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THE APPROXIMATION OF HIGHER-ORDER INTEGRALS OF THE CALCULUS OF VARIATIONS AND THE LAVRENTIEV PHENOMENON*

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Abstract. We prove the following approximation theorem: given a function $x : [a, b] \to \mathbb{R}^N$ in the Sobolev space $\mathbf{W}^{\nu+1,1}$, $\nu \geq 1$, and $\epsilon > 0$, there exists a function x_ϵ in $\mathbf{W}^{\nu+1,\infty}$ such that

$$\int_a^b \sum_{i=1}^m L_i(x_\epsilon^{(\nu)}, x_\epsilon^{(\nu+1)})\psi_i(t, x_\epsilon, x'_\epsilon, \ldots, x_\epsilon^{(\nu)}) < \int_a^b \sum_{i=1}^m L_i(x^{(\nu)}, x^{(\nu+1)})\psi_i(t, x, x', \ldots, x^{(\nu)}) + \epsilon,$$

$$\begin{aligned} x_\epsilon(a) &= x(a), & x_\epsilon(b) &= x(b),\\ x'_\epsilon(a) &= x'(a), & x'_\epsilon(b) &= x'(b),\\ &\vdots \\ x_\epsilon^{(\nu)}(a) &= x^{(\nu)}(a), & x_\epsilon^{(\nu)}(b) &= x^{(\nu)}(b), \end{aligned}$$

provided that, for every i in $\{1, \ldots, m\}$, $L_i\psi_i$ is continuous in a neighborhood of x, L_i is convex in its second variable, and ψ_i evaluated along x has positive sign. We discuss the optimality of our assumptions comparing them with an example of Sarychev [*J. Dynam. Control Systems*, 3 (1997), pp. 565–588].

As a consequence, we obtain the nonoccurrence of the Lavrentiev phenomenon. In particular, the integral functional $\int_a^b L(x^{(\nu)}, x^{(\nu+1)})$ does not exhibit the Lavrentiev phenomenon for any given boundary values $x(a) = A$, $x(b) = B$, $x'(a) = A'$, $x'(b) = B'$, ..., $x^{(\nu)}(a) = A^{(\nu)}$, $x^{(\nu)}(b) = B^{(\nu)}$.

Furthermore, we prove the following necessary condition: an action functional with Lagrangian of the form $\sum_{i=1}^m L_i(x^{(\nu)}, x^{(\nu+1)})\psi_i(t, x, x', \ldots, x^{(\nu)})$, with $\nu \geq 0$, exhibiting the Lavrentiev phenomenon takes the value $+\infty$ in any neighborhood of a minimizer.

Key words. calculus of variations, Lavrentiev phenomenon, reparameterization

AMS subject classifications. 49J30, 49N45, 49N60

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1. Introduction.

In 1926, Lavrentiev [11] proposed an example of a first-order integral functional of the calculus of variations, $\mathcal{I}(x) = \int_a^b L(t, x, x')$, whose infimum taken over the space of the absolutely continuous functions $\mathbf{W}^{1,1}(a, b)$ is strictly less than the infimum taken over the space of Lipschitz continuous functions $\mathbf{W}^{1,\infty}(a, b)$, with $x(a) = A$ and $x(b) = B$. Later, Manià [13] published a simpler example of the same phenomenon where the Lagrangian is

$$L_1(x')\psi_1(t, x) = |x'|^6(x^3 - t)^2.$$

Several papers have been devoted to the problem of finding conditions under which the Lavrentiev phenomenon does not occur: Angell [2], Clarke, Vinter [8], Ball, Mizel [3], Lowen [12], Alberti, Serra Cassano [1]. In a recent paper by Cellina, Ferriero, and Marchini [5] a large class of Lagrangians of the form $L_1(x, x')\psi_1(t, x)$ has been treated, including the autonomous and some nonautonomous cases, under no additional conditions besides the convexity of L_1 in x' and the positivity of ψ_1.

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Besides the first-order case, the Lavrentiev phenomenon occurs as well in the case with $(\nu+1)$-order derivatives, $\mathcal{I}(x) = \int_a^b L(t, x, x', \dots, x^{(\nu+1)})$. For $\nu = 1$, in 1994 Cheng and Mizel [7] described a restricted Lavrentiev phenomenon in which the gap occurs for a dense subset of the absolutely continuous nonnegative functions, and they proved that even autonomous Lagrangian $L(x, x', x'')$ can exhibit it. Some years later Sarychev [15] proved that a class of Lagrangians of the form

$$L_1(x'')\psi_1(x, x') + L_2(x'')$$

exhibits the Lavrentiev phenomenon provided that $\psi_1(x, x') = \phi(kx - k|x'-1|^{k-1} - (k-1)|x'-1|^k)$ for appropriate constants k, that L_1, L_2, ϕ satisfy certain growth conditions, and that $\phi(0) = 0$. For example, $L_1(x'') = |x''|^7$, $L_2(x'') = \alpha|x''|^{3/2}$, $\phi_1(\cdot) = (\cdot)^2$, $k = 3$, and $\alpha > 0$ sufficiently small yield a Lagrangian whose integral exhibits the Lavrentiev phenomenon when the boundary values are $x(0) = 0$, $x(1) = 5/3$, $x'(0) = 1$, $x'(1) = 2$.

The Lagrangians proposed by Manià and Sarychev have the property that L_1 evaluated along the minimizer x is not integrable (this is possible because there exists at least one point t in $[a, b]$ such that ψ_1 evaluated along x in t is 0). A condition avoiding the occurrence of this fact will turn out, in this paper, to be essential for the nonoccurrence of the Lavrentiev phenomenon.

We prove the following general approximation theorem: let $x : [a, b] \to \mathbb{R}^N$ be a function in $\mathbf{W}^{\nu+1,1}$ (independently on whether is a minimizer or not), then the integrability of L_i evaluated along x (or the assumption that $\psi_i > 0$), for every i, implies that, given $\epsilon > 0$, there exists a function x_ϵ in $\mathbf{W}^{\nu+1,\infty}$ with the same boundary values of x in a and in b, i.e., $x_\epsilon(a) = x(a)$, $x_\epsilon(b) = x(b)$, $x'_\epsilon(a) = x'(a)$, $x'_\epsilon(b) = x'(b)$, $\dots$, $x_\epsilon^{(\nu)}(a) = x^{(\nu)}(a)$, $x_\epsilon^{(\nu)}(b) = x^{(\nu)}(b)$, such that

$$\int_a^b \sum_{i=1}^m L_i(x_\epsilon^{(\nu)}, x_\epsilon^{(\nu+1)})\psi_i(t, x_\epsilon, x'_\epsilon, \dots, x_\epsilon^{(\nu)})$$
$$< \int_a^b \sum_{i=1}^m L_i(x^{(\nu)}, x^{(\nu+1)})\psi_i(t, x, x', \dots, x^{(\nu)}) + \epsilon.$$

We underline that an application of this result is the nonoccurrence of the Lavrentiev phenomenon for a class of functionals of the calculus of variations with $(\nu+1)$-order derivatives, $\nu \geq 1$. (The case $\nu = 0$, $m = 1$ has already been treated in [5]. The case $\nu = 0$, $m > 1$ can be obtained modifying slightly the proof of the main result of [5]; see [10].) Moreover, we infer a necessary condition for the Lavrentiev phenomenon.

In section 2 we state our results, we discuss the optimality of the assumptions, and we infer the nonoccurrence of the Lavrentiev phenomenon. In section 3 we prove the main result. In section 4 we deal with a necessary condition for the Lavrentiev phenomenon: a functional

$$\mathcal{I}(x) = \int_a^b \sum_{i=1}^m L_i(x^{(\nu)}, x^{(\nu+1)})\psi_i(t, x, x', \dots, x^{(\nu)}),$$

with $\nu \geq 0$, exhibiting the Lavrentiev phenomenon takes the value $+\infty$ in any neighborhood of a minimizer $\bar{x}$.

2. The main result and the Lavrentiev phenomenon. For $\delta > 0$, $B[c, \delta]$ denotes the closed ball in $\mathbb{R}^N$ centered in c with radius δ. For a function x in $\mathbf{C}^\nu[a, b]$, with values in $\mathbb{R}^N$, the closed δ-tube along $(x,\ldots,x^{(\nu)})$

$$\mathsf{T}^{\nu}_{\delta}[x] = \{(t,z_0,\ldots,z_\nu)\in[a,b]\times\mathbb{R}^{(\nu+1)N}:\\ (z_0,\ldots,z_\nu)\in B[x(t),\delta]\times\cdots\times B[x^{(\nu)}(t),\delta],\, t\in[a,b]\}$$

and the closed δ-neighborhood of the image $\mathrm{Im}(x^{(\nu)})$ of $x^{(\nu)}$

$$\mathsf{I}_\delta[x^{(\nu)}] = \{z\in\mathbb{R}^N : \mathrm{dist}(z,\mathrm{Im}(x^{(\nu)}))\le\delta\}$$

are compact sets.

We recall that the space $\mathbf{W}^{\nu+1,p}(a,b)$ can be seen as the space of functions x in $\mathbf{C}^{\nu}[a,b]$ such that $x^{(\nu)}$ is absolutely continuous with derivative in $\mathbf{L}^p(a,b)$, $p\ge 1$.

The following approximation theorem is our main result.

THEOREM 2.1. *Let x be a function in $\mathbf{W}^{\nu+1,1}(a,b)$, $\nu\ge1$, and let the real-valued functions $L_1,\ldots,L_m$ and $\psi_1,\ldots,\psi_m$ be continuous on $\mathsf{I}_\delta[x^{(\nu)}]\times\mathbb{R}^N$ and on $\mathsf{T}^{\nu}_{\delta}[x]$, respectively, for some $\delta>0$.*

Assume that, for every i in $\{1,\ldots,m\}$,

- *$L_i(\xi,\cdot)$ is convex, for every ξ in $\mathsf{I}_\delta[x^{(\nu)}]$,*
- *ψ_i is nonnegative, and $\psi_i(t,x(t),x'(t),\ldots,x^{(\nu)}(t))>0$, for every t in $[a,b]$.*

Then

(i) $\mathcal{I}(x) = \int_a^b \sum_{i=1}^m L_i(x^{(\nu)}(t),x^{(\nu+1)}(t))\psi_i(t,x(t),x'(t),\ldots,x^{(\nu)}(t))dt > -\infty;$

(ii) *given any $\epsilon>0$, there exists a function x_ϵ in $\mathbf{W}^{\nu+1,\infty}(a,b)$ such that*

$$\mathcal{I}(x_\epsilon) < \mathcal{I}(x)+\epsilon,$$

and

$$\begin{aligned} x_\epsilon(a) &= x(a), & x_\epsilon(b) &= x(b),\\ x'_\epsilon(a) &= x'(a), & x'_\epsilon(b) &= x'(b),\\ &\vdots \\ x^{(\nu)}_\epsilon(a) &= x^{(\nu)}(a), & x^{(\nu)}_\epsilon(b) &= x^{(\nu)}(b). \end{aligned}$$

As a corollary we obtain the nonoccurrence of the Lavrentiev phenomenon.

THEOREM 2.2. *Let $\Omega_0,\ldots,\Omega_\nu$ be open sets in $\mathbb{R}^N$, $\nu\ge1$, such that the set $E=\{x\in\mathbf{W}^{\nu+1,1}(a,b) : x(t)\in\Omega_0,\,\ldots,\,x^{(\nu)}(t)\in\Omega_\nu\ \forall t\in[a,b]\}$ is nonempty.*

Let $L_1,\ldots,L_m:\Omega_\nu\times\mathbb{R}^N\to\mathbb{R}$ and $\psi_1,\ldots,\psi_m:[a,b]\times\Omega_0\times\cdots\times\Omega_\nu\to(0,+\infty)$ be continuous and such that $L_i(\xi,\cdot)$ is convex, for any ξ in Ω_ν, and any i in $\{1,\ldots,m\}$.

Then, for all boundary values $A,B\in\Omega_0$, $A^{(1)},B^{(1)}\in\Omega_1$, $\ldots$, $A^{(\nu)},B^{(\nu)}\in\Omega_\nu$, the infimum of

$$\mathcal{I}(x) = \int_a^b \sum_{i=1}^m L_i(x^{(\nu)}(t),x^{(\nu+1)}(t))\psi_i(t,x(t),x'(t),\ldots,x^{(\nu)}(t))dt$$

over the space $E_{a,b}=\{x\in E : x(a)=A,\ x(b)=B,\ x'(a)=A^{(1)},\ x'(b)=B^{(1)},\ \ldots,\ x^{(\nu)}(a)=A^{(\nu)},\ x^{(\nu)}(b)=B^{(\nu)}\}$ is equal to the infimum of the same functional $\mathcal{I}$ over the space $E_{a,b}\cap\mathbf{W}^{\nu+1,\infty}(a,b)$.

Proof. Let $\{x_n\}_n\subset E_{a,b}$ be a minimizing sequence for $\mathcal{I}$: by the fact that $\psi_i>0$, for every i, the theorem follows from Theorem 2.1 applied to any x_n, with $\epsilon=1/n$. ∎

Setting $m = 1$, $\psi_1 = 1$, and $L_1 = L$, we obtain that a Lagrangian depending only on $x^{(\nu)}$ and $x^{(\nu+1)}$ satisfies the assumptions of Theorem 2.2. Hence, the integral functional

$$\int_a^b L(x^{(\nu)}(t), x^{(\nu+1)}(t))dt$$

does not exhibit the Lavrentiev phenomenon, for any boundary values

$$\begin{aligned} x(a) &= A, & x(b) &= B, \\ x'(a) &= A^{(1)}, & x'(b) &= B^{(1)}, \\ &\vdots \\ x^{(\nu)}(a) &= A^{(\nu)}, & x^{(\nu)}(b) &= B^{(\nu)}. \end{aligned}$$

This extends some previous results ([1], [4]), where functionals without boundary conditions, or with boundary conditions only in a, have been considered.

We point out that the assumption $\psi_i(t, x(t), x'(t), \ldots, x^{(\nu)}(t)) \neq 0$ $\forall t \in [a, b]$ in Theorem 2.1 will be used only to infer that $\int_a^b L_i(x^{(\nu)}, x^{(\nu+1)})$ is finite, provided that $\mathcal{I}(x)$ is finite (point (a) in the proof). The theorem holds under the weaker assumption $\int_a^b |L_i(x^{(\nu)}, x^{(\nu+1)})| < +\infty$, for every i.

To verify how sharp our assumptions are, consider the following example of A. V. Sarychev [15]: for $\nu = 1$, $m = 1$, minimize the functional

$$\int_0^1 |x''(t)|^7 [3x(t) - 3|x'(t) - 1|^2 - 2|x'(t) - 1|^3]^2 dt,$$

with boundary conditions $x(0) = 0$, $x(1) = 5/3$, $x'(0) = 1$, $x'(1) = 2$. He proved that the infimum taken over the space $\mathbf{W}^{2,1}(0, 1)$, assumed in $\bar{x}(t) = (2/3)\sqrt[2]{t^3} + t$, is strictly lower than the infimum taken over the space $\mathbf{W}^{2,\infty}(0, 1)$.

The assumption $\int_a^b |L_1(x', x'')| < +\infty$ along $\bar{x}$ is not verified. Indeed, setting $\psi_1(t, x, \xi) = [3x - 3|\xi - 1|^2 - 2|\xi - 1|^3]^2$ and $L_1(\xi, w) = |w|^7$, we see that $\psi_1 \geq 0$ (but, for example, $\psi_1(0, x(0), x'(0)) = 0$) and that

$$\int_0^1 |\bar{x}''(t)|^7 dt = \int_0^1 \frac{1}{(2\sqrt{t})^7} dt = +\infty.$$

3. Proof of the main theorem. In what follows, $\mathbf{x}$ denotes the matrix $(x, \ldots, x^{(\nu-1)})$ and $\mathsf{x} = x^{(\nu-1)}$, so that $\mathsf{x}' = x^{(\nu)}$, $\mathsf{x}'' = x^{(\nu+1)}$ (similarly, $\mathbf{z} = (z, \ldots, z^{(\nu-1)})$, and $\mathsf{z} = z^{(\nu-1)}$). The Lagrangian we consider takes the form

$$\sum_{i=1}^m L_i(\mathsf{x}'(t), \mathsf{x}''(t))\psi_i(t, \mathbf{x}(t), \mathsf{x}'(t)).$$

(In case $\nu = 1$, $\mathbf{x}$, x', x'' coincide with x, x', x'', respectively.)

(i) For every $t \in [a, b]$, $L_i(\mathsf{x}'(t), \mathsf{x}''(t)) \geq L_i(\mathsf{x}'(t), 0) + \langle p_0(t), \mathsf{x}''(t)\rangle$, where $p_0(t)$ is any selection from the subdifferential $\partial_w L_i(\mathsf{x}'(t), 0)$ of L_i with respect to its second variable. Set $E_i = \{t \in [a, b] : [L_i(\mathsf{x}'(t), \mathsf{x}''(t))]^- \neq 0\}$, so that

$$\begin{aligned} &\int_a^b [L_i(\mathsf{x}'(t), \mathsf{x}''(t))\psi_i(t, \mathbf{x}(t), \mathsf{x}'(t))]^- dt \\ &\leq -\int_{E_i} [L_i(\mathsf{x}'(t), 0) + \langle p_0(t), \mathsf{x}''(t)\rangle]\psi_i(t, \mathbf{x}(t), \mathsf{x}'(t))dt, \end{aligned}$$

for any i. Since ψ_i is bounded and, by Proposition 2 in [5], $p_0(t)$ is bounded, the claim follows by Hölder's inequality.

(ii) Fix $\epsilon > 0$; set $\bar{\epsilon} = \epsilon/m$. Without loss of generality, we shall assume $\epsilon < 1$, and also $\delta < 1$.

In case $\int_a^b L_i(\mathsf{x}'(t), \mathsf{x}''(t))\psi_i(t, \mathbf{x}(t), \mathsf{x}'(t))dt = +\infty$, for some i, any Lipschitz function x_ϵ satisfying the boundary conditions is acceptable. Hence we can assume, for every i,

$$\int_a^b L_i(\mathsf{x}'(t), \mathsf{x}''(t))\psi_i(t, \mathbf{x}(t), \mathsf{x}'(t))dt < +\infty.$$

The proof is in three steps. In Step (1) of the proof we introduce the new functions $\tilde{L}_i$ such that $\tilde{L}_i = L_i$+const and such that their polar functions $\tilde{L}_i^*$ (with respect to the second variable) are nonnegative. In Step (3) we define a variation z_n in $\mathbf{W}^{\infty,1}(a, b)$, with the same boundary values of x in a and in b, such that $\mathcal{I}(z_n) < \mathcal{I}(x) + \epsilon$. In order to define z_n, in Step (2) we define a sequence of reparameterizations s_n of $[a, b]$.

Step (1). We claim that there exists functions $\tilde{L}_i$ and a constant η such that $\tilde{L}_i = L_i + \eta$ and $\tilde{L}_i^* \geq 0$, for any i.

In fact, consider the set

$$V_i = \{(\xi, p) : \xi \in \mathsf{I}_\delta[x^{(\nu)}],\, p \in \partial_w L_i(\xi, w),\, |w| \leq 1\}.$$

By Proposition 2 in [5], arguing by contradiction, we obtain that V_i is compact. Let $L_i^*(\xi, p) = \sup_{w\in\mathbb{R}^N} \langle p, w\rangle - L_i(\xi, w)$ be the polar function of L_i with respect to its second variable. Then, $\min_{V_i} L_i^*$ is attained and is finite. Applying Proposition 3 in [5], we obtain that $L_i^*(\xi, p) \geq \min_{V_i} L_i^*$, for every $\xi \in \mathsf{I}_\delta[x^{(\nu)}]$, for every $p \in \partial_w L_i(\xi, w)$ and for every $w \in \mathbb{R}^N$. Set $\eta = \min\{\min_{V_1} L_1^*, \ldots, \min_{V_m} L_m^*\}$.

Consider $\tilde{L}_i(\xi, w) = L_i(\xi, w) + \eta$. Since $\partial_w L_i(\xi, w) = \partial_w \tilde{L}_i(\xi, w)$, we have that $\tilde{L}_i^*(\xi, p) \geq 0$, for any i. (We denote $\tilde{\mathcal{I}}_i$ the functional $\int_a^b \tilde{L}_i\psi_i$.)

(a) We set some preliminary constants, depending on $\bar{\epsilon}$ fixed, that we shall use in the following steps.

By the condition on ψ_i, there exists $c > 0$ such that $\psi_i(t, \mathbf{x}(t), \mathsf{x}'(t)) \geq c$, for every t in $[a, b]$, and we obtain

$$\begin{aligned} +\infty &> \int_a^b |L_i(\mathsf{x}'(t), \mathsf{x}''(t))\psi_i(t, \mathbf{x}(t), \mathsf{x}'(t))|dt + \eta \int_a^b \psi_i(t, \mathbf{x}(t), \mathsf{x}'(t))dt \\ &\geq \int_a^b |\tilde{L}_i(\mathsf{x}'(t), \mathsf{x}''(t))\psi_i(t, \mathbf{x}(t), \mathsf{x}'(t))|dt \geq c \int_a^b |\tilde{L}_i(\mathsf{x}'(t), \mathsf{x}''(t))|dt. \end{aligned}$$

Set $\ell_i = \int_a^b |\tilde{L}_i(\mathsf{x}'(s), \mathsf{x}''(s))|ds$, $\ell = \max\{\ell_1, \ldots, \ell_m\}$, and Ψ and $\tilde{\mathbf{L}}$ the maximum value of $|\psi_1|, \ldots, |\psi_m|$ over $\mathsf{T}_\delta^\nu[x]$ and of $|\tilde{L}_1|, \ldots, |\tilde{L}_m|$ over $\mathsf{I}_\delta[x^{(\nu)}] \times B[0, |\mathsf{x}''(\tau)| + \delta]$, respectively. Denote $\alpha = \max\{1, (b - a)^\nu\}$.

From the uniform continuity of $\psi_1, \ldots, \psi_m$ on $\mathsf{T}_\delta^\nu[x]$, we infer that we can fix $h \in \mathbb{N}$, $1/2^h < \delta$, such that whenever $(t_1, \mathbf{x}_1, \xi_1)$, $(t_2, \mathbf{x}_2, \xi_2) \in \mathsf{T}_\delta^\nu[x]$ and

$$|t_1 - t_2| \leq \frac{b - a}{2^h}, \quad |\mathbf{x}_{1,j} - \mathbf{x}_{2,j}| \leq \frac{1}{2^h} \;\; \forall j \in \{0, \ldots, \nu - 1\}, \quad |\xi_1 - \xi_2| \leq \frac{1}{2^h},$$

we have

$$|\psi_i(t_1, \mathbf{x}_1, \xi_1) - \psi_i(t_2, \mathbf{x}_2, \xi_2)| < \min\left\{\frac{\bar{\epsilon}}{8(\ell + \tilde{\mathbf{L}} + 1)}, \frac{\bar{\epsilon}}{2(|\eta| + 1)(b - a)}\right\},$$

for any i.

Let $\theta : \mathbb{R} \to [0,1]$ be a $\mathbf{C}^\infty$ increasing function with value 0 on $(-\infty, 0]$ and 1 on $[1, +\infty)$. Observe that $1 \leq ||\theta^{(j)}||_\infty \leq ||\theta^{(j+1)}||_\infty$, for any $j \geq 0$. Set $\Theta = ||\theta^{(\nu+1)}||_\infty$.

There exists a point τ in (a,b) that is a Lebesgue point for the functions $\tilde{L}_1(\mathsf{x}'(\cdot), \mathsf{x}''(\cdot))\psi_1(\cdot, \mathbf{x}(\cdot), \mathsf{x}'(\cdot))$, $\ldots$, $\tilde{L}_m(\mathsf{x}'(\cdot), \mathsf{x}''(\cdot))\psi_m(\cdot, \mathbf{x}(\cdot), \mathsf{x}'(\cdot))$ and x'', $\mathsf{x}''(\tau)$ in $\mathbb{R}^N$. By definition of Lebesgue point, there exists a positive number ρ less than

$$\min\left\{\frac{1}{2^{h+4}(\nu+2)(\nu+1)\nu\Theta\alpha^2}, \frac{\bar{\epsilon}}{32\tilde{\mathbf{L}}\Psi}\right\}$$

such that, for any λ^-, λ^+ in $(0, \rho)$,

$$\int_{\tau-\lambda^-}^{\tau+\lambda^+} |\tilde{L}_i(\mathsf{x}'(t), \mathsf{x}''(t))\psi_i(t, \mathbf{x}(t), \mathsf{x}'(t)) - \tilde{L}_i(\mathsf{x}'(\tau), \mathsf{x}''(\tau))\psi_i(\tau, \mathbf{x}(\tau), \mathsf{x}'(\tau))|dt \\ \leq (\lambda^+ + \lambda^-)\bar{\epsilon}$$

for any i, and

$$\int_{\tau-\lambda^-}^{\tau+\lambda^+} |\mathsf{x}''(t) - \mathsf{x}''(\tau)|dt \leq (\lambda^+ + \lambda^-)\frac{1}{2^{h+4}(\nu+1)\nu\Theta\alpha}.$$

Fix $t_0^- = (b-a)v^-/2^\gamma$, $t_0^+ = (b-a)v^+/2^\gamma$, where $\gamma \in \mathbb{N}$, $v^-, v^+ \in \{0, 1, \ldots, 2^\gamma\}$, $v^- < v^+$, are such that $\tau \in (\tau^-, \tau^+) \subset (\tau - \rho, \tau + \rho)$.

We define the absolutely continuous function $\mathsf{z}' : [a,b] \to \mathbb{R}^N$ by $\mathsf{z}'(t) = x^{(\nu)}(a) + \int_a^t \mathsf{z}''$, where

$$\mathsf{z}''(t) = \begin{cases} x^{(\nu+1)}(\tau) + \dfrac{1}{\tau^+ - \tau^-}\displaystyle\int_{\tau^-}^{\tau^+} [\mathsf{x}'' - \mathsf{x}''(\tau)], & t \in [\tau^-, \tau^+], \\ x^{(\nu+1)}(t), & \text{otherwise.} \end{cases}$$

By definition, $\mathsf{z}''(t) = \mathsf{x}''(t)$, $\mathsf{z}'(t) = \mathsf{x}'(t)$, for any t in $[a, \tau^-] \cup [\tau^+, b]$. For any t in $[\tau^-, \tau^+]$, we have that $\mathsf{z}''(t) \in B[0, |\mathsf{x}''(\tau)| + \delta/2]$ and

$$|\mathsf{z}'(t) - \mathsf{x}'(t)| \leq 2\int_{\tau^-}^{\tau^+} |\mathsf{x}''(\tau) - \mathsf{x}''| < (\tau^+ - \tau^-)\frac{1}{2^{h+3}(\nu+1)\nu\Theta\alpha}.$$

Step (2). Our purpose is to show that there exists a sequence of reparameterizations s_n of $[a,b]$ into itself such that $\mathsf{z}' \circ s_n$ is Lipschitz continuous on $[a,b]$.

From the uniform continuity of $x, \ldots, x^{(\nu)}$ on $[a, \tau^-] \cup [\tau^+, b]$, we infer that we can fix $k \in \mathbb{N}$, such that whenever $|s_1 - s_2| \leq (b-a)/2^k$, we have $|x^{(j)}(s_1) - x^{(j)}(s_2)| < (\tau^+ - \tau^-)^{\nu+2}$, for any j in $\{1, \ldots, \nu\}$.

For $v = 0, \ldots, 2^k - 1$, set $I_v = [(b-a)v/2^k, (b-a)(v+1)/2^k]$, $H_v = \int_{I_v} |\mathsf{z}''(s)|ds$, $\mu = \max\{2^{k+1}H_v/(b-a) : v = 0, \ldots, 2^k - 1\}$, and

$$T_{H_v} = \left\{s \in I_v : |\mathsf{z}''(s)| \leq \frac{2^{k+1}H_v}{b-a}\right\};$$

we have that $|T_{H_v}| \geq (b-a)/2^{k+1}$.

Since $\{(\mathsf{z}'(s), \mathsf{z}''(s)) : s \in \bigcup_{v=0}^{2^k-1} T_{H_v}\}$ belongs to a compact set and $L_1, \ldots, L_m$ are continuous, there exists a constant M, such that

$$\left| \tilde{L}_i(\mathsf{z}'(s) + \xi, 2\mathsf{z}''(s) + w)\frac{1}{2} - \tilde{L}_i\Big(\mathsf{z}'(s) + \xi, \mathsf{z}''(s) + \frac{w}{2}\Big) \right| \leq M,$$

for any $s \in \bigcup_{v=0}^{2^k-1} T_{H_v}$, any $|\xi| \leq \delta$, any $|w| \leq \delta$, and any i.

For every $n \in \mathbb{N}$, set $S_n^v = \{s \in I_v : |\mathsf{z}''(s)| > n\}$. From the integrability of z'' it follows that $\int_{S_n^v} (|\mathsf{z}''(s)|/n - 1)ds$ converges to 0, as n goes to ∞. Hence, we can fix a subset Σ_n^v of T_{H_v} such that $|\Sigma_n^v| = 2\int_{S_n^v}(|\mathsf{z}''(s)|/n - 1)ds$.

We define the absolutely continuous functions t_n by $t_n(s) = a + \int_a^s t_n'$, where

$$t_n'(s) = \begin{cases} 1 + (|\mathsf{z}''(s)|/n - 1), & s \in S_n = \bigcup_{v=0}^{2^k-1} S_n^v, \\ 1 - 1/2, & s \in \Sigma_n = \bigcup_{v=0}^{2^k-1} \Sigma_n^v, \\ 1, & \text{otherwise.} \end{cases}$$

One verifies that t_n admits inverse function s_n on the interval $[a, b]$. Furthermore, for any v in $\{0, \ldots, 2^k - 1\}$, the restriction of t_n to I_v maps I_v onto itself. Hence, $|t_n(s) - s| \leq (b - a)/2^k$, for any s in $[a, b]$. If n is greater than $|\mathsf{x}''(\tau)| + \delta/2$, the restriction of t_n to $[\tau^-, \tau^+]$ is the identity.

The function $\mathsf{z}' \circ s_n$ is Lipschitz continuous on $[a, b]$. In fact, fix t where $s_n'(t)$ exists: we obtain

$$\left| \frac{d(\mathsf{z}' \circ s_n)}{dt}(t) \right| = |\mathsf{z}''(s_n(t))s_n'(t)| \begin{cases} = n, & t \in S_n, \\ \leq \mu, & t \in \Sigma_n, \\ \leq n, & \text{otherwise.} \end{cases}$$

Step (3). We construct a function $z_n : [a, b] \to \mathbb{R}^N$, with the same boundary values of x in a and b, such that z_n belongs to $\mathbf{W}^{\nu+1,\infty}(a, b)$ and $\tilde{\mathcal{I}}_i(z_n) < \tilde{\mathcal{I}}_i(x) + \bar{\epsilon}/2$.

Set $f'(t) = \theta((t - \tau^-)/(\tau^+ - \tau^-))$, for any t in $[a, b]$ (the function θ as defined in point (a)): then f' is identically 0 on $[a, \tau^-]$, it is identically 1 on $[\tau^+, b]$, and $||f^{(j+1)}||_\infty = ||\theta^{(j)}||_\infty/(\tau^+ - \tau^-)^j$, for any $j \geq 0$.

We define ν absolutely continuous functions $z_{n,\nu-1}, \ldots, z_{n,0} : [a, b] \to \mathbb{R}^N$ by

$$\begin{aligned} z_{n,\nu-1}(t) &= x^{(\nu-1)}(a) + \int_a^t \mathsf{z}' \circ s_n + f'(t)D_{\nu-1}, \\ z_{n,\nu-2}(t) &= x^{(\nu-2)}(a) + \int_a^t z_{n,\nu-1} + f'(t)D_{\nu-2}, \\ &\vdots \\ z_{n,0}(t) &= x(a) + \int_a^t z_{n,1} + f'(t)D_0, \end{aligned}$$

where, for any j in $\{0, \cdots, \nu - 2\}$,

$$D_j = x^{(j)}(b) - x^{(j)}(a) - \int_a^b z_{n,j+1}, \qquad D_{\nu-1} = x^{(\nu-1)}(b) - x^{(\nu-1)}(a) - \int_a^b \mathsf{z}' \circ s_n.$$

Set $z_n = z_{n,0}$. The derivatives of z_n up to the order $\nu+1$ are

$$\begin{aligned}
&z_n'(t) = z_{n,1}(t) + f''(t)D_0,\\
&z_n''(t) = z_{n,2}(t) + f'''(t)D_0 + f''(t)D_1,\\
&\vdots\\
&z_n^{(\nu-1)}(t) = z_{n,\nu-1}(t) + \textstyle\sum_{j=0}^{\nu-2} f^{(\nu-j)}(t)D_j,\\
&z_n^{(\nu)}(t) = \mathsf{z}'(s_n(t)) + \textstyle\sum_{j=0}^{\nu-1} f^{(\nu-j+1)}(t)D_j,\\
&z_n^{(\nu+1)}(t) = \mathsf{z}''(s_n(t))s_n'(t) + \textstyle\sum_{j=0}^{\nu-1} f^{(\nu-j+2)}(t)D_j.
\end{aligned}$$

We denote by H' the function $\sum_{j=0}^{\nu-1} f^{(\nu-j+1)}D_j$. By the properties of $f^{(j)}$ and s_n, we have that z_n belongs to $\mathbf{W}^{\nu+1,\infty}(a,b)$, with $||z_n^{(\nu+1)}||_\infty \le n + ||\mathsf{H}''||_\infty$ (where $||\cdot||_\infty$ is the essential supremum on (a,b)), and it has the same boundary values of x in a and b.

(b) We claim that $||z_n^{(j)} - x^{(j)}||_\infty \le 1/2^h$ and $||z_n^{(j)} \circ t_n - x^{(j)}||_\infty \le 1/2^h$, for any j in $\{0,\dots,\nu\}$, eventually in n.

In fact, for any n greater than $|\mathsf{x}''(\tau)| + \delta/2$, we have

$$\begin{aligned}
|D_{\nu-1}| &\le \int_a^{\tau^-} |\mathsf{x}' - \mathsf{x}'\circ s_n| + \int_{\tau^-}^{\tau^+} |\mathsf{x}' - \mathsf{z}'| + \int_{\tau^+}^{b} |\mathsf{x}' - \mathsf{x}'\circ s_n|\\
&\le (\tau^+ - \tau^-)^2 \left[3\alpha(\tau^+ - \tau^-)^\nu + \frac{1}{2^{h+3}(\nu+1)\nu\Theta\alpha}\right]\\
&\le (\tau^+ - \tau^-)^2 \frac{1}{2^{h+2}(\nu+1)\nu\Theta\alpha},\\
|D_{\nu-2}| &\le \int_a^{\tau^+} \left|\mathsf{x}'(t) - \mathsf{x}'(a) - \int_a^t \mathsf{z}'\circ s_n - f'(t)D_{n,\nu-1}\right| dt\\
&\quad + \int_{\tau^+}^{b} \left|\mathsf{x}'(t) - \mathsf{x}'(b) + \int_t^b \mathsf{z}'\circ s_n - [1 - f'(t)]D_{n,\nu-1}\right| dt\\
&\le \int_a^{\tau^+}\int_a^t |\mathsf{x}' - \mathsf{z}'\circ s_n| dt + (\tau^+ - \tau^-)|D_{n,\nu-1}| + \int_{\tau^+}^{b}\int_t^b |\mathsf{x}' - \mathsf{z}'\circ s_n| dt\\
&\le (\tau^+ - \tau^-)^3 \left[4\alpha(\tau^+ - \tau^-)^{\nu-1} + \frac{1}{2^{h+3}(\nu+1)\nu\Theta\alpha}\right] + (\tau^+ - \tau^-)|D_{\nu-1}|\\
&\le (\tau^+ - \tau^-)^3 \frac{2}{2^{h+2}(\nu+1)\nu\Theta\alpha},\\
&\vdots\\
|D_j| &\le (\tau^+ - \tau^-)^{\nu-j+1} \frac{\nu - j}{2^{h+2}(\nu+1)\nu\Theta\alpha} \le (\tau^+ - \tau^-)^{\nu-j+1} \frac{1}{2^{h+2}(\nu+1)\Theta}\\
&\qquad\qquad \forall j \in \{0,\dots,\nu-1\},
\end{aligned}$$

so that $||\mathsf{H}'||_\infty \le \sum_{j=0}^{\nu-1} ||f^{(\nu-j+1)}||_\infty (\tau^+ - \tau^-)^{\nu-j+1}/[2^{h+2}(\nu+1)\Theta] \le (\tau^+ - \tau^-)/2^{h+2}$,

$||\mathsf{H}''||_\infty \leq 1/2^{h+2}$, and

$$\begin{aligned}
|\mathsf{z}'(s_n(t)) - \mathsf{x}'(t)| &\leq (\tau^+ - \tau^-)\left[3\alpha(\tau^+ - \tau^-)^{\nu+1} + \frac{1}{2^{h+3}(\nu+1)\nu\Theta\alpha}\right] \\
&\leq \frac{1}{2^{h+2}(\nu+1)\nu\Theta\alpha}, \\
|z_{n,\nu-1}(t) - x^{(\nu-1)}(t)| &\leq \int_a^b |\mathsf{z}'\circ s_n - \mathsf{x}'| + (b-a)|D_{\nu-1}| \leq (1+b-a)|D_{\nu-1}| \\
&\leq \frac{2\alpha}{2^{h+2}(\nu+1)\nu\Theta\alpha}, \\
&\vdots \\
|z_{n,j}(t) - x^{(j)}(t)| &\leq \frac{(\nu-j+1)\alpha}{2^{h+2}(\nu+1)\nu\Theta\alpha} \leq \frac{1}{2^{h+2}} \qquad \forall j \in \{0,\ldots,\nu-1\}.
\end{aligned}$$

Hence, we can fix n such that $M\Psi|\Sigma_n| < \bar{\epsilon}/8$, $||z_n^{(j)} - x^{(j)}||_\infty \leq 1/2^{h+1}$, and

$$||z_n^{(j)}\circ t_n - x^{(j)}||_\infty \leq ||z_n^{(j)}\circ t_n - x^{(j)}\circ t_n||_\infty + ||x^{(j)}\circ t_n - x^{(j)}||_\infty \leq 1/2^h,$$

for any j in $\{0,\ldots,\nu\}$. The graph of the function $(\mathbf{z}_n, \mathsf{z}_n')$ is included in $\mathsf{T}_\delta^\nu[x]$, and $\mathsf{z}_n''(t) \in B[0, |\mathsf{x}''(\tau)| + \delta]$, for any t in $[\tau^-, \tau^+]$. (From what follows, it turns out that z_n is the sought variation x_ϵ.)

(c) We show that $\tilde{\mathcal{I}}_i(z_n) < \tilde{\mathcal{I}}_i(x) + \bar{\epsilon}/2$, for any i.

Using the change of variable formula [16], we compute $\tilde{\mathcal{I}}_i(z_n) - \tilde{\mathcal{I}}_i(x)$ as the sum of the following three appropriate terms:

$$\begin{aligned}
&\int_a^b \tilde{L}_i(\mathsf{z}_n'(t_n(s)), \mathsf{z}_n''(t_n(s)))\psi_i(t_n(s), \mathbf{z}_n(t_n(s)), \mathsf{z}_n'(t_n(s)))t_n'(s)ds \\
&- \int_a^b \tilde{L}_i(\mathsf{x}'(s), \mathsf{x}''(s))\psi_i(s, \mathbf{x}(s), \mathsf{x}'(s))ds \\
&= \int_a^b \left[\tilde{L}_i\left(\mathsf{z}_n'(t_n(s)), \mathsf{z}_n''(t_n(s))\right)t_n'(s) - \tilde{L}_i(\mathsf{z}_n'(t_n(s)), \mathsf{z}''(s) + t_n'(s)\mathsf{H}''(t_n(s)))\right] \\
&\qquad \times \psi_i(t_n(s), \mathbf{z}_n(t_n(s)), \mathsf{z}_n'(t_n(s)))ds \\
&\quad + \int_a^b \tilde{L}_i(\mathsf{z}_n'(t_n(s)), \mathsf{z}''(s) + t_n'(s)\mathsf{H}''(t_n(s))) \\
&\qquad \times [\psi_i(t_n(s), \mathbf{z}_n(t_n(s)), \mathsf{z}_n'(t_n(s))) - \psi_i(s, \mathbf{x}(s), \mathsf{x}'(s))]ds \\
&\quad + \int_a^b \left[\tilde{L}_i\left(\mathsf{z}_n'(t_n(s)), \mathsf{z}''(s) + t_n'(s)\mathsf{H}''(t_n(s))\right) - \tilde{L}_i(\mathsf{x}'(s), \mathsf{x}''(s))\right]\psi_i(s, \mathbf{x}(s), \mathsf{x}'(s))ds \\
&= I_i^1 + I_i^2 + I_i^3.
\end{aligned}$$

To estimate I_i^1, it is enough to estimate its integrand over the sets S_n and Σ_n (because it is identically 0 elsewhere). Since $\Sigma_n \subset T$ and $||\mathsf{H}''||_\infty \leq \delta$, we obtain that

$$\begin{aligned}
&\tilde{L}_i\left(\mathsf{z}'(s) + \mathsf{H}'(t_n(s)), 2\mathsf{z}''(s) + \mathsf{H}''(t_n(s))\right)\frac{1}{2} \\
&- \tilde{L}_i\left(\mathsf{z}'(s) + \mathsf{H}'(t_n(s)), \mathsf{z}''(s) + \frac{\mathsf{H}''(t_n(s))}{2}\right) \leq M,
\end{aligned}$$

for every s in Σ_n. By Propositions 3 and 4 in [5], for every s in S_n,

$$\tilde{L}_i\left(\mathsf{z}'_n(t_n(s)), n\frac{\mathsf{z}''(s)+t'_n(s)\mathsf{H}''(t_n(s))}{|\mathsf{z}''(s)|}\right)\frac{|\mathsf{z}''(s)|}{n}$$
$$-\tilde{L}_i(\mathsf{z}'_n(t_n(s)), \mathsf{z}''(s)+t'_n(s)\mathsf{H}''(t_n(s))) \le -\left(\frac{|\mathsf{z}''(s)|}{n}-1\right)\tilde{L}^*_i(\mathsf{z}'_n(t_n(s)), p) \le 0,$$

where $p \in \partial_w L_i(\mathsf{z}'_n(t_n(s)), n(\mathsf{z}''(s)+t'_n(s)\mathsf{H}''(t_n(s)))/|\mathsf{z}''(s)|)$. Using the fact that ψ_i is positive and bounded by Ψ, we have $I^1_i \le M\Psi|\Sigma_n| < \bar{\epsilon}/8$.

To estimate I^2_i, we observe that

$$\tilde{L}_i(\mathsf{z}'_n(t_n(s)), \mathsf{z}''(s)+t'_n(s)\mathsf{H}''(t_n(s))) = \begin{cases}\tilde{L}_i(\mathsf{z}'_n(s), \mathsf{z}''(s)+\mathsf{H}''(s)), & s\in[\tau^-,\tau^+],\\ \tilde{L}_i(\mathsf{x}'(s), \mathsf{x}''(s)), & \text{otherwise.}\end{cases}$$

By the fact that $|\psi_i(t_n(s), \mathbf{z}_n(t_n(s)), \mathsf{z}'_n(t_n(s))) - \psi_i(s, \mathbf{x}(s), \mathsf{x}'(s))| \le \bar{\epsilon}/[8(\ell+\tilde{\mathbf{L}}+1)]$, for any s in $[a,b]$, and that $\mathsf{z}''+\mathsf{H}'' \in B[0, |\mathsf{x}''(\tau)|+\delta]$ on $[\tau^-,\tau^+]$, we have $I^2_i \le \bar{\epsilon}/8$.

To estimate I^3_i, it is enough to estimate the integrals over $[\tau^-,\tau^+]$ (because it is identically 0 elsewhere). Recalling that τ is a Lebesgue point for $\tilde{L}_i(\mathsf{x}'(\cdot), \mathsf{x}''(\cdot))\psi_i(\cdot, \mathbf{x}(\cdot), \mathsf{x}'(\cdot))$, we have

$$\begin{aligned}I^3_i &\le \int_{\tau^-}^{\tau^+}[\tilde{L}_i(\mathsf{z}'_n(s), \mathsf{z}''(s)+\mathsf{H}''(s))\psi_i(s, \mathbf{x}(s), \mathsf{x}'(s))\\ &\qquad - \tilde{L}_i(\mathsf{x}'(\tau), \mathsf{x}''(\tau))\psi_i(\tau, \mathbf{x}(\tau), \mathsf{x}'(\tau))]ds + \frac{\bar{\epsilon}}{8}\\ &\le 4\rho\tilde{\mathbf{L}}\Psi + \frac{\bar{\epsilon}}{8} < \frac{\bar{\epsilon}}{4}.\end{aligned}$$

Hence, $I^1_i + I^2_i + I^3_i < \bar{\epsilon}/2$, for any i.

Conclusion. We have obtained

$$\begin{aligned}&\int_a^b L_i(\mathsf{z}'_n(t), \mathsf{y}''_n(t))\psi_i(t, \mathbf{z}_n(t), \mathsf{z}'_n(t))dt - \int_a^b L_i(\mathsf{x}'(t), \mathsf{x}''(t))\psi_i(t, \mathbf{x}(t), \mathsf{x}'(t))dt\\ &< \int_a^b [L_i(\mathsf{z}'_n(t), \mathsf{z}''_n(t))+\eta]\psi_i(t, \mathbf{z}_n(t), \mathsf{z}'_n(t))dt\\ &\quad - \int_a^b [L_i(\mathsf{x}'(t), \mathsf{x}''(t))+\eta]\psi_i(t, \mathbf{x}(t), \mathsf{x}'(t))dt + \frac{\bar{\epsilon}}{2}\\ &= \int_a^b \tilde{L}_i(\mathsf{z}'_n(t), \mathsf{z}''_n(t))\psi_i(t, \mathbf{z}_n(t), \mathsf{z}'_n(t))dt - \int_a^b \tilde{L}_i(\mathsf{x}'(s), \mathsf{x}''(s))\psi_i(s, \mathbf{x}(s), \mathsf{x}'(s))ds + \frac{\bar{\epsilon}}{2}\\ &< \bar{\epsilon}.\end{aligned}$$

Hence, $\mathcal{I}(z_n) - \mathcal{I}(x) < \sum_{i=1}^m \bar{\epsilon} = \epsilon$.

So, setting $x_\epsilon = z_n$, we have proved the theorem.

4. A necessary condition for the Lavrentiev phenomenon. The content of this section is provided to show the following necessary condition: a functional

$$\mathcal{I}(x) = \int_a^b \sum_{i=1}^m L_i(x^{(\nu)}, x^{(\nu+1)})\psi_i(t, x, x', \dots, x^{(\nu)}),$$

with $\nu \geq 0$, exhibiting the Lavrentiev phenomenon takes the value $+\infty$ in any neighborhood of a minimizer $\bar{x}$; or equivalently if $\mathcal{I}$ assumes only finite values on a neighborhood of $\bar{x}$, then $\mathcal{I}$ does not exhibit the Lavrentiev phenomenon.

This is proved in the following corollary to Theorem 2.1 and Theorem 1 in [5].

COROLLARY 4.1. *Let $\Omega_0, \ldots, \Omega_\nu$ be open sets in $\mathbb{R}^N$, $\nu \geq 0$, such that the set $E = \{x \in \mathbf{W}^{\nu+1,1}(a,b) : x(t) \in \Omega_0, \ldots, x^{(\nu)}(t) \in \Omega_\nu \ \forall t \in [a,b]\}$ is nonempty. Let $A, B \in \Omega_0$, $A^{(1)}, B^{(1)} \in \Omega_1, \ldots, A^{(\nu)}, B^{(\nu)} \in \Omega_\nu$ be given boundary values.*

Let $L_1, \ldots, L_m : \Omega_\nu \times \mathbb{R}^N \to \mathbb{R}$ and $\psi_1, \ldots, \psi_m : [a,b] \times \Omega_0 \times \cdots \times \Omega_\nu \to [0,+\infty)$ be continuous and such that $L_i(\xi, \cdot)$ is convex, for any ξ in Ω_ν, any i in $\{1, \ldots, m\}$.

Let

$$\mathcal{I}(x) = \int_a^b \sum_{i=1}^m L_i(x^{(\nu)}(t), x^{(\nu+1)}(t))\psi_i(t, x(t), x'(t), \ldots, x^{(\nu)}(t))dt$$

be a functional exhibiting the Lavrentiev phenomenon, and let $\bar{x}$ be a minimum of $\mathcal{I}$ over $E_{a,b} = \{x \in E : x(a) = A,\ x(b) = B,\ x'(a) = A^{(1)},\ x'(b) = B^{(1)}, \ldots, x^{(\nu)}(a) = A^{(\nu)},\ x^{(\nu)}(b) = B^{(\nu)}\}$.

Assume that for any $\delta > 0$ there exists $\sigma_\delta > 0$ such that $\sigma_\delta \to 0$, for $\delta \to 0$, and that ψ_i restricted to $\mathsf{T}^\nu_\delta[\bar{x}]$ may vanish only on the graph of $(\bar{x}, \bar{x}', \ldots, \bar{x}^{(\nu)})$ or on a σ_δ-neighborhood of $(a, A, \ldots, A^{(\nu)})$ or on a σ_δ-neighborhood of $(b, B, \ldots, B^{(\nu)})$, for any i in $\{1, \ldots, m\}$.

Then, for any $\epsilon > 0$, there exists x_ϵ in $E_{a,b}$ such that the graph of $(x_\epsilon, x'_\epsilon, \ldots, x^{(\nu)}_\epsilon)$ is included in $\mathsf{T}^\nu_\epsilon[\bar{x}]$ and $\mathcal{I}(x_\epsilon) = +\infty$.

Proof. Fix $\epsilon > 0$. From Theorem 2.1 and Theorem 1 in [5], it follows that $\int_a^b |L_i(\bar{x}^{(\nu)}, \bar{x}^{(\nu+1)})| = +\infty$, for at least one i in $\{1, \ldots, m\}$.

Without loss of generality, we suppose that $\int_a^{(a+b)/2} |L_i(\bar{x}^{(\nu)}, \bar{x}^{(\nu+1)})| = +\infty$.

Let $g : (-\infty, +\infty) \to [0,1]$ be a $\mathbf{C}^\infty$ increasing function with value 1 on $[b, +\infty)$ and 0 on $(-\infty, (a+b)3/4]$. We define the integrable function $x_{\delta,\nu+1} : [a,b] \to \mathbb{R}^N$ by

$$x_{\delta,\nu+1}(t) = \begin{cases} 0, & t \in [a, a+\sigma_\delta), \\ \bar{x}^{(\nu+1)}(t - \sigma_\delta), & \text{otherwise}, \end{cases}$$

and ν absolutely continuous functions $x_{\delta,j}(t) = A^{(j)} + \int_a^t x_{\delta,j+1} + g(t)D_{\delta,j}$, for any t in $[a,b]$, where $D_{\delta,j} = B^{(j)} - A^{(j)} - \int_a^b x_{\delta,j+1}$, for any j in $\{0, \ldots, \nu\}$.

Set $x_\delta = x_{\delta,0}$. The derivatives of x_δ up to the order $\nu + 1$ are

$$\begin{aligned} x'_\delta(t) &= x_{\delta,1}(t) + g'(t)D_{\delta,0}, \\ x''_\delta(t) &= x_{\delta,2}(t) + g''(t)D_{\delta,0} + g'(t)D_{\delta,1}, \\ &\vdots \\ x^{(\nu+1)}_\delta(t) &= x_{\delta,\nu+1}(t) + \textstyle\sum_{j=0}^\nu g^{(\nu-j+1)}(t)D_{\delta,j}. \end{aligned}$$

By definition, x_δ belongs to $\mathbf{W}^{\nu+1,1}(a,b)$, it has the same boundary values of $\bar{x}$ in a and in b, and, for j in $\{\nu, \nu+1\}$, for any t in $[a+\sigma_\delta, (a+b)3/4]$, we have $x^{(j)}_\delta(t) = \bar{x}^{(j)}(t - \sigma_\delta)$. Furthermore, there exist constants c_j, d_j, independent on δ, such that $|D_{\delta,j}| \leq c_j \int_{b-\sigma_\delta}^b |\bar{x}^{(\nu+1)}|$ and $||x_{\delta,j} - x^{(j)}_\delta||_\infty \leq d_j \int_{b-\sigma_\delta}^b |\bar{x}^{(\nu+1)}|$. Hence, for any j in $\{0, \ldots, \nu\}$,

$$||\bar{x}^{(j)} - x^{(j)}_\delta||_\infty \leq \left(c_j + ||g^{(\nu+1)}||_\infty \sum_{j=0}^\nu d_j \right) \int_{b-\sigma_\delta}^b |\bar{x}^{(\nu+1)}|.$$

By hypothesis, we can choose $\bar\delta > 0$ such that $(c_j + ||g^{(\nu+1)}||_\infty \sum_{j=0}^{\nu} d_j) \int_{b-\sigma_{\bar\delta}}^{b} |\bar x^{(\nu+1)}| < \epsilon$ and $\sigma_{\bar\delta} < (b-a)/4$.

Set $\Psi_i = \min\{\psi_i(t, x_{\bar\delta}(t), \dots, x_{\bar\delta}^{(\nu)}(t)) : t \in [a+\sigma_{\bar\delta}, (a+b)3/4]\}$: by hypothesis, Ψ_i is positive. We have obtained that the graph of $(x_{\bar\delta}, x'_{\bar\delta}, \dots, x_{\bar\delta}^{(\nu)})$ belongs to $\mathsf{T}_\epsilon^\nu[\bar x]$ and

$$\begin{aligned}
&\int_a^b |L_i(x_{\bar\delta}^{(\nu)}(t), x_{\bar\delta}^{(\nu+1)}(t))\psi_i(t, x_{\bar\delta}(t), \dots, x_{\bar\delta}^{(\nu)}(t))|dt \\
&\geq \int_{a+\sigma_{\bar\delta}}^{(a+b)3/4} |L_i(\bar x^{(\nu)}(t-\sigma_{\bar\delta}), \bar x^{(\nu+1)}(t-\sigma_{\bar\delta}))|\psi_i(t, x_{\bar\delta}(t), \dots, x_{\bar\delta}^{(\nu)}(t))dt \\
&\geq \Psi_i \int_a^{(a+b)/2} |L_i(\bar x^{(\nu)}(t), \bar x^{(\nu+1)}(t))|dt = +\infty.
\end{aligned}$$

From (i) in the proof of Theorem 2.1 and Theorem 1 in [5], we infer that $\mathcal{I}(x_{\bar\delta}) = +\infty$.

So, setting $x_\epsilon = x_{\bar\delta}$, we have proved the corollary. ◻

The corollary above applies to the functionals of Manià and Sarychev, for instance, and to the examples of functionals exhibiting the Lavrentiev phenomenon proposed in [3], [4], [11], [12], [13], [14], and [15].

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