Value Function and Optimal Trajectories for Regional Control Problems via Dynamic Programming and Pontryagin Maximum Principles

G. Barles, A. Briani [*], E. Trélat [†]

Abstract

In this paper we study the optimal trajectories for regional, deterministic optimal control problems, i.e., problems where the dynamics and the cost functional can be completely different in two regions of the state space and therefore present discontinuities at their interface. Our first aim is to prove the classical link between the study of optimality conditions and the Bellman approach in this framework, and then exploit it to recover the value function under the assumption that optimal trajectories have only a countable number of switchings between the different regions. From an application point of view this is a very reasonable assumption because in practice one never implements chattering trajectories (i.e., enjoying the Zeno phenomenon). As a consequence of our description we obtain a new regularity result for the value function in the framework of hybrid optimal control problems.

Key-words: Optimal control, discontinuous dynamics, Bellman equation, Pontryagin maximum principle.
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1 Introduction

In this article, we focus on the study of optimal trajectories of regional optimal control problems in finite dimension, regional meaning that the dynamics and the costs functional may depend on the region of the state space and therefore present discontinuities at the interface between these different regions. Our aim is to describe these trajectories exploiting both the pure control approach (i.e., via the Pontryagin maximum principle) and the Dynamic Programming approach (we define the value function and use that it is the viscosity solution of the corresponding Hamilton-Jacobi equation). The study of the relationship between these two approaches for regional problems is new and is one of the objectives of this paper.

[*]Laboratoire de Mathématiques et Physique Théorique (UMR CNRS 7350), Fédération Denis Poisson (FR CNRS 2964), Université François Rabelais, Parc de Grandmont, 37200 Tours, France. email: Guy.Barles@lmpt.univ-tours.fr, ariela.briani@lmpt.univ-tours.fr

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[†]Laboratoire Jacques-Louis Lions,Universit Pierre et Marie Curie (Paris 6) CNRS, UMR 7598, 4 place Jussieu, Paris, France. email:emmanuel.trelat@upmc.fr

Before entering in the description of our results we recall that regional optimal control problems have been widely studied in the existing literature, within the context of hybrid systems (see, e.g., [11, 19, 20, 21, 30, 35, 38] and references therein), or with a Bellman approach in [8, 9, 10, 12, 23, 24, 31, 32] where the Hamilton-Jacobi equation associated with the stratified optimal control problem is studied with the machinery of viscosity solutions. We also refer to [1, 25, 28, 36] for the related subject of Hamilton-Jacobi equations on networks.

In order to point out the main ideas, we present our study in the following simplified framework with only two different regions. Let $N \in \mathbb{N}^*$. We assume that

$$\mathbb{R}^N = \Omega_1 \cup \Omega_2 \cup \mathcal{H}, \qquad \Omega_1, \Omega_2 \text{ open}, \quad \Omega_1 \cap \Omega_2 = \emptyset,$$
$$\mathcal{H} = \partial\Omega_1 = \partial\Omega_2 \text{ is a } C^1\text{-submanifold of } \mathbb{R}^N,$$

and we consider a nonlinear optimal control problem in $\mathbb{R}^N$, stratified according to the above partition. We write the optimal control problem in the form

$$\dot{\mathbb{X}}(t) = f(\mathbb{X}(t), a(t)),$$
$$\mathbb{X}(t^0) = x^0, \quad \mathbb{X}(t^f) = x^f,$$
$$\inf \int_{t^0}^{t^f} \ell(\mathbb{X}(t), a(t))\, dt,$$

where the dynamics f and the running cost ℓ are defined as follows. If $x \in \Omega_i$ for $i = 1$ or $i = 2$ then

$$f(x,a) = f_i(x,a), \qquad \ell(x,a) = l_i(x,a),$$

where $f_i : \mathbb{R}^N \times \mathbb{R}^m \to \mathbb{R}^N$ and $l_i : \mathbb{R}^N \times \mathbb{R}^m \to \mathbb{R}$ are C^1-mappings. If $x \in \mathcal{H}$ then

$$f(x,a) = f_{\mathcal{H}}(x,a), \qquad \ell(x,a) = l_{\mathcal{H}}(x,a),$$

where $f_{\mathcal{H}} : \mathbb{R}^N \times \mathbb{R}^m \to \mathbb{R}^N$ and $\ell_{\mathcal{H}} : \mathbb{R}^N \times \mathbb{R}^m \to \mathbb{R}$ are C^1-mappings. The set $\mathcal{H}$ is called the *interface* between the two open regions Ω_1 and Ω_2 (see figures 1 and 2).

The class of controls that we consider also depends on the region. As long as $\mathbb{X}(t) \in \Omega_i$, we assume that $a \in L^\infty(t^0, t^f; A_i)$, where A_i is a measurable subset of $\mathbb{R}^m$. Accordingly, as long as $\mathbb{X}(t) \in \mathcal{H}$, we assume that $a \in L^\infty(t^0, t^f; A_{\mathcal{H}})$, $A_{\mathcal{H}}$ is a measurable subset of $\mathbb{R}^m$.

The terminal times t^0 and t^f and the terminal points x^0 and x^f may be fixed or free according to the problem under consideration. For instance, if we fix t^0, x^0, t^f, x^f, we define $U(t^0, x^0, t^f, x^f)$ as being the infimum of the cost functional over all possible admissible trajectories steering the control system from (t^0, x^0) to (t^f, x^f).

The main objective of our study is to show that this value function can be recovered studying *classical* (i.e., continuous, non-hybrid) optimal control problems. Our analysis is based on studying what are the possible structures of optimal trajectories. We recall that, for regional optimal control problems, existence of an optimal control and Cauchy uniqueness results are derived using Filippov-like arguments, allowing one to tackle the discontinuities of the dynamics and of the cost functional (see, e.g. [10, 12]).

In what follows, we assume that the regional optimal control problem under consideration admits at least one optimal solution. We consider such an optimal trajectory $\mathbb{X}(\cdot)$ associated with a control $a(\cdot)$ on $[t^0, t^f]$. Assuming that $x^0 \in \Omega_1$ and $x^f \in \Omega_2$, we can consider different structures.

The first possibility is that the trajectory $\mathbb{X}(\cdot)$ consists of two arcs $([t^0, t^1], X_1(\cdot))$ and $([t^1, t^f], X_2(\cdot))$, living respectively in Ω_1 and Ω_2, with $X_1(t^1) = X_2(t^1) \in \mathcal{H}$. Such optimal trajectories are studied in [21] under the assumption of a transversal crossing and an explicit jump condition is given for the adjoint vector obtained by applying the Pontryagin maximum principle. This is the simplest possible trajectory structure, and we denote it by 1-2 (see figure 1).

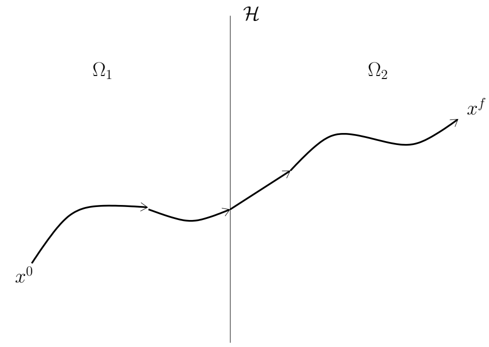

Figure 1: An example of structure 1-2.

The second possibility is that the trajectory $\mathbb{X}(\cdot)$ consists of three arcs $([t^0, t^1], X_1(\cdot))$, $([t^1, t^2], X_{\mathcal{H}}(\cdot))$ and $([t^2, t^f], X_2(\cdot))$, living respectively in Ω_1, $\mathcal{H}$ and Ω_2. The middle arc $X_{\mathcal{H}}$ lives on the interface. Such a structure is denoted by 1-$\mathcal{H}$-2 (see figure 2).

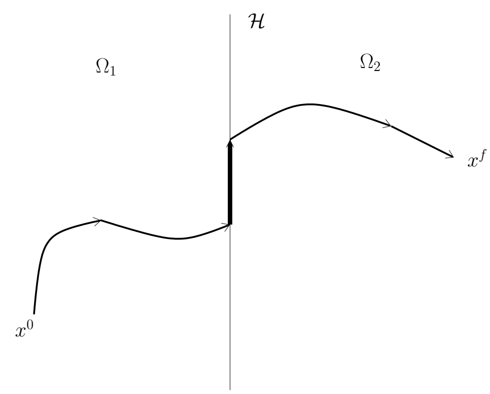

Figure 2: An example of structure 1-$\mathcal{H}$-2.

Accordingly, on can consider the structures 1-2-$\mathcal{H}$-1, 1-$\mathcal{H}$-1-2, 1-2-$\mathcal{H}$-2, etc. For every such fixed structure, we can define a specific optimal control problem, consisting of finding the optimal trajectory steering the system from the initial point to the desired target point, and minimizing the cost functional over all admissible trajectories having exactly such a structure. Denoting by U_{12}, $U_{1\mathcal{H}2}$, etc; the corresponding value functions, it is clear that $U = \inf(U_{12}, U_{1\mathcal{H}2}, \ldots)$.

In this paper, we show that, thanks to a duplication argument developed in [19], each of the above value functions (restricted to some fixed structure) can actually be written as the projection/restriction of the value function of some *classical* optimal control problem in higher dimension

(say p), the projection being considered along some coordinates, and the restriction being done to some submanifolds of the higher dimensional space $\mathbb{R}^p$. Note that the word "duplicated" is used here because each arc of the trajectory gives two components of the dynamics of the problem in higher dimension.
We exploit this technique in the following way. We characterize the value function as a viscosity solution of an Hamilton-Jacobi equation and we apply the classical Pontryagin maximum principle. We thus provide an explicit relationship between the gradient of the value function evaluated along the optimal trajectory and the adjoint vector. This allows us to derive conditions at the interface: a continuity condition on the Hamiltonian and a jump condition on the adjoint vector.

In Section 4 we analyze the structure 1-$\mathcal{H}$-2: we construct the duplicated problem and we prove the main result Theorem 4.2. We only detail this construction because one can similarly derive analogous results for different structures 1-$\mathcal{H}$-1-2, 1-2-$\mathcal{H}$-2, etc. Indeed, this can be done by the construction of a duplicated problem of dimension two times the number of arcs of the structure (see Section 5).
The value function U is the infimum over all possible structures. However, the duplication technique can be applied only to a countable number of possibilities. In general, it might happen that the structure of switchings have a fractal structure, and thus the set of switching points be uncountable. We are not aware of any result providing sufficient conditions for hybrid optimal control problems, under which the number of switchings is finite or even only countable. The *Zeno phenomenon* is a chattering phenomenon, meaning that the control switches an infinite number of times over a compact interval of times. One can find an analysis of this phenomenon, for instance, in [26, 27, 42]. We also refer to [4, 22] for classes of hybrid control problems for which necessary and/or sufficient conditions for the occurrence of the Zeno phenomenon have been derived. Although the Zeno phenomenon may occur in general, restricting to trajectories having only a finite (or even countable) number of switchings is a reasonable assumption for practical implementation (see [14]). In practice, when chattering occurs, it is rather reasonable to design trajectory allowing only a countable number of switchings.

Under this assumption, an important consequence of our results is that the regularity of the value function U is the same (i.e., not more degenerate) than the one of the classical optimal control problem that lifts the problem. Indeed, we prove that each value function $U_{12}, U_{1\mathcal{H}2}, \dots$ is the restriction to some submanifold of the value function of a classical optimal control problem in higher dimension (Theorem 4.7). Therefore, if for instance all value functions above are Lipschitz then the value function of the regional optimal control problem is Lipschitz as well. This kind of result is new in the framework of hybrid optimal control problems (see [41] for the classical case).

This paper is organized as follows. In Section 2 we precisely state some assumptions and we define the separated trajectories, i.e., that stay only in one region (Ω_1, Ω_2 or $\mathcal{H}$) and can touch its boundary only at initial or final time. In Section 3 we recall the result for structure 1-2. In Section 4 we analyze in detail the structure 1-$\mathcal{H}$-2: we construct the duplicated problem and we give the main results of this analysis (Theorem 4.2 and Theorem 4.7). In Section 5 we describe the general case, we give some possible generalizations and we provide a simple illustrating example.

2 Problem and main assumptions

We assume that:

(H$\mathcal{H}$) $\mathbb{R}^N = \Omega_1 \cup \Omega_2 \cup \mathcal{H}$ *with* $\Omega_1 \cap \Omega_2 = \emptyset$ *and* $\mathcal{H} = \partial\Omega_1 = \partial\Omega_2$ *being a* C^1*-submanifold. More precisely, there exists a function* $\Psi : \mathbb{R}^N \to \mathbb{R}$ *of class* C^1 *such that* $\mathcal{H}$ *can be written as* $\mathcal{H} := \{x \in \mathbb{R}^N \mid \Psi(x) = 0\}$.

We consider here the problem of minimizing the cost of trajectories going from x^0 to x^f in time $t^f - t^0$. These trajectories follows different dynamics $f_i, f_\mathcal{H}$ if they are in $\Omega_i, \mathcal{H}$, and pay different costs l_i, $l_\mathcal{H}$ on $\Omega_i, \mathcal{H}$ $(i = 1, 2)$.

Notation. In order to describe our problem on $\mathcal{H}$ we shall consider the tangent bundle

$$T\mathcal{H} := \bigcup_{z\in\mathcal{H}} \Big(\{z\} \times T_z\mathcal{H}\Big)$$

where $T_z\mathcal{H}$ is the tangent space to $\mathcal{H}$ at z (which is essentially $\mathbb{R}^{N-1}$). Thus, if $\phi \in C^1(\mathcal{H})$, and $x \in \mathcal{H}$, we denote by $\nabla_\mathcal{H}\phi(x)$ the gradient of ϕ at x, which belongs to $T_x\mathcal{H}$.
The scalar product in $T_z\mathcal{H}$ will be denoted by $\big\langle u, v\big\rangle_\mathcal{H}$. In this definition, both vectors u, v should belong to $T_z\mathcal{H}$ for this definition to make sense. Hence, to be precise we should use the orthogonal projection $P_z : \mathbb{R}^N \to T_z\mathcal{H}$ when at least one of the vectors u, v lives in $\mathbb{R}^N$, but we will omit this point since no confusion is feared. The notation $\big\langle u, v\big\rangle$ will refer to the usual euclidian scalar product in $\mathbb{R}^N$.

Our precise assumptions are:

(HA) *The sets* A_1, A_2 *and* $A_\mathcal{H}$ *are measurable subsets of* $\mathbb{R}^m$.

(Hg) *Let* $\mathcal{M}$ *be a submanifold of* $\mathbb{R}^N$, *the function* $g : \mathcal{M} \times A \to \mathbb{R}^N$ *is a continuous bounded function,* C^1 *and with Lipschitz continuous derivative with respect to the first variable. More precisely, there exists* $M > 0$ *such that for any* $x \in \mathcal{M}$ *and* $\alpha \in A$

$$|g(x, \alpha)| \leq M \ .$$

Moreover, there exist $L, L^1 > 0$ *such that for any* $z, z' \in \mathcal{M}$ *and* $\alpha \in A$

$$|g(z, \alpha) - g(z', \alpha)| \leq L\, |z - z'| \ ,$$

$$\Big|\frac{\partial}{\partial z_j} g(z, \alpha) - \frac{\partial}{\partial z_j} g(z', \alpha)\Big| \leq L^1 |z - z'| \ , \ j = 1, \ldots, N \ .$$

(Hfl$_i$) *We assume that* $f_i : \Omega_i \times A_i \to \mathbb{R}^N$, $l_i : \Omega_i \times A_i \to \mathbb{R}$, *$(i = 1, 2)$ fulfill assumption (Hg) with a suitable choice of constants* M, L *and* L^1.

(Hfl$_\mathcal{H}$) *We assume that* $(x, f_\mathcal{H}(x, a_\mathcal{H})) : \mathcal{H} \times A_\mathcal{H} \to T\mathcal{H}$ *and* $l_\mathcal{H} : \mathcal{H} \times A_\mathcal{H} \to \mathbb{R}$ *fulfill assumption (Hg) with a suitable choice of constants* M, L *and* L^1.

In this paper we consider optimal trajectories that are decomposed on arcs that stay only on Ω_1, Ω_2 or $\mathcal{H}$ and can touch the boundary of Ω_1, or Ω_2, only at initial or final time. To describe these trajectories we first need to set the problems separately.

The problem in region Ω_1.
We consider here only trajectories $X_1 : \mathbb{R}^+ \to \mathbb{R}^N$ fulfilling

$$\dot{X}_1(t) = f_1(X_1(t), \alpha_1(t)) , \quad X_1(t) \in \Omega_1 \ \ \forall t \in (t^0, t^f) \tag{2.1}$$

$$X_1(t^0) = x^0 \ , \ X_1(t^f) = x^f \ \text{ with } x^0 \neq x^f \in \overline{\Omega}_1 . \tag{2.2}$$

The value function $S_1 : \overline{\Omega}_1 \times \mathbb{R}^+ \times \overline{\Omega}_1 \times \mathbb{R}^+ \to \mathbb{R}$ is

$$S_1(x^0, t^0; x^f, t^f) := \inf \left\{ \int_{t^0}^{t^f} l_1(X_1(t), \alpha_1(t))\, dt \ : \ X_1 \text{ fullfills } (2.1) - (2.2) \right\} .$$

We define the Hamiltonian $\widetilde{H_1} : \Omega_1 \times \mathbb{R}^N \times \mathbb{R} \times A_1 \to \mathbb{R}$ as

$$\widetilde{H_1}(X_1, Q_1, p_0, \alpha_1) := \big\langle Q_1, f_1(X_1, \alpha_1) \big\rangle + p_0\, l_1(X_1, \alpha_1) ,$$

and $H_1 : \Omega_1 \times \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ as

$$H_1(X_1, Q_1, p_0) := \sup_{\alpha_1 \in A_1} \widetilde{H_1}(X_1, Q_1, p_0, \alpha_1) .$$

The problem in region Ω_2.
We consider here only trajectories $X_2 : \mathbb{R}^+ \to \mathbb{R}^N$ fulfilling

$$\dot{X}_2(t) = f_2(X_2(t), \alpha_2(t)) , \quad X_2(t) \in \Omega_2 \ \ \forall t \in (t^0, t^f) \tag{2.3}$$

$$X_2(t^0) = x^0 \ , \ X_2(t^f) = x^f \ \text{ with } x^0 \neq x^f \in \overline{\Omega}_2 . \tag{2.4}$$

The value function $S_2 : \overline{\Omega}_2 \times \mathbb{R}^+ \times \overline{\Omega}_2 \times \mathbb{R}^+ \to \mathbb{R}$ is

$$S_2(x^0, t^0; x^f, t^f) := \inf \left\{ \int_{t^0}^{t^f} l_2(X_2(t), \alpha_2(t))\, dt \ : \ X_2 \text{ fullfills } (2.3) - (2.4) \right\} .$$

We define the Hamiltonians $\widetilde{H_2} : \Omega_2 \times \mathbb{R}^N \times \mathbb{R} \times A_2 \to \mathbb{R}$ as

$$\widetilde{H_2}(X_2, Q_2, p_0, \alpha_2) := \big\langle Q_2, f_2(X_2, \alpha_2) \big\rangle + p_0\, l_2(X_2, \alpha_2) ,$$

and $H_2 : \Omega_2 \times \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ as

$$H_2(X_2, Q_2, p_0) := \sup_{\alpha_2 \in A_2} \widetilde{H_2}(X_2, Q_2, p_0, \alpha_2) .$$

The problem along the interface $\mathcal{H}$.
We consider here only trajectories $X_{\mathcal{H}} : \mathbb{R}^+ \to \mathcal{H}$ fulfilling

$$\dot{X}_{\mathcal{H}}(t) = f_{\mathcal{H}}(X_{\mathcal{H}}(t), a_{\mathcal{H}}(t)) , \quad X_{\mathcal{H}}(t) \in \mathcal{H} \ \ \forall t \in (t^0, t^f) \tag{2.5}$$

$$X_{\mathcal{H}}(t^0) = x^0 \ , \ X_{\mathcal{H}}(t^f) = x^f \ \text{ with } x^0 \neq x^f \in \mathcal{H}\,. \tag{2.6}$$

The value function $S_{\mathcal{H}} : \mathcal{H} \times \mathbb{R}^+ \times \mathcal{H} \times \mathbb{R}^+ \to \mathbb{R}$ is

$$S_{\mathcal{H}}(x^0, t^0; x^f, t^f) := \inf \left\{ \int_{t^0}^{t^f} l_{\mathcal{H}}(X_{\mathcal{H}}(t), a_{\mathcal{H}}(t))\, dt \ : \ X_{\mathcal{H}} \text{ fullfills } (2.5) - (2.6) \right\}.$$

We define the Hamiltonians $\widetilde{H_{\mathcal{H}}} : T\mathcal{H} \times \mathbb{R} \times A_{\mathcal{H}} \to \mathbb{R}$ as

$$\widetilde{H_{\mathcal{H}}}(X_{\mathcal{H}}, Q_{\mathcal{H}}, p_0, a_{\mathcal{H}}) := \big\langle Q_{\mathcal{H}}, f_{\mathcal{H}}(X_{\mathcal{H}}, a_{\mathcal{H}}) \big\rangle_{\mathcal{H}} + p_0\, l_{\mathcal{H}}(X_{\mathcal{H}}, a_{\mathcal{H}}) \ ,$$

and $H_{\mathcal{H}} : T\mathcal{H} \times \mathbb{R} \to \mathbb{R}$ as

$$H_{\mathcal{H}}(X_{\mathcal{H}}, Q_{\mathcal{H}}, p_0) := \sup_{a_{\mathcal{H}} \in A_{\mathcal{H}}} \widetilde{H_{\mathcal{H}}}(X_{\mathcal{H}}, Q_{\mathcal{H}}, p_0, a_{\mathcal{H}})\,.$$

3 Structure 1-2: the transversality crossing condition

We recall here the result for the first possible structure: the trajectory consists of two arcs living respectively in Ω_1, Ω_2 and traversing the interface $\mathcal{H}$ at a given time (see figure 1). More precisely we assume:

(H 1-2) There exist a time $t_c \in (t^0, t^f)$ and an optimal trajectory that: starts from Ω_1, stays in Ω_1 in the interval $[t^0, t_c)$, does not arrive tangentially at time t_c on $\mathcal{H}$ and stays in Ω_2 on the interval $(t_c, t^f]$.

These trajectories are described as follows: for each initial and final data $(x^0, t^0, x^f, t^f) \in \Omega_1 \times \mathbb{R}^+ \times \Omega_2 \times \mathbb{R}^+_*$, the trajectory is given by the vector $\mathbb{X}(t) = (X_1(t), X_2(t)) : \mathbb{R}^+ \to \mathbb{R}^N \times \mathbb{R}^N$ Lipschitz solution of the following system

$$\begin{cases} \dot{X}_1(t) &= f_1(X_1(t), \alpha_1(t)) \quad t \in (t^0, t_c) \\ \dot{X}_2(t) &= f_2(X_2(t), \alpha_2(t)) \quad t \in (t_c, t^f) \end{cases} \tag{3.1}$$

completed with the following mixed conditions

$$X_1(t_0) = x^0\,, \ X_1(t_c) = X_2(t_c)\,, \ X_2(t^f) = x^f\,, \tag{3.2}$$

non tangential conditions

$$\big\langle \nabla\Psi(X_1(t_c^-)), f_1(X_1(t_c^-), \alpha_1(t_c^-)) \big\rangle \neq 0 \quad \big\langle \nabla\Psi(X_2(t_c^+)), f_2(X_2(t_c^+), \alpha_2(t_c^+)) \big\rangle \neq 0\,, \tag{3.3}$$

and state constraints

$$X_1(t) \in \Omega_1 \ \ \forall t \in (t^0, t_c)\,, \qquad X_2(t) \in \Omega_2 \ \ \forall t \in (t_c, t^f)\,. \tag{3.4}$$

The cost of these trajectories is

$$C(x^0, t^0; x^f, t^f; \mathbb{X}) := \int_{t^0}^{t_c} l_1(X_1(t), \alpha_1(t))\, dt + \int_{t_c}^{t^f} l_2(X_2(t), \alpha_2(t))\, dt\,.$$

Our aim is to study the value function $S_{1,2} : \mathbb{R}^N \times \mathbb{R}^+ \times \mathbb{R}^N \times \mathbb{R}^+ \to \mathbb{R}$

$$S_{1,2}(x^0, t^0; x^f, t^f) := \inf \Big\{ C(x^0, t^0; x^f, t^f; \mathbb{X}) \,:\, \mathbb{X} \text{ fullfils } (3.1) - (3.2) - (3.3) - (3.4) \ , \ t^f > t_c > t^0 \ \Big\}.$$

We recall that, see [20, 21, 35, 38], under the assumptions (H$\mathcal{H}$), (HA), (Hfl$_i$), (Hfl$_{\mathcal{H}}$) for any $(x^0, t^0; x^f, t^f) \in \Omega_1 \times \mathbb{R}^+ \times \Omega_2 \times \mathbb{R}^+$ we have

$$S_{1,2}(x^0, t^0; x^f, t^f) = \min \Big\{ S_1(x^0, t^0; x_c, t_c) + S_2(x_c, t_c; x^f, t^f) \ : t^0 < t_c < t^f \ , \ x_c \in \mathcal{H} \Big\} \ .$$

If $\mathbb{X}(\cdot)$ is an optimal trajectory for the value function $S_{1,2}(x^0, t^0; x^f, t^f)$ and $P(\cdot)$ is the corresponding adjoint vector given by the Pontryagin maximum principle we have the following continuity conditions on the Hamiltonans

$$H_1(X_1(t_c^-), P(t_c^-), p_0) = H_2(X_2(t_c^+), P(t_c^+), p_0).$$

Moreover, the following *jump condition* holds

$$P_2(t_c^+) - P_1(t_c^-) = \frac{\big\langle P_1(t_c^-), f_1(t_c^-) - f_2(t_c^+) \big\rangle + p^0(l_1(t_c^-) - l_2(t_c^+))}{\big\langle \nabla \Psi(X_2(t_c^+)), f_2(t_c^+) \big\rangle} \nabla \Psi(X_1(t_c^-))$$

where the short notation $f_i(t_c^\pm)$ stands for $f_i(X_i(t_c^\pm), t_c, \alpha_i(t_c^\pm))$ and $l_i(t_c^\pm)$ stands for $l_i(x_i(t_c^\pm), t_c, \alpha_i(t_c^\pm))$, $(i = 1, 2)$.

4 Analysis of the structure 1-$\mathcal{H}$-2

In this section we analyze the structure with tree arcs described in Figure 2. Precisely, given $(x^0, t^0, x^f, t^f) \in \Omega_1 \times \mathbb{R}^+ \times \Omega_2 \times \mathbb{R}^+$ with $x^0 \neq x^f$ we assume:

(H 1$\mathcal{H}$2) There exist $t^0 < t_1 < t_2 < t^f$ and an optimal trajectory that starts from Ω_1, stays in Ω_1 in the interval $[t^0, t_1)$, stays on $\mathcal{H}$ on a time interval $[t_1, t_2]$ and stays in Ω_2 in the interval $(t_2, t^f]$.

Roughly speaking our aim is here to minimize along the times t_1, t_2 such that there exists a trajectory that stay on $\mathcal{H}$ on the time interval $[t_1, t_2]$.

These trajectories are described as follows: for each initial and final data $(x^0, t^0, x^f, t^f) \in \Omega_1 \times \mathbb{R}^+ \times \Omega_2 \times \mathbb{R}^+_*$, the trajectory will be given by the vector $\mathbb{X}(t) = (X_1(t), X_{\mathcal{H}}(t), X_2(t)) : \mathbb{R}^+ \to \mathbb{R}^N \times \mathcal{H} \times \mathbb{R}^N$ Lipschitz solution of the following system

$$\left\{ \begin{array}{lll} \dot{X}_1(t) & = f_1(X_1(t), \alpha_1(t)) & t \in (t^0, t_1) \\ \dot{X}_{\mathcal{H}}(t) & = f_{\mathcal{H}}(X_{\mathcal{H}}(t), a_{\mathcal{H}}(t)) & t \in (t_1, t_2) \\ \dot{X}_2(t) & = f_2(X_2(t), \alpha_2(t)) & t \in (t_2, t^f) \end{array} \right. \tag{4.1}$$

completed with the following mixed conditions:

$$X_1(t_0) = x^0 \, , \ X_1(t_1) \in \mathcal{H} \, , \ X_1(t_1) = X_{\mathcal{H}}(t_1) \, , \ X_{\mathcal{H}}(t_2) \in \mathcal{H} \, , \ X_2(t_2) = X_{\mathcal{H}}(t_2) \, , \ X_2(t^f) = x^f \tag{4.2}$$

and state constraints

$$X_1(t) \in \Omega_1 \ \ \forall t \in (t^0, t_1) \, , \quad X_{\mathcal{H}} \in \mathcal{H} \ \ \forall t \in (t_1, t_2) \, , \quad X_2(t) \in \Omega_2 \ \ \forall t \in (t_2, t^f) \, . \tag{4.3}$$

The cost of these trajectories is

$$C(x^0,t^0;x^f,t^f;\mathbb{X}) := \int_{t^0}^{t_1} l_1(X_1(t),\alpha_1(t))\,dt + \int_{t_1}^{t_2} l_{\mathcal{H}}(X_{\mathcal{H}}(t),a_{\mathcal{H}}(t))\,dt + \int_{t_2}^{t^f} l_2(X_2(t),\alpha_2(t))\,dt\,.$$

Our aim is then to characterize the value function $S_{1,\mathcal{H},2} : \Omega_1 \times \mathbb{R}^+ \times \Omega_2 \times \mathbb{R}^+ \to \mathbb{R}$

$$S_{1,\mathcal{H},2}(x^0,t^0;x^f,t^f) := \inf\Big\{ C(x^0,t^0;x^f,t^f;\mathbb{X}) \,:\, \mathbb{X} \text{ fullfills } (4.1)-(4.2)-(4.3)\ ,\ t^f > t_2 > t_1 > t^0 \Big\}. \tag{4.4}$$

Remark 4.1. *Note that this definition does not include the special cases when $x^0 \in \mathcal{H}$ and/or $x^f \in \mathcal{H}$. In these cases it is easy to see how the definition could be modified in order to involve only vectors X_1, X_2 or $X_{\mathcal{H}}$. However, remark that if both $x^0, x^f \in \mathcal{H}$ then $S_{1,\mathcal{H},2}(x^0,t^0;x^f,t^f) = S_{\mathcal{H}}(x^0,t^0;x^f,t^f)$.*

Notation: partial derivatives and gradient.
In order to set the notation for the partial derivatives of $S_{1,\mathcal{H},2}$, S_1, S_2 we consider generic a function $u(x^0,t^0;x^f,t^f) : \big(\Omega_1 \times \mathbb{R}^+ \times \Omega_2 \times \mathbb{R}^+_*\big) \to \mathbb{R}$.
We will respectively denote by $\nabla_{x^0} u$, $\nabla_{x^f} u$ the gradients with respect to the first and the second state variable, so $\nabla_{x^0} u$ and $\nabla_{x^f} u$ take values in $\mathbb{R}^N$.
We will respectively denote by u_{t^0} and u_{t^f} the partial derivatives with respect to the first and the second time variable, so u_{t^0} and u_{t^f} take values in $\mathbb{R}$.
If $(x^0,t^0;x^f,t^f) \in \Omega_1 \times \mathbb{R}^+ \times \mathcal{H} \times \mathbb{R}^+_*$ we define $\nabla^{\mathcal{H}}_{x^f} u$ such that $(x^f, \nabla^{\mathcal{H}}_{x^f} u) \in T\mathcal{H}$.
If $(x^0,t^0;x^f,t^f) \in \mathcal{H} \times \mathbb{R}^+ \times \Omega_2 \times \mathbb{R}^+_*$ we define $\nabla^{\mathcal{H}}_{x^0} u$ such that $(x^0, \nabla^{\mathcal{H}}_{x^0} u) \in T\mathcal{H}$.

4.1 The duplicated problem

The main ingredient of our analysis is the construction of the duplicated problem (following [19]), the advantage being that the last will be a more regular problem, even if in a higher dimension. The basic idea is to change the time variable to let the possible optimal trajectories evolve "at the same time" on the three arcs: the one on Ω_1, the one on $\mathcal{H}$ and the one on Ω_2. In this duplicated optimal control problem we will not need to ask the mixed conditions (4.2) and the state constrains (4.3) to be fulfilled. Therefore, we will easily characterize the value function by an Hamilton-Jacobi equation, apply the Pontryagin maximum principle and exploit the classical link between them.

We set $V := (A_1 \times [0,T]) \times (A_{\mathcal{H}} \times [0,T]) \times (A_2 \times [0,T])$, for $T > 0$ big enough.
Fixed $T_0, T_1 \in \mathbb{R}^+$ the *admissible controls* will be a subset of the functions
$\mathbb{V}(\tau) := (v_1(\tau), w_1(\tau), v_{\mathcal{H}}(\tau), w_{\mathcal{H}}(\tau), v_2(\tau), w_2(\tau)) \in L^\infty([T_0,T_1];V)$.
The *admissible trajectories* will be Lipschitz continuous vector functions

$$\mathbb{Z}(\tau) := (Y_1(\tau), \rho_1(\tau), Y_{\mathcal{H}}(\tau), \rho_{\mathcal{H}}(\tau), Y_2(\tau), \rho_2(\tau)) : (T_0,T_1) \to \Omega_1 \times \mathbb{R}^+_* \times \mathcal{H} \times \mathbb{R}^+_* \times \Omega_2 \times \mathbb{R}^+_*$$

solutions of

$$\begin{cases} Y_1'(\tau) &= f_1(Y_1(\tau), v_1(\tau))w_1(\tau) & \tau \in (T_0,T_1) \\ \rho_1'(\tau) &= w_1(\tau) & \tau \in (T_0,T_1) \\ Y_{\mathcal{H}}'(\tau) &= f_{\mathcal{H}}(Y_{\mathcal{H}}(\tau), v_{\mathcal{H}}(\tau))w_{\mathcal{H}}(\tau) & \tau \in (T_0,T_1) \\ \rho_{\mathcal{H}}'(\tau) &= w_{\mathcal{H}}(\tau) & \tau \in (T_0,T_1) \\ Y_2'(\tau) &= f_2(Y_2(\tau), v_2(\tau))w_2(\tau) & \tau \in (T_0,T_1) \\ \rho_2'(\tau) &= w_2(\tau) & \tau \in (T_0,T_1) \end{cases} \tag{4.5}$$

with initial and final conditions

$$\mathbb{Z}(T_0) = Z_0 \ , \ \mathbb{Z}(T_1) = Z_1 \, . \tag{4.6}$$

Note that to take into account the mixed conditions on the original problem, we will allow initial and final state Z_0, Z_1 in $\overline{\Omega}_1 \times \mathbb{R}^+_* \times \mathcal{H} \times \mathbb{R}^+_* \times \overline{\Omega}_2 \times \mathbb{R}^+_*$.
More precisely, given (Z_0, T_0), $(Z_1, T_1) \in (\overline{\Omega}_1 \times \mathbb{R}^+_* \times \mathcal{H} \times \mathbb{R}^+_* \times \overline{\Omega}_2 \times \mathbb{R}^+_*) \times \mathbb{R}^+$ we consider the subset of admissible trajectories

$$\mathcal{Z}_{(Z_0,T_0),(Z_1,T_1)} := \Big\{ \mathbb{Z} \in Lip((T_0,T_1); \Omega_1 \times \mathbb{R}^+_* \times \mathcal{H} \times \mathbb{R}^+_* \times \Omega_2 \times \mathbb{R}^+_*) : \text{ there exists an admissible control}$$
$$\mathbb{V} \in L^\infty([T_0,T_1]; V) \text{ such that } \mathbb{Z} \text{ is a solution of (4.5) fulfilling (4.6)} \Big\} \, .$$

For each admissible trajectory $\mathbb{Z}$ we will consider the cost functional

$$\mathcal{C}(\mathbb{Z}) := \int_{T_0}^{T_1} \Big(l_1(Y_1(\tau), v_1(\tau)) w_1(\tau) + l_{\mathcal{H}}(Y_{\mathcal{H}}(\tau), v_{\mathcal{H}}(\tau)) w_{\mathcal{H}}(\tau) + l_2(Y_2(\tau), v_2(\tau)) w_2(\tau) \Big) \, d\tau$$

therefore the value function $\Sigma : \big(\overline{\Omega}_1 \times \mathbb{R}^+_* \times \mathcal{H} \times \mathbb{R}^+_* \times \overline{\Omega}_2 \times \mathbb{R}^+_* \times \mathbb{R}^+\big)^2 \to \mathbb{R}$ is naturally defined by

$$\Sigma(Z_0, T_0; Z_1, T_1) := \inf \Big\{ \mathcal{C}(\mathbb{Z}) \ : \ \mathbb{Z} \in \mathcal{Z}_{(Z_0,T_0),(Z_1,T_1)} \Big\} \, . \tag{4.7}$$

To state the basic link between the original and the duplicated problem, given $(x^0, t^0; x^f, t^f) \in \Omega_1 \times \mathbb{R}^+_* \times \Omega_2 \times \mathbb{R}^+_*$ we define the following submanifold of $\mathbb{R}^{6(N+1)}$:

$$\mathcal{M}(x^0, t^0; x^f, t^f) := \Big\{ (Z_0, Z_1) \in \mathbb{R}^{6(N+1)} \ : \ Z_0 = (x^0, t^0, x_1, t_1, x_2, t_2) \, ,$$
$$Z_1 = (x_1, t_1, x_2, t_2, x^f, t^f) \text{ with } x_1 \in \mathcal{H}, x_2 \in \mathcal{H}, \text{ and } t^f > t_2 > t_1 > t^0 \Big\}.$$

The result says that the original value function is indeed the minimum over the value functions $\Sigma(Z_0, T_0; Z_1, T_1)$ restricted to $\mathcal{M}(x^0, t^0; x^f, t^f)$. More precisely:

Proposition 4.2. *Assume (H$\mathcal{H}$), (HA), (Hfl$_i$) and (Hfl$_{\mathcal{H}}$).*
Given $(x^0, t^0; x^f, t^f) \in \Omega_1 \times \mathbb{R}^+ \times \Omega_2 \times \mathbb{R}^+$ we have

$$S_{1,\mathcal{H},2}(x^0, t^0; x^f, t^f) = \min \Big\{ \Sigma(Z_0, T_0; Z_1, T_1) \ : \ (Z_0, Z_1) \in \mathcal{M}(x^0, t^0; x^f, t^f) \Big\} \tag{4.8}$$

where $0 \le T_0 < T_1$ can be arbitrary chosen.

Proof. Fix $(x^0, t^0; x^f, t^f) \in \overline{\Omega}_1 \times \mathbb{R}^+_* \times \overline{\Omega}_2 \times \mathbb{R}^+_*$, with $x^0 \neq x^f$ and t_1, t_2 such that $t^f > t_2 > t_1 > t^0$. Let $\mathbb{X}$ be the corresponding trajectory solution of (4.1)-(4.2)-(4.3). We construct three C^1-diffeomorphisms with strictly positive first derivatives:

$$\rho_1 : [T_0, T_1] \to [t^0, t_1] \, , \quad \rho_{\mathcal{H}} : [T_0, T_1] \to [t_1, t_2] \, , \quad \rho_2 : [T_0, T_1] \to [t_2, t^f]$$

with $0 \le T_0 < T_1$ arbitrary chosen. We solve then the state equation (4.5) with controls

$$w_1(\tau) := \rho_1'(\tau) \, , \quad w_2(\tau) := \rho_2'(\tau), \, , \quad w_{\mathcal{H}}(\tau) := \rho_{\mathcal{H}}'(\tau)$$

$$v_1(\tau) := \alpha_1(\rho_1(\tau)) = \alpha_1(t)\ , \quad v_{\mathcal{H}}(\tau) := a_{\mathcal{H}}(\rho_{\mathcal{H}}(\tau)) = a_{\mathcal{H}}(t), \quad v_2(\tau) := \alpha_2(\rho_2(\tau)) = \alpha_2(t)$$

and initial and final data:

$$\bar{Z}_0 := (x^0, t^0, X_1(t_1), t_1, X_2(t_2), t_2)\ ,\ \bar{Z}_1 := (X_1(t_1), t_1, X_2(t_2), t_2, x^f, t^f)\ .$$

Therefore $(\bar{Z}_0, \bar{Z}_1) \in \mathcal{M}(x^0, t^0; x^f, t^f)$ and the duplicated trajectory is such that

$$Y_1(\tau) = X_1(\rho_1(\tau)) = X_1(t)\ , \quad Y_{\mathcal{H}}(\tau) = X_{\mathcal{H}}(\rho_{\mathcal{H}}(\tau)) = X_{\mathcal{H}}(t)\ , \quad Y_2(\tau) = X_2(\rho_2(\tau)) = X_2(t)\ ,$$

for any $t \in (t^0, t^f)$, $\tau \in (T_0, T_1)$. Moreover, by the above change of time variable we have

$$\int_{t^0}^{t_1} l_1(X_1(t), \alpha_1(t))\, dt + \int_{t_1}^{t_2} l_{\mathcal{H}}(X_{\mathcal{H}}(t), a(t))\, dt + \int_{t_2}^{t^f} l_2(X_2(t), \alpha_2(t))\, dt =$$

$$= \int_{T_0}^{T_1} \Big(l_1(Y_1(\tau), v_1(\tau)) w_1(\tau) + l_{\mathcal{H}}(Y_{\mathcal{H}}(\tau), v_{\mathcal{H}}(\tau)) w_{\mathcal{H}}(\tau) + l_2(Y_2(\tau), v_2(\tau)) w_2(\tau) \Big)\, d\tau\ .$$

That is $C(x^0, t^0; x^f, t^f; \mathbb{X}) = \mathcal{C}(\mathbb{Z})$. Vice versa, since the time change of variable ρ is invertible given $(Z_0, Z_1) \in \mathcal{M}(x^0, t^0; x^f, t^f)$ and a corresponding admissible trajectory $\mathbb{Z}$ we can construct a trajectory $\mathbb{X}$ such that $\mathcal{C}(\mathbb{Z}) = C(x^0, t^0; x^f, t^f; \mathbb{X})$ and the proof is completed. $\square$

Notation. Partial derivatives and gradients.
In order to write the partial derivatives of Σ at points $\mathbb{Z} \in \Omega_1 \times \mathbb{R}^+_* \times \mathcal{H} \times \mathbb{R}^+_* \times \Omega_2 \times \mathbb{R}^+_*$ we will enumerate the space variables as follows $(\mathbb{Z}_0, \mathbb{Z}_1) = \big((1,2,3,4,5,6),(7,8,9,10,11,12)\big)$ therefore $\partial_i \Sigma$ takes values in $\mathbb{R}$ for $i = 2,4,6,8,10,12$; $\nabla_i \Sigma$ takes values in $\mathbb{R}^N$ for $i = 1,5,7,11$ and $\nabla_{\mathcal{H},i}\Sigma$ in $T\mathcal{H}$ for $i = 3,9$. We will set

$$\nabla_{\mathbb{Z}_0} := (\nabla_1\Sigma, \partial_2\Sigma, \nabla_{\mathcal{H},3}\Sigma, \partial_4\Sigma, \nabla_5\Sigma, \partial_6\Sigma)\ , \quad \Sigma_{t_0}(\mathbb{Z}_0, t_0, \mathbb{Z}_1, t_1) := -\frac{\partial}{\partial t_0}\Sigma(\mathbb{Z}_0, t_0, \mathbb{Z}_1, t_1).$$

$$\nabla_{\mathbb{Z}_1}\Sigma := (\nabla_7\Sigma, \partial_8\Sigma, \nabla_{\mathcal{H},9}\Sigma, \partial_{10}\Sigma, \nabla_{11}\Sigma, \partial_{12}\Sigma)\ , \quad \Sigma_{t_1}(\mathbb{Z}_0, t_0, \mathbb{Z}_1, t_1) := -\frac{\partial}{\partial t_1}\Sigma(\mathbb{Z}_0, t_0, \mathbb{Z}_1, t_1).$$

Moreover, we will respectively denote by $D^+_{\mathbb{Z}_0}\Sigma$ and $D^-_{\mathbb{Z}_0}\Sigma$ (or $D^+_{\mathbb{Z}_1}\Sigma$ and $D^-_{\mathbb{Z}_1}\Sigma$) the classical super and sub differential in the space variables $1,3,5,7,9$.
Let us now define the **Hamiltonian**.
If $\mathbb{V} = (v_1, w_1, v_{\mathcal{H}}, w_{\mathcal{H}}, v_2, w_2) \in V$, $\mathbb{Z} = (Y_1, \rho_1, Y_{\mathcal{H}}, \rho_{\mathcal{H}}, Y_2, \rho_2) \in \mathbb{R}^N \times \mathbb{R} \times T\mathcal{H} \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}$ and $\mathbb{Q} = (Q_1, Q_2, Q_3, Q_4, Q_5, Q_6) \in \mathbb{R}^N \times \mathbb{R} \times T_{Y_{\mathcal{H}}}\mathcal{H} \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}$, we set

$$\begin{aligned}\widetilde{\mathcal{H}}(\mathbb{Z}, \mathbb{Q}, p_0, \mathbb{V}) := &\big\langle Q_1, f_1(Y_1, v_1) w_1 \big\rangle + Q_2\, w_1 + p_0\, l_1(Y_1, v_1) w_1 + \\ &+ \big\langle Q_3, f_{\mathcal{H}}(Y_{\mathcal{H}}, v_{\mathcal{H}}) w_{\mathcal{H}} \big\rangle_{\mathcal{H}} + Q_4\, w_{\mathcal{H}} + p_0\, l_{\mathcal{H}}(Y_{\mathcal{H}}, v_{\mathcal{H}}) w_{\mathcal{H}} + \\ &+ \big\langle Q_5, f_2(Y_2, v_2) w_2 \big\rangle + Q_6\, w_2 + p_0\, l_2(Y_2, v_2) w_2\ ,\end{aligned}$$

and

$$\mathcal{H}(\mathbb{Z}, \mathbb{Q}, p_0) := \sup_{\mathbb{V} \in V} \big\{ \big\langle Q_1, f_1(Y_1, v_1) w_1 \big\rangle + Q_2\, w_1 + p_0\, l_1(Y_1, v_1) w_1$$

$$+ \big\langle Q_3, f_{\mathcal{H}}(Y_{\mathcal{H}}, v_{\mathcal{H}}) w_{\mathcal{H}} \big\rangle_{\mathcal{H}} + Q_4\, w_{\mathcal{H}} + p_0\, l_{\mathcal{H}}(Y_{\mathcal{H}}, v_{\mathcal{H}}) w_{\mathcal{H}} + \big\langle Q_5, f_2(Y_2, v_2) w_2 \big\rangle + Q_6\, w_2 + p_0\, l_2(Y_2, v_2) w_2 \big\}\ .$$

We first apply the standard Pontryagin maximum principle to the duplicated problem.

Lemma 4.3. The Pontryagin maximum priniciple.
Assume (H$\mathcal{H}$), (HA), (Hfl$_i$) and (Hfl$_{\mathcal{H}}$). Let (Z_0, T_0), $(Z_1, T_1) \in (\Omega_1 \times \mathbb{R}^+_* \times \mathcal{H} \times \mathbb{R}^+_* \times \Omega_2 \times \mathbb{R}^+_*) \times \mathbb{R}^+$ *and let* $\mathbb{Z}(\cdot) \in \mathcal{Z}_{(Z_0,T_0),(Z_1,T_1)}$ *be an optimal trajectory for the value function* $\Sigma(Z_0, T_0; Z_1, T_1)$ *defined in* (4.7). *Assume that* $\mathbb{V}(\cdot)$ *is the corresponding optimal control.*
Then, there exist $p^0 \leq 0$ *and a piecewise absolutely continuous mapping*

$$\mathbb{P}_{\mathbb{Z}}(\cdot) := (P_{Y_1}(\cdot), P_{\rho_1}(\cdot), P_{Y_{\mathcal{H}}}(\cdot), P_{\rho_{\mathcal{H}}}(\cdot), P_{Y_2}(\cdot), P_{\rho_2}(\cdot)) : \mathbb{R}^+ \to \mathbb{R}^N \times \mathbb{R} \times T_{Y_{\mathcal{H}}}\mathcal{H} \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}$$

called an adjoint vector, *with* $(\mathbb{P}_{\mathbb{Z}}(\cdot), p^0) \neq (0, 0)$, *such that the so-called extremal lift* $(\mathbb{Z}(\cdot), \mathbb{P}_{\mathbb{Z}}(\cdot), p^0, \mathbb{V}(\cdot))$ *is, for almost every* $\tau \in (T_0, T_1)$, *a solution of*

$$\mathbb{Z}'(\tau) = \frac{\partial \widetilde{\mathcal{H}}}{\partial P}(\mathbb{Z}(\tau), \mathbb{P}_{\mathbb{Z}}(\tau), p_0, \mathbb{V}(\tau)) \quad , \quad \mathbb{P}'_{\mathbb{Z}}(\tau) = -\frac{\partial \widetilde{\mathcal{H}}}{\partial Z}(\mathbb{Z}(\tau), \mathbb{P}_{\mathbb{Z}}(\tau), p_0, \mathbb{V}(\tau)).$$

Moreover, the maximization condition

$$\widetilde{\mathcal{H}}(\mathbb{Z}(\tau), \mathbb{P}_{\mathbb{Z}}(\tau), p_0, \mathbb{V}(\tau)) = \max_{\mathbb{V} \in V} \widetilde{\mathcal{H}}(\mathbb{Z}(\tau), \mathbb{P}_{\mathbb{Z}}(\tau), p_0, \mathbb{V}) \ \ (= \mathcal{H}(\mathbb{Z}(\tau), \mathbb{P}_{\mathbb{Z}}(\tau), p_0)\,) \tag{4.9}$$

holds for almost every $\tau \in (T_0, T_1)$.
If $Z_0 = (x^0, t^0, x_1, t_1, x_2, t_2)$, $Z_1 = (x_1, t_1, x_2, t_2, x^f, t^f) \in \mathcal{M}(x^0, t^0; x^f, t^f)$ *then the following transversality condition holds: there exist* $\nu_1, \nu_2 \in \mathbb{R}$ *such that*

$$P_{\rho_{\mathcal{H}}}(T_0) = P_{\rho_1}(T_1) \tag{4.10}$$
$$P_{\rho_2}(T_0) = P_{\rho_{\mathcal{H}}}(T_1) \tag{4.11}$$
$$P_{Y_{\mathcal{H}}}(T_0) = P_{Y_1}(T_1) + \nu_1 \nabla\Psi(x_1) \tag{4.12}$$
$$P_{Y_2}(T_0) = P_{Y_{\mathcal{H}}}(T_1) + \nu_2 \nabla\Psi(x_2)\,. \tag{4.13}$$

Proof. This result is completely standard and can be found in the classical book of Pontriagyn [29]. If $Z_0, Z_1 \in \mathcal{M}(x^0, t^0; x^f, t^f)$, the classical transversality condition holds (see [2, Theorem 12.15] or [39]):

$$(-\mathbb{P}_{\mathbb{Z}}(T_0), \mathbb{P}_{\mathbb{Z}}(T_1)) \perp T_{(\mathbb{Z}(T_0),\mathbb{Z}(T_1))}\ \mathcal{M}(x^0, t^0; x^f, t^f).$$

Now, if $\mathbb{Z}(T_0) = Z_0 = (x^0, t^0, x_1, t_1, x_2, t_2)$ and $\mathbb{Z}(T_1) = Z_1 = (x_1, t_1, x_2, t_2, x^f, t^f)$ the above relation reads:
- $t_1 = Z_0^4 = Z_1^2$ implies $P_{\rho_{\mathcal{H}}}(T_0) = P_{\rho_1}(T_1)$
- $t_2 = Z_0^6 = Z_1^4$ implies $P_{\rho_2}(T_0) = P_{\rho_{\mathcal{H}}}(T_1)$
- $x_1 = Z_0^3 = Z_1^1$ and $x_1 \in \mathcal{H}$ imply $P_{Y_{\mathcal{H}}}(T_0) = P_{Y_1}(T_1) + \nu_1 \nabla\Psi(x_1)$
- $x_2 = Z_0^5 = Z_1^3$ and $x_2 \in \mathcal{H}$ imply $P_{Y_2}(T_0) = P_{Y_{\mathcal{H}}}(T_1) + \nu_2 \nabla\Psi(x_2)$

Therefore, equalities (4.10)-(4.13) are fulfilled. □

In order to prove the basic link between the adjoint vector and the gradient of the value function Σ we need to assume the uniqueness of the extremal lift:

(Hu) We assume that the optimal trajectory $\mathbb{Z}(\cdot)$ in Theorem 4.3 admits a unique extremal lift $(\mathbb{Z}(\cdot), \mathbb{P}_{\mathbb{Z}}(\cdot), p^0, \mathbb{V}(\cdot))$ which is moreover normal, i.e., $p_0 = -1$.

Remark 4.4. *The assumption on the uniqueness of the solution of the optimal control problem and on the uniqueness of its extremal lift (which is then moreover normal) is closely related to the differentiability properties of the value function. We refer to [5, 18] for precise results, and to [13, 33, 34, 37] for results on the size of the set where the value function is differentiable. For instance for control-affine systems the singular set of the value function has Hausdorff $(N-1)$-dimension zero, whenever there is no optimal singular trajectory (see [34]), and is a stratified submanifold of $\mathbb{R}^N$ of positive codimension in an analytic context (see [40]). These results eventually say that the value function is of class C^1 at generic points. Moreover, note that the property of having a unique extremal lift, that is moreover normal, is generic in the sense of the Whitney topology for control-affine systems (see [15, 16]).*

Theorem 4.5. *Assume (H$\mathcal{H}$), (HA), (Hfl$_i$) and (Hfl$_\mathcal{H}$).*
Let (Z_0,T_0), $(Z_1,T_1) \in \left(\overline{\Omega}_1 \times \mathbb{R}^+_* \times \mathcal{H} \times \mathbb{R}^+_* \times \overline{\Omega}_2 \times \mathbb{R}^+_*\right) \times \mathbb{R}^+$ and let $\mathbb{Z}(\cdot) \in \mathcal{Z}_{(Z_0,T_0),(Z_1,T_1)}$ be an optimal trajectory for the value function $\Sigma(Z_0,T_0;Z_1,T_1)$ defined in (4.7). Assume that $\mathbb{P}_\mathbb{Z}$ is the corresponding absolutely continuous adjoint vector given by Theorem 4.3.
Then, the following holds:

(i) For any time τ in the closed interval $[T_0,T_1]$ we have

$$D^-_{Z_0}\Sigma(\mathbb{Z}(\tau),\tau;Z_1,T_1) \subseteq -\mathbb{P}_\mathbb{Z}(\tau) \subseteq D^+_{Z_0}\Sigma(\mathbb{Z}(\tau),\tau;Z_1,T_1) \tag{4.14}$$

in the sense that either $D^-_{Z_0}\Sigma(\mathbb{Z}(\tau),\tau;Z_1,T_1)$ is empty or the function $\tau \mapsto \Sigma(\mathbb{Z}(\tau),\tau;Z_1,T_1)$ is differentiable and then $D^-_{Z_0}\Sigma = D^+_{Z_0}\Sigma$ at this point.
Moreover, when assumption (Hu) holds the function $\tau \mapsto \Sigma(\mathbb{Z}(\tau),\tau;Z_1,T_1)$ is differentiable for every time in $[T_0,T_1]$, thus

$$\nabla_{Z_0}\Sigma(\mathbb{Z}(\tau),\tau;Z_1,T_1) = -\mathbb{P}_\mathbb{Z}(\tau) \qquad \forall\tau \in [T_0,T_1]\,. \tag{4.15}$$

(ii) For any time τ in the closed interval $[T_0,T_1]$ we have

$$D^-_{Z_1}\Sigma(Z_0,T_0;\mathbb{Z}(\tau),\tau) \subseteq \mathbb{P}_\mathbb{Z}(\tau) \subseteq D^+_{Z_1}\Sigma(Z_0,T_0;\mathbb{Z}(\tau),\tau) \tag{4.16}$$

in the sense that either $D^-_{Z_1}\Sigma(Z_0,T_0;\mathbb{Z}(\tau),\tau)$ is empty or the function $\tau \mapsto \Sigma(Z_0,T_0;\mathbb{Z}(\tau),\tau)$ is differentiable and then $D^-_{Z_1}\Sigma = D^+_{Z_1}\Sigma$ at this point.
Moreover, when assumption (Hu) holds the function $\tau \mapsto \Sigma(Z_0,T_0;\mathbb{Z}(\tau),\tau)$ is differentiable for every time in $[T_0,T_1]$, thus

$$\nabla_{Z_1}\Sigma(Z_0,T_0;\mathbb{Z}(\tau),\tau) = \mathbb{P}_\mathbb{Z}(\tau) \qquad \forall\tau \in [T_0,T_1]\,. \tag{4.17}$$

Proof. The idea is to apply the theory of viscosity solution for Hamilton-Jacobi equation. To this aim we define two different value functions by considering separately the case when we fix the initial data (Z_0,T_0) and we consider a function the final data (Z_1,T_1) or vice versa.
Precisely, to prove (i) we fix (T_1,Z_1) and for any $T_0 \in \mathbb{R}^+_*$, $Z_0 \in \overline{\Omega}_1 \times \mathbb{R}^+_* \times \mathcal{H} \times \mathbb{R}^+_* \times \overline{\Omega}_2 \times \mathbb{R}^+_*$ we define $\Sigma^0(Z_0,T_0) := \Sigma(Z_0,T_0,Z_1,T_1)$.
Similarly to prove (ii), (T_0,Z_0) is given and for any $T_1 \in \mathbb{R}^+_*$, $Z_1 \in \overline{\Omega}_1 \times \mathbb{R}^+_* \times \mathcal{H} \times \mathbb{R}^+_* \times \overline{\Omega}_2 \times \mathbb{R}^+_*$ we set $\Sigma^1(Z_1,T_1) := \Sigma(Z_0,T_0,Z_1,T_1)$.

Notation. In order to write the partial derivatives of Σ^0 and Σ^1 we consider a generic function $u(\mathbb{Z},t):\big(\Omega_1\times\mathbb{R}^+_*\times\mathcal{H}\times\mathbb{R}^+_*\times\Omega_2\times\mathbb{R}^+_*\big)\times\mathbb{R}^+\to\mathbb{R}$ and we will enumerate the variables as follows $\mathbb{Z}=(1,2,3,4,5,6)$. Therefore $\partial_i u(\mathbb{Z})$ takes values in $\mathbb{R}$ for $i=2,4,6$, $\nabla_i\Sigma(\mathbb{Z})$ takes values in $\mathbb{R}^N$ for $i=1,5$ and $\nabla_{\mathcal{H},i}\Sigma$ in $T\mathcal{H}$ for $i=3$. We will set

$$\nabla u := (\nabla_1 u, \partial_2 u, \nabla_{\mathcal{H},3}u, \partial_4 u, \nabla_5 u, \partial_6 u) \qquad u_t(\mathbb{Z},t) := -\frac{\partial u}{\partial t}(\mathbb{Z},t).$$

Moreover, we will respectively denote by D^+u and D^-u the classical super and sub differential in the space variables $1,3,5$. Note that we have

$$\nabla_{Z_0}\Sigma(Z_0,T_0;Z_1,T_1)=\nabla\Sigma^0(Z_0,T_0) \quad \text{and} \quad \nabla_{Z_1}\Sigma(Z_0,T_0;Z_1,T_1)=\nabla\Sigma^1(Z_1,T_1)\,.$$

By applying the standard theory of viscosity solution (see for instance, in Bardi Capuzzo Dolcetta [6] Proposition 3.1 and Proposition 3.5) we easily know that $\Sigma^0(Z_0,T_0)$ is a bounded, Liptschitz continuous viscosity solution of

$$-\frac{\partial u}{\partial t}(\mathbb{Z},t)+\mathcal{H}\Big(\mathbb{Z}\,,-\nabla u,-1\Big)=0 \quad \text{in } (\Omega_1\times\mathbb{R}^+_*\times\mathcal{H}\times\mathbb{R}^+_*\times\Omega_2\times\mathbb{R}^+_*)\times(T_0,T_1)\,,$$

and $\Sigma^1(Z_1,T_1)$ is a bounded, Liptschitz continuous viscosity solution of

$$\frac{\partial u}{\partial t}(\mathbb{Z},t)+\mathcal{H}\Big(\mathbb{Z}\,,\nabla u,-1\Big)=0 \quad \text{in } (\Omega_1\times\mathbb{R}^+_*\times\mathcal{H}\times\mathbb{R}^+_*\times\Omega_2\times\mathbb{R}^+_*)\times(T_0,T_1)\,.$$

Therefore, we can apply for example Corollary 3.45 in [6] to obtain (4.14) and (4.16). Now, if assumption (Hu) holds, one can prove that the two functions $\tau\mapsto\Sigma^0(\mathbb{Z}(\tau),\tau)$ and $\tau\mapsto\Sigma^1(\mathbb{Z}(\tau),\tau)$ are differentiable (see [13, Theorem 7.4.16] or [5, 18]) thus (4.15) and (4.17) easily follow. $\square$

Remark 4.6. *It is useful to write equalities* (4.15) *and* (4.17) *as a unique equality. We have indeed*

$$-\nabla_{Z_0}\Sigma(\mathbb{Z}(\tau),\tau;Z_1,T_1)=\mathbb{P}_{\mathbb{Z}}(\tau)=\nabla_{Z_1}\Sigma(Z_0,T_0;\mathbb{Z}(\tau),\tau) \qquad \forall\tau\in[T_0,T_1]\,. \tag{4.18}$$

which is, more precisely

$$-\nabla_1\Sigma(\mathbb{Z}(\tau),\tau;Z_1,T_1)=P_{Y_1}(\tau)=\nabla_7\Sigma(Z_0,T_0;\mathbb{Z}(\tau),\tau) \qquad \forall\tau\in[T_0,T_1] \tag{4.19}$$
$$-\partial_2\Sigma(\mathbb{Z}(\tau),\tau;Z_1,T_1)=P_{\rho_1}(\tau)=\partial_8\Sigma(Z_0,T_0;\mathbb{Z}(\tau),\tau) \qquad \forall\tau\in[T_0,T_1] \tag{4.20}$$
$$-\nabla_3^{\mathcal{H}}\Sigma(\mathbb{Z}(\tau),\tau;Z_1,T_1)=P_{Y_{\mathcal{H}}}(\tau)=\nabla_9^{\mathcal{H}}\Sigma(Z_0,T_0;\mathbb{Z}(\tau),\tau) \qquad \forall\tau\in[T_0,T_1] \tag{4.21}$$
$$-\partial_4\Sigma(\mathbb{Z}(\tau),\tau;Z_1,T_1)=P_{\rho_{\mathcal{H}}}(\tau)=\partial_{10}\Sigma(Z_0,T_0;\mathbb{Z}(\tau),\tau) \qquad \forall\tau\in[T_0,T_1] \tag{4.22}$$
$$-\nabla_5\Sigma(\mathbb{Z}(\tau),\tau;Z_1,T_1)=P_{Y_2}(\tau)=\nabla_{11}\Sigma(Z_0,T_0;\mathbb{Z}(\tau),\tau) \qquad \forall\tau\in[T_0,T_1] \tag{4.23}$$
$$-\partial_6\Sigma(\mathbb{Z}(\tau),\tau;Z_1,T_1)=P_{\rho_2}(\tau)=\partial_{12}\Sigma(Z_0,T_0;\mathbb{Z}(\tau),\tau) \qquad \forall\tau\in[T_0,T_1]\,. \tag{4.24}$$

Note that at times T_0,T_1 the gradients are naturally defined as the limits of the gradients in the open interval (T_0,T_1).

4.2 Back on the original problem

We prove now the analogous result to the one obtained for the structure 1-2. We first remark that for this structure one can not directly define a global adjoint vector, therefore his role will be played by the limit of the gradient of the value function (vectors Q_1, Q_2, $Q_{\mathcal{H}}$ below).
The result is the following.

Theorem 4.7. *Assume (H$\mathcal{H}$), (HA), (Hfl$_i$), (Hfl$_{\mathcal{H}}$) and (Hu).*
For any $(x^0, t^0; x^f, t^f) \in \Omega_1 \times \mathbb{R}^+ \times \Omega_2 \times \mathbb{R}^+_*$ we have

$$S_{1,\mathcal{H},2}(x^0, t^0; x^f, t^f) = \min_{0<t_1<t_2<t^f \; x_1,x_2\in\mathcal{H}} \Big\{ S_1(x^0, t^0; , x_1, t_1) + S_{\mathcal{H}}(x_1, t_1; x_2, t_2) + S_2(x_2, t_2; x^f, t^f) \Big\} .$$

Let $\mathbb{X}(\cdot)$ be an optimal trajectory for the value function $S_{1,\mathcal{H},2}(x^0, t^0; x^f, t^f)$ defined in (4.4) *and let us set*

$$Q_1(t_1^-) := - \lim_{t\to t_1^-} \nabla_{x^0}\, S_{1,\mathcal{H},2}\big(X_1(t), t; x^f, t^f\big).$$

$$Q_2(t_2^+) := \lim_{t\to t_2^+} \nabla_{x^f}\, S_{1,\mathcal{H},2}\big(x^0, t^0; X_2(t), t\big).$$

$$Q_{\mathcal{H}}(t_1^+) := - \lim_{t\to t_1^+} \nabla^{\mathcal{H}}_{x^0}\, S_{1,\mathcal{H},2}\big(X_{\mathcal{H}}(t), t; x^f, t^f\big)$$

$$Q_{\mathcal{H}}(t_2^-) := \lim_{t\to t_2^-} \nabla^{\mathcal{H}}_{x^f}\, S_{1,\mathcal{H},2}\big(x^0, t^0; X_{\mathcal{H}}(t), t\big) .$$

We have the following continuity conditions on the Hamiltonans:

$$H_1(X_1(t_1^-), Q_1(t_1^-), p_0) = H_{\mathcal{H}}(X_{\mathcal{H}}(t_1^+), Q_{\mathcal{H}}(t_1^+), p_0) \tag{4.25}$$

$$H_{\mathcal{H}}(X_{\mathcal{H}}(t_2^-), Q_{\mathcal{H}}(t_2^-), p_0) = H_2(X_2(t_2^+), Q_2(t_2^+), p_0). \tag{4.26}$$

Moreover, there exists $\nu_1, \nu_2 \in \mathbb{R}$ such that

$$Q_{\mathcal{H}}(t_1^+) = Q_1(t_1^-) + \nu_1\, \nabla\Psi(X_1(t_1^-)) \quad \text{and} \quad Q_2(t_2^+) = Q_{\mathcal{H}}(t_2^-) + \nu_2\, \nabla\Psi(X_2(t_2^+)) . \tag{4.27}$$

If, in particular $\big\langle \nabla\Psi(X_1(t_1^-), f_1(t_1^-)\big\rangle \neq 0$ and $\big\langle \nabla\Psi(X_2(t_2^+)), f_2(t_2^+)\big\rangle \neq 0$ then ν_1 and ν_2 are respectively given by

$$\nu_1 = \frac{\big\langle Q_{\mathcal{H}}(t_1^+), f_1(t_1^-)\big\rangle - \big\langle Q_1(t_1^-), f_{\mathcal{H}}(t_1^+)\big\rangle_{\mathcal{H}} + p_0\,(l_1(t_1^-) - l_{\mathcal{H}}(t_1^+))}{\big\langle \nabla\Psi(X_1(t_1^-)), f_1(t_1^-)\big\rangle} \tag{4.28}$$

and

$$\nu_2 = \frac{\big\langle Q_{\mathcal{H}}(t_2^-), f_{\mathcal{H}}(t_2^-)\big\rangle_{\mathcal{H}} - \big\langle Q_{\mathcal{H}}(t_2^-), f_2(t_2^+)\big\rangle + p_0\,(l_{\mathcal{H}}(t_2^-) - l_2(t_2^+))}{\big\langle \nabla\Psi(X_2(t_2^+)), f_2(t_2^+)\big\rangle} \tag{4.29}$$

where we used the short notation $f_i(t_i^\pm)$ for $f_i(X_i(t_i^\pm), \alpha_i(t_i^\pm))$ and $l_i(t_i^\pm)$ for $l_i(X_i(t_i^\pm), \alpha_i(t_i^\pm))$, $i = 1, 2, \mathcal{H}$.

Proof. Fix $(x^0, t^0; x^f, t^f) \in \Omega_1 \times \mathbb{R}^+ \times \Omega_2 \times \mathbb{R}^+_*$.
To obtain the first result we rewrite equality (4.8) in Proposition 4.2 as follows

$$S_{1,\mathcal{H},2}(\lambda^0, \lambda^f) = \inf_{\chi=(x_1,t_1,x_2,t_2)} \left\{ \Sigma((\lambda^0, \chi); (\chi, \lambda^f)) \ : \ \Psi(x_1) = 0 \, , \Psi(x_2) = 0 \ , t^f > t_2 > t_1 > t^0 \right\} \tag{4.30}$$

where we set $\lambda := (\lambda^0, \lambda^f) := (x^0, t^0; x^f, t^f)$ and $\chi := (x_1, t_1, x_2, t_2)$. Thus, by the construction of the duplicated value function Σ one can easily see that

$$S_{1,\mathcal{H},2}(x^0, t^0; x^f, t^f) = \min_{0<t_1<t_2<t^f \ x_1,x_2\in\mathcal{H}} \left\{ S_1(x^0, t^0; , x_1, t_1) + S_{\mathcal{H}}(x_1, t_1; x_2, t_2) + S_2(x_2, t_2; x^f, t^f) \right\} .$$

Always thanks to (4.8) in Proposition 4.2 we can consider now $(Z_0, Z_1) \in \mathcal{M}(x^0, t^0; x^f, t^f)$ such that $S_{1,\mathcal{H},2}(x^0, t^0; x^f, t^f) = \Sigma(Z_0, T_0; Z_1, T_1)$ for an optimal trajectory $\mathbb{Z}(\cdot) \in \mathcal{Z}_{(Z_0,T_0),(Z_1,T_1)}$ (note that we have $S_{1,\mathcal{H},2}(x^0, t^0; x^f, t^f) = \Sigma(Z_0, T_0; Z_1, T_1) = C(\mathbb{Z})$).
Let $\mathbb{P}_{\mathbb{Z}}(\cdot)$ be the adjoint vector given by Theorem 4.3, the maximality condition (4.9) implies that

$$\left\langle P_{Y_1}(\tau), f_1(Y_1(\tau), v_1(\tau)) \right\rangle + P_{\rho_1}(\tau) \ + p_0 \, l_1(Y_1(\tau), v_1(\tau)) \ \ = 0 \tag{4.31}$$

$$\left\langle P_{Y_{\mathcal{H}}}(\tau), f_{\mathcal{H}}(Y_{\mathcal{H}}(\tau), v_{\mathcal{H}}(\tau)) \right\rangle_{\mathcal{H}} + P_{\rho_{\mathcal{H}}}(\tau) \ + p_0 \, l_{\mathcal{H}}(Y_{\mathcal{H}}(\tau), v_{\mathcal{H}}(\tau)) \ \ = 0 \tag{4.32}$$

$$\left\langle P_{Y_2}(\tau), f_2(Y_2(\tau), v_2(\tau)) \right\rangle + P_{\rho_2}(\tau) \ + p_0 \, l_2(Y_2(\tau), v_2(\tau)) \ \ = 0 \tag{4.33}$$

for almost every $\tau \in (T_0, T_1)$. Moreovever, by the transversality condition in Theorem 4.3, there exists $\nu_1, \nu_2 \in \mathbb{R}$ such that

$$P_{\rho_{\mathcal{H}}}(T_0) \ = \ P_{\rho_1}(T_1) \tag{4.34}$$

$$P_{\rho_2}(T_0) \ = \ P_{\rho_{\mathcal{H}}}(T_1) \tag{4.35}$$

$$P_{Y_{\mathcal{H}}}(T_0) \ = \ P_{Y_1}(T_1) + \nu_1 \, \nabla\Psi(x_1) \tag{4.36}$$

$$P_{Y_2}(T_0) \ = \ P_{Y_{\mathcal{H}}}(T_1) + \nu_2 \, \nabla\Psi(x_2) \, . \tag{4.37}$$

Or aim is now to read all these equalities on the original problem. By definition of the duplicated problem, we can construct an optimal trajectory $\mathbb{X}(\cdot)$ for $S_{1,\mathcal{H},2}(x^0, t^0; x^f, t^f)$, such that

$$\begin{aligned} Y_1(\tau) = X_1(\rho_1(\tau)) = X_1(t) \quad \forall\tau \in (T_0, T_1) \quad \forall t \in (t^0, t_1) \\ Y_{\mathcal{H}}(\tau) = X_{\mathcal{H}}(\rho_{\mathcal{H}}(\tau)) = X_{\mathcal{H}}(t) \quad \forall\tau \in (T_0, T_1) \quad \forall t \in (t_1, t_2) \\ Y_2(\tau) = X_2(\rho_2(\tau)) = X_2(t) \quad \forall\tau \in (T_0, T_1) \quad \forall t \in (t_2, t^f) \end{aligned}$$

(indeed, we recall that by construction $C(\mathbb{X}) = C(\mathbb{Z}) = S_{1,\mathcal{H},2}(x^0, t^0; x^f, t^f) = \Sigma(Z_0, T_0; Z_1, T_1)$).
We set now:

$$\begin{aligned} P_1(t) := P_{Y_1}(\tau) = P_{Y_1}(\rho_1(t)) \quad \forall\tau \in (T_0, T_1) \quad \forall t \in (t^0, t_1) \\ P_{\mathcal{H}}(t) := P_{Y_{\mathcal{H}}}(\tau) = P_{Y_{\mathcal{H}}}(\rho_{\mathcal{H}}(t)) \quad \forall\tau \in (T_0, T_1) \quad \forall t \in (t_1, t_2) \\ P_2(t) := P_{\rho_2}(\tau) = P_{\rho_2}(\rho_2(\tau)) \quad \forall\tau \in (T_0, T_1) \quad \forall t \in (t_2, t^f) \, . \end{aligned}$$

Therefore, by definition of the Hamitonians $\widetilde{H_1}$, $\widetilde{H_{\mathcal{H}}}$ and $\widetilde{H_2}$ equalities (4.31)-(4.33) give us

$$\begin{aligned} \widetilde{H_1}(X_1(t), P_1(t), p_0, \alpha_1(t)) = -P_{\rho_1}(\tau) \\ \widetilde{H_{\mathcal{H}}}(X_{\mathcal{H}}(t), P_{\mathcal{H}}(t), p_0, a_{\mathcal{H}}(t)) = -P_{\rho_{\mathcal{H}}}(\tau) \\ \widetilde{H_2}(X_2(t), P_2(t), p_0, \alpha_2(t)) = -P_{\rho_2}(\tau). \end{aligned}$$

for almost every $t \in (t^0, t^f)$, $\tau \in (T_0, T_1)$.
To obtain the continuity conditions on the Hamiltoinans we look now at the above equalities at times t_1, t_2. By construction of the time change of variable and the continuity of the adjoint vector we can write

$$\widetilde{H_1}(X_1(t_1^-), P_1(t_1^-), p_0, \alpha_1(t_1^-)) = \lim_{t\to t_1^-} \widetilde{H_1}(X_1(t), P_1(t), p_0, \alpha_1(t)) = \lim_{\tau\to T_1} (-P_{\rho_1}(\tau)) = -P_{\rho_1}(T_1)$$

$$\widetilde{H_\mathcal{H}}(X_\mathcal{H}(t_1^+), P_\mathcal{H}(t_1^+), p_0, a_\mathcal{H}(t_1^+)) = \lim_{t\to t_1^+} \widetilde{H_\mathcal{H}}(X_\mathcal{H}(t), P_\mathcal{H}(t), p_0, a_\mathcal{H}(t)) = \lim_{\tau\to T_0} (-P_{\rho_\mathcal{H}}(\tau)) = -P_{\rho_\mathcal{H}}(T_0)$$

$$\widetilde{H_\mathcal{H}}(X_\mathcal{H}(t_2^-), P_\mathcal{H}(t_2^-), p_0, a_\mathcal{H}(t_2^-)) = \lim_{t\to t_2^-} \widetilde{H_\mathcal{H}}(X_\mathcal{H}(t), P_\mathcal{H}(t), p_0, a_\mathcal{H}(t)) = \lim_{\tau\to T_1} (-P_{\rho_\mathcal{H}}(\tau)) = -P_{\rho_\mathcal{H}}(T_1)$$

$$\widetilde{H_2}(X_2(t_2^+), P_2(t_2^+), p_0, \alpha_2(t_2^+)) = \lim_{t\to t_2^+} \widetilde{H_2}(X_2(t), P_2(t), p_0, \alpha_2(t)) = \lim_{\tau\to T_0} (-P_{\rho_2}(\tau)) = -P_{\rho_2}(T_0)\,.$$

Since by (4.34), (4.35) we have $P_{\rho_\mathcal{H}}(T_0) = P_{\rho_1}(T_1)$ and $P_{\rho_\mathcal{H}}(T_1) = P_{\rho_2}(T_0)$, the above equalities give

$$\widetilde{H_1}(X_1(t_1^-), P_1(t_1^-), p_0, \alpha_1(t_1^-)) = \widetilde{H_\mathcal{H}}(X_\mathcal{H}(t_1^+), P_\mathcal{H}(t_1^+), p_0, a_\mathcal{H}(t_1^+))$$
$$\widetilde{H_\mathcal{H}}(X_\mathcal{H}(t_2^-), P_\mathcal{H}(t_2^-), p_0, a_\mathcal{H}(t_2^-)) = \widetilde{H_2}(X_2(t_2^+), P_2(t_2^+), p_0, \alpha_2(t_2^+))\,,$$

therefore, by the optimality of the trajectory, we can conclude that

$$H_1(X_1(t_1^-), P_1(t_1^-), p_0) = H_\mathcal{H}(X_\mathcal{H}(t_1^+), P_\mathcal{H}(t_1^+), p_0) \tag{4.38}$$

$$H_\mathcal{H}(X_\mathcal{H}(t_2^-), P_\mathcal{H}(t_2^-), p_0) = H_2(X_2(t_2^+), P_2(t_2^+), p_0)\,. \tag{4.39}$$

To obtain the jump conditions on the adjoint vector we aim to exploit the transversality conditions on the duplicated problem: (4.12) and (4.13) in Theorem 4.3. By applying the usual change of variable in (4.36) and (4.37) we have

$$P_\mathcal{H}(t_1^+) = P_1(t_1^-) + \nu_1\, \nabla\Psi(X_1(t_1^-)) \quad \text{and} \quad P_2(t_2^+) = P_\mathcal{H}(t_2^-) + \nu_2\, \nabla\Psi(X_2(t_2^+))\,. \tag{4.40}$$

Note now that by definition of $\widetilde{H_1}$, $\widetilde{H_2}$, $\widetilde{H_\mathcal{H}}$ the continuity conditions (4.38)-(4.39) read

$$\left\langle P_1(t_1^-), f_1(t_1^-)\right\rangle + p_0\, l_1(t_1^-) = \left\langle P_\mathcal{H}(t_1^+), f_\mathcal{H}(t_1^+)\right\rangle_\mathcal{H} + p_0\, l_\mathcal{H}(t_1^+) \tag{4.41}$$

$$\left\langle P_\mathcal{H}(t_2^-), f_\mathcal{H}(t_2^-)\right\rangle_\mathcal{H} + p_0\, l_\mathcal{H}(t_2^-) = \left\langle P_2(t_2^+), f_2(t_2^+)\right\rangle + p_0\, l_2(t_2^+) \tag{4.42}$$

where we used the short notation $f_i(t_i^\pm)$ for $f_i(X_i(t_i^\pm), \alpha_i(t_i^\pm))$ and $l_i(t_i^\pm)$ for $l_i(X_i(t_i^\pm), \alpha_i(t_i^\pm))$, $i = 1, 2, \mathcal{H}$. By using twice $P_\mathcal{H}(t_1^+) = P_1(t_1^-) + \nu_1\, \nabla\Psi(X_1(t_1^-))$ and by recalling that by construction $\left\langle \nabla\Psi(X_1(t_1^-)), f_\mathcal{H}(t_1^+)\right\rangle_\mathcal{H} = 0$ equality (4.41) becomes

$$\nu_1 \left\langle \nabla\Psi(X_1(t_1^-)), f_1(t_1^-)\right\rangle = \left\langle P_\mathcal{H}(t_1^+), f_1(t_1^-)\right\rangle - \left\langle P_1(t_1^-), f_\mathcal{H}(t_1^+)\right\rangle_\mathcal{H} + p_0\, (l_1(t_1^-) - l_\mathcal{H}(t_1^+))$$

thus

$$\nu_1 = \frac{\left\langle P_\mathcal{H}(t_1^+), f_1(t_1^-)\right\rangle - \left\langle P_1(t_1^-), f_\mathcal{H}(t_1^+)\right\rangle_\mathcal{H} + p_0\, (l_1(t_1^-) - l_\mathcal{H}(t_1^+))}{\left\langle \nabla\Psi(X_1(t_1^-)), f_1(t_1^-)\right\rangle}\,.$$

by assumption $\langle \nabla\Psi(X_1(t_1^-)), f_1(t_1^-)\rangle \neq 0$. Similarly, if we replace $P_2(t_2^+) = P_{\mathcal{H}}(t_2^-) + \nu_2 \nabla\Psi(X_2(t_2^+))$ in (4.42) we obtain

$$\langle P_{\mathcal{H}}(t_2^-), f_{\mathcal{H}}(t_2^-)\rangle_{\mathcal{H}} + p_0\, l_{\mathcal{H}}(t_2^-) = \langle P_{\mathcal{H}}(t_2^-) + \nu_2\, \nabla\Psi(X_2(t_2^+)), f_2(t_2^+)\rangle + p_0\, l_2(t_2^+)\,.$$

Thus

$$\nu_2 = \frac{\langle P_{\mathcal{H}}(t_2^-), f_{\mathcal{H}}(t_2^-)\rangle_{\mathcal{H}} - \langle P_{\mathcal{H}}(t_2^-), f_2(t_2^+)\rangle + p_0\,(l_{\mathcal{H}}(t_2^-) - l_2(t_2^+))}{\langle \nabla\Psi(X_2(t_2^+)), f_2(t_2^+)\rangle}$$

thanks to the assumption $\langle \nabla\Psi(X_2(t_2^+)), f_2(t_2^+)\rangle \neq 0$.

In order to conclude the proof we need, roughly speaking, to replace P_1, P_2 $P_{\mathcal{H}}$ by Q_1, Q_2 and $Q_{\mathcal{H}}$. To this aim we need to compute the relation between P_1, P_2 $P_{\mathcal{H}}$ and the derivatives of $S_{1,\mathcal{H},2}$, for simplicity, this is done separately in Lemma 4.8 below.

The conclusion goes as follows. By (4.44) in Lemma 4.8 we have

$$\nabla_{x^0}\, S_{1,\mathcal{H},2}\big(X_1(t), t; x^f, t^f\big) = \nabla_1\Sigma\big(\mathbb{Z}(\tau), \tau; Z_1, T_1\big) \quad \forall\tau\in(T_0,T_1) \quad \forall t\in(t^0,t_1),$$

therefore, by the continuity of the adjoint vector and (4.19), we have

$$P_1(t_1^-) = \lim_{t\to t_1^-} P_1(t) = \lim_{\tau\to T_1} P_{Y_1}(\tau) = \lim_{\tau\to T_1} -\nabla_1\Sigma\big(\mathbb{Z}(\tau), \tau; Z_1, T_1\big) = \lim_{t\to t_1^-} -\nabla_{x^0} S_{1,\mathcal{H},2}\big(X_1(t), t; x^f, t^f\big).$$

That is $P_1(t_1^-) = Q_1(t_1^-)$.
In a completely similar way, by Lemma 4.8 below, equalities (4.21)-(4.23) and the continuity of the adjoint vector, one can prove that $P_2(t_2^+) = Q_2(t_2^+)$, $P_{\mathcal{H}}(t_2^-) = Q_{\mathcal{H}}(t_2^-)$ and conclude the proof.

$\square$

Lemma 4.8. *Assume (H$\mathcal{H}$), (HA), (Hfl$_i$), (Hfl$_{\mathcal{H}}$) and (Hu).*
Given $(x^0, t^0; x^f, t^f) \in \Omega_1 \times \mathbb{R}^+_* \times \Omega_2 \times \mathbb{R}^+_*$. If $\chi = (x_1, t_1, x_2, t_2) = \chi((x^0, t^0; x^f, t^f))$ is a minimum point in (4.30), we have :

$$\frac{\partial}{\partial t^0}\, S_{1,\mathcal{H},2}\big(x^0, t^0; x^f, t^f\big) = \partial_2\Sigma\big((x^0, t^0, \chi), (\chi, x^f, t^f)\big) \tag{4.43}$$

$$\begin{aligned}
\frac{\partial}{\partial t^f}\, S_{1,\mathcal{H},2}\big(x^0, t^0; x^f, t^f\big) &= \partial_{12}\Sigma\big((x^0, t^0, \chi), (\chi, x^f, t^f)\big)\\
\nabla_{x^0}\, S_{1,\mathcal{H},2}\big(x^0, t^0; x^f, t^f\big) &= \nabla_1\Sigma\big((x^0, t^0, \chi), (\chi, x^f, t^f)\big)\\
\nabla_{x^f}\, S_{1,\mathcal{H},2}\big(x^0, t^0; x^f, t^f\big) &= \nabla_{11}\Sigma\big((x^0, t^0, \chi), (\chi, x^f, t^f)\big)\,.
\end{aligned} \tag{4.44}$$

Moreover, if $(x^0, t^0; x^f, t^f) \in \Omega_1 \times \mathbb{R}^+_* \times \mathcal{H} \times \mathbb{R}^+_*$ then

$$\nabla^{\mathcal{H}}_{x^f}\, S_{1,\mathcal{H},2}\big(x^0, t^0; x^f, t^f\big) = \nabla^{\mathcal{H}}_9\Sigma\big((x^0, t^0, \chi), (\chi, x^f, t^f)\big)$$

and if $(x^0, t^0; x^f, t^f) \in \mathcal{H} \times \mathbb{R}^+_* \times \Omega_2 \times \mathbb{R}^+_*$ then

$$\nabla^{\mathcal{H}}_{x^0}\, S_{1,\mathcal{H},2}\big(x^0, t^0; x^f, t^f\big) = \nabla^{\mathcal{H}}_3\Sigma\big((x^0, t^0, \chi), (\chi, x^f, t^f)\big)\,.$$

Proof. Given $\lambda = (x^0, t^0; x^f, t^f)$, let $\chi := (x_1, t_1, x_2, t_2) = (x_1(\lambda), t_1(\lambda), x_2(\lambda), t_2(\lambda))$ be a minimum point in (4.30) we can then write

$$S_{1,\mathcal{H},2}(x^0, t^0; x^f, t^f) = \Sigma\Big(\big(x^0, t^0, x_1(\lambda), t_1(\lambda), x_2(\lambda), t_2(\lambda)\big)\,;\big(x_1(\lambda), t_1(\lambda), x_2(\lambda), t_2(\lambda), x^f, t^f\big)\Big)\,.$$

We first remark that putting together (4.10)-(4.13) in Theorem 4.3 and (4.18) in Remark 4.6 we have

$$\begin{array}{rcl}(\partial_4\Sigma + \partial_8\Sigma)\big((\lambda^0, \chi(\lambda)); (\chi(\lambda), \lambda^f)\big) &=& 0\\ (\partial_6\Sigma + \partial_{10}\Sigma)\big((\lambda^0, \chi(\lambda)); (\chi(\lambda), \lambda^f)\big) &=& 0\\ (\nabla_3\Sigma + \nabla_7\Sigma)\big((\lambda^0, \chi(\lambda)); (\chi(\lambda), \lambda^f)\big) &=& \nu_1\nabla\Psi(x_1)\\ (\nabla_5\Sigma + \nabla_9\Sigma)\big((\lambda^0, \chi(\lambda)); (\chi(\lambda), \lambda^f)\big) &=& \nu_2\nabla\Psi(x_2)\\ \Psi(x_1(\lambda)) &=& 0\\ \Psi(x_2(\lambda)) &=& 0\,.\end{array} \tag{4.45}$$

We will only detail the proof of (4.43) and (4.44), the other proofs being completely similar. If we set $\bar\chi := \big((\lambda^0, \chi(\lambda)), (\chi(\lambda), \lambda^f)\big)$ by simple computations we get :

$$\begin{aligned}\frac{\partial S}{\partial t^0}(\lambda) = \quad & \partial_2\Sigma(\bar\chi) + \quad < \nabla_3\Sigma(\bar\chi), \frac{\partial x_1}{\partial t^0}(\lambda) > + \partial_4\Sigma(\bar\chi)\,\frac{\partial t_1}{\partial t^0}(\lambda) + < \nabla_5\Sigma(\bar\chi), \frac{\partial x_2}{\partial t^0}(\lambda) > + \partial_6\Sigma(\bar\chi)\,\frac{\partial t_2}{\partial t^0}(\lambda)\\ & + \quad < \nabla_7\Sigma(\bar\chi), \frac{\partial x_1}{\partial t^0}(\lambda) > + \partial_8\Sigma(\bar\chi)\,\frac{\partial t_1}{\partial t^0}(\lambda) + < \nabla_9\Sigma(\bar\chi), \frac{\partial x_2}{\partial t^0}(\lambda) > + \partial_{10}\Sigma(\bar\chi)\,\frac{\partial t_2}{\partial t^0}(\lambda).\end{aligned}$$

Therefore, thanks to (4.45),

$$\frac{\partial S}{\partial t^0}(\lambda) = \partial_2\Sigma(\bar\chi) + < \mu_1\nabla\Psi(x_1(\lambda)), \frac{\partial x_1}{\partial t^0}(\lambda) > + < \mu_2\nabla\Psi(x_2(\lambda)), \frac{\partial x_2}{\partial t^0}(\lambda) >$$

moreover, since differentiating conditions $\Psi(x_1(\lambda)) = 0,\ \Psi(x_2(\lambda)) = 0$ in (4.45) we obtain

$$< \nabla\Psi(x_1(\lambda)), \frac{\partial x_1}{\partial t^0}(\lambda) >= 0 \quad \text{ and } \ < \nabla\Psi(x_2(\lambda)), \frac{\partial x_2}{\partial t^0}(\lambda) >= 0$$

we can conclude that $\dfrac{\partial S}{\partial t^0}(\lambda) = \partial_2\Sigma(\bar\chi)$.

Similarly

$$\begin{aligned}\frac{\partial S}{\partial x^0}(\lambda) = \quad & \nabla_1\Sigma(\bar\chi) + \quad < \nabla_3\Sigma(\bar\chi), \frac{\partial x_1}{\partial x^0}(\lambda) > + \partial_4\Sigma(\bar\chi)\,\frac{\partial t_1}{\partial x^0}(\lambda) + < \nabla_5\Sigma(\bar\chi), \frac{\partial x_2}{\partial x^0}(\lambda) > + \partial_6\Sigma(\bar\chi)\,\frac{\partial t_2}{\partial x^0}(\lambda)\\ & + \quad < \nabla_7\Sigma(\bar\chi), \frac{\partial x_1}{\partial x^0}(\lambda) > + \partial_8\Sigma(\bar\chi)\,\frac{\partial t_1}{\partial x^0}(\lambda) + < \nabla_9\Sigma(\bar\chi), \frac{\partial x_2}{\partial x^0}(\lambda) > + \partial_{10}\Sigma(\bar\chi)\,\frac{\partial t_2}{\partial x^0}(\lambda).\end{aligned}$$

thanks to (4.45), this reads

$$\frac{\partial S}{\partial x^0}(\lambda) = \nabla_1\Sigma(\bar\chi) + < \nu_1\nabla\Psi(x_1(\lambda)), \frac{\partial x_1}{\partial x^0}(\lambda) > + < \nu_2\nabla\Psi(x_2(\lambda)), \frac{\partial x_2}{\partial x^0}(\lambda) >$$

by differentiating conditions $\Psi(x_1(\lambda)) = 0,\ \Psi(x_2(\lambda)) = 0$ in (4.45) we have

$$< \nabla\Psi(x_1(\lambda)), \frac{\partial x_1}{\partial x^0}(\lambda) >= 0 \quad \text{ and } \ < \nabla\Psi(x_2(\lambda)), \frac{\partial x_2}{\partial x^0}(\lambda) >= 0$$

therefore we can conclude that $\dfrac{\partial S}{\partial x^0}(\lambda) = \nabla_1\Sigma(\bar\chi)$. $\square$

5 Global result and possible generalizations

5.1 An example

As an example we describe here a very simple regional optimal control problem where it is easy to see that a trajectory 1-$\mathcal{H}$-2 is the best possible choice. The idea is to model situations where it is optimal to move along the interface $\mathcal{H}$ as long as possible. One can think, for example of a pedestrian walking on Ω_1 and Ω_2 but with the possibility of taking a tram at any point of the interface $\mathcal{H}$. Or, more generally this example models any problem where moving along a direction is much fast and/or cheaper than along others.

In $\mathbb{R}^2$ we set $\Omega_1 := \{(x,y) \,:\, y < 0\}$, $\Omega_2 := \{(x,y) \,:\, y > 0\}$ and $\mathcal{H} := \{(x,y) \,:\, y = 0\}$. We choose the dynamics

$$f_1(X_1, \alpha_1) = \begin{pmatrix} \cos(\alpha_1) \\ \sin(\alpha_1) \end{pmatrix}, \quad f_{\mathcal{H}}(X_{\mathcal{H}}, \alpha_{\mathcal{H}}) = 10\,, \quad f_2(X_2, \alpha_2) = \begin{pmatrix} \cos(\alpha_2) \\ \sin(\alpha_2) \end{pmatrix}$$

where the controls α_i take values on $[-\pi, \pi]$. We consider the minimal time problem, therefore our aim is to compute the following value function

$$U(x^0, 0; x^f) = \inf \left\{ t^f \,:\, \dot{\mathbb{X}}(t) = f(\mathbb{X}(t), a(t)) \quad \text{with} \quad \mathbb{X}(t^0) = x^0, \quad \mathbb{X}(t^f) = x^f \right\},$$

where the dynamics f takes values f_1,f_2, $f_{\mathcal{H}}$ in Ω_1, Ω_2, $\mathcal{H}$, respectively.
We will analyze the case when we start from a point (x_0, y_0) in Ω_1 and we aim to reach a point (x_1, y_1) in Ω_2 with $x_1 > x_0$. In Ω_1, the dynamics f_1 allows us to move with constant velocity equal to one in any direction, therefore it is clear that our best choice is to go "towards $\mathcal{H}$ but also in the direction of x_1". Indeed, if we compare in figure 3 below the dotted trajectory and the black one, they spend the same time in Ω_1, but on $\mathcal{H}$ the dotted is not the minimal time. Therefore the black one is a better choice. For this reason, and since the problem is symmetric, it is not restrictive to

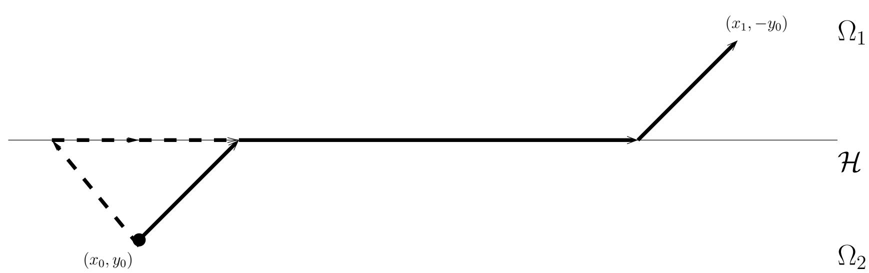

Figure 3: Why we do not go "on the left".

assume that $y_1 = -y_0$ and that trajectories with the structure 1-$\mathcal{H}$-2 are like the ones described in figure 4 with $0 \le a \le \dfrac{x_1 - x_0}{2}$.

For each trajectory steering (x_0, y_0) to $(x_1, -y_0)$ by a simple computation we obtain the following cost (as a function of the parameter a):

$$C(a) = 2\sqrt{y_0^2 + a^2} + \frac{x_1 - x_0}{10} - \frac{a}{5}$$

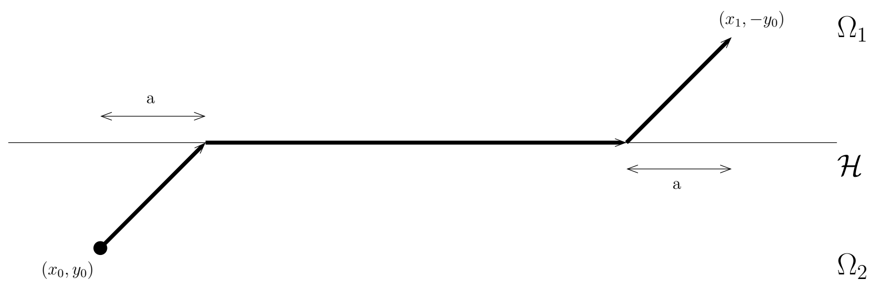

Figure 4: The trajectory 1-$\mathcal{H}$-2 .

therefore, to obtain our value function it suffices to compute

$$U(x^0,0;x^f) = \min_{0\le a\le \frac{x_1-x_0}{2}} \left(2\sqrt{y_0^2+a^2} + \frac{x_1-x_0}{10} - \frac{a}{5}\right) .$$

It is easy to conclude that:

If $\dfrac{x_1-x_0}{2} > \dfrac{|y_0|}{3\sqrt{11}}$ then the best choice is a trajectory with structure 1-$\mathcal{H}$-2 with $a = \frac{|y_0|}{3\sqrt{11}}$ and $t_f = \dfrac{19}{3\sqrt{11}} - \dfrac{x_1-x_0}{10}$.

If $\dfrac{x_1-x_0}{2} \le \dfrac{|y_0|}{3\sqrt{11}}$ then the best possible choice is a trajectory with structure 1-2 that corresponds to $a = \dfrac{x_1-x_0}{2}$ and $t_f = 2\sqrt{y_0^2 + \dfrac{(x_1-x_0)^2}{4}}$ (see figure 5).

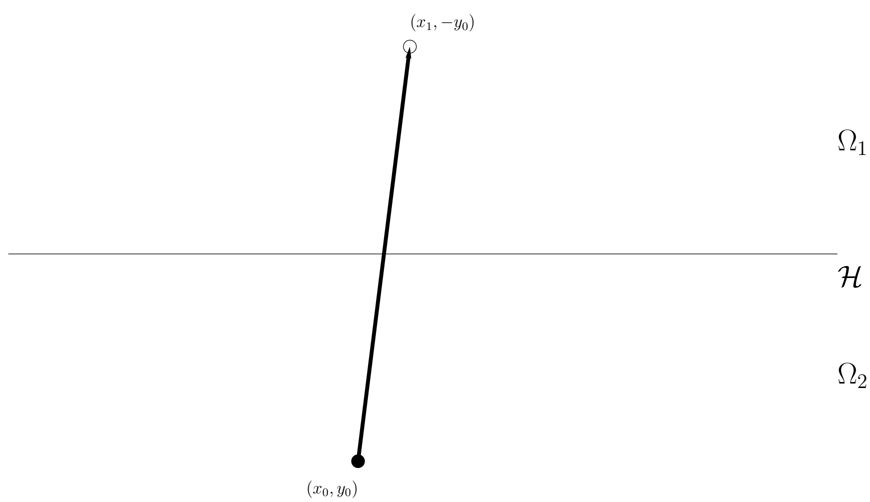

Figure 5: The trajectory 1-2.

To conclude with this example, we note that, although the example is very simple, it is paradigmatic and illustrates many possible situations where one has, in more general, two regions of the space (with specific dynamics), separated by an interface along which the dynamics is much more quicker than in the two regions. In this sense, the above example can be easily adapted and complexified to represent some real-life situations.

5.2 Generalization and further comments

In order to derive a general result we first remark that, for each type of structure with a similar construction of the duplicated problem, one can easily derive the analogue of Theorem 4.2. For example, suppose we consider optimal trajectories with the following structure: start from Ω_2, stay in Ω_2 for a time interval $[t^0, t_1[$, stay on $\mathcal{H}$ for time $[t_1, t_2]$ then go back to Ω_2 for time $]t_2, t_3]$ and finally stay in Ω_1 in the time interval $]t_3, t^f]$. Then, the duplicated problem has four arcs and is settled in dimension 8, but nothing changes in the analysis hence the analogue of Theorem 4.7 (and of Theorem 4.2) can be easily derived. It is then clear that all possible structures that can be described as composed of different arcs, can be analyzed by the duplication technique and similar results can be derived. If the structure has N arcs then the duplicated problem is settled in dimension $2N$.

As we write in the introduction, from an application point of view it is very reasonable to assume that optimal trajectories have only a finite number of switchings, and then to minimize over a finite number of structures. Then, we can write the original value function as a minimum,

$$U(x^0, t^0) = \min \left\{ S_{1,\mathcal{H},2}(x^0, t^0; x^f_{1,\mathcal{H},2}, t^f) \,, S_{1,2,\mathcal{H},2}(x^0, t^0; x^f_{1,2,\mathcal{H},2}, t^f) \,, S_{1,2}(x^0, t^0; x^f_{1,2}, t^f) \,, \ldots \right\} . \tag{5.1}$$

An important consequence is that the regularity of the value function U is the same (i.e., not more degenerate) than the one of the classical optimal control problem that lifts the problem. Indeed, this lifting technique is a kind of desingularization, showing that the value function of a regional optimal control problem is the minimum over all possible structures of value functions associated with optimal control problems over fixed structures, each of them being the restriction to some submanifold of the value function of a classical optimal control problem in higher dimension. Therefore, if for instance all value functions above are Lipschitz then the value function of the regional optimal control problem is Lipschitz as well. This result is known in the classical case (see for instance the survey in [41]) but new in our framework, for regional optimal control problems.

In the general case, when trajectories can oscillate between Ω_1 and Ω_2 an infinite number of times in a finite time interval, the *Zeno problem* manifests itself by an infinite number of possibilities in the infimum making up the value function, resulting in a possible degeneracy of its regularity. It might even happen that the set of switching points has a fractal structure. However, we are not aware of any explicit example and we recall that in the case of affine systems this cannot happen (see [3]).

We describe here two generalizations.

1) For the sake of simplicity we analyzed the simplest possible decomposition of $\mathbb{R}^N$, i.e., under assumption (H$\mathcal{H}$). Since all arguments are local we can apply the same description to a more general stratification. When we can write the whole space as $\mathbb{R}^N = M^0 \cup M^1 \cup \ldots M^N$, where for each $j = 0, \dots N$, M^j is a j-dimensional embedded submanifold of $\mathbb{R}^N$, the M^j are assumed to be disjoint. For a precise description of optimal control problems on stratified domaines see [10, 12].

2) Assuming the same regularity as in assumptions (Hfl_i)-(Hf$l_{\mathcal{H}}$) we extend the result to time dependent dynamics and running costs. Moreover, Ω_i may depend on time, always assuming at least a C^1-dependence.

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