BREZIS-BOURGAIN-MIRONESCU FORMULA FOR MAGNETIC OPERATORS

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Abstract. We prove a Brezis-Bourgain-Mironescu type formula for a class of nonlocal magnetic spaces, which builds a bridge between a fractional magnetic operator recently introduced and the classical theory.

1. Introduction

Let $s \in (0,1)$ and $N > 2s$. If $A : \mathbb{R}^N \to \mathbb{R}^N$ is a smooth function, the nonlocal operator

$$(-\Delta)^s_A u(x) = c(N,s) \lim_{\varepsilon \searrow 0} \int_{B^c_\varepsilon(x)} \frac{u(x) - e^{\mathrm{i}(x-y)\cdot A\left(\frac{x+y}{2}\right)} u(y)}{|x-y|^{N+2s}} dy, \qquad x \in \mathbb{R}^N,$$

has been recently introduced in [6], where the ground state solutions of $(-\Delta)^s_A u + u = |u|^{p-2}u$ in the three dimensional setting have been obtained via concentration compactness arguments. If $A = 0$, then the above operator is consistent with the usual notion of fractional Laplacian. The motivations that led to its introduction are carefully described in [6] and rely essentially on the Lévy-Khintchine formula for the generator of a general Lévy process. We point out that the normalization constant $c(N,s)$ satisfies

$$\lim_{s \nearrow 1} \frac{c(N,s)}{1-s} = \frac{4N\Gamma(N/2)}{2\pi^{N/2}},$$

where Γ denotes the Gamma function. For the sake of completeness, we recall that different definitions of nonlocal magnetic operator are viable, see e.g. [8, 9]. All these notions aim to extend the well-know definition of the magnetic Schrödinger operator

$$\big(\nabla - \mathrm{i}A(x)\big)^2 u = -\Delta u + 2\mathrm{i}A(x)\cdot\nabla u + |A(x)|^2 u + \mathrm{i}u\,\mathrm{div}A(x),$$

namely the differential of the energy functional

$$\mathcal{E}_A(u) = \int_{\mathbb{R}^N} |\nabla u - \mathrm{i}A(x)u|^2 dx,$$

for which we refer the reader to [1,2,11] and the included references. In order to corroborate the justification for the introduction of $(-\Delta)^s_A$, in this note we prove that a well-known formula due to Brezis, Bourgain and Mironescu (see [3, 4, 10]) for the limit of the Gagliardo semi-norm of $H^s(\Omega)$ as $s \nearrow 1$ extends to the magnetic setting. As a consequence, in a suitable sense, from the nonlocal to the local regime, it holds

$$(-\Delta)^s_A u \rightsquigarrow \big(\nabla - \mathrm{i}A(x)\big)^2 u, \qquad \text{for } s \nearrow 1.$$

Precisely, we have the following

Theorem 1.1 (Magnetic Brezis-Bourgain-Mironescu). *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set with Lipschitz boundary and $A \in C^2(\mathbb{R}^N)$. Then, for every $u \in H^1_A(\Omega)$, we have*

$$\lim_{s \nearrow 1} (1-s) \int_\Omega \int_\Omega \frac{|u(x) - e^{\mathrm{i}(x-y)\cdot A\left(\frac{x+y}{2}\right)} u(y)|^2}{|x-y|^{N+2s}} dxdy = K_N \int_\Omega |\nabla u - \mathrm{i}A(x)u|^2 dx,$$

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where

$$K_N = \frac{1}{2}\int_{\mathbb{S}^{N-1}} |\omega\cdot\mathbf{e}|^2 d\mathcal{H}^{N-1}(\omega), \tag{1.1}$$

being $\mathbb{S}^{N-1}$ the unit sphere and $\mathbf{e}$ any unit vector in $\mathbb{R}^N$.

As a variant of Theorem 1.1, if $H^1_{0,A}(\Omega)$ denotes the closure of $C_c^\infty(\Omega)$ in $H^1_A(\Omega)$, there holds

Theorem 1.2. *Let $\Omega\subset\mathbb{R}^N$ be an open bounded set with Lipschitz boundary and $A\in C^2(\mathbb{R}^N)$. Then, for every $u\in H^1_{0,A}(\Omega)$, we have*

$$\lim_{s\nearrow 1}(1-s)\int_{\mathbb{R}^{2N}}\frac{|u(x)-e^{\mathrm{i}(x-y)\cdot A\left(\frac{x+y}{2}\right)}u(y)|^2}{|x-y|^{N+2s}}dxdy = K_N\int_\Omega |\nabla u-\mathrm{i}A(x)u|^2dx.$$

Notations. Let $\Omega\subset\mathbb{R}^N$ be an open set. We denote by $L^2(\Omega,\mathbb{C})$ the Lebesgue space of complex valued functions with summable square. For $s\in(0,1)$, the magnetic Gagliardo semi-norm is

$$[u]_{H^s_A(\Omega)} := \sqrt{\int_\Omega\int_\Omega \frac{|u(x)-e^{\mathrm{i}(x-y)\cdot A\left(\frac{x+y}{2}\right)}u(y)|^2}{|x-y|^{N+2s}}dxdy}.$$

We denote by $H^s_A(\Omega)$ the space of functions $u\in L^2(\Omega,\mathbb{C})$ such that $[u]_{H^s_A(\Omega)}<\infty$ endowed with

$$\|u\|_{H^s_A(\Omega)} := \sqrt{\|u\|^2_{L^2(\Omega)}+[u]^2_{H^s_A(\Omega)}}.$$

We also consider

$$[u]_{H^1_A(\Omega)} := \sqrt{\int_\Omega |\nabla u-\mathrm{i}A(x)u|^2dx},$$

and define $H^1_A(\Omega)$ as the space of functions $u\in L^2(\Omega,\mathbb{C})$ such that $[u]_{H^1_A(\Omega)}<\infty$ endowed with

$$\|u\|_{H^1_A(\Omega)} := \sqrt{\|u\|^2_{L^2(\Omega)}+[u]^2_{H^1_A(\Omega)}}.$$

We denote by $B(x_0,R)$ the ball in $\mathbb{R}^N$ of center x_0 and radius $R>0$. For any set $E\subset\mathbb{R}^N$ we will denote by E^c the complement of E. For $A,B\subset\mathbb{R}^N$ open and bounded, $A\Subset B$ means $\bar{A}\subset B$.

2. Preliminary results

We start with the following Lemma.

Lemma 2.1. *Assume that $A:\mathbb{R}^N\to\mathbb{R}^N$ is locally bounded. Then, for any compact $V\subset\mathbb{R}^N$ with $\Omega\Subset V$, there exists $C=C(A,V)>0$ such that*

$$\int_{\mathbb{R}^N}|u(y+h)-e^{\mathrm{i}h\cdot A\left(y+\frac{h}{2}\right)}u(y)|^2dy\le C|h|^2\|u\|^2_{H^1_A(\mathbb{R}^N)},$$

for all $u\in H^1_A(\mathbb{R}^N)$ such that $u=0$ on V^c and any $h\in\mathbb{R}^N$ with $|h|\le 1$.

Proof. Assume first that $u\in C_0^\infty(\mathbb{R}^N)$ with $u=0$ on V^c. Fix $y,h\in\mathbb{R}^N$ and define

$$\varphi(t) := e^{\mathrm{i}(1-t)h\cdot A\left(y+\frac{h}{2}\right)}u(y+th), \quad t\in[0,1].$$

Then we have

$$u(y+h)-e^{\mathrm{i}h\cdot A\left(y+\frac{h}{2}\right)}u(y)=\varphi(1)-\varphi(0)=\int_0^1\varphi'(t)dt,$$

and since

$$\varphi'(t)=e^{\mathrm{i}(1-t)h\cdot A\left(y+\frac{h}{2}\right)}\,h\cdot\Big(\nabla_y u(y+th)-\mathrm{i}A\Big(y+\frac{h}{2}\Big)u(y+th)\Big),$$

by Hölder inequality we get

$$|u(y+h)-e^{\mathrm{i}h\cdot A\left(y+\frac{h}{2}\right)}u(y)|^2\le |h|^2\int_0^1\Big|\nabla_y u(y+th)-\mathrm{i}A\Big(y+\frac{h}{2}\Big)u(y+th)\Big|^2dt.$$

Therefore, integrating with respect to y over $\mathbb{R}^N$ and using Fubini's Theorem, we get

$$\begin{aligned}\int_{\mathbb{R}^N} |u(y+h) - e^{\mathrm{i}h\cdot A\left(y+\frac{h}{2}\right)}u(y)|^2 dy &\le |h|^2 \int_0^1 dt \int_{\mathbb{R}^N} \left|\nabla_y u(y+th) - \mathrm{i}A\Big(y+\frac{h}{2}\Big)u(y+th)\right|^2 dy \\ &= |h|^2 \int_0^1 dt \int_{\mathbb{R}^N} \left|\nabla_z u(z) - \mathrm{i}A\Big(z+\frac{1-2t}{2}h\Big)u(z)\right|^2 dz \\ &\le 2|h|^2 \int_{\mathbb{R}^N} |\nabla_z u(z) - \mathrm{i}A\,(z)\,u(z)|^2 dz \\ &+ 2|h|^2 \int_V \left|A\Big(z+\frac{1-2t}{2}h\Big) - A(z)\right|^2 |u(z)|^2 dz.\end{aligned}$$

Then, since A is bounded on the set V, we have for some constant $C>0$

$$\begin{aligned}\int_{\mathbb{R}^N} |u(y+h) - e^{\mathrm{i}h\cdot A\left(y+\frac{h}{2}\right)}u(y)|^2 dy &\le C|h|^2 \left(\int_{\mathbb{R}^N} |\nabla_z u(z) - \mathrm{i}A\,(z)\,u(z)|^2 dz + \int_{\mathbb{R}^N} |u(z)|^2 dz\right) \\ &= C|h|^2 \|u\|^2_{H^1_A(\mathbb{R}^N)}.\end{aligned}$$

When dealing with a general u we can argue by a density argument. □

Lemma 2.2. *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set with Lipschitz boundary, $V \subset \mathbb{R}^N$ a compact set with $\Omega \Subset V$ and $A : \mathbb{R}^N \to \mathbb{R}^N$ locally bounded. Then there exists $C(\Omega, V, A) > 0$ such that for any $u \in H^1_A(\Omega)$ there exists $Eu \in H^1_A(\mathbb{R}^N)$ such that $Eu = u$ in Ω, $Eu = 0$ in V^c and*

$$\|Eu\|_{H^1_A(\mathbb{R}^N)} \le C(\Omega, V, A)\|u\|_{H^1_A(\Omega)}.$$

Proof. Observe that, for any bounded set $W \subset \mathbb{R}^N$ there exist $C_1(A,W), C_2(A,W) > 0$ with

$$C_1(A,W)\|u\|_{H^1(W)} \le \|u\|_{H^1_A(W)} \le C_2(A,W)\|u\|_{H^1(W)}, \quad \text{for any } u \in H^1(W).$$

This follows easily, via simple computations, by the definition of the norm of $H^1_A(W)$ and in view of the local boundedness assumption on the potential A, see [1, Lemma 2.3] for further details. Now, by the standard extension property for $H^1(\Omega)$ (see e.g. [7, Theorem 1, p.254]) there exists $C(\Omega, V) > 0$ such that for any $u \in H^1(\Omega)$ there exists a function $Eu \in H^1(\mathbb{R}^N)$ such that $Eu = u$ in Ω, $Eu = 0$ in V^c and $\|Eu\|_{H^1(\mathbb{R}^N)} \le C(\Omega, V)\|u\|_{H^1(\Omega)}$. Then, for any $u \in H^1_A(\Omega)$, we get

$$\begin{aligned}\|Eu\|_{H^1_A(\mathbb{R}^N)} &= \|Eu\|_{H^1_A(V)} \le C_2(A,V)\|Eu\|_{H^1(V)} = C_2(A,V)\|Eu\|_{H^1(\mathbb{R}^N)} \\ &\le C(\Omega,V)C_2(A,V)\|u\|_{H^1(\Omega)} \le C(\Omega,V)C_2(A,V)C_1^{-1}(A,\Omega)\|u\|_{H^1_A(\Omega)},\end{aligned}$$

which concludes the proof. □

We can now prove the following result:

Lemma 2.3. *Let $A : \mathbb{R}^N \to \mathbb{R}^N$ be locally bounded. Let $u \in H^1_A(\Omega)$ and $\rho \in L^1(\mathbb{R}^N)$ with $\rho \ge 0$. Then*

$$\int_\Omega \int_\Omega \frac{|u(x) - e^{\mathrm{i}(x-y)\cdot A\left(\frac{x+y}{2}\right)}u(y)|^2}{|x-y|^2}\rho(x-y)\,dxdy \le C\|\rho\|_{L^1}\|u\|^2_{H^1_A(\Omega)}$$

where C depends only on Ω and A.

Proof. Let $V \subset \mathbb{R}^N$ be a fixed compact set with $\Omega \Subset V$. Given $u \in H^1_A(\Omega)$, by Lemma 2.2, there exists a function $\tilde{u} \in H^1_A(\mathbb{R}^N)$ with $\tilde{u} = u$ on Ω and $\tilde{u} = 0$ on V^c. By Lemma 2.1 and 2.2,

$$\int_{\mathbb{R}^N} |\tilde{u}(y+h) - e^{\mathrm{i}h\cdot A\left(y+\frac{h}{2}\right)}\tilde{u}(y)|^2 dy \le C|h|^2\|\tilde{u}\|^2_{H^1_A(\mathbb{R}^N)} \le C|h|^2\|u\|^2_{H^1_A(\Omega)}, \tag{2.1}$$

for some positive constant C depending on Ω and A. Then, in light of (2.1), we get

$$\begin{aligned}\int_\Omega\int_\Omega \frac{|u(x)-e^{\mathrm{i}(x-y)\cdot A\left(\frac{x+y}{2}\right)}u(y)|^2}{|x-y|^2}\rho(x-y)\,dxdy &\leq \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\rho(h)\frac{|\tilde u(y+h)-e^{\mathrm{i}h\cdot A\left(y+\frac{h}{2}\right)}\tilde u(y)|^2}{|h|^2}dydh\\ &= \int_{\mathbb{R}^N}\frac{\rho(h)}{|h|^2}\Big(\int_{\mathbb{R}^N}|\tilde u(y+h)-e^{\mathrm{i}h\cdot A\left(y+\frac{h}{2}\right)}\tilde u(y)|^2dy\Big)dh\\ &\leq C\|\rho\|_{L^1}\|u\|^2_{H^1_A(\Omega)},\end{aligned}$$

which concludes the proof. □

Lemma 2.4. *Let $A:\mathbb{R}^N\to\mathbb{R}^N$ be locally bounded and let $u\in H^1_{0,A}(\Omega)$. Then, we have*

$$(1-s)\int_{\mathbb{R}^{2N}}\frac{|u(x)-e^{\mathrm{i}(x-y)\cdot A\left(\frac{x+y}{2}\right)}u(y)|^2}{|x-y|^{N+2s}}dxdy\leq C\|u\|^2_{H^1_A(\Omega)}$$

where C depends only on Ω and A.

Proof. Given $u\in C^\infty_c(\Omega)$, by Lemma 2.1 we have

$$\int_{\mathbb{R}^N}|u(y+h)-e^{\mathrm{i}h\cdot A\left(y+\frac{h}{2}\right)}u(y)|^2dy\leq C|h|^2\|u\|^2_{H^1_A(\Omega)},$$

for some $C>0$ depending on Ω and A and all $h\in\mathbb{R}^N$ with $|h|\leq 1$. Then, we get

$$\begin{aligned}&(1-s)\int_{\mathbb{R}^{2N}}\frac{|u(x)-e^{\mathrm{i}(x-y)\cdot A\left(\frac{x+y}{2}\right)}u(y)|^2}{|x-y|^{N+2s}}\,dxdy\leq (1-s)\int_{\mathbb{R}^{2N}}\frac{|u(y+h)-e^{\mathrm{i}h\cdot A\left(y+\frac{h}{2}\right)}u(y)|^2}{|h|^{N+2s}}dydh\\ &=(1-s)\int_{\{|h|\leq 1\}}\frac{1}{|h|^{N+2s}}\Big(\int_{\mathbb{R}^N}|u(y+h)-e^{\mathrm{i}h\cdot A\left(y+\frac{h}{2}\right)}u(y)|^2dy\Big)dh\\ &+4(1-s)\int_{\{|h|\geq 1\}}\frac{1}{|h|^{N+2s}}dh\|u\|^2_{L^2(\Omega)}\\ &\leq (1-s)\int_{\{|h|\leq 1\}}\frac{1}{|h|^{N+2s-2}}dh\|u\|^2_{H^1_A(\Omega)}+C\|u\|^2_{L^2}\leq C\|u\|^2_{H^1_A(\Omega)}.\end{aligned}$$

The assertion then follows by a density argument. □

Assuming now that $A\in C^2(\mathbb{R}^N)$, we are able to give the following result.

Theorem 2.5. *Assume that $A\in C^2(\mathbb{R}^N)$. Let $u\in H^1_A(\Omega)$ and consider a sequence $\{\rho_n\}_{n\in\mathbb{N}}$ of nonnegative radial functions in $L^1(\mathbb{R}^N)$ with*

$$\lim_{n\to\infty}\int_0^\infty\rho_n(r)r^{N-1}dx=1,\tag{2.2}$$

and such that, for every $\delta>0$,

$$\lim_{n\to\infty}\int_\delta^\infty\rho_n(r)r^{N-1}dr=\lim_{n\to\infty}\int_0^\delta\rho_n(r)r^N dr=\lim_{n\to\infty}\int_0^\delta\rho_n(r)r^{N+1}dr=0.\tag{2.3}$$

Then, we have

$$\lim_{n\to\infty}\int_\Omega\int_\Omega\frac{|u(x)-e^{\mathrm{i}(x-y)\cdot A\left(\frac{x+y}{2}\right)}u(y)|^2}{|x-y|^2}\rho_n(x-y)\,dxdy=2K_N\int_\Omega|\nabla u-\mathrm{i}A(x)u|^2\,dx\tag{2.4}$$

being K_N the constant introduced in (1.1).

Proof. We follow the main lines of the proof in [3]. Setting

$$F^u_n(x,y):=\frac{u(x)-e^{\mathrm{i}(x-y)\cdot A\left(\frac{x+y}{2}\right)}u(y)}{|x-y|}\rho_n^{1/2}(x-y),\qquad x,y\in\Omega,\ n\in\mathbb{N},$$

by virtue of Lemma 2.3, for all $u,v\in H^1_A(\Omega)$, recalling (2.2) we have

$$\big|\|F^u_n\|_{L^2(\Omega\times\Omega)}-\|F^v_n\|_{L^2(\Omega\times\Omega)}\big|\leq\|F^u_n-F^v_n\|_{L^2(\Omega\times\Omega)}\leq C\|u-v\|_{H^1_A(\Omega)},$$

for some $C > 0$ depending on Ω and A. This allows to prove (2.4) for $u \in C^2(\bar{\Omega})$. If we set

$$\varphi(y) := e^{\mathrm{i}(x-y)\cdot A\left(\frac{x+y}{2}\right)}u(y),$$

since

$$\nabla_y\varphi(y) = e^{\mathrm{i}(x-y)\cdot A\left(\frac{x+y}{2}\right)}\Big(\nabla_y u(y) - \mathrm{i}A\Big(\frac{x+y}{2}\Big)u(y) + \frac{\mathrm{i}}{2}u(y)(x-y)\cdot\nabla_y A\Big(\frac{x+y}{2}\Big)\Big),$$

if $x \in \Omega$, a second order Taylor expansion gives (since $u, A \in C^2$, then $\nabla^2_y\varphi$ is bounded on $\bar{\Omega}$)

$$u(x) - e^{\mathrm{i}(x-y)\cdot A\left(\frac{x+y}{2}\right)}u(y) = \varphi(x) - \varphi(y) = (\nabla u(x) - \mathrm{i}A(x)u(x))\cdot(x-y) + \mathcal{O}(|x-y|^2).$$

Hence, for any fixed $x \in \Omega$,

$$\frac{\left|u(x) - e^{\mathrm{i}(x-y)\cdot A\left(\frac{x+y}{2}\right)}u(y)\right|}{|x-y|} = \left|(\nabla u(x) - \mathrm{i}A(x)u(x))\cdot\frac{x-y}{|x-y|}\right| + \mathcal{O}(|x-y|). \tag{2.5}$$

Fix $x \in \Omega$. If we set $R_x := \mathrm{dist}(x, \partial\Omega)$, integrating with respect to y, we have

$$\begin{aligned}\int_\Omega \frac{|u(x) - e^{\mathrm{i}(x-y)\cdot A\left(\frac{x+y}{2}\right)}u(y)|^2}{|x-y|^2}\rho_n(x-y)\,dy &= \int_{B(x,R_x)} \frac{|u(x) - e^{\mathrm{i}(x-y)\cdot A\left(\frac{x+y}{2}\right)}u(y)|^2}{|x-y|^2}\rho_n(x-y)\,dy \\ &\quad + \int_{\Omega\setminus B(x,R_x)} \frac{|u(x) - e^{\mathrm{i}(x-y)\cdot A\left(\frac{x+y}{2}\right)}u(y)|^2}{|x-y|^2}\rho_n(x-y)\,dy.\end{aligned} \tag{2.6}$$

The second integral goes to zero by the first limit of conditions (2.3), since

$$\lim_{n\to\infty}\int_{\Omega\setminus B(x,R_x)} \frac{|u(x) - e^{\mathrm{i}(x-y)\cdot A\left(\frac{x+y}{2}\right)}u(y)|^2}{|x-y|^2}\rho_n(x-y)\,dy \le C\lim_{n\to\infty}\int_{B^c(0,R_x)}\rho_n(z)dz = 0.$$

Now, in light of (2.5), following [3] we compute

$$\begin{aligned}\int_{B(x,R_x)} \frac{|u(x) - e^{\mathrm{i}(x-y)\cdot A\left(\frac{x+y}{2}\right)}u(y)|^2}{|x-y|^2}\rho_n(x-y)\,dy &= Q_N|\nabla u(x) - \mathrm{i}A(x)u(x)|^2\int_0^{R_x} r^{N-1}\rho_n(r)dr \\ &\quad + \mathcal{O}\left(\int_0^{R_x} r^N\rho_n(r)dr\right) + \mathcal{O}\left(\int_0^{R_x} r^{N+1}\rho_n(r)dr\right),\end{aligned}$$

where we have set

$$Q_N = \int_{\mathbb{S}^{N-1}} |\omega\cdot\mathbf{e}|^2 d\mathcal{H}^{N-1}(\omega),$$

being $\mathbf{e} \in \mathbb{R}^N$ a unit vector. Letting $n \to \infty$ in (2.6), the result follows by dominated convergence. □

3. Proofs of Theorem 1.1 and 1.2

3.1. Proof of Theorem 1.1. If $r_\Omega := \mathrm{diam}(\Omega)$, we consider a radial cut-off $\psi \in C_c^\infty(\mathbb{R}^N)$, $\psi(x) = \psi_0(|x|)$ with $\psi_0(t) = 1$ for $t < r_\Omega$ and $\psi_0(t) = 0$ for $t > 2r_\Omega$. Then, by construction, $\psi_0(|x-y|) = 1$, for every $x, y \in \Omega$. Furthermore, let $\{s_n\}_{n\in\mathbb{N}} \subset (0,1)$ be a sequence with $s_n \nearrow 1$ as $n \to \infty$ and consider the sequence of radial functions in $L^1(\mathbb{R}^N)$

$$\rho_n(|x|) = \frac{2(1-s_n)}{|x|^{N+2s_n-2}}\psi_0(|x|), \qquad x \in \mathbb{R}^N,\ n \in \mathbb{N}. \tag{3.1}$$

Notice that (2.2) holds, since

$$\lim_{n\to\infty}\int_0^{r_\Omega}\rho_n(r)r^{N-1}dr = \lim_{n\to\infty}2(1-s_n)\int_0^{r_\Omega}\frac{1}{r^{2s_n-1}}dr = \lim_{n\to\infty}r_\Omega^{2-2s_n} = 1,$$

and

$$\lim_{n\to\infty}\int_{r_\Omega}^{2r_\Omega}\rho_n(r)r^{N-1}dr = \lim_{n\to\infty}2(1-s_n)\int_{r_\Omega}^{2r_\Omega}\frac{\psi_0(r)}{t^{2s_n-1}}dr = 0.$$

In a similar fashion, for any $\delta > 0$, there holds

$$\lim_{n\to\infty}\int_\delta^\infty \rho_n(r)r^{N-1}dr \le \lim_{n\to\infty} 2(1-s_n)\int_\delta^{2r_\Omega}\frac{1}{t^{2s_n-1}}dt = 0,$$
$$\lim_{n\to\infty}\int_0^\delta \rho_n(r)r^{N}dr \le \lim_{n\to\infty} 2(1-s_n)\int_0^{\delta}\frac{1}{t^{2s_n-2}}dt = 0,$$
$$\lim_{n\to\infty}\int_0^\delta \rho_n(r)r^{N+1}dr \le \lim_{n\to\infty} 2(1-s_n)\int_0^{\delta}\frac{1}{t^{2s_n-3}}dt = 0.$$

Then Theorem 1.1 follows directly from Theorem 2.5 using ρ_n as defined in (3.1). □

3.2. **Proof of Theorem 1.2.** In light of Theorem 1.1 and since $u = 0$ on Ω^c, we have

$$\lim_{s\nearrow 1}(1-s)\int_{\mathbb{R}^{2N}}\frac{|u(x)-e^{\mathrm{i}(x-y)\cdot A\left(\frac{x+y}{2}\right)}u(y)|^2}{|x-y|^{N+2s}}dxdy = K_N\int_\Omega |\nabla u - \mathrm{i}A(x)u|^2dx + \lim_{s\nearrow 1}R_s,$$

where we have set

$$R_s := 2(1-s)\int_\Omega\int_{\mathbb{R}^N\setminus\Omega}\frac{|u(x)|^2}{|x-y|^{N+2s}}dxdy.$$

On the other hand, arguing as in the proof of [5, Proposition 2.8], we get $R_s \to 0$ as $s \nearrow 1$ when $u \in C_c^\infty(\Omega)$ and, on account of Lemma 2.4, for general function in $H^1_{0,A}(\Omega)$ by a density argument. □

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