STRONG SOLUTIONS OF A CONTINUUM MODEL FOR EPITAXIAL GROWTH WITH ELASTICITY ON VICINAL SURFACES

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Abstract. In this paper we analyze the parabolic equation

$$u_t = -\left[H(u_x) + \log(u_{xx} + a) + (u_{xx} + a)^2/2\right]_{xx}, \qquad u(0) = u^0, \quad (1)$$

where $a > 0$ is a given parameter, and H denotes the Hilbert transform. Equation (1) arises from a continuum model for heteroepitaxial growth organizing according to misfit elasticity forces, derived by Xiang (SIAM J. Appl. Math. 63:241–258, 2002), and subsequently studied by Dal Maso, Fonseca and Leoni (Arch. Rational Mech. Anal. 212: 1037–1064, 2014). Then, it was proven by Fonseca, Leoni and the author, that (1) admits a unique weak solution, which is also Lipschitz regular in time (Commun. Part. Diff. Eq. 40(10):1942–1957, 2015). The aim of this paper is to prove existence, uniqueness and regularity of strong solutions of (1).

Keywords: epitaxial growth, vicinal surfaces, evolution equations, Hilbert transform, monotone operators

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1. Introduction

In this paper we study existence, uniqueness, and regularity of strong solutions of

$$u_t = -\left[H(u_x) + \Phi_a'(u_{xx})\right]_{xx}, \qquad u(0) = u^0, \qquad (2)$$

where u^0 is the initial datum, $a > 0$ is a given constant, Φ_a is defined by

$$\Phi_a(\xi) := \Phi(\xi + a),$$

$$\Phi : \mathbb{R} \longrightarrow (-\infty, +\infty], \quad \Phi(\xi) := \begin{cases} +\infty & \text{if } \xi < 0, \\ 0 & \text{if } \xi = 0, \\ \xi \log \xi + \xi^3/6 & \text{if } \xi > 0, \end{cases}$$

and H denotes the Hilbert transform. That is,

$$H(f)(x) := \frac{1}{2\pi} PV \int_I \frac{f(x-y)}{\tan(y/2)}\, \mathrm{d}y, \qquad I := (-\pi, \pi),$$

with PV denoting the Cauchy principal value. Equation (2) arises in the context of heteroepitaxial growth. It was proven, by Dal Maso, Fonseca and Leoni in [7], that (2) is equivalent to

$$h_t = -\left[H(h_x) + \left(\frac{1}{h_x} + h_x\right) h_{xx}\right]_{xx}. \tag{3}$$

The latter is (upon space inversion) the continuum variant derived by Xiang in [14] of the discrete models describing heteroepitaxial growth proposed by Duport, Politi and Villain in [8], and by Tersoff, Phang, Zhang and Lagally in [13]. We refer the interested reader to the related works by Xiang and E [15], and by Xu and Xiang [16].

The choice (in [7]) to study (2) on $I = (-\pi, \pi)$, and then extend to $\mathbb{R}$ by periodicity, is based on the fact that only the function u and its derivatives (and never the space coordinate x alone) appear explicitly in (2). The main advantage to study (2) on $I = (-\pi, \pi)$ is the ability to use Poincaré's inequality.

It was also proven by Dal Maso, Fonseca and Leoni ([7, Theorems 1 and 2]) that (2) admits a weak solution u satisfying some particular variational inequalities. Subsequently, the existence of a Lipschitz regular weak solution of (2) was proven by Fonseca, Leoni and the author [9]. More precisely, [9, Theorem 1] states that given $T, a > 0$, and $u^0 \in W^{2,2}_{\mathrm{per}_0}(I)$ (defined in (6) below) satisfying

- there exists $z^0 \in L^2_{\mathrm{per}_0}(I)$ such that
$$\int_I [z^0 v - H(u^0_{xx})v_x + \Phi_a(v_{xx}) - \Phi_a(u^0_{xx})]\,\mathrm{d}x \geq 0 \tag{4}$$
for any $v \in W^{2,3}_{\mathrm{per}_0}(I)$,

then there exists a unique solution u of (2) in the sense that

$$\int_0^T \int_I u_t(t)\varphi(t)\,\mathrm{d}x\,\mathrm{d}t = \int_0^T \int_I [H(u_{xx}(t))\varphi_x(t) - \Phi'_a(u_{xx}(t))\varphi_{xx}(t)]\,\mathrm{d}x\,\mathrm{d}t \tag{5}$$

for any test function $\varphi \in C^\infty_c((0,T)\times I; \mathbb{R})$. Moreover, the solution u satisfies

$$u \in L^\infty(0,T; W^{2,3}_{\mathrm{per}_0}(I)) \cap C^0([0,T]; L^2_{\mathrm{per}_0}(I)),$$
$$u_t \in L^\infty(0,T; L^2_{\mathrm{per}_0}(I)), \quad u(0) = u^0.$$

Here, and for future reference, given $k \in \mathbb{N}$, $p \in [1, +\infty]$, $W^{k,p}_{\mathrm{per}_0}(I)$ and $L^p_{\mathrm{per}_0}(I)$ are defined by

$$W^{k,p}_{\mathrm{per}_0}(I) := \left\{ f \in W^{k,p}_{\mathrm{loc}}(\mathbb{R}) : f \text{ is } 2\pi\text{-periodic and } \int_I f\,\mathrm{d}x = 0 \right\}, \tag{6}$$
$$L^p_{\mathrm{per}_0}(I) := \left\{ f \in L^p_{\mathrm{loc}}(\mathbb{R}) : f \text{ is } 2\pi\text{-periodic and } \int_I f\,\mathrm{d}x = 0 \right\},$$

and endowed with the standard norm of $W^{k,p}_{\mathrm{loc}}(\mathbb{R})$ and $L^p_{\mathrm{loc}}(\mathbb{R})$ respectively. Moreover, it can be shown, by straightforward computation, that $W^{k,p}_{\mathrm{per}_0}(I)$ and $L^p_{\mathrm{per}_0}(I)$ are reflexive for all $k \in \mathbb{N}$ and $p \in (1,+\infty)$.

It has been suggested by Leoni that, for sufficiently regular initial datum, equation (2) should admit a strong solution. The aim of this paper is to study existence, uniqueness and regularity of strong solutions of (2). The main results are:

Theorem 1. *(Existence and regularity) Given T, $a > 0$, an initial datum $u^0 \in W^{2,2}_{\mathrm{per}_0}(I)$ such that*

$$H(u^0_x)_{xx} + \Phi'_a(u^0_{xx})_{xx} \in L^2_{\mathrm{per}_0}(I), \tag{7}$$

then there exists a strong solution

$$u \in L^\infty(0,T; W^{2,2}_{\mathrm{per}_0}(I)) \cap C^0([0,T]; L^2_{\mathrm{per}_0}(I)), \qquad u_t \in L^\infty(0,T; L^2_{\mathrm{per}_0}(I)),$$

such that

$$u_t(t) = -H(u_x(t))_{xx} - \Phi'_a(u_{xx}(t))_{xx} \quad \text{for a.e. } t \in [0,T], \quad u(0) = u^0, \tag{8}$$

and

$$\|u_t\|_{L^\infty(0,T;L^2_{\mathrm{per}_0}(I))} \le \|H(u^0_x)_{xx} + \Phi'_a(u^0_{xx})_{xx}\|_{L^2_{\mathrm{per}_0}(I)}. \tag{9}$$

Theorem 2. *(Uniqueness and stability) Under the same hypotheses of Theorem 1, the strong solution given by Theorem 1 is unique.*

Moreover, denote by u the unique strong solution of (2), and let

$$\begin{aligned} D &:= \{z \in L^2_{\mathrm{per}_0}(I) : H(z_x)_{xx} + \Phi'_a(z_{xx})_{xx} \in L^2_{\mathrm{per}_0}(I)\}, \\ Y &:= L^2(0,T; W^{2,2}_{\mathrm{per}_0}(I)) \cap C^0([0,T]; L^2_{\mathrm{per}_0}(I)), \\ \|\cdot\|_Y &:= \|\cdot\|_{L^2(0,T;W^{2,2}_{\mathrm{per}_0}(I))} + \|\cdot\|_{C^0([0,T];L^2_{\mathrm{per}_0}(I))}. \end{aligned} \tag{10}$$

Then the function

$$\sigma : (D, \|\cdot\|_{L^2(I)}) \longrightarrow (Y, \|\cdot\|_Y), \qquad \sigma(u^0) := u, \tag{11}$$

which maps an initial datum u^0 into the solution u of (2), is 2-Lipschitz continuous.

Corollary 3. *(Exponential decay for speed) Under the same hypotheses of Theorem 1, denote by u the unique solution of (2). Then, setting,*

$$B : W^{2,2}_{\mathrm{per}_0}(I) \longrightarrow (W^{2,2}_{\mathrm{per}_0}(I))', \qquad Bu := H(u_x)_{xx} + \Phi'_a(u_{xx})_{xx}, \tag{12}$$

it holds

$$\|u_t(t)\|_{L^2(I)} \le e^{-t}\|Bu^0\|_{L^2(I)}.$$

The theory for evolution equations governed by symmetric maximal monotone operators (see Definition 4 below) is quite rich (see for instance Brézis [2] and references therein). Similarly, the theory for evolution equations governed by accretive operators (see Definition 5 below) is also quite rich (see for instance Barbu [1], Crandall and Liggett [5], Crandall and Pazy [6]). However, there are essentially two main difficulties in our analysis:

(1) first, the operator B governing (2) is not symmetric, unbounded, and not accretive.
(2) Second, the domain of B is *not* the entire space $W^{2,2}_{\mathrm{per}_0}(I)$, due to the non-definition of Φ on $(-\infty, 0)$.

To overcome these issues, we will exploit heavily the variational structure of (2), and the monotonicity of B. Crucial steps are Lemma 6 and Proposition 7.

2. Preliminary results

The main aim of this section is to present the setting of our problem, and to prove the crucial estimates in Lemma 6 and Proposition 7. As done by Dal Maso, Fonseca and Leoni in [7], and by Fonseca, Leoni and the author in [9], we will study (2) on the space domain $I = (-\pi, \pi)$ (and then extend by periodicity).

Let

$$V := W^{2,2}_{\mathrm{per}_0}(I), \qquad U := L^2_{\mathrm{per}_0}(I).$$

Endow U with the standard inner product of $L^2(I)$

$$\langle u^*, u\rangle_{U',U} := \int_I u^* u \,\mathrm{d}x, \qquad u^* \in U',\ u \in U,$$

and identify U with its dual U'. Endow V with the norm $\|v\|_V := \|v_{xx}\|_{L^2(I)}$. It is straightforward to check that U, V are reflexive. The duality pairing on V will be denoted by $\langle , \rangle_{V',V}$. More explicitly, given $v^* \in V'$, $v \in V$, it holds

$$\langle v^*, v\rangle_{V',V} = \int_I v^* v \,\mathrm{d}x.$$

Note that the embeddings $V \hookrightarrow U \hookrightarrow V'$ are compact, hence (V, U, V') is a Gelfand triple. Since the underlying space V is reflexive, it is straightforward to check (by direct computation, without using Aubin-Lions lemma) that the embeddings

$$L^2(0, T; V) \hookrightarrow L^2(0, T; U) \hookrightarrow L^2(0, T; V')$$

are also continuous. For future references, given a Banach space X and an operator $A : X \longrightarrow X'$, $\mathrm{dom}_X(A)$ denotes the "domain" of A in X. That is,

$$\mathrm{dom}_X(A) := \{x \in X : Ax \in X'\}.$$

We recall the following classical definitions (see for instance [1]).

Definition 4. *Given a Banach space X, denote by $\langle , \rangle_{X',X}$ the duality pairing between X' and X. A single-valued operator $A : X \longrightarrow X'$ is:*

(1) **monotone** *if for any $u, v \in \mathrm{dom}_X(A)$, it holds*

$$\langle Au - Av, u - v\rangle_{X',X} \geq 0.$$

Similarly, a set $G \subseteq X \times X'$ is "monotone" *if for any pair (u, u'), $(v, v') \in G$, it holds*

$$\langle u' - v', u - v\rangle_{X',X} \geq 0;$$

(2) **maximal monotone** *if the graph*

$$\Gamma_A := \{(u, Au) : u \in X\} \subseteq X \times X'$$

is not a proper subset of any monotone set;

(3) **hemi-continuous** *if for any $u, v, w \in X$ the mapping*

$$t \longmapsto \langle A(u + tv), w\rangle_{X',X}$$

is continuous.

Definition 5. *Given a Banach space X, a single-valued operator $\tilde{A} : X \longrightarrow X$, its graph $\Gamma_{\tilde{A}}(X) := \{(x, \tilde{A}x) : x \in X$ such that $\tilde{A}x \in X)\}$ is:*

(1) **accretive** *if for any couple $(x, \tilde{A}x)$, $(y, \tilde{A}y)$, there exists an element $z \in J_X(x - y)$ such that $\langle z, \tilde{A}x - \tilde{A}y\rangle_{X',X} \geq 0$, where $J_X : X \to X'$ denotes the duality mapping;*

(2) **demi-closed** *if for any sequence $(x_n) \subseteq X$, such that $x_n \to x$ strongly in X, and $\tilde{A}x_n \rightharpoonup \xi \in X$, it holds $(x, \xi) \in \Gamma_{\tilde{A}}(X)$.*

The next lemma proves some key properties of B.

Lemma 6. *The operator $B : V \longrightarrow V'$ satisfies the following properties:*

(i) B is maximal monotone,

(ii) (coercivity) for any $u, v \in \mathrm{dom}_V(B)$ it holds

$$\langle Bu - Bv, u - v\rangle_{V',V} \geq \|u - v\|_V^2,$$

(iii) the graph of B is demi-closed in $V \times V'$. That is, given a sequence $(x_k) \subseteq V$ such that $x_k \to x$ strongly in V, $Bx_k \to y$ in V', then (x, y) belongs to the graph of B, and $y = Bx$.

Proof. To prove *(i)* and *(ii)*, we use the same arguments from [9, Lemma 6]. For completeness, we report the proof. Set

$$\begin{aligned}
\tilde{B} : V \longrightarrow V', &\qquad \langle \tilde{B}u, v\rangle_{V',V} := \int_I [2u_{xx}v_{xx} - H(u_{xx})v_x]\,\mathrm{d}x, \\
\Psi_a : \mathbb{R} \longrightarrow (-\infty+\infty], &\qquad \Psi_a(\xi) := \Phi_a(\xi) - \xi^2, \\
\psi : V \to (-\infty+\infty], &\qquad \psi(u) := \begin{cases} \int_I \Psi_a(u_{xx})\,\mathrm{d}x & \text{if } u \in V, \\ +\infty & \text{otherwise.} \end{cases}
\end{aligned}$$

Direct computation gives $B = \tilde{B} + \partial\psi$. Here, and for future reference, "∂" denotes the sub-gradient operator. More precisely, it is easily checked (by direct computation) that $\partial\psi(z) = \{\Psi_a'(z_{xx})_{xx}\}$ for all $z \in \mathrm{dom}_V(\partial\psi)$. Note also that $\mathrm{dom}_V(\partial\psi) \subseteq \mathrm{dom}_V(\psi) \subseteq V$. Since

$$\Psi_a''(\xi) = \xi + a + \frac{1}{\xi + a} - 2 \geq 0 \qquad \text{for any } \xi > -a,$$

Ψ_a is convex on $(-a, +\infty)$. Consequently, ψ is convex. By construction $B = \tilde{B} + \partial\psi$ is hemi-continuous. To prove monotonicity, note that [4, Proposition 9.1.9] and $\int_I u_{xx}\,\mathrm{d}x = 0$ give

$$\|H(u_{xx})\|_U = \|u_{xx}\|_U + \frac{1}{2\pi}\left(\int_I u_{xx}\,\mathrm{d}x\right)^2 = \|u_{xx}\|_U, \tag{13}$$

while $\|u_x\|_U \leq \|u_{xx}\|_U$ holds in view of [10, Section 7.7]. Hence

$$\|H(u_{xx})\|_U\|u_x\|_U \leq \|u_{xx}\|_U^2, \tag{14}$$

and

$$\begin{aligned}
\langle \tilde{B}(u-v), u-v\rangle_{V',V} &= \int_I (2|u_{xx} - v_{xx}|^2 - H(u-v)_{xx}(u-v)_x)\,\mathrm{d}x \\
&= 2\|u_{xx} - v_{xx}\|_U^2 - \int_I H(u-v)_{xx}(u-v)_x\,\mathrm{d}x \\
&\geq 2\|u_{xx} - v_{xx}\|_U^2 - \|H(u-v)_{xx}\|_U\|(u-v)_x\|_U \\
&\overset{(14)}{\geq} \|u_{xx} - v_{xx}\|_U^2.
\end{aligned}$$

As ψ is convex (hence $\partial\psi$ is monotone), combining (14) and (13) gives

$$\begin{aligned}
&\langle Bu - Bv, u-v\rangle_{V',V} \\
&\qquad = \langle \tilde{B}(u-v), u-v\rangle_{V',V} + \langle \partial\psi(u) - \partial\psi(v), u-v\rangle_{V',V} \\
&\qquad \geq \|u-v\|_V^2.
\end{aligned}$$

Thus B is monotone and hemi-continuous, hence (by [3, Theorem 1.2]) maximal monotone.

Statement *(iii)* follows from the well-known result stating that the graph of any maximal monotone operator is demi-closed. For further details, we refer to [12, Theorem 1, Remarks 3-4]. □

3. An existence result

The next proposition is a refinement of the existence result from [11, Section 5]. Due to its relevance to our arguments, we dedicate an entire section to its proof.

Proposition 7. *Let $\tilde{B}$ and ψ be the functionals from Lemma 6. Let $u^0 \in \mathrm{dom}_U(B)$ be a given initial datum, satisfying*

$$u^0 \in \mathrm{dom}_U(B), \qquad Bu^0 \in U. \tag{15}$$

Then there exists a function

$$u \in L^\infty(0,T;V) \cap C^0([0,T];U), \qquad u_t \in L^\infty(0,T;U) \tag{16}$$

such that $u(0) = u^0$ and

$$\langle u_t(t), v - u(t)\rangle_{U',U} + \langle \tilde{B}u(t), v - u(t)\rangle_{V',V} + \psi(v) - \psi(u(t)) \geq 0 \tag{17}$$

for a.e. time $t \in (0,T)$, and all $v \in V$. Moreover, it holds

$$\|u_t\|_{L^\infty(0,T;U)} \leq \|Bu^0\|_U. \tag{18}$$

Remark. The main improvement is that we only assume that the initial datum u^0 satisfies (15), instead of

- "$u^0 \in W^{2,2}_{\mathrm{per}_0}(I)$ and there exists $z^0 \in L^2_{\mathrm{per}_0}(I)$ satisfying

  $$\int_I [z^0 v - H(u^0_{xx})v_x + \Phi_a(v_{xx}) - \Phi_a(u^0_{xx})]\,\mathrm{d}x \geq 0$$

  for any $v \in W^{2,3}_{\mathrm{per}_0}(I)$".

We note that such functions u^0 satisfying (15) exist: for instance, since

$$Bu^0 = H(u^0_x)_{xx} + [\log(u^0_{xx} + a) + (u^0_{xx} + a)^2/2]_{xx},$$

all the functions of the form $u^0(x) := b\sin x$, with $|b| < a$, satisfy (15).

Proof. (of **Proposition 7**) The proof is essentially divided into three steps:

(1) first, using the classic method of time discretization, we construct a sequence of piece-wise linear approximate solutions $u^\varepsilon : [0,T] \longrightarrow V$;
(2) then we prove that $(u^\varepsilon)_\varepsilon$ is uniformly bounded in $L^\infty(0,T;V) \cap W^{1,\infty}([0,T];U)$, and we obtain a (weak) limit function

$$u \in L^\infty(0,T;V) \cap C^0([0,T];U), \qquad u_t \in L^\infty(0,T;U);$$

(3) finally, we prove that such u is solution of (17).

Step 1. Let $\varepsilon > 0$ be given. Consider the partition

$$0 = t_0 < t_1 < \cdots < t_{n_\varepsilon - 1} < t_{n_\varepsilon} \le T \le t_{n_\varepsilon} + \varepsilon,$$
$$t_j - t_{j-1} = \varepsilon, \qquad j = 1, \cdots, n_\varepsilon := \lfloor T/\varepsilon \rfloor,$$

where $\lfloor \cdot \rfloor$ denotes the integer part mapping. Construct the recursive sequence $(u_{\varepsilon,i})$ in the following way: $u_{\varepsilon,0} := u^0$, and given $u_{\varepsilon,i-1} \in V$, let $u_{\varepsilon,i} \in V$ be a solution of

$$\left\langle \frac{u_{\varepsilon,i} - u_{\varepsilon,i-1}}{t_i - t_{i-1}} + Bu_{\varepsilon,i}, v - u_{\varepsilon,i} \right\rangle_{V',V} \ge 0 \qquad \text{for all } v \in V.$$

Observe that this is equivalent to find $u_{\varepsilon,i} \in V$ such that

$$\langle (\mathrm{id} + \varepsilon B) u_{\varepsilon,i}, v - u_{\varepsilon,i} \rangle_{V',V} \ge \langle u_{\varepsilon,i-1}, v - u_{\varepsilon,i} \rangle_{V',V} \qquad \text{for all } v \in V. \quad (19)$$

Since B is maximal monotone, $\mathrm{id} + \varepsilon B : \mathrm{dom}_U(B) \longrightarrow U'$ is surjective for all $\varepsilon > 0$, hence there exists $u_{\varepsilon,i} \in \mathrm{dom}_U(B) \subseteq V$ (since $\tilde{B}$ – from Lemma 6 – is bounded and linear, and $\partial\psi$ is well-defined only on V) such that $u_{\varepsilon,i-1} = (\mathrm{id} + \varepsilon B) u_{\varepsilon,i}$. Moreover, $\mathrm{id} + \varepsilon B$ is also injective since B is monotone, hence $u_{\varepsilon,i} = (\mathrm{id} + \varepsilon B)^{-1} u_{\varepsilon,i-1}$ is unique. Thus $u_{\varepsilon,i} \in \mathrm{dom}_U(B) \subseteq V$ is solution of (19). Define the piece-wise linear functions u^ε satisfying

$$u^\varepsilon : [0, T] \longrightarrow V, \qquad u^\varepsilon(k\varepsilon) := u_{\varepsilon,k}, \quad k = 0, \cdots, \lfloor T/\varepsilon \rfloor.$$

Step 2. By construction, $u_{\varepsilon,i} = (\mathrm{id} + \varepsilon B)^{-1} u_{\varepsilon,i-1}$, thus

$$\begin{aligned} u_{\varepsilon,i} - u_{\varepsilon,i-1} &= (\mathrm{id} + \varepsilon B)^{-1} u_{\varepsilon,i-1} - (\mathrm{id} + \varepsilon B)^{-1} u_{\varepsilon,i-2} \\ &\Longrightarrow \|u_{\varepsilon,i} - u_{\varepsilon,i-1}\|_U \le \|u_{\varepsilon,i-1} - u_{\varepsilon,i-2}\|_U, \end{aligned} \quad (20)$$

since $(\mathrm{id} + \varepsilon B)^{-1} : U \longrightarrow \mathrm{dom}_U(B)$ is non-expansive as B is maximal monotone. Note that, by construction $u_{\varepsilon,1} = (\mathrm{id} + \varepsilon B)^{-1} u^0$, and we get

$$u_{\varepsilon,1} - u^0 = (\mathrm{id} + \varepsilon B)^{-1} u^0 - (\mathrm{id} + \varepsilon B)^{-1} (\mathrm{id} + \varepsilon B) u^0,$$

hence

$$\begin{aligned} \|u_{\varepsilon,1} - u^0\|_U &= \|(\mathrm{id} + \varepsilon B)^{-1} u^0 - (\mathrm{id} + \varepsilon B)^{-1} (\mathrm{id} + \varepsilon B) u^0\|_U \\ &\le \|u^0 - (\mathrm{id} + \varepsilon B) u^0\|_U = \varepsilon \|Bu^0\|_U, \end{aligned}$$

which in turn gives

$$\frac{\|u_{\varepsilon,1} - u^0\|_U}{\varepsilon} \le \|Bu^0\|_U. \quad (21)$$

Combining (20) and (21) gives

$$\frac{\|u_{\varepsilon,i} - u_{\varepsilon,i-1}\|_U}{\varepsilon} \le \|Bu^0\|_U \qquad \text{for all } \varepsilon > 0, \ i = 0, \cdots, \lfloor T/\varepsilon \rfloor.$$

Since

$$\frac{\|u_{\varepsilon,i} - u_{\varepsilon,i-1}\|_U}{\varepsilon} = \|u_t^\varepsilon(t)\|_U \qquad \text{for } t \in ((i-1)\varepsilon, i\varepsilon),$$

it follows

$$\|u_t^\varepsilon\|_U \le \|Bu^0\|_U \Longrightarrow \sup_\varepsilon \left(\sup_{t\in[0,T]} \|u^\varepsilon(t) - u^0\|_U \right) \le T\|Bu^0\|_U. \tag{22}$$

To estimate $\|u^\varepsilon(t)\|_V$, note that $(\mathrm{id} + \varepsilon B)u_{\varepsilon,i} = u_{\varepsilon,i-1}$ implies

$$\begin{aligned} \|u_{\varepsilon,i}\|_V^2 &\le \left| \langle Bu_{\varepsilon,i}, u_{\varepsilon,i}\rangle_{V',V} \right| = \left| \left\langle \frac{u_{\varepsilon,i} - u_{\varepsilon,i-1}}{\varepsilon}, u_{\varepsilon,i} \right\rangle_{V',V} \right| \\ &\le \frac{\|u_{\varepsilon,i} - u_{\varepsilon,i-1}\|_U}{\varepsilon} \|u_{\varepsilon,i}\|_U \\ &\le \|Bu^0\|_U (T\|Bu^0\|_U + \|u^0\|_U) \end{aligned}$$

therefore

$$\sup_\varepsilon \left(\sup_{t\in[0,T]} \|u^\varepsilon(t)\|_V^2 \right) \le \|Bu^0\|_U (T\|Bu^0\|_U + \|u^0\|_U). \tag{23}$$

Step 3. Consider an arbitrary sequence $\varepsilon_n \to 0$. In view of (22) and (23), there exists (upon subsequence, which we do not relabel) a function $u \in L^\infty(0,T;V)$ such that

$$u^{\varepsilon_n} \overset{*}{\rightharpoonup} u \text{ in } L^\infty(0,T;V), \qquad u_t^{\varepsilon_n} \overset{*}{\rightharpoonup} u_t \text{ in } L^\infty(0,T;U), \tag{24}$$

where "$\overset{*}{\rightharpoonup}$" denotes the convergence in the weak-* topology. Combining (22), (23) and (24) gives (16) and (18).

Fix an arbitrary $p \in (2, +\infty)$. In view of (24), we have

$$u^{\varepsilon_n} \rightharpoonup u \text{ in } L^p(0,T;V), \qquad u_t^{\varepsilon_n} \rightharpoonup u_t \text{ in } L^p(0,T;U),$$

In particular, $u^{\varepsilon_n} \rightharpoonup u$ in $L^p(t_1,t_2;V)$ and $u_t^{\varepsilon_n} \rightharpoonup u_t$ in $L^p(t_1,t_2;U)$ for any $0 \le t_1 < t_2 \le T$. The main advantage of working with $p \in (2,+\infty)$ (instead of $p = \infty$) is that the functional ψ is weakly sequentially lower semi-continuous. This will be crucial for the proof of (27) below.

By construction, each u^{ε_n} satisfies

$$\langle u_t^{\varepsilon_n}(t) + Bu^{\varepsilon_n}(t), v - u^{\varepsilon_n}(t)\rangle_{V',V} \ge 0 \tag{25}$$

for a.e. $t \in [0,T]$, and all $v \in V$. Since $B = \tilde{B} + \partial\psi$, with $\tilde{B}$ and ψ from Lemma 6, and ψ is convex, (25) gives

$$\left\langle u_t^{\varepsilon_n}(t) + \tilde{B}u^{\varepsilon_n}(t), v - u^{\varepsilon_n}(t) \right\rangle_{V',V} + \psi(v) - \psi(u^{\varepsilon_n}(t)) \ge 0$$

for a.e. $t \in [0,T]$, and all $v \in V$. Integrating on an arbitrary time set (t_1,t_2) with $0 \le t_1 < t_2 \le T$ gives

$$\int_{t_1}^{t_2} \left[\left\langle u_t^{\varepsilon_n}(t) + \tilde{B}u^{\varepsilon_n}(t), v - u^{\varepsilon_n}(t) \right\rangle_{V',V} + \psi(v) - \psi(u^{\varepsilon_n}(t)) \right] \mathrm{d}t \ge 0 \tag{26}$$

for all $v \in V$. Next, we claim

$$\limsup_{n\to+\infty} -\int_{t_1}^{t_2} \psi(u^{\varepsilon_n}(t))\,\mathrm{d}t \le -\int_{t_1}^{t_2} \psi(u(t))\,\mathrm{d}t, \tag{27}$$

$$\lim_{n\to+\infty} \int_{t_1}^{t_2} \left\langle \tilde{B}u^{\varepsilon_n}(t), v - u^{\varepsilon_n}(t)\right\rangle_{V',V} \,\mathrm{d}t = \int_{t_1}^{t_2} \left\langle \tilde{B}u(t), v - u(t)\right\rangle_{V',V} \,\mathrm{d}t, \tag{28}$$

$$\lim_{n\to+\infty} \int_{t_1}^{t_2} \langle u_t^{\varepsilon_n}(t), v - u^{\varepsilon_n}(t)\rangle_{V',V} \,\mathrm{d}t = \int_{t_1}^{t_2} \langle u_t(t), v - u(t)\rangle_{V',V} \,\mathrm{d}t. \tag{29}$$

To prove (27), it suffices to note that $-\psi$ is concave, hence weak upper-semicontinuous, and $u^{\varepsilon_n} \rightharpoonup u$ in $L^p(t_1, t_2; V)$.

Substep 3.1: proof of (28). Note that

$$\begin{aligned}\int_{t_1}^{t_2} \left\langle \tilde{B}u^{\varepsilon_n}(t), v - u^{\varepsilon_n}(t)\right\rangle_{V',V} \,\mathrm{d}t = &\int_{t_1}^{t_2} \left\langle \tilde{B}u^{\varepsilon_n}(t), v - u(t)\right\rangle_{V',V} \,\mathrm{d}t \\ &+ \int_{t_1}^{t_2} \left\langle \tilde{B}u^{\varepsilon_n}(t), u(t) - u^{\varepsilon_n}(t)\right\rangle_{V',V} \,\mathrm{d}t,\end{aligned}$$

where

$$\lim_{n\to+\infty} \int_{t_1}^{t_2} \left\langle \tilde{B}u^{\varepsilon_n}(t), v - u(t)\right\rangle_{V',V} \,\mathrm{d}t = \int_{t_1}^{t_2} \left\langle \tilde{B}u(t), v - u(t)\right\rangle_{V',V} \,\mathrm{d}t \tag{30}$$

due to the boundedness and linearity of $\tilde{B}$. To prove

$$\lim_{n\to+\infty} \int_{t_1}^{t_2} \left\langle \tilde{B}u^{\varepsilon_n}(t), u(t) - u^{\varepsilon_n}(t)\right\rangle_{V',V} \,\mathrm{d}t = 0, \tag{31}$$

observe that

$$u^{\varepsilon_n} \rightharpoonup u \text{ in } L^p(0,T;V), \quad u_t^{\varepsilon_n} \rightharpoonup u_t \text{ in } L^p(0,T;U),$$

and the embeddings $V \hookrightarrow W^{1,2}_{\mathrm{per}_0}(I) \hookrightarrow U$ are all compact. Thus Aubin-Lions lemma gives that $u^{\varepsilon_n} \to u$ strongly in $L^p(0,T;W^{1,2}_{\mathrm{per}_0}(I))$. Therefore,

$$\begin{aligned}&\int_{t_1}^{t_2} |\langle \tilde{B}u^{\varepsilon_n}(t), u(t) - u^{\varepsilon_n}(t)\rangle_{V',V}| \,\mathrm{d}t \\ &\quad = \int_{t_1}^{t_2}\int_I |Hu^{\varepsilon_n}_{xx}(t,x)(u_x(t,x) - u^{\varepsilon_n}_x(t,x))| \,\mathrm{d}x\,\mathrm{d}t \\ &\quad \le \int_{t_1}^{t_2} \|Hu^{\varepsilon_n}_{xx}(t)\|_U \|u_x(t) - u^{\varepsilon_n}_x(t)\|_U \,\mathrm{d}t \\ &\quad \overset{(13)}{=} \int_{t_1}^{t_2} \|u^{\varepsilon_n}_{xx}(t)\|_U \|u_x(t) - u^{\varepsilon_n}_x(t)\|_U \,\mathrm{d}t \\ &\quad \le \|u^{\varepsilon_n}(t)\|_{L^\infty(0,T;V)} \int_{t_1}^{t_2} \|u_x(t) - u^{\varepsilon_n}_x(t)\|_U \,\mathrm{d}t \\ &\quad \le \|u^{\varepsilon_n}(t)\|_{L^\infty(0,T;V)} |t_2 - t_1|^{1-1/p} \|u_x(t) - u^{\varepsilon_n}_x(t)\|_{L^p(0,T;U)} \overset{n\to+\infty}{\to} 0.\end{aligned}$$

Thus (31) is proven. Combining (30) and (31) gives (28).

Substep 3.2: proof of (29). Since $(u_t^{\varepsilon_n})_n$ is bounded in $L^\infty(0,T;U)$, it follows

$$\begin{aligned}\int_{t_1}^{t_2} &\left|\langle u_t^{\varepsilon_n}(t), u(t)-u^{\varepsilon_n}(t)\rangle_{V',V}\right| \mathrm{d}t\\ &\le \int_{t_1}^{t_2} \|u_t^{\varepsilon_n}(t)\|_U \|u(t)-u^{\varepsilon_n}(t)\|_U \,\mathrm{d}t\\ &\le \|u_t^{\varepsilon_n}(t)\|_{L^\infty(0,T;U)}|t_2-t_1|^{1-1/p}\|u(t)-u^{\varepsilon_n}(t)\|_{L^p(0,T;U)} \overset{n\to+\infty}{\to} 0, \end{aligned} \quad (32)$$

and

$$\int_{t_1}^{t_2} \langle u_t^{\varepsilon_n}(t), v-u(t)\rangle_{V',V}\ \mathrm{d}t \to \int_{t_1}^{t_2} \langle u_t(t), v-u(t)\rangle_{V',V}\ \mathrm{d}t. \quad (33)$$

Combining (32) and (33) gives (29).

Combining (27), (28) and (29) gives

$$\begin{aligned}\int_{t_1}^{t_2} &\left[\left\langle u_t(t)+\tilde{B}(t), v-u(t)\right\rangle_{V',V} + \psi(v)-\psi(u(t))\right] \mathrm{d}t\\ &\ge \limsup_{n\to+\infty}\int_{t_1}^{t_2}\left[\left\langle u_t^{\varepsilon_n}(t)+\tilde{B}u^{\varepsilon_n}(t), v-u^{\varepsilon_n}(t)\right\rangle_{V',V} + \psi(v)-\psi(u^{\varepsilon_n}(t))\right] \mathrm{d}t\\ &\ge 0.\end{aligned}$$

The arbitrariness of t_1, t_2 gives (17), concluding the proof. □

4. Proof of the main results

Now we are ready to prove that the function u given by Proposition 7 is the desired solution.

The proof of Theorem 1 uses some ideas from [1]. However, it is noted that $B: V \longrightarrow V'$ is not accretive, thus crucial monotonicity estimates have to be achieved differently.

Proof. (of **Theorem 1**) Let u be a solution of (17) given by Proposition 7. Since, in Proposition 7, u_t was a (weak-*) limit $L^\infty(0,T;U)$ of $u_t^{\varepsilon_n}$ satisfying $\sup_n \|u_t^{\varepsilon_n}\|_{L^\infty(0,T;U)} \le \|Bu^0\|_U$, it follows

$$\|u_t\|_{L^\infty(0,T;U)} \le \liminf_{n\to+\infty}\|u_t^{\varepsilon}\|_{L^\infty(0,T;U)} \le \|Bu^0\|_U,$$

which proves (9). By construction, u satisfies also

$$u \in L^\infty(0,T;V)\cap C^0([0,T];U), \qquad u_t \in L^\infty(0,T;U). \quad (34)$$

We need to check that such u satisfies

$$u_t(t) = -Bu(t) \quad \text{for a.e. } t\in[0,T], \qquad u(0)=u^0. \quad (35)$$

Consider $t > 0$ such that

$$u(t-h) = u(t) - hu_t(t) - hg(h), \qquad h > 0, \tag{36}$$

for some function $g(h)$ satisfying

$$\lim_{h\to 0} \|g(h)\|_U = 0. \tag{37}$$

In view of (34), the set of times such that (36) holds for some g satisfying (37) has full measure. Since $\operatorname{id} + hB : U \longrightarrow U$ is bijective, we define

$$x^h := (\operatorname{id} + hB)^{-1} u(t-h) \in \operatorname{dom}_U(B).$$

Thus we get

$$u(t) - x^h = h[Bx^h + u_t(t) + g(h)]. \tag{38}$$

Multiplying both sides by $u(t) - x^h$ gives

$$\begin{aligned} &\langle u(t) - x^h, u(t) - x^h \rangle_{V',V} \\ &\qquad = h\langle Bx^h + u_t(t), u(t) - x^h \rangle_{U',U} + h\langle g(h), u(t) - x^h \rangle_{V',V}. \end{aligned} \tag{39}$$

Next, we claim

$$\langle Bx^h + u_t(t), u(t) - x^h \rangle_{U',U} \le 0. \tag{40}$$

Since u is a solution of (17), taking $v = x^h$ gives

$$\langle u_t(t), x^h - u(t) \rangle_{U',U} + \langle \tilde{B}u(t), x^h - u(t) \rangle_{V',V} + \psi(x^h) - \psi(u(t)) \ge 0,$$

hence, due to the convexity of ψ and the monotonicity of $\tilde{B}$, we get

$$\begin{aligned} 0 &\le \langle u_t(t), x^h - u(t) \rangle_{U',U} + \langle \tilde{B}u(t) + \partial\psi(x^h), x^h - u(t) \rangle_{V',V} \\ &= \langle u_t(t), x^h - u(t) \rangle_{U',U} + \langle \tilde{B}x^h + \partial\psi(x^h), x^h - u(t) \rangle_{V',V} \\ &\quad + \langle \tilde{B}u(t) - \tilde{B}x^h, x^h - u(t) \rangle_{V',V} \\ &\le \langle u_t(t), x^h - u(t) \rangle_{U',U} + \langle \tilde{B}x^h + \partial\psi(x^h), x^h - u(t) \rangle_{V',V} \\ &= \langle u_t(t), x^h - u(t) \rangle_{U',U} + \langle Bx^h, x^h - u(t) \rangle_{V',V}, \end{aligned}$$

which proves (40). Thus (39) gives

$$\begin{aligned} \langle u(t) - x^h, u(t) - x^h \rangle_{V',V} &= h\langle Bx^h + u_t(t) + g(h), u(t) - x^h \rangle_{V',V} \\ &\le h\langle g(h), u(t) - x^h \rangle_{V',V}, \end{aligned}$$

hence $\|u(t) - x^h\|_U / h \to 0$ as $h \to 0$. Note that, by construction, we have

$$Bx^h = \frac{u(t-h) - x^h}{h} = \frac{u(t) - x^h}{h} + \frac{u(t-h) - u(t)}{h},$$

hence

$$Bx^h = \frac{u(t-h) - x^h}{h} = \frac{u(t) - x^h}{h} + \frac{u(t-h) - u(t)}{h} \to -u_t(t),$$

strongly in U. Summing up, we proved that $x^h \to u(t)$, $Bx^h \to -u(t)$, and

$$\{(w, Bw) : w \in \mathrm{dom}_U(B), Bw \in U\}$$

is demi-closed in $U \times U$, thus we infer (by [12, Theorem 1, Remarks 3-4]) $Bu(t) = -u(t)$. Since this argument holds for a.e. $t \in [0,T]$, (35) is proven. □

Proof. (of **Theorem 2**) From [10, Section 7.7] we get $\|v\|_U \le \|v\|_V$ for all $v \in V$. Consider initial data $u^{0,1}$, $u^{0,2} \in D$ (with D defined in (10)), and let u^1, u^2 be corresponding solutions to (2) given by Theorem 1. Therefore, it holds

$$u_t^1(t) + \tilde{B}u^1(t) + \partial\psi(u^1(t)) = u_t^2(t) + \tilde{B}u^2(t) + \partial\psi(u^2(t)) = 0$$

for a.e. $t \in [0,T]$, hence

$$u_t^1(t) - u_t^2(t) + \tilde{B}(u^1(t) - u^2(t)) + \partial\psi(u^1(t)) - \partial\psi(u^2(t)) = 0$$

for a.e. $t \in [0,T]$. Multiplying both sides by $u^1(t) - u^2(t)$ gives

$$\begin{aligned}\langle u_t^1(t) - u_t^2(t), u^1(t) - u^2(t)\rangle_{V',V} + \langle \tilde{B}(u^1(t) - u^2(t)), u^1(t) - u^2(t)\rangle_{V',V} \\ + \langle \partial\psi(u^1(t)) - \partial\psi(u^2(t)), u^1(t) - u^2(t)\rangle_{V',V} = 0\end{aligned}$$

for a.e. $t \in [0,T]$. Note that

$$\begin{aligned}\langle u_t^1(t) - u_t^2(t), u^1(t) - u^2(t)\rangle_{V',V} &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u^1(t) - u^2(t)\|_U^2,\\ \langle \tilde{B}(u^1(t) - u^2(t)), u^1(t) - u^2(t)\rangle_{V',V} &\ge \frac{1}{2}\|u^1(t) - u^2(t)\|_V^2,\\ \langle \partial\psi(u^1(t)) - \partial\psi(u^2(t)), u^1(t) - u^2(t)\rangle_{V',V} &\ge 0,\end{aligned}$$

which gives

$$0 \ge \frac{\mathrm{d}}{\mathrm{d}t}\|u^1(t) - u^2(t)\|_U^2 + \|u^1(t) - u^2(t)\|_V^2, \tag{41}$$

hence

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|u^1(t) - u^2(t)\|_U^2 \le -\|u^1(t) - u^2(t)\|_V^2 \le -\|u^1(t) - u^2(t)\|_U^2 \\ \Longrightarrow \|u^1(t) - u^2(t)\|_U^2 \le e^{-t}\|u^{0,1} - u^{0,2}\|_U^2.\end{aligned} \tag{42}$$

Integrating (41) on $[0,s]$ (for arbitrarily chosen $s \in (0,T]$) gives

$$\begin{aligned}\int_0^s \|u^1(t) - u^2(t)\|_V^2 \,\mathrm{d}t &\le -\int_0^s \frac{\mathrm{d}}{\mathrm{d}t}\|u^1(t) - u^2(t)\|_U^2 \,\mathrm{d}t\\ &= \|u^{0,1} - u^{0,2}\|_U^2 - \|u^1(s) - u^2(s)\|_U^2\\ &\le \|u^{0,1} - u^{0,2}\|_U^2.\end{aligned} \tag{43}$$

Choosing $u^{0,1} = u^{0,2}$ proves that the solution given by Theorem 1 is unique.

Given $u^0 \in D$, a sequence $(u^{0,n})_n \subseteq D$ such that $u^{0,n} \to u^0$ strongly in U, estimate (42) gives

$$u^n \to u \qquad \text{strongly in } C^0([0,T];U),$$

while (43) gives

$$u^n \to u \qquad \text{strongly in } L^2(0,T;V),$$

where u^n (resp. u) denotes the (unique) solution of (2) associated to the initial datum $u^{0,n}$ (resp. u^0). Thus the map σ defined in (11) is continuous. Combining (42) and (43), and setting

$$u^1 := \sigma(u^{0,1}), \qquad u^2 := \sigma(u^{0,2}),$$

we get

$$\begin{aligned}\|u^1 - u^2\|_{L^2(0,T;V)} &+ \|u^1 - u^2\|_{C^0([0,T];U)} \\ &= \left(\int_0^T \|u^1(t) - u^2(t)\|_V^2 \,\mathrm{d}t\right)^{1/2} + \sup_{t\in[0,T]} \|u^1(t) - u^2(t)\|_U \\ &\leq 2\|u^{0,1} - u^{0,2}\|_U,\end{aligned}$$

thus σ is 2-Lipschitz continuous, concluding the proof. □

Remark. Theorems 1 and 2 give the existence and uniqueness of a strong solution $u : [0,T] \longrightarrow V$. In particular, it is also a solution in the weak sense, i.e.

$$\int_0^T \int_I u_t(t)\varphi(t)\,\mathrm{d}x\,\mathrm{d}t = \int_0^T \int_I [H(u_{xx}(t))\varphi_x(t) - \Phi_a'(u_{xx}(t))\varphi_{xx}(t)]\,\mathrm{d}x\,\mathrm{d}t \tag{44}$$

for any test function $\varphi \in C_c^\infty((0,T)\times I;\mathbb{R})$. Thus, if the initial datum u^0 satisfies the variational inequality (4) for some $z^0 \in U$, then by [9, Theorem 1], the following (stronger) regularity result holds:

$$u \in L^\infty(0,T;W^{2,3}_{\mathrm{per}_0}(I)) \cap C^0([0,T];U), \quad u_t \in L^\infty(0,T;U).$$

Proof. (of **Corollary 3**) Let u be the (unique) strong solution given by Theorem 1. Recall that, in the proof of Proposition 7, the sequence u^ε was defined as the unique piece-wise linear function with nodes $u_{\varepsilon,i}$, $i = 0,\cdots,\lfloor T/\varepsilon\rfloor$, such that $u_{\varepsilon,i} = (\mathrm{id} + \varepsilon B)^{-1}u_{\varepsilon,i-1}$. In particular, we get

$$\langle u_{\varepsilon,i} - u_{\varepsilon,i-1} + \varepsilon B u_{\varepsilon,i}, v - u_{\varepsilon,i}\rangle_{V',V} \geq 0, \tag{45}$$

$$\langle u_{\varepsilon,i-1} - u_{\varepsilon,i-2} + \varepsilon B u_{\varepsilon,i-1}, v - u_{\varepsilon,i-1}\rangle_{V',V} \geq 0, \tag{46}$$

for all $v \in V$. Choosing $v = u_{\varepsilon,i-1}$ in (45) and $v = u_{\varepsilon,i}$ in (46) gives

$$\langle u_{\varepsilon,i} - u_{\varepsilon,i-1} + \varepsilon B u_{\varepsilon,i}, u_{\varepsilon,i-1} - u_{\varepsilon,i}\rangle_{V',V} \geq 0,$$
$$\langle u_{\varepsilon,i-1} - u_{\varepsilon,i-2} + \varepsilon B u_{\varepsilon,i-1}, u_{\varepsilon,i} - u_{\varepsilon,i-1}\rangle_{V',V} \geq 0,$$

and summing both sides gives

$$\begin{aligned}\|u_{\varepsilon,i}-u_{\varepsilon,i-1}\|_U^2 &\le \varepsilon\langle Bu_{\varepsilon,i}-Bu_{\varepsilon,i-1}, u_{\varepsilon,i-1}-u_{\varepsilon,i}\rangle_{V',V}\\ &\quad+\langle u_{\varepsilon,i-1}-u_{\varepsilon,i-2}, u_{\varepsilon,i}-u_{\varepsilon,i-1}\rangle_{V',V}\\ &\le -\varepsilon\|u_{\varepsilon,i}-u_{\varepsilon,i-1}\|_U^2+\|u_{\varepsilon,i}-u_{\varepsilon,i-1}\|_U\|u_{\varepsilon,i-2}-u_{\varepsilon,i-1}\|_U,\end{aligned}$$

i.e.,

$$\|u_{\varepsilon,i}-u_{\varepsilon,i-1}\|_U \le -\varepsilon\|u_{\varepsilon,i}-u_{\varepsilon,i-1}\|_U+\|u_{\varepsilon,i-2}-u_{\varepsilon,i-1}\|_U,$$

which gives

$$\|u_{\varepsilon,i}-u_{\varepsilon,i-1}\|_U \le (1+\varepsilon)^{-1}\|u_{\varepsilon,i-2}-u_{\varepsilon,i-1}\|_U. \tag{47}$$

Taking $i=1$, $v=u^0$ in (45) yields

$$\langle u_{\varepsilon,1}-u^0+\varepsilon Bu_{\varepsilon,1}, u^0-u_{\varepsilon,1}\rangle_{V',V}\ge 0,$$

which gives

$$\begin{aligned}\|u_{\varepsilon,1}-u^0\|_U^2 &\le \varepsilon\langle Bu_{\varepsilon,1}, u^0-u_{\varepsilon,1}\rangle_{V',V}\\ &=\varepsilon\langle Bu_{\varepsilon,1}-Bu^0, u^0-u_{\varepsilon,1}\rangle_{V',V}+\varepsilon\langle Bu^0, u^0-u_{\varepsilon,1}\rangle_{V',V}\\ &\le \varepsilon\|Bu^0\|_U\|u_{\varepsilon,1}-u^0\|_U,\end{aligned}$$

hence $\|u_{\varepsilon,1}-u^0\|_U\le\varepsilon\|Bu^0\|_U$. Combining with (47) gives

$$\frac{\|u_{\varepsilon,i}-u_{\varepsilon,i-1}\|_U}{\varepsilon}\le(1+\varepsilon)^{-(i-1)}\|Bu^0\|_U. \tag{48}$$

By construction it holds $\|u_t^\varepsilon(s)\|_U=\|u_{\varepsilon,i}-u_{\varepsilon,i-1}\|_U/\varepsilon$ for every $s\in((i-1)\varepsilon, i\varepsilon)$. Thus, for $t\in[0,T]$ such that $t/\varepsilon\notin\mathbb{N}$, it holds

$$\begin{aligned}\|u_t^\varepsilon(t)\|_U &= \frac{\|u_{\varepsilon,\lfloor t/\varepsilon\rfloor+1}-u_{\varepsilon,\lfloor t/\varepsilon\rfloor}\|_U}{\varepsilon}\\ &\le(1+\varepsilon)^{-\lfloor t/\varepsilon\rfloor}\|Bu^0\|_U\le(1+\varepsilon)^{1-t/\varepsilon}\|Bu^0\|_U.\end{aligned}$$

Since (upon subsequence) $u_t^\varepsilon \overset{*}{\rightharpoonup} u_t$ in $L^\infty(t-\delta,t+\delta;U)$ for any $\delta>0$, we get

$$\operatorname{esssup}_{s\in(t-\delta,t+\delta)}\|u_t(s)\|_U\le\|Bu^0\|_U\lim_{\varepsilon\to0}(1+\varepsilon)^{1-(t-\delta)/\varepsilon}=e^{-t+\delta}\|Bu^0\|_U,$$

and we conclude by the arbitrariness of δ. □

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