Metric differentiability of Lipschitz maps defined on Wiener spaces

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This note is devoted to the differentiability properties of $\mathcal{H}$-Lipschitz maps defined on abstract Wiener spaces and with values in metric spaces, so we start by recalling some basic definitions related to the Wiener space structure.

Let $(E, \|\cdot\|)$ be a separable Banach space endowed with a Gaussian measure γ. Recall that a *Gaussian measure* γ on E equipped with its Borel $\sigma-$algebra $\mathscr{B}$ is a probability measure on $(E, \mathscr{B})$ such that the law (pushforward measure) of each continuous linear functional on E is Gaussian, that is, $\gamma \circ (e^*)^{-1}$ is a Gaussian measure on $\mathbb{R}$ for each $e^* \in E^* \setminus \{0\}$, possibly a Dirac mass. If we assume, as we shall do, that γ is not supported in a proper subspace of E, then all such measures are Gaussian measures. We shall also assume, for the sake of simplicity, that γ is centered, i.e. $\int_E x d\gamma(x) = 0$.

The Cameron Martin space associated to (E, γ) can be defined, as a vector space, by

$$\mathcal{H} := \Big\{ \int_E x\phi(x) d\gamma(x) : \phi \in L^2(\gamma) \Big\},$$

where the integral above, well defined thanks to Fernique's exponential integrability theorem (see [L, 4.1]), has to be understood as a Bochner integral. By Hölder's inequality we have $\| \int_E x\phi(x) d\gamma\| \leq c\|\phi\|_{L^2(\gamma)}$ with $c = c(\gamma)$.

We denote by $i : L^2(\gamma) \to \mathcal{H} \subset E$ the map $\phi \to \int_E x\phi(x) d\gamma(x)$, and by K the kernel of i. Let us observe that, since i is continuous, K is closed in $L^2(\gamma)$, and so we can define the Cameron-Martin norm

$$\|i(\phi)\|_{\mathcal{H}} = \min_{\psi \in K} \|\phi - \psi\|_{L^2(\gamma)},$$

whose induced scalar product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ satisfies

$$\langle i(\phi), i(\psi) \rangle_{\mathcal{H}} = \int_E \phi\psi d\gamma \quad \forall \phi \in L^2(\gamma), \forall \psi \in K^{\perp} \quad (*).$$

Observe that $\mathcal{H}$ is a Hilbert space which is continuously injected in E, because the continuity of i gives $\|h\| \leq c\|h\|_{\mathcal{H}}$. Since γ is not supported in proper subspaces of E it follows that $\mathcal{H}$ is a dense subset of E. However, since i is not injective in general, it is often more convenient to work with the map $j^* : E^* \to \mathcal{H}$, dual of the inclusion map $j : \mathcal{H} \to E$ (i.e. $j^*(e^*)$ is defined by $\langle j^*(e^*), h\rangle_{\mathcal{H}} = \langle e^*, h\rangle$):

$$E^* \xrightarrow{j^*} \mathcal{H}^* \sim \mathcal{H} \xrightarrow{j} E.$$

It is easy to check that the set $j^*(E^*)$ is dense in $\mathcal{H}$ for the norm $\|\cdot\|_{\mathcal{H}}$ and the fact that j is dense implies that j^* is injective.

The triple $(E, \mathcal{H}, \gamma)$ is called an *abstract Wiener space.*

For the scopes of this paper the following characterization of $\mathcal{H}$ is of fundamental importance:

Theorem 1 (Cameron-Martin theorem) *Let $v \in E$ and let $T_v\gamma(B) = \gamma(B + v)$ be the shifted measure. Then $T_v\gamma \ll \gamma$ if and only if $v \in \mathcal{H}$.*

Definition 2 Let $(E, \mathcal{H}, \gamma)$ be an abstract Wiener space and let (Y, d_Y) be a metric space. A mapping $f : E \to Y$ is said to be *$\mathcal{H}-$Lipschitzian at x* with constant C if

$$d_Y(f(x + h), f(x)) \leq C\|h\|_{\mathcal{H}}, \quad \forall h \in \mathcal{H}.$$

If, for some constant C, f is $\mathcal{H}-$Lipschitzian at x with constant C for $\gamma-$a.e. x, then we say that f is $\mathcal{H}-$Lipschitzian with constant C.

We state in the next theorem two properties of $\mathcal{H}-$Lipschitzian functions; the first one corresponds, in this context, to Rademacher's theorem.

Theorem 3 [ES],[B, 5.11.8] *Let $f : E \to \mathbb{R}$ be $\mathcal{H}$-Lipschitzian. Then*

(i) there exists a Borel $\gamma-$negligible set $N \subset E$ such that, for all $x \in E \setminus N$, the map $h \mapsto f(x + h)$ is Gâteaux differentiable at 0;

(ii) there exists a modification $\tilde{f}$ of f in a $\gamma-$negligible set which is $\mathcal{H}-$Lipschitzian at all $x \in E$.

In addition, we mention the relation between real-valued $\mathcal{H}-$Lipschitzian functions and the Sobolev space $W^{1,\infty}_{\mathcal{H}}(E, \gamma)$ for Gaussian measures (see [B, Section 5.2]). In what follows, the weak $\mathcal{H}-$derivative for the Sobolev spaces will be denoted by $\nabla_{\mathcal{H}}$.

Theorem 4 *If $f \in W^{1,\infty}_{\mathcal{H}}(E, \gamma)$, then there exists a modification $\tilde{f}$ of f in a $\gamma-$negligible set which is $\mathcal{H}-$Lipschitzian at all $x \in E$, with constant $C =$ ess-sup $|\nabla f|_{\mathcal{H}}$. Conversely, all $\mathcal{H}-$Lipschitzian functions $f : E \to \mathbb{R}$ belong to $W^{1,\infty}_{\mathcal{H}}(E, \gamma)$.*

We are going to study the differentiability properties of $\mathcal{H}-$Lipschitz functions $f : E \to Y$, where $(E, \mathcal{H}, \gamma)$ is an abstract Wiener space and Y is a separable metric space or the dual of a separable Banach space. To this aim, following the same approach of [A] and [R], we introduce the Sobolev class $W^{1,\infty}_{\mathcal{H}}(E, \gamma, Y)$, where Y is any metric space, via the connection with the $\mathbb{R}-$valued Sobolev space $W^{1,\infty}_{\mathcal{H}}(E, \gamma)$.

Definition 5 *Let $(E, \mathcal{H}, \gamma)$ be an abstract Wiener space, let (Y, d_Y) be a metric space and let $\mathcal{F}$ be the collection of all $1-$Lipschitz maps between Y and $\mathbb{R}$. Then, a Borel function $f : E \to Y$ belongs to $W^{1,\infty}_{\mathcal{H}}(E, \gamma, Y)$ if the following two condition hold:*

(i) $\phi \circ f \in W^{1,\infty}_{\mathcal{H}}(E, \gamma)$ for each $\phi \in \mathcal{F}$.

(ii) There exists $C \geq 0$ such that $\|\nabla_{\mathcal{H}}(\phi \circ f)\|_\infty \leq C$ for each $\phi \in \mathcal{F}$.

Recall that any separable metric space (Y, d_Y) embeds isometrically in duals of separable Banach spaces, for example in $\ell_\infty(\mathbb{N}) = (\ell_1(\mathbb{N}))^*$. A possible embedding is for instance given by the map

$$x \to \{d_Y(x, x_i) - d_Y(x_0, x_i)\}$$

where $\{x_i\}$ is a dense sequence in Y and $x_0 \in Y$ is a base point. The next result provides an extension of Theorem 4 when the target is the dual of a separable Banach space. By the above-mentioned isometric embedding theorem, the result applies also to maps with values in separable metric spaces.

Proposition 6 *Let $(E, \mathcal{H}, \gamma)$ be an abstract Wiener space and let $Y = G^*$ be a dual Banach space, with G separable. If $f \in W^{1,\infty}_{\mathcal{H}}(E, \gamma, Y)$, then f has a Borel modification $\widetilde{f}$ in a $\gamma-$negligible set with*

$$\|\widetilde{f}(x + h) - \widetilde{f}(x)\|_Y \leq C\|h\|_{\mathcal{H}}, \quad \forall h \in \mathcal{H} \quad \forall x \in E.$$

Proof. Let $D \subset G$ be a dense and countable vector space over $\mathbb{Q}$. First, we define the function

$$\begin{array}{rcccl} \varphi_g & : & Y & \longrightarrow & \mathbb{R} \\ & & x & \longrightarrow & \langle x, g\rangle, \end{array}$$

which is $\|g\|-$Lipschitz for any $g \in D$. Since $f \in W^{1,\infty}_{\mathcal{H}}(E, \gamma, Y)$, the function $f_g = \varphi_g \circ f \in W^{1,\infty}_{\mathcal{H}}(\gamma)$ for each $g \in D$. We know that $W^{1,\infty}_{\mathcal{H}}(E, \gamma)$ can be canonically identified with the space of $\mathcal{H}-$Lipschitzian functions. Moreover, by Theorem 3(ii) we have that there exists a modification $\widetilde{f_g}$ of f_g which is $\mathcal{H}-$Lipschitz at each $x \in E$.

Let us denote

$$N_{g,g'} = \{x \in E : \widetilde{f_{g+g'}}(x) \neq \widetilde{f_g}(x) + \widetilde{f_{g'}}(x)\},$$

which is, thanks to the identity $f_{g+g'} = f_g + f_{g'}$, a $\gamma-$negligible set. Now, we are going to construct a full measure set $F_{g,g'} \subset E \setminus N_{g,g'}$ such that $F_{g,g'}$ is $\mathcal{H}$-invariant, that is, $F_{g,g'} + \mathcal{H} =$

$F_{g,g'}$. Let us take an orthonormal basis $\{e_n\}$ in $\mathcal{H}$. Denote by $\{h_n\}$ the countable set of all finite linear combinations of the vectors e_i with rational coefficients. The set

$$\Omega_n = \{x \in E \setminus N_{g,g'} : x + h_n \in E \setminus N_{g,g'}\},$$

has full measure. If we put $F_{g,g'} = \bigcap_{n\in\mathbb{N}} \Omega_n$, then $F_{g,g'}$ has full measure as well and it is $\mathcal{H}-$invariant. Indeed, let $x \in F_{g,g'}$ and let $h \in \mathcal{H}$. We have to check that $x + h \in F_{g,g'}$. Let us choose a sequence $\{h_n\}$ converging to h in the norm of $\mathcal{H}$ such that

$$\widetilde{f_{g+g'}}(x + h_n) = \widetilde{f_g}(x + h_n) + \widetilde{f_{g'}}(x + h_n).$$

Now, since $\widetilde{f_{g+g'}}, \widetilde{f_g}, \widetilde{f_g'}$ are $\mathcal{H}-$Lipschitz functions we have that

$$\begin{aligned}|\widetilde{f_{g+g'}}(x+h) - \widetilde{f_g}(x+h) - \widetilde{f_{g'}}(x+h)| =& |\widetilde{f_{g+g'}}(x+h) - \widetilde{f_g}(x+h) - \widetilde{f_{g'}}(x+h) \\ &+ \widetilde{f_{g+g'}}(x+h_n) - \widetilde{f_g}(x+h_n) - \widetilde{f_{g'}}(x+h_n)| \\ \leq& 3C\|h_n - h\|_{\mathcal{H}}\end{aligned}$$

If we let n tend to infinity, we get that

$$\widetilde{f_{g+g'}}(x + h) = \widetilde{f_g}(x + h) + \widetilde{f_{g'}}(x + h),$$

as wanted. Observe that it $F_{g,g'}$ is $\mathcal{H}$-invariant, then $E \setminus F_{g,g'}$ is also $\mathcal{H}$-invariant. Since D is countable, the Borel set $N := \bigcup_{g,g'\in D}(E \setminus F_{g,g'})$ is $\gamma-$negligible and $\mathcal{H}$-invariant (since $X \setminus N = \bigcap_{g,g'} E \setminus F_{g,g'}$ is an intersection of $\mathcal{H}$-invariant sets).

Now, consider the functional

$$T : g \in D \longrightarrow \widetilde{f_g}(x),$$

which is $\mathbb{Q}-$linear in D for each $x \notin N$. In addition, we have that T is continuous. Indeed,

$$|\widetilde{f_g}(x)| \leq \sup_{x\in E} |\widetilde{f_g}(x)| = \|\widetilde{f_g}\|_\infty = \|f_g\|_\infty = \sup_{x\in E}\langle f(x), g\rangle = \langle \|f\|_\infty, g\rangle \leq C'\|g\|,$$

for each $g \in D$. Hence, it is the restriction to D of a linear continuous functional on G. Now, define $\widetilde{f}(x)$ as the unique element in G^* such that

$$\langle \widetilde{f}(x), g\rangle = \widetilde{f_g}(x) \quad \text{if } x \notin N$$

and $\widetilde{f}(x) = 0$ if $x \in N$. In order to prove that $\widetilde{f}$ is $\mathcal{H}-$Lipschitzian for each $x \in E$, just observe that

$$|\langle \widetilde{f}(x + h) - \widetilde{f}(x), g\rangle| = |\widetilde{f_g}(x + h) - \widetilde{f_g}(x)| \leq C\|h\|_{\mathcal{H}}\|g\| \quad \text{if } x \notin N.$$

Now, since by hypothesis the $\mathcal{H}-$Lipschitz constant is uniformly bounded for each $g \in G$, we have, upon taking the supremum over G, that $\widetilde{f}$ is an $\mathcal{H}-$Lipschitz function at each $x \in E$. □

Now we discuss the differentiability properties of these maps. First of all notice that, if we consider a mapping taking values in a metric space admitting no linear structure, then the differential properties cannot be interpreted in classical terms. It turns out that, the local behavior of mappings of $\mathbb{R}^n$ into metric spaces can also be read in terms of the so called *metric differential* introduced in [K] (see also [KS]).

Definition 7 Let $f : \mathbb{R}^k \to Y$, where (Y, d_Y) is any metric space. We shall denote

$$mdf_x(u) = \lim_{t \to 0} \frac{d_Y(f(x+tu), f(x))}{|t|} \quad (*),$$

wherever this limit exists. f is *metrically differentiable* at x if $(*)$ exists for all $u \in \mathbb{R}^k$ and $mdf_x(\cdot)$ is a continuous seminorm on $\mathbb{R}^k$.

A Lipschitz function from an interval to a Banach space need not be differentiable somewhere, but the notion of metric differentiability allows to give a generalization of the classical Rademacher's theorem.

Theorem 8 [AK, 3.2],[K],[KS] *Any Lipschitz function $f : \mathbb{R}^k \to Y$ is metrically differentiable at $\mathscr{L}^k$-a.e. $x \in \mathbb{R}^k$.*

In [D] this theorem has been generalized to mappings between Banach spaces, when the domain is separable. Another different generalization has been given in [P], when the domain is a Carnot group. In that work, a metric differentiability theorem is obtained, as it is natural in that context, along the "horizontal" directions.

The property we look for is the natural transposition in our context of the one given in Definition 7:

Definition 9 Let $(E, \mathcal{H}, \gamma)$ be an abstract Wiener space and let (Y, d_Y) be a metric space. We say that $f : E \to Y$ is *metrically differentiable* at x if there exists a continuous seminorm $mdf_x(\cdot)$ in $\mathcal{H}$ such that

$$mdf_x(h) = \lim_{t \to 0} \frac{d_Y(f(x+th), f(x))}{|t|} \qquad \forall h \in \mathcal{H}.$$

As we have mentioned before, using an isometric embedding of Y in a dual space, we reduce ourselves to the case of duals of separable Banach spaces; the linear structure we gain allows us to give a metric differentiability theorem through a weaker version of differentiability for maps with values in dual Banach spaces, namely w^*−differentiability. It seems that this notion goes back to [HM].

Definition 10 Let $(E, \mathcal{H}, \gamma)$ be an abstract Wiener space and let $Y = G^*$ be a dual Banach space. A function $f : E \to Y$ is *w^*−differentiable* at $x \in E$ if there exists a continuous linear map $wdf_x : \mathcal{H} \to Y$ satisfying

$$\frac{f(x+th) - f(x) - t \cdot wdf_x(h)}{t} \xrightarrow{w^*} 0 \ \text{ as } \ t \to 0, \quad \forall h \in \mathcal{H}.$$

The following simple Lemma will be useful in the sequel.

Lemma 11 *Let N be a Borel set in E and let $h \in j^*(E^*)$ be a vector with unit norm. If $\mathscr{L}^1(\{t \in \mathbb{R} : x + th \in N\}) = 0$ for each $x \in E$ then $\gamma(N) = 0$.*

Before proving the Lemma above, we recall an useful tool which allows us to decompose a measure into more elementary components. This process involves the notion of *conditional measure*. Let $h \in j^*(E^*) \subset \mathcal{H}$ be a vector with unit norm. We can define the following linear projection:

$$\begin{array}{rccl} \pi & : & K \oplus \mathbb{R}h & \longrightarrow & \mathbb{R}h \\ & & x & \longrightarrow & \pi(x) = \langle e^*, x\rangle h \end{array}$$

where K is the kernel of e^*, and we can identify E with $K \oplus \mathbb{R}h$ (since $\langle e^*, h\rangle = \langle h, h\rangle_{\mathcal{H}} = 1$ we obtain that $\pi \circ \pi = \pi$). Now, we define the natural projection $\pi_K : E \to K$ by $x \mapsto x - \pi(x)$ and denote by ν the image of the measure γ under the projection π_K. In [B, 3.10.2] it is proved that the *conditional measures* γ^y $(y \in K)$, characterized up to $\nu-$negligible sets by the property of being concentrated on $y + \mathbb{R}h$ and by

$$\gamma(B) = \int_K \gamma^y(B) d\nu(y) \qquad \forall B \in \mathscr{B}(E),$$

can be explicitly represented by

$$\gamma^y(B) = \gamma_1(\{t \in \mathbb{R} : \ y + th \in B\})$$

where γ_1 denotes the standard Gaussian measure on $\mathbb{R}$.

Proof of Lemma 11. Using the disintegration of the measure γ described above we have that

$$\begin{aligned} \gamma(N) &= \int_K \gamma^y(N) d\nu(y) = \int_K \gamma_1(\{t \in \mathbb{R} : \ th + y \in N\}) d\nu(y) \\ &= \int_K 0 d\nu(y) = 0 \qquad (\gamma_1 \ll \mathscr{L}^1). \end{aligned}$$

□

Now, we are in a position to prove a metric differentiability theorem in the context of abstract Wiener spaces.

Theorem 12 *Let (E, H, γ) be an abstract Wiener space and let $Y = G^*$ be a dual Banach space, with G separable. Let $f : E \to Y$ be $\mathcal{H}-$Lipschitz. Then $f : E \to Y$ is w^*-differentiable and metrically differentiable and*

$$mdf_x(h) = \|wdf_x(h)\|_Y \quad \forall h \in \mathcal{H}$$

for $\gamma-$a.e. $x \in E$.

Proof. We denote by $\bar{N}$ a Borel $\gamma-$negligible set such that f is $\mathcal{H}-$Lipschitz, with constant C, at all $x \in E \setminus \bar{N}$.

Let $D \subset G$ be a dense and countable vector space over $\mathbb{Q}$. First, we define the function

$$\begin{array}{rcccl} f_g & : & E & \longrightarrow & \mathbb{R} \\ & & x & \longrightarrow & \langle f(x), g \rangle, \end{array}$$

which is $\mathcal{H}-$Lipschitz for any $g \in D$. Indeed, for $x \in E \setminus \bar{N}$ we have

$$|\langle f(x+h) - f(x), g \rangle| \leq \|f(x+h) - f(x)\|_Y \|g\|_G \leq C\|h\|_{\mathcal{H}} \|g\|_G \qquad \forall h \in \mathcal{H}.$$

By Stroock and Enchev's Rademacher's (Theorem 3(i)), there exists a Borel $\gamma-$negligible set $N_g \supset \bar{N}$ such that f_g is $\mathcal{H}$-differentiable (i.e. Gateaux differentiable, along the directions in $\mathcal{H}$) at all $x \in E \setminus N_g$. Since D is countable, the Borel set $N := \bigcup_{g \in D} N_g$ is $\gamma-$negligible as well and f_g is $\mathcal{H}-$differentiable at any $x \in E \setminus N$ for any $g \in D$.

Now, fix $h \in \mathcal{H}$ and consider the functional

$$L_h : g \longrightarrow L_h(g) = \nabla_h f_g(x),$$

where $\nabla_h f_g(x)$ denotes the directional derivative of f_g in the direction of h at x, that is,

$$\nabla_h f_g(x) := \lim_{t \to 0} \frac{f_g(x+th) - f_g(x)}{t}.$$

The functional L_h is $\mathbb{Q}-$linear in D and since L_h is continuous (because, for each $g \in D$, $\|\nabla_h f_g(x)\|_{\mathcal{H}} \leq C\|h\|_{\mathcal{H}}\|g\|_G$) it is the restriction to D of a linear continuous functional on G, that we represent by a vector $\beta_h \in Y$, with $\|\beta_h\|_Y \leq C\|h\|_{\mathcal{H}}$. Once more, $h \mapsto \beta_h$ is additive and continuous, so it corresponds to a continuous linear functional $\nabla f(x) : \mathcal{H} \to Y$. Summing up, for $x \in E \setminus N$ we have a continuous linear functional $\nabla f(x) : \mathcal{H} \to Y$ satisfying

$$\nabla_h f_g(x) = \langle \nabla f(x)(h), g \rangle \qquad \forall h \in \mathcal{H}, \ g \in D.$$

Using the definition of differentiability, we have that

$$\lim_{t \to 0} \left\langle \frac{f(x+th) - f(x) - t \cdot \nabla f(x)(h)}{t}, g \right\rangle = \lim_{t \to 0} \frac{f_g(x+th) - f_g(x) - t \cdot \nabla_g f_g(x)}{t} = 0,$$

for each $x \in E \setminus N$, $h \in \mathcal{H}$ and $g \in D$. Since D is dense in G we obtain that the w^*-limit of the difference quotients is 0 for each $h \in \mathcal{H}$, and so f is w^*-differentiable at any $x \in E \setminus N$ and $wdf_x = \nabla f(x)$.

As a supremum of w^*-continuous functions, every dual norm is a w^*-lower semicontinuous function. Using this fact, we have that

$$\|wdf_x(h)\|_Y \leq \liminf_{t \to 0^+} \frac{\|f(x+th) - f(x)\|_Y}{t} \quad \forall h \in \mathcal{H}. \tag{1}$$

Now, let D' be a countable dense set in the unit sphere of $\mathcal{H}$. Let us see that, given $h \in \mathcal{H}$, for $\gamma-$a.e.x it holds

$$\mathscr{L}^1(\{\tau \in \mathbb{R} : \ x + \tau h \in N\}) = 0.$$

Indeed,

$$\begin{aligned}\int_E \mathscr{L}^1(\{\tau :\ x+\tau h\in N\})d\gamma(x) &= \int_{\mathbb{R}} \gamma(\{x :\ x+\tau h \in N\})d\tau \qquad (\text{ Fubini's Theorem })\\ &= \int_{\mathbb{R}} T_{\tau h}(\gamma)(\{x :\ x\in N\})d\tau\\ &= \int_{\mathbb{R}} 0 d\tau = 0 \qquad (T_{\tau h}\gamma \ll \gamma).\end{aligned}$$

Hence, if we set $N^h := \{x : \mathscr{L}^1(\{\tau :\ x+\tau h\in N\}) > 0\}$ and $N' := \bigcup_{h\in D'} N^h$, it is obvious that $\gamma(N') = 0$.

By the fundamental Theorem of calculus for Lipschitz functions we obtain that

$$f_g(x+th) - f_g(x) = \int_0^t \frac{d}{d\tau} f_g(x+\tau h)d\tau \overset{(*)}{=} \int_0^t \nabla_h f_g(x+\tau h)d\tau,$$

for any $t > 0$, $h \in D'$, $g \in D$ and $x \in E \setminus (N \cup N')$. Observe that the identity $(*)$ above makes sense because we have chosen x outside the set N', and so the integrands in the two integrals are equal $\mathscr{L}^1-$a.e. in $\mathbb{R}$ for each $h \in D'$ and each $g \in D$. Moreover, we have that

$$\lim_{\varrho\to 0} \frac{1}{\varrho}\int_0^{\varrho} \|\nabla f(x+\tau h)(h)\|_Y d\tau = \|\nabla f(x)(h)\|_Y \quad (**),$$

outside a $\gamma-$negligible set $N'' \subset E$ for every $h \in D'$ and $g \in D$. Indeed, if we denote

$$\mathcal{N}_h := \{x \in E \setminus N :\ (**) \text{ does not hold }\},$$

we know by the Lebesgue differentiation theorem that $\mathscr{L}^1(\{t : x+th \in \mathcal{N}_h\}) = 0$ for each $x \in E \setminus (N \cup N')$. Now, by Lemma 11 we obtain that $\gamma(\mathcal{N}_h) = 0$ and if we set $N'' := \bigcup_{h\in D'} \mathcal{N}_h$ the assertion follows.

We have that

$$\begin{aligned}|\langle f(x+th) - f(x), g\rangle| =& |f_g(x+th) - f_g(x)| = \Big|\int_0^t \nabla_h f_g(x+\tau h)d\tau\Big|\\ &\le \int_0^t |\nabla_h f_g(x+\tau h)|d\tau = \int_0^t |\langle \nabla f(x+\tau h)(h), g\rangle| d\tau\end{aligned}$$

for any $t > 0$, $h \in D'$, $g \in D$ and $x \in E \setminus (N \cup N')$. By density, and taking the supremum over all $g \in G$ in the extreme parts of the previous inequality we obtain that

$$\|f(x+th) - f(x)\|_Y \le \int_0^t \|\nabla f(x+\tau h)(h)\|_Y d\tau.$$

If $x \notin (N \cup N' \cup N'')$ and $h \in D'$ we can divide both sides by t and let t tend to zero to obtain

$$\limsup_{t\to 0^+} \frac{\|f(x+th) - f(x)\|_Y}{t} \le \|\nabla f(x)(h)\|_Y = \|wdf_x(h)\|_Y \quad \forall h \in D'.$$

Again, by density of D' in the unit sphere and $1-$homogeneity of directional derivatives, the inequality above holds for any $h \in \mathcal{H}$. This, combined with (1), proves the metric differentiability of f at $\gamma-$a.e. x. □

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