# Integral representation and Γ-convergence of variational integrals with p(x)-growth

*Alessandra Coscia & Domenico Mucci*

*We study the integral representation properties of limits of sequences of integral functionals like* $\int f(x, Du)\, dx$ *under nonstandard growth conditions of* $(p, q)$*-type: namely, we assume that*

$$|z|^{p(x)} \leq f(x,z) \leq L(1 + |z|^{p(x)})\,.$$

*Under weak assumptions on the continuous function* $p(x)$*, we prove* $\Gamma$*-convergence to integral functionals of the same type. We also analyse the case of integrands* $f(x, u, Du)$ *depending explicitly on* $u$*; finally we weaken the assumption allowing* $p(x)$ *to be discontinuous on nice sets.*



## 1 Introduction

The aim of this paper is the study of the Γ-convergence and integral representation properties for sequences of integral functionals of the type

$$F(u, \Omega) := \int_\Omega f(x, u(x), Du(x))\, dx\,, \tag{1.1}$$

where $\Omega$ is an open subset of $\mathbf{R}^n$ and $f$ is a non-negative Borel function defined on $\Omega \times \mathbf{R}^N \times \mathbf{R}^{nN}$. Under the assumption of $p$-growth

$$|z|^p \leq f(x, u, z) \leq L(1 + |z|^p) \tag{1.2}$$

existence and integral representation of the Γ-limit with respect to the strong topology of $L^p$ of a sequence of functionals as (1.1) was proved in the scalar case in [16], [11], [15], and in the vector-valued case in [21], under suitable assumptions on the dependence of $f$ on $u$, see also [14], [10].

In the context of regularity theory for minimizers, ten years ago Marcellini [22] replaced (1.2) with the more flexible $(p, q)$-growth assumption

$$|z|^p \leq f(x, u, z) \leq L(1 + |z|^q)\,, \qquad q \geq p > 1\,; \tag{1.3}$$

the theory of integrals with $(p, q)$-growth received contributions from various authors, see the references in [23].

Dealing with the passage to the limit for variational problems, i.e., convergence of energies as $j \to +\infty$, and in the context of Γ-convergence, Zhikov [27] proved several results, under the $(p, q)$-growth assumption (1.3), when $N = 1$ and $f(x, u, \cdot)$ is convex. If (1.3) is satisfied, the functional $F$ is coercive if regarded on the Sobolev space $W^{1,p}(\Omega; \mathbf{R}^N)$, whereas it is bounded (and, in case of convexity, continuous) if regarded on the smaller space $W^{1,q}(\Omega; \mathbf{R}^N)$. This fact is responsible for the presence of the so called Lavrentiev effect, due to the lack of $W^{1,q}$-density of smooth functions in $W^{1,p}(\Omega; \mathbf{R}^N)$, see [28]. On the other hand, for the same reason it is not clear how to obtain existence of the Γ-limit with respect to $L^p$-convergence in the whole space, see e.g. [10, Ch. 21].

In the context of cavitation and related theories, dealing with integral functionals satisfying a $q$-growth condition from above and taken to be $+\infty$ outside $W^{1,q}(\Omega; \mathbf{R}^N)$, measure representation of the relaxation with respect to weak $W^{1,p}$ convergence is obtained in [6] and [1], assuming $z \mapsto f(x, z)$ is convex and $p > q - q/n$.

A borderline case lying between (1.2) and (1.3) is the one of $p(x)$-growth:

$$|z|^{p(x)} \leq f(x, u, z) \leq L(1 + |z|^{p(x)})\,, \qquad p(x) \geq 1\,. \tag{1.4}$$

This kind of growth was first considered by Zhikov in the context of homogenization, see [31], and in recent years the subject gained importance by providing variational models for many problems from Mathematical Physics. For instance, recently Rajagopal and Růžička elaborated a model for the electrorheological fluids; they are special non-Newtonian fluids which are characterized by their ability to change their mechanical properties when in presence of an electromagnetic field, see [25] and [26]. Other models of this type arise for fluids whose viscosity is influenced in a similar way by the temperature, see [30]; the mathematical model for Zhikov's thermistor problem includes equations like

$$-\mathrm{div}(p(x)|Du|^{p(x)-2}Du) = 0\,,$$

whose solutions correspond to minimizers of $\int_\Omega |Du|^{p(x)}\,dx$.

For the regularity of minimizers of functionals with $p(x)$-growth, we refer to [22], [28], [20], [4], [12], [13], [2] and [3]. In particular, Zhikov proved higher integrability of the gradient under the following condition about the modulus of continuity $\omega(R)$ of $p(x)$

$$\limsup_{R\to 0^+} \omega(R)\,\log(1/R) < +\infty\,, \tag{1.5}$$

a condition which is sharp since, in general, dropping it causes the loss of any type of regularity of minimizers, see [29].

Moreover, condition (1.5) seems to play a central role in the theory of functionals with $p(x)$-growth since Zhikov proved in [29] that such functionals exhibit the Lavrentiev phenomenon if (1.5) is violated. On the other side, in [1] it is proved that the singular part of the measure representation of relaxed functionals with growth (1.4) disappears if (1.5) holds true.

In this paper we show that if the growth exponent $p(x)$ satisfies a local continuity estimate of the type (1.5), then integral representation holds for non-negative local functionals satisfying a $p(x)$-growth condition from above in the set $W^{1,p(x)}(\Omega;\mathbf{R}^N)$ of functions $u\in W^{1,1}$ with $|Du|^{p(x)}\in L^1$.

Also, we prove $\Gamma$-compactness in $W^{1,p(x)}(\Omega;\mathbf{R}^N)$ for sequences of local functionals satisfying the $p(x)$-growth condition (1.4). Moreover, the $\Gamma$-limit turns out to be the integral of a quasi-convex function in the sense of Morrey [24], satisfying the same $p(x)$-growth condition from above. More precisely, we organize the paper as follows.

After giving the notation and some preliminary results in Sec. 2, we analyse in Sec. 3 the properties of functionals with $p(x)$-growth, in particular recalling a density result due to Zhikov in the set $W^{1,p(x)}(\Omega;\mathbf{R}^N)$, see Proposition 3.18. Here a central role is played by the local assumption (1.5) on the modulus of continuity of $p(x)$. In Sec. 4 we prove an integral representation result of the type $\int f(x,Du)\,dx$ for non-negative local functionals satisfying a $p(x)$-growth condition from above, see Theorem 4.1. Then, in Sec. 5 we prove existence and integral representation of the $\Gamma(L^1)$-limit of sequences of local functionals with $p(x)$-growth, see Theorems 5.1 and 5.2. Sec. 6 is dedicated to the more general case of integrands $f(x,u,Du)$ with explicit dependence on $u$, obtaining the same conclusions under the assumption of a continuous dependence of $f$ on $u$, see Theorems 6.1 and 6.2. Finally, in Sec. 7 we generalize the results allowing the growth exponent $p(x)$ to be discontinuous on a negligible set given by the interfaces between nice subsets of the domain $\Omega$, see Theorems 7.1 and 7.2. We finally remark that, in the particular case of relaxation (i.e., the $\Gamma$-limit of a constant sequence) we expect the relaxed functional to be the integral of the quasi-convex envelope of $f$: see Corollary 7.3 for the case of $p(x)$ piecewise constant.

# 2 Notation and preliminaries

In the sequel $\Omega$ is a fixed bounded open subset of $\mathbf{R}^n$ and $\mathcal{A}$ is the family of its open subsets; if $A,B\in\mathcal{A}$, by $A\subset\subset B$ we mean that the closure $\overline{A}$ of $A$ is a compact set contained in $B$, and by $\mathcal{A}_0$ we denote the class of all $A\in\mathcal{A}$ such that $A\subset\subset\Omega$. Also, $B_\delta(x)$ denotes the ball of radius $\delta>0$ centred at $x\in\mathbf{R}^n$.

As usual, $L^p(\Omega;\mathbf{R}^N)$ and $W^{1,p}(\Omega;\mathbf{R}^N)$ will denote the standard Lebesgue and Sobolev spaces of functions $u:\Omega\to\mathbf{R}^N$, for any $p\geq 1$. Finally, we will denote by $p^*$ the Sobolev conjugate of $p$, i.e., $p^* := np/(n-p)$ if $1\leq p<n$, and $p^*=+\infty$ if $p\geq n$. We recall that if $u\in L^1(\Omega;\mathbf{R}^N)$ and $Du\in L^p(\Omega;\mathbf{R}^{nN})$, then $u\in W^{1,p}(\Omega;\mathbf{R}^N)$ provided $\Omega$ has Lipschitz boundary (also weaker conditions are sufficient, see e.g. [4]). For such sets $\Omega$, if a sequence $\{u_j\}$ is bounded in $L^1(\Omega;\mathbf{R}^N)$ and $\{Du_j\}$ is bounded in $L^p(\Omega;\mathbf{R}^{nN})$, then $\{u_j\}$ is bounded in $W^{1,p}(\Omega;\mathbf{R}^N)$.

We recall the main definitions and results from the theory of $\Gamma$-convergence. We refer to [14] or [10] for a more extensive introduction to the subject.

**Definition 2.1** Let $F_j : X \to \overline{\mathbf{R}}$ be a sequence of functions defined on a metric space $(X, d)$. We say that $\{F_j\}$ is $\Gamma(d)$-converging to $F : X \to \overline{\mathbf{R}}$ if for all $x \in X$ we have

i) (*lim inf inequality*) for every sequence $\{x_j\}$, if $d(x_j, x) \to 0$, then

$$F(x) \leq \liminf_{j \to +\infty} F_j(x_j)\,;$$

ii) (*existence of a recovery sequence*) there exists a sequence $\{x_j\}$, with $d(x_j, x) \to 0$, such that

$$F(x) = \lim_{j \to +\infty} F_j(x_j)\,.$$

The function $F$ is called the $\Gamma(d)$-limit of $\{F_j\}$ and we write $F = \Gamma(d)\text{-}\lim_{j \to +\infty} F_j$.

**Remark 2.2** It is well known that if $F_j \equiv G$ for all $j \in \mathbf{N}$, then $\Gamma(d)\text{-}\lim_j F_j = \overline{G}$, where $\overline{G}$ is the $d$-lower semicontinuos envelope of $G$ (or $d$-relaxed functional), given by

$$\overline{G}(x) := \inf\{\liminf_{j \to +\infty} G(x_j) \mid \{x_j\} \subset X\,, \quad d(x_j, x) \to 0\}\,.$$

Since in general the $\Gamma$-limit may not exist, we introduce the $\Gamma(d)$-upper and lower limits.

**Definition 2.3** Let $F_j : X \to \overline{\mathbf{R}}$ and $x \in X$. We define the $\Gamma(d)$-upper limit of $\{F_j\}$ at $x$ as

$$\Gamma(d)\text{-}\limsup_{j \to +\infty} F_j(x) := \inf\{\limsup_{j \to +\infty} F_j(x_j) \mid \{x_j\} \subset X\,, \quad d(x_j, x) \to 0\}\,.$$

Similarly, we define the $\Gamma(d)$-lower limit of $\{F_j\}$ at $x$ as

$$\Gamma(d)\text{-}\liminf_{j \to +\infty} F_j(x) := \inf\{\liminf_{j \to +\infty} F_j(x_j) \mid \{x_j\} \subset X\,, \quad d(x_j, x) \to 0\}\,.$$

Therefore, the upper and lower $\Gamma$-limits always exist and, since the two infima in the definition above are actually minima, the $\Gamma$-limit exists at $x$ if and only if $\Gamma(d)\text{-}\liminf_j F_j(x) = \Gamma(d)\text{-}\limsup_j F_j(x)$. Anyway, for $\Gamma$-convergence the following compactness property holds.

**Proposition 2.4** *If $(X, d)$ is a separable metric space and $F_j : X \to \overline{\mathbf{R}}$, $j \in \mathbf{N}$, are given functions, there exists an increasing sequence $\{j_k\}$ such that $\Gamma(d)\text{-}\lim_k F_{j_k}(x)$ exists for all $x \in X$.*

We also recall the following facts about set functions.

**Definition 2.5** A function $\alpha : \mathcal{A} \to [0, +\infty]$ is called an increasing set function if $\alpha(\emptyset) = 0$ and $\alpha(A) \leq \alpha(B)$ if $A \subseteq B$. An increasing set function $\alpha$ is said to be subadditive if

$$\alpha(A \cup B) \leq \alpha(A) + \alpha(B)$$

for all $A, B \in \mathcal{A}$, and it is said to be superadditive if

$$\alpha(A \cup B) \geq \alpha(A) + \alpha(B)$$

for all $A, B \in \mathcal{A}$ with $A \cap B = \emptyset$; finally, $\alpha$ is said to be inner regular if for all $A \in \mathcal{A}$

$$\alpha(A) = \sup\{\alpha(B) \mid B \in \mathcal{A}\,, \quad B \subset\subset A\}\,.$$

In this paper we will consider functionals defined on the Lebesgue space $L^1(\Omega; \mathbf{R}^N)$. We will first prove an integral representation theorem and then we will apply the direct method of $\Gamma$-convergence, which consists in proving general abstract compactness results, ensuring the existence of $\Gamma$-converging sequences, and then recovering enough information on the structure of the $\Gamma$-limits to obtain a representation in a suitable form.

For both problems we make use of the localization method, which consists in considering at the same time the dependence of the $\Gamma$-limit on the function and on the open set.

More precisely, if $F_j : L^1(\Omega;\mathbf{R}^N)\times\mathcal{A}\to[0,+\infty]$ is a sequence of functionals, $u\in L^1(\Omega;\mathbf{R}^N)$ and $A\in\mathcal{A}$, we denote by $F'(u,A)$ and $F''(u,A)$ the lower and upper $\Gamma$-limits

$$\begin{aligned} F'(u,A) &=: \inf\{\liminf_{j\to+\infty} F_j(u_j,A) \mid \{u_j\}\subset L^1(\Omega;\mathbf{R}^N),\ u_j\to u \text{ in } L^1(\Omega;\mathbf{R}^N)\}, \\ F''(u,A) &=: \inf\{\limsup_{j\to+\infty} F_j(u_j,A) \mid \{u_j\}\subset L^1(\Omega;\mathbf{R}^N),\ u_j\to u \text{ in } L^1(\Omega;\mathbf{R}^N)\}, \end{aligned}$$

so that with the previous notation $d$ is the metric of $L^1(\Omega;\mathbf{R}^N)$ and $X=L^1(\Omega;\mathbf{R}^N)$.

The following compactness theorem holds for the $\Gamma$-limit of localized functionals, see [17].

**Proposition 2.6** *Let $F_j : L^1(\Omega;\mathbf{R}^N)\times\mathcal{A}\to[0,+\infty]$ be a sequence of functionals. Suppose that for every $u\in L^1(\Omega;\mathbf{R}^N)$ the lower and upper $\Gamma$-limits*

$$\begin{aligned} \alpha'(A) &:= F'(u,A) = \Gamma(L^1(\Omega;\mathbf{R}^N))\text{-}\liminf_{j\to+\infty} F_j(u,A) \\ \alpha''(A) &:= F''(u,A) = \Gamma(L^1(\Omega;\mathbf{R}^N))\text{-}\limsup_{j\to+\infty} F_j(u,A) \end{aligned}$$

*define inner regular increasing set functions. Then there exists a subsequence $\{j_k\}$ such that the $\Gamma$-limit*

$$F(u,A) = \Gamma(L^1(\Omega;\mathbf{R}^N))\text{-}\lim_{k\to+\infty} F_{j_k}(u,A)$$

*exists for all $A\in\mathcal{A}$ and $u\in L^1(\Omega;\mathbf{R}^N)$.*

# 3 Functionals with p(x)-growth

In this paper we consider functionals $F : L^1(\Omega;\mathbf{R}^N)\times\mathcal{A}\to[0,+\infty]$ satisfying a growth condition of the form

$$0\le F(u,A)\le\beta\int_A (a(x)+|Du|^{p(x)})\,dx$$

for all $A\in\mathcal{A}$ and all $u\in L^1(\Omega;\mathbf{R}^N)$ such that $\int_\Omega(|u|^{p(x)}+|Du|^{p(x)})\,dx<+\infty$, where $\beta$ is a positive constant, $a(x)\in L^1(\Omega)$ and $p:\Omega\to\mathbf{R}$ are given functions with $p(x)\ge 1$.

It is natural to define, for every $A\in\mathcal{A}$, the sets

$$\begin{aligned} L^{p(x)}(A;\mathbf{R}^N) &:= \{u : A\to\mathbf{R}^N \mid \int_A |u|^{p(x)}\,dx<+\infty\}, \\ W^{1,p(x)}(A;\mathbf{R}^N) &:= \{u\in L^{p(x)}(A;\mathbf{R}^N) \mid Du\in L^{p(x)}(A;\mathbf{R}^{nN})\}, \\ W^{1,p(x)}_{\rm loc}(A;\mathbf{R}^N) &:= \{u : A\to\mathbf{R}^N \mid u_{|B}\in W^{1,p(x)}(B;\mathbf{R}^N)\quad \forall B\in\mathcal{A},\ B\subset\subset A\}, \end{aligned}$$

which coincide with the spaces $L^p(A;\mathbf{R}^N)$, $W^{1,p}(A;\mathbf{R}^N)$ and $W^{1,p}_{\rm loc}(A;\mathbf{R}^N)$ in case $p(x)\equiv p\ge 1$. In the sequel, the target space $\mathbf{R}^N$ will be omitted when it is clear from the context, for example within proofs.

Note that in general $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$ are not vector spaces: taking e.g. $p(x)$ so that $\int_\Omega 2^{p(x)}\,dx = +\infty$, clearly $u\equiv 1\in L^{p(x)}(\Omega)$ but $2u\notin L^{p(x)}(\Omega)$. It is easy to show that they become vector spaces if $\sup_\Omega p(x)<+\infty$, thus $W^{1,p(x)}_{\rm loc}(\Omega)$ is a vector space if $p(x)$ is locally bounded (as e.g. if $p(x)$ is continuous).

In any case, since $|\cdot|^{p(x)}$ is a convex function if $p(x)\ge 1$, it comes out that $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$ are convex sets.

We will work with continuous exponent functions $p(x)$, or in the last section with a particular class of discontinuous functions. We introduce the following assumptions, which we shall use only when needed.

**Definition 3.1** A family $\{\Omega_i\}$ is a locally finite regular partition of an open set $\Omega$ if each $\Omega_i$ is an open set with Lipschitz boundary and

$$\Omega = \Sigma\cup\bigcup_{i=1}^{+\infty}\Omega_i$$

where $|\Sigma|=0$, $\Omega_i\cap\Omega_j=\emptyset$ if $i\ne j$, and if $A\in\mathcal{A}_0$ yields $A\cap\Omega_i=\emptyset$ except for a finite number of indices.

**Definition 3.2** A function $p : \Omega \to [1, +\infty)$ is a regular piecewise continuous exponent if there exist a locally finite regular partition $\{\Omega_i\}$ of $\Omega$ and, for every $i$, a uniformly continuous function $p_i : \Omega_i \to [1, +\infty)$ such that $p(x) = p_i(x)$ for every $x \in \Omega_i$.

**Remark 3.3** If $p(x)$ is a regular piecewise continuous exponent satisfying

$$\inf_{x \in \Omega_i} p_i(x) > 1 \qquad \forall\, i \tag{3.1}$$

then for every $A \in \mathcal{A}_0$ there exist $\overline{p}$, $\overline{q}$ such that $1 < \overline{p} \le p(x) \le \overline{q} < +\infty$ a.e. in $A$.

Since we need to work with open sets $A$ which do not satisfy the assumptions of the Sobolev embedding theorem, if $u \in L^1(A)$ and $\int_A |Du|^{p(x)}\, dx < +\infty$ we cannot expect that $|u|^{p(x)}$ belongs to $L^1(A)$, see Example 3.5. However, we can prove the following

**Lemma 3.4** *Let $p : \Omega \to [1, +\infty)$ be continuous in $\Omega$. Then, for every function $u \in L^1(\Omega; \mathbf{R}^N)$ such that $\int_\Omega |Du|^{p(x)}\, dx < +\infty$ and every $A \in \mathcal{A}_0$ we have*

$$\int_A |u|^{p(x)}\, dx < +\infty\,. \tag{3.2}$$

*In addition, if $\Omega$ has Lipschitz boundary and $p(x)$ is uniformly continuous in $\Omega$, then (3.2) holds for every $A \in \mathcal{A}$ and in particular $|u|^{p(x)} \in L^1(\Omega)$.*

Proof: let us fix $A \in \mathcal{A}_0$; for every $x_0 \in \overline{A}$, choose $\varepsilon_{x_0} > 0$ so small that

$$1 \le p(x_0) - \varepsilon_{x_0} < p(x_0) + \varepsilon_{x_0} < (p(x_0) - \varepsilon_{x_0})^* \qquad \text{if} \quad p(x_0) > 1\,,$$

or just $p(x_0) + \varepsilon_{x_0} < 1^*$ if $p(x_0) = 1$. Now, by the continuity of $p(x)$ we can find $\delta = \delta(x_0, \varepsilon_{x_0}) > 0$ such that $B_\delta(x_0) \subset\subset \Omega$ and in $\overline{B}_\delta(x_0)$ either $|p(x) - p(x_0)| < \varepsilon_{x_0}$ if $p(x_0) \le n$, or $p(x) \ge n$ if $p(x_0) > n$. By the compactness of $\overline{A}$, we can extract a finite cover $\bigcup\limits_{i=1}^{m} B_{\delta_i}(x_i)$ of $A$, where $x_i \in \overline{A}$ and $\delta_i = \delta(x_i, \varepsilon_{x_i})$ for each $i = 1, \ldots, m$. Denoting now

$$B_i := B_{\delta_i}(x_i)\,, \qquad p^i_{\min} := \min_{x \in \overline{B}_i} p(x)\,, \qquad p^i_{\max} := \max_{x \in \overline{B}_i} p(x)\,,$$

we have

$$\int_{B_i} |Du|^{p^i_{\min}}\, dx \le \int_{B_i} (1 + |Du|^{p(x)})\, dx < +\infty\,,$$

hence $u \in W^{1, p^i_{\min}}(B_i)$ for each $i = 1, \ldots, m$. Since by construction either $p^i_{\min} \ge n$ or $p^i_{\max} < (p^i_{\min})^*$, the Sobolev embedding theorem yields that $u \in L^{p^i_{\max}}(B_i)$ for each $i = 1, \ldots, m$. Then we get

$$\int_A |u|^{p(x)}\, dx \le \sum_{i=1}^{m} \int_{B_i} (1 + |u|^{p^i_{\max}})\, dx < +\infty\,,$$

which proves (3.2). Finally, if $\Omega$ has Lipschitz boundary and $p(x)$ is uniformly continuous in $\Omega$, once $p(x)$ is extended in a continuous way to the closure $\overline{\Omega}$, to obtain $|u|^{p(x)} \in L^1(\Omega)$ we can argue exactly as above, simply replacing $B_{\delta_i}(x_i)$ with $B_{\delta_i}(x_i) \cap \overline{\Omega}$, provided the radii $\delta_i$ are chosen so that all these sets have Lipschitz boundary. $\square$

**Example 3.5** Remark that Lemma 3.4 may fail to hold if $p(x)$ is a generic discontinuous function. Indeed, if $x_0 \in \Omega$ and $p(x)$ is any function such that $\int_{B_\varepsilon(x_0)} 2^{p(x)}\, dx = +\infty$ for every $\varepsilon$, then the function $u(x) \equiv 2$ is in $L^1(\Omega)$ and $\int_\Omega |Du|^{p(x)}\, dx = 0$, but $|u|^{p(x)} \notin L^1_{\mathrm{loc}}(\Omega)$ since $\int_{B_\varepsilon(x_0)} |u|^{p(x)}\, dx = +\infty$. However, the following result clarifies the situation.

**Corollary 3.6** *If $p : \Omega \to [1, +\infty)$ is a regular piecewise continuous exponent, then from $u \in L^1(\Omega; \mathbf{R}^N)$ and $\int_\Omega |Du|^{p(x)}\, dx < +\infty$ we deduce $\int_A |u|^{p(x)}\, dx < +\infty$ for every $A \in \mathcal{A}_0$.*

To prove this result, it is enough to enclose $A$ in an open set $A' \in \mathcal{A}_0$ such that $A' \cap \Omega_i$ has Lipschitz boundary for every $i$, and to apply Lemma 3.4.

**Remark 3.7** It follows from Lemma 3.4 that if $p(x)$ is uniformly continuous in $A \in \mathcal{A}$, then

$$W^{1,p(x)}(A;\mathbf{R}^N) = \{u \in L^1(A;\mathbf{R}^N) \mid \int_A |Du|^{p(x)}\,dx < +\infty\}$$

if $A$ has Lipschitz boundary, or at least the cone property (see [4]). In any case, if $p(x)$ is continuous in $\Omega$ we have that

$$W^{1,p(x)}_{\mathrm{loc}}(A;\mathbf{R}^N) = \{u \in L^1_{\mathrm{loc}}(A;\mathbf{R}^N) \mid |Du|^{p(x)} \in L^1_{\mathrm{loc}}(A)\}\,. \tag{3.3}$$

In the sequel we will also need the following result.

**Lemma 3.8** *Let $p : \Omega \to [1,+\infty)$ be continuous in $\Omega$. If $u \in L^1_{\mathrm{loc}}(\Omega;\mathbf{R}^N)$, then for every open set $A \in \mathcal{A}_0$ with Lipschitz boundary and for every sequence $\{u_j\}$ converging to $u$ in $L^1(A;\mathbf{R}^N)$ with*

$$\sup_{j\in\mathbf{N}} \int_A |Du_j|^{p(x)}\,dx < +\infty\,, \tag{3.4}$$

*we have*

$$\lim_{j\to+\infty} \int_A |u_j - u|^{p(x)}\,dx = 0\,.$$

PROOF: if $\bigcup_{i=1}^m B_{\delta_i}(x_i)$ is a finite cover of $A$ defined as in Lemma 3.4, but with the radii $\delta_i$ chosen so that $A_i := B_{\delta_i}(x_i) \cap A$ has Lipschitz boundary, we deduce by (3.4)

$$\int_{A_i} |Du_j|^{p^i_{\min}}\,dx \le C \qquad \forall\, i = 1,\dots,m\,, \quad \forall\, j \in \mathbf{N}\,,$$

thus $\{u_j\}$ is bounded in $W^{1,p^i_{\min}}(A_i)$, for each $i = 1,\dots,m$. By Rellich's theorem we obtain that $u_j \to u$ in $L^{p^i_{\max}}(A_i)$ for each $i$, hence $u \in L^{p(x)}(A)$ and

$$\lim_{j\to+\infty} \int_A |u_j - u|^{p(x)}\,dx \le \lim_{j\to+\infty} \sum_{i=1}^m \int_{A_i} |u_j - u|^{p(x)}\,dx \le \lim_{j\to+\infty} \sum_{i=1}^m \int_{A_i} (|u_j - u|^{p^i_{\min}} + |u_j - u|^{p^i_{\max}})\,dx = 0$$

and the proof is complete. □

We have an analogous of this result for discontinuous exponents:

**Corollary 3.9** *The result of Lemma 3.8 holds again for regular piecewise continuous exponents $p : \Omega \to [1,+\infty)$.*

**Remark 3.10** By Remark 3.7 the sequence $\{u_j\}$ of Lemma 3.8 belongs to $W^{1,p(x)}(A;\mathbf{R}^N)$; in addition, by Proposition 3.13 below we also have $u \in W^{1,p(x)}(A;\mathbf{R}^N)$.

If $A', A$ are open sets in $\mathcal{A}$, with $A' \subset\subset A$, a cut-off function between $A'$ and $A$ is a smooth function $\varphi \in C^\infty_0(\Omega)$ with $\operatorname{spt}\varphi \subset A$, $0 \le \varphi \le 1$ and $\varphi \equiv 1$ on $A'$. In order to prove the fundamental estimate for a family of functionals with $p(x)$-growth (compare Proposition 5.4) we will also need the following

**Lemma 3.11** *Let $p : \Omega \to [1,+\infty)$ be continuous in $\Omega$. Also, let $A', A, B \in \mathcal{A}$ with $A' \subset\subset A$, and let $u \in W^{1,p(x)}_{\mathrm{loc}}(A;\mathbf{R}^N)$ and $v \in W^{1,p(x)}_{\mathrm{loc}}(B;\mathbf{R}^N)$. Then for every cut-off function $\varphi$ between $A'$ and $A$ we have that $\varphi u + (1-\varphi)v \in W^{1,p(x)}_{\mathrm{loc}}(A' \cup B;\mathbf{R}^N)$.*

PROOF: clearly $\varphi u + (1-\varphi)v \in L^1_{\mathrm{loc}}(A' \cup B)$; in addition, by setting $K := \operatorname{spt}\varphi$ and $q := \sup_K p(x) < +\infty$, for every $C \in \mathcal{A}$ with $C \subset\subset A' \cup B$ we have

$$\int_C |D(\varphi u + (1-\varphi)v)|^{p(x)}\,dx = \int_C |\varphi Du + (1-\varphi)Dv + D\varphi \otimes (u-v)|^{p(x)}\,dx =$$
$$= \int_{C\cap A'} |Du|^{p(x)}\,dx + \int_{C\setminus K} |Dv|^{p(x)}\,dx + \int_{(C\cap K)\setminus A'} |\cdot|^{p(x)}\,dx =: I + II + III\,.$$

Since $C \cap A' \subset\subset A$ and $C \setminus K \subset\subset B$, the terms $I$ and $II$ are finite; also,

$$III \leq 4^{q-1} \int_{(C\cap K)\setminus A'} [(|Du|^{p(x)} + |Dv|^{p(x)}) + (1 + \|D\varphi\|_\infty^q)(|u|^{p(x)} + |v|^{p(x)})]\, dx$$

which is finite, since $(C \cap K) \setminus A' \subset\subset A \cap B$, and by (3.3) the proof is complete. □

**Corollary 3.12** *The result of Lemma* 3.11 *holds again for regular piecewise continuous exponents* $p : \Omega \to [1, +\infty)$.

In order to preserve the $p(x)$-growth condition in the $\Gamma$-limit (see Theorem 5.1 and Proposition 5.3) we need to study some measure-theoretic and lower semicontinuity properties of the functionals $\Psi_{p(x)}, \widetilde{\Psi}_{p(x)} : L^1(\Omega; \mathbf{R}^N) \times \mathcal{A} \to [0, +\infty]$ given by

$$\Psi_{p(x)}(u, A) \quad := \quad \begin{cases} \displaystyle\int_A |Du(x)|^{p(x)}\, dx & \text{if } u \in W^{1,p(x)}_{\mathrm{loc}}(A; \mathbf{R}^N) \\ +\infty & \text{elsewhere on } L^1(\Omega; \mathbf{R}^N), \end{cases} \tag{3.5}$$

$$\widetilde{\Psi}_{p(x)}(u, A) \quad := \quad \begin{cases} \displaystyle\int_A (|u|^{p(x)} + |Du(x)|^{p(x)})\, dx & \text{if } u \in W^{1,p(x)}_{\mathrm{loc}}(A; \mathbf{R}^N) \\ +\infty & \text{elsewhere on } L^1(\Omega; \mathbf{R}^N), \end{cases} \tag{3.6}$$

where $p : \Omega \to [1, +\infty)$ is a given function (possibly discontinuous).

It is easy to verify that $\Psi_{p(x)}(u, \cdot)$ and $\widetilde{\Psi}_{p(x)}(u, \cdot)$ are measures on $\mathcal{A}$ for every $u \in L^1(\Omega; \mathbf{R}^N)$. We remark that if we replace $W^{1,p(x)}_{\mathrm{loc}}(A)$ with $W^{1,p(x)}(A)$ in (3.5), then $\Psi_{p(x)}(u, \cdot)$ may fail to be inner regular, and hence a measure, even if $p(x) \equiv p > 1$, since we may find $u \in L^1(\Omega) \setminus L^p(A)$ with $Du \in L^p(A)$ when the boundary of $A$ is not smooth. In the definition (3.6) of $\widetilde{\Psi}_{p(x)}$ we can replace $W^{1,p(x)}_{\mathrm{loc}}(A)$ with $W^{1,p(x)}(A)$ still obtaining exactly the same functional, due to the presence of the term $\int_A |u|^{p(x)}\, dx$. In addition we are able to prove lower semicontinuity of the functionals $\Psi_{p(x)}$ and $\widetilde{\Psi}_{p(x)}$.

**Proposition 3.13** *If* $p : \Omega \to (1, +\infty)$ *is continuous and such that* $p(x) \geq p > 1$ *on* $\Omega$*, then for every* $A \in \mathcal{A}$ *the functional* $\Psi_{p(x)}(\cdot, A)$ *defined in* (3.5) *is lower continuous in* $L^1(\Omega; \mathbf{R}^N)$*. The same result holds for the functional* $\widetilde{\Psi}_{p(x)}(\cdot, A)$ *defined in* (3.6) *without any continuity assumption on* $p(x)$.

Proof: let $u_j \to u$ in $L^1(\Omega)$ and $A \in \mathcal{A}$: we have to prove that

$$\Psi_{p(x)}(u, A) \leq \liminf_{j\to+\infty} \Psi_{p(x)}(u_j, A). \tag{3.7}$$

We may assume that the right-hand side is finite and that we are working on a subsequence, which we still label $\{u_j\}$, such that the lower limit in (3.7) is a limit. Then $\{u_j\} \subset W^{1,p(x)}_{\mathrm{loc}}(A)$ and $\int_A |Du_j|^{p(x)}\, dx \leq c$, which implies that $\{Du_j\}$ is bounded in $L^p(A)$ since

$$\int_A |Du_j|^p \leq \int_A (1 + |Du_j|^{p(x)})\, dx \leq c.$$

Thus we may also suppose that $\{Du_j\}$ converges weakly in $L^p(A)$, but $u_j \to u$ in $L^1(\Omega)$, so it is easy to verify that $Du \in L^p(A)$ and $Du_j \rightharpoonup Du$ in $L^p(A)$. Now we may apply De Giorgi's lower semicontinuity theorem [7, Thm. 2.3.1], which holds without any regularity assumption on $A$ and by which the functional $G(v, w) = \int_A |w|^{p(x)}\, dx$, defined for $v \in L^1(A; \mathbf{R}^N)$, $w \in L^p(A; \mathbf{R}^{nN})$, is lower semicontinuous with respect to the strong topology in $L^1(A; \mathbf{R}^N)$ and the weak topology in $L^p(A; \mathbf{R}^{nN})$, obtaining that

$$\int_A |Du|^{p(x)}\, dx \leq \lim_{j\to+\infty} \int_A |Du_j|^{p(x)}\, dx = \lim_{j\to+\infty} \Psi_{p(x)}(u_j, A) < +\infty.$$

Then $\int_A |Du|^{p(x)}\, dx < +\infty$ and by the continuity assumption on $p(x)$ we may apply Lemma 3.4 to conclude that $u \in W^{1,p(x)}_{\mathrm{loc}}(A)$ and $\Psi_{p(x)}(u, A) = \int_A |Du|^{p(x)}\, dx$, which yields (3.7) and completes the proof.

To prove lower semicontinuity of the functional $\widetilde{\Psi}_{p(x)}(\cdot, A)$, we apply De Giorgi's theorem to the functional $G(v,w) = \int_A (|v|^{p(x)} + |w|^{p(x)})\, dx$ and we may conclude without applying Lemma 3.4, thus the continuity of $p(x)$ is not needed. $\square$

**Corollary 3.14** *The result of Proposition* 3.13 *holds again for regular piecewise continuous exponents, provided* (3.1) *holds.*

**Example 3.15** Remark that $\Psi_{p(x)}(\cdot, A)$ may fail to be $L^1(\Omega)$-l.s.c. if $p(x)$ is not continuous. Indeed, if we take $p(x)$ as in Example 3.5, but with

$$\int_\Omega (2-\delta)^{p(x)}\, dx < +\infty \qquad \forall\, \delta > 0\,,$$

then the sequence $u_j(x) := 2 - 1/j$ converges to $u(x) := 2$ in $L^1(\Omega)$ and $Du_j \equiv 0$, thus $\Psi_{p(x)}(u_j, \Omega) = 0$, but $\Psi_{p(x)}(u, \Omega) = +\infty$ since $u \notin W^{1,p(x)}_{\mathrm{loc}}(\Omega)$.

For our purposes, mainly to prove the integral representation theorem 4.1, we need to introduce a suitable notion of strong convergence on the set $W^{1,p(x)}_{\mathrm{loc}}(\Omega)$, which in particular coincides with the $W^{1,p}_{\mathrm{loc}}$ convergence in the constant case $p(x) \equiv p \geq 1$. More precisely, we give the following

**Definition 3.16** We say that a sequence $\{u_j\} \subset W^{1,p(x)}_{\mathrm{loc}}(\Omega; \mathbf{R}^N)$ converges to $u \in W^{1,p(x)}_{\mathrm{loc}}(\Omega; \mathbf{R}^N)$ strongly in $W^{1,p(x)}_{\mathrm{loc}}(\Omega; \mathbf{R}^N)$ if

$$\lim_{j \to +\infty} \int_A (|u_j - u|^{p(x)} + |Du_j - Du|^{p(x)})\, dx = 0 \tag{3.8}$$

for every $A \in \mathcal{A}_0$.

**Remark 3.17** If $u_j \to u$ in $W^{1,p(x)}_{\mathrm{loc}}(\Omega; \mathbf{R}^N)$, then $|Du_j|^{p(x)} \to |Du|^{p(x)}$ in $L^1_{\mathrm{loc}}(\Omega)$, as an easy consequence of the estimate $|Du_j|^{p(x)} \leq 2^{p(x)-1}(|Du_j - Du|^{p(x)} + |Du|^{p(x)})$ and of the dominated convergence theorem.

We will make use of the following density result with respect to the convergence above, which is essentially due to Zhikov [30], compare also [1, Lemma 4.5].

**Proposition 3.18** *Let $p : \Omega \to [1, +\infty)$ be a continuous function satisfying the following local estimate about the modulus of continuity:*

$$\forall\, A \in \mathcal{A}_0 \quad \exists\, \gamma_A > 0 : \quad |p(x) - p(y)| \leq \frac{\gamma_A}{|\log|x-y||} \quad \forall\, x, y \in A\,, \quad 0 < |x-y| < \frac{1}{2}\,. \tag{3.9}$$

*Then for every $u \in W^{1,p(x)}_{\mathrm{loc}}(\Omega; \mathbf{R}^N)$ there exists a sequence of smooth functions $\{u_j\} \subset C^\infty_0(\Omega; \mathbf{R}^N)$ such that $u_j \to u$ in $W^{1,p(x)}_{\mathrm{loc}}(\Omega; \mathbf{R}^N)$. If in addition $u \in W^{1,p(x)}(\Omega; \mathbf{R}^N)$, then $u_j \to u$ also in $L^1(\Omega; \mathbf{R}^N)$.*

PROOF: let $\rho(x)$ be a standard mollifier with support in the unit ball $B_1(0) \subset \mathbf{R}^n$, for every $j \in \mathbf{N}$ let $\rho_j(x) := j^n \rho(j\, x)$, and set

$$u_j(x) := (\overline{u}_j * \rho_j)(x)\,, \qquad \overline{u}_j(x) := \begin{cases} u(x) & \text{if } \operatorname{dist}(x, \partial\Omega) > 1/j \\ 0 & \text{otherwise}\,. \end{cases}$$

Then it is well known that $u_j \in C^\infty_0(\Omega)$, $u_j \to u$ in $W^{1,1}_{\mathrm{loc}}(\Omega)$, and $u_j \to u$ in $L^1(\Omega)$ if $u \in L^1(\Omega)$. To prove that $|Du_j - Du|^{p(x)} \to 0$ in $L^1_{\mathrm{loc}}(\Omega)$, fix $A \in \mathcal{A}_0$, choose $0 < \varepsilon_0 < 1$ so that $\operatorname{dist}(A, \partial\Omega) \geq 2\varepsilon_0$, call $q_0 := \sup_A p(x)$ and try to apply the Lebesgue dominated convergence theorem to the sequence $\{|Du_j - Du|^{p(x)}\}_{j > 2/\varepsilon_0}$. Since a.e. in $A$ we have $Du_j = Du * \rho_j \to Du$ and $|Du_j - Du|^{p(x)} \leq 2^{q_0 - 1}(|Du_j|^{p(x)} + |Du|^{p(x)})$, it is enough to estimate from above the sequence $\{|Du_j|^{p(x)}\}_{j > 2/\varepsilon_0}$ with some sequence strongly convergent in $L^1(A)$. To do this, let us set

$$A' := \{x \in \Omega \mid \operatorname{dist}(x, A) < \varepsilon_0\}\,, \quad 1 \leq p := \inf_{x \in A'} p(x)\,, \quad q := \sup_{x \in A'} p(x) < +\infty\,,$$
$$g(x) := |Du|^{p(x)}, \quad p_j(x) := \min\{p(y) \mid |x - y| \leq 1/j\} \quad \text{and} \quad \varphi_j(x, z) := |z|^{p_j(x)},\ z \in \mathbf{R}^{nN}.$$

Using the fact that $\varphi_j(x, Du(y)) \le 1 + |Du(y)|^{p(y)}$ if $|x-y| \le 1/j$, we are able to estimate $|Du_j|^{p_j(x)}$. Indeed, since $\varphi_j$ is a Carathéodory function convex with respect to $z$, by Jensen inequality we have

$$\begin{aligned} \varphi_j(x, Du_j(x)) &\le \int_{|x-y|\le 1/j} \rho_j(x-y)\varphi_j(x, Du(y))\,dy \\ &\le \int_{|x-y|\le 1/j} \rho_j(x-y)(1+|Du(y)|^{p(y)})\,dy = 1 + (g * \rho_j)(x)\,. \end{aligned} \tag{3.10}$$

In addition, using Hölder inequality if $p>1$, and simply estimating $\rho_j$ if $p=1$, we obtain

$$\begin{aligned} |Du_j(x)| &\le \int_{|x-y|\le 1/j} |\rho_j(x-y)\,Du(y)|\,dy \\ &\le \|Du\|_{L^p(A')} \cdot j^{n-n(p-1)/p} \cdot \|\rho\|_{L^{p/(p-1)}(B_1(0))} = c(p)\,\|Du\|_{L^p(A')} \cdot j^{n/p}. \end{aligned} \tag{3.11}$$

Therefore, by (3.10) and (3.11) we have

$$\begin{aligned} |Du_j(x)|^{p(x)} &= |Du_j(x)|^{p(x)-p_j(x)} \cdot \varphi_j(x, Du_j(x)) \\ &\le (1 + c(p)^{q-p}\,\|Du\|^{q-p}_{L^p(A')}) \cdot (j^{n/p})^{p(x)-p_j(x)} \cdot (1 + (g*\rho_j)(x))\,. \end{aligned}$$

Finally, if $x \in A$, estimate (3.9) means that $0 \le p(x) - p_j(x) \le \gamma_A/\log j$, hence

$$(j^{n/p})^{p(x)-p_j(x)} \le (j^{\gamma_A/\log j})^{n/p} = e^{n\gamma_A/p}$$

and then we obtain

$$|Du_j(x)|^{p(x)} \le c\,(1 + (g*\rho_j)(x))$$

for all $x \in A$, where $c>0$ only depends on $n$, $p$, $q$, $\gamma_A$ and $u$. Now, since $g * \rho_j \to g$ in $L^1(A)$, we can apply the Lebesgue dominated convergence theorem to conclude at the same time that $|Du_j(x)|^{p(x)} \to |Du(x)|^{p(x)}$ and $|Du_j(x) - Du(x)|^{p(x)} \to 0$ in $L^1(A)$, as required.

Finally, it follows directly from Lemma 3.8 that $\int_A |u_j - u|^{p(x)}\,dx \to 0$, thus $u_j \to u$ in $W^{1,p(x)}_{\mathrm{loc}}(\Omega)$ and the proof is complete. □

We recall now that a function $u \in L^1(\Omega; \mathbf{R}^N)$ is piecewise affine in $\Omega$ if there exists a finite family $\{\Omega_i\}_{i\in I}$ of disjoint open subsets of $\Omega$ and a Borel subset $N$ of $\Omega$ with $|N|=0$ such that $\Omega = (\bigcup_{i\in I}\Omega_i)\bigcup N$ and $u_{|\Omega_i}$ is affine on each $\Omega_i$. The following density result holds.

**Proposition 3.19** *For every $u \in W^{1,p(x)}(\Omega; \mathbf{R}^N)$ there exists a sequence $\{u_j\} \subset W^{1,p(x)}(\Omega; \mathbf{R}^N)$ of functions which are piecewise affine on $\Omega$ and such that $u_j \to u$ both in $L^1(\Omega; \mathbf{R}^N)$ and in $W^{1,p(x)}_{\mathrm{loc}}(\Omega; \mathbf{R}^N)$.*

PROOF: by Proposition 3.18 for every $u \in W^{1,p(x)}(\Omega)$ there exists a sequence $\{u_j\} \subset C_0^\infty(\Omega)$ converging to $u$ both in $L^1(\Omega)$ and in $W^{1,p(x)}_{\mathrm{loc}}(\Omega)$. On the other side, by [19, Ch. X, Prop. 2.1], given $u \in C_0^\infty(\Omega)$ there exists a sequence $\{u_j\}$ of functions in $C_0^0(\Omega)$ which are piecewise affine on $\Omega$ and such that $u_j \to u$ and $Du_j \to Du$ uniformly in $\Omega$. Then $u_j \in W^{1,p(x)}(\Omega)$ and, since uniform convergence implies that $\int_\Omega (|u_j - u|^{p(x)} + |Du_j - Du|^{p(x)})\,dx \to 0$, the desired sequence is obtained by a diagonal procedure. □

# 4 An integral representation result

In this section we state and prove an integral representation theorem for a class of functionals with $p(x)$-growth. More precisely, we have the following

**Theorem 4.1** *Let $p : \Omega \to [1, +\infty)$ be a continuous function satisfying* (3.9). *Let $F : L^1(\Omega; \mathbf{R}^N) \times \mathcal{A} \to [0, +\infty]$ satisfy the following conditions:*

i) (locality) $F$ *is local, i.e.,* $F(u,A)=F(v,A)$ *for every* $A\in\mathcal{A}$ *and* $u,v\in L^1(\Omega;\mathbf{R}^N)$ *with* $u=v$ *a.e. on* $A$*;*

ii) (measure property) *for all* $u\in L^1(\Omega;\mathbf{R}^N)$ *the set function* $F(u,\cdot)$ *is increasing, and is the trace on* $\mathcal{A}$ *of a Borel measure;*

iii) (growth conditions) *there exist* $\beta>0$ *and* $a(x)\in L^1_{\rm loc}(\Omega)$ *such that*

$$0\le F(u,A)\le\beta\int_A(a(x)+|Du|^{p(x)})\,dx$$

*for all* $u\in W^{1,p(x)}(\Omega;\mathbf{R}^N)$ *and* $A\in\mathcal{A}$*;*

iv) (translation invariance in $u$) $F(u+c,A)=F(u,A)$ *for all* $u\in L^1(\Omega;\mathbf{R}^N)$, $A\in\mathcal{A}$, $c\in\mathbf{R}^N$*;*

v) (lower semicontinuity) $F(\cdot,A)$ *is sequentially lower semicontinuous with respect to the strong convergence in* $L^1(\Omega;\mathbf{R}^N)$ *for all* $A\in\mathcal{A}$*.*

*Then there exists a Carathéodory function* $f:\Omega\times\mathbf{R}^{nN}\to[0,+\infty)$ *such that*

$$F(u,A)=\int_A f(x,Du(x))\,dx \tag{4.1}$$

*for every* $A\in\mathcal{A}$ *and for every* $u\in L^1(\Omega;\mathbf{R}^N)$ *such that* $u_{|A}\in W^{1,p(x)}_{\rm loc}(A;\mathbf{R}^N)$*; in addition, the function* $f(x,\cdot)$ *is quasi-convex on* $\mathbf{R}^{nN}$ *for a.e.* $x\in\Omega$ *and satisfies the growth condition*

$$0\le f(x,z)\le\beta\,(a(x)+|z|^{p(x)}) \tag{4.2}$$

*for a.e.* $x\in\Omega$ *and all* $z\in\mathbf{R}^{nN}$*.*

Proof: the proof of Theorem 4.1 works as in the classical case of Sobolev spaces, see [9, Thm. 1.1], with the exception of Steps 4, 5 and 7; for more details on the other steps see e.g. [14, Thm. 20.1] and [10, Thm. 9.1].
*Step 1: definition of $f$.*
For every $z\in\mathbf{R}^{nN}$ denote by $u_z:\Omega\to\mathbf{R}^N$ the linear function $u_z(x)=zx$. By ii), $F(u_z,\cdot)$ can be extended to a Borel measure on $\Omega$ which, by iii), is absolutely continuous with respect to the Lebesgue measure. Hence, there exists a density function $g_z\in L^1_{\rm loc}(\Omega)$ such that

$$F(u_z,A)=\int_A g_z(x)\,dx$$

for all $A\in\mathcal{A}$. We set $f(x,z):=g_z(x)$ for all $x\in\Omega$ and $z\in\mathbf{R}^{nN}$. Then it is easy to verify that $f$ is a Borel function satisfying (4.2) for a.e. $x\in\Omega$ and all $z\in\mathbf{R}^{nN}$.
*Step 2: integral representation on piecewise affine functions.*
By the measure property of $F$ it is easy to show that (4.1) holds for all $A\in\mathcal{A}$ and all $u\in W^{1,p(x)}(\Omega)$ which are piecewise affine on $\Omega$.
*Step 3: rank-one-convexity of $f$.*
It can be shown that for every fixed $x\in\Omega$ the function $f(x,\cdot)$ is rank-one-convex, that is

$$f(x,tz_2+(1-t)z_1)\le tf(x,z_2)+(1-t)f(x,z_1)$$

for all $z_1,z_2\in\mathbf{R}^{nN}$ with $\mathrm{rank}(z_1-z_2)\le1$ and for every $t\in(0,1)$. In particular it is well known that a rank-one-convex function with a growth condition of order $p$ satisfies a local Lipschitz condition, hence $f(x,\cdot)$ is locally Lipschitz for every $x\in\Omega$ and then $f$ is a Carathéodory function.
*Step 4: the inequality* $F(u,A)\le\int_A f(x,Du(x))\,dx$ *for* $u\in W^{1,p(x)}(\Omega;\mathbf{R}^N)$ *and* $A\in\mathcal{A}$*.*
As $f$ is a Carathéodory function (see Step 3) satisfying the growth conditions (4.2), we have that for every $A'\in\mathcal{A}_0$ the functional

$$u\mapsto\int_{A'}f(x,Du(x))\,dx \tag{4.3}$$

is continuous with respect to the $W^{1,p(x)}_{\mathrm{loc}}(\Omega)$ convergence of Definition 3.16 (use the dominated convergence theorem on subsequences).

Now, let $u \in W^{1,p(x)}(\Omega)$ and $A \in \mathcal{A}$. By Proposition 3.19 there exists a sequence $\{u_j\}$ of functions in $W^{1,p(x)}(\Omega)$ which are piecewise affine on $\Omega$ and such that $u_j \to u$ in $L^1(\Omega)$ and in $W^{1,p(x)}_{\mathrm{loc}}(\Omega)$. Then by lower semicontinuity v) of $F$, Step 2 and the continuity of the functional (4.3) in $W^{1,p(x)}_{\mathrm{loc}}(\Omega)$, we obtain for every $A' \in \mathcal{A}_0$, $A' \subset\subset A$, that

$$F(u, A') \leq \liminf_{j\to+\infty} F(u_j, A') = \lim_{j\to+\infty} \int_{A'} f(x, Du_j(x))\, dx = \int_{A'} f(x, Du(x))\, dx\, .$$

Since $F(u,\cdot)$ is a measure, taking the limit as $A' \nearrow A$ we get by the monotone convergence theorem

$$F(u, A) \leq \int_A f(x, Du(x))\, dx \tag{4.4}$$

for every $u \in W^{1,p(x)}(\Omega)$ and for every $A \in \mathcal{A}$.

*Step 5: the equality* $F(u, A) = \displaystyle\int_A f(x, Du(x))\, dx$ *for* $u \in W^{1,p(x)}(\Omega; \mathbf{R}^N)$ *and* $A \in \mathcal{A}$.

Fix $u \in W^{1,p(x)}(\Omega)$ and let $A, A' \in \mathcal{A}$ with $A' \subset\subset A$. We modify the function $u$ in the following way: take $A'' \in \mathcal{A}_0$ such that $A' \subset\subset A'' \subset\subset \Omega$, let $\varphi$ be a cut-off function between $A'$ and $A''$ and set $\widetilde{u} := \varphi u$. It is easy to verify that $\widetilde{u} \in W^{1,p(x)}(\Omega)$ and that $\widetilde{u} + v \in W^{1,p(x)}(\Omega)$ for every $v \in W^{1,p(x)}(\Omega)$, since $\widetilde{u}$ has compact support.

Consider the functional $G : L^1(\Omega) \times \mathcal{A} \to [0, +\infty]$ defined by

$$G(v, B) := F(v + \widetilde{u}, B)\, .$$

Then $G$ satisfies all hypotheses of Theorem 4.1. Indeed, i), ii), iv) and v) are trivially satisfied, whereas for all $v \in W^{1,p(x)}(\Omega)$ and all $B \in \mathcal{A}$ we have

$$0 \leq G(v, B) = F(v + \widetilde{u}, B) \leq \beta \int_B (a(x) + |D\widetilde{u} + Dv|^{p(x)})\, dx \leq \gamma \int_B (b(x) + |Dv|^{p(x)})\, dx$$

where $\gamma = 2^{q-1}\beta$, with $q = \sup_{x\in A''} p(x) < +\infty$, and $b(x) = a(x) + |D\widetilde{u}(x)|^{p(x)} \in L^1_{\mathrm{loc}}(\Omega)$. Therefore from Steps 1-4 above it follows that there exists a Carathéodory function $g : \Omega \times \mathbf{R}^{nN} \to [0, +\infty)$, satisfying (4.2) with $\gamma$ and $b(x)$ instead of $\beta$ and $a(x)$, such that

$$G(v, B) \leq \int_B g(x, Dv(x))\, dx \qquad \forall\, v \in W^{1,p(x)}(\Omega)\, , \quad \forall\, B \in \mathcal{A}\, , \tag{4.5}$$

with equality for $v$ piecewise affine in $\Omega$. In addition, arguing as for (4.3), we can prove that for every $B' \in \mathcal{A}_0$ the functional

$$v \mapsto \int_{B'} g(x, Dv(x))\, dx \tag{4.6}$$

is continuous in $W^{1,p(x)}_{\mathrm{loc}}(\Omega)$. We now prove that

$$F(u, A') = \int_{A'} f(x, Du(x))\, dx\, ; \tag{4.7}$$

since $F(u,\cdot)$ is a measure, taking $A' \nearrow A$ we will obtain (4.1) for all $A \in \mathcal{A}$ and $u \in W^{1,p(x)}(\Omega)$.

By Proposition 3.19 there exists a sequence $\{u_j\}$ of functions in $W^{1,p(x)}(\Omega)$, piecewise affine in $\Omega$, such that $u_j \to \widetilde{u}$ in $L^1(\Omega)$ and in $W^{1,p(x)}_{\mathrm{loc}}(\Omega)$. Then, using the locality i) of $F$, Step 4, Step 2, (4.5) and the continuity of the functionals (4.3) and (4.6), we obtain

$$\begin{aligned}
\int_{A'} g(x, 0)\, dx &= G(0, A') = F(\widetilde{u}, A') = F(u, A') \leq \int_{A'} f(x, Du)\, dx \\
&= \int_{A'} f(x, D\widetilde{u})\, dx = \lim_{j\to+\infty} \int_{A'} f(x, Du_j)\, dx = \lim_{j\to+\infty} F(u_j, A') \\
&= \lim_{j\to+\infty} G(u_j - \widetilde{u}, A') \leq \lim_{j\to+\infty} \int_{A'} g(x, D(u_j - \widetilde{u}))\, dx = \int_{A'} g(x, 0)\, dx
\end{aligned}$$

and (4.7) is proved.

*Step 6: the equality* $F(u,A) = \int_A f(x, Du(x))\,dx$ *for* $u_{|A} \in W^{1,p(x)}_{\mathrm{loc}}(A;\mathbf{R}^N)$ *and* $A \in \mathcal{A}$.

If $u \in L^1(\Omega)$, $A \in \mathcal{A}$ and $u_{|A} \in W^{1,p(x)}_{\mathrm{loc}}(A)$, then for every $A' \in \mathcal{A}_0$, $A' \subset\subset A$, we can find a function $v \in W^{1,p(x)}(\Omega)$ such that $v_{|A'} = u_{|A'}$ (it suffices to take $v = \varphi u$, where $\varphi \in C_0^\infty(\Omega)$ is a cut-off function between $A'$ and $A''$, with $A' \subset\subset A'' \subset\subset A$). Then, by the locality of $F$ and Step 5 we have

$$F(u,A') = F(v,A') = \int_{A'} f(x, Dv(x))\,dx = \int_{A'} f(x, Du(x))\,dx$$

and we obtain the assertion as $A' \nearrow A$, by the measure property of $F$.

*Step 7: quasi-convexity of* $f$.

It is enough to prove that, for every $A \in \mathcal{A}_0$ with Lipschitz boundary, $f(x,\cdot)$ is quasi-convex on $\mathbf{R}^{nN}$ for a.e. $x \in A$. If $A \in \mathcal{A}_0$ is fixed and $q := \sup_A p(x) < +\infty$, the restriction $f : A \times \mathbf{R}^{nN} \to [0,+\infty)$ is a Carathéodory integrand with

$$0 \le f(x,z) \le \beta(a(x) + |z|^{p(x)}) \le \beta(1 + a(x) + |z|^q) \le b(x)(1 + |z|^q)$$

where $b(x) := \beta(1 + a(x)) \in L^1(A)$ . In addition, by lower semicontinuity v) it is easy to verify that the functional (4.3) is sequentally weakly l.s.c. on $W^{1,q}(A)$; hence we can apply [7, Thm. 4.1.5] to obtain that $f(x,\cdot)$ is quasi-convex in $\mathbf{R}^{nN}$ for a.e. $x \in A$. □

# 5 A Γ-convergence result

In this section we state and prove the main result of the paper, which characterizes the Γ-limit in the $L^1(\Omega;\mathbf{R}^N)$ topology of a class of functionals with $p(x)$-growth. We will consider functions satisfying

$$\alpha|z|^{p(x)} \le f(x,z) \le \beta(a(x) + |z|^{p(x)}) \tag{5.1}$$

for a.e. $x \in \Omega$ and all $z \in \mathbf{R}^{nN}$, where $0 < \alpha \le \beta$ are positive constants and $a(x) \in L^1(\Omega)$. More precisely, we are able to prove the following results.

**Theorem 5.1** *Let* $p : \Omega \to (1,+\infty)$ *be a continuous function with* $p(x) \ge p > 1$ *for all* $x \in \Omega$. *Let* $f_j : \Omega \times \mathbf{R}^{nN} \to [0,+\infty)$ *be Borel functions satisfying* (5.1) *for a.e.* $x \in \Omega$ *and all* $z \in \mathbf{R}^{nN}$. *Finally let* $F_j : L^1(\Omega;\mathbf{R}^N) \times \mathcal{A} \to [0,+\infty]$ *be the variational functionals defined by*

$$F_j(u,A) := \begin{cases} \displaystyle\int_A f_j(x, Du(x))\,dx & \text{if } u \in W^{1,p(x)}_{\mathrm{loc}}(A;\mathbf{R}^N) \\ +\infty & \text{elsewhere on } L^1(\Omega;\mathbf{R}^N)\,. \end{cases} \tag{5.2}$$

*Then there exists a subsequence* $\{j_k\}$ *such that the* Γ*-limit*

$$F(u,A) := \Gamma(L^1(\Omega;\mathbf{R}^N))\text{-}\lim_{k\to+\infty} F_{j_k}(u,A) \tag{5.3}$$

*exists for all* $u \in L^1(\Omega;\mathbf{R}^N)$ *and* $A \in \mathcal{A}$, *with estimates*

$$\alpha\,\Psi_{p(x)}(u,A) \le F(u,A) \le \beta\left(\int_A a(x)\,dx + \Psi_{p(x)}(u,A)\right) \tag{5.4}$$

*for all* $u \in L^1(\Omega;\mathbf{R}^N)$ *and* $A \in \mathcal{A}$, *where the functional* $\Psi_{p(x)}$ *is defined in* (3.5).

**Theorem 5.2** *Let* $F(u,A)$ *be the as in* (5.3) *and, in addition to the hypotheses of Theorem* 5.1, *suppose that the function* $p$ *satisfies* (3.9). *Then there exists a Carathéodory function* $f : \Omega \times \mathbf{R}^{nN} \to [0,+\infty)$, *satisfying the growth estimate* (5.1) *for a.e.* $x \in \Omega$ *and all* $z \in \mathbf{R}^{nN}$, *with* $f(x,\cdot)$ *quasi-convex for a.e.* $x \in \Omega$, *such that for every* $A \in \mathcal{A}$

$$F(u,A) = \begin{cases} \displaystyle\int_A f(x,Du(x))\,dx & \text{if } u \in W^{1,p(x)}_{\mathrm{loc}}(A;\mathbf{R}^N) \\ +\infty & \text{elsewhere on } L^1(\Omega;\mathbf{R}^N)\,. \end{cases} \tag{5.5}$$

Proof of Theorem 5.1: the proof is divided in two steps: first we prove existence of the $\Gamma$-limit and then we show measure property.

*Step 1: existence of the* $\Gamma$*-limit.*
First of all, we prove a growth estimates for the upper and lower $\Gamma$-limits $F''(u,A)$ and $F'(u,A)$.

**Proposition 5.3** *If* $p:\Omega\to(1,+\infty)$ *is a continuous function with* $p(x)\geq p>1$ *in* $\Omega$, *and* $\{F_j\}$ *is given by* (5.2), *then for every* $A\in\mathcal{A}$ *and* $u\in L^1(\Omega;\mathbf{R}^N)$ *we have*

$$\alpha\,\Psi_{p(x)}(u,A) \leq F'(u,A) \leq F''(u,A) \leq \beta\left(\int_A a(x)\,dx + \Psi_{p(x)}(u,A)\right)\,. \tag{5.6}$$

Proof: since the functional $F_j$ satisfies (5.4), and since by Proposition 3.13 the functional $\Psi_{p(x)}(\cdot,A)$ is $L^1(\Omega)$-l.s.c. for every $A\in\mathcal{A}$, applying [14, Prop. 6.7] we obtain (5.6) for every $u\in L^1(\Omega)$ and $A\in\mathcal{A}$. □

Remark that Proposition 5.3 yields that the $\Gamma$-limit $F(u,A)$ exists and is equal to $+\infty$ for every $A\in\mathcal{A}$ and every $u\in L^1(\Omega)$ such that $u\notin W^{1,p(x)}_{\mathrm{loc}}(A)$ or $u\in W^{1,p(x)}_{\mathrm{loc}}(A)$ but $\int_A |Du|^{p(x)}\,dx=+\infty$.

To prove existence of the $\Gamma$-limit for every $u$ and $A$, we want to apply the compactness result of Proposition 2.6; thus we need to show that for fixed $u\in L^1(\Omega)$ the upper and lower $\Gamma$-limits $F''(u,\cdot)$ and $F'(u,\cdot)$ define inner regular increasing set functions. Now, the increasing property is inherited by the same property of the integral functionals $F_j$. To show inner regularity, we use the following uniform fundamental $L^{p(x)}$ estimate for the sequence $\{F_j\}$.

**Proposition 5.4 (Uniform fundamental estimate)** *If* $p:\Omega\to[1,+\infty)$ *is continuous in* $\Omega$, *for all open sets* $A,A',B\in\mathcal{A}$, *with* $A'\subset\subset A$, *and for every* $\sigma>0$, *there exists a constant* $M_\sigma>0$, *depending on* $\alpha$, $\beta$, $p(x)$ *and* $a(x)$, *such that for every* $u,v\in L^1(\Omega;\mathbf{R}^N)$ *there exists a cut-off function* $\varphi$ *between* $A'$ *and* $A$ *such that for any* $j\in\mathbf{N}$

$$F_j(\varphi u+(1-\varphi)v,A'\cup B) \leq (1+\sigma)(F_j(u,A)+F_j(v,B)) + M_\sigma\int_{A\cap B}|u-v|^{p(x)}\,dx+\sigma\,. \tag{5.7}$$

Proof: fix $A,A',B\in\mathcal{A}$, with $A'\subset\subset A$, $\sigma>0$ and choose $u,v\in L^1(\Omega)$ so that the right-hand side of (5.7) is finite. Then by (5.2) we have $u\in W^{1,p(x)}_{\mathrm{loc}}(A)$, $v\in W^{1,p(x)}_{\mathrm{loc}}(B)$, $|Du|^{p(x)}\in L^1(A)$ and $|Dv|^{p(x)}\in L^1(B)$. If $2\,\delta:=\mathrm{dist}(A',\partial A)$, $0<\nu<\delta$, $0<r<\delta-\nu$, $d(x):=\mathrm{dist}(x,A')$, let $\varphi$ be a cut-off function between $\{x\in A \mid d(x)<r\}$ and $\{x\in A\mid d(x)<\nu+r\}$, with $\|D\varphi\|_\infty\leq 2/\nu$. Since by Lemma 3.11 the function $\varphi u+(1-\varphi)v$ belongs to $W^{1,p(x)}_{\mathrm{loc}}(A'\cup B)$, by (5.2) one has

$$\begin{aligned} F_j(\varphi u+(1-\varphi)v,A'\cup B) &= \int_{A'\cup B} f_j(x,D(\varphi u+(1-\varphi)v))\,dx \\ &\leq F_j(u,A)+F_j(v,B)+\int_{B^\nu_r} f_j(x,D(\varphi u+(1-\varphi)v))\,dx\,, \end{aligned} \tag{5.8}$$

where

$$B^\nu_r := \{x\in B \mid r<d(x)<r+\nu\}\,.$$

We need to estimate the right-hand side in (5.8) independently of $r$ and $\nu$, which will be chosen at the end. To this purpose note that for any possible choice of $r$ and $\nu$ we have $B^\nu_r\subset K_\delta:=\{x\in A\cap B\mid 0<d(x)<\delta\}$, with $K_\delta\in\mathcal{A}_0$. Therefore, if we set $q:=\sup_{K_\delta}p(x)$, which is independent of $r$ and $\nu$, by (5.1) we obtain

(for $2/\nu > 1$)

$$\int_{B_r^\nu} f_j(x, D(\varphi u + (1-\varphi)v))\,dx \le \beta\, 2^{q-1} \int_{B_r^\nu} (a(x) + |Du|^{p(x)} + |Dv|^{p(x)} + |D\varphi \otimes (u-v)|^{p(x)})\,dx$$
$$\le \mu(B_r^\nu) + \beta\, 2^{2q-1}\,\nu^{-q} \int_{A\cap B} |u-v|^{p(x)}\,dx\,,$$

where

$$\mu(E) := \beta\, 2^{q-1} \int_E (a(x) + |Du|^{p(x)} + |Dv|^{p(x)})\,dx\,.$$

Now, if $N \in \mathbf{N}$ is large enough that

$$N \ge \max\left\{ \frac{\beta\, 2^{q-1}}{\sigma\alpha}\,,\ \frac{\beta\, 2^{q-1}}{\sigma} \int_{A\cap B} a(x)\,dx \right\}\,,$$

there exists $k \in \{1, \ldots, N\}$ such that, for $r = \delta\,(k-1)/N$ and $\nu = \delta/N$, we have

$$\mu(B_r^\nu) \le \frac{1}{N}\,\mu(K_\delta) \le \frac{1}{N}\,\mu(A\cap B) \le \sigma\,\alpha \int_{A\cap B} (|Du|^{p(x)} + |Dv|^{p(x)})\,dx + \sigma < +\infty\,.$$

In addition, by (5.1) we obtain

$$\alpha \int_{A\cap B} (|Du|^{p(x)} + |Dv|^{p(x)})\,dx \le \int_{A\cap B} (f_j(x,Du) + f_j(x,Dv))\,dx \le F_j(u,A) + F_j(v,B)\,.$$

Therefore we obtain (5.7) with $M_\sigma = \beta 2^{2q-1} N^q \delta^{-q}$, which only depends on $\alpha$, $\beta$, $p(x)$, $a(x)$, $A'$, $A$ and $B$, as required. $\square$

We are now able to prove the following

**Proposition 5.5** *For every $u \in L^1(\Omega; \mathbf{R}^N)$, the lower and upper $\Gamma$-limits $F'(u,\cdot)$ and $F''(u,\cdot)$ define inner regular set functions on $\mathcal{A}$.*

Proof: we prove inner regularity of the lower $\Gamma$-limit; the same proof works for the upper $\Gamma$-limit. For any fixed $u \in L^1(\Omega)$ and $C \in \mathcal{A}$, we have to show that

$$F'(u,C) \le \sup\{F'(u,A) \mid A \in \mathcal{A}\,, \quad A \subset\subset C\}\,, \tag{5.9}$$

since the other inequality follows immediately from the monotonicity of $F'(u,\cdot)$. If the right-hand side of (5.9) is equal to $+\infty$ there is nothing to prove; else, by (5.6) and (3.5) we have $\sup\{\int_A |Du|^{p(x)}\,dx \mid A \in \mathcal{A},\ A \subset\subset C\} < +\infty$, thus $|Du|^{p(x)} \in L^1(C)$ and hence $u \in W^{1,p(x)}_{\mathrm{loc}}(C)$ by (3.3).

Now fix a compact subset $K \subset C$, choose $A', A \in \mathcal{A}_0$ such that $K \subset A' \subset\subset A \subset\subset C$, $A$ with Lipschitz boundary, and define $B := C \setminus K$. By definition of $F'(u,A) < +\infty$ there exists a sequence $\{u_j\}$ converging to $u$ in $L^1(\Omega)$ such that $F'(u,A) = \liminf_{j\to+\infty} F_j(u_j,A)$. As usual, we may assume that the lower limit is a limit and that

$$\sup_{j\in\mathbf{N}} \int_A |Du_j|^{p(x)}\,dx < +\infty\,, \tag{5.10}$$

so $\{u_j\} \subset W^{1,p(x)}(A)$ by Lemma 3.4. By the fundamental estimate (5.7), applied to $u_j$ on $A$ and $u$ on $B$, and by (5.1), for any $\sigma > 0$ we can find $M_\sigma > 0$ and a sequence $\{\varphi_j\}$ of cut-off functions between $A'$ and $A$ such that

$$F_j(\varphi_j u_j + (1-\varphi_j)u, A' \cup B) \le (1+\sigma)(F_j(u_j,A) + F_j(u,B)) + M_\sigma \int_{A\cap B} |u_j - u|^{p(x)}\,dx + \sigma$$
$$\le (1+\sigma)\left(F_j(u_j,A) + \beta \int_B (a(x) + |Du|^{p(x)})\,dx\right) + M_\sigma \int_A |u_j - u|^{p(x)}\,dx + \sigma\,.$$

Now, since $\varphi_j u_j + (1-\varphi_j)u \to u$ in $L^1(\Omega)$, using (5.10) and Lemma 3.8 we obtain

$$
\begin{aligned}
F'(u,C) &= F'(u,A'\cup B) \le \liminf_{j\to+\infty} F_j(\varphi_j u_j + (1-\varphi_j)u, A'\cup B) \\
&\le (1+\sigma)\left(F'(u,A) + \beta\int_{C\setminus K}(a(x)+|Du|^{p(x)})\,dx\right) + \sigma
\end{aligned}
$$

that is, by the arbitrariness of $\sigma > 0$,

$$
F'(u,C) \le F'(u,A) + \beta\int_{C\setminus K}(a(x)+|Du|^{p(x)})\,dx\,. \tag{5.11}
$$

Since $|Du|^{p(x)} \in L^1(C)$, the last term in (5.11) can be taken arbitrarily small as $|C\setminus K| \to 0$ and the proof is complete. □

Applying Proposition 2.6, we obtain a subsequence $\{j_k\}$ such that the $\Gamma$-limit $F(u,A)$ in (5.3) exists for all $u \in L^1(\Omega)$ and $A \in \mathcal{A}$; in addition $F(u,\cdot)$ is an increasing set function on $\mathcal{A}$ for every $u \in L^1(\Omega)$ and $F(u,A)$ satisfies the estimates (5.4) by Proposition 5.3.

*Step 2: measure property of the* $\Gamma$*-limit.*
We now show that for all $u \in L^1(\Omega;\mathbf{R}^N)$ the $\Gamma$-limit $F(u,\cdot)$ is the trace on $\mathcal{A}$ of a Borel measure.

In order to apply the De Giorgi-Letta criterion [18], we have to prove that for every fixed $u \in L^1(\Omega)$ the increasing set function $F(u,\cdot)$ is subadditive, superadditive and inner regular. Since we have just proved (Proposition 5.5) that both the upper and lower $\Gamma$-limits are inner regular, and it is well known that the lower $\Gamma$-limit $F'(u,\cdot)$ is superadditive (see e.g. [14, Prop. 16.12]), we only have to show that the upper $\Gamma$-limit $F''(u,\cdot)$ is subadditive.

**Proposition 5.6** *For every* $u \in L^1(\Omega;\mathbf{R}^N)$ *we have*

$$
F''(u,A\cup B) \le F''(u,A) + F''(u,B) \tag{5.12}
$$

*for every* $A, B \in \mathcal{A}$.

The subadditivity (5.12) of $F''(u,\cdot)$ follows, by means of inner regularity, from the following weak subadditivity property:

$$
F''(u,A'\cup B) \le F''(u,A) + F''(u,B) \tag{5.13}
$$

for every $A', A, B \in \mathcal{A}$ with $A' \subset\subset A$ and $B$ with Lipschitz boundary.

We first prove (5.13) under the additional assumptions that $A$ belongs to $\mathcal{A}_0$ and $A\cap B$ has Lipschitz boundary. To this purpose let us fix $u \in L^1(\Omega)$ and let $A', A, B \in \mathcal{A}$, with $A' \subset\subset A$, $A \in \mathcal{A}_0$, $B$ and $A\cap B$ with Lipschitz boundary, such that the right-hand side of (5.13) is finite. By definition of upper $\Gamma$-limit, there exist two sequences $\{u_j\}$ and $\{v_j\}$, both converging to $u$ in $L^1(\Omega)$, such that

$$
F''(u,A) = \limsup_{j\to+\infty} F_j(u_j,A) < +\infty \quad \text{and} \quad F''(u,B) = \limsup_{j\to+\infty} F_j(v_j,B) < +\infty
$$

and hence by (5.2) and (5.1)

$$
\sup_{j\in\mathbf{N}}\int_A |Du_j|^{p(x)}\,dx < +\infty \quad \text{and} \quad \sup_{j\in\mathbf{N}}\int_B |Dv_j|^{p(x)}\,dx < +\infty\,. \tag{5.14}
$$

By the fundamental estimate (Proposition 5.4) applied to $u_j$ and $v_j$, for any $\sigma > 0$ we can find $M_\sigma > 0$ and a sequence $w_j := \varphi_j u_j + (1-\varphi_j)v_j$, where the $\varphi_j$ are cut-off functions between $A'$ and $A$, such that

$$
\begin{aligned}
&F_j(w_j, A'\cup B) \le (1+\sigma)(F_j(u_j,A) + F_j(v_j,B)) + M_\sigma\int_{A\cap B}|u_j - v_j|^{p(x)}\,dx + \sigma \\
&\le (1+\sigma)(F_j(u_j,A) + F_j(v_j,B)) + M_\sigma\, 2^{q-1}\int_{A\cap B}(|u_j-u|^{p(x)} + |v_j-u|^{p(x)})\,dx + \sigma
\end{aligned}
$$

where $q := \sup_{x\in A\cap B} p(x) < +\infty$. Since $w_j \to u$ in $L^1(\Omega)$, $A\cap B$ belongs to $\mathcal{A}_0$, $A\cap B$ has Lipschitz boundary and (5.14) holds, by definition of upper $\Gamma$-limit and Lemma 3.8 we obtain

$$F''(u, A'\cup B) \le \limsup_{j\to+\infty} F_j(w_j, A'\cup B) \le (1+\sigma)(F''(u,A) + F''(u,B)) + \sigma$$

and hence (5.13) holds by the arbitrariness of $\sigma > 0$. To extend (5.13) to any $A \in \mathcal{A}$, it is enough to take $C \in \mathcal{A}_0$ such that $A' \subset\subset C \subset\subset A$ and $C \cap B$ has Lipschitz boundary, and by the monotonicity of $F''(u,\cdot)$ we obtain

$$F''(u, A'\cup B) \le F''(u,C) + F''(u,B) \le F''(u,A) + F''(u,B)\,.$$

Now, to prove (5.12) we fix $u \in L^1(\Omega)$ and $A, B \in \mathcal{A}$ such that the right-hand side of (5.12) is finite, which in particular gives $F''(u, A\cup B) < +\infty$ by (5.6). By inner regularity, it is clear that (5.13) yields (5.12) for any $A, B \in \mathcal{A}$, provided $B$ has Lipschitz boundary. In fact, for any $C \in \mathcal{A}$ with $C \subset\subset A\cup B$, by enlarging a bit the subset $C\setminus \overline{B}$, we can find $A' \subset\subset A$ such that $C \subset A' \cup B$, which yields

$$F''(u,C) \le F''(u, A'\cup B) \le F''(u,A) + F''(u,B)$$

and by inner regularity (5.12). Finally, to prove (5.12) for any $B \in \mathcal{A}$, since $F''(u, A\cup B) < +\infty$, by inner regularity for each small $\varepsilon > 0$ we can find $C \subset\subset A\cup B$ such that

$$F''(u,C) \ge F''(u, A\cup B) - \varepsilon\,.$$

By enlarging a bit the set $C\setminus\overline{A}$, we can find an open subset $\widetilde{B}$ of $B$ with Lipschitz boundary and such that $C \subset A\cup\widetilde{B}$. Then one has

$$F''(u,A\cup B) \le F''(u,C) + \varepsilon \le F''(u, A\cup\widetilde{B}) + \varepsilon \le F''(u,A) + F''(u,\widetilde{B}) + \varepsilon \le F''(u,A) + F''(u,B) + \varepsilon$$

and hence (5.12), by letting $\varepsilon \to 0^+$. □

Proof of Theorem 5.2: we can apply Theorem 4.1 to the $\Gamma$-limit functional $F(u,A)$ given by (5.3). Indeed the locality property is well known (see e.g. [14, Prop. 16.15]), the measure property ii) was proved in Step 2 of Theorem 5.1, we saw in Step 1 that the $\Gamma$-limit $F$ satisfies (5.4), which in particular gives the growth condition iii), and conditions iv) and v) are trivially satisfied. Therefore, Theorem 4.1 implies that (4.1) holds for all $u \in W^{1,p(x)}_{\mathrm{loc}}(A)$ and $A \in \mathcal{A}$, with $f$ quasi-convex. Finally, it is easy to conclude that (5.4) and (3.5) yield the growth estimate (5.1) for $f$ and the integral representation (5.5) on all of $L^1(\Omega;\mathbf{R}^N)$.□

# 6 Integrands depending on u

In this section we extend the results previously obtained to the case of functionals with explicit dependence on $u$. We will then consider functions satisfying

$$\alpha(|u|^{p(x)} + |z|^{p(x)}) \le f(x,u,z) \le \beta(a(x) + |u|^{p(x)} + |z|^{p(x)}) \tag{6.1}$$

for a.e. $x \in \Omega$ and for all $u \in \mathbf{R}^N$ and $z \in \mathbf{R}^{nN}$, where $0 < \alpha \le \beta$ and $a(x) \in L^1(\Omega)$. We are able to prove the following results.

**Theorem 6.1** *Let $p : \Omega \to (1,+\infty)$ be a continuous function with $p(x) \ge p > 1$ for all $x \in \Omega$. Let $f_j : \Omega\times\mathbf{R}^N\times\mathbf{R}^{nN} \to [0,+\infty)$ be Borel functions satisfying (6.1) for a.e. $x \in \Omega$ and for all $u \in \mathbf{R}^N$ and $z \in \mathbf{R}^{nN}$. Finally let $F_j : L^1(\Omega;\mathbf{R}^N)\times\mathcal{A} \to [0,+\infty]$ be the variational functionals defined by*

$$F_j(u,A) := \begin{cases} \displaystyle\int_A f_j(x,u(x),Du(x))\,dx & \textit{if } u \in W^{1,p(x)}_{\mathrm{loc}}(A;\mathbf{R}^N) \\ +\infty & \textit{elsewhere on } L^1(\Omega;\mathbf{R}^N)\,. \end{cases}$$

*Then there exists a subsequence $\{j_k\}$ such that the $\Gamma$-limit*

$$F(u,A) := \Gamma(L^1(\Omega;\mathbf{R}^N))\text{-}\lim_{k\to+\infty} F_{j_k}(u,A) \tag{6.2}$$

*exists for all $u \in L^1(\Omega;\mathbf{R}^N)$ and $A \in \mathcal{A}$, with estimates*

$$\alpha\, \widetilde{\Psi}_{p(x)}(u,A) \le F(u,A) \le \beta\left(\int_A a(x)\,dx + \widetilde{\Psi}_{p(x)}(u,A)\right), \tag{6.3}$$

*where $\widetilde{\Psi}_{p(x)}$ is the local functional given by* (3.6).

**Theorem 6.2** *Let $F(u,A)$ be as in* (6.2) *and, in addition to the hypotheses of Theorem* 6.1, *suppose that the function $p$ satisfies* (3.9) *and that for every $A \in \mathcal{A}_0$*

$$|f_j(x,u_1,z) - f_j(x,u_2,z)| \le \omega_A(|u_1-u_2|)(b_A(x) + |z|^{p(x)}) \tag{6.4}$$

*holds for a.e. $x \in A$, where $b_A(x) \in L^1(A)$ and $\omega_A : \mathbf{R}^+ \to \mathbf{R}^+$ is an increasing continuous function with $\omega_A(0) = 0$. Then there exists a Carathéodory function $f : \Omega \times \mathbf{R}^N \times \mathbf{R}^{nN} \to [0,+\infty)$, satisfying the growth estimate* (6.1) *for a.e. $x \in \Omega$ and all $u \in \mathbf{R}^N$ and $z \in \mathbf{R}^{nN}$, with $f(x,u,\cdot)$ quasi-convex in $\mathbf{R}^{nN}$ for a.e. $x \in \Omega$ and all $u \in \mathbf{R}^N$, such that for every $A \in \mathcal{A}$*

$$F(u,A) := \begin{cases} \displaystyle\int_A f(x,u(x),Du(x))\,dx & \textit{if } u \in W^{1,p(x)}(A;\mathbf{R}^N) \\ +\infty & \textit{elsewhere on } L^1(\Omega;\mathbf{R}^N). \end{cases} \tag{6.5}$$

In proving Theorems 6.1 and 6.2, we will only outline the main differences from the case when there is no dependence on $u$ of the integrands $f_j$. Condition (6.4) is required to rely on the classical integral representation theorem of Buttazzo and Dal Maso, see [8, Thm. 1.10]. In fact, we will first write the $\Gamma$-limit $F(u,A)$ as an integral functional only for $u \in W^{1,q}(A;\mathbf{R}^N)$ and $A \in \mathcal{A}_0$, where $q = \sup_{x\in A} p(x)$, and then we will extend the integral representation.

Proof of Theorem 6.1: since each of the local functionals $F_j$ satisfies (6.3), and by Proposition 3.13 the functional $\widetilde{\Psi}_{p(x)}(\cdot,A)$ is $L^1(\Omega)$-l.s.c. for every $A \in \mathcal{A}$, we obtain

$$\alpha\, \widetilde{\Psi}_{p(x)}(u,A) \le F'(u,A) \le F''(u,A) \le \beta\left(\int_A a(x)\,dx + \widetilde{\Psi}_{p(x)}(u,A)\right) \tag{6.6}$$

for every $u \in L^1(\Omega)$ and $A \in \mathcal{A}$. In particular, the $\Gamma$-limit $F(u,A)$ is equal to $+\infty$ if and only if $u \in L^1(\Omega) \setminus W^{1,p(x)}(A)$.

Moreover, the uniform fundamental $L^{p(x)}$ estimate of Proposition 5.4 holds again. In fact, the upper estimate of $F_j(\varphi u + (1-\varphi)v, (A' \cup B) \cap \{0 < \varphi < 1\})$, due to (6.1), contains the extra term

$$\beta \int_{B_r^\nu} |\varphi u + (1-\varphi)v|^{p(x)}\,dx \le \beta \int_{B_r^\nu} (|u|^{p(x)} + |v|^{p(x)})\,dx\,.$$

Then it suffices to choose

$$\mu(E) := \beta\, 2^{q-1} \int_E (a(x) + |u|^{p(x)} + |Du|^{p(x)} + |v|^{p(x)} + |Dv|^{p(x)})\,dx$$

and remark that by (6.1) one has

$$\alpha \int_{A\cap B} (|u|^{p(x)} + |Du|^{p(x)})\,dx \le \int_{A\cap B} f_j(x,u(x),Du(x))\,dx\,.$$

Monotonicity of the lower and upper $\Gamma$-limits $F'(u,\cdot)$ and $F''(u,\cdot)$ is trivial. Moreover, to show inner regularity, we reduce to prove (5.9) in case the right-hand side is finite, which yields that $u \in W^{1,p(x)}(C)$ by

(6.6) and (3.6). Following Proposition 5.5, by means of the fundamental $L^{p(x)}$ estimate we deduce, instead of (5.11),

$$F'(u,C) \le F'(u,A) + \beta \int_{C\setminus K} (a(x) + |u|^{p(x)} + |Du|^{p(x)})\, dx$$

and again, as $K \nearrow C$, we obtain (5.9), and hence inner regularity. By Proposition 2.6, we then have existence of the $\Gamma$-limit $F(u,A)$ with the growth estimates (6.3) given by (6.6).

To show the measure property of the $\Gamma$-limit $F(u,\cdot)$, following Step 2 in Theorem 5.1, it suffices to recover subadditivity of the upper $\Gamma$-limit $F''(u,\cdot)$: this holds since the right-hand side of (5.13) is finite for $u \in W^{1,p(x)}(A\cup B)$, and the proof of Proposition 5.6 actually relies on the fundamental $L^{p(x)}$ estimate. $\square$

Proof of Theorem 6.2: to avoid overcrowding the paper with indices, we relabel $F_j$ the $\Gamma$-converging subsequence. To obtain the integral representation of the $\Gamma$-limit, take any sequence $\mathcal{O}_i \nearrow \Omega$ such that $\mathcal{O}_i \in \mathcal{A}_0$ and $\mathcal{O}_i$ has Lipschitz boundary for every $i$. Then (6.4) holds for a.e. $x \in \mathcal{O}_i$ whereas, by continuity of $p(x)$,

$$q_i := \sup_{x\in\mathcal{O}_i} p(x) < +\infty \qquad \forall\, i\,. \tag{6.7}$$

Denoting by $\mathcal{A}_i$ the family of open subsets of $\mathcal{O}_i$, we now consider the functional $G_i : W^{1,q_i}(\mathcal{O}_i)\times\mathcal{A}_i \to [0,+\infty]$ given by

$$G_i(u,A) := F(\overline{u},A)\,,$$

where $\overline{u} \in W^{1,p(x)}(\Omega)$ is any extension of $u$. From now on we drop all the indices $i$: we will recover the full notation later. By (6.3) and (6.7) we have

$$0 \le G(u,A) \le \beta \int_A (2 + a(x) + |u|^q + |Du|^q)\, dx$$

for all $u \in W^{1,q}(\mathcal{O})$ and $A\in\mathcal{A}$, and hence $G$ satisfies a growth condition of order $q$. Moreover, locality of $F$ implies that also $G$ is local on $\mathcal{A}$. Also, since $F(\cdot,A)$ is $L^1(\Omega)$-l.s.c., and $\mathcal{O}$ has Lipschitz boundary, for every $A\in\mathcal{A}$ the functional $u\mapsto G(u,A)$ is sequentially $w^*$-$W^{1,\infty}(\mathcal{O})$-l.s.c. and strongly $W^{1,q}(\mathcal{O})$-l.s.c.. Finally, since $G(u,\cdot)$ is a measure on $\mathcal{A}$ for every $u\in W^{1,q}(\mathcal{O})$, in order to apply [8, Thm. 1.10] we have to show that $G$ also satisfies a weak condition $\omega$. More precisely, we show the existence of a sequence of functions $\omega_m(x,r)$, integrable in $x$, increasing and continuous in $r$ and with $\omega_m(x,0)\equiv 0$, such that

$$|G(u+s,A) - G(u,A)| \le \int_A \omega_m(x,|s|)\, dx \tag{6.8}$$

for every $m\in\mathbf{N}$, $A\in\mathcal{A}$, $s\in\mathbf{R}^N$ and $u\in C^1(\overline{\mathcal{O}})$ with $|u|$, $|u+s|$, $|Du|\le m$ in $\mathcal{O}$. In fact, assume e.g. that $G(u+s,A)\ge G(u,A)$, let $\overline{u}\in W^{1,p(x)}(\Omega)$ be an extension of $u$ and let $\{u_j\}$ be such that $u_j\to\overline{u}$ in $L^1(\Omega)$ and $F_j(u_j,A)\to F(\overline{u},A) = G(u,A)$. Then by (6.4) we have

$$\begin{aligned}
G(u+s,A) - G(u,A) &= F(\overline{u}+s,A) - F(\overline{u},A) \le \liminf_{j\to+\infty}(F_j(u_j+s,A) - F_j(u_j,A))\\
&\le \liminf_{j\to+\infty}\int_A |f_j(x,u_j+s,Du_j) - f_j(x,u_j,Du_j)|\, dx \le \liminf_{j\to+\infty}\int_A \omega_{\mathcal{O}}(|s|)(b_{\mathcal{O}}(x) + |Du_j|^{p(x)})\, dx\\
&\le \liminf_{j\to+\infty}\omega_{\mathcal{O}}(|s|)\left(\int_A b_{\mathcal{O}}(x)\,dx + \frac{1}{\alpha}\,F_j(u_j,A)\right) = \omega_{\mathcal{O}}(|s|)\left(\int_A b_{\mathcal{O}}(x)\,dx + \frac{1}{\alpha}\,F(\overline{u},A)\right)\\
&\le \omega_{\mathcal{O}}(|s|)\int_A\left(b_{\mathcal{O}}(x) + \frac{\beta}{\alpha}(2 + a(x) + |u|^q + |Du|^q)\right)\, dx\,,
\end{aligned}$$

where we used (6.3), and we obtain (6.8) with $\omega_m(x,r) := \omega(r)\left(b_{\mathcal{O}}(x) + \dfrac{\beta}{\alpha}(2 + a(x) + 2m^q)\right)$.

Then, by [8, Thm. 1.10], there exists a Carathéodory integrand $g(x,u,z) : \mathcal{O}\times\mathbf{R}^N\times\mathbf{R}^{nN}\to[0,+\infty)$, quasi-convex in $z$ for a.e. $x\in\mathcal{O}$ and all $u\in\mathbf{R}^N$, and satisfying a growth condition of order $q$, such that

$$G(u,A) = \int_A g(x,u(x),Du(x))\, dx \tag{6.9}$$

holds for every $u \in W^{1,q}(\mathcal{O})$ and $A \in \mathcal{A}$. We now recover the notation with the index $i$. In particular, (6.9) holds for the restriction $u_{|\mathcal{O}_i}$ of every function $u \in C_0^\infty(\Omega)$, and thus for $i < j$ large enough that $\operatorname{spt}(u) \subset \mathcal{O}_i$ we have $G_i(u, \mathcal{O}_i) = F(u, \mathcal{O}_i) = G_j(u, \mathcal{O}_i)$, so that $g_i(\cdot, u, z) \equiv g_j(\cdot, u, z)$ for a.e. $x \in \mathcal{O}_i$. Setting $f(x, u, z) := g_i(x, u, z)$ if $x \in \mathcal{O}_i$, we have that

$$F(u, A) = \int_A f(x, u(x), Du(x))\, dx \tag{6.10}$$

for every $u \in C_0^\infty(\Omega)$ and all $A \in \mathcal{A}_0$ and hence, by the measure property, for all $A \in \mathcal{A}$.

To extend the integral representation (6.10) to all $u \in W^{1,p(x)}(\Omega)$, we first remark that (6.3) and (6.10) yield that the integrand $f$ satisfies the $p(x)$-growth condition (6.1). Consequently, since $f$ is a Carathéodory function satisfying (6.1), we have that for every $A \in \mathcal{A}_0$ the functional $u \mapsto \int_A f(x, u, Du)\, dx$ is continuous with respect to the $W^{1,p(x)}_{\mathrm{loc}}(\Omega)$ convergence, compare Step 4 in Sec. 3. Now fix $u \in W^{1,p(x)}(\Omega)$ and, by the density result in Proposition 3.18, let $\{u_j\} \subset C_0^\infty(\Omega)$ be a smooth sequence converging to $u$ both in $W^{1,p(x)}_{\mathrm{loc}}(\Omega)$ and in $L^1(\Omega)$: by the lower semicontinuity of $F$

$$F(u, A) \le \liminf_{j\to+\infty} F(u_j, A) = \lim_{j\to+\infty} \int_A f(x, u_j, Du_j)\, dx = \int_A f(x, u, Du)\, dx\,, \tag{6.11}$$

hence one inequality holds in (6.10) for all $u \in W^{1,p(x)}(\Omega)$ and $A \in \mathcal{A}_0$. To obtain equality, let $\overline{u} \in W^{1,p(x)}(\Omega)$ be an extension of $u$ outside $A$, with compact support in $\Omega$, see Step 5 of Sec. 3, let $q_0$ be the maximum of $p$ on the support of $\overline{u}$, and consider the functional $G : L^1(\Omega) \times \mathcal{A} \to [0, +\infty]$ given by

$$G(v, B) := F(v + \overline{u}, B) + \alpha\, \widetilde{\Psi}(\overline{u}, B)\,.$$

It is easy to show that $G$ satisfies hypotheses similar to (6.1), since in particular by (6.1)

$$\begin{aligned} G(v, B) \;&\ge\; \alpha \int_B (|\overline{u} + v|^{p(x)} + |\overline{u}|^{p(x)} + |D\overline{u} + Dv|^{p(x)} + |D\overline{u}|^{p(x)})\, dx \\ &\ge\; \alpha \int_B 2^{1-p(x)}(|v|^{p(x)} + |Dv|^{p(x)})\, dx \;\ge\; \alpha\, 2^{1-q_0}\, \widetilde{\Psi}(v, B) \end{aligned}$$

and similarly

$$G(v, B) \le \beta \int_B (\widetilde{a}(x) + |v|^{p(x)} + |Dv(x)|^{p(x)})\, dx$$

for all $v \in W^{1,p(x)}(\Omega)$ and $B \in \mathcal{A}$, with $\widetilde{a}(x) \in L^1(\Omega)$. Then, by the previous argument, there exists a Carathéodory function $g : \Omega \times \mathbf{R}^N \times \mathbf{R}^{nN} \to [0, +\infty)$, satisfying $p(x)$-growth estimates similar to (6.1), such that

$$G(v, B) \le \int_B g(x, v(x), Dv(x))\, dx\,, \qquad \forall\, v \in W^{1,p(x)}(\Omega)\,,\ B \in \mathcal{A}_0\,, \tag{6.12}$$

with equality holding for $v \in C_0^\infty(\Omega)$. As we remarked above for $f$, also the functional $v \mapsto \int_B g(x, u, Dv)\, dx$ is strongly continuous in $W^{1,p(x)}_{\mathrm{loc}}$. If $\{u_j\} \subset C_0^\infty(\Omega)$ is the same sequence we chose to get (6.11), and $\overline{\alpha} := \alpha\, \widetilde{\Psi}(\overline{u}, A)$, we have

$$\begin{aligned} \int_A g(x, 0, 0)\, dx - \overline{\alpha} &= F(\overline{u}, A) = F(u, A) \le \int_A f(x, u, Du)\, dx = \lim_{j\to+\infty} \int_A f(x, u_j, Du_j)\, dx \\ &= \lim_{j\to+\infty} F(u_j, A) = \lim_{j\to+\infty} G(u_j - \overline{u}, A) - \overline{\alpha} \le \lim_{j\to+\infty} \int_A g(x, u_j - \overline{u}, D(u_j - \overline{u}))\, dx - \overline{\alpha} \\ &= \lim_{j\to+\infty} \int_A g(x, u_j - u, D(u_j - u))\, dx - \overline{\alpha} = \int_A g(x, 0, 0)\, dx - \overline{\alpha} \end{aligned}$$

and therefore equality in (6.10) holds for all $u \in W^{1,p(x)}(\Omega)$ and $A \in \mathcal{A}_0$ and hence, by the measure property of $F$, for all $A \in \mathcal{A}$.

Finally, we argue as in Step 6 in Sec. 3 to extend the integral representation (6.10) to all $u \in L^1(\Omega)$ with $u_{|A} \in W^{1,p(x)}(A)$; since by (6.3) we have $F(u, A) = +\infty$ if $u_{|A} \notin W^{1,p(x)}(A)$, we obtain (6.5) and the proof is complete. □

# 7 Integrands with discontinuous p(x)-growth exponent

In this section we extend the results previously obtained to the more general case of functionals with $p(x)$-growth where $p(x)$ is allowed to be discontinuous in a negligible set $\Sigma \subset \Omega$: henceforward, $p(x)$ is a regular piecewise continuous exponent, according to Definition 3.2.

In the sequel, if $A \in \mathcal{A}$ we will denote $A_i := A \cap \Omega_i$ the intersection of $A$ with $\Omega_i$. We will again obtain existence and integral representation of $\Gamma$-limits. More precisely, we are going to prove the following results.

**Theorem 7.1** *Let $p : \Omega \to (1, +\infty)$ be a regular piecewise continuous exponent satisfying* (3.1)*, and let $f_j : \Omega \times \mathbf{R}^{nN} \to [0, +\infty)$ be Borel functions satisfying* (5.1) *for a.e. $x \in \Omega$ and all $z \in \mathbf{R}^{nN}$. If $F_j$ is as in* (5.2) *there exists a subsequence $\{j_k\}$ such that the $\Gamma$-limit* (5.3) *exists for all $u \in L^1(\Omega; \mathbf{R}^N)$ and $A \in \mathcal{A}$, with estimates* (5.4)*.*

**Theorem 7.2** *Under the assumptions of Theorem* 7.1*, suppose that each function $p_i$ satisfies* (3.9) *in $\Omega_i$. Then there exists a Carathéodory function $f : \Omega \times \mathbf{R}^{nN} \to [0, +\infty)$, with $f(x, \cdot)$ quasi-convex for a.e. $x \in \Omega$, $f$ satisfying the growth estimates* (5.1)*, such that the integral representation* (5.5) *holds for all $A \in \mathcal{A}$.*

In the particular case when $p(x)$ is constant on each region $\Omega_i$, as in the case of standard $p$-growth we obtain the precise expression of the relaxed functional, compare [7, 4.4.5]. To this aim, if $f : \Omega \times \mathbf{R}^{nN} \to [0, +\infty)$ is a Borel function satisfying the growth condition (5.1), and $\mathcal{F} : L^1(\Omega; \mathbf{R}^N) \times \mathcal{A} \to [0, +\infty]$ is the variational functional given by

$$\mathcal{F}(u, A) := \begin{cases} \displaystyle\int_A f(x, Du(x))\, dx & \text{if } u \in W^{1,p(x)}_{\mathrm{loc}}(A; \mathbf{R}^N) \\ +\infty & \text{elsewhere on } L^1(\Omega; \mathbf{R}^N) \end{cases}$$

for all $A \in \mathcal{A}$, we denote by $\overline{\mathcal{F}}(u, A)$ the relaxed functional of $\mathcal{F}(u, A)$ in the $L^1$ topology,

$$\overline{\mathcal{F}}(u, A) := \inf\{\liminf_{k \to +\infty} \mathcal{F}(u_k, A) \mid \{u_k\} \subset L^1(\Omega; \mathbf{R}^N), \quad u_k \to u \text{ in } L^1(\Omega; \mathbf{R}^N)\}\,.$$

Also, we recall that if $g : \mathbf{R}^{nN} \to [0, +\infty)$ is a Borel function, the quasi-convex envelope of $g$, denoted by $Qg$, is the greatest quasi-convex function which is lower than or equal to $g$; if $f$ is defined on $\Omega \times \mathbf{R}^{nN}$, we denote by $Qf$ the quasi-convex envelope of $z \mapsto f(x, z)$. We will now prove the following

**Corollary 7.3** *Under the assumptions of Theorem* 7.1*, suppose in particular that there exist constants $p_i \geq 1$ such that $p(x) \equiv p_i$ for each $x \in \Omega_i$ and all $i$. Then, if $f(x, \cdot)$ is upper semicontinuous in $\mathbf{R}^{nN}$ for a.e. $x \in \Omega$, we have*

$$\overline{\mathcal{F}}(u, A) := \begin{cases} \displaystyle\int_A Qf(x, Du(x))\, dx & \text{if } u \in W^{1,p(x)}_{\mathrm{loc}}(A; \mathbf{R}^N) \\ +\infty & \text{elsewhere on } L^1(\Omega; \mathbf{R}^N) \end{cases} \tag{7.1}$$

*for all $A \in \mathcal{A}$, where $Qf(x, \cdot)$ is the quasi-convex envelope of $f(x, \cdot)$, for a.e. $x \in \Omega$.*

Proof of Corollary 7.3: by Theorem 7.2 (which still holds if $p_i(x) \equiv 1$ for some $i$) we obtain the integral representation (7.1) with a generic integrand $\varphi(x, z)$ quasi-convex in $\mathbf{R}^{nN}$ for a.e. $x \in \Omega$ and satisfying

$$\alpha |z|^{p_i} \leq \varphi(x, z) \leq \beta(1 + |z|^{p_i}) \tag{7.2}$$

a.e. in $\Omega_i$, see Remark 7.4 below.

Now the integral functional $u \mapsto \int_A \varphi(x, Du)\, dx$ is $L^1(\Omega)$-lower semicontinuous in $W^{1,p(x)}_{\mathrm{loc}}(A)$, for all $A \in \mathcal{A}$. In fact, if $\{u_k\} \subset W^{1,p(x)}_{\mathrm{loc}}(A)$ is such that $u_k \to u$ in $L^1(\Omega)$ with

$$\liminf_{k \to +\infty} \int_A \varphi(x, Du_k)\, dx < +\infty\,, \tag{7.3}$$

(7.2) yields $\sup_k \|Du_k\|_{L^{p_i}(A_i)} < +\infty$. Then, passing to a subsequence which we relabel $\{u_j\}$ we can suppose that $u_j \rightharpoonup u$ in $W^{1,p_i}(A_i)$, or in the weak $BV$ sense if $p_i = 1$, and that the lower limit in (7.3) is actually a limit.

Recall now that quasi-convexity of $\varphi$ yields lower semicontinuity of the functional $u \mapsto \int_{A_i} \varphi(x, Du)\, dx$ with respect to $w$-$W^{1,p_i}(A_i)$ convergence (weak $BV$ if $p_i = 1$). We then deduce

$$\int_A \varphi(x, Du)\, dx = \sum_{i=1}^{+\infty} \int_{A_i} \varphi(x, Du)\, dx \le \sum_{i=1}^{+\infty} \liminf_{j\to+\infty} \int_{A_i} \varphi(x, Du_j)\, dx$$
$$\le \liminf_{j\to+\infty} \sum_{i=1}^{+\infty} \int_{A_i} \varphi(x, Du_j)\, dx = \lim_{j\to+\infty} \sum_{i=1}^{+\infty} \int_{A_i} \varphi(x, Du_j)\, dx = \lim_{j\to+\infty} \int_A \varphi(x, Du_j)\, dx$$

and hence $L^1(\Omega)$-lower semicontinuity. Therefore, by definition of relaxed functional we obtain

$$\int_A Qf(x, Du(x))\, dx \le \int_A \varphi(x, Du(x))\, dx \le \int_A f(x, Du(x))\, dx$$

for all $u \in W^{1,p(x)}_{\mathrm{loc}}(A)$ and $A \in \mathcal{A}$. Finally, since $Qf$ and $\varphi$ are Carathéodory integrands, and $f(x, \cdot)$ is upper semicontinuous, we have

$$Qf(x, z) \le \varphi(x, z) \le f(x, z)$$

for a.e. $x \in \Omega$ and all $z \in \mathbf{R}^{nN}$, and hence $\varphi = Qf$ by the quasi-convexity of $\varphi$. □

**Remark 7.4** The estimate from below in (7.2) holds even if $p_i = 1$ for some $i$. In fact, if $u_k \to u_z(x) := zx$ in $L^1(\Omega)$ with $F(u_k, A) \to \overline{\mathcal{F}}(u_z, A) < +\infty$, $A \subset \Omega_i$ and $p_i = 1$, up to a subsequence we have $u_k \rightharpoonup u_z$ in the weak $BV$ sense so that, by lower semicontinuity of the total variation, we have

$$\alpha \int_A |z|\, dx \le \liminf_{k\to+\infty} \alpha \int_A |Du_k|\, dx \le \liminf_{k\to+\infty} \mathcal{F}(u_k, A) = \overline{\mathcal{F}}(u_z, A) = \int_A \varphi(x, z)\, dx\,.$$

Proof of Theorem 7.1: we first remark that by Corollary 3.14 we again obtain (5.6), whereas the fundamental estimate (Proposition 5.4) holds too, since it relies on Corollary 3.12 and on the fact that $q := \sup_{x \in K_\delta \setminus \Sigma} p(x) < +\infty$, which holds again by Remark 3.3, as $K_\delta \in \mathcal{A}_0$.

Inner regularity of the upper and lower $\Gamma$-limits holds again, since Proposition 5.5 actually relies on (5.6), (3.3), which follows from Corollary 3.6, on Proposition 5.4 and Corollary 3.9. We then obtain $\Gamma$-convergence by Proposition 2.6, with the estimates (5.4).

Subadditivity (5.12) of the upper $\Gamma$-limit $F''(u, A)$ holds again, since Proposition 5.6 relies on inner regularity, on (5.4), on the fundamental estimate and on Corollary 3.9. Then, by the De Giorgi-Letta criterion, we obtain the measure property of the $\Gamma$-limit $F(u, \cdot)$. □

Proof of Theorem 7.2: conditions i), ii), iii), iv) and v) in Theorem 4.1 are easily verified. However, the growth exponent $p(x)$ satisfies the local estimate (3.9) for each open set $A \subset\subset \Omega_i$. Then for every $i$ we obtain a quasi-convex function $f_i : \Omega_i \times \mathbf{R}^{nN} \to [0, +\infty)$, satisfying growth condition (4.2), such that (4.1) holds, with $f = f_i$, for each $u \in W^{1,p(x)}_{\mathrm{loc}}(A)$ and for all open sets $A \in \mathcal{A}$ with $A \subset \Omega_i$.

To extend the integral representation to all open sets $A \in \mathcal{A}$, we define $f(x, z) := f_i(x, z)$ if $x \in \Omega_i$ and $z \in \mathbf{R}^{nN}$, and set

$$\Sigma_r := \{x \in \Omega \mid \mathrm{dist}(x, \Sigma) < r\}\,.$$

Then, for every $u \in W^{1,p(x)}_{\mathrm{loc}}(A)$ and $A \in \mathcal{A}_0$, if $r > 0$ is small, by increasing property and subadditivity we have

$$F(u, A \setminus \overline{\Sigma}_r) \le F(u, A) \le F(u, A \setminus \overline{\Sigma}_r) + F(u, \Sigma_{2r} \cap A)\,. \tag{7.4}$$

Now we have

$$F(u, A \setminus \overline{\Sigma}_r) = \sum_{i=1}^{+\infty} F(u, A_i \setminus \overline{\Sigma}_r) = \sum_{i=1}^{+\infty} \int_{A_i \setminus \overline{\Sigma}_r} f_i(x, Du)\, dx = \int_{A \setminus \overline{\Sigma}_r} f(x, Du)\, dx\,.$$

Moreover since $A \in \mathcal{A}_0$ we have $|\Sigma_{2r} \cap A| = \sum_{i=1}^{+\infty} |\Sigma_{2r} \cap A_i| \to 0$ as $r \to 0^+$, and hence by growth condition (5.4) and by absolute continuity we get

$$F(u, \Sigma_{2r} \cap A) \leq \beta \int_{\Sigma_{2r} \cap A} (a(x) + |Du|^{p(x)})\, dx \to 0$$

as $r \to 0^+$. Finally, since

$$\int_{A \setminus \overline{\Sigma}_r} f(x, Du)\, dx \to \int_A f(x, Du)\, dx\,,$$

letting $r \to 0^+$ in (7.4) we obtain (4.1) for $A \in \mathcal{A}_0$ and hence, by the measure property, for all $A \in \mathcal{A}$ and $u \in W^{1,p(x)}_{\mathrm{loc}}(A)$. Finally, (5.6) yields growth condition (5.1) for $f$ and integral representation (5.5) on all $L^1(\Omega)$. $\square$

# References


[1] ACERBI E., BOUCHITTÉ G., FONSECA I., Relaxation of convex functionals: the gap phenomenon, *to appear.*

[2] ACERBI E., MINGIONE G., Regularity results for a class of functionals with non standard growth, *Arch. Rat. Mech. Anal.* **156** 2 (2001), 121-140.

[3] ACERBI E., MINGIONE G., Regularity results for a class of quasiconvex functionals with non standard growth, *Ann. Scuola Norm. Sup. Pisa* (2001).

[4] ADAMS R.A., *Sobolev spaces*, Academic Press, New York 1975.

[5] ALKUTOV Yu. A., The Harnack inequality and the Hölder property of solutions of nonlinear elliptic equations with a nonstandard growth condition, *Diff. Eq.* **33** (1998), 1653-1663.

[6] BOUCHITTÉ G., FONSECA I., MALÝ The effective bulk energy of the relaxed energy of multiple integrals below the growth exponent, *Proc. Roy. Soc. Edinburgh* **128A** (1988), 463-479.

[7] BUTTAZZO G., *Semicontinuity, relaxation and integral representation in the calculus of variations,* Pitman Res. Notes in Math., 207, Longman, Harlow 1989.

[8] BUTTAZZO G., DAL MASO G., A characterization of nonlinear functionals on Sobolev spaces which admit an integral representation with a Carathéodory integrand, *J. Math. Pures Appl.* **64** (1985), 337-361.

[9] BUTTAZZO G., DAL MASO G., Integral representation and relaxation of local functionals, *Nonlinear Anal.* **9** (1985), 515-532.

[10] BRAIDES A., DEFRANCESCHI A., *Homogenization of multiple integrals.* Oxford Lecture series in Maths. and its Appl. **12**, Oxford University Press, Oxford 1998.

[11] CARBONE L., SBORDONE C., Some properties of $\Gamma$-limits of integral functionals, *Ann. Mat. Pura e Appl.* (iv) **122** (1979), 1-60.

[12] CHIADÒ PIAT V., COSCIA A., Hölder continuity of minimizers of functionals with variable growth exponent, *Manuscripta Math.* **93** (1997), 283-299.

[13] COSCIA A., MINGIONE G., Hölder continuity of the gradient of $p(x)$-harmonic mappings, *C.R. Acad. Sci. Paris* **328** (1999), 363-368.

[14] DAL MASO G., *An introduction to* $\Gamma$*-convergence*, Prog. in Nonl. Diff. Eq. and their Appl. **8**, Birkäuser, Boston 1993.

[15] DAL MASO G., MODICA L., A general theory for variational functionals, *Topics in Functional Analysis*, Quaderno S.N.S. Pisa 1982.

[16] DE GIORGI E., Sulla convergenza di alcune successioni di integrali di tipo dell'area, *Rend. Mat. Univ. Roma* **8** (1975), 277-294.

[17] DE GIORGI E., FRANZONI T., Su un tipo di convergenza variazionale, *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Nat.* **58** (1975), 842-850.

[18] DE GIORGI E., LETTA G., Une notion générale de convergence faible pour des fonctions croissantes d'ensemble, *Ann. S.N.S. Pisa Cl. Sci.* **4** (1977), 61-99.

[19] EKELAND I., TEMAM R., *Convex analysis and variational problems*, North Holland, Amsterdam 1978.

[20] FAN XIANGLING, ZHAO DUN, A class of De Giorgi type and Hölder continuity, *Nonlinear Anal. T.M.A.* **36** (A)(1999), 295-318.

[21] FUSCO N., On the convergence of integral functionals depending on vector-valued functions, *Ricerche Mat.* **32** (2)(1983), 321-339.

[22] MARCELLINI P., Regularity and existence of solutions of elliptic equations with $p, q$-growth conditions, *J. Differential Equations* **90** (1991), 1-30.

[23] MARCELLINI P., Regularity for some scalar variational problems under general growth conditions, *J. Optim. Theory Appl.* **90** (1996), 161-181.

[24] MORREY C.B., Quasi-convexity and semicontinuity of multiple integrals, *Pacific J. Math.* **2** (1952), 25-53.

[25] RAJAGOPAL K.R., RŮŽIČKA M., Mathematical modelling of electrorheological fluids, *Cont. Mech. Therm.* **13**, (2001), 59-78.

[26] RŮŽIČKA M., *Electrorheological fluids: modeling and mathematical theory,* Lecture Notes in Math. **1748**, Springer, Berlin 2000.

[27] ZHIKOV V.V., On the passage to the limit in nonlinear variational problems, *Russian Acad. Sci. Sb. Math.* **76** (2) (1993), 427-459.

[28] ZHIKOV V.V., On Lavrentiev's phenomenon, *Russian J. Math. Physics* **3** (1995), 249-269.

[29] ZHIKOV V.V., On some variational problems, *Russian J. Math. Physics* **5** (1997), 105-116.

[30] ZHIKOV V.V., Meyers type estimates for solving the non linear Stokes system, *Differential Equations* **33** (1) (1997), 107-114.

[31] ZHIKOV V.V., KOZLOV S.M., OLEINIK O.A., *Homogenization of differential operators and integral functionals*, Springer, Berlin 1994.

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