# On the viscous Cahn-Hilliard equation with singular potential and inertial term

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**Abstract**

We consider a relaxation of the viscous Cahn-Hilliard equation induced by the second-order inertial term $u_{tt}$. The equation also contains a semilinear term $f(u)$ of "singular" type. Namely, the function $f$ is defined only on a bounded interval of $\mathbb{R}$ corresponding to the physically admissible values of the unknown $u$, and diverges as $u$ approaches the extrema of that interval. In view of its interaction with the inertial term $u_{tt}$, the term $f(u)$ is difficult to be treated mathematically. Based on an approach originally devised for the strongly damped wave equation, we propose a suitable concept of weak solution based on duality methods and prove an existence result.

**Key words:** Cahn-Hilliard equation, inertia, weak formulation, maximal monotone operator, duality.

**AMS (MOS) subject classification:** 35K67, 35L85, 46A20, 47H05, 80A22.

## 1 Introduction

The celebrated Cahn-Hilliard equation was proposed to describe phase separation phenomena in binary systems [9]. Its "standard" version has the form of a semilinear parabolic fourth order equation, namely

$$u_t - \Delta(-\Delta u + f(u)) = 0. \tag{1.1}$$

Here the unknown $u$ stands for the relative concentration of one phase, or component, in a binary material, and $f$ is the derivative of a non-convex potential $F$ whose minima represent the energetically more favorable configurations usually attained in correspondance, or in proximity, of pure phases or concentrations. In view of the fact that $u$ is an order parameter, often it is normalized in such a way that the pure states correspond to the values $u = \pm 1$, whereas $-1 < u < 1$ denotes the (local) presence of a mixture. We will also adopt this convention. In such setting the values $u \notin [-1, 1]$ are generally interpreted as "nonphysical" and should be somehow excluded. In view of the fourth-order character of (1.1), no maximum principle is available for $u$. Hence, the *constraint* $u \in [-1, 1]$ is generally enforced by assuming $F$ to be defined only for $u \in (-1, 1)$ (or for $u \in [-1, 1]$; both choices are admissible under proper structure conditions) and to be identically $+\infty$ outside the interval $[-1, 1]$. A relevant example is given by the so-called *logarithmic potential*

$$F(u) = (1 - u)\log(1 - u) + (1 + u)\log(1 + u) - \frac{\lambda}{2}u^2, \quad \lambda \geq 0, \tag{1.2}$$

where the last term may induce nonconvexity. Such a kind of potential is generally termed as a *singular* one and its occurrence may give rise to mathematical difficulties in the analysis of the system. For this reason, singular potentials are often replaced by "smooth" approximations like the so-called *double-well* potential taking, after normalization, the form $F(u) = (1-u^2)^2$. Of course, in the presence of a smooth double-well potential, solutions are no longer expected to satisfy the physical constraint $u \in [-1,1]$.

The mathematical literature devoted to (1.1) is huge and the main properties of the solutions in terms of regularity, qualitative behavior, and asymptotics are now well-understood, also in presence of singular potentials like (1.2) (cf., e.g., [20,21] and the references therein). Actually, in recent years, the attention has moved to more sophisticated versions of (1.1) related to specific physical situations. Among these, we are interested here in the so-called *hyperbolic relaxation* of the equation. This can be written as

$$\alpha u_{tt} + u_t - \Delta(-\Delta u + f(u)) = 0, \tag{1.3}$$

where $\alpha > 0$ is a (small) relaxation parameter and the new term accounts for the occurrence of "inertial" effects. Equation (1.3) may be used in order to describe strongly non-equilibrium decomposition generated by deep supercooling into the spinodal region occurring in certain materials (e.g., glasses), see [11,12]. From the mathematical point of view, equation (1.3) carries many similarities with the semilinear (damped) wave equation, but is, however, much more delicate to deal with. For instance, in space dimension $N = 3$ the existence of global in time strong solutions is, up to our knowledge, an open issue also in the case when $f$ is a globally Lipschitz (nonlinear) function [17], whereas for $N = 2$ the occurrence of a critical exponent is observed in case $f$ has a polynomial growth [16,18]. The situation is somehow more satisfactory in space dimension $N = 1$ (cf., e.g., [26,27]) due to better Sobolev embeddings (in particular all solutions taking values in the "energy space" are also uniformly bounded). It is however worth noting that, in the case when $f$ is singular, even the existence of (global) *weak* solutions is a mathematically very challenging problem. Indeed, at least up to our knowledge, this seems to be an open issue even in one space dimension.

The picture is only partially more satisfactory when one considers a further relaxation of the equation containing a "strong damping" (or "viscosity") term, namely

$$\alpha u_{tt} + u_t - \Delta(\delta u_t - \Delta u + f(u)) = 0, \tag{1.4}$$

with $\delta > 0$ (a physical justification for this equation is given, e.g., in [22]). The new term induces additional regularity and some parabolic smoothing effects, and, for this reason, (1.4) is mathematically more tractable in comparison to (1.3). Indeed, existence, regularity and large time behavior of solutions have been analyzed in a number of papers (cf., e.g., [6,7,13,15,19] and references therein). In all these contributions, however, $f$ is taken as a smooth function of at most polynomial growth at infinity. Here, instead, we will consider (1.4) with the choice of a singular function $f$.

To explain the related difficulties, the main point stands, of course, in the low number of available a-priori estimates. This is a general feature of equations of the second order in time, and, as a consequence, approximating sequences satisfy very poor compactness properties. In particular, the second order term $u_{tt}$ can be only controlled in a space like $L^1(0,T;X)$, where $X$ is a Sobolev space of negative order. In view of the bad topological properties of $L^1$, this implies that in the limit the term $u_t$ cannot be shown to be (and, in fact, is not expected to be) continuous in time, but only of bounded variation. In particular, it may present *jumps* with respect to the time variable. In turn, the occurrence of these jumps is strictly connected to the fact that it is no longer possible to compute the singular term $f(u)$ in the "pointwise" sense.

Indeed, in the weak formulation $f(u)$ is suitably reinterpreted in the distributional sense, and, in particular, concentration phenomena may occur. This idea comes from the theory of convex integrals in Sobolev spaces introduced in the celebrated paper by Brezis [8] and later developed and adapted to cover a number of different situations (cf., e.g., [3,4,23] and references therein). In our former paper in collaboration with E. Bonetti and E. Rocca [5] we have shown that this method can be adapted to treat equations of the second order in time. Actually, using duality methods in Sobolev spaces of *parabolic type* (i.e., depending both on space and on time variables), we may provide the required relaxation of the term $f(u)$ accounting for the possible occurrence of concentration phenomena with respect to time. The reader is referred to [5] for further considerations and extended comments and examples.

Equation (1.4) will be considered here in the simplest mathematical setting. Namely, we will settle it in a smooth bounded domain $\Omega \subset \mathbb{R}^N$, $N \leq 3$ (we remark however that the results could be easily extended to any spatial dimension), in a fixed reference interval $(0, T)$ of arbitrary length, and with homogeneous Dirichlet boundary conditions. Then, existence of weak solutions will be proved by suitably adapting the approach of [5]. It is worth observing that, as happens for the mentioned strongly damped wave equation and for other similar models, an alternative weak formulation could be given by restating the problem in the form of a variational inequality. However, as noted in [5], we believe the concept of solution provided here to be somehow more flexible. In particular, with this method we may provide an explicit characterization of the (relaxed) term $f(u)$ (which may be thought as a physical quantity representing the vincular reaction provided by the constraint) in terms of regularity (for instance, for equation (1.4) concentration phenomena are expected to occur *only* with respect to the time variable $t$). Moreover, we can prove that at least *some* weak solutions satisfy a suitable form of the energy inequality. This can be seen as a sort of selection principle for "physical" solutions (note, indeed, that uniqueness is not expected to hold).

The plan of the paper is the following: in the next Section 2 we introduce our assumptions on coefficients and data and state our main result regarding existence of at least one solution to a suitable weak formulation of equation (1.4). The proof of this theorem is then carried out in Section 3 by means of an approximation – a priori estimates – compactness argument.

# 2 Main result

## 2.1 Preliminaries

We consider the viscous Cahn-Hilliard equation with inertia:

$$\begin{cases} \alpha u_{tt} + u_t - \Delta w = 0, \\ w = \delta u_t - \Delta u + \beta(u) - \lambda u. \end{cases} \tag{2.1}$$

Here the coefficients $\alpha$ and $\delta$ are strictly positive constants, whereas $\lambda \geq 0$. Moreover, $\beta$ is a maximal monotone operator in $\mathbb{R} \times \mathbb{R}$ satisfying

$$\overline{D(\beta)} = [-1, 1], \quad 0 \in \beta(0). \tag{2.2}$$

Actually, $\beta$ represents the monotone part of $f(u)$ (cf. (1.3)). The *domain* $D(\beta)$ has been normalized just for mathematical convenience. Following [2], there exists a convex and lower semicontinuous function $j : \mathbb{R} \to [0, +\infty]$ such that $\beta = \partial j$, $\overline{D(j)} = [-1, 1]$, and $j(0) = \min j = 0$. For all $\epsilon \in (0, 1)$ we denote by $j^\epsilon : \mathbb{R} \to [0, +\infty)$ the *Moreau-Yosida regularization* of $j$, and by $\beta^\epsilon := \partial j^\epsilon = (j^\epsilon)'$ the corresponding *Yosida approximation* of $\beta = \partial j$.

By a direct check (cf. also [20, Appendix A]), one may prove, based on (2.2), that there exist constants $c_1 > 0$ and $c_2 \geq 0$ independent of $\epsilon$, such that

$$\beta^\epsilon(r) r \geq c_1 |\beta^\epsilon(r)| - c_2. \tag{2.3}$$

Let us also introduce some functional spaces: we set $H := L^2(\Omega)$ and $V := H^1_0(\Omega)$, so that $V' = H^{-1}(\Omega)$. Moreover, we put

$$\mathcal{V} := H^1(0, T; H),$$

and, for all $t \in (0, T]$,

$$\mathcal{V}_t := H^1(0, t; H).$$

We denote by $(\cdot, \cdot)$ and $\langle \cdot, \cdot \rangle$ the scalar product in $H$ and the duality pairing between $V'$ and $V$, respectively. The scalar products on $L^2(0, T; H)$ and on $L^2(0, t; H)$, for $t \in (0, T)$, are indicated respectively by

$$((\cdot, \cdot)) \quad \text{and by} \quad ((\cdot, \cdot))_t.$$

Correspondingly, the duality products between $\mathcal{V}$ and $\mathcal{V}'$ and between $\mathcal{V}_t$ and $\mathcal{V}'_t$ are noted as

$$\langle\langle \cdot, \cdot \rangle\rangle \quad \text{and} \quad \langle\langle \cdot, \cdot \rangle\rangle_t,$$

respectively.

Next, we indicate by $A : D(A) \to H$, with domain $D(A) := H^2(\Omega) \cap H_0^1(\Omega)$, the Laplace operator with homogeneous Dirichlet boundary condition seen as an unbounded linear operator on $H$. Hence, $A$ is strictly positive and its powers $A^s$ are well defined for all $s \in \mathbb{R}$. In particular, $D(A^{1/2}) = H_0^1(\Omega) = V$. Moreover, $A$ may be extended to the space $V$ and it turns out that $A : V \to V'$ is an isomorphism. In particular, $V'$ is a Hilbert space when endowed with the scalar product

$$(u, v)_* := \langle v, A^{-1}u \rangle = \langle u, A^{-1}v \rangle \quad \text{for } \ u, v \in V'.$$

The associated norm is then given by $\|u\|_{V'}^2 = (u, u)_*$ for $u \in V'$. Correspondingly, the scalar products of the spaces $L^2(0, T; V')$ and $L^2(0, t; V')$ are denoted by

$$(\!(\cdot, \cdot)\!)_* \qquad (\!(\cdot, \cdot)\!)_{*,t}$$

respectively. In particular, we have

$$(\!(u, v)\!)_* = \int_0^T \langle v, A^{-1}u \rangle \, \mathrm{d}t \quad \text{for } \ u, v \in L^2(0, T; V'),$$

with a similar characterization holding for $(\!(\cdot, \cdot)\!)_{*,t}$.

## 2.2 Relaxation of the constraint

We now provide a brief sketch of the relaxation of $\beta$ mentioned in the introduction, referring to [5, Sec. 2] for additional details. First of all, we introduce the functional $J : H \to [0, +\infty]$, $J(u) := \int_\Omega j(u) \, \mathrm{d}x$ for all $u \in H$, whose value is intended to be $+\infty$ if $j(u) \notin L^1(\Omega)$. Moreover it is convenient to define

$$\mathcal{J}(u) := \int_0^T \int_\Omega j(u) \, \mathrm{d}x \, \mathrm{d}t \qquad \forall\, u \in L^2((0, T) \times \Omega), \tag{2.4}$$

and its counterpart on $(0, t)$, namely

$$\mathcal{J}_t(u) := \int_0^t \int_\Omega j(u) \, \mathrm{d}x \, \mathrm{d}s \qquad \forall\, u \in L^2((0, t) \times \Omega). \tag{2.5}$$

Then, the relaxed version of $\beta$ will be intended as a maximal monotone operator in the duality couple $\mathcal{V} \times \mathcal{V}'$. Indeed, we first introduce $\mathcal{J}_\mathcal{V} := \mathcal{J}\llcorner_\mathcal{V}$, the restriction of $\mathcal{J}$ to $\mathcal{V}$. Then, we consider its subdifferential $\partial\mathcal{J}_\mathcal{V}$ with respect to the duality pairing between $\mathcal{V}$ and $\mathcal{V}'$. Namely, for $\xi \in \mathcal{V}'$ and $u \in \mathcal{V}$, we say that

$$\xi \in \partial\mathcal{J}_\mathcal{V}(u) \iff \mathcal{J}_\mathcal{V}(z) \ge \langle\!\langle \xi, z - u \rangle\!\rangle + \mathcal{J}_\mathcal{V}(u) \quad \forall\, z \in \mathcal{V}. \tag{2.6}$$

In order to emphasize that $\partial\mathcal{J}_\mathcal{V}$ consists in a relaxation of $\beta$, we will simply note $\partial\mathcal{J}_\mathcal{V} =: \beta_w$ ($w$ standing for “weak”). Proceeding in a similar way for the functional $\mathcal{J}_t$, we define the subdifferential $\partial\mathcal{J}_{t,\mathcal{V}_t}$ of the operator $\mathcal{J}_{t,\mathcal{V}_t} := \mathcal{J}_t\llcorner_{\mathcal{V}_t}$. This will be indicated simply by $\beta_{w,t}$.

In this setting it is not true anymore that an *element* $\xi$ of the *set* $\beta_w(u)$ (recall that $\beta$ is a multivalued operator and, as a consequence, $\beta_w$ may be multivalued as well) admits a “pointwise” interpretation as “$\xi(t, x) = \beta(u(t, x))$”. Indeed, $\xi$ belongs to the negative order Sobolev space $\mathcal{V}'$ and concentration phenomena are expected to occur. Nevertheless, the maps $\beta^\epsilon$ still provide a suitable approximation of $\beta_w$. Referring the reader to [5,24] for additional details and comments, we just mention here some basic facts. First of all, let us define $J^\epsilon(u) := \int_\Omega j^\epsilon(u) \, \mathrm{d}x$ and $\mathcal{J}^\epsilon(u) := \int_0^T \int_\Omega j^\epsilon(u) \, \mathrm{d}x \, \mathrm{d}t$. Then, one may prove that the functionals $\mathcal{J}^\epsilon$ converge to $\mathcal{J}$ in the sense of Mosco-convergence with respect to the topology of $L^2(0, T; H)$. Moreover, their restrictions to $\mathcal{V}$ Mosco-converge to $\mathcal{J}_\mathcal{V}$ in the topology of $\mathcal{V}$. The analogue of these properties also holds for restrictions to time subintervals $(0, t)$. Referring the reader to [1, Chap. 3] for the definition and basic properties of Mosco-convergence, here we just recall that this convergence notion for functionals implies (and is in fact equivalent to) a related notion of convergence for their subdifferentials, called graph-convergence (or G-convergence). Namely, noting that the function $\beta^\epsilon$ represents the subdifferential of $\mathcal{J}^\epsilon$ both with respect to the topology of $L^2(0, T; H)$ and to that of $\mathcal{V}$, it turns out that the operators $\beta^\epsilon$, if identified with their graphs, G-converge to $\beta$ in the topology of $L^2(0,T;H) \times L^2(0,T;H)$ and G-converge to $\beta_w$ in the topology of $\mathcal{V} \times \mathcal{V}'$. As a consequence of the latter property, we may apply the so-called *Minty's trick* in the duality between $\mathcal{V}$ and $\mathcal{V}'$. This argument will be the main tool we will use in order to take the limit in the approximation of the problem and can be simply stated in this way: once one deals with a sequence $\{v^\epsilon\} \subset \mathcal{V}$ satisfying $v_\epsilon \rightharpoonup v$ weakly in $\mathcal{V}$ and $\beta^\epsilon(v_\epsilon) \rightharpoonup \xi$ weakly in $\mathcal{V}'$, then the inequality

$$\limsup_{\epsilon\searrow 0} \langle\!\langle \xi_\epsilon, v_\epsilon \rangle\!\rangle \leq \langle\!\langle \xi, v \rangle\!\rangle \tag{2.7}$$

implies that $\xi \in \beta_w(v)$. In other words, $\xi$ is identified as an element of the set $\beta_w(v) \subset \mathcal{V}'$.

## 2.3 Statement of the main result

We start with presenting our basic concept of weak solution, which can be seen as an adaptation of [5, Def. 2.2].

**Definition 2.1.** *A couple $(u,\eta)$ is called a weak solution to the initial-boundary value problem for the viscous Cahn-Hilliard equation with inertia whenever the following conditions hold:*

*(a) There hold the regularity properties*

$$u_t \in BV(0,T;H^{-4}(\Omega)) \cap L^\infty(0,T;V') \cap L^2(0,T;H), \tag{2.8}$$

$$u \in L^\infty(0,T;V) \cap L^2(0,T;D(A)), \tag{2.9}$$

$$\eta \in \mathcal{V}'. \tag{2.10}$$

*(b) For any test function $\varphi \in \mathcal{V}$, there holds the following weak version of* (2.1)*:*

$$\begin{aligned}\alpha(u_t(T),\varphi(T))_* - \alpha(u_1,\varphi(0))_* - \alpha(\!(u_t,\varphi_t)\!)_* + (\!(u_t,\varphi)\!)_* \\ + \delta(\!(u_t,\varphi)\!) + (\!(A^{1/2}u, A^{1/2}\varphi)\!) + \langle\!\langle \eta,\varphi\rangle\!\rangle - \lambda(\!(u,\varphi)\!) = 0.\end{aligned} \tag{2.11}$$

*Moreover, for all $t \in [0,T]$ there exists $\eta_{(t)} \in \mathcal{V}'$ such that*

$$\begin{aligned}\alpha(u_t(t),\varphi(t))_* - \alpha(u_1,\varphi(0))_* - \alpha(\!(u_t,\varphi_t)\!)_{*,t} + (\!(u_t,\varphi)\!)_{*,t} \\ + \delta(\!(u_t,\varphi)\!)_t + (\!(A^{1/2}u, A^{1/2}\varphi)\!)_t + \langle\!\langle \eta_{(t)},\varphi\rangle\!\rangle_t - \lambda(\!(u,\varphi)\!)_t = 0,\end{aligned} \tag{2.12}$$

*for all $\varphi \in \mathcal{V}_t$.*

*(c) The functionals $\eta$ and $\eta_{(t)}$ satisfy*

$$\eta \in \beta_w(u), \qquad \eta_{(t)} \in \beta_{w,t}(u_{\llcorner(0,t)}) \quad \text{for all } t \in (0,T), \tag{2.13}$$

*and the following compatibility condition holds true:*

$$\langle\!\langle \eta_{(t)},\varphi\rangle\!\rangle_t = \langle\!\langle \eta,\bar\varphi\rangle\!\rangle \quad \text{for all } \varphi \in \mathcal{V}_{t,0} \quad \text{and all } t \in [0,T), \tag{2.14}$$

*where $\mathcal{V}_{t,0} := \{\varphi \in \mathcal{V}_t : \ \varphi(t) = 0\}$ and $\bar\varphi$ is the trivial extension of $\varphi \in \mathcal{V}_{t,0}$ to $\mathcal{V}$, i.e., $\bar\varphi(s) = \varphi(t) = 0$ for all $s \in (t,T]$.*

*(d) There holds the Cauchy condition*

$$u|_{t=0} = u_0 \quad \text{a.e. in } \Omega. \tag{2.15}$$

Correspondingly, we conclude this section with our main result, stating existence of at least one weak solution.

**Theorem 2.2.** *Let $T > 0$ and let the initial data satisfy*

$$u_0 \in V, \quad j(u_0) \in L^1(\Omega), \qquad u_1 \in H. \tag{2.16}$$

*Then, there exists a solution* $(u,\eta)$ *to the viscous Cahn-Hilliard equation with inertia in the sense of* Def. 2.1. *Moreover,* $u$ *satisfies the* energy inequality

$$\begin{aligned}&\frac{\alpha}{2}\|u_t(t_2)\|_{V'}^2+\frac{1}{2}\|A^{1/2}u(t_2)\|_H^2+J(u(t_2))-\frac{\lambda}{2}\|u(t_2)\|_H^2+\int_{t_1}^{t_2}\left(\delta\|u_t\|_H^2+\|u_t\|_{V'}^2\right)\,\mathrm{d}s\\&\quad\le\frac{\alpha}{2}\|u_t(t_1)\|_{V'}^2+\frac{1}{2}\|A^{1/2}u(t_1)\|_H^2+J(u(t_1))-\frac{\lambda}{2}\|u(t_1)\|_H^2,\end{aligned}\tag{2.17}$$

*for* almost every $t_1\in[0,T)$ *(surely including* $t_1=0$*) and* every $t_2\in(t_1,T]$.

# 3 Proof of Theorem 2.2

## 3.1 Approximation

We consider a regularization of system (2.1), namely for $\epsilon\in(0,1)$ we denote by $(u^\epsilon,w^\epsilon)$ the solution to

$$\alpha u_{tt}^\epsilon+u_t^\epsilon+Aw^\epsilon=0,\tag{3.1}$$

$$w^\epsilon=\delta u_t^\epsilon+Au^\epsilon+\beta^\epsilon(u^\epsilon)-\lambda u^\epsilon,\tag{3.2}$$

coupled with the initial conditions

$$u^\epsilon|_{t=0}=u_0^\epsilon\ \text{ and }\ u_t^\epsilon|_{t=0}=u_1^\epsilon,\quad\text{a.e. in }\Omega.\tag{3.3}$$

Recall that $\beta^\epsilon$ was defined in Subsec. 2.1. The following result provides existence of a unique smooth solution to (3.1)-(3.3) once the initial data are suitably regularized:

**Theorem 3.1.** *Let* $T>0$, $u_0^\epsilon\in D(A)=H^2(\Omega)\cap V$, $u_1^\epsilon\in D(A^{1/2})=V$. *Then there exists a unique function* $u^\epsilon$ *with*

$$u^\epsilon\in W^{1,\infty}(0,T;H)\cap H^1(0,T;V)\cap L^\infty(0,T;D(A)),\tag{3.4}$$

$$u_t^\epsilon\in W^{1,\infty}(0,T;D(A^{-1})),\tag{3.5}$$

*satisfying* (3.1)-(3.3). *Moreover, for every* $t_1,t_2\in[0,T]$*, there holds the approximate energy balance*

$$\begin{aligned}&\frac{\alpha}{2}\|u_t^\epsilon(t_2)\|_{V'}^2+\frac{1}{2}\|A^{1/2}u^\epsilon(t_2)\|_H^2+J^\epsilon(u^\epsilon(t_2))-\frac{\lambda}{2}\|u^\epsilon(t_2)\|_H^2+\int_{t_1}^{t_2}\left(\delta\|u_t^\epsilon\|_H^2+\|u_t^\epsilon\|_{V'}^2\right)\,\mathrm{d}s\\&\quad=\frac{\alpha}{2}\|u_t^\epsilon(t_1)\|_{V'}^2+\frac{1}{2}\|A^{1/2}u^\epsilon(t_1)\|_H^2+J^\epsilon(u^\epsilon(t_1))-\frac{\lambda}{2}\|u^\epsilon(t_1)\|_H^2.\end{aligned}\tag{3.6}$$

The proof of the above result is standard (see, e.g., [14, Thm. 2.1]). Actually, one can replicate the a-priori estimates corresponding to the regularity properties (3.4)-(3.5) by multiplying (3.1) by $u_t^\epsilon$, (3.2) by $Au_t^\epsilon$, and using the Lipschitz continuity of $\beta^\epsilon$. The regularity of $\beta^\epsilon$ is also essential for having uniqueness, as one can show via standard contractive methods. Then, to prove the energy equality it is sufficient to test (3.1) by $A^{-1}u_t^\epsilon$, (3.2) by $u_t^\epsilon$, and integrate the results with respect to the time and space variables. It is worth observing that these test functions are *admissible* thanks to the regularity properties (3.4)-(3.5). As a consequence of this fact, we can apply standard chain-rule formulas to obtain that (3.6) holds with the *equal* sign, which will no longer be the case in the limit.

As a first step in the proof of Theorem 2.2, we need to specify the required regularization of the initial data:

**Lemma 3.2.** *Let* (2.16) *hold. Then there exist two families* $\{u_0^\epsilon\}\subset D(A)\cap V$ *and* $\{u_1^\epsilon\}\subset V$, $\epsilon\in(0,1)$*, satisfying*

$$J^\epsilon(u_0^\epsilon)\le J(u_0)\ \ \forall\epsilon>0\qquad\text{and}\ \ u_0^\epsilon\to u_0\ \text{ in }V,\tag{3.7}$$

$$u_1^\epsilon\to u_1\ \text{ in }H.\tag{3.8}$$

Also the above lemma is standard. Indeed, one can construct $u_0^\epsilon$, $u_1^\epsilon$ by simple singular perturbation methods (see, e.g., [23, Sec. 3]). Let us then consider the solutions $u^\epsilon$ to the regularized system (3.1)-(3.3) with the initial data provided by Lemma 3.2. Then, taking a test function $\varphi \in \mathcal{V}$, multiplying (3.1) by $A^{-1}\varphi$, (3.2) by $\varphi$, and performing standard manipulations, one can see that $u^\epsilon$ also satisfies the weak formulation (compare with (2.11))

$$\begin{aligned}
&\alpha(u_t^\epsilon(T), \varphi(T))_* - \alpha(u_1^\epsilon, \varphi(0))_* - \alpha(\!(u_t^\epsilon, \varphi_t)\!)_* + (\!(u_t^\epsilon, \varphi)\!)_* \\
&\quad + \delta(\!(u_t^\epsilon, \varphi)\!) + (\!(A^{1/2}u^\epsilon, A^{1/2}\varphi)\!) + (\!(\beta^\epsilon(u^\epsilon), \varphi)\!) - \lambda(\!(u^\epsilon, \varphi)\!) = 0. \qquad (3.9)
\end{aligned}$$

Correspondingly, the analogue over subintervals $(0,t)$ also holds. Namely, for $\varphi \in \mathcal{V}_t$ one has (compare with (2.12))

$$\begin{aligned}
&\alpha(u_t^\epsilon(t), \varphi(t))_* - \alpha(u_1^\epsilon, \varphi(0))_* - \alpha(\!(u_t^\epsilon, \varphi_t)\!)_{*,t} + (\!(u_t^\epsilon, \varphi)\!)_{*,t} \\
&\quad + \delta(\!(u_t^\epsilon, \varphi)\!)_t + (\!(A^{1/2}u^\epsilon, A^{1/2}\varphi)\!)_t + (\!(\beta^\epsilon(u^\epsilon), \varphi)\!)_t - \lambda(\!(u^\epsilon, \varphi)\!)_t = 0. \qquad (3.10)
\end{aligned}$$

## 3.2 A priori estimates

We now establish some a-priori estimates for $u^\epsilon$. The estimates will be uniform in $\epsilon$ and permit us to take $\epsilon \searrow 0$ at the end. First of all, the energy balance (3.6) and the uniform bounded properties (3.7)-(3.8) of approximating initial data provide the existence of a constant $M > 0$, independent of $\epsilon$, such that the following bounds hold true:

$$\begin{aligned}
&\|u^\epsilon\|_{L^\infty(0,T;V)} \le M, && (3.11a)\\
&\|u^\epsilon\|_{H^1(0,T;V')} \le M, && (3.11b)\\
&\delta^{1/2}\|u^\epsilon\|_{H^1(0,T;H)} \le M, && (3.11c)\\
&\alpha^{1/2}\|u^\epsilon\|_{W^{1,\infty}(0,T;V')} \le M, && (3.11d)\\
&\|j^\epsilon(u^\epsilon)\|_{L^\infty(0,T;L^1(\Omega))} \le M, && (3.11e)
\end{aligned}$$

for all $\epsilon \in (0,1)$. More precisely, thanks to the fact that, for every (fixed) $\epsilon \in (0,1)$, $u_t^\epsilon$ lies in $C^0([0,T];V')$ by (3.4)-(3.5), we are allowed to evaluate $u_t^\epsilon$ pointwise in time. Hence, (3.11d) may be complemented by

$$\|u_t^\epsilon(t)\|_{V'} \le M \qquad \text{for every } \ t \in [0,T], \qquad (3.12)$$

and in particular, for $t = T$. Analogously, thanks to $u^\epsilon \in C^0([0,T];V)$, in addition to (3.11a) we also have

$$\|u^\epsilon(t)\|_V \le M \quad \text{for every } \ t \in [0,T]. \qquad (3.13)$$

Next, taking $\varphi = u^\epsilon$ in (3.9) and rearranging terms, we infer

$$\begin{aligned}
\int_0^T \int_\Omega \beta^\epsilon(u^\epsilon)u^\epsilon \,\mathrm{d}x\,\mathrm{d}t &\le \alpha\|u_t^\epsilon(T)\|_{V'}\|u^\epsilon(T)\|_{V'} + \alpha\|u_1^\epsilon\|_{V'}\|u_0^\epsilon\|_{V'} \\
&+ \alpha\|u_t^\epsilon\|^2_{L^2(0,T;V')} + \|u_t^\epsilon\|_{L^2(0,T;V')}\|u^\epsilon\|_{L^2(0,T;V')} \\
&+ \delta\|u_t^\epsilon\|_{L^2(0,T;H)}\|u^\epsilon\|_{L^2(0,T;H)} + \|A^{1/2}u^\epsilon\|^2_{L^2(0,T;H)} + \lambda\|u^\epsilon\|^2_{L^2(0,T;H)}. \qquad (3.14)
\end{aligned}$$

Then, thanks to estimates (3.11), (3.12) and (3.13), we may check that the right hand side of (3.14) is bounded uniformly with respect to $\epsilon$. Consequently, using also (2.3), we infer

$$\|\beta^\epsilon(u^\epsilon)\|_{L^1(0,T;L^1(\Omega))} \le M. \qquad (3.15)$$

Now, since we assumed $N \le 3$, we know that $L^1(\Omega) \subset D(A^{-1})$, the latter being a closed subspace of $H^{-2}(\Omega)$. Moreover, $A$ can be extended to a bounded linear operator $A : D(A^{-1}) \to D(A^{-2}) \subset H^{-4}(\Omega)$. Then, letting $X := H^{-4}(\Omega)$ (note that for $N > 3$ the argument still works up to suitably modifying the choice of $X$) and rewriting (3.1)-(3.2) as a single equation, i.e.,

$$\alpha u_{tt}^\epsilon + u_t^\epsilon + \delta A u_t^\epsilon + A^2 u^\epsilon + A(\beta^\epsilon(u^\epsilon)) - \lambda A u^\epsilon = 0, \qquad (3.16)$$

we may check by a comparison of terms that

$$\alpha \|u_t^\epsilon\|_{W^{1,1}(0,T;X)} \le M. \tag{3.17}$$

Actually, we used here the estimates (3.11) together with (3.15).

Next, thanks to the last of (3.4), we are allowed to multiply (3.1) by $u^\epsilon$ and (3.2) by $Au^\epsilon$. Using the monotonicity of $\beta^\epsilon$ and the bounds (3.11), standard arguments lead us to the additional estimate

$$\|u^\epsilon\|_{L^2(0,T;D(A))} \le M, \tag{3.18}$$

still holding for $M > 0$ independent of $\epsilon$.

Finally, for all $\varphi \in \mathcal{V}_t$ we can compute from (3.10)

$$\begin{aligned}\left| \int_0^t \langle \beta^\epsilon(u^\epsilon), \varphi \rangle \, \mathrm{d}s \right| &\le \alpha \|u_t^\epsilon(t)\|_{V'} \|\varphi(t)\|_{V'} + \alpha \|u_1^\epsilon\|_{V'} \|\varphi(0)\|_{V'} + \alpha \|u_t^\epsilon\|_{L^2(0,t;V')} \|\varphi_t\|_{L^2(0,t;V')} \\ &+ \|u_t^\epsilon\|_{L^2(0,t;V')} \|\varphi\|_{L^2(0,t;V')} + \delta \|u_t^\epsilon\|_{L^2(0,t;H)} \|\varphi\|_{L^2(0,t;H)} \\ &+ \|u^\epsilon\|_{L^2(0,t;D(A))} \|\varphi\|_{L^2(0,t;H)} + \lambda \|u^\epsilon\|_{L^2(0,t;H)} \|\varphi\|_{L^2(0,t;H)}\end{aligned} \tag{3.19}$$

and the right-hand side, by (3.11), (3.12) and (3.18), is less or equal than $C\|\varphi\|_{\mathcal{V}_t}$, with $C$ depending only on the (controlled) norms of $u^\epsilon$. Hence it follows that there exists a constant $M > 0$ independent of $\epsilon$ such that

$$\|\beta^\epsilon(u^\epsilon)\|_{\mathcal{V}_t'} \le M, \tag{3.20}$$

for every $t \in (0, T]$. In particular, $\|\beta^\epsilon(u^\epsilon)\|_{\mathcal{V}'} \le M$.

## 3.3 Passage to the limit

Using the estimates obtained above, we now aim to pass to the limit as $\epsilon \searrow 0$ in the weak formulation (3.9). Firstly, (3.11), (3.18) and (3.20) imply that there exist $u \in W^{1,\infty}(0,T;V') \cap H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;D(A))$ and $\eta \in \mathcal{V}'$ such that

$$u^\epsilon \rightharpoonup u \quad \text{weakly star in } W^{1,\infty}(0,T;V') \text{ and weakly in } L^2(0,T;D(A)), \tag{3.21a}$$

$$u^\epsilon \rightharpoonup u \quad \text{weakly star in } L^\infty(0,T;V) \text{ and weakly in } H^1(0,T;H), \tag{3.21b}$$

$$u_t^\epsilon \rightharpoonup u_t \quad \text{weakly star in } BV(0,T;X), \tag{3.21c}$$

$$\beta^\epsilon(u^\epsilon) \rightharpoonup \eta \quad \text{weakly in } \mathcal{V}'. \tag{3.21d}$$

Here and below all convergence relations are implicitly intended to hold up to extraction of a (non relabeled) subsequence of $\epsilon \searrow 0$.

Thanks to (3.21a)-(3.21b) and (3.13) we also infer

$$u^\epsilon(t) \rightharpoonup u(t) \quad \text{weakly in } V \quad \text{for all } t \in [0,T]. \tag{3.21e}$$

Next, condition (3.21a) implies, thanks to the Aubin-Lions lemma, that

$$u^\epsilon \to u \quad \text{strongly in } L^2(0,T;V). \tag{3.21f}$$

A generalized version of the same lemma [25, Cor. 4, Sec. 8] implies, thanks to (3.21b) and (3.21c),

$$u_t^\epsilon \to u_t \quad \text{strongly in } L^2(0,T;V'). \tag{3.21g}$$

From (3.21c) and a proper version of the Helly selection principle [10, Lemma 7.2], we infer

$$u_t^\epsilon(t) \rightharpoonup u_t(t) \quad \text{weakly in } X \quad \text{for all } t \in [0,T]. \tag{3.21h}$$

Combining this with (3.12), we obtain more precisely

$$u_t^\epsilon(t) \to u_t(t) \quad \text{weakly in } V' \text{ and strongly in } D(A^{-1}) \quad \text{for all } t \in [0,T]. \tag{3.21i}$$

Hence, using (3.21), we can take $\epsilon \searrow 0$ in (3.9) and get back (2.11). Indeed, it is not difficult to check that all terms pass to the limit. Notice however that, in view of (3.21d), the $L^2$-*scalar product* $(\!(\beta^\epsilon(u^\epsilon), \varphi)\!)$ is replaced by the $\mathcal{V}'$-$\mathcal{V}$ duality $\langle\!\langle \eta, \varphi \rangle\!\rangle$ in the limit.

Let us now consider the weak formulation on subintervals. Taking $\varphi \in \mathcal{V}_t$, $t \in [0, T]$, we may rearrange terms in (3.10) to get

$$(\!(\beta^\epsilon(u^\epsilon), \varphi)\!)_t = -\alpha(u^\epsilon_t(t), \varphi(t))_* + \alpha(u^\epsilon_1, \varphi(0))_* + \alpha(\!(u^\epsilon_t, \varphi_t)\!)_{*,t} \\ - (\!(u^\epsilon_t, \varphi)\!)_{*,t} - \delta(\!(u^\epsilon_t, \varphi)\!)_t - (\!(A^{1/2}u^\epsilon, A^{1/2}\varphi)\!)_t + \lambda(\!(u^\epsilon, \varphi)\!)_t = 0. \tag{3.22}$$

Now, *without extracting further subsequences*, it can be checked that, as a consequence of (3.21), the right hand side tends to

$$- \alpha(u_t(t), \varphi(t))_* + \alpha(u_1, \varphi(0))_* + \alpha(\!(u_t, \varphi_t)\!)_{*,t} - (\!(u_t, \varphi)\!)_{*,t} \\ - \delta(\!(u_t, \varphi)\!)_t - (\!(A^{1/2}u, A^{1/2}\varphi)\!)_t + \lambda(\!(u, \varphi)\!)_t =: \langle\!\langle \eta_{(t)}, \varphi \rangle\!\rangle_t. \tag{3.23}$$

Hence we have proved (2.11) and (2.12). The compatibility property (2.14) is also a straighforward consequence of this argument.

Next, to prove (2.13), according to (2.7), we need to show

$$\limsup_{\epsilon \searrow 0} \langle\!\langle \beta^\epsilon(u^\epsilon), u^\epsilon \rangle\!\rangle \le \langle\!\langle \eta, u \rangle\!\rangle. \tag{3.24}$$

Thanks to (3.9) with $\varphi = u^\epsilon$, we have

$$\langle\!\langle \beta^\epsilon(u^\epsilon), u^\epsilon \rangle\!\rangle = -\alpha(u^\epsilon_t(T), u^\epsilon(T))_* + \alpha(u^\epsilon_1, u^\epsilon_0)_* + \alpha(\!(u^\epsilon_t, u^\epsilon_t)\!)_* \\ - (\!(u^\epsilon_t, u^\epsilon)\!)_* - \delta(\!(u^\epsilon_t, u^\epsilon)\!) - \|A^{1/2}u^\epsilon\|^2_{L^2(0,T;H)} + \lambda\|u^\epsilon\|^2_{L^2(0,T;H)}. \tag{3.25}$$

Then, we take the lim sup of the above expression as $\epsilon \searrow 0$. Then, using relations (3.21) and standard lower semicontinuity arguments we infer that the lim sup of the above expression is less or equal than

$$- \alpha(u_t(T), u(T))_* + \alpha(u_1, u_0)_* + \alpha(\!(u_t, u_t)\!)_* - (\!(u_t, u)\!)_* \\ - \delta(\!(u_t, u)\!) - \|A^{1/2}u\|^2_{L^2(0,T;H)} + \lambda\|u\|^2_{L^2(0,T;H)} = \langle\!\langle \eta, u \rangle\!\rangle, \tag{3.26}$$

the last equality following from (2.11) with the choice $\varphi = u$. Combining (3.25) with (3.26) we obtain (3.24), whence the first of (2.13). The same argument applied to the subinterval $(0, t)$ entails $\eta_{(t)} \in \beta_w(u_{\llcorner(0,t)})$, for all $t \in (0, T]$, as desired.

Finally, we need to prove the energy *inequality* (2.17). To this aim, we consider the approximate energy balance (3.6) and take its lim inf as $\epsilon \searrow 0$.

Then, by standard lower semicontinuity arguments, it is clear that the left hand side of (2.17) is less or equal than the lim inf of the left hand side of (3.6). The more delicate point stands, of course, in dealing with the right hand sides. Indeed, we claim that there exists the limit

$$\lim_{\epsilon \searrow 0} \Big( \frac{\alpha}{2}\|u^\epsilon_t(t_1)\|^2_{V'} + \frac{1}{2}\|A^{1/2}u^\epsilon(t_1)\|^2_H + J^\epsilon(u^\epsilon(t_1)) - \frac{\lambda}{2}\|u^\epsilon(t_1)\|^2_H \Big) \\ = \Big( \frac{\alpha}{2}\|u_t(t_1)\|^2_{V'} + \frac{1}{2}\|A^{1/2}u(t_1)\|^2_H + J(u(t_1)) - \frac{\lambda}{2}\|u(t_1)\|^2_H \Big), \tag{3.27}$$

at least for almost every $t_1 \in [0, t)$, surely including $t_1 = 0$. We just sketch the proof of this fact, which follows closely the lines of the argument given in [5, Section 3] to which we refer the reader for more details.

First, we observe that the last summand passes to the limit in view of (3.21e) and the compact embedding $V \subset H$. Next, the convergence

$$\Big( \frac{\alpha}{2}\|u^\epsilon_t(t_1)\|^2_{V'} + \frac{1}{2}\|A^{1/2}u^\epsilon(t_1)\|^2_H \Big) \to \Big( \frac{\alpha}{2}\|u_t(t_1)\|^2_{V'} + \frac{1}{2}\|A^{1/2}u(t_1)\|^2_H \Big)$$

holds for *almost* every choice of $t_1$ and up to extraction of a further subsequence of $\epsilon \searrow 0$ in view of (3.21f) and (3.21g) (indeed, because these are just $L^2$-bounds with respect to time, we cannot hope to get convergence for *every* $t_1 \in [0,T)$). Finally, we need to show

$$J^\epsilon(u^\epsilon(t_1)) \to J(u(t_1)).$$

This is the most delicate part, which proceeds exactly as in [5, Section 3], to which the reader is referred. Note, finally, that (3.27) for $t_1 = 0$ can be easily proved as a direct consequence of Lemma 3.2 (again, we refer the reader to [5] for details). The proof is concluded.

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