NONLOCAL SCHRÖDINGER–KIRCHHOFF EQUATIONS WITH EXTERNAL MAGNETIC FIELD

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Abstract. The paper deals with the existence and multiplicity of solutions of the fractional Schrödinger–Kirchhoff equation involving an external magnetic potential. As a consequence, the results can be applied to the special case

$$(a + b[u]_{s,A}^{2\vartheta-2})(-\Delta)_A^s u + V(x)u = f(x,|u|)u \quad \text{in } \mathbb{R}^N,$$

where $s \in (0,1)$, $N > 2s$, $a \in \mathbb{R}_0^+$, $b \in \mathbb{R}_0^+$, $\vartheta \in [1, N/(N-2s))$, $A : \mathbb{R}^N \to \mathbb{R}^N$ is a magnetic potential, $V : \mathbb{R}^N \to \mathbb{R}^+$ is an electric potential, $(-\Delta)_A^s$ is the fractional magnetic operator. In the super– and sub–linear cases, the existence of least energy solutions for the above problem is obtained by the mountain pass theorem, combined with the Nehari method, and by the direct methods respectively. In the superlinear–sublinear case, the existence of infinitely many solutions is investigated by the symmetric mountain pass theorem.

1. Introduction and main result

The paper deals with the existence of solutions of the fractional *Schrödinger–Kirchhoff* problem

$$M([u]_{s,A}^2)(-\Delta)_A^s u + V(x)u = f(x,|u|)u \quad \text{in } \mathbb{R}^N, \tag{1.1}$$

where hereafter $s \in (0,1)$, $N > 2s$,

$$[u]_{s,A} = \left(\iint_{\mathbb{R}^{2N}} \frac{|u(x) - e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})}u(y)|^2}{|x-y|^{N+2s}} dxdy \right)^{1/2},$$

$M : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is a Kirchhoff function, $V : \mathbb{R}^N \to \mathbb{R}^+$ is a scalar potential, $A : \mathbb{R}^N \to \mathbb{R}^N$ is a magnetic potential, and $(-\Delta)_A^s$ is the associated fractional magnetic operator which, up to a normalization constant, is defined as

$$(-\Delta)_A^s \varphi(x) = 2 \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{\varphi(x) - e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})}\varphi(y)}{|x-y|^{N+2s}} \, dy, \quad x \in \mathbb{R}^N,$$

along functions $\varphi \in C_0^\infty(\mathbb{R}^N, \mathbb{C})$. Henceforward $B_\varepsilon(x)$ denotes the ball of $\mathbb{R}^N$ centered at $x \in \mathbb{R}^N$ and radius $\varepsilon > 0$. For details on fractional magnetic operators we refer to [14] and to the references [21–24] for the physical background.

The operator $(-\Delta)_A^s$ is consistent with the definition of fractional Laplacian $(-\Delta)^s$ when $A \equiv 0$. For further details on $(-\Delta)^s$, we refer the interested reader to [16]. Nonlocal operators can be seen as the infinitesimal generators of Lévy stable diffusion processes [1]. Moreover, they allow us to develop a generalization of quantum mechanics and also to describe the motion of a chain or an array of particles that are connected by elastic springs as well as unusual diffusion processes in turbulent fluid motions and material transports in fractured media (for more details

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see for example [1, 5, 6] and the references therein). Indeed, the literature on nonlocal fractional operators and on their applications is quite large, see for example the recent monograph [33], the extensive paper [17] and the references cited there.

The paper was motivated by some works appeared in recent years concerning the magnetic Schrödinger equation

$$-(\nabla - \mathrm{i}A)^2 u + V(x)u = f(x, |u|)u \quad \text{in } \mathbb{R}^N, \tag{1.2}$$

which has been extensively studied (see [2, 11, 15, 28, 42]). The magnetic Schrödinger operator is defined as

$$-(\nabla - \mathrm{i}A)^2 u = -\Delta u + 2\mathrm{i}A(x)\cdot\nabla u + |A(x)|^2 u + \mathrm{i}u \,\mathrm{div}A(x).$$

As stated in [43] (see also [36, 37]), up to correcting the operator by the factor $(1-s)$, it follows that $(-\Delta)^s_A u$ converges to $-(\nabla u - \mathrm{i}A)^2 u$ as $s \uparrow 1$. Thus, up to normalization, the nonlocal case can be seen as an approximation of the local case (see Section 2 for further details). As $A = 0$ and $M = 1$, equation (1.1) becomes the fractional Schrödinger equation

$$(-\Delta)^s u + V(x)u = f(x, |u|)u \quad \text{in } \mathbb{R}^N,$$

introduced by Laskin [29, 30]. Here the nonlinearity f satisfies general conditions. We refer, for instance, to [18, 19, 41] and the references therein for recent results.

Throughout the paper, without explicit mention, we also assume that $A : \mathbb{R}^N \to \mathbb{R}^N$ *and* $V : \mathbb{R}^N \to \mathbb{R}^+$ *are continuous functions, and that* V *satisfies*,

(V_1) *there exists* $V_0 > 0$ *such that* $\inf_{\mathbb{R}^N} V \geq V_0$.

The Kirchhoff function $M : \mathbb{R}^+_0 \to \mathbb{R}^+_0$ is assumed to be *continuous and to verify*

(M_1) *for any* $\tau > 0$ *there exists* $\kappa = \kappa(\tau) > 0$ *such that* $M(t) \geq \kappa$ *for all* $t \geq \tau$;

(M_2) *there exists* $\vartheta \in [1, 2^*_s/2)$ *such that* $tM(t) \leq \vartheta \mathscr{M}(t)$ *for all* $t \geq 0$, $\mathscr{M}(t) = \int_0^t M(\tau)d\tau$.

A simple typical example of M is given by $M(t) = a + b\,t^{\vartheta-1}$ for $t \in \mathbb{R}^+_0$, where $a \in \mathbb{R}^+_0$, $b \in \mathbb{R}^+_0$ and $a + b > 0$. When M is of this type, problem (1.1) is said to be *non–degenerate* if $a > 0$, while it is called *degenerate* if $a = 0$.

Clearly, assumptions (M_1) and (M_2) cover the degenerate case. It is worth pointing out that the degenerate case is rather interesting and is treated in well–known papers in Kirchhoff theory, see for example [13]. In the large literature on degenerate Kirchhoff problems, the transverse oscillations of a stretched string, with nonlocal flexural rigidity, depends continuously on the Sobolev deflection norm of u via $M(\|u\|^2)$. From a physical point of view, the fact that $M(0) = 0$ means that the base tension of the string is zero, a very realistic model. The presence of the nonlinear coefficient M is crucial to be considered when the changes in tension during the motion cannot be neglected. In the case of linear string vibrations, the tension is constant that is $M(t) \equiv M(0) > 0$. When the inertial effects of longitudinal modes can be neglected, the tension is spatially uniform along the string and can be directly computed from the elongation of the string according to the Hooke law and arriving to the form of M proposed by Kirchhoff and derived properly by Carrier. Again the case $M(0) = 0$ means that the base tension of the string is zero, a very lifelike prototype.

After the model proposed in 1883 by Kirchhoff in [26] several physicists also considered such equations for their researches in the theory of nonlinear vibrations theoretically or experimentally, see [8,9,34,35]. Carrier [8,9] developed a more rigorous approach to model transverse vibration via the coupled governing equation of planar vibration and recovered the nonlinear integro–partial–differential equation, without quoting Kirchhoff. Narasimha [34] also obtained the equation, called nowadays *the Kirchhoff string equation* in the literature, using another approach.

For fractional degenerate Kirchhoff problems we refer to [3, 7, 32, 40, 45] and the references therein for more details in bounded domains and in the whole space. Recent existence results of solutions for fractional non–degenerate Kirchhoff problems are given, for example, in [20,38,44,46].

Assumptions (M_1) and (M_2) on the Kirchhoff function M are enough to assure the existence of solutions of (1.1). However, to get the existence of ground states, we assume also the further mild request

(M_3) *there exists* $m_0 > 0$ *such that* $M(t) \geq m_0 t^{\vartheta-1}$ *for all* $t \in [0, 1]$,

where ϑ *is the number given in* (M_2) *when* (M_2) *is assumed, otherwise* ϑ *is any number greater than or equal to 1.*

Of course, (M_3) is satisfied also in the model case, even when $M(0) = 0$, that is in the degenerate case. In [40], condition (M_3) was also applied to investigate the existence of entire solutions for the stationary Kirchhoff type equations driven by the fractional p–Laplacian operator in $\mathbb{R}^N$.

Superlinear nonlinearities f satisfy

(f_1) $f \in \mathbb{R}^N \times \mathbb{R}^+ \to \mathbb{R}$ *is a Carathéodory function and there exist* $C > 0$ *and* $p \in (2\vartheta, 2_s^*)$ *such that*

$$|f(x,t)| \leq C(1 + |t|^{p-2}) \quad \textit{for all } (x,t) \in \mathbb{R}^N \times \mathbb{R}^+;$$

(f_2) *There exists a constant* $\mu > 2\vartheta$ *such that*

$$0 < \mu F(x,t) \leq f(x,t)t^2, \quad F(x,t) = \int_0^t f(x,\tau)\tau d\tau,$$

whenever $x \in \mathbb{R}^N$ *and* $t \in \mathbb{R}^+$;

(f_3) $f(x,t) = o(1)$ *as* $t \to 0^+$, *uniformly for* $x \in \mathbb{R}^N$;

(f_4) $\inf_{x \in \mathbb{R}^N} F(x,1) > 0$.

A typical example of f, verifying (f_1)–(f_4), is given by $f(x,|u|) = |u|^{p-2}$, with $2\vartheta < p < 2_s^*$. The fractional solution spaces $\mathscr{H}_{A,V}(\mathbb{R}^N, \mathbb{C})$ and $H^s_{A,V}(\mathbb{R}^N, \mathbb{C})$ are introduced precisely in Section 3. We say that $u \in \mathscr{H}_{A,V}(\mathbb{R}^N, \mathbb{C})$ (resp. $u \in H^s_{A,V}(\mathbb{R}^N, \mathbb{C})$) is a (weak) *solution* of (1.1), if

$$\Re\left[M([u]^2_{s,A}) \iint_{\mathbb{R}^{2N}} \frac{\left[u(x) - e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})}u(y)\right] \cdot \overline{\left[\varphi(x) - e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})}\varphi(y)\right]}}{|x-y|^{N+2s}} dxdy + \int_{\mathbb{R}^N} V u\overline{\varphi} dx\right]$$
$$= \Re \int_{\mathbb{R}^N} f(x,|u|)u\overline{\varphi}dx,$$

for all $\varphi \in \mathscr{H}_{A,V}(\mathbb{R}^N, \mathbb{C})$ (resp. $\varphi \in H^s_{A,V}(\mathbb{R}^N, \mathbb{C})$).

Now we are in a position to state the first existence result.

Theorem 1.1 (Superlinear case)**.** *Assume that* V *satisfies* (V_1), f *satisfies* (f_1)–(f_4) *and* M *fulfills* (M_1)–(M_2). *Then* (1.1) *admits a nontrivial radial mountain pass solution* $u_0 \in \mathscr{H}_{A,V}(\mathbb{R}^N, \mathbb{C})$. *Furthermore, if* M *satisfies* (M_1)–(M_3), *then* (1.1) *has a ground state* $u \in \mathscr{H}_{A,V}(\mathbb{R}^N, \mathbb{C})$ *with positive energy.*

Sublinear nonlinearities f verify

(f_5) *There exist* $q \in (1, 2)$ *and* $a \in L^{\frac{2}{2-q}}(\mathbb{R}^N)$ *such that*

$$|f(x,t)| \leq a(x)t^{q-2} \quad \textit{for all } (x,t) \in \mathbb{R}^N \times \mathbb{R}^+.$$

(f_6) *There exist* $q \in (1, 2)$, $\delta > 0$, $a_0 > 0$ *and a nonempty open subset* Ω *of* $\mathbb{R}^N$ *such that*

$$F(x, t) \geq a_0 t^q \quad \text{for all } (x, t) \in \Omega \times (0, \delta).$$

A typical example of f, verifying (f_5)–(f_6), is $f(x, |u|) = (1 + |x|^2)^{(q-2)/2}|u|^{q-2}$ with $1 < q < 2$. The second result reads as follows.

Theorem 1.2 (Sublinear case)**.** *Assume that* V *satisfies* (V_1)*,* f *satisfies* (f_5)*–*(f_6) *and* M *is continuous in* $\mathbb{R}^+_0$ *and satisfies* (M_1) *and* (M_3)*, with* $\vartheta \geq 1$*. Then* (1.1) *admits a nontrivial solution* $u \in H^s_{A,V}(\mathbb{R}^N, \mathbb{C})$*, which is a ground sate of* (1.1)*.*

To get infinitely many solutions for equation (1.1) in the local sublinear–superlinear case, we also assume

(V_2) *There exists* $h > 0$ *such that*

$$\lim_{|y|\to\infty} \mathscr{L}^N\big(\{x \in B_h(y) : V(x) \leq c\}\big) = 0$$

for all $c > 0$.

(f_7) $F(x, t) \geq 0$ *for all* $(x, t) \in \mathbb{R}^N \times \mathbb{R}^+_0$*, and there exist* $q \in (1, 2)$*, a nonempty open subset* Ω *of* $\mathbb{R}^N$ *and* $a_1 > 0$ *such that*

$$F(x, t) \geq a_1 t^q \quad \text{for all } (x, t) \in \Omega \times \mathbb{R}^+.$$

An example of f, which satisfies assumptions (f_1) and (f_7), is

$$f(x, t) = (1 + |x|^2)^{(q-2)/2}t^{q-2} + t^{p-2} \quad \text{for all } (x, t) \in \mathbb{R}^N \times \mathbb{R}^+_0,$$

when $1 < q < 2 \leq 2\vartheta < p < 2^*_s$.

Theorem 1.3 (Multiplicity – local superlinear–sublinear case)**.** *Assume that* V *satisfies* (V_1)*–*(V_2)*, that* f *fulfills* (f_1) *and* (f_7) *and that* M *is a continuous function in* $\mathbb{R}^+_0$*, verifying* (M_1) *and* (M_3)*, with* $\vartheta \geq 1$*. Then* (1.1) *admits a sequence* $(u_k)_k$ *of nontrivial solutions.*

Remark 1.4. (i) Condition (V_2), which is weaker than the coercivity assumption: $V(x) \to \infty$ as $|x| \to \infty$, was first proposed by *Bartsch* and *Wang* in [4] to overcome the lack of compactness.

(ii) To our best knowledge, Theorem 1.3 is the first result for the Schrödinger–Kirchhoff equations involving concave–convex nonlinearities in the fractional setting. We also refer to [45] for some related multiplicity results.

Remark 1.5. As it is pointed out in [22], in place of the *midpoint prescription*

$$(x, y) \mapsto A\left(\frac{x + y}{2}\right),$$

other (physically justified [22]) prescriptions are viable such as the *averaged prescription*

$$(x, y) \mapsto \int_0^1 A\left((1 - \vartheta)x + \vartheta y\right) d\vartheta =: A_\sharp(x, y).$$

If $(-\Delta)^s_A$ and $(-\Delta)^s_{A_\sharp}$ are the fractional operators associated with the potentials $A((x + y)/2)$ and $A_\sharp(x, y)$ respectively it follows that $(-\Delta)^s_{A_\sharp}$ is Gauge-covariant, while $(-\Delta)^s_A$ is not, namely

$$(-\Delta)^s_{(A+\nabla\phi)_\sharp} = e^{\mathrm{i}\phi}(-\Delta)^s_{A_\sharp}e^{-\mathrm{i}\phi}, \qquad \text{for all } \phi \in \mathscr{S}(\mathbb{R}^n).$$

This is usually relevant for Schrödinger type operators. The results and proofs in this paper carry on in the same way for the operator with averaged prescription $A_\sharp$. Furthermore, the result of [43] extends to the case of $A_\sharp$ with the same proof, that is

$$\lim_{s\nearrow 1}(1-s)\int_\Omega\int_\Omega \frac{|u(x)-e^{\mathrm{i}(x-y)\cdot A_\sharp(x,y)}u(y)|^2}{|x-y|^{n+ps}}dxdy = K_N\int_\Omega |\nabla u - \mathrm{i}A(x)u|^2dx,$$

see the discussion in Section 2 for $A((x+y)/2)$.

The paper is organized as follows. In Section 2 we provide a few remarks about the singular limit as $s\uparrow 1$. In Section 3, we recall some necessary definitions and properties for the functional setting. In Section 4, we obtain some preliminary results. In Section 5, the existence of ground states of (1.1) is obtained by using the mountain pass theorem together with the Nehari method, and by the direct methods respectively. In Section 6, the existence of infinitely many solutions of (1.1) is obtained by using the symmetric mountain pass theorem.

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2. Remarks on the singular limit as $s\uparrow 1$

The functional framework investigated in the paper admits a very nice consistency property with more familiar local problems, in the singular limit as the fractional diffusion parameter s approaches 1. Let Ω be a nonempty open subset of $\mathbb{R}^N$. We denote by $L^2(\Omega,\mathbb{C})$ the Lebesgue space of complex valued functions with summable square, endowed with the norm $\|u\|_{L^2(\Omega,\mathbb{C})}$. We indicate by $H^s_A(\Omega)$ the space of functions $u\in L^2(\Omega,\mathbb{C})$ with finite magnetic Gagliardo semi–norm, given by

$$[u]_{H^s_A(\Omega)} = \left(\iint_{\Omega\times\Omega}\frac{|u(x)-e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})}u(y)|^2}{|x-y|^{N+2s}}dxdy\right)^{1/2}.$$

The space $H^s_A(\Omega)$ is equipped with the norm

$$\|u\|_{H^s_A(\Omega)} = \left(\|u\|^2_{L^2(\Omega,\mathbb{C})}+[u]^2_{H^s_A(\Omega)}\right)^{1/2}.$$

The space $H^s_{0,A}(\Omega)$ is the completion of $C^\infty_c(\Omega,\mathbb{C})$ in $H^s_A(\Omega)$.

Indeed, in the recent paper [43], the following theorem was proved, which is a Bourgain–Brezis–Mironescu type result in the framework of magnetic Sobolev spaces.

Proposition 2.1 (Theorems 1.1 and 1.2 of [43])**.** *Let Ω be an open bounded subset of $\mathbb{R}^N$, with Lipschitz boundary and let A be of class C^2 over $\overline{\Omega}$. Then,*

$$\lim_{s\uparrow 1}(1-s)\iint_{\Omega\times\Omega}\frac{|u(x)-e^{\mathrm{i}(x-y)\cdot A\left(\frac{x+y}{2}\right)}u(y)|^2}{|x-y|^{N+2s}}dxdy = K_N\int_\Omega |\nabla u-\mathrm{i}A(x)u|^2dx$$

for every $u \in H^1_A(\Omega)$*, where*

$$K_N = \frac{1}{2}\int_{\mathbb{S}^{N-1}} |\omega\cdot\mathbf{e}|^2 d\mathcal{H}^{N-1}(\omega),$$

and $\mathbb{S}^{N-1}$ *is the unit sphere of* $\mathbb{R}^N$ *and* $\mathbf{e}$ *any unit vector of* $\mathbb{R}^N$*. Furthermore,*

$$\lim_{s\uparrow 1}(1-s)\iint_{\mathbb{R}^{2N}} \frac{|u(x)-e^{\mathrm{i}(x-y)\cdot A\left(\frac{x+y}{2}\right)}u(y)|^2}{|x-y|^{N+2s}}dxdy = K_N\int_\Omega |\nabla u - \mathrm{i}A(x)u|^2dx$$

for every $u \in H^1_{0,A}(\Omega)$*.*

Problem (1.1) could be treated in an arbitrary smooth open bounded subset Ω of $\mathbb{R}^N$, provided that the solution space is W, which consists of all functions u in $H^s_A(\mathbb{R}^N)$, with $u=0$ in $\mathbb{R}^N\setminus\Omega$. More precisely, consider the non–degenerate model case

$$M(t) = a(s) + b(s)t, \qquad \text{where } a(s)\approx 1-s \ \text{ and } \ b(s)\approx (1-s)^2 b_0 \quad \text{as } s\uparrow 1.$$

Then the corresponding problem (1.1) in Ω writes as

$$\begin{cases} \left(1+(1-s)b_0\displaystyle\iint_{\mathbb{R}^{2N}} \frac{|u(x)-e^{\mathrm{i}(x-y)\cdot A\left(\frac{x+y}{2}\right)}u(y)|^2}{|x-y|^{N+2s}}dxdy\right)\widehat{(-\Delta)^s_A}u + V(x)u = f(x,|u|)u & \text{in } \Omega,\\ u=0 & \text{in } \mathbb{R}^N\setminus\Omega, \end{cases}$$

where u belongs to the solution space W and

$$\widehat{(-\Delta)^s_A}u = (1-s)(-\Delta)^s_A u.$$

This is natural since the Gagliardo semi–norms are typically multiplied by normalizing constants which vanish at the rate of $1-s$. Since by Proposition 2.1

$$\begin{aligned} (1-s)\iint_{\mathbb{R}^{2N}} \frac{|u(x)-e^{\mathrm{i}(x-y)\cdot A\left(\frac{x+y}{2}\right)}u(y)|^2}{|x-y|^{N+2s}}dxdy &\approx \int_\Omega |\nabla u-\mathrm{i}A(x)u|^2dx \quad \text{as } s\uparrow 1,\\ \widehat{(-\Delta)^s_A}u = (1-s)(-\Delta)^s_A u &\approx -(\nabla u - \mathrm{i}A)^2u, \quad \text{as } s\uparrow 1, \end{aligned}$$

the above problem converges to the local problem

$$\begin{cases} -\left(1+b_0\displaystyle\int_\Omega |\nabla u-\mathrm{i}A(x)u|^2dx\right)(\nabla u-\mathrm{i}A)^2u + V(x)u = f(x,|u|)u & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega, \end{cases}$$

which as $A\to O$ reduces to

$$\begin{cases} -\left(1+b_0\displaystyle\int_\Omega |\nabla u|^2dx\right)\Delta u + V(x)u = f(x,|u|)u & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega. \end{cases}$$

This is the classical model of a Schrödinger–Kirchhoff equation. When $b_0=0$, the last two problems become the classical Schrödinger Dirichlet problems with or without external magnetic potential A.

3. Functional setup

We first provide some basic functional setting that will be used in the next sections. The critical exponent 2_s^* is defined as $2N/(N-2s)$. Let $L^2(\mathbb{R}^N, V)$ denote the Lebesgue space of real valued functions with $V(x)|u|^2 \in L^1(\mathbb{R}^N)$, equipped with norm

$$\|u\|_{2,V} = \left(\int_{\mathbb{R}^N} V(x)|u|^2 dx\right)^{1/2} \quad \text{for all } u \in L^2(\mathbb{R}^N, V).$$

The fractional Sobolev space $H_V^s(\mathbb{R}^N)$ is then defined as

$$H_V^s(\mathbb{R}^N) = \{u \in L^2(\mathbb{R}^N, V) : [u]_s < \infty\},$$

where $[u]_s$ is the Gagliardo semi–norm

$$[u]_s = \left(\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} dxdy\right)^{1/2}.$$

The space $H_V^s(\mathbb{R}^N)$ is endowed with the norm

$$\|u\|_s = \left(\|u\|_{2,V}^2 + [u]_s^2\right)^{1/2}.$$

The localized norm, on a compact subset K of $\mathbb{R}^N$, for the space $H_V^s(K)$, is denoted by

$$\|u\|_{s,K} = \left(\int_K V(x)|u|^2 dx + \iint_{K\times K} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} dxdy\right)^{1/2}. \tag{3.1}$$

The embedding $H_V^s(\mathbb{R}^N) \hookrightarrow L^\nu(\mathbb{R}^N)$ is continuous for any $\nu \in [2, 2_s^*]$ by [16, Theorem 6.7], namely there exists a positive constant C such that

$$\|u\|_{L^\nu(\mathbb{R}^N)} \le C\|u\|_s \quad \text{for all } u \in H_V^s(\mathbb{R}^N).$$

Let us set

$$H_{r,V}^s(\mathbb{R}^N) = \{u \in H_V^s(\mathbb{R}^N) : u(x) = u(|x|) \text{ for all } x \in \mathbb{R}^N\}.$$

To prove the existence of radial weak solutions of (1.1), we shall use the following embedding theorem due to P.L. Lions.

Theorem 3.1 (Compact embedding, I – Théorème II.1 of [31])**.** *Let $N \ge 2$. For any $\alpha \in (2, 2_s^*)$ the embedding $H_{r,V}^s(\mathbb{R}^N) \hookrightarrow\hookrightarrow L^\alpha(\mathbb{R}^N)$ is compact.*

Furthermore, we also have

Theorem 3.2 (Compact embedding, II – Theorem 2.1 of [39])**.** *Assume that conditions* (V_1)–(V_2) *hold. Then, for any $\nu \in (2, 2_s^*)$ the embedding $H_V^s(\mathbb{R}^N) \hookrightarrow\hookrightarrow L^\nu(\mathbb{R}^N)$ is compact.*

Let $L_V^2(\mathbb{R}^N, \mathbb{C})$ be the Lebesgue space of functions $u : \mathbb{R}^N \to \mathbb{C}$ with $V|u|^2 \in L^1(\mathbb{R}^N)$, endowed with the (real) scalar product

$$\langle u, v\rangle_{L^2,V} = \Re \int_{\mathbb{R}^N} V(x) u\bar{v} dx \quad \text{for all } u, v \in L^2(\mathbb{R}^N, \mathbb{C}),$$

where $\bar{z}$ denotes complex conjugation of $z \in \mathbb{C}$. Consider now, according to [14], the magnetic Gagliardo semi–norm given by

$$[u]_{s,A} = \left(\iint_{\mathbb{R}^{2N}} \frac{|u(x) - e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})} u(y)|^2}{|x-y|^{N+2s}} dxdy\right)^{1/2}.$$

Define $H^s_{A,V}(\mathbb{R}^N)$ as the closure of $C_c^\infty(\mathbb{R}^N,\mathbb{C})$ with respect to the norm

$$\|u\|_{s,A}=\left(\|u\|^2_{L^2,V}+[u]^2_{s,A}\right)^{1/2}.$$

A scalar product on $H^s_{A,V}(\mathbb{R}^N)$ is given by

$$\langle u,v\rangle_{s,A}=\langle u,v\rangle_{L^2,V}+\Re\iint_{\mathbb{R}^{2N}}\frac{\left[u(x)-e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})}u(y)\right]\cdot\overline{\left[v(x)-e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})}v(y)\right]}}{|x-y|^{N+2s}}dxdy.$$

Arguing as in [14, Proposition 2.1], we see that $\left(H^s_{A,V}(\mathbb{R}^N),\langle\cdot,\cdot\rangle_{s,A}\right)$ is a real Hilbert space.

Lemma 3.3. *For each $u\in H^s_{A,V}(\mathbb{R}^N,\mathbb{C})$*

$$|u|\in H^s_V(\mathbb{R}^N)\quad\text{and}\quad \big\||u|\big\|_s\le\|u\|_{s,A}.$$

Proof. The assertion follows directly from the pointwise diamagnetic inequality

$$\big||u(x)|-|u(y)|\big|\le\left|u(x)-e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})}u(y)\right|,$$

for a.e. $x,y\in\mathbb{R}^N$, see [14, Lemma 3.1, Remark 3.2]. □

Following Lemma 3.3 and using the same discussion of [14, Lemma 3.5], we have

Lemma 3.4. *The embedding*

$$H^s_{A,V}(\mathbb{R}^N,\mathbb{C})\hookrightarrow L^p(\mathbb{R}^N,\mathbb{C})$$

is continuous for all $p\in[2,2^*_s]$. Furthermore, for any compact subset $K\subset\mathbb{R}^N$ and all $p\in[1,2^*_s)$ the embeddings

$$H^s_{A,V}(\mathbb{R}^N,\mathbb{C})\hookrightarrow H^s_V(K,\mathbb{C})\hookrightarrow\hookrightarrow L^p(K,\mathbb{C})$$

are continuous and the latter is compact, where $H^s_V(K,\mathbb{C})$ is endowed with (3.1).

Define now

$$\mathscr{H}_{A,V}(\mathbb{R}^N,\mathbb{C})=\left\{u\in H^s_{A,V}(\mathbb{R}^N,\mathbb{C}):u(x)=u(|x|),\ x\in\mathbb{R}^N\right\}.$$

By Theorems 3.1–3.2 and Lemma 3.3, we have the following lemma (cf. also [14, Lemma 4.1]).

Lemma 3.5. *Let V satisfy (V_1). Let $(u_n)_n$ be a bounded sequence in $\mathscr{H}_{A,V}(\mathbb{R}^N,\mathbb{C})$. Then, up to a subsequence, $(|u_n|)_n$ converges strongly to some function u in $L^p(\mathbb{R}^N)$ for all $p\in(2,2^*_s)$.*

Moreover, if V satisfies (V_1)–(V_2), then for all bounded sequence $(u_n)_n$ in $H^s_{A,V}(\mathbb{R}^N,\mathbb{C})$ the sequence $(|u_n|)_n$ admits a subsequence converging strongly to some u in $L^p(\mathbb{R}^N)$ for all $p\in[2,2^*_s)$.

4. Preliminary results

The functional $\mathcal{I}:\mathscr{H}_{A,V}(\mathbb{R}^N,\mathbb{C})\to\mathbb{R}$, associated with equation (1.1), is defined by

$$\mathcal{I}(u)=\frac{1}{2}\mathscr{M}([u]^2_{s,A})+\frac{1}{2}\|u\|^2_{L^2,V}-\int_{\mathbb{R}^N}F(x,|u|)dx.$$

It is easy to see that $\mathcal{I}$ is of class $C^1(\mathscr{H}_{A,V}(\mathbb{R}^N,\mathbb{C}),\mathbb{R})$ and

$$\begin{aligned}\langle\mathcal{I}'(u),v\rangle=&\Re\Bigg[M([u]^2_{s,A})\iint_{\mathbb{R}^{2N}}\frac{(u(x)-e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})}u(y))\overline{(v(x)-e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})}v(y))}}{|x-y|^{N+2s}}dxdy\\&+\int_{\mathbb{R}^N}Vu\bar{v}dx\Bigg]-\Re\int_{\mathbb{R}^N}f(x,|u|)u\bar{v}dx,\end{aligned}$$

for all $u, v \in \mathscr{H}_{A,V}(\mathbb{R}^N, \mathbb{C})$. Hereafter, $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $\big(\mathscr{H}_{A,V}(\mathbb{R}^N, \mathbb{C})\big)'$ and $\mathscr{H}_{A,V}(\mathbb{R}^N, \mathbb{C})$.

Hence, the critical points of $\mathcal{I}$ are exactly the weak solutions of (1.1). Moreover, $\mathscr{M}([u]_{s,A}^2)$ is weakly lower semi–continuous in $\mathscr{H}_{A,V}(\mathbb{R}^N, \mathbb{C})$ by the weak lower semi–continuity of $u \mapsto [u]_{s,A}^2$ jointly with the monotonicity and continuity of $\mathscr{M}$. Hence, $\mathcal{I}$ is weakly lower semi–continuous in $\mathscr{H}_{A,V}(\mathbb{R}^N, \mathbb{C})$, being $\int_{\mathbb{R}^N} F(x, |u|)dx$ weakly continuous in $\mathscr{H}_{A,V}(\mathbb{R}^N, \mathbb{C})$.

Definition 4.1. We say that $\mathcal{I}$ satisfies the (PS) *condition* in $\mathscr{H}_{A,V}(\mathbb{R}^N, \mathbb{C})$, if any (PS) sequence $(u_n)_n \subset \mathscr{H}_{A,V}(\mathbb{R}^N, \mathbb{C})$, namely a sequence such that $(\mathcal{I}(u_n))_n$ is bounded and $\mathcal{I}'(u_n) \to 0$ as $n \to \infty$, admits a strongly convergent subsequence in $\mathscr{H}_{A,V}(\mathbb{R}^N, \mathbb{C})$.

Lemma 4.2 (Palais–Smale condition)**.** *Let* (M_1)*–*(M_2) *and* (f_1)*–*(f_3) *hold. Then* $\mathcal{I}$ *satisfies the* (PS) *condition in* $\mathscr{H}_{A,V}(\mathbb{R}^N, \mathbb{C})$*.*

Proof. Let $(u_n)_n$ be a (PS) sequence in $\mathscr{H}_{A,V}(\mathbb{R}^N, \mathbb{C})$. Then there exists $C > 0$ such that $|\mathcal{I}(u_n)| \leq C$ and $|\langle \mathcal{I}'(u_n), u_n \rangle| \leq C\|u_n\|_{s,A}$ for all n. As in Lemma 4.5 of [7], see also [12], we divide the proof into two parts.

• *Case* $\inf_{n\in\mathbb{N}}[u_n]_{s,A} = d > 0$. By (M_1), there exists $\kappa = \kappa(d) > 0$ with $M(t) \geq \kappa > 0$ for all $t \geq d$. Thus, (M_2) and (f_2) yield

$$
\begin{aligned}
C + C\|u_n\|_{s,A} &\geq \mathcal{I}(u_n) - \frac{1}{\mu}\langle \mathcal{I}'(u_n), u_n \rangle \\
&= \frac{1}{2}\mathscr{M}([u_n]_{s,A}^2) - \frac{1}{\mu}M([u_n]_{s,A}^2)[u_n]_{s,A}^2 + \left(\frac{1}{2} - \frac{1}{\mu}\right)\|u_n\|_{L^2,V}^2 \\
&\quad - \frac{1}{\mu}\int_{\mathbb{R}^N}(\mu F(x, |u_n|) - f(x, |u_n|)|u_n|^2)dx \\
&\geq \frac{1}{2}\mathscr{M}([u_n]_{s,A}^2) - \frac{1}{\mu}M([u_n]_{s,A}^2)[u_n]_{s,A}^2 + \left(\frac{1}{2} - \frac{1}{\mu}\right)\|u_n\|_{L^2,V}^2 \\
&\geq \left(\frac{1}{2\vartheta} - \frac{1}{\mu}\right)M([u_n]_{s,A}^2)[u_n]_{s,A}^2 + \left(\frac{1}{2} - \frac{1}{\mu}\right)\|u_n\|_{L^2,V}^2 \\
&\geq \kappa\left(\frac{1}{2\vartheta} - \frac{1}{\mu}\right)[u_n]_{s,A}^2 + \left(\frac{1}{2} - \frac{1}{\mu}\right)\|u_n\|_{L^2,V}^2.
\end{aligned}
\tag{4.1}
$$

This implies at once that $(u_n)_n$ is bounded in $\mathscr{H}_{A,V}(\mathbb{R}^N, \mathbb{C})$, being $\mu > 2\vartheta$. Going if necessary to a subsequence, thanks to Lemmas 3.4 and 3.5, we have

$$
\begin{aligned}
&u_n \rightharpoonup u \ \text{ in } \mathscr{H}_{A,V}(\mathbb{R}^N, \mathbb{C}), \quad u_n \to u \ \text{ a.e. in } \mathbb{R}^N, \\
&|u_n| \to |u| \ \text{ in } L^p(\mathbb{R}^N), \\
&|u_n| \leq h \ \text{ a.e. in } \mathbb{R}^N, \quad \text{for some } h \in L^p(\mathbb{R}^N).
\end{aligned}
\tag{4.2}
$$

To prove that $(u_n)_n$ converges strongly to u in $\mathscr{H}_{A,V}(\mathbb{R}^N, \mathbb{C})$ as $n \to \infty$, we first introduce a simple notation. Let $\varphi \in \mathscr{H}_{A,V}(\mathbb{R}^N, \mathbb{C})$ be fixed and denote by $L(\varphi)$ the linear functional on $\mathscr{H}_{A,V}(\mathbb{R}^N, \mathbb{C})$ defined by

$$
\langle L(\varphi), v \rangle = \Re \iint_{\mathbb{R}^{2N}} \frac{(\varphi(x) - e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})}\varphi(y))}{|x-y|^{N+2s}} \overline{(v(x) - e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})}v(y))}dxdy,
\tag{4.3}
$$

for all $v \in \mathscr{H}_{A,V}(\mathbb{R}^N, \mathbb{C})$. Clearly, by the Hölder inequality, $L(\varphi)$ is continuous, being

$$
|\langle L(\varphi), v \rangle| \leq \|\varphi\|_{s,A}\|v\|_{s,A}.
$$

Hence the weak convergence in (4.2) gives

$$\lim_{n\to\infty}\langle L(u), u_n - u\rangle = 0.$$

Furthermore, by the boundedness of $M([u_n]_{s,A}^2)$ we have

$$\lim_{n\to\infty} M([u_n]_{s,A}^2)\langle L(u), u_n - u\rangle = 0. \tag{4.4}$$

By (f_1) and (f_3), for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$|f(x,t)t| \le \varepsilon|t| + C_\varepsilon|t|^{p-1} \quad \text{for all } x \in \mathbb{R}^N \text{ and } t \in \mathbb{R}^+. \tag{4.5}$$

Using the Hölder inequality, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^N} \left|(f(x,|u_n|)u_n - f(x,|u|)u)(\overline{u_n - u})\right| dx \\
&\qquad \le \int_{\mathbb{R}^N} [\varepsilon(|u_n| + |u|) + C_\varepsilon(|u_n|^{p-1} + |u|^{p-1})]|u_n - u| dx \\
&\qquad \le \varepsilon(\|u_n\|_{L^2} + \|u\|_{L^2})\|u_n - u\|_{L^2} + C_\varepsilon(\|u_n\|_{L^p}^{p-1} + \|u\|_{L^p}^{p-1})\|u_n - u\|_{L^p} \\
&\qquad \le C\varepsilon + CC_\varepsilon\|u_n - u\|_{L^p}.
\end{aligned} \tag{4.6}$$

The Brézis–Lieb lemma and the fact that $|u_n| \to |u|$ in $L^p(\mathbb{R}^N)$ give

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} |u_n - u|^p dx = \lim_{n\to\infty}\int_{\mathbb{R}^N} \big(|u_n|^p - |u|^p\big) dx = 0.$$

Inserting this in (4.6), we get

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} (f(x,|u_n|)u_n - f(x,|u|)u)(\overline{u_n - u}) dx = 0, \tag{4.7}$$

since ε is arbitrary. Of course, $\langle \mathcal{I}'(u_n) - \mathcal{I}'(u), u_n - u\rangle \to 0$ as $n \to \infty$, since $u_n \rightharpoonup u$ in $\mathscr{H}_{A,V}(\mathbb{R}^N,\mathbb{C})$ and $\mathcal{I}'(u_n) \to 0$ in the dual space of $\mathscr{H}_{A,V}(\mathbb{R}^N,\mathbb{C})$. Thus,

$$\begin{aligned}
o(1) &= \langle \mathcal{I}'(u_n) - \mathcal{I}'(u), u_n - u\rangle \\
&= M([u_n]_{s,A}^2)\langle L(u_n) - L(u), u_n - u\rangle + \|u_n - u\|_{L^2,V}^2 \\
&+ \big(M([u_n]_{s,A}^2) - M([u]_{s,A}^2)\big)\langle L(u), u_n - u\rangle - \Re\int_{\mathbb{R}^N} (f(x,|u_n|)u_n - f(x,|u|)u)(\overline{u_n - u}) dx,
\end{aligned}$$

this, together with (4.4) and (4.7), implies that

$$\lim_{n\to\infty}\Big(M([u_n]_{s,A}^2)\langle L(u_n) - L(u), u_n - u\rangle + \|u_n - u\|_{L^2,V}^2\Big) = 0,$$

which yields $u_n \to u$ in $\mathscr{H}_{A,V}(\mathbb{R}^N,\mathbb{C})$, since $M([u_n]_{s,A}^2) \ge \kappa > 0$ for all $n \ge 1$.

• *Case* $\inf_{n\in\mathbb{N}}[u_n]_{s,A} = 0$. If 0 is an isolated point for $([u_n]_{s,A})_n$, then there is a subsequence $([u_{n_k}]_{s,A})_k$ such that $\inf_{k\in\mathbb{N}}[u_{n_k}]_{s,A} = d > 0$ and one can proceed as before. If, instead, 0 is an accumulation point for $([u_n]_{s,A})_n$, there is a subsequence, still labeled as $(u_n)_n$, such that

$$[u_n]_{s,A} \to 0, \;\; u_n \to 0 \text{ in } L^{2_s^*}(\mathbb{R}^N) \text{ and a.e. in } \mathbb{R}^N. \tag{4.8}$$

We claim that $(u_n)_n$ converges strongly to 0 in $\mathscr{H}_{A,V}(\mathbb{R}^N,\mathbb{C})$. To this aim, we need only to show that $\|u_n\|_{2,V} \to 0$ thanks to (4.8). Now, (4.1) and (4.8) yield that as $n \to \infty$

$$C + C\|u_n\|_{2,V} + o(1) \ge \left(\frac{1}{2} - \frac{1}{\mu}\right)\|u_n\|_{2,V}^2 + o(1).$$

Hence, $(u_n)_n$ is bounded in $L^2(\mathbb{R}^N, V)$ and so in $\mathscr{H}_{A,V}(\mathbb{R}^N, \mathbb{C})$. Thus, by (4.8) and Lemma 3.4

$$u_n \rightharpoonup 0 \text{ in } \mathscr{H}_{A,V}(\mathbb{R}^N, \mathbb{C}) \text{ and } u_n \to 0 \text{ in } L^p(\mathbb{R}^N), \tag{4.9}$$

being $p \in (2, 2_s^*)$. Clearly, by (4.5) and (4.9), for every $\varepsilon > 0$

$$\left| \int_{\mathbb{R}^N} f(x, |u_n|) u_n^2 dx \right| \le \varepsilon \|u_n\|_2^2 + C_\varepsilon \|u_n\|_p^p = \varepsilon C + o(1)$$

as $n \to \infty$. Thus,

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} f(x, |u_n|) u_n^2 dx = 0, \tag{4.10}$$

being $\varepsilon > 0$ arbitrary. Obviously, $\langle \mathcal{I}'(u_n), u_n \rangle \to 0$ as $n \to \infty$, by (4.9) and the fact that $\mathcal{I}'(u_n) \to 0$ in $\big(\mathscr{H}_{A,V}(\mathbb{R}^N, \mathbb{C})\big)'$. Hence, by the continuity of M and (4.8)–(4.10), we have

$$\begin{aligned} o(1) = \langle \mathcal{I}'(u_n), u_n \rangle &= M([u_n]_{s,A}^2)[u_n]_{s,A}^2 + \|u_n\|_{2,V}^2 - \int_{\mathbb{R}^N} f(x, |u_n|) u_n^2 dx \\ &= \|u_n\|_{2,V}^2 + o(1) \end{aligned}$$

as $n \to \infty$. This shows the claim.

Therefore, $\mathcal{I}$ satisfies the (PS) condition in $\mathscr{H}_{A,V}(\mathbb{R}^N, \mathbb{C})$ also in this second case and this completes the proof. □

Before going to the proof of Theorem 1.1, we give some useful preliminary results.

Lemma 4.3 (Mountain Pass Geometry I)**.** *Assume that* (M_1)*–*(M_2)*,* (f_1) *and* (f_3) *hold. Then there exist constant* $\varrho, \alpha > 0$ *such that* $\mathcal{I}(u) \ge \alpha$ *for all* $u \in \mathscr{H}_{A,V}(\mathbb{R}^N, \mathbb{C})$ *with* $\|u\|_{s,A} = \varrho$*.*

Proof. It follows from (f_3) that for any $\varepsilon \in (0, 1)$ there exists $\delta = \delta(\varepsilon) > 0$ such that $|f(x, t)| \le \varepsilon$ for all $x \in \mathbb{R}^N$ and $t \in [0, \delta]$. On the other hand, (f_1) yields that $|f(x, t)| \le C\big(1 + \delta^{2-p}\big)|t|^{p-2}$ for all $x \in \mathbb{R}^N$ and $t > \delta$. In conclusion,

$$|f(x, t)| \le \varepsilon + C\big(1 + \delta^{2-p}\big)|t|^{p-2} \quad \text{for all } x \in \mathbb{R}^N \text{ and } t \in \mathbb{R}_0^+. \tag{4.11}$$

Whence, for some $C_\varepsilon > 0$, we get

$$|F(x, t)| \le \int_0^t |f(x, \tau)\tau| d\tau \le \frac{\varepsilon}{2} t^2 + C_\varepsilon t^p, \tag{4.12}$$

for all $x \in \mathbb{R}^N$ and $t \ge 0$. Moreover, (M_2) gives

$$\mathscr{M}(t) \ge \mathscr{M}(1) t^\vartheta \quad \text{for all } t \in [0, 1], \tag{4.13}$$

while (M_1) implies that $\mathscr{M}(1) > 0$. Thus, using (4.12), (4.13) and the Hölder inequality, we obtain for all $u \in \mathscr{H}_{A,V}(\mathbb{R}^N, \mathbb{C})$, with $\|u\|_{s,A} \leq 1$,

$$\begin{aligned}
\mathcal{I}(u) &= \frac{1}{2}\mathscr{M}(\|u\|_{s,A}^2) + \frac{1}{2}\|u\|_{L^2,V}^2 - \int_{\mathbb{R}^N} F(x,|u|)dx \\
&\geq \frac{\mathscr{M}(1)}{2}[u]_{s,A}^{2\vartheta} + \frac{1}{2}\|u\|_{L^2,V}^2 - \frac{\varepsilon}{2}\int_{\mathbb{R}^N}|u|^2dx - C_\varepsilon\int_{\mathbb{R}^N}|u|^pdx \\
&\geq \frac{\mathscr{M}(1)}{2}[u]_{s,A}^{2\vartheta} + \Big(\frac{1}{2} - \frac{\varepsilon}{2V_0}\Big)\|u\|_{L^2,V}^2 - C_\varepsilon C_p^p\|u\|_{s,A}^p \\
&\geq \min\left\{\frac{\mathscr{M}(1)}{2}, \frac{V_0-\varepsilon}{2V_0}\right\}([u]_{s,A}^{2\vartheta} + \|u\|_{L^2,V}^2) - C_\varepsilon C_p^p\|u\|_{s,A}^p \\
&\geq \min\left\{\frac{\mathscr{M}(1)}{2}, \frac{V_0-\varepsilon}{2V_0}\right\}([u]_{s,A}^{2\vartheta} + \|u\|_{L^2,V}^{2\vartheta}) - C_\varepsilon C_p^p\|u\|_{s,A}^p \\
&\geq 2^{1-\vartheta}\min\left\{\frac{\mathscr{M}(1)}{2}, \frac{V_0-\varepsilon}{2V_0}\right\}([u]_{s,A}^{2} + \|u\|_{L^2,V}^{2})^\vartheta - C_\varepsilon C_p^p\|u\|_{s,A}^p \\
&= \left(2^{1-\vartheta}\min\left\{\frac{\mathscr{M}(1)}{2}, \frac{V_0-\varepsilon}{2V_0}\right\} - C_\varepsilon C_p^p\|u\|_{s,A}^{p-2\vartheta}\right)\|u\|_{s,A}^{2\vartheta},
\end{aligned}$$

where C_p is the embedding constant of $\mathscr{H}_{A,V}(\mathbb{R}^N, \mathbb{C})$ into $L^p(\mathbb{R}^N, \mathbb{C})$ given by Lemma 3.4. Here we used that $\|u_n\|_{L^2,V} \leq \|u_n\|_{s,A} \leq 1$ and the inequality $(a+b)^\vartheta \leq 2^{\vartheta-1}(a^\vartheta + b^\vartheta)$ for all $a, b \geq 0$. Choosing $\varepsilon = V_0/2$ and taking $\|u\|_{s,A} = \varrho \in (0,1)$ so small that

$$2^{1-\vartheta}\min\left\{\frac{\mathscr{M}(1)}{2}, \frac{V_0}{4}\right\} - C_{V_0/2}C_p^p\varrho^{p-2\vartheta} > 0,$$

we have

$$\mathcal{I}(u) \geq \alpha = \left(2^{1-\vartheta}\min\left\{\frac{\mathscr{M}(1)}{2}, \frac{V_0}{4}\right\} - C_{V_0/2}C_p^p\varrho^{p-2\vartheta}\right)\varrho^{2\vartheta} > 0,$$

for all $u \in \mathscr{H}_{A,V}(\mathbb{R}^N, \mathbb{C})$, with $\|u\|_{s,A} = \varrho$. □

Lemma 4.4 (Mountain Pass Geometry II)**.** *Assume that (M_1)–(M_2) and (f_1)–(f_4) hold. Then there exists $e \in C_c^\infty(\mathbb{R}^N, \mathbb{C})$, with $\|e\|_{s,A} \geq 2$, such that $\mathcal{I}(e) < 0$. In particular, $\|e\|_{s,A} > \rho$, where $\rho > 0$ is the number introduced in Lemma 4.3*

Proof. For any $x \in \mathbb{R}^N$, set $k(t) = F(x,t)t^{-\mu}$ for all $t \geq 1$. Condition (f_2) implies that k is nondecreasing on $[1,\infty)$. Therefore, $k(t) \geq k(1)$ for any $t \geq 1$, that is,

$$F(x,t) \geq F(x,1)t^\mu \geq c_F|t|^\mu \quad \text{for all } x \in \mathbb{R}^N \text{and } t \geq 1, \tag{4.14}$$

where $c_F = \inf_{x\in\mathbb{R}^N} F(x,1) > 0$ by assumption (f_4). From (f_3) there exists $\delta \in (0,1)$ such that $|f(x,t)t| \leq t$ for all $x \in \mathbb{R}^N$ and $t \in [0,\delta]$. Furthermore, $|f(x,t)| \leq 2C$ for all $x \in \mathbb{R}^N$ and all t, with $\delta < t \leq 1$, thanks to (f_1). Hence, the above inequalities imply that $f(x,t)t \geq -(1+2C)t$ for $x \in \mathbb{R}^N$ and $t \in [0,1]$. Thus,

$$F(x,t) = \int_0^t f(x,\tau)\tau d\tau \geq -\frac{1+2C}{2}t^2 \quad \text{for all } x \in \mathbb{R}^N \text{ and } t \in [0,1]. \tag{4.15}$$

Combining (4.14) with (4.15), we obtain

$$F(x,t) \geq c_F|t|^\mu - C_F|t|^2 \ \ \text{for all } x \in \mathbb{R}^N \text{ and } t \geq 0, \tag{4.16}$$

where $C_F = c_F + (1+2C)/2$. Again (M_2) gives

$$\mathscr{M}(t) \le \mathscr{M}(1)t^{\vartheta} \quad \text{for all } t \ge 1, \tag{4.17}$$

with $\mathscr{M}(1) > 0$ by (M_1). Fix $u \in C_c^\infty(\mathbb{R}^N, \mathbb{C})$, with $[u]_{s,A} = 1$. By (4.16) and (4.17) as $t \to \infty$

$$\begin{aligned}
\mathcal{I}(tu) &= \frac{1}{2}\mathscr{M}([tu]_{s,A}^2) + \frac{1}{2}\|tu\|_{L^2,V}^2 - \int_{\mathbb{R}^N} F(x, t|u|)dx \\
&\le \frac{\mathscr{M}(1)}{2} t^{2\vartheta}[u]_{s,A}^{2\vartheta} + \frac{1}{2}\|tu\|_{L^2,V}^2 - c_F t^\mu \|u\|_{L^\mu(\mathbb{R}^N)}^\mu + \frac{M_1}{V_0} t^2 \|u\|_{L^2,V}^2 \\
&\le \frac{\mathscr{M}(1)}{2} t^{2\vartheta} - c_F C_\mu^\mu t^\mu \|u\|_{s,A}^\mu + \left(\frac{M_1}{V_0} + \frac{1}{2}\right) t^2 \|u\|_{L^2,V}^2 \\
&\le \frac{\mathscr{M}(1)}{2} t^{2\vartheta} - c_F C_\mu^\mu t^\mu [u]_{s,A}^\mu + \left(\frac{M_1}{V_0} + \frac{1}{2}\right) t^2 \|u\|_{L^2,V}^2 \\
&= \frac{\mathscr{M}(1)}{2} t^{2\vartheta} - c_F C_\mu^\mu t^\mu + \left(\frac{M_1}{V_0} + \frac{1}{2}\right) t^2 \|u\|_{L^2,V}^2 \to -\infty,
\end{aligned}$$

since $2 \le 2\vartheta < \mu$. The assertion follows at once, taking $e = T_0 u$, with $T_0 > 0$ large enough. $\square$

5. Proof of Theorems 1.1 and 1.2

The following standard Mountain Pass Theorem will be used to get our main result.

Theorem 5.1. *Let J be a functional on a real Banach space E and of class $C^1(E, \mathbb{R})$. Let us assume that there exists α, $\rho > 0$ such that*
(i) $J(u) \ge \alpha$ for all $u \in E$ with $\|u\| = \rho$,
(ii) $J(0) = 0$ and $J(e) < \alpha$ for some $e \in E$ with $\|e\| > \rho$.
Let us define $\Gamma = \{\gamma \in C([0,1]; E) : \gamma(0) = 0, \gamma(1) = e\}$, and

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J(\gamma(t)).$$

Then there exists a sequence $(u_n)_n$ in E such that $J(u_n) \to c$ and $J'(u_n) \to 0$ in E', the dual space of E, as $n \to \infty$.

5.1. **Proof of Theorem 1.1.** Taking into account Lemmas 4.3 and 4.4, by Theorem 5.1 there exists a sequence $(u_n)_n \subset \mathscr{H}_{A,V}(\mathbb{R}^N, \mathbb{C})$ such that $\mathcal{I}(u_n) \to c > 0$ and $\mathcal{I}'(u_n) \to 0$ as $n \to \infty$. Then, in view of Lemma 4.2, there exists a nontrivial critical point $u_0 \in \mathscr{H}_{A,V}(\mathbb{R}^N, \mathbb{C})$ of $\mathcal{I}$ with $\mathcal{I}(u_0) = c > 0 = \mathcal{I}(0)$.

Set $\mathscr{N} = \{u \in \mathscr{H}_{A,V}(\mathbb{R}^N, \mathbb{C}) \setminus \{0\} : \mathcal{I}'(u) = 0\}$. Then $u_0 \in \mathscr{N} \ne \emptyset$. Next we show that $\mathcal{I}$ is coercive and bounded from below on $\mathscr{N}$. Indeed, by $\mathcal{I}'(u) = 0$ and (f_2), we get

$$\int_{\mathbb{R}^N} F(x, |u|)dx \le \frac{1}{\mu} \int_{\mathbb{R}^N} f(x, |u|)|u|^2 dx = \frac{1}{\mu}\big(M([u]_{s,A}^2)[u]_{s,A}^2 + \|u\|_{L^2,V}^2\big). \tag{5.1}$$

By using (5.1), (M_2) and the fact that $2 \le 2\vartheta < \mu$, for all $u \in \mathscr{N}$, we have

$$\begin{aligned}
\mathcal{I}(u) &\ge \frac{1}{2}\mathscr{M}(\|u\|_{s,A}^2) + \frac{1}{2}\|u\|_{L^2,V}^2 - \frac{1}{\mu}(M([u]_{s,A}^2)[u]_{s,A}^2 + \|u\|_{L^2,V}^2) \\
&= \left(\frac{1}{2\vartheta} - \frac{1}{\mu}\right) M([u]_{s,A}^2)[u]_{s,A}^2 + \left(\frac{1}{2} - \frac{1}{\mu}\right) \|u\|_{L^2,V}^2 \ge 0.
\end{aligned}$$

Hence, by (M_1) and (M_3) for $u \in \mathscr{N}$

$$\mathcal{I}(u) \geq \left(\frac{1}{2\vartheta} - \frac{1}{\mu}\right) \cdot \left(\|u\|_{L^2,V}^2 + \begin{cases} \kappa[u]_{s,A}^2, & \text{if } [u]_{s,A} \geq 1 \\ m_0[u]_{s,A}^{2\vartheta}, & \text{if } [u]_{s,A} \leq 1 \end{cases}\right), \tag{5.2}$$

where $\kappa = \kappa(1) > 0$ by (M_1). Hence in all cases, for all $u \in \mathscr{N}$

$$\mathcal{I}(u) \geq \min\{\kappa, m_0\} \left(\frac{1}{2\vartheta} - \frac{1}{\mu}\right) \|u\|_{s,A}^2 - 1,$$

by the elementary inequality $t^\vartheta \geq t - 1$ for all $t \in \mathbb{R}_0^+$. In particular, $\mathcal{I}$ is coercive and bounded from below on $\mathscr{N}$.

Define $c_{\min} = \inf\{\mathcal{I}(u) : u \in \mathscr{N}\}$. Clearly, $0 \leq c_{\min} \leq \mathcal{I}(u_0) = c$. Let $(u_n)_n$ be a minimizing for $c_{\min}$, namely $\mathcal{I}(u_n) \to c_{\min}$ and $\langle \mathcal{I}'(u_n), u_n\rangle = 0$. Then, since $\mathscr{N}$ is a complete metric space, by Ekeland's variational principle we can find a new minimizing sequence, still denoted by $(u_n)_n$, which is a (PS) sequence for $\mathcal{I}$ at the level $c_{\min}$. Moreover, Lemma 4.2 implies that $(u_n)_n$ has a convergence subsequence, which we still denote by $(u_n)_n$, such that $u_n \to u$ in $\mathscr{H}_{A,V}(\mathbb{R}^N, \mathbb{C})$. Thus $c_{\min} = \mathcal{I}(u)$ and $\langle \mathcal{I}'(u), u\rangle = 0$.

We claim that $c_{\min} > 0$. Otherwise, there is $(u_n)_n \subset \mathscr{H}_{A,V}(\mathbb{R}^N, \mathbb{C}) \setminus \{0\}$ with $\mathcal{I}'(u_n) = 0$ and $\mathcal{I}(u_n) \to 0$. This via (5.2) implies that $\|u_n\|_{s,A} \to 0$. On the other hand, by (4.11), we have for any $\varepsilon \in (0, V_0)$

$$M([u_n]_{s,A}^2)[u_n]_{s,A}^2 + \|u_n\|_{L^2,V}^2 = \int_{\mathbb{R}^N} f(x, |u_n|)|u_n|^2 dx \leq \frac{\varepsilon}{V_0}\|u_n\|_{L^2,V}^2 + C_\varepsilon C_p^p \|u_n\|_{s,A}^p.$$

Thus, $M([u_n]_{s,A}^2)[u_n]_{s,A}^2 + \Big(1 - \varepsilon/V_0\Big)\|u_n\|_{L^2,V}^2 \leq C_\varepsilon C_p^p \|u_n\|_{s,A}^p$. Now take N_1 so large that $\|u_n\|_{s,A} \leq 1$ for all $n \geq N_1$. Hence, (M_3) implies that for all $n \geq N_1$

$$m_0[u_n]_{s,A}^{2\vartheta} + \big(1 - \varepsilon/V_0\big)\|u_n\|_{L^2,V}^{2\vartheta} \leq C_\varepsilon C_p^p \|u_n\|_{s,A}^p,$$

that is

$$\min\big\{m_0, \big(1 - \varepsilon/V_0\big)\big\} \leq C_\varepsilon C_p^p \|u_n\|_{s,A}^{p-2\vartheta}.$$

This is a contradiction since $2\vartheta < p$ and proves the claim.

Thus, u is a nontrivial critical point of $\mathcal{I}$, with $\mathcal{I}(u) = c_{\min} > 0$. Therefore, u is a ground state solution of (1.1). □

5.2. **Proof of Theorem 1.2.** By (f_5), (V_1) and the Hölder inequality, for all $u \in H_{s,A}^s(\mathbb{R}^N, \mathbb{C})$ we have

$$\begin{aligned}
\mathcal{I}(u) &\geq \frac{1}{2}\mathscr{M}([u]_{s,A}^2) + \frac{1}{2}\|u\|_{L^2,V}^2 - \int_{\mathbb{R}^N} a(x)|u|^q dx \\
&\geq \frac{1}{2}\mathscr{M}([u]_{s,A}^2) + \frac{1}{2}\|u\|_{L^2,V}^2 - \|a\|_{L^{\frac{2}{2-q}}}\|u\|_{L^2}^q \\
&\geq \frac{1}{2}\mathscr{M}([u]_{s,A}^2) + \frac{1}{4}\|u\|_{L^2,V}^2 + \frac{V_0}{4}\|u\|_{L^2}^2 - \|a\|_{L^{\frac{2}{2-q}}}\|u\|_{L^2}^q \\
&\geq \frac{1}{2}\mathscr{M}([u]_{s,A}^2) + \frac{1}{4}\|u\|_{L^2,V}^2 - C_0, \\
C_0 &= \frac{\|a\|_{L^{\frac{2}{2-q}}}}{2q}(2q-1)\left(\frac{2\|a\|_{L^{\frac{2}{2-q}}}}{qV_0}\right)^{q/(2-q)}.
\end{aligned}$$

As shown in (5.2), this, (M_1) and (M_3) imply at once that for all $u \in H^s_{s,A}(\mathbb{R}^N, \mathbb{C})$

$$\mathcal{I}(u) \geq \frac{\min\{\kappa, m_0\}}{4}\|u\|^2_{s,A} - 1 - C_0,$$

$\kappa = \kappa(1)$. Hence $\mathcal{I}$ is coercive and bounded below on $H^s_{s,A}(\mathbb{R}^N, \mathbb{C})$. Set

$$J(u) = \frac{1}{2}\mathscr{M}([u]^2_{s,A}) + \frac{1}{2}\|u\|^2_{L^2,V}, \quad H(u) = \int_{\mathbb{R}^N} F(x, |u|)dx$$

for all $u \in H^s_{A,V}(\mathbb{R}^N)$. Then J is weakly lower semi–continuous in $H^s_{A,V}(\mathbb{R}^N)$, since $\mathscr{M}$ is continuous and monotone non–decreasing in $\mathbb{R}^+_0$. Moreover, by using a similar discussion as [40, Lemma 2.3], one can show that H is weakly continuous on $H^s_{A,V}(\mathbb{R}^N)$ under condition (f_5). Thus, $\mathcal{I}(u) = J(u) - H(u)$ is weakly lower semi–continuous in $H^s_{A,V}(\mathbb{R}^N)$. Then there exists $u_0 \in H^s_{A,V}(\mathbb{R}^N)$ such that

$$\mathcal{I}(u_0) = \inf\{\mathcal{I}(u) : u \in H^s_{A,V}(\mathbb{R}^N)\}.$$

Next we show $u_0 \neq 0$. Let $x_0 \in \Omega$ and let $R > 0$ such that $B_R(x_0) \subset \Omega$. Fix $\varphi \in C^\infty_0(B_R(x_0))$ with $0 \leq \varphi \leq 1$, $\|\varphi\|_{s,A} \leq C(R)$ and $\|\varphi\|_{L^q(B_R(x_0))} \neq 0$. Then, by (f_6) for all $t \in (0, \delta)$

$$\begin{aligned}
\mathcal{I}(t\varphi) &\leq \frac{t^2}{2}\left(\sup_{0\leq\xi\leq(\delta C(R))^2} M(\xi)\right)[\varphi]^2_{s,A} + \frac{t^2}{2}\|\varphi\|^2_{L^2,V} - t^q \int_{B_R(x_0)} a_0|\varphi|^q dx \\
&\leq \frac{t^2}{2}\left(\sup_{0\leq\xi\leq(\delta C(R))^2} M(\xi) + 1\right)\|\varphi\|^2_{s,A} - t^q a_0 \|\varphi\|_{L^q(B_R(x_0))}.
\end{aligned}$$

Since $1 < q < 2$, we get $\mathcal{I}(\bar{t}\varphi) < 0$ by taking $\bar{t} > 0$ small enough. Hence $\mathcal{I}(u_0) \leq \mathcal{I}(\bar{t}\varphi) < 0$, and so u_0 is a nontrivial critical point. In other words, u_0 is a nontrivial solution of (1.1). □

6. Proof of Theorem 1.3

We first recall the following symmetric mountain pass theorem in [25].

Theorem 6.1. *Let X be an infinite dimensional real Banach space. Suppose that J is in $C^1(X, \mathbb{R})$ and satisfies the following condition:*
(a) J is even, bounded from below, $J(0) = 0$ and J satisfies the (PS) condition;
(b) For each $k \in \mathbb{N}$ there exists $E_k \subset \Gamma_k$ such that $\sup_{u\in E_k} J(u) < 0$, where

$$\Gamma_k = \{E : E \text{ is closed symmetric subset of } X \text{ and } 0 \notin E,\ \gamma(E) \geq k\}$$

and $\gamma(E)$ is a genus of a closed symmetric set E. Then J admits a sequence of critical points $(u_k)_k$ such that $J(u_k) \leq 0$, $u_k \neq 0$ and $\|u_k\| \to 0$ as $k \to \infty$.

Let $h \in C^1(\mathbb{R}^+_0, \mathbb{R})$ be a radial decreasing function such that $0 \leq h(t) \leq 1$ for all $t \in \mathbb{R}^+_0$, $h(t) = 1$ for $0 \leq t \leq 1$ and $h(t) = 0$ for $t \geq 2$. Let $\phi(u) = h(\|u\|^2_{s,A})$. Following the idea of [20], we consider the truncation functional

$$\mathcal{I}(u) = \frac{1}{2}\mathscr{M}([u]^2_{s,A}) + \frac{1}{2}\|u\|^2_{L^2,V} - \phi(u)\int_{\mathbb{R}^N} F(x, |u|)dx.$$

Clearly, $\mathcal{I} \in C^1(H^s_{A,V}(\mathbb{R}^N,\mathbb{C}),\mathbb{R})$ and

$$\begin{aligned}\langle \mathcal{I}'(u), v\rangle &= M([u]^2_{s,A})\langle L(u), v\rangle + \Re\int_{\mathbb{R}^N} V(x)u\overline{v}dx\\ &\quad - 2\phi'(u)\int_{\mathbb{R}^N} F(x,|u|)dx\cdot\langle L(u), v\rangle - \phi(u)\Re\int_{\mathbb{R}^N} f(x,|u|)u\overline{v}dx\end{aligned}$$

for all $u, v \in H^s_{A,V}(\mathbb{R}^N,\mathbb{C})$. Here $L(u)$ is the linear functional on $H^s_{A,V}(\mathbb{R}^N,\mathbb{C})$, introduced in (4.3).

6.1. **Proof of Theorem 1.3.** For all $u \in H^s_{A,V}(\mathbb{R}^N,\mathbb{C})$, with $\|u\|_{s,A} \geq 2$, we get

$$\mathcal{I}(u) \geq \frac{1}{2}\mathscr{M}([u]^2_{s,A}) + \frac{1}{2}\|u\|^2_{L^2,V} \geq \frac{1}{2}\min\{\kappa,\, m_0\}\|u\|^2_{s,A},$$

by (M_1) and (M_3), where $\kappa = \kappa(1)$, as in the proof of Theorem 1.2. Hence $\mathcal{I}(u) \to \infty$ as $\|u\|_{s,A} \to \infty$ and $\mathcal{I}$ is coercive and bounded from below on $H^s_{A,V}(\mathbb{R}^N,\mathbb{C})$.

Let $(u_n)_n$ be a (PS) sequence, i.e. $\mathcal{I}(u_n)$ is bounded and $\mathcal{I}'(u_n) \to 0$ as $n \to \infty$. Then the coercivity of $\mathcal{I}$ implies that $(u_n)_n$ is bounded in $H^s_{A,V}(\mathbb{R}^N,\mathbb{C})$. Without loss of generality, we assume that $u_n \rightharpoonup u$ in $H^s_{A,V}(\mathbb{R}^N,\mathbb{C})$ and $u_n \to u$ a.e. in $\mathbb{R}^N$. We now claim that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}(f(x,|u_n|)u_n - f(x,|u|)u)(\overline{u_n - u})dx = 0. \tag{6.1}$$

Clearly, $|f(x,t)t| \leq C(|t|+|t|^{p-1})$ for all $x \in \mathbb{R}^N$ and $t \in \mathbb{R}^+_0$ by (f_1). Using the Hölder inequality, we obtain

$$\begin{aligned}&\int_{\mathbb{R}^N} |(f(x,|u_n|)u_n - f(x,|u|)u)(\overline{u_n-u})|dx\\ &\leq \int_{\mathbb{R}^N} C[|u_n| + |u| + |u_n|^{p-1} + |u|^{p-1}]|u_n - u|dx\\ &\leq C(\|u_n\|_{L^2} + \|u\|_{L^2})\|u_n - u\|_{L^2} + C(\|u_n\|^{p-1}_{L^p} + \|u\|^{p-1}_{L^p(\mathbb{R}^N)})\|u_n - u\|_{L^p(\mathbb{R}^N)}\\ &\leq C(\|u_n - u\|_{L^2} + \|u_n - u\|_{L^p}).\end{aligned} \tag{6.2}$$

Lemma 3.5 guarantees that $|u_n| \to |u|$ in $L^p(\mathbb{R}^N)$ and $|u_n| \to |u|$ in $L^2(\mathbb{R}^N)$. Hence, $u_n \to u$ in $L^p(\mathbb{R}^N,\mathbb{C})$ and in $L^2(\mathbb{R}^N,\mathbb{C})$ by the Brézis–Lieb lemma. Inserting these facts in (6.2), we get the desired claim (6.1).

Now, $\langle \mathcal{I}'(u_n) - \mathcal{I}'(u), u_n - u\rangle \to 0$, since $\mathcal{I}'(u_n) \to 0$ and $u_n \rightharpoonup u$ in $H^s_{A,V}(\mathbb{R}^N,\mathbb{C})$. By (6.1), we have as $n \to \infty$

$$\begin{aligned}o(1) = \langle \mathcal{I}'(u_n) - \mathcal{I}'(u), u_n - u\rangle &= M([u_n]^2_{s,A})\langle L(u_n), u_n - u\rangle - M([u]^2_{s,A})\langle L(u), u_n - u\rangle\\ &+ \Re\int_{\mathbb{R}^N} V(x)(u_n - u)\overline{(u_n - u)}dx - 2\phi'(u_n)\int_{\mathbb{R}^N} F(x,|u_n|)dx\cdot\langle L(u_n), u_n - u\rangle\\ &- 2\phi'(u)\int_{\mathbb{R}^N} F(x,|u|)dx\cdot\langle L(u), u_n - u\rangle - \phi(u_n)\Re\int_{\mathbb{R}^N} f(x,|u_n|)u_n(\overline{u_n - u})dx\\ &- \phi(u)\Re\int_{\mathbb{R}^N} f(x,|u|)u(\overline{u_n - u})dx.\end{aligned}$$

From (f_7) and the facts that $u_n \rightharpoonup u$ in $H^s_{A,V}(\mathbb{R}^N,\mathbb{C})$ and $\phi' \leq 0$ it follows that

$$0 \leq M([u_n]^2_{s,A})\langle L(u_n) - L(u), u_n - u\rangle + \Re\int_{\mathbb{R}^N} V(x)(u_n - u)\overline{(u_n - u)}dx \leq o(1). \tag{6.3}$$

We divide the proof into two parts.

Case $\inf_{n\in\mathbb{N}}[u_n]_{s,A} = d > 0$. By (M_1), there exists $\kappa = \kappa(d) > 0$ with $M(t) \geq \kappa > 0$ for all $t \geq d$. This, together with (6.3), implies that

$$\lim_{n\to\infty}\left[\iint_{\mathbb{R}^{2N}} \frac{|u_n(x) - u(x) - e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})}(u_n(y) - u(y))|^2}{|x-y|^{N+2s}}dxdy + \int_{\mathbb{R}^N} V(x)|u_n - u|^2 dx\right] = 0.$$

Hence $u_n \to u$ in $H^s_{A,V}(\mathbb{R}^N, \mathbb{C})$.

Case $\inf_{n\in\mathbb{N}}[u_n]_{s,A} = 0$. If 0 is an isolated point for $([u_n]_{s,A})_n$, then there is a subsequence $([u_{n_k}]_{s,A})_k$ such that $\inf_{k\in\mathbb{N}}[u_{n_k}]_{s,A} = d > 0$ and one can proceed as before.

If, instead, 0 is an accumulation point for $([u_n]_{s,A})_n$, there is a subsequence, still labeled as $(u_n)_n$, such that $[u_n]_{s,A} \to 0$ and $u_n \to 0$ in $L^{2^*_s}(\mathbb{R}^N)$ as $n \to \infty$ and again (6.3) implies at once that $u_n \to 0$ in $H^s_{A,V}(\mathbb{R}^N, \mathbb{C})$, since $\langle L(u_n) - L(u), u_n - u\rangle \to 0$ and $M([u_n]^2_{s,A}) \to M(0) \geq 0$.

In conclusion, $\mathcal{I}$ satisfies the (PS) condition in $H^s_{A,V}(\mathbb{R}^N, \mathbb{C})$. For each $k \in \mathbb{N}$, we take k disjoint open sets K_i such that $\bigcup_{i=1}^k K_i \subset \Omega$. For each $i = 1, \ldots, k$ let $u_i \in (H^s_{A,V}(\mathbb{R}^N, \mathbb{C}) \bigcap C_0^\infty(K_i, \mathbb{C}))\backslash\{0\}$, with $\|u_i\|_{s,A} = 1$, and $W_k = \operatorname{span}\{u_1, u_2, \ldots, u_k\}$. Therefore, for any $u \in W_k$, with $\|u\|_{s,A} = \rho \leq 1$ small enough, we obtain by (f_7), being $q \in (1, 2)$,

$$\begin{aligned}
\mathcal{I}(u) &\leq \frac{1}{2}\left(\max_{0\leq t\leq 1} M(t)\right)[u]^2_{s,A} + \frac{1}{2}\|u\|^2_{L^2,V} - \int_\Omega a_1|u|^q dx \\
&\leq \frac{1}{2}\left(1 + \max_{0\leq t\leq 1} M(t)\right)\|u\|^2_{s,A} - C_k^q a_1\|u\|^q_{s,A} \\
&= \frac{1}{2}\left(1 + \max_{0\leq t\leq 1} M(t)\right)\rho^2 - C_k^q a_1\rho^q < 0,
\end{aligned}$$

where $C_k > 0$ is a constant such that $\|u\|_{L^q(\mathbb{R}^N,\mathbb{C})} \leq C_k\|u\|_{s,A}$ for all $u \in W_k$, since all norms on W_k are equivalent. Therefore, we deduce

$$\{u \in W_k : \|u\|_{s,A} = \rho\} \subset \{u \in W_k : \mathcal{I}(u) < 0\}.$$

Obviously, $\gamma(\{u \in W_k : \|u\|_{s,A} = \rho\}) = k$, see [10]. Hence by the monotonicity of the genus γ, cf. [27], we obtain

$$\gamma(u \in W_k : \mathcal{I}(u) < 0) \geq k.$$

Choosing $E_k = \{u \in W_k : \mathcal{I}(u) < 0\}$, we have $E_k \subset \Gamma_k$ and $\sup_{u\in\Gamma_k}\mathcal{I}(u) < 0$. Thus, all the assumptions of Theorem 6.1 are satisfied, Hence, there exists a sequence $(u_k)_k$ such that

$$\mathcal{I}(u_k) \leq 0, \quad \mathcal{I}'(u_k) = 0, \quad \text{and} \quad \|u_k\|_{s,A} \to 0 \text{ as } k \to \infty.$$

Therefore, we can take k so large that $\|u_k\|_{s,A} \leq 1$, and so these infinitely many functions u_k are solutions of (1.1). □

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