Abstract. An analytical validation is obtained for the evolution equation

\[ h_t = \Delta [F^{-1}(-aE_F(h)) - r/h^2 - \Delta h], \]

introduced in [18] by W.T. Tekalign and B.J. Spencer to describe the heteroepitaxial growth of a two-dimensional thin film on an elastic substrate. In the expression above, \( h \) denotes the surface height of the film, \( F \) is the Fourier transform, and \( a, E, r \) are positive material constants. Existence, uniqueness, and Lipschitz regularity in time for weak solutions are proved, under suitable assumptions on the initial datum.

Introduction

The epitaxial deposition of a thin film on a relatively thick substrate has gained much interest in the recent years due to its applications to semiconductor electronics and quantum dots. Roughly speaking, the morphology of the film is known to be the result of a competition between the elastic energy associated to the mismatch between film and substrate, and the surface mass transport due to the film deposition. An extensive mathematical analysis of the mechanism associated to epitaxial film growth has been carried out in [2, 3, 10, 11, 15, 16] in the context of plane linear elasticity, and regularity results have been established for volume-constrained minimizers.

Short time existence for a surface diffusion type geometric evolution equation keeping into account elasticity has first been analyzed in [12] in a two-dimensional setting (see also [17]). The previous result has been recently extended in [13] to the three-dimensional case.

The central aim of this work is to study existence and Lipschitz regularity in time of weak solutions to the 2+1 dimensional evolution equation

\[ h_t = \Delta [F^{-1}(-aE_F(h)) - r/h^2 - \Delta h] \]  

(0.1)
derived by W.T. Tekalign and B.J. Spencer in [18], where \( h \) denotes the surface height of the film and \( F \) is the Fourier transform. The quantities \( a, E, r \) are positive material constants. To be precise, \( a \) (resp. \( r \)) is the wavenumber (resp. wetting coefficient) associated to the equation, and \( E \) is defined as

\[ E := \frac{2\mu^F(1+\nu^F)(1-\nu^S)}{(1-\nu^F)\mu^S}, \]

where \( \mu^F \) and \( \nu^F \) (resp. \( \mu^S \) and \( \nu^S \)) are the elastic shear modulus and the Poisson’s

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ratio of the film (resp. substrate). Equation (0.1) arises in the context of growth of an epitaxially strained, dislocation-free, thin solid film on a deformable substrate in the absence of vapor deposition, and under the assumption of thin film approximation. We focus in particular on the case in which \( h > 0 \), namely the substrate is never exposed.

Denoting by \( T := (0, 1) \times (0, 1) \) the space domain, we assume the film to be \( T \)-periodic and we set our study in the space

\[ V := \{ v \in L^2(T; \mathbb{R}^2) : \nabla \nabla \cdot v \in L^2(T; \mathbb{R}^2), \text{ with } \int_T \nabla \cdot u \, dx = 0 \text{ and } \nabla \cdot v|_{\partial T} = 0 \text{ on } \partial T \}. \]

Our analysis moves from the strategy employed in [8] and [14] to study the evolution equation of vicinal surfaces in heteroepitaxial growth. The key idea is to transform (0.1) into an equivalent parabolic evolution equation, with the observation that, formally, to any function \( u : [0, T] \times T \to \mathbb{R}^2 \) satisfying

\[ u_t = \left[ F^{-1}(-aE \nabla \cdot u - c) - \frac{r}{\sqrt{\nabla \cdot u^0 + c}^2} - \Delta \nabla \cdot u \right], \]

and

\[ \nabla \cdot u + c > 0 \quad \text{in } T \]

for some \( c > 0 \), we can associate a map \( h := \nabla \cdot u + c \) solving (1), and such that \( h > 0 \) in \( T \).

Our main result is the following.

**Theorem 1.** Let \( T, c, r, a, E > 0 \). Let \( u^0 \in V \) be such that

\[ \nabla \left(- aE \nabla \cdot u^0 + \frac{r}{(\nabla \cdot u^0 + c)^2} + \Delta \nabla \cdot u^0 \right) \in L^2_{\text{per}}(T; \mathbb{R}^2), \]

with

\[ \nabla \cdot u^0 + c > 0 \quad \text{in } T. \]

Then there exists a unique map \( u \in W^{1, \infty}([0, T]; L^2_{\text{per}}(T; \mathbb{R}^2)) \cap L^\infty([0, T]; V) \) satisfying \( u(0) = u^0 \), and

\[ \int_0^T \int_T \left( u_t \varphi + aE \nabla \cdot u \nabla \cdot \varphi + \frac{r \nabla \cdot \varphi}{\nabla \cdot u + c} + \nabla \cdot u \nabla \nabla \cdot \varphi \right) \, dx \, dt = 0, \]

for any test function \( \varphi \in C^\infty_c((0, T) \times T; \mathbb{R}^2) \).

With respect to previous contributions to the analysis of the evolution equations associated to thin film deposition, the novelty of our result is twofold.

First, the great generality of the model analyzed in [12, 13] leads the authors to work with a gradient-flow formulation in \( W^{-1, 2} \). The careful analysis required to tackle such problem allows to prove local in time existence of solutions. The specialization of the model discussed here to the case of small thickness of the film, on the contrary, allows us to deduce global in-time existence.

Second, the proof strategy is close to the one in [14] but its implementation presents some peculiarities due to the different model and the 2+1 nature of the problem, which makes the analysis much more delicate.

The paper is organized as follows: in Section 1 we derive equation (0.3) and we collect some preliminary lemmas. Section 2 is devoted to the proof of Theorem 1.
1. Setting of the problem and preliminary results

Let $T > 0$ be a given positive time, and let $[0, T]$ be the time domain. Our first step is to replace the problem of finding a solution to

$$h_t = \Delta [F^{-1}(-aE, F(h)) - r/h^2 - \Delta h]$$

with solving a parabolic evolution equation. In (1.1), the map $F : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ denotes the Fourier transform, defined as

$$F(v)(\xi) := \int_{\mathbb{R}^2} \chi_T(x)v(x)e^{-2\pi i \langle x, \xi \rangle} \, dx,$$

for every $v \in L^2(\mathbb{T})$ and for almost every $\xi \in \mathbb{R}^2$, where $\chi$ is the characteristic function of the subscripted set.

Let $U := L^2_{\text{per}}(\mathbb{T}; \mathbb{R}^2)$ and identify it with its dual $U'$. Let $V$ be the space defined in (0.2), endowed with the scalar product

$$\langle v, u \rangle_V := \langle v, u \rangle_U + \langle \nabla \cdot \nabla v, \nabla \cdot u \rangle_U.$$

Note that the embeddings $V \hookrightarrow U \hookrightarrow V'$ are compact and that $V$ is reflexive. We will say that a sequence $\{u_n\}$ converges to $u$ strongly in $V$ if $u_n \to u$ strongly in $U$ and $\nabla \cdot u_n \to \nabla \cdot u$ strongly in $U$.

Let $u \in V$ be a function satisfying $\nabla \cdot u + c = h$ for some $c > 0$. For $h$ regular enough the existence of such $u$ follows by the standard theory of elliptic PDEs, by solving

$$\left\{ \begin{array}{l}
-\Delta \psi = -h + c \\
\psi \text{ periodic in } \mathbb{T},
\end{array} \right.$$ pointwise in time, and by setting $u = \nabla \psi$. Notice that $c$ and $h$ are related as $\int_{\mathbb{T}} h \, dx = c$ and $h = c$ on $\partial \mathbb{T}$. Then (0.1) becomes

$$(\nabla \cdot u)_t = \Delta \left[ F^{-1}(-aE, F(\nabla \cdot u + c)) - \frac{r}{|\nabla \cdot u + c|^2} - \Delta \nabla \cdot u \right],$$

hence, formally, for any function $u : [0, T] \times \mathbb{T} \to \mathbb{R}^2$ satisfying

$$u_t = \nabla \left[ F^{-1}(-aE, F(\nabla \cdot u + c)) - \frac{r}{|\nabla \cdot u + c|^2} - \Delta \nabla \cdot u \right],$$

(1.2)

the map $h = \nabla \cdot u + c$ is a solution to (1). In the following, we will focus on equation (1.2).

In the remaining part of this section we collect a few lemmas and a proposition which will allow us to reformulate (0.3) as the evolution equation

$$u_t = Au + \partial \Phi_c(u) = 0,$$

where $A$ is a maximal monotone operator and $\partial \Phi_c$ is the subdifferential of a convex $c$-dependent map $\Phi_c$.

We first prove an integration by parts formula.

**Lemma 2.** For any $u, v \in V$, $c > 0$, there holds

$$\begin{align*}
(i) & \quad \langle \nabla \Delta \nabla \cdot u, v \rangle_V = \int_{\mathbb{T}} \langle \nabla \cdot u, \nabla \nabla \cdot v \rangle_V \\
(ii) & \quad \int_{\mathbb{T}} \langle \nabla F^{-1}(-aE, F(\nabla \cdot u + c)), v \rangle \, dx = \int_{\mathbb{T}} F^{-1}(aE, F(\nabla \cdot u)) \nabla \cdot v \, dx.
\end{align*}$$
Proof. Statement (i) follows from the density of $C_0^\infty(T;\mathbb{R}^2)$ in $V$. Indeed, let $w \in C_0^\infty(V;\mathbb{R}^2)$. Then

$$
\langle \nabla \Delta \cdot u, w \rangle_{W^{-2,2}(T;\mathbb{R}^2), W_0^{2,2}(T;\mathbb{R}^2)} = \int_T \langle \nabla \nabla \cdot u \nabla \cdot w \rangle \, dx.
$$

Let $v \in V$, and let $\{v_n\} \subset C_0^\infty(T;\mathbb{R}^2)$ be such that $v_n \to v$ strongly in $V$. We have

$$
\langle \nabla \Delta \cdot u, v \rangle_V = \lim_{n \to +\infty} \langle \nabla \Delta \cdot u, v_n \rangle_{W^{-2,2}(T;\mathbb{R}^2), W_0^{2,2}(T;\mathbb{R}^2)} = \lim_{n \to +\infty} \int_T \langle \nabla \nabla \cdot u, \nabla \cdot v_n \rangle \, dx
$$

and the thesis follows from the inequality

$$
|\langle \nabla \Delta \cdot u, v_n - v \rangle_V| \leq \lim_{n \to +\infty} \|\nabla \nabla \cdot u\|_{L^2(T;\mathbb{R}^2)} \|\nabla \cdot (v_n - v)\|_{L^2(T;\mathbb{R}^2)} \\
\leq \|\nabla \nabla \cdot u\|_{L^2(T;\mathbb{R}^2)} \|v_n - v\|_V.
$$

To prove (ii) note that, by the definition of $V$,

$$
\int_T \langle \nabla F^{-1}(-(aE,F(\nabla \cdot u)), v) \rangle \, dx = \int_T F^{-1}(aE,F(\nabla \cdot u)) \nabla \cdot v \, dx \\
= -\int_{\partial T} F^{-1}(aE,F(\nabla \cdot u)) \langle v, n \rangle \, d\mathcal{H}^1
$$

$$
= \int_T F^{-1}(aE,F(\nabla \cdot u)) \nabla \cdot v \, dx,
$$

where $n$ is the outer unit normal to $\partial T$. Thus, for any $c > 0$,

$$
\int_T \langle \nabla F^{-1}(-(aE,F(\nabla \cdot u + c)), v) \rangle \, dx = \int_T F^{-1}(aE,F(\nabla \cdot u + c)) \nabla \cdot v \, dx
$$

$$
= \int_T F^{-1}(aE,F(\nabla \cdot u)) \nabla \cdot v \, dx + \int_T F^{-1}(aE,F(c)) \nabla \cdot v \, dx
$$

$$
= \int_T F^{-1}(aE,F(\nabla \cdot u)) \nabla \cdot v \, dx,
$$

where in the last equality we have used the fact that $F^{-1}(aE,F(c))$ is a constant, and

$$
\int_T \nabla \cdot v \, dx = 0.
$$

This completes the proof of the lemma. \qed

To exploit the variational structure of (0.3), and to account for the fact that the film height $h$ is positive (and hence $\nabla \cdot u + c > 0$ everywhere in $T$), we introduce the (convex) function

$$
\phi : \mathbb{R} \to (0, +\infty], \quad \phi(t) := \begin{cases} r/t & \text{if } t > 0, \\
+\infty & \text{if } t \leq 0.
\end{cases}
$$

For any $k > 0$, let

$$
\phi_k : \mathbb{R} \to (0, +\infty], \quad \phi_k(r) := \phi(r + k),
$$

and define

$$
\Phi_k : V \to (0, +\infty], \quad \Phi_k(w) := \int_T \phi_k(\nabla \cdot w) \, dx.
$$
By construction, $\Phi_k$ is a convex functional. Denoting by $\partial \Phi_k$ its subdifferential, for every $w \in V$ satisfying $\nabla \cdot w + k > 0$ in $\mathbb{T}$ there holds
\begin{equation*}
\partial \Phi_k(w) = \nabla \left( \frac{r}{\nabla \cdot w + k} \right).
\end{equation*}

We define the operator
\begin{equation*}
A : V \rightarrow V', \quad Av := -aE \nabla \nabla \cdot v + \nabla \Delta \cdot v \quad \text{for every } v \in V.
\end{equation*}

In view of Lemma 2,
\begin{equation*}
\langle Aw, v \rangle_{V', V} := \int_{\mathbb{T}} \left[ F^{-1}(aE F(\nabla \cdot w)) \nabla \cdot v + (\nabla \nabla \cdot w, \nabla \nabla \cdot v) \right] \, dx
\end{equation*}
for every $w$ and $v$ in $V$, and finding a solution to (0.3) is equivalent to solving
\begin{equation*}
u_t + Au + \partial \Phi_c(u) = 0, \quad u(0) = u^0.
\end{equation*}

We now recall a few definitions.

**Definition 3.** An operator $B : V \rightarrow V'$ is:
- **monotone** if for any $w, v \in V$, there holds
  \begin{equation*}
  \langle Bw - Bv, w - v \rangle_{V', V} \geq 0.
  \end{equation*}
  Similarly, a set $G \subseteq V \times V'$ is “monotone” if for any pair $(w, w'), (v, v') \in G$, there holds
  \begin{equation*}
  \langle w' - v', w - v \rangle_{V', V} \geq 0.
  \end{equation*}
- **maximal monotone** if the graph
  \begin{equation*}
  \Gamma_B := \{(w, Bw) : w \in V\} \subseteq V \times V'
  \end{equation*}
  is not a proper subset of any monotone set.
- **hemi-continuous** if for any $u, w, v \in V$ the mapping $t \mapsto \langle B(u + tv), w \rangle_{V', V}$ is continuous.

In the following, given an operator $B : V \rightarrow V'$, we will denote by $\text{dom}_U(B)$ the set
\begin{equation*}
\text{dom}_U(B) := \{u \in V : Bu \in U'\},
\end{equation*}
and we will define the unbounded operator $T_B : U \rightarrow 2^{U'}$ as
\begin{equation*}
T_B(u) := \begin{cases} Bu & \text{if } u \in \text{dom} B, \\ \emptyset & \text{otherwise}. \end{cases}
\end{equation*}

We conclude this section by providing a characterization of the operator
\begin{equation*}
T_{A + \partial \Phi_c} : U \rightarrow 2^{U'}.
\end{equation*}

**Proposition 4.** The operator $T_{A + \partial \Phi_c} : U \rightarrow 2^{U'}$ is maximal monotone.

**Proof.** We divide the proof into two steps.

**Step 1:** We first show that the operator $A : V \rightarrow V'$ is maximal monotone.

Indeed, the hemi-continuity of $A$ follows from its linearity. To prove that $A$ is a monotone operator note that for any $w \in V$ there holds $F^{-1}(aE F(\nabla \cdot w)) = aE \nabla \cdot w$. Thus
\begin{equation}
\langle Aw, w \rangle_{V', V} = aE \| \nabla \cdot w \|^2_{L^2(\mathbb{T})} + \| \nabla \nabla \cdot w \|^2_{L^2(\mathbb{T}; \mathbb{R}^2)}.
\end{equation}
The maximal monotonicity of $A$ is a consequence of [5, Theorem 1.2].
Since $A$ is a bounded linear operator and the sub-differential of convex functionals is maximal monotone, it follows (see for instance Browder [7, 6]) that

$$A + \partial \Phi_c : V \to V'$$

is maximal monotone as well.

**Step 2:** we show now that $T_{A + \partial \Phi_k} : U \to 2^{U'}$ is maximal monotone. Indeed the graph of $T_{A + \partial \Phi_k}$ is just $T_{A + \partial \Phi_k}(\text{dom } U(A + \partial \Phi_k))$, hence it coincides with the graph of $A + \partial \Phi_k$. Since $A + \partial \Phi_k$ is maximal monotone, we infer that $T_{A + \partial \Phi_k}(\text{dom } U(A + \partial \Phi_k))$, and hence $T_{A + \partial \Phi_k}$, are also maximal monotone. □

2. **Existence and uniqueness of solutions**

This section is devoted to the proof of Theorem 1. We first prove a preliminary compactness lemma.

**Lemma 5.** Let $u^0 \in U$ be such that $Au^0 + \partial \Phi_c(u^0) \in U$, and let $(u^\varepsilon)_\varepsilon \subseteq W^{1,\infty}([0,T];U)$ be such that

$$u^\varepsilon(0) := u^0, \quad \|u^\varepsilon\|_{L^\infty((0,T);U)} \leq C < +\infty$$

for some constant $C$. Then there exists a sequence $\varepsilon_n \to 0$ and a limit function $u \in W^{1,\infty}([0,T];U)$ such that $u^\varepsilon \rightharpoonup^* u$ weakly* in $W^{1,\infty}([0,T];U)$, and $u^\varepsilon(t) \to u(t)$ weakly in $U$ for every $t \in [0,T]$.

**Proof.** By (2.1), the sequence $(u^\varepsilon)_\varepsilon$ is uniformly bounded in $W^{1,\infty}([0,T];U)$. Hence, up to the extraction of a subsequence there holds

$$u^\varepsilon_n \rightharpoonup^* u \quad \text{weakly* in } W^{1,\infty}([0,T];U).$$

The thesis follows by applying [1, Proposition 3.3.1], with $d$ being the distance in $V$, $\sigma$ the weak topology of $U$, $\mathcal{S} := U$, and $K := \{g \in U : \|g - u^0\|_U \leq CT\}$. □

In what follows, let $\text{id}_U : U \to U'$ be the duality mapping, and denote by $\tilde{A}$ the operator $\tilde{A} := A + \partial \Phi_c$. To make the notation easier to follow we will assume henceforth that $r = 1$ and we will identify the operator $\tilde{A}$ with its extension $T_{\tilde{A}}$.

The unboundedness of $\partial \Phi_c(u)$ makes it unclear whether the spatial regularity of the initial datum is preserved in time by (0.3). Proposition 6 gives a positive answer in this direction.

**Proposition 6.** Let $u^0 \in \text{dom } U(\partial \Phi_c) \subseteq V$ be such that $Au^0 \in U$. Then there exists a function $u : [0,T] \to V$ such that $u \in W^{1,\infty}([0,T];U)$, $u(0) = u^0$, $\nabla \cdot u \in L^\infty((0,T);U)$, and

$$\langle u_t(t), v - u(t) \rangle_{V',V} + \langle Au(t), v - u(t) \rangle_{V',V} + \Phi_c(u(t)) - \Phi_c(u(t)) \geq 0$$

(2.2)

for a.e. $t \in (0,T)$, and for all $v \in V$. In addition,

$$\nabla \cdot u(t) + c > 0 \quad \text{in } \mathbb{T} \quad \text{for every } t \in [0,T],$$

(2.3)

and

$$\|u_t\|_{L^\infty((0,T);U)} \leq \|\tilde{A}u^0\|_U.$$

**Proof.** The proof is divided into three steps. In Step 1, using the classic method of time discretization, we construct a sequence of piecewise linear approximate solutions $u^\varepsilon : [0,T] \to V$. In Step 2 we prove that the maps $u^\varepsilon$ are uniformly Lipschitz, and by Lemma 5 we obtain a limit function $u \in W^{1,\infty}([0,T];U)$. Step 3 is devoted to proving that $u$ solves (2.2).
Step 1. Fix $\varepsilon > 0$. Consider the time partition
\[
0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = T \leq t_n + \varepsilon,
\]
where $\lfloor \cdot \rfloor$ denotes the least integer part. We preliminarily observe that the maximal monotonicity of the operator $T_{\tilde{A}}$ (see Proposition 4) guarantees (by [4, Theorem 1]) the surjectivity of the map $\text{id} + \varepsilon \tilde{A} : \text{dom} \partial \Phi_c \to U'$. To prove the injectivity of $\text{id} + \varepsilon \tilde{A}$, assume that there exist $u_1, u_2 \in V$ such that $u_1 \neq u_2$ but $u_1 + \varepsilon \tilde{A}u_1 = u_2 + \varepsilon \tilde{A}u_2$. The monotonicity of $\tilde{A}$ yields
\[
\|u_1 - u_2\|_V^2 \leq \langle u_1 - u_2, u_1 - u_2 \rangle_V + \langle \tilde{A}u_1 - \tilde{A}u_2, u_1 - u_2 \rangle_{V',V} = 0
\]
which in turn provides a contradiction.

We construct the recursive sequence $(u_{\varepsilon,i})$ in the following way: we set $u_{\varepsilon,0} := u^0$, and given $u_{\varepsilon,i-1} \in \text{dom} \partial \Phi_c$, we define $u_{\varepsilon,i} \in \text{dom} \partial \Phi_c \subset V$ as the unique solution to
\[
u_{\varepsilon,i-1} = (\text{id} + \varepsilon \tilde{A})u_{\varepsilon,i}.
\]
We consider both the piecewise linear functions $u^\varepsilon : [0,T] \to V$ satisfying
\[
u(t) = u_{\varepsilon,k-1} + (t - \varepsilon \lfloor t/\varepsilon \rfloor) \left( \frac{u_{\varepsilon,k} - u_{\varepsilon,k-1}}{\varepsilon} \right)
\]
for every $t \in [t_{k-1}, t_k)$, and the piecewise constant interpolants
\[
u^\varepsilon(t) := u_{\varepsilon,k-1}
\]
for every $t \in [t_{k-1}, t_k)$, $\nu^\varepsilon(T) = u_{\varepsilon,\lfloor T/\varepsilon \rfloor}$.

Observe that
\[
u^\varepsilon(t) = \nu^\varepsilon(t) - \varepsilon(t/\varepsilon - \lfloor t/\varepsilon \rfloor) \nu^\varepsilon(t)
\]
for every $t \in [0,T]$. (2.4)

In addition,
\[
\nabla \cdot \nu^\varepsilon(t) + c > 0 \quad \text{and} \quad \nabla \cdot \nu^\varepsilon(t) + c > 0
\]
in $T$, for every $t \in [0,T]$.

Step 2. In order to apply Lemma 5 to the sequence $(\nu^\varepsilon)$, it suffices to check that the maps $\nu^\varepsilon$ are uniformly Lipschitz. By construction, $u_{\varepsilon,i} = (\text{id} + \varepsilon \tilde{A})^{-1}u_{\varepsilon,i-1}$, thus
\[
u_{\varepsilon,i} - u_{\varepsilon,i-1} = (\text{id} + \varepsilon \tilde{A})^{-1}u_{\varepsilon,i-1} - (\text{id} + \varepsilon \tilde{A})^{-1}u_{\varepsilon,i-2}.
\]
In view of the maximal monotonicity of $T_{\tilde{A}}$, the operator $(\text{id} + \varepsilon \tilde{A})^{-1}$ is non-expansive. Indeed, for every $u, v \in \text{dom} \partial \Phi_c$ there holds
\[
\|(\text{id} + \varepsilon \tilde{A})u - (\text{id} + \varepsilon \tilde{A})v\|_U^2 = \langle u - v + \varepsilon(\tilde{A}u - \tilde{A}v), u - v + \varepsilon(\tilde{A}u - \tilde{A}v) \rangle_{U',U}
\]
\[
= \|u - v\|_U^2 + \varepsilon\langle \tilde{T}_{\tilde{A}}u - \tilde{T}_{\tilde{A}}v, u - v \rangle_{U',U} + \varepsilon^2\|\tilde{T}_{\tilde{A}}u - \tilde{T}_{\tilde{A}}v\|_U^2
\]
\[
\geq \|u - v\|_U^2.
\]
Therefore
\[
\|u_{\varepsilon,i} - u_{\varepsilon,i-1}\|_U \leq \|u_{\varepsilon,i-1} - u_{\varepsilon,i-2}\|_U.
\]
(2.6)

On the other hand, by construction, $u_{\varepsilon,1} = (\text{id} + \varepsilon \tilde{A})^{-1}u^0$, hence
\[
u_{\varepsilon,1} - u^0 = (\text{id} + \varepsilon \tilde{A})^{-1}u^0 - (\text{id} + \varepsilon \tilde{A})^{-1}(\text{id} + \varepsilon \tilde{A})u^0,
\]
and
\[ \| u_{\varepsilon,1} - u_0 \|_U = \|(id + \varepsilon \tilde{A})^{-1} u_0 - (id + \varepsilon \tilde{A})^{-1}(id + \varepsilon \tilde{A})u_0 \|_U \]
\[ \leq \| u_0 - (id + \varepsilon \tilde{A})u_0 \|_U = \varepsilon \| \tilde{A}u_0 \|_U, \]
which in turn gives
\[ \frac{\| u_{\varepsilon,1} - u_0 \|_U}{\varepsilon} \leq \| \tilde{A}u_0 \|_U < +\infty. \] (2.7)
Combining (2.6) and (2.7) we obtain
\[ \frac{\| u_{\varepsilon,i} - u_{\varepsilon,i-1} \|_U}{\varepsilon} \leq \| \tilde{A}u_0 \|_U < +\infty \] (2.8)
for all \( \varepsilon > 0, i = 0, \ldots, [T/\varepsilon] \). Thus \( u^\varepsilon : [0, T] \rightarrow U \) is Lipschitz continuous for every \( \varepsilon \), with Lipschitz constant not greater than \( \| \tilde{A}u_0 \|_U \). By Lemma 5, up to the extraction of a (non relabeled) subsequence there exists \( u \in W^{1,\infty}([0, T]; U) \) such that
\[ u^\varepsilon(t) \rightharpoonup u(t) \text{ weakly in } U \text{ for every } t \in [0, T], \] (2.9)
\[ u^\varepsilon \rightharpoonup^* u \text{ weakly* in } W^{1,\infty}((0, T); U). \] (2.10)
In particular, by (2.8)
\[ \| u_t \|_{L^\infty([0, T]; U)} \leq \| \tilde{A}u_0 \|_U, \]
and by (2.4),
\[ \bar{u}^\varepsilon \rightharpoonup^* u \text{ weakly* in } L^\infty((0, T), U). \] (2.11)
In view of (1.3), and recalling that \( (id + \varepsilon \tilde{A})u_{\varepsilon,i} = u_{\varepsilon,i-1} \), for every \( t \in [t_{i-1}, t_i) \) there holds
\[ \| \nabla \nabla \cdot u^\varepsilon(t) \|_U^2 \leq \left| \left( \tilde{A}u_{\varepsilon,i}, u_{\varepsilon,i} \right)_{U', U} \right| = \left| \left( \frac{u_{\varepsilon,i} - u_{\varepsilon,i-1}}{\varepsilon}, u_{\varepsilon,i} \right)_{U', U} \right| \leq \frac{\| u_{\varepsilon,i} - u_{\varepsilon,i-1} \|_U}{\varepsilon} \| u_{\varepsilon,i} \|_U < +\infty, \]
hence
\[ \sup_{\varepsilon > 0} \sup_{t \in [0, T]} \| \nabla \nabla \cdot u^\varepsilon(t) \|_U < +\infty. \] (2.12)
This proves that \( \{ \nabla \nabla \cdot u^\varepsilon \} \) is bounded in \( L^\infty((0, T); U) \). By (2.10), we conclude that up to the extraction of a (not relabeled) subsequence \( \nabla \nabla \cdot u^\varepsilon \rightharpoonup^* \nabla \nabla \cdot u \) in \( L^\infty((0, T); U) \), and
\[ \| \nabla \nabla \cdot u \|_{L^\infty([0, T]; L^\infty(U, U))} \leq \liminf_{\varepsilon \to 0} \| \nabla \nabla \cdot u^\varepsilon \|_{L^\infty([0, T]; L^\infty(U, U))}. \]
In view of (2.12) and Poincaré inequality, since \( u^\varepsilon(t) \in V \) for every \( t \in [0, T] \), we have
\[ \sup_{\varepsilon > 0} \sup_{t \in [0, T]} \| \nabla \cdot u^\varepsilon(t) \|_U < +\infty, \] (2.13)
and property (2.11) yields
\[ \nabla \cdot \bar{u}^\varepsilon(t) \rightharpoonup \nabla \cdot u(t) \text{ weakly in } W^{1,2}(T), \] (2.14)
for every \( t \in [0, T] \). Thus, by (2.5) we deduce (2.3).

**Step 3.** By construction, \( u_{\varepsilon,i} \) solves
\[ \langle (id + \varepsilon \tilde{A})u_{\varepsilon,i}, v - u_{\varepsilon,i} \rangle_{U', U} = \langle u_{\varepsilon,i-1}, v - u_{\varepsilon,i} \rangle_{U', U} \text{ for all } v \in U, \]
which in turn is equivalent to
\[
\left\langle \frac{u_{\varepsilon,i} - u_{\varepsilon,i-1}}{\varepsilon} + \tilde{A}u_{\varepsilon,i}, v - u_{\varepsilon,i} \right\rangle_{U', U} = 0 \quad \text{for all } v \in U,
\]
that is,
\[
\left\langle \frac{u^\varepsilon(i\varepsilon) - u^\varepsilon(i\varepsilon - \varepsilon)}{\varepsilon} + Au^\varepsilon(i\varepsilon) + \partial \Phi_c(u^\varepsilon(i\varepsilon)), v - u^\varepsilon(i\varepsilon) \right\rangle_{U', U} = 0 \quad \text{for all } v \in U.
\]  
(2.15)

In view of the convexity of \( \Phi_c \) and by (2.15) we deduce the inequality
\[
\left\langle \frac{u^\varepsilon(i\varepsilon) - u^\varepsilon(i\varepsilon - \varepsilon)}{\varepsilon} + Au^\varepsilon(i\varepsilon), v - u^\varepsilon(i\varepsilon) \right\rangle_{U', U} + \Phi_c(v) - \Phi_c(u^\varepsilon(i\varepsilon)) \geq 0 \quad \text{for all } v \in U,
\]
namely
\[
\langle u^\varepsilon_t(t) + Au^\varepsilon(t), v - \tilde{u}^\varepsilon(t) \rangle_{U', U} + \Phi_c(v) \geq \Phi_c(\tilde{u}^\varepsilon(t)) \quad \text{for all } v \in U,
\]  
(2.16)

and for every \( t \in [0, T] \). In order to prove (2.2) we need to pass into the limit in (2.16). To this purpose, let \( \varphi \in L^1(0, T) \), with \( \varphi(t) \geq 0 \) for every \( t \in (0, T) \). By (2.10) and (2.11) there holds
\[
\lim_{\varepsilon \to 0} \int_0^T \langle u^\varepsilon_t(t) + Au^\varepsilon(t), v \rangle_{U', U} \varphi(t) dt = \int_0^T \langle u_t(t) + Au(t), v \rangle_{U', U} \varphi(t) dt.
\]  
(2.17)

Since \( \Phi_c \) is convex, it is weakly upper semi-continuous. Thus, (2.14) and Fatou lemma yield
\[
\liminf_{\varepsilon \to 0} \int_0^T \Phi_c(\tilde{u}^\varepsilon(t)) \varphi(t) dt \geq \int_0^T \Phi_c(u(t)) \varphi(t) dt.
\]  
(2.18)

In addition, by combining (2.16)–(2.18), we obtain
\[
\limsup_{\varepsilon \to 0} \int_0^T \langle u^\varepsilon_t(t) + Au^\varepsilon(t), \tilde{u}^\varepsilon(t) \rangle_{U', U} \varphi(t) dt - \int_0^T \langle u_t(t) + Au(t), v \rangle_{U', U} \varphi(t) dt 
\]  
(2.19)

\[
+ \int_0^T \{ \langle u_t(t) + Au(t), v \rangle_{U', U} + \Phi_c(v) - \Phi_c(u(t)) \} \varphi(t) dt \geq 0
\]

for all \( v \in V \). Therefore, to prove (2.2) it remains to show that
\[
\int_0^T \langle u_t(t) + Au(t), u(t) \rangle_{U', U} \varphi(t) dt \leq \liminf_{\varepsilon \to 0} \int_0^T \langle u^\varepsilon_t(t) + Au^\varepsilon(t), \tilde{u}^\varepsilon(t) \rangle_{U', U} \varphi(t) dt.
\]  
(2.20)

By the monotonicity of \( A \) a classical argument yields
\[
\langle Au^\varepsilon(t), \tilde{u}^\varepsilon(t) \rangle_{U', U} - (Au(t), u(t))_{U', U}
\]  
\[
= \langle Au^\varepsilon(t) - Au(t), \tilde{u}^\varepsilon(t) - u(t) \rangle_{U', U} + \langle Au^\varepsilon(t) - Au(t), u(t) \rangle_{U', U} + (Au(t), \tilde{u}^\varepsilon(t) - u(t))_{U', U}
\]
\[
\geq \langle Au^\varepsilon(t) - Au(t), u(t) \rangle_{U', U} + \langle Au(t), \tilde{u}^\varepsilon(t) - u(t) \rangle_{U', U}.
\]

Hence, in view of (2.11),
\[
\liminf_{\varepsilon \to 0} \int_0^T \langle Au^\varepsilon(t), \tilde{u}^\varepsilon(t) \rangle_{U', U} \varphi(t) dt \geq \int_0^T \langle Au(t), u(t) \rangle_{U', U} \varphi(t) dt.
\]  
(2.21)

Finally, by (2.9) and (2.11),
\[
\lim_{\varepsilon \to 0} \int_0^T \langle u^\varepsilon_t(t), \tilde{u}^\varepsilon(t) \rangle_{U', U} \varphi(t) dt = \int_0^T \langle u_t(t), u(t) \rangle_{U', U} \varphi(t) dt.
\]  
(2.22)
The thesis follows by combining (2.19), (2.20), (2.21), and (2.22), and by the arbitrariness of \( \varphi \).

We are in a position to prove that the map \( u \) provided by Proposition 6 is a solution to (0.4) in the sense of Theorem 1. Given \( \varphi \in C^\infty_c((0, T) \times \mathbb{T}; \mathbb{R}^2) \), the idea is to test (2.2) with \( v = u \pm \varepsilon \varphi \). The delicate point of this argument is the fact that \( \nabla \cdot (u \pm \varepsilon \varphi) + c \) may fail to be nonnegative, as Proposition 6 guarantees only that \( \nabla \cdot u + c \geq 0 \). An ad hoc construction is hence required.

**Proof of Theorem 1. Existence of solutions.** Let \( \varphi \in C^\infty_c((0, T) \times \mathbb{T}; \mathbb{R}^2) \) and assume, without loss of generality, that \( r = 1 \). We subdivide the proof into two steps.

**Step 1.** We first prove that

\[
\frac{1}{|\nabla \cdot u + c|^2} \in L^1((0, T) \times \mathbb{T}).
\]  

(2.23)

Indeed, fix an arbitrary time \( t \) such that (2.2) holds, and let \( 0 < \delta < 1 \). By Proposition 6 there holds

\[
\langle u_t(t) + Au(t), -\delta u(t) \rangle_{U', U} + \Phi_c((1 - \delta)u(t)) - \Phi_c(u(t)) \geq 0.
\]

In view of Proposition 6, it follows that \((1 - \delta)\nabla \cdot u(t) + c > 0 \) in \( \mathbb{T} \). Indeed, if \( x \) is such that \( \nabla \cdot u(t, x) \geq 0 \), then

\[
(1 - \delta)\nabla \cdot u(t, x) + c \geq c > 0,
\]

whereas if \( \nabla \cdot u(t, x) < 0 \), then

\[
(1 - \delta)\nabla \cdot u(t, x) + c > \nabla \cdot u(t, x) + c > 0.
\]

Hence, by the definition of \( \Phi_c \) we have

\[
\langle u_t(t) + Au(t), u(t) \rangle_{U', U} \leq \frac{\Phi_c((1 - \delta)u(t)) - \Phi_c(u(t))}{\delta} 
\]

(2.24)

\[
= \int_\mathbb{T} \left[ \frac{1}{(1 - \delta)\nabla \cdot u(t) + c} - \frac{1}{\nabla \cdot u(t) + c} \right] d x 
\]

\[
= \int_{\mathbb{T}} \frac{\nabla \cdot u(t)}{(1 - \delta)\nabla \cdot u(t) + c} d x 
\]

\[
= \left( \int_{\{ \nabla \cdot u(t) \geq -c/2 \}} \frac{\nabla \cdot u(t)}{(1 - \delta)\nabla \cdot u(t) + c} d x 
\]

\[
+ \int_{\{ \nabla \cdot u(t) < -c/2 \}} \frac{\nabla \cdot u(t)}{(1 - \delta)\nabla \cdot u(t) + c} d x \right). 
\]

In view of the dominated convergence theorem, the first term in the right-hand side of (2.24) satisfies

\[
\lim_{\delta \to 0} \int_{\{ \nabla \cdot u(t) \geq -c/2 \}} \frac{\nabla \cdot u(t)}{(1 - \delta)\nabla \cdot u(t) + c} d x 
\]

(2.25)

\[
= \int_{\{ \nabla \cdot u(t) \geq -c/2 \}} \frac{\nabla \cdot u(t)}{(\nabla \cdot u(t) + c)^2} d x.
\]
By Fatou’s lemma, the second term in the right-hand side of (2.24) is bounded by

\[
\lim_{\delta \to 0} \int_{\nabla \cdot u(t) < -c/2} \frac{\nabla \cdot u(t)}{(1 - \delta)^2 \nabla \cdot u(t) + c(\nabla \cdot u(t) + c)} \, dx \\
\leq -\frac{c}{2} \lim_{\delta \to 0} \int_{\nabla \cdot u(t) < -c/2} \frac{1}{(1 - \delta)^2 \nabla \cdot u(t) + c(\nabla \cdot u(t) + c)} \, dx \\
\leq -\frac{c}{2} \lim_{\delta \to 0} \int_{\nabla \cdot u(t) < -c/2} \frac{1}{(1 - \delta)^2 \nabla \cdot u(t) + c(\nabla \cdot u(t) + c)} \, dx \\
= -\frac{c}{2} \int_{\nabla \cdot u(t) < -c/2} \frac{1}{(\nabla \cdot u(t) + c)^2} \, dx.
\]

By combining (2.24)–(2.26), we deduce the inequality

\[
\langle u_t(t) + Au(t), u(t) \rangle_{U', U} \\
\leq -\frac{c}{2} \int_{\nabla \cdot u(t) < -c/2} \frac{1}{(\nabla \cdot u(t) + c)^2} \, dx + \int_{\{\nabla \cdot u(t) \geq -c/2\}} \frac{\nabla \cdot u}{(\nabla \cdot u(t) + c)^2} \, dx,
\]

whereas (2.13) and (2.14) yield

\[
\int_{\{\nabla \cdot u(t) \geq -c/2\}} \frac{\nabla \cdot u}{(\nabla \cdot u(t) + c)^2} \, dx \leq \frac{4}{c^2} \int_T |\nabla \cdot u| \, dx \leq C.
\]

Estimates (2.27) and (2.28) imply

\[
\frac{1}{(\nabla \cdot u(t) + c)^2} \in L^1(T)
\]

for almost every \( t \in [0, T] \). On the other hand, since \( u \in W^{1, \infty}([0, T]; U) \), \( \nabla \cdot u \in L^\infty([0, T]; U) \), and \( A \) is a monotone operator, we have

\[
\int_0^T \langle u_t(t) + Au(t), u(t) \rangle_{U', U} \, dt > -\infty.
\]

By (2.28), and by integrating (2.27) on \([0, T]\), we obtain

\[
\int_0^T \int_{\nabla \cdot u(t) < -c/2} \frac{1}{(\nabla \cdot u(t) + c)^2} \, dx \, dt < +\infty.
\]

Claim (2.23) follows by (2.13), (2.29) and by integrating (2.28) with respect to time.

**Step 2.** Let \( t \in [0, T] \) be such that (2.2) holds true. Set

\[
E^\delta := \{ x \in \mathbb{T} : -c < \nabla \cdot u(t, x) < -c + \delta \},
\]

and note that \( \cap_{\delta > 0} E^\delta = \emptyset \), and \( E^{\delta_1} \subset E^{\delta_2} \) for every \( \delta_1 < \delta_2 \). Therefore

\[
\lim_{\delta \to 0} \mathcal{L}^2(E^\delta) = 0.
\]

Let \( \psi^\delta : \mathbb{T} \to \mathbb{R} \) be defined as follows

\[
\psi^\delta := \begin{cases} 
-c + \delta & \text{if } \nabla \cdot u(t) < -c + \delta, \\
\left(1 - \frac{\int_{E^\delta} (\delta - \nabla \cdot u(t)) \, dx}{\int_{\{\nabla \cdot u(t) < 0\}} (\nabla \cdot u(t)) \, dx}\right) \nabla \cdot u(t) & \text{if } \nabla \cdot u(t) > 0, \\
\nabla \cdot u(t) & \text{otherwise}.
\end{cases}
\]

Since \( u \in V \), we have that \( \int_\mathbb{T} \nabla \cdot u(t) \, dx = 0 \). As a result, it follows that \( \psi^\delta \in W^{1,2}(\mathbb{T}) \), \( \psi^\delta = 0 \) on \( \partial \mathbb{T} \), and \( \int_\mathbb{T} \psi^\delta(x) \, dx = \int_\mathbb{T} \nabla \cdot u(t) \, dx = 0 \).
Let $u^\delta : \mathbb{T} \to \mathbb{T}$ be a solution to the Dirichlet problem
\[
\begin{cases}
\Delta u^\delta = \psi^\delta & \text{in } \mathbb{T}, \\
u^\delta = 0 & \text{on } \partial \mathbb{T},
\end{cases}
\]
extended by periodicity, and set $u^\delta := \nabla v^\delta$. By the definition of $\psi^\delta$ and the periodicity of $v^\delta$ it follows that $u^\delta \in V$ and
\[
\nabla \cdot u^\delta + c \geq \delta \quad \text{for every } x \in \mathbb{T}.
\]
Fix $\varphi \in C_c^\infty(\mathbb{T}; \mathbb{R}^2)$, and let $\varepsilon$ be defined as
\[
\varepsilon := \delta \left(1 + \frac{1}{M_\varphi}\right),
\]
where $M_\varphi := \sup_{\mathbb{T}} |\nabla \varphi|$. Note that $u^\delta \pm \varepsilon \varphi \in \text{dom} \, U \partial \Phi_c$. By (2.2) there holds
\[
\langle u_t(t) + Au(t), u^\delta - u(t) + \varepsilon \varphi \rangle_{U', U} + \Phi_c(u^\delta + \varepsilon \varphi) - \Phi_c(u(t)) \geq 0.
\]
More explicitly,
\[
\langle u_t(t) + Au(t), u^\delta - u(t) \rangle_{U', U} + aE(\nabla \cdot u(t), \nabla \cdot (u^\delta - u(t)))_{U', U} + \langle \nabla \nabla \cdot u(t), \nabla \nabla \cdot (u^\delta - u(t)) \rangle_{U', U}
\]
\[
+ \varepsilon \left(\langle u_t(t), \varphi \rangle_{U', U} + aE(\nabla \cdot u(t), \nabla \cdot \varphi)_{U', U} + \langle \nabla \nabla \cdot u(t), \nabla \nabla \cdot \varphi \rangle_{U', U}\right)
\]
\[
+ \Phi_c(u^\delta + \varepsilon \varphi) - \Phi_c(u(t)) \geq 0.
\]

We claim that the following properties hold true
\[
\lim_{\delta \to 0} \frac{aE(\nabla \cdot u(t), \nabla \cdot (u^\delta - u(t)))_{U', U} + \langle \nabla \nabla \cdot u(t), \nabla \nabla \cdot (u^\delta - u(t)) \rangle_{U', U}}{\delta} \leq 0,
\]
and
\[
\lim_{\delta \to 0} \frac{\langle u_t(t), u^\delta - u(t) \rangle_{U', U}}{\delta} \leq 0.
\]

Indeed,
\[
\langle \nabla \cdot u(t), \nabla \cdot (u^\delta - u(t)) \rangle_{U', U} \leq \int_\mathbb{T} |\nabla \cdot u(t)\nabla \cdot (u^\delta - u(t))| \, dx
\]
\[
\leq \delta \int_{E^\delta} |\nabla \cdot u(t)| \, dx + \delta L^2(E^\delta) \frac{||\nabla \cdot u(t)||_{L^2(\mathbb{T})}^2}{\int_{\{\nabla \cdot u(t) > 0\}} \nabla \cdot u(t) \, dx},
\]
and
\[
\langle \nabla \nabla \cdot u(t), \nabla \nabla \cdot (u^\delta - u(t)) \rangle_{U', U}
\]
\[
\leq - \int_{E^\delta} |\nabla \nabla \cdot u(t)|^2 \, dx + \delta L^2(E^\delta) \int_{\{\nabla \cdot u(t) > 0\}} \nabla \cdot u(t) \, dx
\]
\[
\leq \delta L^2(E^\delta) \frac{||\nabla \nabla \cdot u(t)||_{L^2(\mathbb{T})}^2}{\int_{\{\nabla \cdot u(t) > 0\}} \nabla \cdot u(t) \, dx}.
\]

Claim (2.32) follows now by combining (2.30), (2.34) and (2.35).
To prove (2.33) we use the stability of solutions of Poisson's equation with respect to the initial datum. By [9, Chapter 6, Theorem 5], if \( \psi \in L^2(T) \), solutions to the Dirichlet problem
\[
\begin{cases}
\Delta v = \psi & \text{in } T, \\
v = 0 & \text{on } \partial T,
\end{cases}
\]
satisfy
\[\|v\|_{W^{2,2}(T)} \leq \|\psi\|_{L^2(T)}.
\]
Now, for \( \delta \neq \delta' \), we have
\[
\begin{cases}
\Delta(v^\delta - v'^{\delta'}) = \psi^\delta - \psi'^{\delta'} & \text{in } T, \\
v^\delta - v'^{\delta'} = 0 & \text{on } \partial T.
\end{cases}
\]
Therefore,
\[\|v^\delta - v'^{\delta'}\|_{W^{2,2}(T)} \leq \|\psi^\delta - \psi'^{\delta'}\|_{L^2(T)} \quad (2.36)
\]
for every \( \delta \neq \delta' \). On the other hand,
\[
\|v^\delta - \nabla \cdot u(t)\|_{L^2(T)} \leq \delta \mathcal{L}^2(E^\delta)^{\frac{1}{2}} + \delta \mathcal{E}^2(E^\delta)^{\frac{1}{2}} \left[ \int_{\{\nabla \cdot u(t) > 0\}} \nabla \cdot u(t) \, dx \right], \quad (2.37)
\]
which converges to zero as \( \delta \to 0 \). Thus \( \{v^\delta\}_\delta \subset W^{2,2}(T) \) is a Cauchy sequence, there exists \( v \in W^{2,2}(T) \) such that
\[v^\delta \to v \quad \text{strongly in } W^{2,2}(T),
\]
and by (2.36) and (2.37),
\[\|v^\delta - v\|_{W^{2,2}(T)} \leq C\delta \mathcal{L}^2(E^\delta)^{\frac{1}{2}}.
\]
In particular, in view of (2.37), and by the uniqueness of solution to the Laplace problem with Dirichlet datum, we deduce that \( u(t) = \nabla v \). Recalling that \( v^\delta = \nabla v^\delta \), and \( u_t \in L^\infty((0,T);U) \), by (2.36) we obtain
\[
|\langle u_t(t), u^\delta - u(t) \rangle_{U^*, U}| \leq \int_T |\langle u_t, u^\delta - u(t) \rangle| \, dx 
\leq \|u_t\|_{L^\infty((0,T);U)} \int_T |u^\delta - u(t)| \, dx 
\leq \|u_t\|_{L^\infty((0,T);U)} \|v^\delta - v\|_{W^{2,2}(T)} \leq \delta \|u_t\|_{L^\infty((0,T);U)} \mathcal{L}^2(E^\delta)^{1/2}.
\]
Claim (2.33) follows now by (2.30).
By the definition of $\Phi_c$ we have
\[
\Phi_c(u^\delta + \varepsilon \varphi) - \Phi_c(u(t)) = \int_T \left( \frac{1}{\nabla \cdot (u^\delta + \varepsilon \varphi) + c} - \frac{1}{\nabla \cdot u(t) + c} \right) dx \tag{2.39}
\]
\[
= \int_T \frac{\nabla \cdot u(t) - \nabla \cdot (u^\delta + \varepsilon \varphi)}{(\nabla \cdot (u^\delta + \varepsilon \varphi) + c)(\nabla \cdot u(t) + c)} dx \\
= \int_T \frac{\nabla \cdot (u(t) - u^\delta)}{(\nabla \cdot (u^\delta + \varepsilon \varphi) + c)(\nabla \cdot u(t) + c)} dx \\
- \varepsilon \int_T \frac{\nabla \cdot \varphi}{(\nabla \cdot (u^\delta + \varepsilon \varphi) + c)(\nabla \cdot u(t) + c)} dx.
\]
By the definition of $\Phi_c$ we have
\[
\Phi_c(u^\delta + \varepsilon \varphi) - \Phi_c(u(t)) = \int_T \left( \frac{1}{\nabla \cdot (u^\delta + \varepsilon \varphi) + c} - \frac{1}{\nabla \cdot u(t) + c} \right) dx \tag{2.39}
\]
\[
= \int_T \frac{\nabla \cdot u(t) - \nabla \cdot (u^\delta + \varepsilon \varphi)}{(\nabla \cdot (u^\delta + \varepsilon \varphi) + c)(\nabla \cdot u(t) + c)} dx \\
= \int_T \frac{\nabla \cdot (u(t) - u^\delta)}{(\nabla \cdot (u^\delta + \varepsilon \varphi) + c)(\nabla \cdot u(t) + c)} dx \\
- \varepsilon \int_T \frac{\nabla \cdot \varphi}{(\nabla \cdot (u^\delta + \varepsilon \varphi) + c)(\nabla \cdot u(t) + c)} dx.
\]
The first line in the right-hand side of (2.39) is bounded from above by
\[
\left| \int_T \frac{\nabla \cdot (u(t) - u^\delta)}{(\nabla \cdot (u^\delta + \varepsilon \varphi) + c)(\nabla \cdot u(t) + c)} \right| \leq \delta \int_{E^\delta} \frac{1}{E^\delta} (\varepsilon \nabla \cdot \varphi + c)(\nabla \cdot u(t) + c) \\
+ \left( \frac{\delta L^2(E^\delta)}{\int_{\nabla \cdot u(t) > 0} \nabla \cdot u(t) dx} \right) \nabla \cdot u(t) dx \\
\int_{\nabla \cdot u(t) > 0} \left[ \left(\frac{1 - \frac{\int_{\nabla \cdot u(t) > 0} \nabla \cdot u(t) dx}{\int_{\nabla \cdot u(t) > 0} \nabla \cdot u(t) dx}}{\delta - c} \right) \nabla \cdot u(t) + \varepsilon \nabla \cdot \varphi + c \right] \nabla \cdot u(t) + c
\]
By the definition of $\varepsilon$, for $\delta$ small enough there holds
\[
\left( \frac{\int_{E^\delta} (\delta - c - \nabla \cdot u(t)) dx}{\int_{\nabla \cdot u(t) > 0} \nabla \cdot u(t) dx} \right) \nabla \cdot u(t) + \varepsilon |\nabla \cdot \varphi| \leq c \tag{2.41}
\]
Hence,
\[
\left( \frac{\int_{E^\delta} (\delta - c - \nabla \cdot u(t)) dx}{\int_{\nabla \cdot u(t) > 0} \nabla \cdot u(t) dx} \right) \nabla \cdot u(t) + \varepsilon \nabla \cdot \varphi + c \geq \nabla \cdot u(t) + \frac{c}{2},
\]
and
\[
\int_{\nabla \cdot u(t) > 0} \left[ \left(\frac{1 - \frac{\int_{\nabla \cdot u(t) > 0} \nabla \cdot u(t) dx}{\int_{\nabla \cdot u(t) > 0} \nabla \cdot u(t) dx}}{\delta - c} \right) \nabla \cdot u(t) + \varepsilon \nabla \cdot \varphi + c \right] \nabla \cdot u(t) + c
\]
By (2.41) we also have
\[
\int_{E^\delta} \frac{1}{E^\delta} ( \varepsilon \nabla \cdot \varphi + c)(\nabla \cdot u(t) + c) dx \leq \int_{E^\delta} \frac{2}{c(\nabla \cdot u(t) + c)} dx \tag{2.43}
\]
Thus, in view of (2.30), (2.40), (2.42), and (2.43) we obtain
\[
\lim_{\delta \to 0} \frac{1}{\delta} \int_T \frac{\nabla \cdot (u(t) - u^\delta)}{(\nabla \cdot (u^\delta + \varepsilon \varphi) + c)(\nabla \cdot u(t) + c)} dx = 0. \tag{2.44}
\]
To estimate the second line of the right-hand side of (2.39) we notice that in the
set \( \{ \nabla \cdot u(t) > 0 \} \) there holds
\[
\nabla \cdot u^\delta(t) + \varepsilon \nabla \cdot \varphi + c = \left( \frac{\int_{E^\varepsilon} (\delta - c - \nabla \cdot u(t)) \, dx}{\int_{\{ \nabla \cdot u(t) > 0 \}} \nabla \cdot u(t) \, dx} \right) \nabla \cdot u(t) + \varepsilon \nabla \cdot \varphi + c
\[
\geq \nabla \cdot u(t) + \frac{c}{2} \geq \frac{c}{2},
\]
hence
\[
\frac{1}{\nabla \cdot u^\delta + \varepsilon \nabla \cdot \varphi} = \frac{2}{c} \text{ for a.e. } x \in \{ \nabla \cdot u(t) > 0 \}.
\]
Analogously, on \( \{ \nabla \cdot u(t) \leq 0 \} \setminus E^\varepsilon \),
\[
\nabla \cdot u^\delta + c = \nabla \cdot u(t) + c \geq \delta,
\]
whereas
\[
\varepsilon |\nabla \cdot \varphi| = \frac{\delta |\nabla \cdot \varphi|}{1 + M_{\varphi}} \leq \delta \frac{M_{\varphi}}{1 + M_{\varphi}} \leq (\nabla \cdot u(t) + c) \frac{M_{\varphi}}{1 + M_{\varphi}}.
\]
Thus,
\[
\nabla \cdot u^\delta + c + \varepsilon \nabla \cdot \varphi \geq (\nabla \cdot u(t) + c) \frac{M_{\varphi}}{1 + M_{\varphi}},
\]
and
\[
\frac{1}{\nabla \cdot u^\delta + \varepsilon \nabla \cdot \varphi + c} \leq \frac{1 + M_{\varphi}}{M_{\varphi}(\nabla \cdot u(t) + c)}.
\]
Setting
\[
\xi^\delta := \frac{\nabla \cdot \varphi}{(\nabla \cdot u(t) + c)^2},
\]
and denoting by \( \chi_{T \setminus E^\delta} \) the characteristic function of the set \( T \setminus E^\delta \), the sequence \( \{ \chi_{T \setminus E^\delta} \xi^\delta \}_\delta \) is uniformly bounded from above by \( \frac{(1 + M_{\varphi})}{M_{\varphi}(\nabla \cdot u(t) + c)} \), which belongs to \( L^1(T) \) owing to Step 1. By (2.30), we have that
\[
\chi_{T \setminus E^\delta} \xi^\delta \rightarrow \frac{\nabla \cdot \varphi}{(\nabla \cdot u(t) + c)^2} \text{ for a.e. } x \in T.
\]
Therefore, by the dominated convergence theorem there holds
\[
\lim_{\delta \to 0} \int_{T \setminus E^\delta} \frac{\nabla \cdot \varphi}{(\nabla \cdot (u^\delta + \varepsilon \varphi) + c)(\nabla \cdot u(t) + c)} \, dx = \int_T \frac{\nabla \cdot \varphi}{(\nabla \cdot u(t) + c)^2} \, dx. \tag{2.45}
\]
In the set \( E^\delta \) the following inequalities hold true
\[
\nabla \cdot u^\delta + c = \delta, \quad \text{and} \quad \varepsilon |\nabla \cdot \varphi| \leq \delta \frac{M_{\varphi}}{1 + M_{\varphi}} \leq \frac{c}{2},
\]
for \( \delta \) small enough. Hence, by (2.30),
\[
\lim_{\delta \to 0} \left| \int_{E^\delta} \frac{\nabla \cdot \varphi}{(\nabla \cdot (u^\delta + \varepsilon \varphi) + c)(\nabla \cdot u(t) + c)} \, dx \right| \leq \lim_{\delta \to 0} \int_{E^\delta} \frac{2|\nabla \cdot \varphi|}{c(\nabla \cdot u(t) + c)} \, dx = 0. \tag{2.46}
\]
Combining (2.45) and (2.46) we deduce
\[
\lim_{\delta \to 0} \int_T \frac{\nabla \cdot \varphi}{(\nabla \cdot (u^\delta + \varepsilon \varphi) + c)(\nabla \cdot u(t) + c)} \, dx = \int_T \frac{\nabla \cdot \varphi}{(\nabla \cdot u(t) + c)^2} \, dx. \tag{2.47}
\]
In view of (2.44) and (2.47), passing to the limit as $\delta \to 0$ in (2.38) yields the inequality
\[
\langle u(t), \varphi \rangle_{U',U} - aE(\nabla \cdot u(t), \nabla \cdot \varphi)_{U',U} + \langle \nabla \nabla \cdot u(t), \nabla \nabla \cdot \varphi \rangle_{U',U} - \int_T \frac{\nabla \cdot \varphi}{(\nabla \cdot u(t) + c)^2} \, dx \geq 0,
\]
that is,
\[
\langle u(t) + Au(t) + \partial \Phi_c(u(t)), \varphi \rangle_{U',U} \geq 0,
\]
for every $\varphi \in C_c^\infty(\mathbb{T}; \mathbb{R}^2)$. Since the above argument holds for a.e. $t \in [0, T]$, integrating in time we obtain
\[
\int_0^T \langle u_t(t) + Au(t) + \partial \Phi_c(u(t)), \varphi \rangle_{U',U} \, dt \geq 0,
\]
for every $\varphi \in C_c^\infty(\mathbb{T}; \mathbb{R}^2)$. Replacing $u^\delta + \varepsilon \varphi$ with $u^\delta - \varepsilon \varphi$, the same procedure yields
\[
\int_0^T \langle u_t(t) + Au(t) + \partial \Phi_c(u(t)), \varphi \rangle_{U',U} \, dt \leq 0.
\]
Therefore, by integration by parts, we have
\[
0 = \int_0^T \langle u_t(t) + Au(t) + \partial \Phi_c(u(t)), \varphi \rangle_{U',U} \, dt
\]
\[
= \int_0^T \int_T \left[ u_t(t) \varphi + aE \nabla \cdot u \nabla \cdot \varphi + \nabla \nabla \cdot u \nabla \nabla \cdot \varphi + \frac{\nabla \cdot \varphi}{(\nabla \cdot u + c)^2} \right] \, dx \, dt,
\]
concluding the proof of the existence part.

**Uniqueness.** Assume that $u^1, u^2 : [0, T] \to V$ are two solutions of (0.3) in the sense of (0.4). Then
\[
u^1_t(t) + T_A u^1(t) = u^2_t(t) + T_A u^2(t) = 0 \quad \text{for a.e. } t \in [0, T].
\]
On the other hand Proposition 4 yields
\[
\frac{1}{2} \frac{d}{dt} \|u^1(t) - u^2(t)\|^2 \leq \langle u^1_t(t) - u^2_t(t), u^1(t) - u^2(t) \rangle_{V',V}
\]
\[
= \langle T_A u^2(t) - T_A u^1(t), u^1(t) - u^2(t) \rangle_{V',V} \leq 0.
\]
Since $u^1(0) = u^2(0) = u^0$, it follows $u^1(t) = u^2(t)$ for a.e. $t \in [0, T]$. \qed

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