (4.27) it follows that for every $k \in \mathbb{N}$

$$|S_{1/k}(T_k(\tilde{v}))| \leq C T_k(\tilde{u}) \quad \text{a.e. on } \mathbb{R}^2_+ \,. \tag{4.28}$$

In particular, $S_{1/k}(T_k(\tilde{v})) = 0$ where $\tilde{u} < 1/(kC)$.

By the properties of $\tilde{u}$ and of $\tilde{\nu}_{\hat{\Gamma}}$, for every k we have

$$[T_k(\tilde{u})\tilde{\nu}_{\hat{\Gamma}}] \cdot \nu_\Gamma = [T_k(\tilde{u})] \nu_\Gamma \cdot \nu_\Gamma = T_k([\overline{u}] \cdot \nu_\Gamma) \quad \text{on } \Gamma \,.$$

The previous equality implies that

$$0 \leq [T_k(\tilde{u})\tilde{\nu}_{\hat{\Gamma}}] \cdot \nu_\Gamma \leq [\overline{u}] \cdot \nu_\Gamma \leq [u] \cdot \nu_\Gamma \quad \text{on } \Gamma \,. \tag{4.29}$$

Taking into account (4.29), with the same technique used to prove Proposition 4.3 we deduce that there exists $q_k \in \mathbb{R}$ such that for every $\varphi \in C^1_c(\mathbb{R}^2)$

$$\int_{\mathbb{R}^2_+ \setminus \Gamma} \mathbb{C}(\mathrm{E}u - \epsilon \mathrm{I}) \cdot \mathrm{E}(\varphi T_k(\tilde{u}) \tilde{\nu}_{\hat{\Gamma}}) \, \mathrm{d}x = q_k \int_\Gamma \varphi [T_k(\tilde{u}) \tilde{\nu}_{\hat{\Gamma}}] \cdot \nu_\Gamma \, \mathrm{d}\mathcal{H}^1 \,. \tag{4.30}$$

We now show that $q_k \to q(\Gamma, V, \epsilon)$. Since $T_k(\tilde{u})\tilde{\nu}_{\hat{\Gamma}} \to \tilde{u}\tilde{\nu}_{\hat{\Gamma}}$ in $H^1(\mathbb{R}^2_+ \setminus \Gamma)$, passing to the limit in (4.30) as $k \to +\infty$ and recalling (4.17), we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^2_+\setminus\Gamma} \mathbb{C}(\mathrm{E}u - \epsilon\mathrm{I}) \cdot \mathrm{E}(\varphi\overline{u})\,\mathrm{d}x = \int_{\mathbb{R}^2_+\setminus\Gamma} \mathbb{C}(\mathrm{E}u - \epsilon\mathrm{I}) \cdot \mathrm{E}(\varphi\tilde{u}\tilde{\nu}_{\hat{\Gamma}})\,\mathrm{d}x \\
&= \lim_k \int_{\mathbb{R}^2_+\setminus\Gamma} \mathbb{C}(\mathrm{E}u - \epsilon\mathrm{I}) \cdot \mathrm{E}(\varphi T_k(\tilde{u})\tilde{\nu}_{\hat{\Gamma}})\,\mathrm{d}x = \lim_k q_k \int_\Gamma \varphi[T_k(\tilde{u})\tilde{\nu}_{\hat{\Gamma}}] \cdot \nu_\Gamma\,\mathrm{d}\mathcal{H}^1 \\
&= \lim_k q_k \int_\Gamma \varphi[\tilde{u}\tilde{\nu}_{\hat{\Gamma}}] \cdot \nu_\Gamma\,\mathrm{d}\mathcal{H}^1 = \lim_k q_k \int_\Gamma \varphi[\overline{u}] \cdot \nu_\Gamma\,\mathrm{d}\mathcal{H}^1 = \lim_k q_k \int_\Gamma \varphi\overline{\varphi}[u] \cdot \nu_\Gamma\,\mathrm{d}\mathcal{H}^1 .
\end{aligned} \tag{4.31}$$

Taking $\varphi = \overline{\varphi}$ in (4.31), by (4.13) of Proposition 4.3 we get

$$q(\Gamma, V, \epsilon) \int_\Gamma \overline{\varphi}^2[u] \cdot \nu_\Gamma\,\mathrm{d}\mathcal{H}^1 = \int_{\mathbb{R}^2_+\setminus\Gamma} \mathbb{C}(\mathrm{E}u - \epsilon\mathrm{I}) \cdot \mathrm{E}(\overline{\varphi}^2 u)\,\mathrm{d}x = \lim_k q_k \int_\Gamma \overline{\varphi}^2[u] \cdot \nu_\Gamma\,\mathrm{d}\mathcal{H}^1 . \tag{4.32}$$

Since (4.26) holds, from (4.32) we deduce that $q_k \to q(\Gamma, V, \epsilon)$.

We now define the scalar function

$$w_k(x) := \begin{cases} \dfrac{S_{1/k}(T_k(\tilde{v}(x)))}{T_k(\tilde{u}(x))} & \text{if } \tilde{u}(x) \neq 0 , \\ 0 & \text{if } \tilde{u}(x) = 0 . \end{cases}$$

Then, by (4.28), $w_k \in H^1(\mathbb{R}^2_+ \setminus \Gamma) \cap L^\infty(\mathbb{R}^2_+)$ and $supp(w_k) \subseteq supp(\tilde{v}) \subset\subset \Omega$. In particular, $w_k = 0$ in Ω^-. Hence, for every k there exists a sequence $(\varphi^j_k)_j$ in $C^1_c(\mathbb{R}^2_+)$ such that $\|\varphi^j_k\|_{\infty,\mathbb{R}^2_+} \leq \|w_k\|_{\infty,\mathbb{R}^2_+}$ and $\varphi^j_k \to w_k$ strongly in $H^1(\Omega^+)$ as $j \to +\infty$.

We consider the sequence $\varphi^j_k T_k(\tilde{u})\tilde{\nu}_{\hat{\Gamma}}$ in $H^1(\mathbb{R}^2_+ \setminus \Gamma)$. By the dominated convergence theorem, we have $\varphi^j_k T_k(\tilde{u})\tilde{\nu}_{\hat{\Gamma}} \to S_{1/k}(T_k(\tilde{v}))\tilde{\nu}_{\hat{\Gamma}}$ strongly in $H^1(\mathbb{R}^2_+ \setminus \Gamma)$ as $j \to +\infty$. Since $S_{1/k}(T_k(\tilde{v}))\tilde{\nu}_{\hat{\Gamma}} \to \tilde{v}\tilde{\nu}_{\hat{\Gamma}}$ strongly in $H^1(\mathbb{R}^2_+ \setminus \Gamma)$ as $k \to +\infty$, by a diagonal argument we find a sequence φ_k in $C^\infty_c(\mathbb{R}^2_+)$ such that $\varphi_k T_k(\tilde{u})\tilde{\nu}_{\hat{\Gamma}} \to \tilde{v}\tilde{\nu}_{\hat{\Gamma}}$ strongly in $H^1(\mathbb{R}^2_+ \setminus \Gamma)$. Therefore, we get

$$\begin{aligned}
&\int_{\mathbb{R}^2_+\setminus\Gamma} \mathbb{C}(\mathrm{E}u - \epsilon\mathrm{I}) \cdot \mathrm{E}v\,\mathrm{d}x = \int_{\mathbb{R}^2_+\setminus\Gamma} \mathbb{C}(\mathrm{E}u - \epsilon\mathrm{I}) \cdot \mathrm{E}(\tilde{v}\tilde{\nu}_{\hat{\Gamma}})\,\mathrm{d}x \\
&\qquad = \lim_k \int_{\mathbb{R}^2_+\setminus\Gamma} \mathbb{C}(\mathrm{E}u - \epsilon\mathrm{I}) \cdot \mathrm{E}(\varphi_k T_k(\tilde{u})\tilde{\nu}_{\hat{\Gamma}})\,\mathrm{d}x = \lim_k q_k \int_\Gamma \varphi_k[T_k(\tilde{u})\tilde{\nu}_{\hat{\Gamma}}] \cdot \nu_\Gamma\,\mathrm{d}\mathcal{H}^1 \\
&\qquad = q(\Gamma, V, \epsilon) \int_\Gamma [\tilde{v}\tilde{\nu}_{\hat{\Gamma}}] \cdot \nu_\Gamma\,\mathrm{d}\mathcal{H}^1 = q(\Gamma, V, \epsilon) \int_\Gamma [v] \cdot \nu_\Gamma\,\mathrm{d}\mathcal{H}^1 ,
\end{aligned}$$

and this concludes the proof. □

Remark 4.10. Integrating by parts, thanks to Proposition 4.9 we get that a solution u of (4.1) satisfies the condition $\boldsymbol{\sigma}(u)\nu_\Gamma = (\sigma(\epsilon) - q(\Gamma, V, \epsilon))\nu_\Gamma$ on $\{[u] \cdot \nu_\Gamma \neq 0\}$, which is the part of the crack Γ occupied by the fluid. Therefore, we can repeat the argument of Remark 4.8 on the set $\{[u] \cdot \nu_\Gamma \neq 0\}$ and we conclude that $p(\Gamma, V, \epsilon) = q(\Gamma, V, \epsilon) - \sigma(\epsilon)$ can be interpreted as the fluid pressure.

We conclude this section considering another static problem. In view of Proposition 4.3 and of Remarks 4.8 and 4.10, we know that to every triple $(\Gamma, V, \epsilon) \in \mathcal{C}_\eta \times [0, +\infty) \times \mathbb{R}$ corresponds a pressure $p(\Gamma, V, \epsilon) = q(\Gamma, V, \epsilon) - \sigma(\epsilon)$, with $q(\Gamma, V, \epsilon) \in [0, +\infty)$.

What we want to do now is to briefly discuss the relationship between Γ, V, ϵ, and p studying the equilibrium problem of an elastic body filling $\mathbb{R}^2_+$ subject to a uniform strain $\epsilon\mathrm{I}$, $\epsilon \in \mathbb{R}$, and with a force $p\nu_\Gamma$ acting on the crack $\Gamma \in \mathcal{C}_\eta$. According to the result presented in Proposition 3.3, in this case the total energy of the system is of the form

$$\mathscr{E}(u, \Gamma, p, \epsilon) := \mathcal{F}(u, \Gamma, \epsilon) - p \int_\Gamma [u] \cdot \nu_\Gamma\,\mathrm{d}\mathcal{H}^1 = \mathcal{E}(u, \Gamma, \epsilon) - (p + \sigma(\epsilon)) \int_\Gamma [u] \cdot \nu_\Gamma\,\mathrm{d}\mathcal{H}^1 , \tag{4.33}$$

where $\mathcal{F}$ is defined in (3.17). The class of admissible displacements is the set $\mathcal{AD}(\Gamma,\epsilon)$ given by formula (3.4). As in (4.1), the equilibrium condition is expressed by the minimum problem

$$\min_{u\in\mathcal{AD}(\Gamma,\epsilon)} \mathscr{E}(u,\Gamma,p,\epsilon)\,. \tag{4.34}$$

The existence of a solution of (4.34) follows by the arguments used to prove Proposition 4.1 and Corollary 4.2. The solution is unique up to a translation along the x_1-axis.

Given u a solution of (4.34), we set

$$V(\Gamma,p,\epsilon) := \int_\Gamma [u]\cdot\nu_\Gamma \,\mathrm{d}\mathcal{H}^1\,, \tag{4.35}$$

the volume between the crack lips. Then, the following proposition holds.

Proposition 4.11. *For every* $\Gamma\in\mathcal{C}_\eta$*, every* $V\in[0,+\infty)$*, and every* $\epsilon\in\mathbb{R}$*, we have*

$$V(\Gamma,p(\Gamma,V,\epsilon),\epsilon)=V\,. \tag{4.36}$$

Proof. During this proof, we denote by u_V a solution of (4.1) associated to (Γ,V,ϵ), and by u_p a solution of (4.34) corresponding to $(\Gamma,p(\Gamma,V,\epsilon),\epsilon)$.

First of all, we notice that, by (4.23), the energy defined in (4.33) reduces to

$$\mathscr{E}(u,\Gamma,p(\Gamma,V,\epsilon),\epsilon)=\mathcal{E}(u,\Gamma,\epsilon)-q(\Gamma,V,\epsilon)\int_\Gamma [u]\cdot\nu_\Gamma\,\mathrm{d}\mathcal{H}^1 \tag{4.37}$$

for every $u\in\mathcal{AD}(\Gamma,\epsilon)$.

If $V=0$, we have, by Remarks 4.8 and 4.10, that $p(\Gamma,V,\epsilon)=-\sigma(\epsilon)$. Hence, it is clear by (3.19) and (4.33) that we can take $u_V=u_p=\epsilon\, id$, and (4.36) is satisfied.

Assume now $V>0$. Let us first show that $V(\Gamma,p(\Gamma,V,\epsilon),\epsilon)>0$. By contradiction, if $V(\Gamma,p(\Gamma,V,\epsilon),\epsilon)=0$, then, up to a horizontal translation, $u_p=\epsilon\, id$. Thus, by (3.19), (4.22), (4.33), (4.37), and by the minimality of u_p, we get that

$$\begin{aligned}
&\mathscr{E}(u_p,\Gamma,p(\Gamma,V,\epsilon),\epsilon)=\kappa\mathcal{H}^1(\Gamma)\\
&\quad\le \mathscr{E}(u_V,\Gamma,p(\Gamma,V,\epsilon),\epsilon)=\frac12\int_{\mathbb{R}^2_+\setminus\Gamma}\mathbb{C}(\mathrm{E}u_V-\epsilon\mathrm{I})\cdot(\mathrm{E}u_V-\epsilon\mathrm{I})\,\mathrm{d}x-q(\Gamma,V,\epsilon)V+\kappa\mathcal{H}^1(\Gamma)\\
&\quad=-\frac12\int_{\mathbb{R}^2_+\setminus\Gamma}\mathbb{C}(\mathrm{E}u_V-\epsilon\mathrm{I})\cdot(\mathrm{E}u_V-\epsilon\mathrm{I})\,\mathrm{d}x+\kappa\mathcal{H}^1(\Gamma)\,,
\end{aligned}$$

which, in view of (3.3), leads to a contradiction. Hence, $V(\Gamma,p(\Gamma,V,\epsilon),\epsilon)>0$.

Arguing as in Proposition 4.3 and Remark 4.7, we can prove that

$$q(\Gamma,V,\epsilon)=\frac{1}{V(\Gamma,p(\Gamma,V,\epsilon),\epsilon)}\int_{\mathbb{R}^2_+\setminus\Gamma}\mathbb{C}(\mathrm{E}u_p-\epsilon\mathrm{I})\cdot(\mathrm{E}u_p-\epsilon\mathrm{I})\,\mathrm{d}x\,. \tag{4.38}$$

Therefore, by the minimality of u_p and by formula (4.37) we have

$$\begin{aligned}
\mathscr{E}(u_p,\Gamma,p(\Gamma,V,\epsilon),\epsilon)&=\mathcal{E}(u_p,\Gamma,\epsilon)-q(\Gamma,V,\epsilon)V(\Gamma,p(\Gamma,V,\epsilon),\epsilon)\\
&\le\mathscr{E}(u_V,\Gamma,p(\Gamma,V,\epsilon),\epsilon)=\mathcal{E}(u_V,\Gamma,\epsilon)-q(\Gamma,V,\epsilon)V\,.
\end{aligned} \tag{4.39}$$

Combining (4.22), (4.38), and (4.39), we get

$$\int_{\mathbb{R}^2_+\setminus\Gamma}\mathbb{C}(\mathrm{E}u_V-\epsilon\mathrm{I})\cdot(\mathrm{E}u_V-\epsilon\mathrm{I})\,\mathrm{d}x\le\int_{\mathbb{R}^2_+\setminus\Gamma}\mathbb{C}(\mathrm{E}u_p-\epsilon\mathrm{I})\cdot(\mathrm{E}u_p-\epsilon\mathrm{I})\,\mathrm{d}x\,,$$

which implies, together with (4.39), that

$$V(\Gamma,p(\Gamma,V,\epsilon),\epsilon)\ge V\,. \tag{4.40}$$

Finally, let us set

$$v:=\frac{V}{V(\Gamma,p(\Gamma,V,\epsilon),\epsilon)}(u_p-\epsilon\, id)+\epsilon\, id\,.$$

Then $v \in \mathcal{A}(\Gamma, V, \epsilon)$ and, by (3.8), (3.19), (4.22), (4.38), (4.40) and by definition of u_V,

$$\mathcal{E}(u_V, \Gamma, \epsilon) \leq \mathcal{E}(v, \Gamma, \epsilon) = \left(\frac{V}{V(\Gamma, p(\Gamma, V, \epsilon), \epsilon)} \right)^2 \mathcal{E}^{el}(u_p, \Gamma, \epsilon) + \kappa \mathcal{H}^1(\Gamma)$$

$$= \frac{V^2\, q(\Gamma, V, \epsilon)}{2V(\Gamma, p(\Gamma, V, \epsilon), \epsilon)} + \kappa \mathcal{H}^1(\Gamma) = \frac{V}{V(\Gamma, p(\Gamma, V, \epsilon), \epsilon)} \mathcal{E}^{el}(u_V, \Gamma, \epsilon) + \kappa \mathcal{H}^1(\Gamma) \leq \mathcal{E}(u_V, \Gamma, \epsilon)\,.$$

Therefore, the only possibility is $V(\Gamma, p(\Gamma, V, \epsilon), \epsilon) = V$, and this concludes the proof. □

Remark 4.12. With the notation used in Proposition 4.11, we also get that u_V and u_p coincide up to a horizontal translation.

Remark 4.13. Let us comment on the meaning of the result obtained in Proposition 4.11. When considering the equilibrium problem for the elastic body $\mathbb{R}^2_+$ subject to a far strain field $\epsilon \mathrm{I}$, $\epsilon \in \mathbb{R}$, with a crack Γ containing an incompressible fluid, we can, in principle, decide to work in two different settings: assume to know either the volume V or the pressure p of the fluid inside Γ. In the first case, we are led to study the minimum problem (4.1), finding, according to Proposition 4.3 and Remark 4.8, the fluid pressure $p(\Gamma, V, \epsilon)$. If, viceversa, we know the pressure p acting on Γ, we can solve the minimum problem (4.34) and deduce from formula (4.35) the volume $V(\Gamma, p, \epsilon)$ of the fluid between the crack lips. The equality (4.36) proved in Proposition 4.11 means that the solutions obtained considering either (4.1) or (4.33) coincide (same volumes, pressures, and displacements). Hence, we are considering the same problem from two different viewpoints. As it will be clear in Section 5 (see Remark 5.4), working with fixed fluid volume (4.1) is better for our purposes.

5. Quasi-static evolution problem

We now describe the quasi-static evolution for our model of hydraulic fracture growth. Given $T > 0$, for every $t \in [0, T]$ the elastic body is subject to a uniform strain field $\epsilon(t)\mathrm{I}$, $\epsilon(t) \in \mathbb{R}$, while a pressure $p_\infty(t) \in \mathbb{R}$ acts on the fluid far from the crack inlet. For technical reasons, we assume $\epsilon, p_\infty \in C([0, T])$, the space of continuous functions from $[0, T]$ to $\mathbb{R}$. We denote by $V(t)$ the volume of fluid injected into the crack at time t.

It is convenient to introduce the *reduced* energy $\mathcal{E}_m(t, \Gamma, V)$ defined for every $t \in [0, T]$, every $\Gamma \in \mathcal{C}_\eta$, and every $V \in [0, +\infty)$ by

$$\mathcal{E}_m(t, \Gamma, V) := \min_{u \in \mathcal{A}(\Gamma, V, \epsilon(t))} \mathcal{E}(u, \Gamma, \epsilon(t)) = \min_{u \in \mathcal{A}(\Gamma, V, \epsilon(t))} \mathcal{E}^{el}(u, \Gamma, \epsilon(t)) + \kappa \mathcal{H}^1(\Gamma)\,. \tag{5.1}$$

Following [17] and [9], we state the problem in the general framework of rate independent processes. The evolution is described by a crack set function $t \mapsto \Gamma(t)$ and a volume function $t \mapsto V(t)$. The Griffith's stability condition is here expressed in a "derivative free" setting in the following way: for every $t \in [0, T]$

$$\mathcal{E}_m(t, \Gamma(t), V(t)) \leq \mathcal{E}_m(t, \Gamma, V(t)) \qquad \text{for every } \Gamma \in \mathcal{C}_\eta \text{ with } \Gamma \supseteq \Gamma(t)\,.$$

Since the process is irreversible, we require $t \mapsto \Gamma(t)$ to be an increasing set function. Moreover, we impose an energy-dissipation balance: the rate of change of the reduced energy (5.1) of the system along a solution equals the power of the pressure forces exerted by the fluid plus the power expended by the far stress field $\sigma(\epsilon(t))$ generated by the strain $\epsilon(t)$ (see (3.6)).

Finally, we have to give an evolution law for the volume function $t \mapsto V(t)$. As we have seen in Proposition 4.3 and Remark 4.8, the presence of a strain $\epsilon(t)\mathrm{I}$ and of a volume $V(t)$ of fluid inside the crack $\Gamma(t)$ produces a pressure $p(t) := p(\Gamma(t), V(t), \epsilon(t))$ acting on the fracture lips, which is also interpreted as the fluid pressure inside the crack (see Remarks 4.8 and 4.10). As a consequence, a pressure difference $p_\infty(t) - p(t)$ is created into the fluid, which drives the evolution of $V(\cdot)$ according to an approximation of the Darcy's law: $\dot{V}(t) = p_\infty(t) - p(t)$.

This leads to the following definition.

Definition 5.1. Let $T>0$, and let $\epsilon, p_\infty \in C([0,T])$. We say that a pair $(\Gamma, V)\colon [0,T] \to \mathcal{C}_\eta \times [0,+\infty)$ is an *irreversible quasi-static evolution for the hydraulic crack problem* if it satisfies the following conditions:

(a) *irreversibility*: Γ is increasing, i.e., $\Gamma(s) \subseteq \Gamma(t)$ for every $0 \le s \le t \le T$;

(b) *global stability*: for every $t \in [0,T]$,

$$\mathcal{E}_m(t,\Gamma(t),V(t)) \le \mathcal{E}_m(t,\Gamma,V(t)) \qquad \text{for every } \Gamma \in \mathcal{C}_\eta \text{ with } \Gamma \supseteq \Gamma(t)\,;$$

(c) *Darcy's law*: the function V is absolutely continuous on the interval $[0,T]$ and

$$\dot V(t) = (p_\infty(t) - p(t))\mathbf{1}_{\{V>0\}}(t)$$

for almost every $t \in [0,T]$, where $p(t) := q(\Gamma(t),V(t),\epsilon(t)) - \sigma(\epsilon(t))$ is the pressure introduced in Remark 4.8;

(d) *energy-dissipation balance*: the function $t \mapsto \mathcal{E}_m(t,\Gamma(t),V(t))$ is absolutely continuous on the interval $[0,T]$ and

$$\frac{d}{dt}\mathcal{E}_m(t,\Gamma(t),V(t)) = \big(p(t) + \sigma(\epsilon(t))\big)\dot V(t) \tag{5.2}$$

for almost every $t \in [0,T]$.

We are now in a position to state the main theorem of this paper.

Theorem 5.2. *Let $\epsilon, p_\infty \in C([0,T])$ and let $\Gamma_0 \in \mathcal{C}_\eta$ and $V_0 \in [0,+\infty)$. Assume that (stability at time $t=0$)*

$$\mathcal{E}_m(0,\Gamma_0,V_0) \le \mathcal{E}_m(0,\Gamma,V_0) \tag{5.3}$$

for every $\Gamma \in \mathcal{C}_\eta$ with $\Gamma \supseteq \Gamma_0$. Then, there exists an irreversible quasi-static evolution (Γ,V) of the hydraulic crack problem, with $\Gamma(0)=\Gamma_0$ and $V(0)=V_0$.

Let us comment on the initial condition of Theorem 5.2.

Remark 5.3. If the pair $(\Gamma_0,V_0) \in \mathcal{C}_\eta \times [0,+\infty)$ does not satisfy the stability condition (5.3), we define a new initial condition (Γ_0^*, V_0), with Γ_0^* solution of (5.3). In particular, Γ_0^* minimizes $\mathcal{E}_m(0,\Gamma,V_0)$ among all $\Gamma \in \mathcal{C}_\eta$ with $\Gamma \supseteq \Gamma_0^*$. Therefore, we can solve the evolution problem in Theorem 5.2 starting from (Γ_0^*, V_0).

A solution of (5.3) can be found by the direct method of the calculus of variations. Indeed, a minimizing sequence $\Gamma_k \in \mathcal{C}_\eta$ has bounded $\mathcal{H}^1$-measure, and thus is bounded in $\mathcal{C}_\eta$. By Proposition 3.2, we may assume that $\Gamma_k \to \Gamma$ in the Hausdorff metric, for a suitable $\Gamma \in \mathcal{C}_\eta$. For every $k \in \mathbb{N}$, there exists a unique (up to a horizontal translation) $u_k \in \mathcal{A}(\Gamma_k, V_0, \epsilon(0))$ solution of (4.1). Since $\mathrm{E}u_k - \epsilon(0)\mathrm{I}$ is bounded in $L^2(\mathbb{R}^2_+)$, by Proposition 4.1 we have $\mathrm{E}u_k - \epsilon(0)\mathrm{I} \rightharpoonup \mathrm{E}v - \epsilon(0)\mathrm{I}$ weakly in $L^2(\mathbb{R}^2_+)$ for some $v \in \mathcal{A}(\Gamma, V_0, \epsilon(0))$, and

$$\mathcal{E}_m(0,\Gamma,V_0) \le \mathcal{E}(v,\Gamma,\epsilon(0)) \le \liminf_k \mathcal{E}_m(0,\Gamma_k,V_0)\,.$$

Thus Γ is a minimizer.

The following remark explains why it is convenient to state the evolution problem in terms of the energy functional $\mathcal{E}$ defined in (3.19) rather than working with $\mathscr{E}$ of formula (4.33).

Remark 5.4. Let us assume for a moment to know a priori the pressure p of the fluid inside the crack $\Gamma \in \mathcal{C}_\eta$. Given $t \in [0,T]$, we may define the reduced energy

$$\mathscr{E}_m(t,\Gamma,p) := \min_{u \in \mathcal{AD}(\Gamma,\epsilon(t))} \mathscr{E}(u,\Gamma,p,\epsilon(t))\,, \tag{5.4}$$

where $\mathscr{E}$ and $\mathcal{AD}(\Gamma,\epsilon(t))$ are defined in (4.33) and (3.4), respectively. The non-interpenetration condition in (3.4) and the presence of the linear term

$$(p + \sigma(\epsilon(t))) \int_\Gamma [u]\cdot\nu_\Gamma \,\mathrm{d}\mathcal{H}^1$$

in (4.33) imply that the reduced energy $\mathscr{E}_m$ is not bounded from below with respect to the crack set variable. Indeed, when we try to repeat the argument of Remark 5.3, it is possible (when $p + \sigma(\epsilon(t)) > 0$) to construct a sequence Γ_k in $\mathcal{C}_\eta$ such that $\mathscr{E}_m(t, \Gamma_k, p) \to -\infty$ and $\mathcal{H}^1(\Gamma_k) \to +\infty$. This means that it would be energetically convenient to have a catastrophic rupture of the elastic body, which is in contrast with the quasi-static nature of the phenomenon we are studying.

On the contrary, the energy $\mathcal{E}_m$ defined in (5.1) is always positive, and this simplifies our analysis.

To prove Theorem 5.2, and in particular to obtain the global stability condition of Definition 5.1, we need the following two technical lemmas. The first one corresponds, in our setting, to the Jump Transfer Theorem [8, Theorem 2.1].

Lemma 5.5. *Let $\Gamma, \Gamma_k, \Gamma_\infty, \hat{\Gamma}_\infty \in \mathcal{C}_\eta$ be such that $\Gamma \subseteq \Gamma_k$, $\Gamma_k \to \Gamma_\infty$ in the Hausdorff metric, and $\Gamma_\infty \subseteq \hat{\Gamma}_\infty$. Let $V_k, V_\infty > 0$ and $t_k, t_\infty \in [0,T]$ with $V_k \to V_\infty$ and $t_k \to t_\infty$, and let $u \in \mathcal{A}(\hat{\Gamma}_\infty, V, \epsilon(t_\infty))$. Then there exist a sequence $\hat{\Gamma}_k$ in $\mathcal{C}_\eta$ and a sequence $u_k \in \mathcal{A}(\hat{\Gamma}_k, V_k, \epsilon(t_k))$ such that $\hat{\Gamma}_k \to \hat{\Gamma}_\infty$ in the Hausdorff metric, $\Gamma_k \subseteq \hat{\Gamma}_k$, $\mathrm{E}u_k - \epsilon(t_k)\mathrm{I} \to \mathrm{E}u - \epsilon(t_\infty)\mathrm{I}$ strongly in $L^2(\mathbb{R}^2_+)$, and $\mathcal{E}(u_k, \hat{\Gamma}_k, \epsilon(t_k)) \to \mathcal{E}(u, \hat{\Gamma}_\infty, \epsilon(t_\infty))$.*

Proof. The proof is carried out following the steps of [18, Lemma 3.7]. The letter C will denote a positive constant, which can possibly change from line to line.

First, we construct the sets $\hat{\Gamma}_k$. Let $a_k, a_\infty > 0$, $\hat{a}_\infty > a_\infty$, $\gamma_k \in C^{1,1}([0,a_k])$, $\gamma_\infty \in C^{1,1}([0,a_\infty])$, and $\hat{\gamma}_\infty \in C^{1,1}([0,\hat{a}_\infty])$ be as in Definition 3.1. In particular, $\Gamma_k = \mathit{graph}(\gamma_k)$, $\Gamma_\infty = \mathit{graph}(\gamma_\infty)$, and $\hat{\Gamma}_\infty = \mathit{graph}(\hat{\gamma}_\infty)$. It is also convenient to define a $W^{2,\infty}$-extension of $\hat{\gamma}_\infty$ to the interval $[0, \hat{a}_\infty + 2\delta]$, for some $\delta > 0$. For instance, this can be done in the following way:

$$\hat{\gamma}_\infty(x_2) := \begin{cases} \hat{\gamma}_\infty(x_2) & \text{if } x_2 \in [0, \hat{a}_\infty], \\ \hat{\gamma}_\infty(\hat{a}_\infty) + (x_2 - \hat{a}_\infty)\hat{\gamma}'_\infty(\hat{a}_\infty) & \text{if } x_2 \in (\hat{a}_\infty, \hat{a}_\infty + 2\delta]. \end{cases}$$

The idea of the construction of $\hat{\Gamma}_k$ is to glue $\mathit{graph}(\gamma_k)$ and $\mathit{graph}(\hat{\gamma}_\infty)$ with a suitable arc of circumference. In view of the Hausdorff convergence of Γ_k to Γ_∞, we have that

$$a_k \to a_\infty\,, \qquad \gamma_k(a_k) \to \gamma_\infty(a_\infty) = \hat{\gamma}_\infty(a_\infty)\,, \qquad \gamma'_k(a_k) \to \gamma'_\infty(a_\infty) = \hat{\gamma}'_\infty(a_\infty)\,. \tag{5.5}$$

Without loss of generality, we may assume that $\gamma'_k(a_k) \geq \hat{\gamma}'_\infty(a_\infty) \geq 0$ (the other cases can be dealt in similar ways). Let $r > (1+\eta^2)^{3/2}/\eta$ and

$$z_k := (\gamma_k(a_k), a_k) - \frac{r}{\sqrt{1 + |\gamma'_k(a_k)|^2}}(1, -\gamma'_k(a_k)) \;\in \mathbb{R}^2_+\,.$$

Let us consider the ball $\mathrm{B}_r(z_k)$, which is tangent to Γ_k in $(\gamma_k(a_k), a_k)$. In a neighborhood of $(\gamma_k(a_k), a_k)$, the circle $\partial\mathrm{B}_r(z_k)$ can be seen as the graph of the function

$$\zeta_k(x_2) := \gamma_k(a_k) - \frac{r}{\sqrt{1+|\gamma'_k(a_k)|^2}} + \sqrt{r^2 - \left(x_2 - a_k - \frac{r\gamma'_k(a_k)}{\sqrt{1+|\gamma'_k(a_k)|^2}}\right)^2}\,.$$

We deduce that there exists $b_k \geq a_k$ such that $\hat{\gamma}'_\infty(a_\infty) = \zeta'_k(b_k)$ and $\hat{\gamma}'_\infty(a_\infty) \leq \zeta'_k(x_2) \leq \gamma'_k(a_k)$ for every $x_2 \in (a_k, b_k)$. Moreover, by (5.5), $b_k \to a_\infty$ and, by the choice of r, we have, at least for k large, $|\zeta''_k(x_2)| \leq \eta$ for $x_2 \in (a_k, b_k)$.

We define

$$\hat{\gamma}_k(x_2) := \begin{cases} \gamma_k(x_2) & \text{if } x_2 \in [0, a_k], \\ \zeta_k(x_2) & \text{if } x_2 \in (a_k, b_k], \\ \hat{\gamma}_\infty(x_2 + a_\infty - b_k) + \zeta_k(b_k) - \hat{\gamma}_\infty(a_\infty) & \text{if } x_2 \in (b_k, \hat{a}_\infty + 2\delta + b_k - a_\infty]. \end{cases} \tag{5.6}$$

For k large enough, we have that $\hat{\gamma}_k$ is well-defined on the interval $[0, \hat{a}_\infty + \delta]$, $\hat{\gamma}_k \in C^{1,1}([0, \hat{a}_\infty + \delta])$ and, by construction of ζ_k,

$$\|\hat{\gamma}_k'\|_{\infty,[0,\hat{a}_\infty+\delta]} \leq \eta \qquad \text{and} \qquad \|\hat{\gamma}_k''\|_{\infty,[0,\hat{a}_\infty+\delta]} \leq \eta\,. \tag{5.7}$$

Moreover, (5.6) and (5.7) imply that $\hat{\gamma}_k \rightharpoonup \hat{\gamma}_\infty$ weakly* in $W^{2,\infty}([0, \hat{a}_\infty + \delta])$. Therefore, if we set $\hat{\Gamma}_k := \mathit{graph}(\hat{\gamma}_k|_{[0,\hat{a}_\infty]})$, we deduce that $\hat{\Gamma}_k \in \mathcal{C}_\eta$, $\Gamma_k \subseteq \hat{\Gamma}_k$, and $\hat{\Gamma}_k \to \hat{\Gamma}_\infty$ in the Hausdorff metric.

Let us fix $\rho > 0$ and let $d_k := \|\hat{\gamma}_k - \hat{\gamma}_\infty\|_{W^{1,\infty}([0,\hat{a}_\infty+\delta])}$. By the weak* convergence in $W^{2,\infty}$ of $\hat{\gamma}_k$ to $\hat{\gamma}_\infty$, we have that $d_k \to 0$. For k large enough (so that $\hat{\Gamma}_k \subseteq \mathcal{I}_\rho(\hat{\Gamma}_\infty)$), we want to construct a $C^{1,1}$-function $\Lambda_{k,\rho}$ such that $\Lambda_{k,\rho}(\hat{\Gamma}_\infty) = \hat{\Gamma}_k$ and $\Lambda_{k,\rho}(x) = x$ for $x \in \mathbb{R}^2 \setminus \mathrm{I}_\rho(\hat{\Gamma}_\infty)$. Let us first fix a function $\vartheta_\rho \in C^\infty_c(\mathbb{R}^2)$ such that $0 \leq \vartheta_\rho \leq 1$, $\vartheta_\rho = 1$ on $\mathcal{I}_{\rho/2}(\hat{\Gamma}_\infty \setminus \Gamma)$, and $\mathit{supp}(\vartheta_\rho) \subset\subset \mathcal{I}_\rho(\hat{\Gamma}_\infty \setminus \Gamma)$. For every $x = (x_1, x_2) \in \mathcal{I}_\rho(\hat{\Gamma}_\infty \setminus \Gamma)$, we define

$$\Lambda_{k,\rho}(x) := x + \begin{pmatrix} \vartheta_\rho(x)\big(\hat{\gamma}_k(x_2) - \hat{\gamma}_\infty(x_2)\big) \\ 0 \end{pmatrix}. \tag{5.8}$$

By the properties of ϑ_ρ, we have that $\Lambda_{k,\rho}(x) = x$ for every $x \notin \mathcal{I}_\rho(\hat{\Gamma}_\infty \setminus \Gamma)$, so that it makes sense to extend $\Lambda_{k,\rho}$ with the identity out of $\mathcal{I}_\rho(\hat{\Gamma}_\infty \setminus \Gamma)$. Moreover, we notice that, $\Lambda_{k,\rho} \in C^{1,1}(\mathbb{R}^2;\mathbb{R}^2)$ and $\Lambda_{k,\rho}(\hat{\Gamma}_\infty) = \hat{\Gamma}_k$.

From (5.8) and the definition of d_k, we deduce that

$$\lim_k \|\Lambda_{k,\rho} - id\|_{W^{1,\infty}(\mathbb{R}^2)} = 0\,, \tag{5.9}$$

$$\limsup_k \|\Lambda_{k,\rho} - id\|_{W^{2,\infty}(\mathbb{R}^2)} \leq C\,, \tag{5.10}$$

where $C > 0$ in (5.10) is independent of ρ. In particular, in view of (5.9), we can apply Hadamard Theorem (see [15, Theorem 6.2.3]), to deduce that $\Lambda_{k,\rho}$ is globally invertible with $\Lambda^{-1}_{k,\rho} \in C^{1,1}(\mathbb{R}^2;\mathbb{R}^2)$ and $\|\Lambda^{-1}_{k,\rho} - id\|_{W^{1,\infty}(\mathbb{R}^2)} \to 0$ as $k \to +\infty$.

We are now in a position to define the approximating functions. Let $u \in \mathcal{A}(\hat{\Gamma}_\infty, V_\infty, \epsilon(t_\infty))$. We set

$$v_{k,\rho} := \big((\mathit{cof}\,\nabla\Lambda_{k,\rho})^{-T}(u - \epsilon(t_\infty)\,id)\big) \circ \Lambda^{-1}_{k,\rho}\,, \tag{5.11}$$

$$u_{k,\rho} := \frac{V_k}{V_\infty} v_{k,\rho} + \epsilon(t_k)\,id\,. \tag{5.12}$$

Thanks to [4, Section 1.7], $u_{k,\rho}$ satisfies the non-interpenetration condition and the volume constraint on $\hat{\Gamma}_k$, hence $u_{k,\rho} \in \mathcal{A}(\hat{\Gamma}_k, V_k, \epsilon(t_k))$. Moreover, (5.9)-(5.12) and Proposition 2.3 imply that

$$\limsup_k \|\mathrm{E}v_{k,\rho}\|_{2,\mathcal{I}_\rho(\hat{\Gamma}_\infty)} \leq C\|u - \epsilon(t_\infty)\,id\|_{H^1(\mathcal{I}_\rho(\hat{\Gamma}_\infty)\setminus\hat{\Gamma}_\infty)}\,, \tag{5.13}$$

$$\mathrm{E}u_{k,\rho} - \epsilon(t_k)\mathrm{I} = \frac{V_k}{V_\infty}(\mathrm{E}u - \epsilon(t_\infty)\mathrm{I}) \qquad \text{in } \mathbb{R}^2_+ \setminus \mathcal{I}_\rho(\hat{\Gamma}_\infty). \tag{5.14}$$

In view of (5.12) and (5.14), we have that

$$\begin{aligned} &|\mathcal{E}(u_{k,\rho}, \hat{\Gamma}_k, \epsilon(t_k)) - \mathcal{E}(u, \hat{\Gamma}_\infty, \epsilon(t_\infty))| \\ &\leq \frac{V_k^2}{2V_\infty^2} \int_{\mathcal{I}_\rho(\hat{\Gamma}_\infty)} \mathbb{C}\mathrm{E}v_{k,\rho} \cdot \mathrm{E}v_{k,\rho}\,\mathrm{d}x + \frac{1}{2}\int_{\mathcal{I}_\rho(\hat{\Gamma}_\infty)} \mathbb{C}(\mathrm{E}u - \epsilon(t_\infty)\mathrm{I}) \cdot (\mathrm{E}u - \epsilon(t_\infty)\mathrm{I})\,\mathrm{d}x \\ &\quad + \frac{1}{2}\Big(\frac{V_k^2}{V_\infty^2} - 1\Big) \int_{\mathbb{R}^2_+\setminus\mathcal{I}_\rho(\hat{\Gamma}_\infty)} \mathbb{C}(\mathrm{E}u - \epsilon(t_\infty)\mathrm{I}) \cdot (\mathrm{E}u - \epsilon(t_\infty)\mathrm{I})\,\mathrm{d}x + |\mathcal{H}^1(\hat{\Gamma}_k) - \mathcal{H}^1(\hat{\Gamma}_\infty)|\,. \end{aligned} \tag{5.15}$$

Recalling that $\mathcal{H}^1(\hat\Gamma_k) \to \mathcal{H}^1(\hat\Gamma_\infty)$, $V_k \to V_\infty$, and that (3.3) and (5.13) hold, we pass to the lim sup in (5.15) as $k \to +\infty$ obtaining

$$\limsup_k |\mathcal{E}(u_{k,\rho}, \hat\Gamma_k, \epsilon(t_k)) - \mathcal{E}(u, \hat\Gamma_\infty, \epsilon(t_\infty))| \le C\|u - \epsilon(t_\infty)\, id\|^2_{H^1(\mathcal{I}_\rho(\hat\Gamma_\infty))} . \tag{5.16}$$

Passing to the limit as $\rho \to 0$ in (5.16), we deduce that

$$\lim_{\rho\to 0} \limsup_k |\mathcal{E}(u_{k,\rho}, \hat\Gamma_k, \epsilon(t_k)) - \mathcal{E}(u, \hat\Gamma_\infty, \epsilon(t_\infty))| = 0 . \tag{5.17}$$

Therefore, in view of (5.13) and (5.17), we can construct a sequence of functions $u_k \in \mathcal{A}(\hat\Gamma_k, V_k, \epsilon(t_k))$ such that $\mathcal{E}(u_k, \hat\Gamma_k, \epsilon(t_k)) \to \mathcal{E}(u, \hat\Gamma_\infty, \epsilon(t_\infty))$ and $\mathrm{E}u_k - \epsilon(t_k)\mathrm{I} \to \mathrm{E}u - \epsilon(t_\infty)\mathrm{I}$ strongly in $L^2(\mathbb{R}^2_+)$. This concludes the proof of the lemma. □

The following lemma will be useful in the proof of the global stability condition (b) of Definition 5.1.

Lemma 5.6. *Let $\Gamma, \Gamma_k, \Gamma_\infty \in \mathcal{C}_\eta$ be such that $\Gamma \subseteq \Gamma_k$ and $\Gamma_k \to \Gamma_\infty$ in the Hausdorff metric. Let $V_k, V_\infty \ge 0$ and $t_k, t_\infty \in [0, T]$ with $V_k \to V_\infty$ and $t_k \to t_\infty$. Assume that*

$$\mathcal{E}_m(t_k, \Gamma_k, V_k) \le \mathcal{E}_m(t_k, \hat\Gamma, V_k) \quad \textit{for every } \hat\Gamma \in \mathcal{C}_\eta \textit{ with } \hat\Gamma \supseteq \Gamma_k . \tag{5.18}$$

Then

$$\mathcal{E}_m(t_\infty, \Gamma_\infty, V_\infty) \le \mathcal{E}_m(t_\infty, \hat\Gamma, V_\infty) \quad \textit{for every } \hat\Gamma \in \mathcal{C}_\eta \textit{ with } \hat\Gamma \supseteq \Gamma_\infty . \tag{5.19}$$

Moreover, let u_k, u_∞ be solutions of (4.1) corresponding to the triples $(\Gamma_k, V_k, \epsilon(t_k))$ and $(\Gamma_\infty, V_\infty, \epsilon(t_\infty))$, and let $p(\Gamma_k, V_k, \epsilon_k)$, $p(\Gamma_\infty, V_\infty, \epsilon_\infty)$ be the corresponding pressures according to Remark 4.8. Then $\mathrm{E}u_k - \epsilon(t_k)\mathrm{I} \to \mathrm{E}u_\infty - \epsilon(t_\infty)\mathrm{I}$ in $L^2(\mathbb{R}^2_+)$, $p(\Gamma_k, V_k, \epsilon_k) \to p(\Gamma_\infty, V_\infty, \epsilon_\infty)$, and $\mathcal{E}_m(t_k, \Gamma_k, V_k) \to \mathcal{E}_m(t_\infty, \Gamma_\infty, V_\infty)$.

Proof. Let us fix $w_0 \in \mathcal{A}(\Gamma, 1, 0)$. Then,

$$w_k := V_k w_0 + \epsilon(t_k)\, id \in \mathcal{A}(\Gamma_k, V_k, \epsilon(t_k))$$

and, by definition of u_k,

$$\mathcal{E}^{el}(u_k, \Gamma_k, \epsilon(t_k)) \le \mathcal{E}^{el}(w_k, \Gamma_k, \epsilon(t_k)) = V_k^2 \mathcal{E}^{el}(w_0, \Gamma_k, 0) . \tag{5.20}$$

In view of (3.3), inequality (5.20) implies that the sequence $\mathrm{E}u_k - \epsilon(t_k)\mathrm{I}$ is bounded in $L^2(\mathbb{R}^2_+)$, hence, applying Proposition 4.1, we deduce that there exists $u_\infty \in \mathcal{A}(\Gamma_\infty, V_\infty, \epsilon(t_\infty))$ such that, up to a subsequence,

$$\mathrm{E}u_k - \epsilon(t_k)\mathrm{I} \rightharpoonup \mathrm{E}u_\infty - \epsilon(t_\infty)\mathrm{I} \quad \text{weakly in } L^2(\mathbb{R}^2_+) . \tag{5.21}$$

Let us prove (5.19). Let $\hat\Gamma \in \mathcal{C}_\eta$, $\hat\Gamma \supseteq \Gamma_\infty$ be fixed. Let us denote by $u_{\hat\Gamma} \in \mathcal{A}(\hat\Gamma, V_\infty, \epsilon(t_\infty))$ a solution to (4.1) associated to $(\hat\Gamma, V_\infty, \epsilon(t_\infty))$. Applying Lemma 5.5 to $\Gamma_k, \Gamma_\infty, \hat\Gamma$, we can find a sequence $\hat\Gamma_k \in \mathcal{C}_\eta$ such that $\hat\Gamma_k \supseteq \Gamma_k$ and $\hat\Gamma_k \to \hat\Gamma$ in the Hausdorff metric, as well as a sequence of functions $v_k \in \mathcal{A}(\hat\Gamma_k, V_k, \epsilon(t_k))$ such that $\mathcal{E}(v_k, \hat\Gamma_k, \epsilon(t_k)) \to \mathcal{E}(u_{\hat\Gamma}, \hat\Gamma, \epsilon(t_\infty))$.

By (5.1), (5.18) and (5.21), we have that

$$\begin{aligned}\mathcal{E}_m(t_\infty, \Gamma_\infty, V_\infty) &\le \mathcal{E}(u_\infty, \Gamma_\infty, \epsilon(t_\infty)) \le \liminf_k \mathcal{E}(u_k, \Gamma_k, \epsilon(t_k)) \\ &= \liminf_k \mathcal{E}_m(t_k, \Gamma_k, V_k) \le \limsup_k \mathcal{E}_m(t_k, \Gamma_k, V_k) \le \limsup_k \mathcal{E}_m(t_k, \hat\Gamma_k, V_k) \\ &\le \lim_k \mathcal{E}(v_k, \hat\Gamma_k, \epsilon(t_k)) = \mathcal{E}(u_{\hat\Gamma}, \hat\Gamma, \epsilon(t_\infty)) = \mathcal{E}_m(t_\infty, \hat\Gamma, V_\infty) ,\end{aligned} \tag{5.22}$$

from which we deduce (5.19). Moreover, taking $\hat\Gamma = \Gamma_\infty$ in (5.22), we get that $u_\infty \in \mathcal{A}(\Gamma_\infty, V_\infty, \epsilon(t_\infty))$ is a solution of (4.1), $\mathrm{E}u_k - \epsilon(t_k)\mathrm{I} \to \mathrm{E}u_\infty - \epsilon(t_\infty)\mathrm{I}$ strongly in $L^2(\mathbb{R}^2_+)$, and $\mathcal{E}_m(t_k, \Gamma_k, V_k) \to \mathcal{E}_m(t_\infty, \Gamma_\infty, V_\infty)$. In view of these convergences, of Remark 4.7, and of formula (4.23), we deduce that $p(\Gamma_k, V_k, \epsilon_k) \to p(\Gamma_\infty, V_\infty, \epsilon_\infty)$, at least in the case $V_\infty > 0$.

It remains to prove that $p(\Gamma_k, V_k, \epsilon_k) \to -\sigma(\epsilon(t_\infty)) = p(\Gamma_\infty, V_\infty, \epsilon(t_\infty))$ if $V_\infty = 0$. Without loss of generality, we may assume $V_k > 0$ for every $k \in \mathbb{N}$. In view of (5.20), we have that

$$\int_{\mathbb{R}^2_+ \setminus \Gamma_k} \mathbb{C}(\mathrm{E}u_k - \epsilon(t_k)\mathrm{I}) \cdot (\mathrm{E}u_k - \epsilon(t_k)\mathrm{I})\,\mathrm{d}x \leq V_k^2 \int_{\mathbb{R}^2_+ \setminus \Gamma_k} \mathbb{C}\mathrm{E}w_0 \cdot \mathrm{E}w_0\,\mathrm{d}x\,,$$

which implies, together with Remark 4.7 and formula (4.23), that

$$0 \leq p(\Gamma_k, V_k, \epsilon_k) + \sigma(\epsilon(t_k)) \leq V_k \int_{\mathbb{R}^2_+ \setminus \Gamma_k} \mathbb{C}\mathrm{E}w_0 \cdot \mathrm{E}w_0\,\mathrm{d}x\,.$$

Since $V_k \to V_\infty = 0$ and $\epsilon(t_k) \to \epsilon(t_\infty)$, we get $p(\Gamma_k, V_k, \epsilon_k) \to -\sigma(\epsilon(t_\infty))$. □

We are now ready to prove Theorem 5.2

Proof of Theorem 5.2. Let ϵ, p_∞, Γ_0, and V_0 be as in the statement of the theorem and let ν_{Γ_0} be the unit normal vector to Γ_0.

The proof is based on a time discretization process, see [9, 17]. For every $k \in \mathbb{N}$, we introduce the time step $\tau_k := T/k$ and a subdivision of the interval $[0,T]$ of the form $t^k_i := i\tau_k$ for $i = 0, \ldots, k$. Let us describe the discrete problems. For every k we define V^k_i and Γ^k_i recursively with respect to i. For $i = 0$, we set $V^k_0 := V_0$, $\Gamma^k_0 := \Gamma_0$, and $p^k_0 := p(\Gamma_0, V_0, \epsilon(0))$ the pressure introduced in Remark 4.8. For $i > 0$, assume that we already know V^k_{i-1}, Γ^k_{i-1}, and $p^k_{i-1} := p(\Gamma^k_{i-1}, V^k_{i-1}, \epsilon(t^k_{i-1}))$. We define

$$V^k_i := \max\{V^k_{i-1} + \big(p_\infty(t^k_{i-1}) - p^k_{i-1}\big)\tau_k, 0\}\,. \tag{5.23}$$

We notice that (5.23) is the discrete approximation of the Darcy's law of Definition 5.1. Then, we set Γ^k_i to be a solution of

$$\min\{\mathcal{E}_m(t^k_i, \Gamma, V^k_i) : \Gamma \in \mathcal{C}_\eta,\, \Gamma \supseteq \Gamma^k_{i-1}\}\,, \tag{5.24}$$

which can be found arguing as in Remark 5.3. In particular, (5.24) is the discrete form of the global stability condition in Definition 5.1.

Finally, we denote by u^k_i a solution of (4.1) with $\Gamma = \Gamma^k_i$, $V = V^k_i$, and $\epsilon = \epsilon(t^k_i)$, and we set $p^k_i := p(\Gamma^k_i, V^k_i, \epsilon(t^k_i))$ to be the corresponding pressure, according to Proposition 4.3 and Remark 4.8. Arguing as in the proof of Lemma 5.6, it is possible to prove that

$$\begin{gathered} \|\mathrm{E}u^k_i - \epsilon(t^k_i)\mathrm{I}\|_{2,\mathbb{R}^2_+} \leq CV^k_i\,, \\ -\sigma(\epsilon(t^k_i)) \leq p^k_i \leq CV^k_i - \sigma(\epsilon(t^k_i))\,, \end{gathered} \tag{5.25}$$

for some constant $C > 0$ independent of k and i.

We introduce the following piecewise constant interpolation functions: for $t \in [t^k_i, t^k_{i+1})$

$$\begin{aligned} &u_k(t) := u^k_i\,, \quad \Gamma_k(t) := \Gamma^k_i\,, \quad V_k(t) := V^k_i\,, \quad \epsilon_k(t) := \epsilon(t^k_i)\,, \\ &p_k(t) := p^k_i\,, \quad p^k_\infty(t) := p_\infty(t^k_i)\,, \quad \sigma_k(t) := \sigma(\epsilon(t^k_i))\,, \end{aligned} \tag{5.26}$$

and, for $t \in (t^k_i, t^k_{i+1}]$, $\overline{V}_k(t) := V^k_{i+1}$. Furthermore, we will also use the piecewise affine function

$$V^k(t) := V^k_{i-1} + \frac{V^k_i - V^k_{i-1}}{\tau_k}(t - t^k_{i-1}) \qquad \text{for } t \in (t^k_{i-1}, t^k_i]\,. \tag{5.27}$$

Since $p^k_i \geq -\sigma(\epsilon(t^k_i))$ for every k and every i, from (5.23) we easily deduce that

$$V^k_i \leq V^k_{i-1} + |p_\infty(t^k_{i-1}) + \sigma(\epsilon(t^k_{i-1}))|\tau_k\,. \tag{5.28}$$

Iterating inequality (5.28), we get

$$V^k_i \leq V_0 + \tau_k \sum_{j=1}^{i} |p_\infty(t^k_{j-1}) + \sigma(\epsilon(t^k_{j-1}))|\,. \tag{5.29}$$

Taking into account the regularity of $t \mapsto p_\infty(t)$ and of $t \mapsto \sigma(\epsilon(t))$, inequality (5.29) implies that

$$\sup_k \|V_k\|_{\infty,[0,T]} < +\infty \,. \tag{5.30}$$

Therefore, from (5.25), (5.30), we obtain that

$$\sup_{i,k} \|\mathrm{E}u_i^k - \epsilon(t_i^k)\mathrm{I}\|_{2,\mathbb{R}^2_+} < \infty \qquad \text{and} \qquad \sup_k \|p_k\|_{\infty,[0,T]} < +\infty \,. \tag{5.31}$$

Moreover, thanks to (5.23) we have that

$$V_{i-1}^k - V_i^k \le |p_\infty(t_{i-1}^k)|\tau_k + |p_{i-1}^k|\tau_k \,. \tag{5.32}$$

Combining (5.28), (5.31), and (5.32), we get that

$$\sup_k \|V^k\|_{W^{1,\infty}(0,T)} < +\infty \,. \tag{5.33}$$

We now prove a discrete energy inequality. By (5.24) we have that

$$\mathcal{E}_m(t_i^k, \Gamma_i^k, V_i^k) \le \mathcal{E}_m(t_i^k, \Gamma_{i-1}^k, V_i^k) \,. \tag{5.34}$$

In order to estimate the right-hand side of (5.34), we fix $w_0 \in \mathcal{A}(\Gamma_0, 1, 0)$ and we define the functions

$$v_i^k := \begin{cases} \dfrac{u_i^k - \epsilon(t_i^k)\, id}{V_i^k} & \text{if } V_i^k \neq 0, \\ w_0 & \text{if } V_i^k = 0. \end{cases}$$

Notice that $v_i^k \in \mathcal{A}(\Gamma_i^k, 1, 0)$ and, by (5.25),

$$\|\mathrm{E}v_i^k\|_{2,\mathbb{R}^2_+} \le M \,, \tag{5.35}$$

where $M \ge \|\mathrm{E}w_0\|_{2,\mathbb{R}^2_+}$.

Since $u_{i-1}^k + (\epsilon(t_i^k) - \epsilon(t_{i-1}^k))id + (V_i^k - V_{i-1}^k)v_{i-1}^k \in \mathcal{A}(\Gamma_{i-1}^k, V_i^k, \epsilon(t_i^k))$, by (5.34) we get

$$\begin{aligned} \mathcal{E}_m(t_i^k, \Gamma_i^k, V_i^k) &\le \mathcal{E}(u_{i-1}^k + (\epsilon(t_i^k) - \epsilon(t_{i-1}^k))id + (V_i^k - V_{i-1}^k)v_{i-1}^k, \Gamma_{i-1}^k, \epsilon(t_i^k)) \\ &= \mathcal{E}(u_{i-1}^k, \Gamma_{i-1}^k, \epsilon(t_{i-1}^k)) + (V_i^k - V_{i-1}^k)\int_{\mathbb{R}^2_+\setminus\Gamma_{i-1}^k} \mathbb{C}(\mathrm{E}u_{i-1}^k - \epsilon(t_{i-1}^k)\mathrm{I})\cdot \mathrm{E}v_{i-1}^k\,\mathrm{d}x \\ &\quad + \frac{(V_i^k - V_{i-1}^k)^2}{2}\int_{\mathbb{R}^2_+\setminus\Gamma_{i-1}^k} \mathbb{C}\mathrm{E}v_{i-1}^k \cdot \mathrm{E}v_{i-1}^k\,\mathrm{d}x \end{aligned} \tag{5.36}$$

Recalling (3.3), (5.35), and formula (4.22) which relates p_i^k to $\sigma(\epsilon(t_i^k))$ and to the quantity $q(\Gamma_i^k, V_i^k, \epsilon(t_i^k))$ introduced in Proposition 4.3, we can continue in (5.36) obtaining

$$\begin{aligned} \mathcal{E}_m(t_i^k, \Gamma_i^k, V_i^k) \le{}& \mathcal{E}_m(t_{i-1}^k, \Gamma_{i-1}^k, V_{i-1}^k) + (p_{i-1}^k + \sigma(\epsilon(t_{i.-1}^k)))\int_{t_{i-1}^k}^{t_i^k} \dot V^k(s)\,\mathrm{d}s \\ &+ \beta\widetilde{V}_k M^2 \int_{t_{i-1}^k}^{t_i^k} |\dot V_k(s)|\,\mathrm{d}s \,, \end{aligned} \tag{5.37}$$

where we have set

$$\widetilde{V}_k := \frac{1}{2}\sup_{j=1,\dots,k} |V_j^k - V_{j-1}^k| \,.$$

Iterating inequality (5.37) we obtain, for $t \in [t_i^k, t_{i+1}^k)$,

$$\begin{aligned} \mathcal{E}_m(t_i^k, \Gamma_k(t), V_k(t)) \le{}& \mathcal{E}_m(0, \Gamma_0, V_0) + \int_0^{t_i^k} (p_k(s) + \sigma_k(s))\dot V^k(s)\,\mathrm{d}s \\ &+ \beta\widetilde{V}_k M^2 \int_0^T |\dot V^k(s)|\,\mathrm{d}s \,. \end{aligned} \tag{5.38}$$

In particular, (5.38) implies that $\mathcal{H}^1(\Gamma_k(t))$ is bounded uniformly with respect to $t \in [0,T]$ and $k \in \mathbb{N}$.

By Theorem 2.2 and Proposition 3.2, we have that, up to a subsequence, $\Gamma_k(t) \to \Gamma(t)$ in the Hausdorff metric for every $t \in [0,T]$, $\mathcal{H}^1(\Gamma_k(t)) \to \mathcal{H}^1(\Gamma(t))$, and the set function $\Gamma\colon [0,T] \to \mathcal{C}_\eta$ is bounded and increasing. Moreover, in view of (5.30) and (5.33), there exists a nonnegative function $V \in W^{1,\infty}([0,T])$ such that, up to a further subsequence, $V^k \rightharpoonup V$ weakly* in $W^{1,\infty}([0,T])$ and $V^k, V_k, \overline{V}_k \to V$ strongly in $L^\infty([0,T])$. Let us also denote by $u(t)$ a solution (unique up to a horizontal translation) to (4.1) associated to the triple $(\Gamma(t), V(t), \epsilon(t))$, and let $p(t) := p(\Gamma(t), V(t), \epsilon(t))$ be the corresponding pressure, according to Proposition 4.3 and Remark 4.8.

Thanks to the previous convergences, from Lemma 5.6 we deduce that for every $t \in [0,T]$ the pair $(\Gamma(t), V(t))$ satisfies the global stability condition (b) of Definition 5.1, that $\mathrm{E}u_k(t) - \epsilon_k(t)\mathrm{I} \to \mathrm{E}u(t) - \epsilon(t)\mathrm{I}$ in $L^2(\mathbb{R}^2_+)$, and that $p_k(t) \to p(t)$.

In order to prove the energy-dissipation balance, we first pass to the limit in (5.38) as $k \to +\infty$. The third term in the right-hand side of (5.38) tends to zero because of (5.33). In view of (5.31), of the pointwise convergence of p_k to p, of the continuity of $\sigma(\epsilon(\cdot))$, and of the weak* convergence in $L^\infty([0,T])$ of $\dot V^k$ to $\dot V$, we get that

$$\mathcal{E}_m(t, \Gamma(t), V(t)) \leq \mathcal{E}_m(0, \Gamma_0, V_0) + \int_0^t \big(p(s) + \sigma(\epsilon(s))\big)\dot V(s)\,\mathrm{d}s\,. \tag{5.39}$$

For the opposite inequality, for every $t \in [0,T]$ we consider a subdivision of the interval $[0,t]$ of the form $s^k_h := \frac{ht}{k}$ for $k, h \in \mathbb{N}$, $k \neq 0$, and $h \leq k$. For every $h = 0, \ldots, k$ we set

$$v^k_h := \begin{cases} \dfrac{u(s^k_h) - \epsilon(s^k_h)\, id}{V(s^k_h)} & \text{if } V(s^k_h) \neq 0, \\ w_0 & \text{if } V(s^k_h) = 0. \end{cases}$$

Therefore, $\|\mathrm{E}v^k_h\|_{2,\mathbb{R}^2_+} \leq M$ and $u(s^k_{h+1}) + (\epsilon(s^k_h) - \epsilon(s^k_{h+1}))id + (V(s^k_h) - V(s^k_{h+1}))v^k_{h+1}$ belongs to $\mathcal{A}(\Gamma(s^k_{h+1}), V(s^k_h), \epsilon(s^k_h))$. Since $\Gamma(\cdot)$ is increasing and satisfies the global stability condition, we have

$$\mathcal{E}_m(s^k_h, \Gamma(s^k_h), V(s^k_h)) \leq \mathcal{E}_m(s^k_h, \Gamma(s^k_{h+1}), V(s^k_h))\,.$$

Hence,

$$\begin{aligned}
&\mathcal{E}_m(s^k_h, \Gamma(s^k_h), V(s^k_h)) \\
&\leq \mathcal{E}(u(s^k_{h+1}) + (\epsilon(s^k_h) - \epsilon(s^k_{h+1}))id + (V(s^k_h) - V(s^k_{h+1}))v^k_{h+1}, \Gamma(s^k_{h+1}), \epsilon(s^k_h)) \\
&= \mathcal{E}_m(s^k_{h+1}, \Gamma(s^k_{h+1}), V(s^k_{h+1})) + (V(s^k_h) - V(s^k_{h+1})) \int_{\mathbb{R}^2_+ \setminus \Gamma(s^k_{h+1})} \mathbb{C}(\mathrm{E}u(s^k_{h+1}) - \epsilon(s^k_{h+1})\mathrm{I}) \cdot \mathrm{E}v^k_{h+1}\,\mathrm{d}x \\
&\quad + \frac{(V(s^k_h) - V(s^k_{h+1}))^2}{2} \int_{\mathbb{R}^2_+ \setminus \Gamma(s^k_{h+1})} \mathbb{C}\mathrm{E}v^k_{h+1} \cdot \mathrm{E}v^k_{h+1}\,\mathrm{d}x \\
&\leq \mathcal{E}_m(s^k_{h+1}, \Gamma(s^k_{h+1}), V(s^k_{h+1})) - \int_{s^k_h}^{s^k_{h+1}} \big(p(s^k_{h+1}) + \sigma(\epsilon(s^k_{h+1}))\big)\dot V(s)\mathrm{d}s + \beta \widehat{V}_k M^2 \int_{s^k_h}^{s^k_{h+1}} |\dot V(s)|\mathrm{d}s\,,
\end{aligned}$$

where β is the constant defined in (3.3) and

$$\widehat{V}_k := \frac{1}{2} \sup_{h=1,\ldots,k} |V(s^k_h) - V(s^k_{h-1})|\,.$$

Iterating the previous inequality for $h = 0, \dots, k$ and setting $p^k(s) := p(s^k_{h+1})$, $\sigma^k(s) := \sigma(\epsilon(s^k_{h+1}))$ for $s \in (s^k_h, s^k_{h+1}]$, we get

$$\mathcal{E}_m(0,\Gamma_0,V_0) \le \mathcal{E}_m(t,\Gamma(t),V(t)) - \int_0^t (p^k(s)+\sigma^k(s))\dot V(s)\,\mathrm{d}s + \beta\widehat{V}_k M^2 \int_0^t |\dot V(s)|\,\mathrm{d}s\,. \tag{5.40}$$

Since $\Gamma\colon [0,T] \to \mathcal{C}_\eta$ is an increasing set function, according to Theorem 2.1 there exists a set $\Theta \subseteq [0,T]$ such that $[0,T]\setminus\Theta$ is at most countable and $\Gamma(\cdot)$ is continuous at every point in Θ. By Lemma 5.6, we have that $s \mapsto \mathrm{E}u(s) - \epsilon(s)\mathrm{I}$ is strongly continuous in $L^2(\mathbb{R}^2_+)$ at every point of Θ and $s \mapsto p(s)$ is continuous at the same points. Thus $p^k(s) \to p(s)$ for every $s \in \Theta$. By the dominated convergence theorem $(p^k+\sigma^k)\dot V \to (p+\sigma(\epsilon))\dot V$ in $L^1([0,t])$ and, passing to the limit in (5.40) as $k \to +\infty$, we obtain

$$\mathcal{E}_m(0,\Gamma_0,V_0) \le \mathcal{E}_m(t,\Gamma(t),V(t)) - \int_0^t \big(p(s)+\sigma(\epsilon(s))\big)\dot V(s)\,\mathrm{d}s\,.$$

Recalling (5.39), this concludes the proof of the energy-dissipation balance (d) of Definition 5.1.

It remains to prove the Darcy's law (c) of Definition 5.1. Let us fix $j \in \mathbb{N}$, $j \ne 0$, and let us set $E_j := \{t \in [0,T] : V(t) \ge 1/j\}$. By the uniform convergences, for k large enough we may assume that $V_k(t), V^k(t), \overline{V}_k(t) > 0$ for every $t \in E_j$. Therefore, in view of (5.23) and of (5.27), for such t we get, using the notation introduced in (5.26),

$$\dot V^k(t) = p^k_\infty(t) - p_k(t)\,. \tag{5.41}$$

In view of (5.41), for every $t \in [0,T]$ we have

$$V^k(t) = V_0 + \int_0^t \dot V^k(s)\,\mathrm{d}s = V_0 + \int_{[0,t]\setminus E_j} \dot V^k(s)\,\mathrm{d}s + \int_{E_j} (p^k_\infty(s) - p_k(s))\,\mathrm{d}s\,. \tag{5.42}$$

Passing to the limit as $k \to +\infty$ in (5.42), by the continuity of p_∞ and by L^1-convergence of p_k to p we obtain that

$$V(t) = V_0 + \int_{[0,t]\setminus E_j} \dot V(s)\,\mathrm{d}s + \int_{E_j} (p_\infty(s) - p(s))\,\mathrm{d}s\,,$$

from which we deduce, passing to the limit as $j \to +\infty$ and recalling that $\dot V = 0$ a.e. in $\{V = 0\}$, that

$$V(t) = V_0 + \int_0^t (p_\infty(s) - p(s))\mathbf{1}_{\{V>0\}}(s)\,\mathrm{d}s\,.$$

This concludes the proof of condition (c) of Definition 5.1. □

6. Derivatives of the energy and Griffith's principle

In this section we discuss some properties of a quasi-static evolution $(\Gamma, V)\colon [0,T] \to \mathcal{C}_\eta \times [0,+\infty)$ given by Definition 5.1. In Theorem 6.4 we show that, under suitable regularity assumptions on the crack set, the reduced energy (5.1) is differentiable with respect to time, to the crack length, and to the fluid volume. The main result of this section is Theorem 6.8, where we prove that the evolution (Γ, V) satisfies the *Griffith's criterion* (see [12]).

Let us start with the computation of the derivatives of the reduced energy (5.1). We will do it in a quite general setting, assuming that the crack path is known a priori: the crack set can only evolve along a curve $\Lambda \in \mathcal{C}_\eta$. For technical reasons, we need Λ to be of class $C^{2,1}$.

Remark 6.1. Since we are interested in the (a posteriori) properties of a quasi-static evolution (Γ, V), we notice that it is not so strange to assume that the crack can only move along a prescribed path. Indeed, once the crack set function $\Gamma\colon [0,T] \to \mathcal{C}_\eta$ is given, it is clear that the fracture grows following $\Gamma(T)$. Hence, the true assumption is that $\Gamma(T)$ (or Λ) is a $C^{2,1}$-curve.

Let $L := \mathcal{H}^1(\Lambda) > 0$, and let $\lambda\colon [0,L] \to \mathbb{R}^2$ be an arc-length parametrization of Λ of class $C^{2,1}$ such that $\lambda(0) = (0,0)$. In what follows, we denote by λ_1 and λ_2 the components of λ. Moreover, for every $s \in [0,L]$, we define

$$\Lambda_s := \{\lambda(\sigma) : \, 0 \le \sigma \le s\} . \tag{6.1}$$

In order to do our computations, we will need to slightly move the crack tip along the prescribed curve Λ. Thus, for $s \in (0,L)$ and δ such that $s+\delta \in [0,L]$, we construct a $C^{2,1}$-diffeomorphism $F_{s,\delta}$ such that $F_{s,\delta}(\mathbb{R}^2_+) = \mathbb{R}^2_+$, $F_{s,\delta}|_\Sigma = id|_\Sigma$, and $F_{s,\delta}(\Lambda_s) = \Lambda_{s+\delta}$. Indeed, by definition of the class $\mathcal{C}_\eta$ and by our regularity assumption, there exists $\lambda_g\colon [0,\lambda_2(L)] \to \mathbb{R}$ of class $C^{2,1}$ such that $\Lambda = \mathit{graph}(\lambda_g) = \{(\lambda_g(x_2), x_2) : \, x_2 \in [0,\lambda_2(L)]\}$.

Choose $\zeta > 0$ and a cut-off function $\vartheta \in C^\infty_c(\mathrm{B}_{\zeta/2}(0))$ with $\vartheta = 1$ on $\overline{\mathrm{B}}_{\zeta/3}(0)$. We define $F_{s,\delta}\colon \mathbb{R}^2 \to \mathbb{R}^2$ by

$$F_{s,\delta}(x) := x + \left(\begin{array}{c} \lambda_g(x_2 + (\lambda_2(s+\delta) - \lambda_2(s))\vartheta(\lambda(s) - x)) - \lambda_g(x_2) \\ (\lambda_2(s+\delta) - \lambda_2(s))\vartheta(\lambda(s) - x) \end{array} \right) \tag{6.2}$$

if $x \in \mathrm{B}_{\zeta/2}(\lambda(s))$, while $F_{s,\delta}(x) := x$ if $x \in \mathbb{R}^2 \setminus \overline{\mathrm{B}}_{\zeta/2}(\lambda(s))$.

In the following lemma, we give some properties of $F_{s,\delta}$ (see, e.g., [14]).

Lemma 6.2. *For every $s \in (0,L)$, there exists $\delta_0 > 0$ such that:*

- (*a*) *$F_{s,\cdot} \in C^{2,1}((-\delta_0,\delta_0) \times \mathbb{R}^2; \mathbb{R}^2)$ and, for every $|\delta| < \delta_0$, the map $F_{s,\delta}$ is a $C^{2,1}$-diffeomorphism. Moreover, $F_{s,\delta}(\mathbb{R}^2_+) = \mathbb{R}^2_+$, $F_{s,\delta}(\lambda(s)) = \lambda(s+\delta)$, $F_{s,\delta}(\Lambda_s) = \Lambda_{s+\delta}$, and $F_{s,\delta}(x) = x$ for every $x \in \mathbb{R}^2 \setminus \mathrm{B}_{\zeta/2}(\lambda(s))$;*
- (*b*) *the norms $\|F_{s,\delta}\|_{C^{2,1}}$ and $\|F^{-1}_{s,\delta}\|_{C^{2,1}}$ are uniformly bounded with respect to δ and there exist c_1, $c_2 > 0$ such that, for every $|\delta| < \delta_0$ and every $x \in \mathbb{R}^2$, we have $c_1 \le \det \nabla F_{s,\delta}(x) \le c_2$;*
- (*c*) *$\|id - F_{s,\delta}\|_{C^2} \to 0$ as $\delta \to 0$;*
- (*d*) *some derivatives:*

$$\rho_s(x) := \partial_\delta(F_{s,\delta}(x))|_{\delta=0} = \lambda_2'(s)\vartheta(\lambda(s) - x) \left(\begin{array}{c} \lambda_g'(x_2) \\ 1 \end{array} \right) ,$$

$$\partial_\delta(\det \nabla F_{s,\delta})|_{\delta=0} = \mathrm{div}\rho_s ,$$

$$\partial_\delta(\nabla F_{s,\delta})|_{\delta=0} = -\partial_\delta(\nabla F_{s,\delta})^{-1}|_{\delta=0} = \nabla\rho_s ,$$

$$\partial_\delta(\mathit{cof}\, \nabla F_{s,\delta})^T|_{\delta=0} = -\partial_\delta(\mathit{cof}\, \nabla F_{s,\delta})^{-T}|_{\delta=0} = \mathrm{div}\rho_s \, \mathrm{I} - \nabla\rho_s .$$

Proof. See [11] for the proof of (a), (b), and (d) in the case of C^∞ maps. The same arguments are applicable with the $C^{2,1}$ regularity of $F_{s,\delta}$. Property (c) follows immediately from the definition (6.2) of $F_{s,\delta}$. □

As we have seen in Corollary 4.2, a solution to the minimum problem (4.1) which defines the reduced energy $\mathcal{E}_m$ exists and is unique up to a horizontal translation. In order to compute the derivatives of $\mathcal{E}_m$ with respect to the crack length s and to the volume V, it is convenient to slightly modify the set of admissible displacements $\mathcal{A}$ defined in (3.18) in such a way that the minimizer of (4.1) is unique. To do this, it is enough, for instance, to fix the mean value of the first component of the displacement in an open set $E \subset\subset \mathbb{R}^2_+ \setminus \Lambda$ with $E \neq \emptyset$. Thus, for every $s \in [0,L]$, every $V \in [0,+\infty)$, and every $\epsilon \in \mathbb{R}$ we define

$$\widetilde{\mathcal{A}}(\Lambda_s, V, \epsilon) := \Big\{ u \in \mathcal{A}(\Lambda_s, V, \epsilon) : \int_E u_1 \, \mathrm{d}x = 0 \Big\} . \tag{6.3}$$

For simplicity of notation, when $\epsilon = 0$ we set $\widetilde{\mathcal{A}}(\Lambda_s, V) := \widetilde{\mathcal{A}}(\Lambda_s, V, 0)$. We notice that

$$\widetilde{\mathcal{A}}(\Lambda_s, V) = \Big\{ u \in LD^2_E(\mathbb{R}^2_+ \setminus \Lambda_s) : \, [u]\cdot\nu_{\Lambda_s} \ge 0 \text{ on } \Lambda_s , \, \int_{\Lambda_s} [u]\cdot\nu_{\Lambda_s} \, \mathrm{d}\mathcal{H}^1 = V \Big\} ,$$

where $LD^2_E(\mathbb{R}^2_+ \setminus \Lambda_s)$ is defined in (2.5).

In view of Corollary 4.2, for every $s \in [0,L]$ and every $V \in [0,+\infty)$ there exists a unique $u^s_V \in \widetilde{\mathcal{A}}(\Lambda_s, V)$ solution of (4.1) for the triple $(\Lambda_s, V, 0)$. In particular, for every $\epsilon \in \mathbb{R}$ we have that

$$\mathcal{E}(u^s_V, \Lambda_s, 0) = \mathcal{E}(u^s_V + \epsilon\, id, \Lambda_s, \epsilon)\,.$$

This implies that, for every $t \in [0,T]$,

$$\mathcal{E}_m(t, \Lambda_s, V) = \min_{u \in \mathcal{A}(\Lambda_s, V, \epsilon(t))} \mathcal{E}(u, \Lambda_s, \epsilon(t)) = \min_{u \in \widetilde{\mathcal{A}}(\Lambda_s, V)} \mathcal{E}(u, \Lambda_s, 0) =: \widetilde{\mathcal{E}}_m(\Lambda_s, V)\,. \tag{6.4}$$

Definition 6.3. For every $s \in (0,L)$ and every $V \in [0,+\infty)$, we set

$$\begin{aligned}\mathcal{G}(s,V) := &\int_{\mathbb{R}^2_+ \setminus \Lambda_s} \mathbb{C}\mathrm{E}u^s_V \cdot \nabla\big((\mathrm{div}\rho_s\, \mathrm{I} - \nabla\rho_s)\, u^s_V\big)\, \mathrm{d}x \\ &+ \int_{\mathbb{R}^2_+ \setminus \Lambda_s} \mathbb{C}\mathrm{E}u^s_V \cdot \big(\nabla u^s_V \nabla\rho_s\big)\, \mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^2_+ \setminus \Lambda_s} \mathbb{C}\mathrm{E}u^s_V \cdot \mathrm{E}u^s_V \mathrm{div}\rho_s\, \mathrm{d}x\end{aligned}$$

We will refer to $\mathcal{G}(s,V)$ as the *energy release rate.*

In the following theorem we give explicit formulas of the derivatives of the reduced energy (5.1) with respect to t, s, and V.

Theorem 6.4. *Let* $t \in [0,T]$, $s \in (0,L)$, *and* $V \in [0,+\infty)$. *Then*

$$\frac{\partial \mathcal{E}_m}{\partial t}(t, \Lambda_s, V) = \frac{\partial \widetilde{\mathcal{E}}_m}{\partial t}(\Lambda_s, V) = 0\,, \tag{6.5}$$

$$\frac{\partial \mathcal{E}_m}{\partial s}(t, \Lambda_s, V) = \frac{\partial \widetilde{\mathcal{E}}_m}{\partial s}(\Lambda_s, V) = \kappa - \mathcal{G}(s,V)\,, \tag{6.6}$$

where κ *is defined in* (3.19).

If, in addition, $V > 0$, *then*

$$\frac{\partial \mathcal{E}_m}{\partial V}(t, \Lambda_s, V) = \frac{\partial \widetilde{\mathcal{E}}_m}{\partial V}(\Lambda_s, V) = p(\Lambda_s, V, \epsilon(t)) + \sigma(\epsilon(t))\,. \tag{6.7}$$

To prove Theorem 6.4 we need to introduce, for every $s \in (0,L)$ and $\delta \in (-\delta_0, \delta_0)$ (see Lemma 6.2), the Piola transformation $P_{s,\delta}$ associated to $F_{s,\delta}$:

$$P_{s,\delta}\, u := (\mathit{cof}\, \nabla F_{s,\delta})^T u \circ F_{s,\delta} \qquad \text{for every } u \in \widetilde{\mathcal{A}}(\Lambda_{s+\delta}, V)\,. \tag{6.8}$$

We refer to [4, Section 1.7] for the main properties of $P_{s,\delta}$. We notice that, at least for $|\delta|$ small, $P_{s,\delta}$ is an isomorphism between $\widetilde{\mathcal{A}}(\Lambda_{s+\delta}, V)$ and $\widetilde{\mathcal{A}}(\Lambda_s, V)$ whose inverse is given by

$$P^{-1}_{s,\delta}\, u := ((\mathit{cof}\, \nabla F_{s,\delta})^{-T} u) \circ F^{-1}_{s,\delta} \qquad \text{for every } u \in \widetilde{\mathcal{A}}(\Lambda_s, V)\,. \tag{6.9}$$

Lemma 6.5. *Let* $s \in (0,L)$ *and let* $u_\delta \in LD^2_E(\mathbb{R}^2_+ \setminus \Lambda)$. *Assume that there exists* $u_0 \in LD^2_E(\mathbb{R}^2_+ \setminus \Lambda)$ *such that* $u_\delta \to u_0$ *in* $LD^2_E(\mathbb{R}^2_+ \setminus \Lambda)$ *as* $\delta \to 0$. *Then the sequences* $u_\delta \circ F_{s,\delta}$, $u_\delta \circ F^{-1}_{s,\delta}$, $P_{s,\delta}\, u_\delta$, *and* $P^{-1}_{s,\delta}\, u_\delta$ *converge to* u_0 *strongly in* $LD^2_E(\mathbb{R}^2_+ \setminus \Lambda)$ *as* $\delta \to 0$.

Proof. Thanks to Proposition 2.4 and to the properties stated in Lemma 6.2, the lemma can be easily proved by using the changes of coordinates $x = F^{-1}_{s,\delta}(y)$ and $x = F_{s,\delta}(y)$. □

Before proving Theorem 6.4, we show the continuity of u^s_V with respect to the parameters s and V.

Lemma 6.6. *Let* $s_k, s \in (0,L)$ *and* $V_k, V \in [0,+\infty)$ *be such that* $s_k \to s$ *and* $V_k \to V$. *Let* $u^{s_k}_{V_k} \in \widetilde{\mathcal{A}}(\Lambda(s_k), V_k)$ *be the sequence of solutions of* (4.1) *corresponding to* s_k *and* V_k. *Then* $u^{s_k}_{V_k} \to u^s_V$ *in* $LD^2_E(\mathbb{R}^2_+ \setminus \Lambda)$.

Proof. Arguing as in the proof of Lemma 5.6, we can show that

$$\|\mathrm{E}u^{s_k}_{V_k}\|_{2,\mathbb{R}^2_+} \leq MV_k \tag{6.10}$$

for some $M \in \mathbb{R}$. Hence, by Propositions 2.4 and 4.1, there exists $u \in \widetilde{\mathcal{A}}(\Lambda_s, V)$ such that, up to a subsequence, $\mathrm{E}u^{s_k}_{V_k} \rightharpoonup \mathrm{E}u$ weakly in $L^2(\mathbb{R}^2_+)$. If $V=0$ we have that $u^s_V = 0$ and, by (6.10), $u^{s_k}_{V_k} \to 0$ in $LD^2_E(\mathbb{R}^2_+ \setminus \Lambda)$.

Assume now that $V>0$. Let us prove that $u = u^s_V$. By Lemma 6.2 and by the properties of the Piola transformation (6.8), for k large enough we have

$$v_k := \frac{V_k}{V} P^{-1}_{s,s_k-s}\, u^s_V \in \widetilde{\mathcal{A}}(\Lambda(s_k), V_k)\,.$$

Thanks to Lemma 6.5, $v_k \to u^s_V$ in $LD^2_E(\mathbb{R}^2_+ \setminus \Lambda)$ as $k \to +\infty$. Thus, by the minimality of $u^{s_k}_{V_k}$ we obtain

$$\begin{aligned}\widetilde{\mathcal{E}}_m(\Lambda_s, V) &\leq \mathcal{E}(u, \Lambda_s, 0) \leq \liminf_{k\to+\infty} \mathcal{E}(u^{s_k}_{V_k}, \Lambda(s_k), 0)\\ &\leq \limsup_{k\to+\infty} \mathcal{E}(u^{s_k}_{V_k}, \Lambda(s_k), 0) \leq \lim_{k\to+\infty} \mathcal{E}(v_k, \Lambda(s_k), 0)\\ &= \mathcal{E}(u^s_V, \Lambda_s, 0) = \widetilde{\mathcal{E}}_m(\Lambda_s, V)\,.\end{aligned} \tag{6.11}$$

From (6.11) we deduce that $u = u^s_V$ and that $u^{s_k}_{V_k} \to u^s_V$ in $LD^2_E(\mathbb{R}^2_+ \setminus \Lambda)$. □

We are now ready to prove Theorem 6.4.

Proof of Theorem 6.4. In view of (6.4), it is clear that $\mathcal{E}_m$ and $\widetilde{\mathcal{E}}_m$ do not depend on t, hence (6.5) holds.

In order to prove (6.6) and (6.7), we use the ideas of [1, 13]. Let us start with (6.6). Let $s \in (0, L)$ and $V \in [0, +\infty)$. Recalling the notation introduced in (6.8) and (6.9), we set

$$u^{s,\delta}_V := (cof\, \nabla F_{s,\delta})^{-T} u^s_V = (P^{-1}_{s,\delta}\, u^s_V) \circ F_{s,\delta}\,. \tag{6.12}$$

By (6.9), we have that $P^{-1}_{s,\delta} u^s_V \in \widetilde{\mathcal{A}}(\Lambda_{s+\delta}, V)$. Hence, by definition of $\widetilde{\mathcal{E}}_m$ and by the change of variables $x = F^{-1}_{s,\delta}(y)$, for $\delta > 0$ small enough we have

$$\begin{aligned}\frac{\widetilde{\mathcal{E}}_m(\Lambda_{s+\delta}, V) - \widetilde{\mathcal{E}}_m(\Lambda_s, V)}{\delta} &\leq \frac{\mathcal{E}(P^{-1}_{s,\delta} u^s_V, \Lambda_{s+\delta}, 0) - \mathcal{E}(u^s_V, \Lambda_s, 0)}{\delta}\\ = \frac{1}{2\delta}\bigg(\int_{\mathbb{R}^2_+\setminus\Lambda_s} &\mathbb{C}\big(\nabla u^{s,\delta}_V\, (\nabla F_{s,\delta})^{-1}\big) \cdot \nabla u^{s,\delta}_V\, (\nabla F_{s,\delta})^{-1} \det \nabla F_{s,\delta}\, \mathrm{d}x\\ &- \int_{\mathbb{R}^2_+\setminus\Lambda_s} \mathbb{C}\mathrm{E}u^s_V \cdot \mathrm{E}u^s_V\, \mathrm{d}x\bigg) + \kappa\,.\end{aligned}$$

Thanks to the properties of $F_{s,\delta}$ stated in Lemma 6.2, applying the dominated convergence theorem we easily get that

$$\limsup_{\delta\searrow 0} \frac{\widetilde{\mathcal{E}}_m(\Lambda_{s+\delta}, V) - \widetilde{\mathcal{E}}_m(\Lambda_s, V)}{\delta} \leq \kappa - \mathcal{G}(s, V)\,. \tag{6.13}$$

On the other hand, if we set $U^{s,\delta}_V := u^{s+\delta}_V \circ F_{s,\delta}$, for $\delta > 0$ small we have, in view of (6.8),

$$\begin{aligned}\frac{\widetilde{\mathcal{E}}_m(\Lambda_{s+\delta}, V) - \widetilde{\mathcal{E}}_m(\Lambda_s, V)}{\delta} &\geq \frac{\mathcal{E}(u^{s+\delta}_V, \Lambda_{s+\delta}, 0) - \mathcal{E}(P_{s,\delta} u^{s+\delta}_V, \Lambda_s, 0)}{\delta}\\ = \frac{1}{2\delta}\bigg(\int_{\mathbb{R}^2_+\setminus\Lambda_s} &\mathbb{C}\big(\nabla U^{s,\delta}_V (\nabla F_{s,\delta})^{-1}\big) \cdot \nabla U^{s,\delta}_V (\nabla F_{s,\delta})^{-1} \det \nabla F_{s,\delta}\, \mathrm{d}x\\ &- \int_{\mathbb{R}^2_+\setminus\Lambda_s} \mathbb{C}\nabla(P_{s,\delta}\, u^{s+\delta}_V) \cdot \nabla(P_{s,\delta}\, u^{s+\delta}_V)\, \mathrm{d}x\bigg) + \kappa\,.\end{aligned} \tag{6.14}$$

By Lemmas 6.5 and 6.6 we have that $U_V^{s,\delta}$ and $P_{s,\delta}u_V^{s+\delta}$ converge to u_V^s in $LD^2_E(\mathbb{R}^2_+ \setminus \Lambda)$. Thus, by the dominated convergence theorem, passing to the limit in (6.14) as $\delta \searrow 0$ and recalling (6.13), we obtain

$$\lim_{\delta\searrow 0} \frac{\widetilde{\mathcal{E}}_m(\Lambda_{s+\delta}, V) - \widetilde{\mathcal{E}}_m(\Lambda_s, V)}{\delta} = \kappa - \mathcal{G}(s, V)\,. \tag{6.15}$$

With the same argument we can prove that

$$\lim_{\delta\nearrow 0} \frac{\widetilde{\mathcal{E}}_m(\Lambda_{s+\delta}, V) - \widetilde{\mathcal{E}}_m(\Lambda_s, V)}{\delta} = \kappa - \mathcal{G}(s, V)\,, \tag{6.16}$$

which, together with (6.15), implies (6.6).

Equality (6.7) can be proved with the same technique. For every $V > 0$, let us show that

$$\lim_{\delta\searrow 0} \frac{\widetilde{\mathcal{E}}_m(\Lambda_s, V+\delta) - \widetilde{\mathcal{E}}_m(\Lambda_s, V)}{\delta} \le p(\Lambda_s, V, \epsilon(t)) + \sigma(\epsilon(t))\,. \tag{6.17}$$

Since $\frac{V+\delta}{V} u_V^s \in \widetilde{\mathcal{A}}(\Lambda_s, V+\delta)$, from (6.4) we deduce that

$$\frac{\widetilde{\mathcal{E}}_m(\Lambda_s, V+\delta) - \widetilde{\mathcal{E}}_m(\Lambda_s, V)}{\delta} \le \frac{1}{2\delta}\left[\left(\frac{V+\delta}{V}\right)^2 - 1\right] \int_{\mathbb{R}^2_+\setminus\Lambda_s} \mathbb{C}\mathrm{E}u_V^s \cdot \mathrm{E}u_V^s \,\mathrm{d}x\,. \tag{6.18}$$

Passing to the lim sup in (6.18) as $\delta \searrow 0$ and taking into account Remarks 4.7 and 4.8, we get (6.17). The rest of the proof can be carried out in a similar way. □

Before stating a Griffith's criterion for our model, we make a comment on formula (6.6) of Theorem 6.4.

Remark 6.7. As we have seen in Proposition 4.3 and Remark 4.8, to every $t \in [0, T]$, $s \in [0, L]$, and $V \in [0, +\infty)$, is associated a pressure $p(\Lambda_s, V, \epsilon(t)) \in [0, +\infty)$ which acts on the fracture lips along the normal ν_{Λ_s}. In order to determine the energy release rate, what is usually done in fracture mechanics (see, e.g., [20]) when a force p is applied to the crack is to compute the derivative of the reduced energy $\mathscr{E}_m$ of (5.4) with respect to the crack length s, keeping p fixed. On the contrary, in (6.6) we have computed the derivative of the reduced energy $\mathcal{E}_m$ of (5.1) with respect to s, keeping the fluid (or crack) volume V fixed.

Let us show that, at least formally, the two derivatives coincide. Indeed, by definition (4.33) of $\mathscr{E}_m$, we notice that, for every $t \in [0, T]$, every $s \in [0, L]$, and every $p \in \mathbb{R}$,

$$\mathscr{E}_m(t, \Lambda_s, p) = \mathcal{E}_m(t, \Lambda_s, V(\Lambda_s, p, \epsilon(t))) - (p + \sigma(\epsilon))V(\Lambda_s, p, \epsilon(t))\,. \tag{6.19}$$

Since $p(\Lambda_s, V(\Lambda_s, p, \epsilon(t)), \epsilon(t)) = p$, computing the derivative of formula (6.19) with respect to s and using (6.6) and (6.7) we obtain

$$\frac{\partial \mathscr{E}_m}{\partial s}(t, \Lambda_s, p) = \kappa - \mathcal{G}(s, V(\Lambda_s, p, \epsilon(t))) = \frac{\partial \mathcal{E}_m}{\partial s}(t, \Lambda_s, V(\Lambda_s, p, \epsilon(t)))\,.$$

We are now ready to state a Griffith's criterion for a quasi-static evolution $(\Gamma, V)\colon [0, T] \to \mathcal{C}_\eta \times [0, +\infty)$ of the hydraulic crack growth problem given by Definition 5.1. In view of the regularity assumption of Theorem 6.4, we have to suppose that the curve $\Gamma(T)$ is of class $C^{2,1}$. Let $L_\Gamma := \mathcal{H}^1(\Gamma(T))$ and let $\gamma\colon [0, L_\Gamma] \to \mathbb{R}^2_+$ be an arc-length parametrization of $\Gamma(T)$ of class $C^{2,1}$. As in (6.1), we set $(\Gamma(T))_s := \gamma([0, s])$ for every $s \in [0, L_\Gamma]$. With this notation, we have the following theorem.

Theorem 6.8. *Let $(\Gamma, V)\colon [0, T] \to \mathcal{C}_\eta \times [0, +\infty)$ be a quasi-static evolution of the hydraulic crack growth problem with the properties stated above. Let $s\colon [0, T] \to [0, L_\Gamma]$ be the function defined by $s(t) := \mathcal{H}^1(\Gamma(t))$ for every $t \in [0, T]$, and let $\mathcal{T}_f := \sup\{t \in [0, T] : s(t) < L_\Gamma\}$. Then the following conditions hold:*

(1) $\dot{s}(t) \ge 0$ *for a.e.* $t \in [0, T]$;
(2) $\mathcal{G}(s(t), V(t)) - \kappa \le 0$ *for every* $t \in [0, \mathcal{T}_f)$;

(3) $(\mathcal{G}(s(t), V(t)) - \kappa)\dot{s}(t) = 0$ *for a.e.* $t \in [0, \mathcal{T}_f)$.

The first condition reflects the irreversibility condition of Definition 5.1. The second condition says that the energy release rate has to be less than or equal to κ during the evolution. Finally, the last condition means that the energy release rate has to be equal to κ when the crack tip moves with a positive velocity. This is the so-called Griffith's criterion in our model.

Proof. Since $t \mapsto s(t)$ is a monotone nondecreasing function, property (1) is clearly satisfied.

Property (2) follows by the global stability condition of Definition 5.1: indeed, for every $t \in [0, \mathcal{T}_f)$ we have that, for $s(t) < \sigma \leq L_\Gamma$,

$$\mathcal{E}_m(t, \Gamma(t), V(t)) \leq \mathcal{E}_m(t, (\Gamma(T))_\sigma, V(t)) . \tag{6.20}$$

Since (6.6) holds, dividing (6.20) by $\sigma - s(t)$ and passing to the limit as $\sigma \searrow s(t)$ we deduce (2).

In order to prove (3), we make more explicit the energy-dissipation balance (5.2): for a.e. $t \in [0, \mathcal{T}_f)$ we have

$$\begin{aligned}\big(p(t) + \sigma(\epsilon(t))\big)\dot{V}(t) &= \frac{d}{dt}\mathcal{E}_m(t, \Gamma(t), V(t)) = \frac{d}{dt}\mathcal{E}_m(t, (\Gamma(T))_{s(t)}, V(t)) \\ &= \big(\kappa - \mathcal{G}(s(t), V(t))\big)\dot{s}(t) + \big(p(t) + \sigma(\epsilon(t))\big)\dot{V}(t) ,\end{aligned}$$

where, in the last equality, we have used the results of Theorem 6.4. □

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