# LIOUVILLE PROPERTIES AND CRITICAL VALUE OF FULLY NONLINEAR ELLIPTIC OPERATORS

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Abstract. We prove some Liouville properties for sub- and supersolutions of fully nonlinear degenerate elliptic equations in the whole space. Our assumptions allow the coefficients of the first order terms to be large at infinity, provided they have an appropriate sign, as in Ornstein-Uhlenbeck operators. We give two applications. The first is a stabilization property for large times of solutions to fully nonlinear parabolic equations. The second is the solvability of an ergodic Hamilton-Jacobi-Bellman equation that identifies a unique critical value of the operator.

## 1. Introduction

We consider fully nonlinear degenerate elliptic partial differential equations

$$F(x, u, Du, D^2u) = 0, \qquad \text{in } \mathbb{R}^N, \tag{1.1}$$

within the theory of viscosity solutions, and look for sufficient conditions for the validity of Liouville type results such as

(1.2) any subsolution (respectively, supersolution) of (1.1) bounded from above (respectively, from below) is a constant.

For solutions of the equation $F(D^2u) = 0$ with $F$ uniformly elliptic, the result follows from the Harnack-type inequalities for such PDEs [16]. For subsolutions, however, such inequalities do not hold and different tools must be used. Cutrì and Leoni [25] proved (1.2) for subsolutions of equations of the form $F(x, D^2u) + h(x)u^p = 0$ by a nonlinear extension of the Hadamard three spheres theorem. Capuzzo Dolcetta and Cutrì [18] and Chen and Felmer [20] studied inequalities with $F$ of the general form (1.1), where the dependence on the first order derivatives is a nontrivial difficulty that is overcome if the coefficients multiplying $Du$ decay at infinity in a suitable way. All these papers assume the uniform ellipticity of $F$.

Our approach is different and requires different assumptions. It is inspired by our paper [8] on a linear equation of Ornstein-Uhlenbeck type modelling stochastic volatility in finance. We suppose the existence of a sort of Lyapunov function $w$ for the operator $F$, namely, a supersolution of (1.1) for $|x| > R_o$, for some $R_o$, and such that $w \to +\infty$ as $|x| \to +\infty$. We search examples of such functions among radial ones. For instance, in the case of a Hamilton-Jacobi-Bellman operator

$$\inf_{\alpha \in A}\{-tr(a(x,\alpha)D^2u) - b(x,\alpha)\cdot Du + c(x,\alpha)u\}$$

$w(x) = |x|^2$ is a Lyapunov function if

$$\sup_{\alpha \in A}(tr\, a(x,\alpha) + b(x,\alpha)\cdot x - c(x,\alpha)|x|^2/2) \le 0 \quad \text{ for } |x| \ge R_o.$$

If $\inf c > 0$ this allows for quite general coefficients $a, b$, whereas for $c \equiv 0$ it is satisfied by a drift $b$ of the kind appearing in Ornstein-Uhlenbeck operators plus a possible perturbation of lower order,

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such as

$$b(x,\alpha) = \gamma(m-x) + \tilde{b}(x,\alpha), \qquad \gamma > 0, \qquad \lim_{x\to\infty} \sup_{\alpha\in A} \frac{\tilde{b}(x,\alpha)\cdot x}{|x|^2} = 0.$$

Note that here it is helpful that $b$ is large for $|x|$ large, provided it has the appropriate sign, that is, it points towards the origin, whereas in [18, 20] $b$ must be small at infinity.

The other main ingredient of our method is a strong maximum principle for the equation (1.1). This is true in the uniformly elliptic case, but also for several degenerate operators, see [10, 11]. Our main example is a quasilinear operator whose principal part is hypoelliptic in Hörmander's sense. This seems to be the first Liouville-type result for subelliptic inequalities with nonlinearities involving $Du$. We refer to Chapter 5.8 of the monograph [15] and the references therein for a survey about Liouville properties for sublaplacians, mostly obtained by Harnack-type inequalities for solutions. We refer also to [17] for results on inequalities of the form $Lu + h(x)u^p \le 0$ with $L$ linear degenerate elliptic, and to [35, 36, 38] for more recent results on linear subelliptic equations.

For uniformly elliptic $F$ with constants $0 < \lambda \le \Lambda$ we exploit the comparison with Pucci maximal and minimal operators $\mathcal{M}^+, \mathcal{M}^-$ associated to $\lambda, \Lambda$ and the Lyapunov function $w(x) = \log|x|$. If

$$F(x,t,p,X) \ge \mathcal{M}^-(X) + \inf_{\alpha\in A}\{c(x,\alpha)t - b(x,\alpha)\cdot p\},$$

we prove the Liouville property for subsolutions under the assumption

$$\sup_{\alpha\in A}(b(x,\alpha)\cdot x - c(x,\alpha)|x|^2\log(|x|)) \le \lambda - (N-1)\Lambda \quad \text{ for } |x| \ge R_o.$$

This e some results in [18]. Let us mention that other results on Liouville type properties for fully nonlinear equations are in the papers [14, 3, 40, 6, 39].

The second part of the paper is devoted to two applications of the Liouville properties, both for uniformly elliptic $F$. The first is the stabilization in the space variables for large times of solutions to the parabolic equation

$$u_t + F(x, Du, D^2u) = 0 \quad \text{ in } [0,+\infty)\times\mathbb{R}^N, \qquad u(0,x) = h(x) \quad \text{ in } \mathbb{R}^n,$$

with $F$ positively 1-homogeneous in $(p, X)$ and $h \in BUC(\mathbb{R}^N)$. We prove that

$$\limsup_{t\to+\infty} u(t,x) = \overline{u} \qquad \text{and} \qquad \liminf_{t\to+\infty} u(t,x) = \underline{u}$$

are constant, a result previously known for $F$ and $h$ periodic in $x$ (and in such a case $\overline{u} = \underline{u}$ and the convergence is uniform). The stabilization to a constant $\overline{u} = \underline{u}$ has been studied by several authors for linear equations under additional conditions on $h$ (see [26] and the references therein), and it is known that even for the heat equation it can be $\overline{u} > \underline{u}$ for some bounded and smooth $h$ [22].

The second application concerns the so-called ergodic HJB equation

$$\inf_{\alpha\in A}\{-tr\, a(x,\alpha)D^2\chi(x) + b(x,\alpha)\cdot D\chi(x) - l(x,\alpha)\} = c, \qquad x \in \mathbb{R}^N,$$

where the unknowns are the critical value $c \in \mathbb{R}$ and $\chi \in C(\mathbb{R}^N)$ that must also satisfy a growth condition as $|x| \to \infty$. This problem arises in ergodic stochastic control (see, e.g., [5, 4, 34, 31, 21] and the references therein), weak KAM theory in the 1st order case $a \equiv 0$ [37, 28], periodic homogenization [37, 27, 2], singular perturbations [1, 8, 7], and long-time behavior of solutions to non-homogeneous parabolic equations (see, e.g., [13, 30, 32, 23] and the references therein). The Liouville property plays a crucial role in the proof of the uniqueness of $\chi$, up to additive constants, and of $c$. The existence of the solution pair is proved by the vanishing discount approximation and using the Krylov-Safonov Hölder estimates, as in [27, 5, 1] for the periodic case and in [9, 19] for HJB equations degenerating at the boundary of a bounded open set. The case of a semilinear uniformly elliptic equation in the whole space, under some dissipativity condition, has been considered in [34, 31], see also references therein. To our knowledge our result is the first for fully nonlinear elliptic equations in the whole $\mathbb{R}^N$ without any periodicity assumption obtained by PDE methods. See [4, 23] for probabilistic results under different conditions.

The paper is organized as follows. In Section 2 we prove Liouville properties a bit more general than (1.2) (the sub- and supersolution can be unbounded, provided they are controlled at infinity by the Lyapunov function) for possibly degenerate HJB operators, and then refine the results for Pucci extremal operators plus lower order terms. Section 3 is devoted to general uniformly elliptic operators and Section 4 to quasilinear HJB inequalities with hypoelliptic principal part. In Section 5 we study the stabilization in space for large times of solutions to fully nonlinear parabolic equations. Finally, Section 6 deals with the unique solvability of the ergodic HJB equation.

## 2. Hamilton-Jacobi-Bellman operators

### 2.1. General HJB operators.

We begin with the concave operators

$$L^\alpha u := tr(a(x,\alpha)D^2u) + b(x,\alpha)\cdot Du, \qquad G[u] := \inf_{\alpha\in A}\{-L^\alpha u + c(x,\alpha)u\}, \tag{2.1}$$

where the coefficients $a, b, c$ are defined in $\mathbb{R}^N \times A$ and are at least continuous, $A$ is a metric space, and $tr$ denotes the trace. Throughout the paper sub- and supersolutions are meant in the viscosity sense. We assume the following conditions:

1. $F(x,t,p,X) = \inf_{\alpha\in A}\{-tr(a(x,\alpha)X) - b(x,\alpha)\cdot p + c(x,\alpha)t\}$ is continuous in $\mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \times S^N$, $c \geq 0$, and $G$ satisfies the Comparison Principle in any bounded open set $\Omega$, i.e., if $u, v$ are, respectively, a sub- and a supersolutions of $G[u] = 0$ in $\Omega$ and $u \leq v$ on $\partial\Omega$, then $u \leq v$ in $\Omega$;
2. $G$ satisfies the Strong Maximum Principle, i.e., any viscosity subsolution in $\mathbb{R}^N$ that attains an interior nonnegative maximum must be constant;
3. there exist $R_o \geq 0$ and $w \in LSC(\mathbb{R}^N)$ such that $G[w] \geq 0$ for $|x| > R_o$ and $\lim_{|x|\to\infty} w(x) = +\infty$.

Sufficient conditions for (1) are well-known and will be recalled later in this section (a general reference is [24]). Sufficient conditions for (2) can be found in [10, 11], we will use the strict ellipticity (2.8) in this section and some form of hypoellipticity in Section 4.

**Theorem 2.1.** *Assume* (1), (2) *and* (3)*. Let* $u \in USC(\mathbb{R}^N)$ *satisfy* $G[u] \leq 0$ *in* $\mathbb{R}^N$ *and*

$$\limsup_{|x|\to\infty} \frac{u(x)}{w(x)} \leq 0. \tag{2.2}$$

*If either* $u \geq 0$ *or* $c(x,\alpha) \equiv 0$ *for every* $x, \alpha$*, then* $u$ *is constant.*

*Proof.* We divide the proof in various steps.

**Step 1** . Define $u_\eta(x) := u(x) - \eta w(x)$, for $\eta > 0$. Possibly increasing $R_o$ we can assume that $u$ is not constant in the ball $\{x \mid |x| \leq R_o\}$, otherwise we are done. Set

$$C_\eta := \max_{|x|\leq R_o} u_\eta(x).$$

First of all we show that under our assumptions, $G[C_\eta] \geq 0$ for every $\eta$ sufficiently small.

Indeed, if $c(x,\alpha) \equiv 0$ then necessarily $G[C_\eta] = 0$. On the other hand, if $c \not\equiv 0$ then $u \geq 0$ and in this case we can assume that for $\eta$ sufficiently small $C_\eta > 0$. In fact, if this were not the case, we could conclude letting $\eta \to 0$ that $u(x) = 0$ for every $|x| \leq R_o$, in contradiction with the fact that we assumed that $u$ is not constant in the ball $\{x \mid |x| \leq R_o\}$.

**Step 2** The growth condition (2.2) implies that $\limsup_{|x|\to\infty} \frac{u_\eta(x)}{w(x)} \leq -\eta < 0$ for all $\eta > 0$, so $\lim_{|x|\to\infty} u_\eta(x) = -\infty$. Then for all $\eta > 0$ there exists $M_\eta > R_o$ such that

$$u_\eta(x) \leq C_\eta \quad \text{ for all } |x| \geq M_\eta.$$

**Step 3** . We prove that for all $\eta > 0$, $u_\eta$ satisfies $G[u_\eta] \leq 0$ in $\{x \mid |x| > R_o\}$.

Fix $\overline{x}$, $|\overline{x}| > R_o$, and a smooth function $\phi$ such that $u_\eta(\overline{x}) - \phi(\overline{x}) = 0$ and $u_\eta - \phi$ has a strict maximum at $\overline{x}$.

Assume by contradiction that $G[\phi(\overline{x})] > 0$. Let $\delta > 0$ sufficiently small such that $G[\phi(\overline{x}) - \delta] > 0$. Indeed since $G[\phi(\overline{x})] = F(\overline{x}, \phi(\overline{x}), D\phi(\overline{x}), D^2\phi(\overline{x})) > 0$, by continuity of $F$, stated in assumption (1), there exists $\delta > 0$ such that $F(\overline{x}, \phi(\overline{x}) - \delta, D\phi(\overline{x}), D^2\phi(\overline{x})) > 0$. So, using again continuity of $F$ and regularity of $\phi$ we get that there exists $0 < r < |\overline{x}| - R_o$ such that $G[\phi - \delta] > 0$ in $B(\overline{x}, r)$.

Since $u_\eta - \phi$ has a strict maximum at $\overline{x}$, there exists $0 < k < \delta$ such that $u_\eta - \phi \leq -k < 0$ on $\partial B(\overline{x}, r)$.

Moreover, we claim that $\eta w+\phi-k$ satisfies $G[\eta w+\phi-k]\geq 0$ in $B(\overline{x},r)$. Indeed take $\tilde{x}\in B(\overline{x},r)$ and $\psi$ smooth such that $\eta w+\phi-k-\psi$ has a minimum at $\tilde{x}$. Using the fact that $w$ is a viscosity supersolution, (3), we get

$$\begin{array}{rcl} 0 & \leq & G[\psi(\tilde{x})-\phi(\tilde{x})+k]=\inf_{\alpha\in A}\{-L^\alpha\psi(\tilde{x})+L^\alpha\phi(\tilde{x})+c(\tilde{x},\alpha)(\psi(\tilde{x})-\phi(\tilde{x})+k)\} \\ & \leq & \inf_{\alpha\in A}\{-L^\alpha\psi(\tilde{x})+c(\tilde{x},\alpha)\psi(\tilde{x})\}-\inf_{\alpha\in A}\{-L^\alpha\phi(\tilde{x})+c(\tilde{x},\alpha)(\phi(\tilde{x})-\delta)\} \\ & = & G[\psi(\tilde{x})]-G[\phi(\tilde{x})-\delta]<G[\psi(\tilde{x})]. \end{array}$$

Therefore $G[\psi(\tilde{x})]\geq 0$, which implies that $G[\eta w+\phi-k]\geq 0$ in $B(\overline{x},r)$.

Since $u\leq \eta w+\phi-k$ on $\partial B(\overline{x},r)$, we can now apply the Comparison Principle and get $u\leq\eta w+\phi-k$ in $B(\overline{x},r)$, in contradiction with the fact that $u(\overline{x})=\eta w(\overline{x})+\phi(\overline{x})$.

**Step 4** . Now we use the Comparison Principle in $\Omega=\{x: R_o<|x|<M_\eta\}$. Since $G[C_\eta]\geq 0$, by Step 1, we get $u_\eta(x)\leq C_\eta$ in $\Omega$, using Step 2 and 3. Therefore

$$u_\eta(x)\leq C_\eta \quad \text{ for all } |x|\geq R_o.$$

By letting $\eta\to 0^+$ we obtain that

$$u(x)\leq \max_{|y|\leq R_o} u(y)$$

and then $u$ attains its maximum $\overline{x}$ over $\mathbb{R}^N$. Now if $u\geq 0$ the Strong Maximum Principle gives the desired conclusion. If, on the other hand $c(x,\alpha)\equiv 0$, then we substitute $u$ with $u+|u(\overline{x})|$ and we conclude. ☐

Now we turn to the study of Liouville properties for supersolutions and we substitute assumption (2) with the following

(2′) $G$ satisfies the Strong Minimum Principle, i.e., any viscosity supersolution in $\mathbb{R}^N$ that attains an interior nonpositive minimum must be constant.

**Remark 2.1.** Let $v\in LSC(\mathbb{R}^N)$ satisfy $G[v]\geq 0$ and $\liminf_{|x|\to\infty}\frac{v(x)}{w(x)}\geq 0$. Assume (1), (2′) and (3) where the condition $G[w]\geq 0$ can be replaced by the much weaker requirement that $-L^\alpha w+c(x,\alpha)w\geq 0$ for some $\alpha$. Then an argument similar to the proof of Theorem 2.1 gives that, if either $v\leq 0$ or $c(x,\alpha)\equiv 0$ for every $x,\alpha$, then $v$ is a constant.

Consider convex operators of the form

$$\tilde{G}[u]:=\sup_{\alpha\in A}\{-L^\alpha u+c(x,\alpha)u\}, \tag{2.3}$$

where $L^\alpha$ is as in (2.1). The assumption (3) will be replaced with the following

$$\exists R_o\geq 0 \text{ and } W\in USC(\mathbb{R}^N) \text{ such that } \tilde{G}[W]\leq 0 \text{ for } |x|>R_o \text{ and } \lim_{|x|\to\infty}W(x)=-\infty. \tag{2.4}$$

**Theorem 2.2.** *Assume* (1), (2′), (2.4), *and let* $v\in LSC(\mathbb{R}^N)$ *be a supersolution to* $\tilde{G}[v]\geq 0$ *in* $\mathbb{R}^N$, *such that*

$$\liminf_{|x|\to+\infty}\frac{v(x)}{W(x)}\leq 0. \tag{2.5}$$

*If either* $v\leq 0$ *or* $c(x,\alpha)\equiv 0$, *then* $v$ *is constant.*

*Proof.* We consider the function $v_\eta(x)=v(x)-\eta W(x)$. As in Step 1 of the proof of Theorem 2.1, we get that $\tilde{G}[c_\eta]\leq 0$ for $\eta$ sufficiently small, where

$$c_\eta:=\min_{|x|\leq R_o} v_\eta(x).$$

Following Step 2, by (2.5) and $W(x)<0$ for $|x|$ large, we get $\lim_{|x|\to\infty}v_\eta(x)=+\infty$ for all $\eta>0$. Then for all $\eta>0$ there exists $M_\eta>R_o$ such that

$$v_\eta(x)\geq c_\eta \quad \text{ for all } |x|\geq M_\eta.$$

Moreover, arguing as in Step 3 of the same proof, it is possible to show that $\tilde{G}[v_\eta]\geq 0$ for $|x|>R_o$. So, as in Step 4, we use the Comparison Principle to conclude that $v_\eta(y)\geq c_\eta$ for $|y|\geq R_o$. Finally, we let $\eta \to 0$ and get $v(y) \geq \min_{|x|\leq R_o} v(x)$ for $|y| \geq R_o$. From this we deduce, using the Strong Minimum Principle, that $v$ is constant. $\square$

Let us recall some standard conditions on the coefficients of $L^\alpha$ that imply (1), (2) and (2′): $a(x,\alpha) = \sigma(x,\alpha)\sigma(x,\alpha)^T$ for some $N \times m$ matrix-valued function $\sigma$ and

$$\forall R > 0\, \exists K_R \text{ such that } \sup_{|x|\leq R} (|\sigma| + |b| + |c|) \leq K_R, \tag{2.6}$$
$$\sup_{|x|,|y|\leq R, \alpha\in A} (|\sigma(x,\alpha) - \sigma(y,\alpha)| + |b(x,\alpha) - b(y,\alpha)|) \leq K_R|x-y|;$$

$$c(x,\alpha) \geq 0 \text{ and } c \text{ is continuous in } x \text{ uniformly in } |x| \leq R, \alpha \in A; \tag{2.7}$$

$$\xi^T a(x,\alpha)\xi \geq |\xi|^2/K_R \quad \forall \xi \in \mathbb{R}^N,\ |x| \leq R. \tag{2.8}$$

**Corollary 2.3.** *Assume the operators $L^\alpha$ satisfy* (2.6), (2.7), (2.8), *and*

$$\sup_{\alpha\in A} (tr\, a(x,\alpha) + b(x,\alpha)\cdot x - c(x,\alpha)|x|^2/2) \leq 0 \quad \textit{for } |x| \geq R_o. \tag{2.9}$$

*Let $u \in USC(\mathbb{R}^N)$ be a viscosity subsolution to $G[u] \leq 0$ such that $\limsup_{|x|\to+\infty} \frac{u(x)}{|x|^2} \leq 0$. Assume either that $u \geq 0$ or $c(x,\alpha) \equiv 0$, then $u$ is a constant.*
*Let $v \in LSC(\mathbb{R}^N)$be a viscosity supersolution to $\tilde{G}[v] \geq 0$ such that $\liminf_{|x|\to+\infty} \frac{v(x)}{|x|^2} \geq 0$. Assume either that $v \leq 0$ or $c(x,\alpha) \equiv 0$, then $v$ is a constant.*

*Proof.* Note that $G$, $\tilde{G}$ are uniformly elliptic in any bounded set by (2.8). Then (2.6), (2.7), and (2.8) imply the Comparison Principle on bounded sets (1), see [33] or [12]. Moreover the Strong Maximum Principle (2) for $G$ and the the Strong Minimum Principle (2′) for $\tilde{G}$ hold by Corollary 2.7 of [11].

Next we check the properties (3) by choosing $w(x) = |x|^2/2$. Since

$$L^\alpha w = tr\, a(x,\alpha) + b(x,\alpha)\cdot x$$

(2.9) implies $\inf_{\alpha\in A}\{-L^\alpha w + c(x,\alpha)|x|^2/2\} \geq 0$ for $|x| \geq R_o$. Note that choosing $W(x) = -|x|^2/2$, the same computation gives that (2.9) implies (2.4).

Thus Theorem 2.1 and Theorem 2.2 give the conclusion. $\square$

**Remark 2.2.** If $c$ is bounded away from 0 for $|x|$ large, condition (2.9) is satisfied if $a = o(|x|^2)$ and $b = o(|x|)$ as $x \to \infty$. If $b$ is bounded and $a = o(|x|)$, $c$ can vanish as $x \to \infty$ provided $c(x,\alpha) \geq c_o/|x|$ with $c_o > 0$.

**Remark 2.3.** The condition

$$\limsup_{|x|\to\infty} \sup_{\alpha\in A} (tr\, a(x,\alpha) + b(x,\alpha)\cdot x) < 0 \tag{2.10}$$

is sufficient for (2.9) in view of (2.7); it means that the vector field $b$ points toward the origin for $|x|$ large enough and all $\alpha$, and its inward component is large enough compared to the diffusion matrix $a$. It is satisfied if $a = o(|x|^2)$ and the drift $b$ is a controlled perturbation of a mean reverting drift of Ornstein-Uhlenbeck type, that is, for some $m \in \mathbb{R}^N, \gamma > 0$,

$$b(x,\alpha) = \gamma(m - x) + \tilde{b}(x,\alpha), \qquad \lim_{x\to\infty} \sup_{\alpha\in A} \frac{\tilde{b}(x,\alpha)\cdot x}{|x|^2} = 0. \tag{2.11}$$

More generally, (2.10) holds if there exist $\delta \geq 0, \gamma > 0$, and $a_o < \gamma$ such that

$$\sup_{\alpha\in A} b(x,\alpha)\cdot x = -\gamma|x|^\delta + o(|x|^\delta), \quad \sup_{\alpha\in A} tr\, a(x,\alpha) \leq a_o|x|^\delta + o(|x|^\delta),$$

as $|x| \to \infty$.

2.2. **Pucci extremal operators.** Important examples of uniformly elliptic HJB operators are the Pucci extremal operators. In particular, the minimal operator $\mathcal{M}^-$ has the form (2.1), i.e., it is concave in $u$. Fix $0 < \lambda \le \Lambda$ and denote with $S^N$ the set of $N \times N$ symmetric matrices. For any $X \in S^N$ define

$$\mathcal{M}^-(X) := \inf\{-\mathrm{tr}(MX) : M \in S^N, \lambda I \le M \le \Lambda I\} \tag{2.12}$$

The operator can also be written as

$$\mathcal{M}^-(X) = -\Lambda \sum_{e_i>0} e_i - \lambda \sum_{e_i<0} e_i,$$

where $e_i$ are the eigenvalues of $X$.

The Pucci maximal operator $\mathcal{M}^+$ is defined as

$$\mathcal{M}^+(X) := \sup\{-\mathrm{tr}(MX) : M \in S^N, \lambda I \le M \le \Lambda I\} = -\lambda \sum_{e_i>0} e_i - \Lambda \sum_{e_i<0} e_i. \tag{2.13}$$

Next we prove the Liouville property for subsolutions of the equation

$$\mathcal{M}^-(D^2u) + H(x,u,Du) = 0, \qquad \text{in } \mathbb{R}^N, \tag{2.14}$$

where

$$H(x,t,p) := \inf_{\alpha\in A}\{c(x,\alpha)t - b(x,\alpha)\cdot p\}, \tag{2.15}$$

and supersolutions to

$$\mathcal{M}^+(D^2u) + \tilde{H}(x,u,Du) = 0, \qquad \text{in } \mathbb{R}^N, \tag{2.16}$$

where

$$\tilde{H}(x,t,p) := \sup_{\alpha\in A}\{c(x,\alpha)t - b(x,\alpha)\cdot p\}.$$

In both cases we assume that the data $b, c$ satisfy conditions (2.6) and (2.7).

For these operators the condition (2.9) reads

$$\sup_{\alpha\in A}(b(x,\alpha)\cdot x - c(x,\alpha)|x|^2/2) \le -N\Lambda \qquad \text{for } |x| \ge R_o.$$

The next result improves a bit Corollary 2.3 in this case, for sub- and supersolutions with growth at infinity controlled by $\log|x|$.

**Corollary 2.4.** *Under the previous conditions on $H$ assume*

$$\sup_{\alpha\in A}(b(x,\alpha)\cdot x - c(x,\alpha)|x|^2\log(|x|)) \le \lambda - (N-1)\Lambda \qquad \textit{for } |x| \ge R_o. \tag{2.17}$$

- *Let $u \in USC(\mathbb{R}^N)$ be a subsolution of* (2.14) *such that* $\limsup_{|x|\to+\infty} \frac{u(x)}{\log|x|} \le 0$. *Assume that either $c(x,\alpha) \equiv 0$ or $u \ge 0$, then $u$ is a constant.*
- *Let $v \in LSC(\mathbb{R}^N)$ be a supersolution of* (2.16) *such that* $\liminf_{|x|\to+\infty} \frac{v(x)}{\log|x|} \ge 0$. *Assume that either $c(x,\alpha) \equiv 0$ or $v \le 0$, then $v$ is a constant.*

*Proof.* We have to check the property (3) (and (2.4)) and we choose $w(x) = \log(|x|)$ (resp. $W(x) = -\log|x|$). We recall that for a function of the form $w(x) = \Phi(|x|)$ with $\Phi : (0,+\infty) \to \mathbb{R}$ of class $C^2$ the eigenvalues of $D^2w(x)$, $x \ne 0$, are $\Phi''(|x|)$, which is simple, and $\Phi'(|x|)/|x|$ with multiplicity $N-1$, see Lemma 3.1 of [25]. Since log is increasing and concave we get

$$M^-(D^2w) = -\Lambda\frac{N-1}{|x|^2} + \frac{\lambda}{|x|^2} \qquad M^+(D^2W) = \Lambda\frac{N-1}{|x|^2} - \frac{\lambda}{|x|^2}.$$

Thus $w$ is a supersolution of (2.14) at all points where

$$\frac{\lambda}{|x|^2} - \Lambda\frac{N-1}{|x|^2} + \inf_{\alpha\in A}\left\{c(x,\alpha)\log(|x|) - \frac{b(x,\alpha)\cdot x}{|x|^2}\right\} \ge 0,$$

and this inequality holds for $|x| \ge R_o$ under condition (2.17). Analogously one checks that condition (2.17) implies that $W$ is a subsolution to (2.16) for $|x| \ge R_o$. Therefore Theorem 2.1 and Theorem 2.2 give the conclusion. $\square$

**Remark 2.4.** A condition that implies (2.17) and therefore the Liouville property is the following

$$\limsup_{|x|\to\infty} \sup_{\alpha\in A} b(x,\alpha)\cdot x < \lambda - (N-1)\Lambda,$$

because $c \geq 0$.

## 3. Uniformly elliptic operators

In this section we consider fully nonlinear equations of the general form

$$F(x, u, Du, D^2u) = 0, \qquad \text{in } \mathbb{R}^N, \tag{3.1}$$

where $F : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \times S^N \to \mathbb{R}$ is continuous and uniformly elliptic, namely, there exist constants $0 < \lambda \leq \Lambda$ such that

$$\lambda tr(Q) \leq F(x,t,p,X) - F(x,t,p,X+Q) \leq \Lambda tr(Q), \tag{3.2}$$

for all $x, p \in \mathbb{R}^N$, $t \in \mathbb{R}$, $X, Q \in S^N$, $Q \geq 0$. Then it is well-known that

$$\mathcal{M}^-(X) \leq F(x,t,p,X) - F(x,t,p,0) \leq \mathcal{M}^+(X).$$

Now assume that

$$F(x,t,p,0) \geq H(x,t,p) \tag{3.3}$$

for some concave Hamiltonian of the form (2.15). Then Corollary 2.4 immediately gives the following.

**Corollary 3.1.** *Assume* (3.2) *and* (3.3) *with $H$ given by* (2.15) *and $b, c$ satisfying* (2.6), (2.7)*, and* (2.17)*. Let $u \in USC(\mathbb{R}^N)$ be a subsolution of* (3.1) *such that* $\limsup_{|x|\to+\infty} \frac{u(x)}{\log|x|} \leq 0$*. Assume that either $c(x,\alpha) \equiv 0$ or $u \geq 0$, then $u$ is a constant.*

*Proof.* It is enough to observe that $u$ satisfies

$$\mathcal{M}^-(D^2u) + H(x,u,Du) \leq 0 \quad \text{ in } \mathbb{R}^N$$

and apply Corollary 2.4. □

The analogous statement holds for supersolutions.

**Corollary 3.2.** *Assume* (3.2) *and*

$$F(x,t,p,0) \leq \tilde{H}(x,t,p) := \sup_{\alpha\in A}\{c(x,\alpha)t - b(x,\alpha)\cdot p\},$$

*where $b, c$ satisfies and $b, c$ satisfying* (2.6), (2.7)*, and* (2.17)*. Let $v \in LSC(\mathbb{R}^N)$ be a supersolution of* (3.1) *such that* $\liminf_{|x|\to+\infty} \frac{v(x)}{\log|x|} \geq 0$*. Assume that either $c(x,\alpha) \equiv 0$ or $v \leq 0$, then $v$ is a constant.*

We specialize the last corollaries to a class of examples that are useful for comparing with the existing literature. Assume

$$F(x,t,p,0) \geq -\overline{b}(x)\cdot p - g(x)|p| + \overline{c}(x)t \tag{3.4}$$

where $\overline{b} : \mathbb{R}^N \to \mathbb{R}^N$ and $g : \mathbb{R}^N \to \mathbb{R}$ are locally Lipschitz, $\overline{c} : \mathbb{R}^N \to \mathbb{R}$ is continuous, and

$$g \geq 0, \qquad \overline{c} \geq 0. \tag{3.5}$$

**Corollary 3.3.** *Assume* (3.2), (3.4), (3.5)*, and*

$$\overline{b}(x)\cdot x + g(x)|x| \leq \overline{c}(x)|x|^2\log(|x|) + \lambda - (N-1)\Lambda \quad \textit{for } |x| \geq R_o. \tag{3.6}$$

*Let $u \in USC(\mathbb{R}^N)$ be a subsolution of* (3.1) *such that* $\limsup_{|x|\to+\infty} \frac{u(x)}{\log|x|} \leq 0$*. Assume that either $\overline{c} \equiv 0$ or $u \geq 0$, then $u$ is a constant.*

*Proof.* We observe that $-|p| = \min_{|\alpha|=1}[-\alpha\cdot p]$, so the right hand side of (3.4) can be written in the form of a concave Hamiltonian (2.15) with $b(x,\alpha) = \overline{b}(x) + g(x)\alpha$, $A = \{\alpha \in \mathbb{R}^N : |\alpha| = 1\}$, and $c(x,\alpha) = \overline{c}(x)$. Then condition (2.17) becomes (3.6), (2.6) and (2.7) are satisfied, and the conclusion follows from Corollary 3.1. □

Analogously we get the result for supersolutions.

**Corollary 3.4.** *Assume* (3.2),

$$F(x,t,p,0) \le -\bar{b}(x)\cdot p + g(x)|p| + \bar{c}(x)t$$

*where* $\bar{b}, g, \bar{c}$ *are as in Corollary 3.3 and satisfy* (3.6)*. Let* $v \in LSC(\mathbb{R}^N)$ *be a supersolution of* (3.1) *such that* $\liminf_{|x|\to+\infty} \frac{v(x)}{\log|x|} \ge 0$*. Assume that either* $\bar{c} \equiv 0$ *or* $v \le 0$*, then* $v$ *is a constant.*

**Remark 3.1.** A set of sufficient conditions for (3.6), containing the case that $\bar{b}$ is the drift of an Ornstein-Uhlenbeck process, is the following: there exist $\delta \ge 0, \gamma > 0$, such that

$$\bar{b}(x)\cdot x = -\gamma|x|^\delta + o(|x|^\delta), \quad g(x) = o(|x|^{\delta-1}), \text{ as } |x|\to\infty.$$

Finally, (3.6) holds also if, instead,

$$\liminf_{|x|\to\infty} \bar{c}(x) > 0, \quad |\bar{b}(x)|, g(x) = o(|x|\log(|x|)), \text{ as } |x|\to\infty.$$

## 4. Quasilinear hypoelliptic operators

In this section we consider equations of the form

$$-tr(a(x)D^2u) + \inf_{\alpha\in A}\{-b(x,\alpha)\cdot Du + c(x,\alpha)u\} = 0, \qquad \text{in } \mathbb{R}^N, \tag{4.1}$$

$$-tr(a(x)D^2u) + \sup_{\alpha\in A}\{-b(x,\alpha)\cdot Du + c(x,\alpha)u\} = 0, \qquad \text{in } \mathbb{R}^N, \tag{4.2}$$

where $a(x) = \sigma(x)\sigma^T(x)$ for some locally Lipschitz $N\times m$ matrix $\sigma = (\sigma_{ij})$ and the coefficients $b, c$ satisfy the conditions (2.6) and (2.7).

Instead of the uniform ellipticity (2.8) we assume first

$$\forall\, R > 0 \quad \text{either} \quad \inf_{|x|\le R, \alpha\in A} c(x,\alpha) > 0 \quad \text{or} \quad \exists i : \inf_{|x|\le R} \sum_{j=1}^m \sigma_{ij}^2(x) > 0, \tag{4.3}$$

which will ensure the Comparison Principle on bounded sets.

Sufficient conditions for the Strong Maximum Principle can be given by means of *subunit vector fields* $\tau$ for the matrix $a$, namely, $\tau : \mathbb{R}^N \to \mathbb{R}^N$ such that $\xi^T a(x)\xi \ge (\tau(x)\cdot\xi)^2$ for all $\xi \in \mathbb{R}^N$. Of course each column of $\sigma$ is a subunit vector field, but also $\eta a_j$, where $a_j$ is the $j$-th column of the matrix $a$ and $\eta > 0$ is small enough, see, e.g., [11]. The second assumption will be

$$\begin{aligned} &\text{there exist subunit vector fields } \tau_j,\ j = 1,\dots,n, \text{ of class } C^\infty \\ &\qquad\qquad \text{and generating a Lie algebra of full rank } N \text{ at each point } x \in \mathbb{R}^N. \end{aligned} \tag{4.4}$$

This classical condition of Hörmander can be weakened: see Remark 4.2 after the next result.

**Corollary 4.1.** *Let the previous assumptions and*

$$|\sigma(x)|^2 + \sup_{\alpha\in A}(b(x,\alpha)\cdot x - c(x,\alpha)|x|^2/2) \le 0 \quad \textit{for } |x| \ge R_o, \tag{4.5}$$

*hold.*

- *Let* $u \in USC(\mathbb{R}^N)$ *be a subsolution of* (4.1) *such that* $\limsup_{|x|\to+\infty} \frac{u(x)}{|x|^2} \le 0$*. If either* $c(x,\alpha) \equiv 0$ *or* $u \ge 0$*, then* $u$ *is a constant.*
- *Let* $v \in LSC(\mathbb{R}^N)$ *be a supersolution of* (4.2) *such that* $\liminf_{|x|\to+\infty} \frac{v(x)}{|x|^2} \ge 0$*. If either* $c(x,\alpha) \equiv 0$ *or* $v \le 0$*, then* $v$ *is a constant.*

*Proof.* The Comparison Principle in bounded sets under the first condition in (4.3) is standard [24], whereas under the second condition it is Corollary 4.1 in [12].

The assumption (4.4) implies the Strong Maximum Principle and the Strong Minimum Principle for both equations (4.1) and (4.2) by the results of [11].

Finally, it is easy to see that $tr\, a(x) = |\sigma(x)|^2$, where $|\sigma|$ denotes the Euclidean norm of the matrix $\sigma$. Then (4.5) is equivalent to (2.9) and so $w(x) = |x|^2/2$ is a supersolution of (4.1) as in the proof of Corollary 2.3, whereas $W(x) = -|x|^2/2$ is a subsolution to (4.2). Thus Theorem 2.1 and Theorem 2.2 give the conclusions. □

**Remark 4.1.** The second condition in (4.3) is satisfied in many classical examples of subelliptic operators, e.g., if the columns of the matrix $\sigma$ are the generators of a Carnot group, see [15]. It can be further relaxed to $\inf_{|x|\le R}\sum_{i=1}^N\sum_{j=1}^m \sigma_{ij}^2(x) > 0$ provided that there exist vector fields $\tilde{b} : \mathbb{R}^N \times A \to \mathbb{R}^m$ such that $b(x,\alpha) = \sigma(x)\tilde{b}(x,\alpha)$, see Corollary 4.1 in [12].

**Remark 4.2.** The general sufficient condition for the Strong Maximum Principle originating in Bony's work and extended to nonlinear operators in [11] is the following. Suppose there exist Lipschitz continuous subunit vector fields $\tau_j$, $j = 1,\dots,n$, and consider the control system

$$\dot{y}(t) = \sum_{j=1}^n \beta_j(t)\tau_j(y(t)),$$

where the control $\beta_j$ take values in a compact neighborhood $B$ of the origin. Assume that each $x \in \mathbb{R}^N$ has a neighborhhod such that all points can be reached by a trajectory of the system starting at $x$, i.e., there exists $r > 0$ such that for all $z$ with $|z - x| < r$ there are measurable $\beta_j : [0,+\infty) \to B$ for which the solution of the system with $y(0) = x$ satisfies $y(t) = z$ for some $t > 0$. Then the strong maximum and minimum principles hold for (4.1) and (4.2). The Hörmander condition (4.4) is sufficient for this reachability property but not necessary. In particular, the smoothness of the vector fields can be relaxed.

**Remark 4.3.** Fully nonlinear HJB equations involving hypoelliptic operators $L^\alpha$ can also be considered. Sufficient conditions for the Strong Maximum Principle are given in [11], but they are not as explicit as (4.4) or the condition described in the preceding Remark 4.2. They still concern a reachability property of a control system, but instead of a deterministic one it is either a diffusion process or a deterministic differential game, and therefore the formulation of such conditions is more technical.

## 5. Large-time stabilization in parabolic equations

We consider the operators with continuous coefficients (2.1) and (2.3) introduced in Section 2. For functions $u : [0,+\infty) \times \mathbb{R}^N \to \mathbb{R}$ we denote with $Du = D_x u$ and $D^2 u = D_x^2 u$ the first and second partial derivatives of $u$ with respect to the space variables.

**Corollary 5.1.** *Assume $G$ satisfies the conditions* (1), (2), (3). *If $u \in USC([0,+\infty) \times \mathbb{R}^N)$ satisfies*

$$u_t + G[u] \le 0 \qquad \textit{in } \mathbb{R}^N \times (0,+\infty),$$
$$\limsup_{|x|\to+\infty} \frac{u(t,x)}{w(x)} \le 0 \qquad \textit{uniformly in } t \in [0,+\infty),$$

*and either $c \equiv 0$ or $u \ge 0$, then*

$$\limsup_{t\to+\infty, y\to x} u(t,y) =: \overline{u}(x)$$

*is constant with respect to $x$.*

*Proof.* Consider the rescaled function $v_\eta(t,x) := u(t/\eta, x)$ and note that it is a subsolution of

$$\eta\frac{\partial v_\eta}{\partial t} + G[v_\eta] \le 0 \qquad \text{in } \mathbb{R}^N \times (0,+\infty).$$

By the stability of viscosity subsolutions, the function

$$\overline{v}(t,x) := \limsup_{\eta\to 0, s\to t, y\to x} v_\eta(s,y)$$

is a subsolution of $G[\overline{v}] \le 0$ in $\mathbb{R}^N \times (0,+\infty)$. On the other hand, by the very definitions, $\overline{v}(t,x) = \overline{u}(x)$ for any $t > 0$. Moreover, it is easy to check that $\limsup_{|x|\to+\infty} \frac{\overline{u}(x)}{w(x)} \le 0$ and that, if $u \ge 0$, also $\overline{u} \ge 0$. Then

$$G[\overline{u}] \le 0 \qquad \text{in } \mathbb{R}^N,$$

and we can use Theorem 2.1 to conclude that $\overline{u}$ is a constant. □

**Remark 5.1.** It is immediate to prove the analogous for supersolution. Assume $\tilde{G}$ satisfies (1), (2′) and (2.4). Let $v$ be a LSC supersolutions of $u_t + \tilde{G}[u] \geq 0$, such that $\liminf_{|x|\to+\infty} \frac{v(t,x)}{W(x)} \leq 0$ uniformly in $t$. Assume moreover that either $c \equiv 0$ or $v \leq 0$. Then $\liminf_{t\to+\infty, y\to x} u(t,y)$ is a constant.

Now we consider the Cauchy problem

$$\begin{cases} u_t + F(x, Du, D^2u) = 0 & \text{in } (0,+\infty)\times\mathbb{R}^N \\ u(0,x) = h(x) & \text{in } \mathbb{R}^n, \end{cases} \tag{5.1}$$

where $F$ is uniformly elliptic and satisfies

$$-b_1(x)\cdot p - g_1(x)|p| \leq F(x,p,0) \leq -b_2(x)\cdot p + g_2(x)|p| \tag{5.2}$$

with $b_i : \mathbb{R}^N \to \mathbb{R}^N$ and $g_i : \mathbb{R}^N \to \mathbb{R}$ bounded and locally Lipschitz, $i = 1,2$.

**Theorem 5.2.** *Assume $F$ satisfies the structural conditions for the comparison principle bewteen a sub- and a supersolution of* (5.1)*, as well as* (3.2)*,* (5.2) *with $g_i \geq 0$ and*

$$b_i(x)\cdot x + g_i(x)|x| \leq \lambda - (N-1)\Lambda \quad \textit{for } |x| \geq R_o, i = 1,2. \tag{5.3}$$

*Suppose also that $h \in BUC(\mathbb{R}^N)$. Then there exist a unique solution $u$ of* (5.1) *Hölder continuous in $[0,+\infty)\times\mathbb{R}^N$ and constants $\overline{u}, \underline{u} \in \mathbb{R}$ such that*

$$\limsup_{t\to+\infty} u(t,x) = \overline{u}, \quad \liminf_{t\to+\infty} u(t,x) = \underline{u}, \quad \textit{for all } x \in \mathbb{R}^N. \tag{5.4}$$

*In particular, if for some $\overline{x}$ the limit $\lim_{t\to+\infty} u(t,\overline{x})$ exists, then $\lim_{t\to+\infty} u(t,x)$ exists for all $x$, it is independent of $x$, and locally uniform.*

*Proof.* We divide the proof in three steps.

**Step 1** . We show that

$$\limsup_{t\to+\infty, y\to x} u(t,y) = \overline{u}(x), \qquad \liminf_{t\to+\infty, y\to x} u(t,y) = \underline{u}(x)$$

are constants, that is $\overline{u}(x) \equiv \overline{u}$ and $\underline{u}(x) \equiv \underline{u}$ for every $x$.

The existence and uniqueness of a solution $u \in BUC([0,+\infty)\times\mathbb{R}^N)$ for all $T > 0$ follows from the comparison principle and Perron's method by standard theory. We must prove global regularity estimates. We will use several times that, by (3.2) and (5.2),

$$\mathcal{M}^-(X) - b_1(x)\cdot p - g_1(x)|p| \leq F(x,p,X) \leq \mathcal{M}^+(X) - b_2(x)\cdot p + g_2(x)|p|. \tag{5.5}$$

First observe that it implies that any constant solves the PDE; consequently, we have the bounds $\inf h \leq u(t,x) \leq \sup h$, for every $x, t$, by the Comparison Principle.

Arguing as in Corollary 5.1. we get that $\limsup_{t\to+\infty, y\to x} u(t,y)$ is a subsolution to

$$\mathcal{M}^-(D^2u) - b_1(x)\cdot Du - g_1(x)|Du| \leq 0$$

and $\liminf_{t\to+\infty, y\to x} u(t,y)$ is a supersolution to

$$\mathcal{M}^+(D^2u) - b_2(x)\cdot Du + g_2(x)|Du| \geq 0.$$

Note that condition (5.3) coincides with (3.6), then we can apply Corollary 3.3 and Corollary 3.4 and conclude that

$$\limsup_{t\to+\infty, y\to x} u(t,y) = \overline{u}(x), \qquad \liminf_{t\to+\infty, y\to x} u(t,y) = \underline{u}(x)$$

are constants, that is $\overline{u}(x) \equiv \overline{u}$ and $\underline{u}(x) \equiv \underline{u}$ for every $x$.

**Step 2**. We show that if $h$ is smooth with bounded first and second derivatives, then the conclusion holds.

We apply the theory of uniformly elliptic equations for $t$ fixed. From the comparison principle we get the estimate

$$|u(t,x) - h(x)| \leq Ct \qquad \text{on } [0,+\infty)\times\mathbb{R}^N,$$

for the constant $C := \sup_x |F(x, Dh, D^2h)|$. By applying again the comparison principle we obtain

$$|u(t+s,x) - u(t,x)| \leq \sup_{x\in\mathbb{R}^N} |u(s,x) - h(x)| \leq Cs \qquad \text{on } [0,+\infty)\times\mathbb{R}^N$$

for all $s > 0$. In particular, we have $|u_t| \le C$ in the viscosity sense. From this, (5.5), and the boundedness of $b_i, g_i$, it is easy to deduce that the partial function $u(t,\cdot)$ satisfies for all $t > 0$

$$\mathcal{M}^-(D^2u) - C_1|Du| \le C, \quad \text{in } \mathbb{R}^N, \tag{5.6}$$

$$\mathcal{M}^+(D^2u) + C_1|Du| \ge -C, \quad \text{in } \mathbb{R}^N \tag{5.7}$$

in the viscosity sense. Then we can apply the estimates of Krylov-Safonov type as stated in Thm. 5.1 of [41]. By (5.6) $u(t,\cdot)$ satisfies a local maximum principle with constants depending only on $N, \lambda, \Lambda, C, C_1$, and $\|h\|_\infty$, whereas by (5.7) $u(t,\cdot)$ satisfies a weak Harnack inequality with constants depending only on the same quantities. The combination of these two estimates with the classical Moser iteration technique implies that the family $u(t,\cdot)$ is equi-Hölder continuous. Since $u$ is Lipschitz continuous in $t$, we conclude that it is Hölder continuous in $[0,+\infty) \times \mathbb{R}^N$. This implies that $\limsup_{t\to+\infty} u(t,x) = \limsup_{t\to+\infty, y\to x} u(t,y)$ and $\liminf_{t\to+\infty} u(t,x) = \liminf_{t\to+\infty,y\to x} u(t,y)$. So, by Step 1, $\limsup_{t\to+\infty} u(t,x) = \overline{u}$, and $\liminf_{t\to+\infty} u(t,x) = \underline{u}$ for every $x \in \mathbb{R}^N$.

**Step 3**. We conclude for general $h \in BUC(\mathbb{R}^N)$.

We mollify $h$ and take a sequence of smooth functions $(h_k)$ with bounded first and second derivatives converging uniformly to $h$. The comparison principle implies that the associated sequence of solutions $(u_k)$ converges uniformly to $u$ on $[0,+\infty) \times \mathbb{R}^N$. Moreover, for each fixed $k$, we proved in Step 2 that $\limsup_{t\to+\infty} u_k(t,x) = \overline{u}_k$ and $\liminf_{t\to+\infty} u_k(t,x) = \underline{u}_k$. Note that both $\overline{u}_k$ and $\underline{u}_k$ are bounded (due to the fact that $(h_k)$ are uniformly bounded), so we can extract a converging subsequence.

Let $t_n \to +\infty$ and $x_n \to x$ such that $\lim_n u(t_n,x_n) = \overline{u}$. By uniform convergence of $u_k$ to $u$ in $[0,+\infty) \times \mathbb{R}^N$, for every $\varepsilon > 0$ there exists $\overline{k}$ such that for every $k \ge \overline{k}$,

$$u_k(t_n,x_n) - \varepsilon \le u(t_n,x_n) \le u_k(t_n,x_n) + \varepsilon \qquad \forall k \ge \overline{k}.$$

Letting $n \to +\infty$ we obtain from the previous inequalities

$$\overline{u} \le \overline{u}_k + \varepsilon \qquad \forall k \ge \overline{k},$$

and then letting $k \to +\infty$, we conclude

$$\overline{u} \le \lim_k \overline{u}_k.$$

Let fix $x$, $k \ge \overline{k}$ and $t_n^k \to +\infty$ such that $\lim_n u_k(t_n^k,x) = \overline{u}_k$. Then there exists $n_k$ such that for every $n \ge n_k$

$$\overline{u}_k \le u_k(t_n^k,x) + \varepsilon \le u(t_n^k,x) + 2\varepsilon.$$

Letting $n \to +\infty$, we get that

$$\overline{u}_k \le \limsup_{t\to+\infty} u(t,x) + 2\varepsilon \le \limsup_{t\to+\infty,y\to x} u(t,y) + 2\varepsilon = \overline{u} + 2\varepsilon.$$

So, letting $k \to +\infty$, we get that $\lim_k \overline{u}_k \le \limsup_{t\to+\infty} u(t,x) \le \overline{u}$. Therefore, we conclude that $\overline{u} = \limsup_{t\to+\infty} u(t,x) = \lim_k \overline{u}_k$.

The same argument gives the result for the lim inf. □

**Remark 5.2.** An example where the last statement of Theorem 5.2 holds true is a linear operator whose coefficients satisfy, for some $R_o, M_o > 0$,

$$tr\, a(x) + b(x) \cdot x \le -M_o \qquad \forall\, |x| > R_o, \tag{5.8}$$

which is equivalent to (2.10) and slightly stronger than (2.9). Then the stochastic process $dX_t = b(X_t)dt + \sqrt{2}\sigma(X_t)dW_t$ generated by the operator $L = tr\, \sigma\sigma^T D^2 + b \cdot D$ is ergodic with a unique invariant probability measure $\mu$, see, e.g., [8]. Moreover $\lim_{t\to+\infty} \mathbb{E}h(X_t) = \int_{\mathbb{R}^N} h(y)d\mu(y)$ locally uniformly in $x = X_0$ (Prop. 4.4 of [8]). Since the solution of the Cauchy problem (5.1) is $u(t,x) = \mathbb{E}h(X_t)$, we have in this case that

$$\overline{u} = \underline{u} = \int_{\mathbb{R}^N} h(y)d\mu(y). \tag{5.9}$$

**Remark 5.3.** Without a dissipativity condition like (5.8) the equality $\overline{u} = \underline{u}$ cannot be true for all bounded initial data $h$, even for the heat equation in dimension $N = 1$, see the example in [22]. For linear equations various authors studied the further averaging properties of $h$ necessary and sufficient for the stabilization to a constant, $\overline{u} = \underline{u}$, see [26] and the references therein.

**Remark 5.4.** For a nonlinear operator $F$ of HJB type one may hope for a formula like (5.9) if an associated optimal control problem with long-time cost or payoff (a so-called ergodic control problem) has an optimal feedback producing an ergodic process with unique invariant measure $\mu$. In principle such a feedback can be synthesized from a stationary HJB equation in $\mathbb{R}^N$ (see next section). So far, this has been done with PDE methods only in some special model problems of the form $F[u] = -\Delta u + |Du|^q + l(x)$ with $q > 1$, see, e.g., [31, 21]. Representation formulas like (5.9) have been obtained by probabilistic methods under appropriate dissipativity conditions on the control system in [4], see also [23]. Results of this kind under our growth assumption (5.2) look considerably harder and are beyond the scope of this paper.

For general operators $F$ of HJB type with all the data $\mathbb{Z}^N$-periodic one can exploit the compactness of the flat torus to show that $\overline{u} = \underline{u}$, although an integral representation as (5.9) of such constant is not available. See [1], where the operators can also be of Isaacs type, i.e., inf sup or sup inf of linear operators. Related results for Ornstein-Uhlenbeck type operators of the form $F[u] = -\Delta u + \alpha x \cdot Du + H(Du) + l(x)$ has been obtained in [30].

## 6. Ergodic HJB equations in $\mathbb{R}^N$

In this section we consider the so-called ergodic HJB equation

$$F(x, D\chi(x), D^2\chi(x)) = c, \qquad x \in \mathbb{R}^N, \tag{6.1}$$

where the unknowns are $(c, \chi) \in \mathbb{R} \times C(\mathbb{R}^N)$, $F$ is of the form

$$F(x,p,X) = \begin{cases} \inf_{\alpha\in A}\{-tr\, a(x,\alpha)X - b(x,\alpha)\cdot p - l(x,\alpha)\} & \forall\, x,p,X, \\ \text{or} \\ \sup_{\alpha\in A}\{-tr\, a(x,\alpha)X - b(x,\alpha)\cdot p - l(x,\alpha)\} & \forall\, x,p,X, \end{cases} \tag{6.2}$$

the coefficients $b, a$ satisfy assumptions (2.6) and (2.8), and the function $l : \mathbb{R}^N \times A \to \mathbb{R}$ is continuous, bounded, and uniformly continuous in $x$, uniformly with respect to $\alpha$.

In order to study the well posedness of (6.1), we need to strengthen assumption (3), by imposing, roughly speaking, that $G[w] \to +\infty$ as $x \to +\infty$, see assumption (6.3) below.

**Theorem 6.1.** *Assume that $F$ is as in* (6.2)*, and that for every $M > 0$ there exists $R > 0$ such that*

$$\sup_{a\in A}\{tr\, a(x,\alpha) + b(x,\alpha)\cdot x\} \le -M \qquad \text{for } |x| \ge R. \tag{6.3}$$

*Then exists a unique constant $c \in \mathbb{R}$ for which* (6.1) *admits a viscosity solution $\chi$ such that*

$$\lim_{|x|\to+\infty} \frac{\chi(x)}{|x|^2} = 0. \tag{6.4}$$

*Moreover $\chi \in C^2(\mathbb{R}^N)$ and is unique up to additive constants among all solutions $v$ to* (6.1) *which satisfy* (6.4).

*Finally, if $a(x,\alpha)$ is bounded in $\mathbb{R}^N \times A$, then $\chi$ is unique up to additive constants also among all solutions $v$ to* (6.1) *with polynomial growth at infinity, that is, for which there exists $k \ge 2$ such that*

$$\lim_{|x|\to+\infty} \frac{v(x)}{|x|^k} = 0.$$

*Proof.* The proof is divided in several steps. For a similar construction in bounded domains with irrelevant boundary we refer to [9] (see also [19]), whereas the periodic case is considered in [5] and [1]. We assume that $F(x,p,X) = \inf_{\alpha\in A}\{-tr\, a(x,\alpha)X - b(x,\alpha)\cdot p - l(x,\alpha)\}$ (the other case can be treated analogoulsy).

**Step 1. For every $h \in (0,1]$ there exists $R_h$ such that**

$$-h\frac{|x|^2}{2} + \min_{|x|\le R_h} u_\delta \le u_\delta(x) \le \max_{|x|\le R_h} u_\delta + h\frac{|x|^2}{2}, \tag{6.5}$$

**where $u_\delta$ is a bounded solution to**

$$\delta u + F(x, Du, D^2u) = 0 \qquad \textbf{in } \mathbb{R}^N, \quad \delta > 0. \tag{6.6}$$

For any $\delta > 0$ consider the value function of a discounted, infinite horizon, stochastic control problem

$$u_\delta(x) := \sup_{\alpha_\cdot \in \mathcal{A}} \mathbb{E}\left[\int_0^{+\infty} e^{-\delta t} l(X_t, \alpha_t)dt\right],$$

where $X_t$ solves

$$dX_t = b(X_t, \alpha_t)dt + \sigma(X_t, \alpha_t)dW_t, \qquad X_0 = x,$$

$W_t$ is an $m-$dimensional Brownian motion, $\mathbb{E}$ is the expectation, and $\mathcal{A}$ denotes the set of admissible controls (i.e., $\alpha_\cdot : [0, +\infty) \to A$ progressively measurable with respect to the filtration associated to $W_\cdot$). It is easy to deduce form the definition that

$$\|u_\delta\|_\infty \le \frac{1}{\delta}\|l\|_\infty. \tag{6.7}$$

Moreover it is known that under the current assumptions $u_\delta$ is continuous and solves (6.6), see, e.g., [29].

Consider $w(x) = |x|^2/2$. Then, we get

$$F(x, Dw, D^2w) \ge -\sup_{a\in A}\{tr\, a(x,\alpha) + b(x,\alpha)\cdot x\} - \|l\|_\infty. \tag{6.8}$$

Fix $h \in (0,1]$ and let $M = \frac{2}{h}\|l\|_\infty + 1$. Choose $R_h > 0$ such that (6.3) holds for such $M$. Then, the function

$$V(x) = h\frac{|x|^2}{2} + \max_{|y|\le R_h} u_\delta$$

is a supersolution to (6.6) in $|x| > R_h$, due to (6.8) and (6.7). Indeed

$$\begin{aligned} \delta V + F(x, DV, D^2V) &\ge \delta h\frac{|x|^2}{2} + \delta \max_{|y|\le R_h} u_\delta - h\sup_{a\in A}\{tr\, a(x,\alpha) + b(x,\alpha)\cdot x\} - \|l\|_\infty \\ &\ge -\|l\|_\infty + Mh - \|l\|_\infty \ge h > 0. \end{aligned} \tag{6.9}$$

Note that $(u_\delta - V)(x) \le 0$ for every $x$ with $|x| \le R_h$, and $\lim_{|x|\to+\infty} u_\delta(x) - V(x) = -\infty$. We claim that $u_\delta(x) - V(x) \le 0$ for every $x$. If it were not the case, there would exist a point $\bar{x}$ such that $|\bar{x}| > R_h$ and $u_\delta(\bar{x}) - V(\bar{x}) = \max u_\delta - V > 0$. But then (6.9) would contradict the fact that $u_\delta$ is a viscosity subsolution to (6.6).

So we get that for every $h \in (0,1]$ there exists $R_h$ such that the second inequality in (6.5) holds, where the first inequality is obtained analogously by considering $v(x) = -h\frac{|x|^2}{2} + \min_{|y|\le R_h} u_\delta$.

**Step 2. The functions $v_\delta = u_\delta - u_\delta(0)$ are equibounded in every compact set $K$.**

Assume by contradiction that there exists $K$ compact such that $(\varepsilon_\delta)^{-1} := \|v_\delta\|_{L^\infty(K)} \to +\infty$. Up to enlarging $K$ we can suppose that $K \supset \{x \mid |x| \le R_1\}$ where $R_1$ has been defined in Step 1.

Define $\psi_\delta = \varepsilon_\delta v_\delta$. Then $\|\psi_\delta\|_{L^\infty(K)} = 1$ and $\psi_\delta(0) = 0$. Moreover, by Step 1, we get that if $x \notin K$, then

$$\psi_\delta(x) \le \frac{\frac{|x|^2}{2} + \max_{|x|\le R_1} u_\delta - u_\delta(0)}{\|u_\delta - u_\delta(0)\|_{L^\infty(K)}} \le 1 + \varepsilon_\delta\frac{|x|^2}{2}$$

and

$$\psi_\delta(x) \ge -1 - \varepsilon_\delta\frac{|x|^2}{2}.$$

Therefore the sequence $\psi_\delta$ is equibounded in every compact subset of $\mathbb{R}^N$. Moreover, since $u_\delta$ solves (6.6), $\psi_\delta$ solves in viscosity sense

$$\delta\psi_\delta + \delta\varepsilon_\delta u_\delta(0) + \inf_{\alpha\in A}\{-tr\, a(x,\alpha)D^2\psi_\delta - b(x,\alpha)\cdot D\psi_\delta - \varepsilon_\delta l(x,\alpha)\} = 0.$$

Since $l$ and $\delta u_\delta(0)$ are bounded (uniformly in $\delta$), we argue as in Step 2 of the proof of Theorem 5.2, and we apply the estimates of Krylov-Safonov type as stated in Thm. 5.1 of [41]. In particular, these imply that the family $\psi_\delta$ is equi-Hölder continuous in every compact set of $\mathbb{R}^N$. Using a diagonal procedure, we can find a sequence $\psi_\delta$ converging locally uniformly in $\mathbb{R}^N$ to $\psi$. Moreover, by stability of viscosity solutions, $\psi$ solves in viscosity sense

$$\inf_{\alpha\in A}\{-tr\, a(x,\alpha)D^2\psi - b(x,\alpha)\cdot D\psi\} \leq 0 \qquad \text{and} \qquad \sup_{\alpha\in A}\{-tr\, a(x,\alpha)D^2\psi - b(x,\alpha)\cdot D\psi\} \geq 0.$$

Moreover, we know that $\|\psi\|_{L^\infty(K)} = 1$ and $|\psi(x)| \leq 1$ for $x \in \mathbb{R}^N \setminus K$ since $|\psi_\delta| \leq 1 + \varepsilon_\delta \frac{|x|^2}{2}$. This implies that $\psi$ attains either a global maximum or a global minimum in $K$, so it is constantly equal to 1 or to $-1$ by the Strong Maximum or Minimum Principle ([10], [11]). This contradicts the fact that $\psi(0) = 0$.

**Step 3. Construction of $c$ and $\chi$ solutions to (6.1).**

Due to (6.7), up to extracting a subsequence, $\delta u_\delta(0)$ converges to $-c$ as $\delta \to 0$. Moreover, by Step 2, $v_\delta$ are equibounded in every compact set of $\mathbb{R}^N$ and are viscosity solutions to

$$\delta v_\delta + \delta u_\delta(0) + F(x, Dv_\delta, D^2 v_\delta) = 0.$$

Using again Krylov-Safonov type estimates, we get that actually $v_\delta$ are equi-Hölder continuous in every compact set of $\mathbb{R}^N$. So, using a diagonal procedure, we can extract a subsequence $v_\delta$ which converges locally uniformly to $\chi$. Moreover by stability of viscosity solutions $c, \chi$ solve (6.1).

**Step 4. Qualitative properites of $\chi$.**

Note that the estimates (6.5) is independent of $\delta$, so it holds also for $\chi$: for every $h \in (0,1]$ there exists $R_h > 0$ such that

$$-h\frac{|x|^2}{2} + \min_{|y|\leq R_h} \chi(y) \leq \chi(x) \leq \max_{|y|\leq R_h} \chi(y) + h\frac{|x|^2}{2}.$$

This implies in particular that $\chi$ satisfies the growth condition (6.4). The regularity of $\chi$ comes from elliptic standard regularity theory, see [41].

**Step 5. Uniqueness of $c$ and of $\chi$ up to additive constants.** Assume that there exist $c_1 \leq c_2$ and two solutions $\chi_1, \chi_2$ to (6.1) with $c = c_1$ and $c = c_2$, respectively, which satisfy (6.4). Then we get

$$\begin{aligned}
&\inf_{\alpha\in A}\{-tr\, a(x,\alpha)(D^2\chi_1 - D^2\chi_2) - b(x,\alpha)\cdot(D\chi_1 - D\chi_2)\} \\
\leq\ &\inf_{\alpha\in A}\{-tr\, a(x,\alpha)D^2\chi_1 - b(x,\alpha)\cdot D\chi_1 - l(x,\alpha)\} \\
-\ &\inf_{\alpha\in A}\{-tr\, a(x,\alpha)D^2\chi_2 - b(x,\alpha)\cdot D\chi_2 - l(x,\alpha)\} = c_1 - c_2 \leq 0,
\end{aligned}$$

where all the equalities and inequalities have to be understood in the viscosity sense. So, by Corollary 2.3 applied to $\chi_1 - \chi_2$, we get that $\chi_1 - \chi_2$ is a constant. This implies in particular that $c_1 = c_2$.

**Step 6. Stronger uniqueness for $a$ bounded.**

Let consider two solutions $\chi_1, \chi_2$ to (6.1) with $c = c_1$ and $c = c_2$ respectively such that there exists $k \geq 2$ with

$$\lim_{|x|\to+\infty} \frac{\chi_i(x)}{|x|^k} = 0.$$

As above, $\chi_1 - \chi_2$ satisfies $\inf_{\alpha\in A}\{-L^\alpha(\chi_1 - \chi_2)\} = c_1 - c_2 \leq 0$.

Consider $w(x) = |x|^k/k$. Then

$$L^\alpha w = |x|^{k-2}\left[tr\, a(x,\alpha) + b(x,\alpha)\cdot x + (k-2)\frac{|\sigma(x,\alpha)\cdot x|^2}{|x|^2}\right].$$

Since $a(x,\alpha)$ is bounded, also $\sigma(x,\alpha)$ is bounded, and then the term $(k-2)\frac{|\sigma(x,\alpha)\cdot x|^2}{|x|^2}$ is bounded. Therefore condition (6.3) implies that there exists $R_o$ such that $L^\alpha w(x) \leq 0$ for $|x| \geq R_o$. So, by the same argument of the proof of Corollary 2.3 applied to the function $|x|^k/k$ instead of $|x|^2/2$, we get that $\chi_1 - \chi_2$ is a constant, and then $c_1 = c_2$. □

**Remark 6.1.** If we strengthen condition (6.3), we can get better estimates on the growth at infinity of the solution $\chi$ to (6.1).

In particular, if we substitute assumption (6.3) with the following: for some $0 < \beta < 2$, for every $M > 0$ there exists $R > 0$ such that

$$\sup_{a\in A}\{tr\, a(x,\alpha) + b(x,\alpha)\cdot x\} \le -M|x|^{2-\beta} \qquad \text{for } |x| \ge R, \tag{6.10}$$

then the same argument of Theorem 6.1, with $w(x) = \frac{|x|^\beta}{\beta}$ in place of $\frac{|x|^2}{2}$, gives that the solution $\chi$ to (6.1) has a strictly sub-quadratic growth at infinity, that is

$$\lim_{|x|\to+\infty} \frac{\chi(x)}{|x|^\beta} = 0. \tag{6.11}$$

In particular, for perturbations of the Ornstein-Uhlenbeck drift as in (2.11) of Remark 2.3 the solution $\chi$ satisfies (6.11) for all $\beta > 0$, so it grows at infinity less than any polynomial.

In the limit case where (6.10) holds with $\beta = 0$ we can use $w(x) = \log|x|$ and get that the solution $\chi$ to (6.1) has sublogarithmic growth at infinity, that is,

$$\lim_{|x|\to+\infty} \frac{\chi(x)}{\log|x|} = 0.$$

**Remark 6.2.** On the other hand, if we weaken assumption (6.3), we get weaker results on the growth at infinity of $\chi$. For example, let us assume that there exist $k > 2$ and $R_o > 0$ such that

$$\sup_{a\in A}\{tr\, a(x,\alpha) + (k-2)\frac{|\sigma(x,\alpha)\cdot x|^2}{|x|^2} + b(x,\alpha)\cdot x\} \le 0 \qquad \text{for } |x| \ge R_o,\ i = 1,2.$$

Then, arguing again as in Theorem 6.1 with $w(x) = |x|^k/k$, we get that the solution $\chi$ to (6.1) satisfies

$$\lim_{|x|\to+\infty} \frac{\chi(x)}{|x|^k} = 0$$

instead of (6.4).

We conclude this section with some results on the possible boundedness of the solution $\chi$ to the ergodic equation (6.1). The next example shows that in general it can be unbounded.

**Example 6.1.** Consider the case of $N = 1$, $A$ a singleton, $b(x) = -x$, $a(x) = 1$ and $l(x) = 2\frac{x^4+2x^2-1}{(x^2+1)^2}$. In this case (6.10) is satisfied for every $\beta < 2$ and the ergodic problem (6.1) reads as follows

$$-\chi'' + x\chi' - 2\frac{x^4+2x^2-1}{(x^2+1)^2} = c.$$

So, by Theorem 6.1 and Remark 6.1, there exists a unique $c$ for which this equation has a solution which satisfies (6.11). It is easy to check that this solution is $c = 0$ and $\chi(x) = \log(1 + x^2)$ up to addition of constants.

On the other hand, the solution $\chi$ to (6.1) is bounded if we strenghten condition (6.10) to the following: there exist $\rho > 0$ and $R > 0$ such that

$$\sup_{a\in A}\{tr\, a(x,\alpha) + b(x,\alpha)\cdot x\} \le -\frac{2|c| + \|l\|_\infty}{\rho}|x|^{2+\rho} \qquad \text{for } |x| \ge R, \tag{6.12}$$

where $c$ is the constant solving the ergodic equation (6.1). This is proved in the following proposition (see also [19] for a similar result in bounded domains).

**Theorem 6.2.** *Let* $(c,\chi)$ *be the solution to* (6.1) *as constructed in Theorem 6.1. If* (6.12) *holds, then*

$$\min_{|y|\le R}\chi(y) + \frac{1}{|x|^\rho} - \frac{1}{R^\rho} \le \chi(x) \le \max_{|y|\le R}\chi(y) - \frac{1}{|x|^\rho} + \frac{1}{R^\rho} \qquad \forall |x| \ge R. \tag{6.13}$$

*In particular,* $\chi \in L^\infty(\mathbb{R}^N)$.

*Proof.* First of all observe that $\chi(x) - ct$ is a solution of

$$u_t + F(x, Du, D^2u) = 0, \qquad \forall x \in \mathbb{R}^N, \ t \in (-\infty, +\infty). \tag{6.14}$$

For $R > 0$ given by assumption (6.12) define $w(x) := R^{-\rho} - |x|^{-\rho}$. Then $w \geq 0$ for $|x| \geq R$ and

$$\sup_{a\in A}\{tr\, a(x,\alpha)D^2w(x) + b(x,\alpha)\cdot Dw(x)\} = \tag{6.15}$$
$$\frac{\rho}{|x|^{\rho+2}} \sup_{a\in A}\left\{tr\, a(x,\alpha) + b(x,\alpha)\cdot x - \frac{\rho+2}{|x|^2}|\sigma(x,\alpha)\cdot x|^2\right\} \leq -2|c| - \|l\|_\infty \qquad |x| > R.$$

Fix now $h \in (0,1)$ and consider $t_h < 0$ such that

$$-|c|t_h \geq \max_{|y|\geq R}\left(\chi(y) - h\frac{|y|^2}{2}\right) - \max_{|z|\leq R}\chi(z), \tag{6.16}$$

where the first maximum exists due to (6.4). Define the function

$$v(t,x) := \max_{|z|\leq R}\chi(z) + h\frac{|x|^2}{2} + R^{-\rho} - |x|^{-\rho} - 2|c|t \qquad t_h \leq t \leq 0, \ x \in \mathbb{R}^N.$$

We claim that $\chi(x) - ct \leq v(t,x)$ for every $t \in [t_h, 0]$ and every $x$ with $|x| \geq R$. First of all observe that the inequality holds at $t = t_h$. Indeed, by our choice of $t_h$, for $|x| \geq R$ we get

$$v(t_h, x) \geq \max_{|z|\leq R}\chi(z) + h\frac{|x|^2}{2} - 2|c|t_h \geq \chi(x) - |c|t_h \geq \chi(x) - ct_h.$$

Moreover, if $|x| = R$ and $t \leq 0$, then $v(t,x) \geq \max_{|z|\leq R}\chi(z) - |c|t \geq \chi(x) - ct$. Now assume by contradiction that $v(s,y) \leq \chi(y) - cs$ for some $|y| \geq R$ and $s \in [t_h, 0]$. Then, using again (6.4), $\max_{|y|\geq R, s\in[t_h,0]}\chi(y) - cs - v(s,y) = \chi(x) - ct - v(t,x) > 0$ for some $|x| > R$ and $t \in (t_h, 0]$. From (6.15) we get

$$\begin{aligned}
v_t(t,x) + F(x, Dv(t,x), D^2v(t,x)) &\geq -2|c| - \left(h + \frac{\rho}{|x|^{\rho+2}}\right)\sup_{a\in A}\{tr\, a(x,\alpha) + b(x,\alpha)\cdot x\} - \|l\|_\infty \\
&\geq -2|c| + 2|c| + \|l\|_\infty + h|x|^{\rho+2}\frac{2|c| + \|l\|_\infty}{\rho} - \|l\|_\infty \\
&\geq h|x|^{\rho+2}\frac{2|c| + \|l\|_\infty}{\rho} > 0,
\end{aligned}$$

which contradicts the fact that $\chi(x) - ct$ is a subsolution to (6.14).

So, in particular, $\chi(x) \leq v(0,x)$, which gives the inequality on the right of (6.13) after letting $h \to 0$. The inequality on the left is obtained similarly, by considering $w(t,x) = \min_{|z|\leq R}\chi(z) - h\frac{|x|^2}{2} - R^{-\rho} + |x|^{-\rho} + 2|c|t$. $\square$

**Remark 6.3.** Assume the matrix $a$ in the operator $F$ has a positive lower bound on the minimal eigenvalue, therefore strenghtening (2.8) to

$$\xi^T a(x,\alpha)\xi \geq \lambda|\xi|^2 \quad \forall \xi, x \in \mathbb{R}^N, \tag{6.17}$$

for some $\lambda > 0$. Note that this implies the first inequality in the uniform ellipticity condition (3.2) for both possible forms of $F$ (6.2). Then the conclusions of Theorem 6.2 remain true if condition (6.12) is replaced by the weaker assumption

$$\sup_{a\in A}\{tr\, a(x,\alpha) + b(x,\alpha)\cdot x - \lambda(2+\rho)\} \leq -\frac{2|c| + \|l\|_\infty}{\rho}|x|^{2+\rho} \quad \text{ for } |x| \geq R. \tag{6.18}$$

In fact, (6.17) implies $|\sigma(x,\alpha)\xi| \geq \lambda|\xi|$, which can be used in (6.15) with (6.18) to get the same conclusion.

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