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HÖLDER CONTINUITY UP TO THE BOUNDARY FOR A CLASS OF FRACTIONAL OBSTACLE PROBLEMS

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Abstract. We deal with the obstacle problem for a class of nonlinear integro-differential operators, whose model is the fractional p-Laplacian with measurable coefficients. In accordance with well-known results for the analog for the pure fractional Laplacian operator, the corresponding solutions inherit regularity properties from the obstacle, both in the case of boundedness, continuity, and Hölder continuity, up to the boundary.

Key Words: Quasilinear nonlocal operators, fractional Sobolev spaces, nonlocal tail, Caccioppoli estimates, obstacle problem.

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1. The fractional obstacle problem

The obstacle problem is a fundamental topic in Partial Differential Equations and Potential Theory, with crucial implications in many contexts in Biology, in Elasticity, in Financial Mathematics, and so on. See for instance the books [12] and [27], where many of these applications are described, as well as the classical literature on this problem. It can be stated in several ways. Roughly speaking, there is an elliptic operator $\mathcal{L}$ and a function h (the obstacle), so that the solution u to the obstacle problem in a domain Ω is a minimal supersolution to $\mathcal{L} = 0$ above the obstacle; i. e.,

$$\begin{cases} \mathcal{L}u \geq 0 & \\ u \geq h & \text{everywhere in the domain } \Omega, \\ (\textit{boundary conditions}). & \end{cases}$$

The operator $\mathcal{L}$ can be a classical second order elliptic operator, an integro-differential operator, and even a nonlinear one. The study of the obstacle problem originated in the context of Elasticity as the equation that models the shape of an elastic membrane which is pushed by an obstacle from one side affecting its shape. The same equation also arises in Control Theory, specifically as the question of finding the optimal stopping time for a stochastic process with payoff function. In origin, in both these cases, possibly after linearization, the involved operator $\mathcal{L}$

coincides with the Laplacian operator. A special very important case is when the operator $\mathcal{L}$ is the fractional Laplacian $(-\Delta)^s$; that is,

$$(-\Delta)^s u(x) = p.v. \int_{\mathbb{R}^n} \big(u(x)-u(y)\big)|x-y|^{-n-2s}\,\mathrm{d}y, \qquad x \in \mathbb{R}^n;$$

see [9]. The obstacle problem involving the fractional Laplacian operator indeed appears in many contexts, such as in the analysis of anomalous diffusion, in the quasi-geostrophic flow problem, and in pricing of American options regulated by assets evolving in relation to jump processes; in particular, this important application in Financial Mathematics made the obstacle problem very important in recent times. A large treatment of the fractional obstacle problem can be found in the important papers by Caffarelli, Figalli, Salsa, and Silvestre (see, e. g., [1–3, 30]); see also [10] for the analysis of families of bilateral obstacle problems involving fractional type energies in aperiodic settings; and the paper [26] for the fractional obstacle problems with drift. However, despite its relatively short history, this problem has already evolved into an elaborate theory with several connections to other branches; the literature is too wide to attempt any reasonable comprehensive treatment in a single paper. We refer the interested reader to the exhaustive lecture notes [29], and to the forthcoming work by Danielli and Salsa ([6]), and the references therein.

2. The nonlinear integro-differential obstacle problem

Here we are interested in a very general class of nonlinear nonlocal obstacle problems; i. e., those related to the operator $\mathcal{L}$ defined on suitable fractional Sobolev functions by

$$\mathcal{L}u(x) = p.\,v. \int_{\mathbb{R}^n} K(x,y)|u(x)-u(y)|^{p-2}\big(u(x)-u(y)\big)\,\mathrm{d}y, \qquad x \in \mathbb{R}^n. \tag{2.1}$$

The nonlinear nonlocal operator $\mathcal{L}$ in the display above is driven by its *kernel* $K : \mathbb{R}^n \times \mathbb{R}^n \to [0,\infty)$, which is a measurable function of differentiability order $s \in (0,1)$ and summability exponent $p > 1$,

$$\Lambda^{-1} \leq K(x,y)|x-y|^{n+sp} \leq \Lambda \quad \text{for a.e. } x, y \in \mathbb{R}^n,$$

for some $\Lambda \geq 1$. Clearly, in the linear case when $p = 2$ and without coefficients when $\Lambda = 1$, we recover the aforementioned fractional Laplacian operator $(-\Delta)^s$.

We need now to recall the definition of *the nonlocal tail* $\mathrm{Tail}(f; z, r)$ *of a function* f *in the ball of radius* $r > 0$ *centered in* $z \in \mathbb{R}^n$; see [7, 8]. For any function f initially defined in $L^{p-1}_{\mathrm{loc}}(\mathbb{R}^n)$,

$$\mathrm{Tail}(f; z, r) := \left(r^{sp} \int_{\mathbb{R}^n \setminus B_r(z)} |f(x)|^{p-1}|x-z|^{-n-sp}\,\mathrm{d}x\right)^{\frac{1}{p-1}}. \tag{2.2}$$

In accordance, we recall the definition of the corresponding *tail space* $L^{p-1}_{sp}(\mathbb{R}^n)$,

$$L^{p-1}_{sp}(\mathbb{R}^n) := \Big\{ f \in L^{p-1}_{\rm loc}(\mathbb{R}^n) \ : \ \mathrm{Tail}(f;z,r) < \infty \quad \forall z \in \mathbb{R}^n, \forall r \in (0,\infty) \Big\};$$

see [17]. As expected, one can check that $W^{s,p}(\mathbb{R}^n) \subset L^{p-1}_{sp}(\mathbb{R}^n)$, where we denoted by $W^{s,p}(\mathbb{R}^n)$ the usual fractional Sobolev space of order (s,p), defined by the norm

$$\begin{aligned}\|v\|_{W^{s,p}(\mathbb{R}^n)} &:= \|v\|_{L^p(\mathbb{R}^n)} + [v]_{W^{s,p}(\mathbb{R}^n)} \\ &= \left(\int_{\mathbb{R}^n} |v|^p \,{\rm d}x\right)^{\frac{1}{p}} + \left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|v(x)-v(y)|^p}{|x-y|^{n+sp}}\,{\rm d}x{\rm d}y\right)^{\frac{1}{p}}.\end{aligned}$$

We finally observe that, since we assume that coefficients are merely measurable, the involved equation has to have a suitable weak formulation. For this, we recall the definitions of sub and supersolutions u to

$$\mathcal{L}u = 0 \ \text{ in } \mathbb{R}^n. \tag{2.3}$$

A function $u \in W^{s,p}_{\rm loc}(\Omega) \cap L^{p-1}_{sp}(\mathbb{R}^n)$ is a *fractional weak p-supersolution* of (2.3) if

$$\begin{aligned}\langle \mathcal{L}u, \eta\rangle \equiv & \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} K(x,y)|u(x)-u(y)|^{p-2}\big(u(x)-u(y)\big)\big(\eta(x)-\eta(y)\big)\,{\rm d}x{\rm d}y \\ & \geq 0\end{aligned}$$

for every nonnegative $\eta \in C^\infty_0(\Omega)$. Notice that the summability assumption of $u \in L^{p-1}_{sp}(\mathbb{R}^n)$ is what one expects in the nonlocal framework considered here; see [15]. A function $u \in W^{s,p}_{\rm loc}(\Omega) \cap L^{p-1}_{sp}(\mathbb{R}^n)$ is a *fractional weak p-subsolution* if $-u$ is a fractional weak p-supersolution. A function u is a *fractional weak p-solution* if it is both fractional weak p-sub and supersolution.

2.1. **The variational framework and first results.** From now on, we report the main facts from [14], to which we refer for a more complete presentation and for detailed proofs. First, we recall that the obstacle problem can be reformulated as a standard problem in the theory of variational inequalities on Banach spaces, by seeking the energy minimizer in convex sets of suitable functions. Let us introduce the variational framework of our problem. Let $\Omega \Subset \Omega'$ be open bounded subsets of $\mathbb{R}^n$, let the obstacle function $h\colon \mathbb{R}^n \to [-\infty,\infty)$ be an extended real-valued function, and let $g \in W^{s,p}(\Omega') \cap L^{p-1}_{sp}(\mathbb{R}^n)$ be the boundary values. We define the (non-empty) set

$$\mathcal{K}_{g,h}(\Omega,\Omega') := \Big\{ u \in W^{s,p}(\Omega') \ : \ u \geq h \ \text{ a.e. in } \Omega, \ u = g \ \text{ a.e. on } \mathbb{R}^n \setminus \Omega \Big\},$$

and the functional $\mathcal{A} : \mathcal{K}_{g,h}(\Omega,\Omega') \to [W^{s,p}(\Omega')]'$ as follows

$$\begin{aligned}\mathcal{A}u(v) \ := \ & \int_{\Omega'}\int_{\Omega'} L(u(x),u(y))\big(v(x)-v(y)\big)K(x,y)\,{\rm d}x{\rm d}y \\ & + 2\int_{\mathbb{R}^n\setminus\Omega'}\int_{\Omega} L(u(x),g(y))v(x)K(x,y)\,{\rm d}x{\rm d}y,\end{aligned}$$

for every $u \in \mathcal{K}_{g,h}(\Omega, \Omega')$ and $v \in W^{s,p}(\Omega')$. Notice that the functional $\mathcal{A}u$ really belongs to the dual of the fractional Sobolev space $W^{s,p}(\Omega')$; see [14, Remark 1]. We are now ready to provide the natural definition of solutions to the obstacle problem in the general nonlocal framework considered here.

Definition 2.1. We say that $u \in \mathcal{K}_{g,h}(\Omega, \Omega')$ is *a solution to the obstacle problem* in $\mathcal{K}_{g,h}(\Omega, \Omega')$ if

$$\mathcal{A}u(v-u) \geq 0$$

whenever $v \in \mathcal{K}_{g,h}(\Omega, \Omega')$.

The existence and uniqueness of the solution to the obstacle problem and the fact that such a solution is a weak supersolution to (2.3) is proven in the following

Theorem 2.2. ([14, Theorem 1]). *There exists a unique solution to the obstacle problem in $\mathcal{K}_{g,h}(\Omega, \Omega')$. Moreover, the solution to the obstacle problem is a weak supersolution to (2.3) in Ω.*

Sketch of the proof. First, one can prove by computations that $\mathcal{A}$ is monotone on $\mathcal{K}_{g,h}(\Omega, \Omega')$. Second, by applying the Hölder inequality together with some basic estimates in [14, Lemmata 1 and 2], and by carefully treating in a different way the superquadratic case when $p \geq 2$ and the subquadratic case when $1 < p < 2$, one can prove the weak continuity of $\mathcal{A}$. Then, by using again the Hölder inequality together with some fractional Sobolev embeddings, one can prove that $\mathcal{A}$ is coercive. This will permit to apply the standard theory of monotone operators in order to deduce the existence of a solution u to the obstacle problem. The uniqueness is easily proven via a contradiction argument. Finally, one can show that the function u is a weak supersolution to (2.3) by noticing that for any nonnegative function $\varphi \in C_0^\infty(\Omega)$, the function $u + \varphi$ belongs to $\mathcal{K}_{g,h}(\Omega, \Omega')$. $\square$

Also, under natural assumptions on the obstacle h, one can prove that the solution to the obstacle problem is fractional harmonic (see [15] for the definition and several related properties) away from the contact set, in clear accordance with the classical results when $s = 1$.

Corollary 2.3. ([14, Corollary 2]). *Let u be the solution to the obstacle problem in $\mathcal{K}_{g,h}(\Omega, \Omega')$. If $B_r \subset \Omega$ is such that*

$$\operatorname*{ess\,inf}_{B_r}(u-h) > 0,$$

then u is a weak solution to (2.3) in B_r. In particular, if u is lower semicontinuous and h is upper semicontinuous in Ω, then u is a weak solution to (2.3) in $\Omega_+ := \big\{x \in \Omega : u(x) > h(x)\big\}$.

The solution to the obstacle problem is the smallest supersolution above the obstacle in the sense precised below.

Proposition 2.4. ([14, Proposition 1]). *Let $\Omega \Subset \Omega'' \subset \Omega'$. Let u be the solution to the obstacle problem in $\mathcal{K}_{g,h}(\Omega,\Omega')$ and let v be a weak supersolution in Ω'' such that $\min\{u,v\} \in \mathcal{K}_{g,h}(\Omega,\Omega')$. Then $u \le v$ almost everywhere.*

Sketch of the proof. It will suffice to notice that, since u is the solution to the obstacle problem and $\min\{u,v\} = u$ in $\mathbb{R}^n \setminus \Omega$, we have

$$0 \le \langle \mathcal{A}u, \min\{u,v\} - u\rangle;$$

and, since v is a weak supersolution in Ω'' and $u - \min\{u,v\} \in W_0^{s,p}(\Omega)$ is non-negative, we have,

$$\begin{aligned} 0 \le \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |v(x)-v(y)|^{p-2}\big(u(x)-u(y)\big)\Big(&u(x) - \min\{u,v\}(x) - u(y) \\ &+ \min\{u,v\}(y)\Big)\, K(x,y)\,\mathrm{d}x\mathrm{d}y. \end{aligned}$$

By summing the preceding inequalities, one can deduce that $|\{u > v\}| = 0$. $\square$

Remark 2.5. It is worth noticing that the obstacle function h is an extended real-valued function. In particular, in our results we are also including the case when $h \equiv -\infty$; i. e., no obstacle at all, whose interpretation is

$$\mathcal{K}_g(\Omega,\Omega') \equiv \mathcal{K}_{g,-\infty}(\Omega,\Omega') := \Big\{u \in W^{s,p}(\Omega') \,:\, u = g \ \text{ a. e. on } \mathbb{R}^n \setminus \Omega\Big\},$$

that is, the class where we are seeking solutions to the Dirichlet boundary value problem. Moreover, in view of Theorem 2.2, by solving the obstacle problem in $\mathcal{K}_{g,-\infty}(\Omega,\Omega')$ one obtains a unique weak solution to (2.3) in Ω with boundary values $g \in W^{s,p}(\Omega') \cap L^{p-1}_{sp}(\mathbb{R}^n)$ in the complement of Ω.

3. Interior regularity results

The regularity of the solution to the obstacle problem inherits the regularity of the obstacle, both in the case of boundedness, continuity, and Hölder continuity.

Theorem 3.1. ([14, Theorem 2]). *Let u be the solution to the obstacle problem in $\mathcal{K}_{g,h}(\Omega,\Omega')$. Assume that $B_r(x_0) \subset \Omega'$ and set*

$$M := \max\left\{ \operatorname*{ess\,sup}_{B_r(x_0)\cap\Omega} h,\ \operatorname*{ess\,sup}_{B_r(x_0)\setminus\Omega} g \right\}.$$

Here the interpretation is that $\operatorname{ess\,sup}_A f = -\infty$ if $A = \emptyset$. If M is finite, then u is essentially bounded from above in $B_{r/2}(x_0)$ and

$$\operatorname*{ess\,sup}_{B_{r/2}(x_0)} (u-m)_+ \le \delta\, \mathrm{Tail}((u-m)_+; x_0, r/2) + c\,\delta^{-\gamma}\left(\fint_{B_r(x_0)} (u-m)_+^t\,\mathrm{d}x\right)^{\frac{1}{t}}$$

holds for all $m \ge M$, $t \in (0,p)$ and $\delta \in (0,1]$ with constants $\gamma \equiv \gamma(n,p,s,t)$ and $c \equiv c(n,p,s,t,\Lambda)$.

Theorem 3.2. ([14, Theorem 3]). *Suppose that h is locally Hölder continuous in Ω. Then the solution u to the obstacle problem in $\mathcal{K}_{g,h}(\Omega,\Omega')$ is locally Hölder continuous in Ω as well. Moreover, for every $x_0 \in \Omega$ there is $r_0 > 0$ such that*

$$\operatorname*{osc}_{B_\rho(x_0)} u \le c\left(\frac{\rho}{r}\right)^{\alpha}\left[\operatorname{Tail}(u-h(x_0);x_0,r) + \left(\fint_{B_r(x_0)} |u-h(x_0)|^p\,\mathrm{d}x\right)^{\frac{1}{p}}\right] + c\int_\rho^r \left(\frac{\rho}{t}\right)^{\alpha}\omega_h\left(\frac{t}{\sigma}\right)\frac{\mathrm{d}t}{t}$$

for every $r \in (0,r_0)$ and $\rho \in (0,r/4]$, where $\omega_h(\rho) \equiv \omega_h(\rho,x_0) := \operatorname{osc}_{B_\rho(x_0)} h$, and α, c and σ depend only on n, p, s, and Λ.

Slightly modifying the proof of the preceding theorem (for which we refer also to Section 5 below), one can easily obtain the following

Theorem 3.3. ([14, Theorem 4]). *Suppose that h is continuous in Ω. Then the solution to the obstacle problem in $\mathcal{K}_{g,h}(\Omega,\Omega')$ is continuous in Ω as well.*

All the results presented in this section are consistent with their counterparts for the obstacle problems in the pure fractional Laplacian case. This said, the related proofs are different and, though we are dealing with a wider class of nonlinear integro-differential operators with coefficients, to a certain extent these proofs are even simpler, since we can make use of the new nonlocal set-up and the recent quantitative estimates in [7, 8], by combining them with some well-known tools from the classical nonlinear Potential Theory. Moreover, since we allow the obstacle function h to be an extended real-valued function, the degenerate case when no obstacle is present does reduce the problem to the standard Dirichlet boundary value problem, so that the results proven in [14] are new even when $\mathcal{L}$ does coincide with the fractional p-Laplacian $(-\Delta)^s_p$. Also, as noticing in Remark 2.5, we assume that the boundary data merely belong to an appropriate tail space $L^{p-1}_{sp}(\mathbb{R}^n)$, so that our results are an improvement with respect to all the previous results in the literature when the data are usually given in the whole $W^{s,p}(\mathbb{R}^n)$.

4. Regularity up to the boundary

The results in the previous sections can be extended up to the boundary of Ω. For this, one has to assume that the complement of Ω satisfies the following measure density condition: there exist $\delta_\Omega \in (0,1)$ and $r_0 > 0$ such that for every $x_0 \in \partial\Omega$

$$\inf_{0<r<r_0} \frac{|(\mathbb{R}^n \setminus \Omega) \cap B_r(x_0)|}{|B_r(x_0)|} \ge \delta_\Omega. \tag{4.1}$$

This requirement is in the same spirit of the classical nonlinear Potential Theory, and – as expected in view of the nonlocality of the involved equations – is translated into an information given on the complement of the set Ω. Also, it is worth noticing that this is an improvement with respect to the previous boundary regularity results in all the fractional literature when much stronger Lipschitz regularity or smoothness of the sets are usually assumed.

When the obstacle and the boundary values are bounded on the boundary, so is the solution to the obstacle problem.

Theorem 4.1. ([14, Theorem 5]). *Suppose that $u \in \mathcal{K}_{g,h}(\Omega,\Omega')$ solves the obstacle problem in $\mathcal{K}_{g,h}(\Omega,\Omega')$. Let $x_0 \in \partial\Omega$ and suppose that*

$$\operatorname*{ess\,sup}_{B_r(x_0)} g + \operatorname*{ess\,sup}_{B_r(x_0)\cap\Omega} h < \infty \quad \textit{and} \quad \operatorname*{ess\,inf}_{B_r(x_0)} g > -\infty$$

for $r \in (0, r_0)$ with $r_0 := \operatorname{dist}(x_0, \partial\Omega')$. Then u is essentially bounded close to x_0.

The regularity of the solution to the obstacle problem inherits the regularity of the obstacle, both in the case of continuity and Hölder continuity, up to the boundary.

Theorem 4.2. ([14, Theorem 6]). *Suppose that u solves the obstacle problem in $\mathcal{K}_{g,h}(\Omega,\Omega')$ and assume $x_0 \in \partial\Omega$ and $B_{2R}(x_0) \subset \Omega'$. If $g \in \mathcal{K}_{g,h}(\Omega,\Omega')$ is Hölder continuous in $B_R(x_0)$ and Ω satisfies (4.1) for all $r \le R$, then u is Hölder continuous in $B_R(x_0)$ as well.*

Theorem 4.3. ([14, Theorem 7]). *Suppose that u solves the obstacle problem in $\mathcal{K}_{g,h}(\Omega,\Omega')$ and assume $x_0 \in \partial\Omega$ and $B_{2R}(x_0) \subset \Omega'$. If $g \in \mathcal{K}_{g,h}(\Omega,\Omega')$ is continuous in $B_R(x_0)$ and Ω satisfies (4.1) for all $r \le R$, then u is continuous in $B_R(x_0)$ as well.*

Remark 4.4. We notice that one has to assume that the datum g belongs to $\mathcal{K}_{g,h}(\Omega,\Omega')$, since otherwise the solution may be discontinuous on every boundary point, as one can see by taking $\Omega = B_1(0)$, $\Omega' = B_2(0)$, and (s,p) such that $sp < 1$. It plainly follows that the characteristic function χ_Ω solves the obstacle problem in $\mathcal{K}_{g,h}(\Omega,\Omega')$ with constant functions $g \equiv 0$ and $h \equiv 1$. Indeed, $\chi_\Omega \in W^{s,p}(\Omega')$, and one can check that it is a weak supersolution. As a consequence, by recalling Proposition 2.4, the function χ_Ω is the solution to the obstacle problem in $\mathcal{K}_{g,h}(\Omega,\Omega')$. See [14, Example 1].

Finally, a few observations are in order. Boundary regularity for nonlocal equations driven by singular, possibly degenerate, operators as in (2.1) seems to be a difficult problem in a general nonlinear framework under natural assumptions on the involved quantities (see [19]). The situation simplifies considerably in the linear case when $p = 2$; see for instance the forthcoming survey [28] and the references therein. Coming back to the nonlinear case, to our knowledge, the solely nonlocal result of global Hölder regularity has been obtained very recently in the interesting paper [20], where the authors deal with the non-homogeneous equation, in the special case when the operator $\mathcal{L}$ in (2.1) does coincide with the nonlinear fractional Laplacian $(-\Delta)_p^s$, by considering exclusively zero Dirichlet boundary data, and by strongly assuming $C^{1,1}$-regularity up to the boundary for the domain Ω.

The proofs there are indeed strongly based on the construction of suitable barriers near $\partial\Omega$, by relying on the fact that the function $x \mapsto x_+^s$ is an explicit solution in the half-space. For this, one cannot expect to plainly extend such a strategy in the general framework considered here, in view of the presence of merely measurable coefficients in (2.1). In [14], nonzero boundary Dirichlet data can be chosen, and the domain Ω has to satisfy only the natural measure density condition given in (4.1). Consequently, a new proof is needed which extend up to the boundary part of the results in [7, 8] together with a careful handling of the tail-type contributions.

5. Some idea from the proofs

As one can expect, the main difficulty into the treatment of the operators $\mathcal{L}$ in (2.1) lies in their very definition, which combines the typical issues given by its nonlocal feature together with the ones given by its nonlinear growth behavior; also, further efforts are needed due to the presence of merely measurable coefficients in the kernel K. For this, some very important tools recently introduced in the nonlocal theory, as the by-now classic s-harmonic extension, the strong three-term commutators estimates to deduce the regularity of weak fractional harmonic maps ([5]), the pseudo-differential commutator and energy estimates in [23–25], and many other successful tricks seem not to be trivially adaptable to the nonlinear framework considered here. Increased difficulties are due to the non-Hilbertian structure of the involved fractional Sobolev spaces $W^{s,p}$ when $p \neq 2$. In spite of that, some related regularity results have been very recently achieved in this context, in [7, 8, 13, 14, 16–18, 20] and many others, where often a fundamental role to understand the nonlocality of the nonlinear operators $\mathcal{L}$ has been played by the nonlocal tail defined by (2.2) in order to obtain fine quantitative controls of the long-range interactions.

Sketch of the proof of Theorem 3.1. In order to prove the boundedness result, one can test the equation with a suitable class of functions, by noticing that for any $m \geq M$ the function $u_m = u - m$ solves the corresponding obstacle problem. Thus, after some careful estimates on the local and the nonlocal contributions in the energy formulation, one can arrive to prove a Caccioppoli-type inequality with tail. For similar approach in order to achieve fractional Caccioppoli-type inequalities, though not taking into account the tail, see also [11, 21, 22]. At this level, by following the strategy in the proof of [7, Theorem 1.1], it yields a local boundedness for the truncated functions u_m. Finally, via a covering argument which goes back to the one in the proof of [8, Theorem 1.1], together with a standard iteration argument, one arrives at the desired result. □

Sketch of the proof of Theorem 3.2. The first step is to prove that for any point x_0 in the contact set, and for for any $r \in (0, R)$, one can find $\sigma \in (0, 1)$ and c, both depending only on n, p, s, Λ, such that

$$
\begin{aligned}
&\operatorname*{osc}_{B_{\sigma r}(x_0)} u + \mathrm{Tail}(u - h(x_0); x_0, \sigma r) \\
&\qquad \leq \frac{1}{2}\left(\operatorname*{osc}_{B_r(x_0)} u + \mathrm{Tail}(u - h(x_0); x_0, r)\right) + c\,\omega_h(r). \qquad (5.1)
\end{aligned}
$$

In order to do this, we combine the weak Harnack estimates in [8, Theorem 1.2] with the boundedness estimate in Theorem 3.1 (applied with $m = d + 2\omega_h(r) \geq \sup_{B_{2\rho}(x_0)} h$ there). Then we choose the parameter δ in (3.1) interpolating between the local and nonlocal terms in a suitable way. This gives (5.1). The subsequent step relies into an iterative argument in order to conclude the analysis on the contact set. Finally, we analyze the continuity outside the contact set. For this, it suffices to apply Corollary 2.3 which assures that u is a weak solution in $B_{r_0}(x_0)$, so that one can use the results in [7] by noticing that the proofs there are valid also by assuming that u merely belongs to $W^{s,p}_{\mathrm{loc}}(\Omega) \cap L^{p-1}_{sp}(\mathbb{R}^n)$ instead than $W^{s,p}(\mathbb{R}^n)$. □

Sketch of the proof of Theorem 4.1. Assume that $x_0 \in \partial\Omega$. Let $w_+ := (u - k_+)_+$ and $w_- := (k_- - u)_+$, where $k_+ \geq \max\left\{\operatorname{ess\,sup}_{B_r(x_0)} g, \operatorname{ess\,sup}_{B_r(x_0)\cap\Omega} h\right\}$ and $k_- \leq \operatorname{ess\,inf}_{B_r(x_0)} g$. We obtain the Caccioppoli-type estimate with tail below, whose proof is a verbatim repetition of the proof of [7, Theorem 1.4] after noticing that $v = u \mp w_\pm \varphi^p$, $\varphi \in C^\infty_0(B_r(x_0))$, $0 \leq \varphi \leq 1$, belongs to $\mathcal{K}_{g,h}(\Omega, \Omega')$ for all indicated $k_\pm$. If follows

$$
\begin{aligned}
&\int_{B_r(x_0)} \int_{B_r(x_0)} |w_\pm(x)\varphi(x) - w_\pm(y)\varphi(y)|^p K(x,y)\,\mathrm{d}x\mathrm{d}y \\
&\quad \leq c \int_{B_r(x_0)} \int_{B_r(x_0)} w_\pm^p(x)|\varphi(x) - \varphi(y)|^p K(x,y)\,\mathrm{d}x\mathrm{d}y \\
&\qquad + c \int_{B_r(x_0)} w_\pm(x)\varphi^p(x)\,\mathrm{d}x \left(\sup_{y\,\in\,\mathrm{supp}\,\varphi} \int_{\mathbb{R}^n\setminus B_r(x_0)} w_\pm^{p-1}(x) K(x,y)\,\mathrm{d}x\right).
\end{aligned}
$$

Then, one can deduce that u is essentially bounded in $B_{r/2}(x_0)$ by extending the proof of Theorem 1.1 in [7] and using the estimate above with $w_\pm$. □

Sketch of the proof of Theorem 4.2. We may assume $x_0 = 0$ and $g(0) = 0$. Moreover, we may choose R_0 such that $\operatorname{osc}_{B_0} g \leq \operatorname{osc}_{B_0} u$ for $B_0 \equiv B_{R_0}(0)$ since otherwise we have nothing to prove, and we define $\omega_0 := 8\,(\operatorname{osc}_{B_0} u + \mathrm{Tail}(u; 0, R_0))$. The proof of the Hölder continuity up to the boundary relies on a logarithmic estimate with tail ([14, Lemma 5]), obtained by suitably choosing test functions and by carefully estimating the local and nonlocal energy contributions separately in the super and subquadratic cases. Such a logarithmic lemma can be subsequently extended to truncations of the solution to the obstacle problem, as follows: let

$B_R \Subset \Omega'$, let $B_r \subset B_{R/2}$ be concentric balls and let

$$\infty > k_+ \geq \max\left\{ \operatorname*{ess\,sup}_{B_R} g, \operatorname*{ess\,sup}_{B_R\cap\Omega} h \right\} \quad \text{and} \quad -\infty < k_- \leq \operatorname*{ess\,inf}_{B_R} g;$$

then the functions $w_\pm := \operatorname{ess\,sup}_{B_R}(u-k_\pm)_\pm - (u-k_\pm)_\pm + \varepsilon$ satisfy the following estimate

$$\int_{B_r}\int_{B_r} \left|\log \frac{w_\pm(x)}{w_\pm(y)}\right|^p K(x,y)\,\mathrm{d}x\mathrm{d}y$$

$$\leq c\, r^{n-sp}\left(1+\varepsilon^{1-p}\left(\frac{r}{R}\right)^{sp} \mathrm{Tail}((w_\pm)_-, x_0, R)^{p-1}\right) \tag{5.2}$$

for every $\varepsilon > 0$. Then, we combine the estimate in the display above with a fractional Poincaré-type inequality ([14, Lemma 7]) together with some estimates for the tail term thanks to an application of the Chebyshev inequality and in view of the result in Theorem 4.1. We arrive to prove the existence of τ_0, σ and θ depending only on n, p, s and δ_Ω, such that if

$$\operatorname*{osc}_{B_r(0)} u + \sigma \mathrm{Tail}(u;0,r) \leq \omega \quad \text{and} \quad \operatorname*{osc}_{B_r(0)} g \leq \frac{\omega}{8}$$

hold for a ball $B_r(0)$ and for $\omega > 0$, then

$$\operatorname*{osc}_{B_{\tau r}(0)} u + \sigma \mathrm{Tail}(u;0,\tau r) \leq (1-\theta)\omega$$

holds for every $\tau \in (0,\tau_0]$. Finally, as we can take $\tau \leq \tau_0$ such that

$$\operatorname*{osc}_{\tau^j B_0} g \leq (1-\theta)^j \frac{\omega_0}{8} \qquad \text{for every } j = 0,1,\dots,$$

an iterative argument will give that u belongs to $C^{0,\alpha}(B_0)$. □

6. Further developments

We conclude this paper by briefly commenting about some open problems that arise in this framework.

A first natural open problem concerns the optimal regularity for the solutions to the nonlinear nonlocal obstacle problem. We recall that for the classical obstacle problem, when $\mathcal{L}$ coincides with the Laplacian operator, the solutions are known to be in $C^{1,1}$. The intuition behind this regularity result is that in the contact set one has $-\Delta u = -\Delta h$, while where $u > h$ one has $-\Delta u = 0$; since the Laplacian jumps from $-\Delta h$ to 0 across the free boundary, the second derivatives of u must have a discontinuity... and thus $C^{1,1}$ is the maximum regularity class that can be expected. Surprisingly, when $\mathcal{L} \equiv (-\Delta)^s$, despite the previous local argument does suggest that the solutions u belong to C^{2s}, the optimal regularity is $C^{1,s}$; that is, the regularity exponent is higher than the order of the equation. In the nonlinear nonlocal framework presented here, starting from the Hölder regularity proven in [14], one still expects higher regularity results. Notwithstanding, in view of the interplay between the local and nonlocal contributions, and without having the possibility to rely on all the linear tools mentioned at the beginning of Section 5, it is not completely clear what the optimal exponent could be as the nonlinear growth does take its part. For preliminary results in this direction, it is worth mentioning the very recent paper [4], where optimal regularity results of the solution to the obstacle problem, and of the free boundary near regular points, have been achieved for integro-differential operators as in (2.1) in the linear case when $p = 2$.

Another interesting open problem concerns the regularity in a generic point of the free boundary, which is known to be analytic in the case of the Laplacian, except on a well-defined set of singular points, and smooth in the case of the fractional Laplacian.

Finally, a natural goal is the investigation of the related parabolic version of the nonlinear nonlocal obstacle problem, as it is inspired in the so-called optimal stopping problem with deadline, by corresponding to the American option pricing problem with expiration at some given time. An extension in the setting presented here could be quite important as it would essentially describe a situation which also takes into account the interactions coming from far together with a natural inhomogeneity. Accordingly with the optimal stopping problem model, a starting point in such an investigation could be the special case when the obstacle h coincides with the boundary value g.

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