SOME REMARKS ON CONTACT VARIATIONS IN THE FIRST HEISENBERG GROUP

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ABSTRACT. We show that in the first sub-Riemannian Heisenberg group there are intrinsic graphs of smooth functions that are both critical and stable points of the sub-Riemannian perimeter under compactly supported variations of contact diffeomorphisms, despite the fact that they are not area-minimizing surfaces. In particular, we show that if $f:\mathbb{R}^2\to\mathbb{R}$ is a $\mathscr{C}^1$-intrinsic function, and $\nabla^f\nabla^f f=0$, then the first contact variation of the sub-Riemannian area of its intrinsic graph is zero and the second contact variation is positive.

CONTENTS

1. INTRODUCTION

We want to address some new features of the subRiemannian perimeter in the Heisenberg group. The notion of subRiemannian perimeter in the Heisenberg group, the so-called *intrinsic perimeter*, has been enstablished as a direct and natural extension from the Euclidean perimeter in $\mathbb{R}^n$. However, in many aspects, there are fundamental differences that lead to new open questions [6, 7, 17, 10, 15].

Before a detailed explanation, let us introduce some basic notions and notations we need in this introduction. The (first) Heisenberg group $\mathbb{H}$ is a

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three dimensional Lie group diffeomorphic to $\mathbb{R}^3$. However, when endowed with a left-invariant subRiemannian distance, it becomes a metric space with Hausdorff dimension equal to four; see [3].

By standard methods of Geometric Measure Theory, one defines the *intrinsic perimeter* $P(E;\Omega)$ of a measurable set $E \subset \mathbb{H}$ in an open set $\Omega \subset \mathbb{H}$. We will denote it also by $\mathscr{A}(\partial E \cap \Omega)$.

Regular surfaces are topological surfaces in $\mathbb{H}$ that admit a continuously varying tangent plane and they play an important role in the theory of sets with finite intrinsic perimeter. They are the subRiemannian counterpart of smooth hypersurfaces in $\mathbb{R}^n$. Regular surfaces are locally graphs of so-called *$\mathscr{C}^1$-intrinsic functions* $\mathbb{R}^2 \to \mathbb{R}$. We will focus on these functions and their graphs.

The space of all $\mathscr{C}^1$-intrinsic functions will be denoted by $\mathscr{C}^1_{\mathbb{W}}$ and the graph of $f : \mathbb{R}^2 \to \mathbb{R}$ by $\Gamma_f \subset \mathbb{H}$. It is well known that $f \in \mathscr{C}^1_{\mathbb{W}}$ if and only if $f \in \mathscr{C}^0(\mathbb{R}^2)$ and the distributional derivative

$$\nabla^f f = \partial_\eta f + f \partial_\tau f$$

is continuous, where we denote by (η, τ) the coordinates on $\mathbb{R}^2$; see [1, 17]. If $\omega \subset \mathbb{R}^2$, the intrinsic area of Γ_f above ω is

$$\mathscr{A}(\Gamma_f \cap (\omega \times \mathbb{R})) = \int_\omega \sqrt{1 + (\nabla^f f)^2}\, \mathrm{d}\eta\, \mathrm{d}\tau.$$

An important open problem concerning $\mathscr{C}^1_{\mathbb{W}}$ is *Bernstein's problem*: If the graph Γ_f of $f \in \mathscr{C}^1_{\mathbb{W}}$ is a locally minimizer of the intrinsic area, is Γ_f a plane? See Section 2.4 for a precise statement and [4, 2, 16, 9] for further reading.

In the study of perimeter minimizers in $\mathbb{H}$, we identify three main issues that mark the gap from the Euclidean theory. First, the differential $f \mapsto \nabla^f f$ is a nonlinear operator. Such non-linearity reflects on the fact that basic function spaces like $\mathscr{C}^1_{\mathbb{W}}$ itself, or the space of functions with bounded intrinsic variation, are not vector spaces. See Remark 2.3 for details.

Second, the area functional is not convex (say on $\mathscr{C}^1(\mathbb{R}^2)$). In particular, there are critical points that are not extremals, see [4]. In other words, a first variation condition

$$\left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0} \mathscr{A}(\Gamma_{f+\epsilon\phi} \cap (\omega \times \mathbb{R})) = 0 \qquad \forall \phi \in \mathscr{C}^\infty_c(\omega) \tag{1}$$

does not characterize minimizers. However, if $f \in \mathscr{C}^1(\mathbb{R}^2)$, a second variation condition $\left.\frac{\mathrm{d}^2}{\mathrm{d}\epsilon^2}\right|_{\epsilon=0} \mathscr{A}(\Gamma_{f+\epsilon\phi} \cap (\omega \times \mathbb{R})) \geq 0$ does, see [9].

Third, there are objects among sets of finite intrinsic perimeter with very low regularity, see Remark 2.3. The standard variational approach as in (1) fails when applied to these objects. More precisely, if $f \in \mathscr{C}^1_{\mathbb{W}}$ and $\phi \in \mathscr{C}^\infty_c(\mathbb{R}^2)$, then $\mathscr{A}(\Gamma_{f+\epsilon\phi} \cap (\omega \times \mathbb{R}))$ may be $+\infty$ for all $\epsilon \neq 0$, all $\omega \subset \mathbb{R}^2$ open and all $\phi \in \mathscr{C}^\infty_c(\omega) \setminus \{0\}$. In another approach, one can consider smooth one-parameter families of diffeomorphisms $\Phi_\epsilon : \mathbb{H} \to \mathbb{H}$ with $\Phi_0 = \mathrm{Id}$ and $\{\Phi_\epsilon \neq \mathrm{Id}\} \subset\subset \Omega$, and take variations of $\mathscr{A}(\Phi_\epsilon(\Gamma_f) \cap \Omega)$. However, it may happen again that $\mathscr{A}(\Phi_\epsilon(\Gamma_f) \cap \Omega) = +\infty$ for all $\epsilon \neq 0$.

After further considerations, one understands that we need to restrict the choice of Φ_ϵ to *contact diffeomorphisms*, see Proposition 5.1. In this setting, we address the question whether, despite this restriction, conditions on the first and second variations with contact diffeomorphisms can sigle out minimal graphs. Our answer is no:

Theorem 1.1. *There is $f \in \mathscr{C}^1_{\mathbb{W}}$ such that, for all $\Omega \subset \mathbb{H}$ open and all smooth one-parameter families of contact diffeomorphisms $\Phi_\epsilon : \mathbb{H} \to \mathbb{H}$ with $\Phi_0 = \mathrm{Id}$ and $\{\Phi_\epsilon = \mathrm{Id}\} \subset\subset \Omega$, it holds*

$$\left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0} \mathscr{A}(\Phi_\epsilon(\Gamma_f) \cap \Omega) = 0 \quad \text{and} \quad \left.\frac{\mathrm{d}^2}{\mathrm{d}\epsilon^2}\right|_{\epsilon=0} \mathscr{A}(\Phi_\epsilon(\Gamma_f) \cap \Omega) \geq 0,$$

but Γ_f is not a minimal surface.

The proof of Theorem 1.1 is based on a "Lagrangian" approach to $\mathscr{C}^1_{\mathbb{W}}$. Indeed, a function $f \in \mathscr{C}^1_{\mathbb{W}}$ is uniquely characterized by the integral curves of the planar vector field $\nabla^f = \partial_\eta + f\partial_\tau$. We will thus take variations of f via smooth one-parameter families of diffeomorphisms $\phi_\epsilon : \mathbb{R}^2 \to \mathbb{R}^2$, i.e., by smoothly varying the integral curves of ∇^f; see Section 4. We will then prove that this approach is equivalent to the use of contact diffeomorphisms $\Phi_\epsilon : \mathbb{H} \to \mathbb{H}$; see Section 5.

Finally, we will consider functions $f \in \mathscr{C}^1_{\mathbb{W}}$ that are solutions to the equation $\nabla^f\nabla^f f = 0$ in a Lagrangian sense, that is, functions such that $\nabla^f f$ is constant along the integral curves of ∇^f. We will characterize such functions as the ones for which the integral curves of ∇^f are parables, or, equivalentely, as the ones whose graph Γ_f is ruled by horizontal straight lines. These functions are the ones appearing in Theorem 1.1.

The paper is organized as follows. Section 2 is devoted to the presentation of all main definitions. In the next Section 3, we study solutions to the equation $\nabla^f\nabla^f f = 0$. We construct a Lagrangian variation of a function $f \in \mathscr{C}^1_{\mathbb{W}}$ in Section 4. In Section 5, we prove some basic properties of contact diffeomorphisms. Section 6 is devoted to the first contact variation and Section 7 to the second contact variation for functions $f \in \mathscr{C}^1_{\mathbb{W}}$. Finally, in Section 8 we prove our main theorem. An Appendix is added as a reference for a few equalities that are applied all over the paper.

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2. Preliminaries

2.1. **The Heisenberg group.** The *first Heisenberg group* $\mathbb{H}$ is the connected, simply connected Lie group associated to the Heisenberg Lie algebra $\mathfrak{h}$. The *Heisenberg Lie algebra* $\mathfrak{h}$ is the only three-dimensional nilpotent Lie algebra that is not commutative. It can be proven that, for any two linearly independent vectors $A, B \in \mathfrak{h} \setminus [\mathfrak{h}, \mathfrak{h}]$, the triple $(A, B, [A, B])$ is a basis of $\mathfrak{h}$ and $[A, [A, B]] = [B, [A, B]] = 0$. The Heisenberg group has the structure of a *stratified Lie group*, i.e., $\mathfrak{h} = \mathrm{span}\{A, B\} \oplus \mathrm{span}\{[A, B]\}$, see [12, 13].

We then identify $\mathbb{H} = (\mathrm{span}\{A, B, [A, B]\}, *)$, where

$$p * q := p + q + \frac{1}{2}[p, q].$$

In the coordinates $(x,y,z) = xA + yB + z[A,B]$, which are the exponential coordinates of first kind, we have

$$(a,b,c)*(x,y,z) = (a+x, b+y, c+z+\frac{1}{2}(ay-bx)).$$

The inverse is $(x,y,z)^{-1} = (-x,-y,-z)$.

The elements $A, B, [A,B] \in \mathfrak{h}$ induce a frame of left-invariant vector fields on $\mathbb{H}$:

$$X := \partial_x - \frac{1}{2}y\partial_z, \qquad Y := \partial_y + \frac{1}{2}x\partial_z, \qquad Z := \partial_z.$$

The *horizontal subbundle* is the vector bundle

$$H := \bigsqcup_{p\in\mathbb{H}} \operatorname{span}\{X(p), Y(p)\} \subset T\mathbb{H}.$$

The maps $\delta_\lambda(x,y,z) := (\lambda x, \lambda y, \lambda^2 z)$, $\lambda > 0$, are called *dilations*. They are group automorphisms of $\mathbb{H}$ and for all $\lambda, \mu > 0$ it holds $\delta_\lambda \circ \delta_\mu = \delta_{\lambda\mu}$.

2.2. **Intrinsic graphs and intrinsic differentials.** A *vertical plane* is a plane containing the z-axis. Explicitly, for $\theta \in \mathbb{R}$,

$$\mathbb{W}_\theta := \{(\eta\sin\theta, \eta\cos\theta, \tau) : \eta, \tau \in \mathbb{R}\} \subset \mathbb{H}.$$

Vertical planes are the only 2-dimensional subgroups of $\mathbb{H}$ that are δ_λ-homogeneous, i.e., $\delta_\lambda(\mathbb{W}_\theta) = \mathbb{W}_\theta$ for all $\lambda > 0$.

The *intrinsic X-graph* (or simply *intrinsic graph*) of a function $f : \mathbb{R}^2 \to \mathbb{R}$ is the set[1]

$$\begin{aligned}\Gamma_f &:= \left\{(0,\eta,\tau)*(f(\eta,\tau),0,0) : \eta,\tau \in \mathbb{R}^2\right\} \\ &= \left\{(f(\eta,\tau), \eta, \tau - \frac{1}{2}\eta f(\eta,\tau)) : \eta,\tau \in \mathbb{R}^2\right\}.\end{aligned}$$

If one look at f as a function $\mathbb{W}_0 \to \operatorname{span}\{A\}$, then $\Gamma_f = \{p*f(p) : p \in \mathbb{W}_0\}$. Left translations and dilations of an intrinsic graph are also intrinsic graphs. For $\alpha \in \mathbb{R}$, the vertical plane $\mathbb{W}_{\arctan(\alpha)}$ is the intrinsic graph of the function $f(\eta,\tau) = \alpha\eta$. We will use the map $\pi_X : \mathbb{H} \to \mathbb{R}^2$, $\pi_X(x,y,z) = (y, z + \frac{1}{2}xy)$. Note that $\pi_X(p * f(p)) = p$.

For $(\eta_0,\tau_0) \in \mathbb{R}^2$ and $f : \mathbb{R}^2 \to \mathbb{R}$ continuous, set $f_0 := f(\eta_0,\tau_0)$ and $p_0 := (0,\eta_0,\tau_0)*(f_0,0,0) = (f_0,\eta_0,\tau_0 - \frac{1}{2}\eta_0 f_0)$. We say that f is *intrinsically* $\mathscr{C}^1$, or *belonging to* $\mathscr{C}^1_{\mathbb{W}}$, with differential $\psi : \mathbb{R}^2 \to \mathbb{R}$, if $\delta_\lambda(p_0^{-1}\Gamma_f)$ converge to $\mathbb{W}_{\arctan(\psi(\eta_0,\tau_0))}$ in the sense of the local Hausdorff convergence of sets as $\lambda \to \infty$, and the convergence is uniform on compacts in (η_0,τ_0).

Notice that $\delta_\lambda(p_0^{-1}\Gamma_f) = \Gamma_{f_{(\eta_0,\tau_0);\lambda}}$, where

$$f_{(\eta_0,\tau_0);\lambda}(\eta,\tau) = \lambda\left(-f_0 + f(\eta_0 + \frac{\eta}{\lambda}, \tau_0 + f_0\frac{\eta}{\lambda} + \frac{\tau}{\lambda^2})\right).$$

Therefore, f belongs to $\mathscr{C}^1_{\mathbb{W}}$ with differential ψ if and only if $f_{(\eta_0,\tau_0);\lambda}$ converge uniformly on compact sets to the function $(\eta,\tau) \mapsto \psi(\eta_0,\tau_0)\eta$, as $\lambda \to +\infty$, and the convergence is uniform on compact sets in (η_0,τ_0). Notice that ψ has to be continuous.

[1] In a different choice of coordinates in $\mathbb{H}$, we can have $(0,\eta,\tau)*(f(\eta,\tau),0,0) = (f(\eta,\tau),\eta,\tau)$. For instance, we will use these coordinates in Section 5.2

The *intrinsic gradient* of a function $f : \mathbb{R}^2 \to \mathbb{R}$ is the vector field on $\mathbb{R}^2$ defined as

$$\nabla^f := \partial_\eta + f\partial_\tau.$$

We can express the intrinsic differentiability in terms of the differentiability of f along the integral curves of ∇^f: from [17, Theorem 4.95] we obtain the following characterisation, which justify the notation $\nabla^f f$ for the differential ψ of $f \in \mathscr{C}^1_{\mathbb{W}}$.

Lemma 2.1. *A continuous function $f : \mathbb{R}^2 \to \mathbb{R}$ is in $\mathscr{C}^1_{\mathbb{W}}$ with differential ψ if and only if for every $p \in \mathbb{R}^2$ there exists a $\mathscr{C}^2$-function $g_p : I \to \mathbb{R}$, where $I \subset \mathbb{R}$ is a neighbourhood of* 0*, such that*

$$\begin{cases} g_p(0) = 0, \\ g_p'(t) = f(p + (t, g_p(t))) & \forall t \in I, \\ g_p''(t) = \psi(p + (t, g_p(t))) & \forall t \in I. \end{cases}$$

Note that $t \mapsto p + (t, g_p(t))$ is an integral curve of ∇^f and that g_p is not unique in general. Another interpretation of these curves will be useful:

Lemma 2.2. *Let $f \in \mathscr{C}^1_{\mathbb{W}}$. A curve $\gamma : I \to \mathbb{R}^2$ of class $\mathscr{C}^1$, where $I \subset \mathbb{R}$ is an interval, is an integral curve of ∇^f if and only if the curve $t \mapsto \gamma(t) * f(\gamma(t)) \in \Gamma_f$ is a curve of class $\mathscr{C}^1$ tangent to the horizontal bundle H.*

Remark 2.3. In [11] it has been shown that there exists $f \in \mathscr{C}^1_{\mathbb{W}}$ whose intrinsic graph Γ_f has Euclidean Hausdorff dimension (seen as a subset of the Euclidean $\mathbb{R}^3$) strictly larger than two. It is possible to prove, for example using Lemma 5.4, that Γ_{f+1} does not have locally finite intrinsic perimeter and in particular $f + 1 \notin \mathscr{C}^1_{\mathbb{W}}$. This shows that $\mathscr{C}^1_{\mathbb{W}}$ is not a vector space.

2.3. **Smooth approximation.** A sequence $\{f_k\}_{k\in\mathbb{N}} \subset \mathscr{C}^1_{\mathbb{W}}$ converges to f in $\mathscr{C}^1_{\mathbb{W}}$ if f_k and $\nabla^{f_k} f_k$ converge to f and $\nabla^f f$ uniformly on compact sets. The following lemma has been proven in [1].

Lemma 2.4. *If $f \in \mathscr{C}^1_{\mathbb{W}}$ then there is a sequence of functions $\{f_k\}_{k\in\mathbb{N}} \subset \mathscr{C}^\infty(\mathbb{R}^2)$ that converges to f in $\mathscr{C}^1_{\mathbb{W}}$.*

2.4. **Perimeter and Bernstein's Problem.** The Lebesgue measure $\mathscr{L}^3$ in $\mathbb{R}^3$ is a Haar measure on $\mathbb{H}$ in the exponential coordinates introduced Section 2.1. Notice that for any measurable set $E \subset \mathbb{H}^1$ and any $\lambda > 0$ it holds $\mathscr{L}^3(\delta_\lambda(E)) = \lambda^4 \mathscr{L}^3(E)$.

Let $\langle \cdot, \cdot \rangle$ be the left-invariant scalar product on the subbundle H such that (X, Y) is an orthonormal frame and set $\|v\| := \sqrt{\langle v, v\rangle}$ for $v \in H$. The *sub-Riemannian perimeter* of a measurable set $E \subset \mathbb{H}^1$ in an open set Ω is

$$P(E; \Omega) := \sup \left\{ \int_E \mathrm{div} V : \ V \in \Gamma(H),\ \mathrm{spt}(V) \subset\subset \Omega,\ \|V\| \le 1 \right\},$$

where $\Gamma(H)$ contains all the smooth sections of the horizontal subbundle and $\mathrm{div}V$ is the divergence of vector fields on $\mathbb{R}^3$. One can show that, for every $V_1, V_2 \in \mathscr{C}^\infty(\mathbb{R}^3)$,

$$\mathrm{div}(V_1 X + V_2 Y) = X V_1 + Y V_2.$$

A set E has *locally finite perimeter* if $P(E; \Omega) < \infty$ for all $\Omega \subset \mathbb{H}$ open and bounded. If E has locally finite perimeter, the function $\Omega \mapsto P(E; \Omega)$

induces a Radon measure $|\partial E|$ on $\mathbb{H}^1$, which is concentrated on the so-called *reduced boundary* $\partial^* E \subset \partial E$. Moreover, up to a set of $|\partial E|$-measure zero and a rotation around the z-axis, $\partial^* E$ is the countable union of intrinsic graphs of $\mathscr{C}^1_{\mathbb{W}}$ functions. See [6] and [7] for further reading.

A measurable set E has *minimal perimeter* if, for every bounded open set $\Omega \subset \mathbb{H}^1$ and every measurable set $F \subset \mathbb{H}^1$ with symmetric difference $E \Delta F \subset\subset \Omega$, we have

$$P(E;\Omega) \le P(F;\Omega).$$

In this case, the reduced boundary $\partial^* E$ of E is called *area-minimizing surface.* We are interested in area minimizers that are global intrinsic graphs.

Conjecture 2.5 (Bernstein's Problem)**.** *If $f \in \mathscr{C}^1_{\mathbb{W}}$ is such that Γ_f is an area-minimizing surface, then Γ_f is a vertical plane up to left-translations.*

Such conjecture has been proven in the case $f \in \mathscr{C}^1(\mathbb{R}^2)$ in [9], while it has been presented a counterexample in [16] with $f \in \mathscr{C}^0(\mathbb{R}^2) \setminus \mathscr{C}^1_{\mathbb{W}}$.

For an open domain $\omega \subset \mathbb{R}^2$, set

$$\Omega := \{(0,\eta,\tau) * (\xi,0,0) : (\eta,\tau) \in \omega,\ \xi \in \mathbb{R}\}.$$

If $f \in \mathscr{C}^1_{\mathbb{W}}$ and $E_f = \{(0,\eta,\tau) * (\xi,0,0) \in \mathbb{R}^2,\ \xi \le f(\eta,\tau)\}$, then

$$P(E_f;\Omega) = \int_\omega \sqrt{1 + (\nabla^f f)^2}\,\mathrm{d}\eta\,\mathrm{d}\tau.$$

If E_f has minimal perimeter, then, for every $g \in \mathscr{C}^1_{\mathbb{W}}$ with $\{f \ne g\} \subset\subset \omega$, it holds

$$\int_\omega \sqrt{1 + (\nabla^f f)^2}\,\mathrm{d}\eta\,\mathrm{d}\tau \le \int_\omega \sqrt{1 + (\nabla^g g)^2}\,\mathrm{d}\eta\,\mathrm{d}\tau.$$

It is not known whether the converse implication holds.

3. Lagrangian solutions to $\Delta^f f = 0$

For $f \in \mathscr{C}^1_{\mathbb{W}}$ and $v \in \mathscr{C}^2(\mathbb{R}^2)$, we define the differential operator

$$\Delta^f v := \partial_\eta^2 v + 2f\partial_\eta\partial_\tau v + f^2\partial_\tau^2 v + \nabla^f f \partial_\tau v. \tag{2}$$

Notice that, if $f \in \mathscr{C}^2(\mathbb{R}^2)$, then

$$\Delta^f v = \nabla^f(\nabla^f v).$$

The next lemma will be a fundamental tool for extending some results beyond the smooth case via approximation. The proof trivially follows from the explicit expressions of the differential operators ∇^f and Δ^f.

Lemma 3.1. *If $\{f_k\}_{k\in\mathbb{N}} \subset \mathscr{C}^1_{\mathbb{W}}$ and $\{v_k\}_{k\in\mathbb{N}} \subset \mathscr{C}^2(\mathbb{R}^2)$ are sequences converging to f and v in their respective spaces, then the sequences $\{\nabla^{f_k} v_k\}_{k\in\mathbb{N}}$ and $\{\Delta^{f_k} v_k\}_{k\in\mathbb{N}}$ converge to $\nabla^f v$ and $\Delta^f v$ uniformly on compact sets.*

If $f \in \mathscr{C}^2(\mathbb{R}^2)$ is such that Γ_f is a minimal surface in $\mathbb{H}$, then one easily shows that f satisfies the differential equation (see [2])

$$\nabla^f\left(\frac{\nabla^f f}{\sqrt{1 + (\nabla^f f)^2}}\right) = 0. \tag{3}$$

Equation (3) is equivalent, for $f \in \mathscr{C}^2(\mathbb{R}^2)$, to

$$\Delta^f f = 0. \tag{4}$$

For a generic $f \in \mathscr{C}^1_{\mathbb{W}}$, equation (4) has not the classical interpretation (2). However, using a "Lagrangian interpretation" of $\nabla^f(\nabla^f f) = 0$, we say that $f \in \mathscr{C}^1_{\mathbb{W}}$ satisfies (4) if $\nabla^f f$ is constant along the integral curves of ∇^f. After a simple technical lemma, the following Lemma 3.3 characterizes such functions by the integral lines of ∇^f.

Lemma 3.2. *Let $A, B \in \mathscr{C}^0(\mathbb{R})$. The map $\mathbb{R}^2 \to \mathbb{R}^2$ given by*

$$(t,\zeta) \mapsto \left(t, \frac{A(\zeta)}{2}t^2 + B(\zeta)t + \zeta\right)$$

is a homeomorphism if and only if, for all $\zeta, \zeta' \in \mathbb{R}$, one of the following holds:

(1) $A(\zeta) = A(\zeta')$ *and* $B(\zeta) = B(\zeta')$,
(2) $2\big(A(\zeta) - A(\zeta')\big)(\zeta - \zeta') > \big(B(\zeta) - B(\zeta')\big)^2$.

Lemma 3.3. *Let $f \in \mathscr{C}^1_{\mathbb{W}}$ be such that $\nabla^f f$ is constant along the integral curves of ∇^f. Then all the integral curves of ∇^f are given by $t \mapsto (t, g(t,\zeta))$, $\zeta \in \mathbb{R}$, where*

$$g(t,\zeta) = \frac{\nabla^f f(0,\zeta)}{2}t^2 + f(0,\zeta)t + \zeta.$$

Moreover, for all $\zeta, \zeta' \in \mathbb{R}$ one of the following holds:

(1) $\nabla^f f(0,\zeta) = \nabla^f f(0,\zeta')$ *and* $f(0,\zeta) = f(0,\zeta')$,
(2) $2(\nabla^f f(0,\zeta) - \nabla^f f(0,\zeta'))(\zeta - \zeta') > (f(0,\zeta) - f(0,\zeta'))^2$.

In particular, the function $\tau \mapsto \nabla^f f(\eta,\tau)$ is non-decreasing, for all $\eta \in \mathbb{R}$.

The graphs of these functions are examples of "graphical strips" as introduced in [4].

Proof of Lemma 3.3. Given a function g_p like in Lemma 2.1, $\nabla^f \nabla^f f = 0$ implies $g_p'' = 0$, i.e., g_p is a polynomial of second degree. Moreover, such a g_p is unique for every p, because it is completely determined by $f(p)$ and $\nabla^f f(p)$. Therefore, all the integral curves are of the form described in the statement and the two conditions holds by Lemma 3.2.

Since $(f(0,\zeta)-f(0,\zeta'))^2 \geq 0$, then $\zeta \mapsto \nabla^f f(0,\zeta)$ is non-decreasing. Since $\nabla^f f(t, g(t,\zeta)) = \nabla^f f(0,\zeta)$ and since, for $t \in \mathbb{R}$ fixed, the map $\zeta \mapsto g(t,\zeta)$ is a ordering-preserving homeomorphism $\mathbb{R} \to \mathbb{R}$, then the map $\tau \mapsto \nabla^f f(\eta,\tau)$ is non-decreasing as well, for all $\eta \in \mathbb{R}$. $\square$

Remark 3.4. Notice that any function $g(t,\zeta)$ as in Lemma 3.3 defines a function $f \in \mathscr{C}^1_{\mathbb{W}}$ with $\Delta^f f = 0$. In particular, for any $A \in \mathscr{C}^0(\mathbb{R})$ non-decreasing, we can define $g(t,\zeta) := A(\zeta)t^2 + \zeta$ and we obtain a well defined $f \in \mathscr{C}^1_{\mathbb{W}}$ with $\Delta^f f = 0$ given by

$$f(t, A(\zeta)t^2 + \zeta) = 2A(\zeta)t.$$

Remark 3.5. Lemma 3.3 states in particular that, if $\Delta^f f = 0$ then Γ_f is foliated by horizontal straight lines. Indeed, notice that any parable $t \mapsto g(t,\zeta)$ in $\mathbb{R}^2$ lifts to a straight line in Γ_f. In [9, Theorem 3.5] Galli and Ritoré are able to prove that, if $f \in \mathscr{C}^1(\mathbb{R}^2)$ and if Γ_f is a minimal surface in $\mathbb{H}$, then Γ_f is foliated by horizontal straight lines, i.e., (4) holds in the Lagrangian interpretation.

Lemma 3.6. *Let $f \in \mathscr{C}^1_{\mathbb{W}}$. If $\Delta^f f = 0$, then there is a sequence $\{f_k\}_{k\in\mathbb{N}} \subset \mathscr{C}^\infty(\mathbb{R}^2)$ converging to f in $\mathscr{C}^1_{\mathbb{W}}$ such that $\Delta^{f_k} f_k = 0$ for all $k \in \mathbb{N}$.*

Proof. Let $\{\rho_\epsilon\}_{\epsilon>0} \subset \mathscr{C}^\infty(\mathbb{R})$ be a family of mollifiers with $\mathrm{spt}(\rho_\epsilon) \subset [-\epsilon, \epsilon]$, $\rho_\epsilon \geq 0$ and $\int_{\mathbb{R}} \rho_\epsilon(r)\,\mathrm{d}r = 1$. Fix $f \in \mathscr{C}^1_{\mathbb{W}}$ with $\Delta^f f = 0$. Set $A(\zeta) := \nabla^f f(0,\zeta)$ and $B(\zeta) := f(0,\zeta)$. Define

$$\begin{aligned}
A_\epsilon(\zeta) &:= \int_{\mathbb{R}} \nabla^f f(0, \zeta - r)\rho_\epsilon(r)\,\mathrm{d}r, \\
B_\epsilon(\zeta) &:= \int_{\mathbb{R}} f(0, \zeta - r)\rho_\epsilon(r)\,\mathrm{d}r, \\
g_\epsilon(t,\zeta) &:= \frac{A_\epsilon(\zeta)}{2} t^2 + B_\epsilon(\zeta) t + \zeta.
\end{aligned}$$

We claim that, for all $\epsilon > 0$, both conditions stated in Lemma 3.2 hold for A_ϵ and B_ϵ. Let $\zeta, \zeta' \in \mathbb{R}$ with $\zeta < \zeta'$. First, suppose that $A_\epsilon(\zeta) = A_\epsilon(\zeta')$. Notice that $A(\zeta - r) - A(\zeta' - r) \leq 0$ for all $r \in \mathbb{R}$, because A is non-decreasing. Thus, we deduce from

$$0 = A_\epsilon(\zeta) - A_\epsilon(\zeta') = \int_{\mathbb{R}} (A(\zeta - r) - A(\zeta' - r))\rho_\epsilon(r)\,\mathrm{d}r$$

that $(B(\zeta - r) - B(\zeta' - r))\rho_\epsilon(r) = 0$ for all $r \in \mathbb{R}$ and therefore that $B_\epsilon(\zeta) = B_\epsilon(\zeta')$.
Second, suppose that $A_\epsilon(\zeta) \neq A_\epsilon(\zeta')$. On the one side, we have

$$\begin{aligned}
&2\big(A_\epsilon(\zeta) - A_\epsilon(\zeta')\big)(\zeta - \zeta') \\
&\quad= \int_{\mathbb{R}} 2\big(A(\zeta - r) - A(\zeta' - r)\big)\big((\zeta - r) - (\zeta' - r)\big)\rho_\epsilon(r)\,\mathrm{d}r \\
&\qquad\qquad > \int_{\mathbb{R}} \big(B(\zeta - r) - B(\zeta' - r)\big)^2 \rho_\epsilon(r)\,\mathrm{d}r.
\end{aligned}$$

On the other side, using Hölder inequality, we have

$$\begin{aligned}
\big(B_\epsilon(\zeta) - B_\epsilon(\zeta')\big)^2 &= \left(\int_{\mathbb{R}} \big(B(\zeta - r) - B(\zeta' - r)\big)\rho_\epsilon(r)\,\mathrm{d}r\right)^2 \\
&\leq \left(\left(\int_{\mathbb{R}} (B(\zeta - r) - B(\zeta' - r))^2 \rho_\epsilon(r)\,\mathrm{d}r\right)^{\frac12} \cdot \left(\int_{\mathbb{R}} \rho_\epsilon(r)\,\mathrm{d}r\right)^{\frac12}\right)^2 \\
&= \int_{\mathbb{R}} \big(B(\zeta - r) - B(\zeta' - r)\big)^2 \rho_\epsilon(r)\,\mathrm{d}r
\end{aligned}$$

So, the second condition is also satisfied.

The functions $G_\epsilon : \mathbb{R}^2 \to \mathbb{R}^2$, $G_\epsilon(t,\zeta) := (t, g_\epsilon(t,\zeta))$, are homeomorphisms and, as $\epsilon \to 0$, they converge to G_0 uniformly on compact sets. It follows that G_ϵ^{-1} also converge to G_0^{-1}, as $\epsilon \to 0$.

For $\epsilon > 0$, define $f_\epsilon \in \mathscr{C}^\infty(\mathbb{R}^2)$ via

$$f_\epsilon(t, g_\epsilon(t,\zeta)) = A_\epsilon(\zeta) t + B_\epsilon(\zeta).$$

By the continuity of G_ϵ and G_ϵ^{-1} in ϵ, f_ϵ and $\nabla^{f_\epsilon} f_\epsilon$ converge to f_0 and $\nabla^{f_0} f_0$ uniformly on compact sets. Finally, $\Delta^{f_\epsilon} f_\epsilon = 0$ by construction. $\square$

4. A Lagrangian approach to contact variations

Proposition 4.1. *Let $\phi = (\phi_1, \phi_2) : \mathbb{R}^2 \to \mathbb{R}^2$ be a $\mathscr{C}^\infty$-diffeomorphism. Let $f \in \mathscr{C}^1_{\mathbb{W}}$ and assume*

$$\nabla^f \phi_1(p) \neq 0 \qquad \forall p \in \mathbb{R}^2. \tag{5}$$

Define $\bar{f} : \mathbb{R}^2 \to \mathbb{R}$ as

$$\bar{f} \circ \phi = \frac{\nabla^f \phi_2}{\nabla^f \phi_1}.$$

Then $\bar{f} \in \mathscr{C}^1_{\mathbb{W}}$ and

$$\nabla^{\bar{f}} \bar{f} \circ \phi = \frac{\Delta^f \phi_2}{(\nabla^f \phi_1)^2} - \frac{\nabla^f \phi_2}{(\nabla^f \phi_1)^3} \Delta^f \phi_1. \tag{6}$$

Notice that, if $f \in \mathscr{C}^1(\mathbb{R}^2)$, then $\bar{f} \in \mathscr{C}^1(\mathbb{R}^2)$ as well.

Remark 4.2. If $\{\phi^\epsilon\}_{\epsilon > 0}$ is a smooth one-parameter family of diffeomorphisms $\phi^\epsilon : \mathbb{R}^2 \to \mathbb{R}^2$ with $\phi^0 = \mathrm{Id}$, then the functions defined by

$$f_\epsilon \circ \phi^\epsilon = \frac{\nabla^f \phi_2^\epsilon}{\nabla^f \phi_1^\epsilon}.$$

belong to $\mathscr{C}^1_{\mathbb{W}}$ and converge to f in $\mathscr{C}^1_{\mathbb{W}}$.

Proof. The idea is to transform via ϕ the integral curves of ∇^f into the ones of $\nabla^{\bar{f}}$. Fix $p = (\eta, \tau)$, let $q := (\bar{\eta}, \bar{\tau}) := \phi(p)$ and let $g_p : I \to \mathbb{R}$ be like in Lemma 2.1. Thanks to the condition $\nabla^f \phi_1 \neq 0$ and the Implicit Function Theorem, there exist two $\mathscr{C}^2$-function $s : I \to \mathbb{R}$ and $\bar{g}_q : s(I) \to \mathbb{R}$, such that

$$q + (s, \bar{g}_q(s)) = \phi(p + (t, g_p(t)), \qquad \forall t \in I.$$

Therefore

$$\begin{cases} s(t) = \phi_1(\eta + t, \tau + g_p(t)) - \bar{\eta} \\ \bar{g}_q(s(t)) = \phi_2(\eta + t, \tau + g_p(t)) - \bar{\tau}. \end{cases}$$

We define

$$\bar{f}(q) := \bar{g}_q'(0).$$

Notice that this value does not depend on the choice of g_p, as far as $t \mapsto (t, g_p(t))$ is an integral curve of ∇^f.

We want to write down $\bar{g}_q'(0)$. Set

$$p_t := (\eta + t, \tau + g_p(t)).$$

First

$$\frac{\mathrm{d}}{\mathrm{d}t} s(t) = \partial_\eta \phi_1(p_t) + \partial_\tau \phi_1(p_t) g_p'(t) = \nabla^f \phi_1(p_t),$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \bar{g}_q(s(t)) = \partial_\eta \phi_2(p_t) + \partial_\tau \phi_2(p_t) g_p'(t) = \nabla^f \phi_2(p_t).$$

Since

$$\frac{\mathrm{d}}{\mathrm{d}t} \bar{g}_q(s(t)) = \bar{g}_q'(s(t)) \cdot \frac{\mathrm{d}}{\mathrm{d}t} s(t),$$

we have for $s = 0 = t$

$$\bar{f}(q) = \frac{\nabla^f \phi_2(p)}{\nabla^f \phi_1(p)}.$$

$\nabla^{\bar f}\bar f(q)$ is the derivative of $\bar f$ along the curve $q+(s,\bar g_q(s))$ at $s=0$, i.e.,

$$\nabla^{\bar f}\bar f(q)=\bar g_q''(0).$$

As above, we want to write down $\bar g_q''(0)$ in a more explicit way.

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}t^2}s(t)|_{t=0}&=\partial_\eta^2\phi_1(p)+\partial_\tau\partial_\eta\phi_1(p)f(p)+\\&\quad+\partial_\eta\partial_\tau\phi_1(p)f(p)+\partial_\tau^2\phi_1(p)(f(p))^2+\partial_\tau\phi_1(p)\nabla^f f(p)=\Delta^f\phi_1(p).\end{aligned}$$

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\bar g_q(s(t))|_{t=0}&=\partial_\eta^2\phi_2(p)+\partial_\tau\partial_\eta\phi_2(p)f(p)+\\&\quad+\partial_\eta\partial_\tau\phi_2(p)f(p)+\partial_\tau^2\phi_2(p)(f(p))^2+\partial_\tau\phi_2(p)\nabla^f f(p)=\Delta^f\phi_2(p)\end{aligned}$$

Since

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\bar g_q(s(t))=\bar g_q''(s(t))\cdot\left(\frac{\mathrm{d}}{\mathrm{d}t}s(t)\right)^2+\bar g_q'(s(t))\cdot\frac{\mathrm{d}^2}{\mathrm{d}t^2}s(t),$$

we have

$$\begin{aligned}\nabla^{\bar f}\bar f(q)=\bar g_q''(0)&=\frac{\frac{\mathrm{d}^2}{\mathrm{d}t^2}\bar g_q|_{t=0}-\bar g_q'(0)\cdot\frac{\mathrm{d}^2}{\mathrm{d}t^2}s|_{t=0}}{(\frac{\mathrm{d}}{\mathrm{d}t}s|_{t=0})^2}\\&=\frac{1}{(\nabla^f\phi_1(p))^2}\cdot\left(\Delta^f\phi_2(p)-\frac{\nabla^f\phi_2(p)}{\nabla^f\phi_1(p)}\cdot\Delta^f\phi_1(p)\right).\end{aligned}$$

By Lemma 2.1, the function $\bar f$ belongs to $\mathscr{C}^1_{\mathbb{W}}$. □

5. Contact transformations

A diffeomorphism $\Phi:\mathbb{H}\to\mathbb{H}$ is a *contact diffeomorphism* if $\mathrm{d}\Phi(H)\subset H$. Contact diffeomorphisms are the only diffeomorphisms that keep the sub-Riemannian perimeter.

Proposition 5.1. *Let $\Phi:\mathbb{H}\to\mathbb{H}$ be a diffeomorphism of class $\mathscr{C}^2$. If, for all $E\subset\mathbb{H}$ measurable and all $\Omega\subset\mathbb{H}$ open, it holds*

$$P(E;\Omega)<\infty\qquad\Rightarrow\qquad P(\Phi(E);\Phi(\Omega))<\infty,\tag{7}$$

then Φ is contact.

We will show in this section that any variation of an intrinsic graph Γ_f via contact diffeomorphisms is equivalent to a variation of f via the transformations of Proposition 4.1 and Remark 4.2.

Proposition 5.2. *Let $\phi:\mathbb{R}^2\to\mathbb{R}^2$ be a $\mathscr{C}^\infty$-diffeomorphism and $f,\bar f\in\mathscr{C}^1_{\mathbb{W}}$ as in Proposition 4.1. Then there is a contact diffeomorphism $\Phi:\Omega\to\Phi(\Omega)$, where Ω and $\Phi(\Omega)$ are open subsets of $\mathbb{H}$ with $\Gamma_f\subset\Omega$, such that $\Phi(\Gamma_f)=\Gamma_{\bar f}$.*

Proposition 5.3. *Let $\Phi^\epsilon:\mathbb{H}\to\mathbb{H}$, $\epsilon\in\mathbb{R}$, be a smooth one-parameter family of contact diffeomorphisms such that there is a compact set $K\subset\mathbb{H}$ with $\Phi^\epsilon|_{\mathbb{H}\setminus K}=\mathrm{Id}$ for all ϵ and $\Phi^0=\mathrm{Id}$. Let $f\in\mathscr{C}^\infty(\mathbb{R}^2)$. Then there is $\epsilon_0>0$ such that for all ϵ with $|\epsilon|<\epsilon_0$, the maps $\phi^\epsilon:\mathbb{R}^2\to\mathbb{R}^2$,*

$$\phi^\epsilon(p):=\pi_X\circ\Phi^\epsilon(p*f(p)),$$

form a smoth family of $\mathscr{C}^\infty$-diffeomorphism of $\mathbb{R}^2$.

Moreover, if f^ϵ is the function defined via f and ϕ^ϵ as in Proposition 4.1, then

$$\Phi^\epsilon(\Gamma_f) = \Gamma_{f^\epsilon}.$$

5.1. **Proof of Proposition 5.1.** We use an argument by contradiction. Assume that Φ is not a contact diffeomorphism. Then there is an open and bounded set $\Omega \subset \mathbb{H}$ such that for all $p \in \Omega$ it holds $\mathrm{d}\Phi(H_p) \not\subset H_{\Phi(p)}$. Thanks to the following lemma and Remark 2.3, we get a contradiction with the property (7).

Lemma 5.4. *Let $\Phi : \mathbb{H} \to \mathbb{H}$ be a diffeomorphism of class $\mathscr{C}^2$. Let $\Omega \subset \mathbb{H}$ be an open and bounded set such that for all $p \in \Omega$*

$$\mathrm{d}\Phi(H_p) \not\subset H_{\Phi(p)}.$$

Let $E \subset \mathbb{H}$ be measurable. If $P(E;\Omega) < \infty$ and $P(\Phi(E);\Phi(\Omega)) < \infty$, then E has finite Riemannian perimeter in Ω.

Proof. We extend the scalar product $\langle\cdot,\cdot\rangle$ to the whole $T\mathbb{H}$ in such a way that (X, Y, Z) is an orthonormal frame. The Riemannian perimeter is defined as

$$P_{\mathscr{R}}(E;\Omega) := \sup\left\{\int_E \mathrm{div} U \,\mathrm{dL}^3 : \ U \in \mathrm{Vec}(T\mathbb{H}),\ \mathrm{spt} U \subset\subset \Omega,\ \|U\| \le 1\right\}.$$

Let $U \in \mathrm{Vec}(T\mathbb{H})$ with $\mathrm{spt}(U) \subset\subset \Omega$ and $\|U\| \le 1$. Then there are $V, W \in \mathrm{Vec}(T\mathbb{H})$ with $\mathrm{spt}(V) \cup \mathrm{spt}(W) = \mathrm{spt}(U)$, $V + W = U$, $V(p) \in H_p$ for all p, $\|V\| \le K$ and $\|W\| \le K$, and $\Phi_* W(p) \in H_p$ for all p, where $K \ge 0$ depends on Φ and Ω, but not on U.

Remind that, if W is a smooth vector field on $\mathbb{H}$, then[2]

$$\mathrm{div}(\Phi_* W) = \mathrm{div}(W) \circ \Phi^{-1} \cdot J(\Phi^{-1}).$$

Therefore $\int_E \mathrm{div} W \,\mathrm{dL}^3 = \int_{\Phi(E)} (\mathrm{div} W) \circ \Phi^{-1} J\Phi^{-1} \,\mathrm{dL}^3 = \int_{\Phi(E)} \mathrm{div}(\Phi_* W) \,\mathrm{dL}^3$. Moreover, since Ω is bounded, we can assume $\|\mathrm{d}\Phi(v)\| \le K\|v\|$ for all $v \in T\Omega$, where $K \ge 0$ is the same constant as above. Therefore

$$\begin{aligned}
\int_E \mathrm{div} U \,\mathrm{dL}^3 &= \int_E \mathrm{div} V \,\mathrm{dL}^3 + \int_E \mathrm{div} W \,\mathrm{dL}^3 \\
&= \int_E \mathrm{div} V \,\mathrm{dL}^3 + \int_{\Phi(E)} \mathrm{div}(\Phi_* W) \,\mathrm{dL}^3 \\
&\le K P(E;\Omega) + K^2 P(\Phi(E);\Phi(\Omega)).
\end{aligned}$$

This implies that $P_{\mathscr{R}}(E;\Omega) \le K P(E;\Omega) + K^2 P(\Phi(E);\Phi(\Omega)) < \infty$. □

5.2. **Proof of Proposition 5.2.** In this case our choice of coordinates is not helpful. So, we consider the exponential coordinates of second kind $(\xi, \eta, \tau) \mapsto \exp(\eta B + \tau C) * \exp(\xi A)$, using the notation of Section 2.1.

We define the map Φ as

$$\Phi(\xi, \eta, \tau) := \left(\frac{\nabla^\xi \phi_2}{\nabla^\xi \phi_1}(\eta, \tau), \phi_1(\eta, \tau), \phi_2(\eta, \tau)\right)$$

[2] A sketch of the proof of this formula: it is clearer to show the dual formula $\mathrm{div}(\Phi^* W) = \mathrm{div}(W) \circ \Phi \cdot J(\Phi)$; consider W as a 2-form and the divergence as the exterior derivative d; remind that $\mathrm{d}\Phi^* = \Phi^* \mathrm{d}$; the formula follows.

Clearly, Φ is well defined and smooth on the open set

$$\Omega := \{(\xi,\eta,\tau) : \nabla^\xi \phi_1(\eta,\tau) \neq 0\},$$

$\Gamma_f \subset \Omega$ by the hypothesis of Proposition 4.1 and $\Phi(\Gamma_f) = \Gamma_{\bar f}$. In these coordinates, the differential of Φ is

$$\mathrm{d}\Phi(\xi,\eta,\tau) = \begin{pmatrix} \partial_\xi\left(\frac{\nabla^\xi\phi_2}{\nabla^\xi\phi_1}\right) & \partial_\eta\left(\frac{\nabla^\xi\phi_2}{\nabla^\xi\phi_1}\right) & \partial_\tau\left(\frac{\nabla^\xi\phi_2}{\nabla^\xi\phi_1}\right) \\ 0 & \partial_\eta\phi_1 & \partial_\tau\phi_1 \\ 0 & \partial_\eta\phi_2 & \partial_\tau\phi_2 \end{pmatrix}$$

Since ϕ is a diffeomorphism, Φ is a diffeomorphism if and only if $\partial_\xi\left(\frac{\nabla^\xi\phi_2}{\nabla^\xi\phi_1}\right) \neq 0$. A short computation shows that

$$\partial_\xi\left(\frac{\nabla^\xi\phi_2}{\nabla^\xi\phi_1}\right) = \frac{\det(\,\mathrm{d}\phi)}{(\nabla^\xi\phi_1)^2},$$

which is non-zero.

Now, we need to show that Φ is a contact diffeomorphism. In this system of coordinates, the left-invariant vector fields X, Y, Z are written as

$$\tilde X(\xi,\eta,\tau) = \partial_\xi, \qquad \tilde Y(\xi,\eta,\tau) = \partial_\eta + \xi\partial_\tau, \qquad \tilde Z(\xi,\eta,\tau) = \partial_\tau.$$

We have

$$\mathrm{d}\Phi\left(\tilde X(\xi,\eta,\tau)\right) = \partial_\xi\left(\frac{\nabla^\xi\phi_2}{\nabla^\xi\phi_1}\right)\tilde X(\Phi(\xi,\eta,\tau)),$$

$$\mathrm{d}\Phi\left(\tilde Y(\xi,\eta,\tau)\right) = \nabla^\xi\left(\frac{\nabla^\xi\phi_2}{\nabla^\xi\phi_1}\right)\tilde X(\Phi(\xi,\eta,\tau)) + \nabla^\xi\phi_1\tilde Y(\Phi(\xi,\eta,\tau)).$$

Therefore, $\Phi(H) \subset H$. □

5.3. **Proof of Proposition 5.3.** The functions $\phi^\epsilon : \mathbb{R}^2 \to \mathbb{R}^2$ are well defined and smooth for all $\epsilon \in \mathbb{R}$. Since Φ^ϵ and all its derivative converge to Id uniformly on $\mathbb{H}$, there exists $\epsilon_0 > 0$ such that for all ϵ with $|\epsilon| < \epsilon_0$, the vector field X is not tangent to $\Phi^\epsilon(\Gamma_f)$ at any point. Therefore, $\det(\,\mathrm{d}\phi^\epsilon) \neq 0$ for all such ϵ. Since $\phi^\epsilon|_{\pi_X(K)} = \mathrm{Id}$, ϕ^ϵ is a covering map and therefore it is a smooth diffeomorphism.

The last statement is a direct consequence Lemma 2.2. □

6. First Contact Variation

Similar formulas for the first and the second variation for the sub-Riemannian perimeter in the Heisenberg group can be found in [4, 5, 8, 14].

In all the formulas below, we set $\psi := \nabla^f f$.

Proposition 6.1. *Let $f \in \mathscr{C}^1_{\mathbb{W}}$ be such that Γ_f is an area-minimizing surface. Then for all $V_1, V_2 \in \mathscr{C}^\infty_c(\mathbb{R}^2)$ it hold*

$$0 = \int_{\mathbb{R}^2} \frac{\psi}{\sqrt{1+\psi^2}}\left(-2\psi\cdot\nabla^f V_1 - f\cdot\Delta^f V_1\right) + \sqrt{1+\psi^2}\,\partial_\eta V_1 \,\mathrm{d}\eta\,\mathrm{d}\tau. \tag{8}$$

and

$$0 = \int_{\mathbb{R}^2} \frac{\psi}{\sqrt{1+\psi^2}}\Delta^f V_2 + \sqrt{1+\psi^2}\,\partial_\tau V_2 \,\mathrm{d}\eta\,\mathrm{d}\tau. \tag{9}$$

Proposition 6.2. *Let $f \in \mathscr{C}^\infty(\mathbb{R}^2)$ be such that for all $V_2 \in \mathscr{C}^\infty_c(\mathbb{R}^2)$ the equation* (9) *holds. Then* (8) *holds as well for all $V_1 \in \mathscr{C}^\infty_c(\mathbb{R}^2)$.*

Proposition 6.3. *A function $f \in \mathscr{C}^\infty(\mathbb{R}^2)$ satisfies* (9) *for all $V_2 \in \mathscr{C}^\infty_c(\mathbb{R}^2)$ if and only if*

$$(\nabla^f + 2\partial_\tau f)\nabla^f\left(\frac{\psi}{\sqrt{1+\psi^2}}\right) = 0. \tag{10}$$

6.1. **Proof of Proposition 6.1.** Let $f \in \mathscr{C}^1_{\mathbb{W}}$, $\omega \subset \mathbb{R}^2$ an open and bounded set and $V = (V_1, V_2) : \mathbb{R}^2 \to \mathbb{R}^2$ a smooth vector field with $\mathrm{spt} V \subset\subset \omega$. Let $\phi^\epsilon : \mathbb{R}^2 \to \mathbb{R}^2$ be a smooth one-parameter family of diffeomorphism such that $\{\phi^\epsilon \neq \mathrm{Id}\} \subset \mathrm{spt} V$ for all $\epsilon > 0$ and, for all $p \in \mathbb{R}^2$,

$$\begin{cases} \phi^0(p) = p \\ \partial_\epsilon \phi^\epsilon(p)|_{\epsilon=0} = V(p). \end{cases}$$

Notice that $\nabla^f \phi_1^\epsilon = \partial_\eta \phi_1^\epsilon + f\partial_\tau \phi_1^\epsilon$ is not zero for ϵ small enough, because $\nabla^f \phi_1^\epsilon$ converges to 1 uniformly as $\epsilon \to 0$. Hence, by Proposition 4.1, there is an interval $I = (-\hat\epsilon, \hat\epsilon)$ such that the function given by

$$f_\epsilon \circ \phi^\epsilon = \frac{\nabla^f \phi_2^\epsilon}{\nabla^f \phi_1^\epsilon} \tag{11}$$

is well defined for all $\epsilon \in I$. Define $\gamma : I \to \mathbb{R}$ as

$$\begin{aligned} \gamma(\epsilon) &:= \int_\omega \sqrt{1 + (\nabla^{f_\epsilon} f_\epsilon)^2}\, \mathrm{d}\eta\, \mathrm{d}\tau \\ &= \int_\omega \sqrt{1 + ((\nabla^{f_\epsilon} f_\epsilon)\circ\phi^\epsilon)^2} J_{\phi^\epsilon}\, \mathrm{d}\eta\, \mathrm{d}\tau, \end{aligned}$$

where we performed a change of coordinates via ϕ^ϵ and

$$J_{\phi^\epsilon} = \partial_\eta \phi_1^\epsilon \partial_\tau \phi_2^\epsilon - \partial_\tau \phi_1^\epsilon \partial_\eta \phi_2^\epsilon$$

is the Jacobian of ϕ^ϵ. Using equality (6) and Lemma 3.1, it is immediate to see that γ is continuous.

Lemma 6.4. *The function $\gamma : I \to \mathbb{R}$ is continuously differentiable and*

$$\begin{aligned} \gamma'(\epsilon) = \int_\omega & \frac{((\nabla^{f_\epsilon} f_\epsilon)\circ\phi^\epsilon)}{\sqrt{1 + ((\nabla^{f_{\infty,\epsilon}} f_{\infty,\epsilon})\circ\phi^\epsilon)^2}} A_f(\epsilon) J_{\phi^\epsilon} + \\ & + \sqrt{1 + ((\nabla^{f_\epsilon} f_\epsilon)\circ\phi^\epsilon)^2}\,\partial_\epsilon J_{\phi^\epsilon}\, \mathrm{d}\eta\, \mathrm{d}\tau, \end{aligned} \tag{12}$$

where

$$\begin{aligned} A_f(\epsilon) := & \frac{\Delta^f \partial_\epsilon \phi_2^\epsilon}{(\nabla^f \phi_1^\epsilon)^2} - 2\frac{\Delta^f \phi_2^\epsilon}{(\nabla^f \phi_1^\epsilon)^3}\nabla^f \partial_\epsilon \phi_1^\epsilon + \\ & - \frac{\nabla^f \partial_\epsilon \phi_2^\epsilon}{(\nabla^f \phi_1^\epsilon)^3}\Delta^f \phi_1^\epsilon + 3\frac{\nabla^f \phi_2^\epsilon}{(\nabla^f \phi_1^\epsilon)^4}\nabla^f \partial_\epsilon \phi_1^\epsilon \cdot \Delta^f \phi_1^\epsilon - \frac{\nabla^f \phi_2^\epsilon}{(\nabla^f \phi_1^\epsilon)^3}\Delta^f \partial_\epsilon \phi_1^\epsilon. \end{aligned} \tag{13}$$

Proof. First, suppose $f \in \mathscr{C}^\infty(\mathbb{R}^2)$. Then $\gamma \in \mathscr{C}^\infty(I)$ and

$$\gamma'(\epsilon) = \int_\omega \frac{((\nabla^{f_\epsilon} f_\epsilon)\circ\phi^\epsilon)}{\sqrt{1 + ((\nabla^{f_\epsilon} f_\epsilon)\circ\phi^\epsilon)^2}} \partial_\epsilon((\nabla^{f_\epsilon} f_\epsilon)\circ\phi^\epsilon) J_{\phi^\epsilon} +$$

$$+\sqrt{1+((\nabla^{f_\epsilon} f_\epsilon)\circ\phi^\epsilon)^2}\,\partial_\epsilon J_{\phi^\epsilon}\,\mathrm{d}\eta\,\mathrm{d}\tau.$$

Applying the formula in Proposition 4.1 and the identity $\nabla^f\partial_\epsilon = \partial_\epsilon\nabla^f$, one obtains

$$\partial_\epsilon((\nabla^{f_\epsilon} f_\epsilon)\circ\phi^\epsilon) = A_f(\epsilon)$$

and thus formula (12) holds in the smooth case.

Next, suppose $f = f_\infty$ is the limit in $\mathscr{C}^1_{\mathbb{W}}$ of a sequence $f_k \in \mathscr{C}^\infty(\mathbb{R}^2)$, as in Lemma 2.4. Notice that $\nabla^{f_k}\phi_1^\epsilon$ is not zero for ϵ small enough and k large enough. Indeed, $|\nabla^{f_k}\phi_1^\epsilon - \nabla^{f_\infty}\phi_1^\epsilon| \le \|f_k - f\|_{\mathrm{L}^\infty(\mathrm{spt}V)}\|\partial_\tau\phi_1^\epsilon\|_{\mathrm{L}^\infty(\mathrm{spt}V)}$ and $\nabla^{f_\infty}\phi_1^\epsilon$ converges to one uniformly on $\mathbb{R}^2$ as $\epsilon\to 0$. Hence, there is an interval $I\subset\mathbb{R}$ centered at zero such that the functions $f_{k,\epsilon}$ as in Proposition 4.1 are well defined for $\epsilon\in I$ and, without loss of generality, for all $k\in\mathbb{N}\cup\{\infty\}$. For $k\in\mathbb{N}\cup\{\infty\}$, define $\gamma_k : I\to\mathbb{R}$ as

$$\gamma_k(\epsilon) := \int_\omega \sqrt{1+(\nabla^{f_{k,\epsilon}} f_{k,\epsilon})^2}\,\mathrm{d}\eta\,\mathrm{d}\tau$$

Define also the function $\eta : I\to\mathbb{R}$ as the right-hand side of (12). From Lemma 3.1, it follows that $\{A_{f_k}\}_{k\in\mathbb{N}}$ converges to A_f uniformly on I. Therefore, we have that $\{\gamma_k\}_{k\in\mathbb{N}}$ and $\{\gamma_k'\}_{k\in\mathbb{N}}$ converge to γ and η uniformly on I. We conclude that $\gamma\in\mathscr{C}^1(I)$ and $\gamma' = \eta$. □

In order to evaluate $\gamma'(0)$, notice that

$$\begin{aligned}
\nabla^f\phi_1^0 &= 1 & \nabla^f\phi_2^0 &= f\\
\nabla^f\partial_\epsilon\phi_1^\epsilon|_{\epsilon=0} &= \nabla^f V_1 & \nabla^f\partial_\epsilon\phi_2^\epsilon|_{\epsilon=0} &= \nabla^f V_2\\
\Delta^f\phi_1^0 &= 0 & \Delta^f\phi_2^0 &= \psi\\
\Delta^f\partial_\epsilon\phi_1^\epsilon|_{\epsilon=0} &= \Delta^f V_1 & \Delta^f\partial_\epsilon\phi_2^\epsilon|_{\epsilon=0} &= \Delta^f V_2.
\end{aligned}$$

Therefore

$$A_f(0) = \Delta^f V_2 - 2\psi\nabla^f V_1 - f\Delta^f V_1.$$

Moreover, using the facts $\partial_\tau\phi_1^0 = \partial_\eta\phi_2^0 = 0$ and $\partial_\eta\phi_1^0 = \partial_\tau\phi_2^0 = 1$ and that the derivatives ∂_ϵ, ∂_η and ∂_τ commute, we have

$$\partial_\epsilon J_{\phi^\epsilon}|_{\epsilon=0} = \partial_\eta V_1 + \partial_\tau V_2.$$

Putting all together, we obtain

$$\gamma'(0) = \int_\omega \frac{\psi}{\sqrt{1+\psi^2}}\left(\Delta^f V_2 - 2\psi\cdot\nabla^f V_1 - f\cdot\Delta^f V_1\right) + \\ + \sqrt{1+\psi^2}(\partial_\eta V_1 + \partial_\tau V_2)\,\mathrm{d}\eta\,\mathrm{d}\tau.$$

Since Γ_f is an area-minimizing surface, then $\gamma'(0) = 0$ for all $V_1, V_2 \in \mathscr{C}_c^\infty(\mathbb{R}^2)$. Since this expression is linear in V, then we obtain both conditions (8) and (9). □

6.2. **Proof of Proposition 6.2.** Let $V_1 \in \mathscr{C}^\infty(\mathbb{R}^2)$ and set $V_2 := fV_1 \in \mathscr{C}_c^\infty(\mathbb{R}^2)$. Then

$$
\begin{aligned}
0 &= \int_{\mathbb{R}^2} \frac{\psi}{\sqrt{1+\psi^2}} \Delta^f V_2 + \sqrt{1+\psi^2}\partial_\tau V_2 \,\mathrm{d}\eta\,\mathrm{d}\tau \\
&= \int_{\mathbb{R}^2} \frac{\psi}{\sqrt{1+\psi^2}} (\nabla^f \psi V_1 + 2\psi\nabla^f V_1 + f\Delta^f V_1) + \\
&\qquad\qquad + \sqrt{1+\psi^2}(\partial_\tau f V_1 + f\partial_\tau V_1)\,\mathrm{d}\eta\,\mathrm{d}\tau \\
&= \int_{\mathbb{R}^2} \frac{\psi}{\sqrt{1+\psi^2}} (2\psi\nabla^f V_1 + f\Delta^f V_1) + \\
&\qquad + \left(\frac{\psi\nabla^f\psi}{\sqrt{1+\psi^2}} + \sqrt{1+\psi^2}\partial_\tau f \right) V_1 + \\
&\qquad\qquad + \sqrt{1+\psi^2}(\nabla^f V_1 - \partial_\eta V_1)\,\mathrm{d}\eta\,\mathrm{d}\tau \\
&= \int_{\mathbb{R}^2} \frac{\psi}{\sqrt{1+\psi^2}} (2\psi\nabla^f V_1 + f\Delta^f V_1) - \sqrt{1+\psi^2}\nabla^f V_1 + \\
&\qquad\qquad + \sqrt{1+\psi^2}(\nabla^f V_1 - \partial_\eta V_1)\,\mathrm{d}\eta\,\mathrm{d}\tau \\
&= \int_{\mathbb{R}^2} \frac{\psi}{\sqrt{1+\psi^2}} (2\psi\nabla^f V_1 + f\Delta^f V_1) - \sqrt{1+\psi^2}\partial_\eta V_1)\,\mathrm{d}\eta\,\mathrm{d}\tau.
\end{aligned}
$$

Hence (8) holds true for V_1 as well. □

6.3. **Proof of Proposition 6.3.** We have for all $V_2 \in \mathscr{C}_c^\infty(\mathbb{R}^2)$

$$
\begin{aligned}
&\int_{\mathbb{R}^2} \frac{\psi}{\sqrt{1+\psi^2}} \nabla^f\nabla^f V_2 + \sqrt{1+\psi^2}\partial_\tau V_2 \,\mathrm{d}\eta\,\mathrm{d}\tau \\
&= -\int_{\mathbb{R}^2} \nabla^f \left(\frac{\psi}{\sqrt{1+\psi^2}} \right) \nabla^f V_2 + \frac{\partial_\tau f \psi}{\sqrt{1+\psi^2}} \nabla^f V_2 + \partial_\tau(\sqrt{1+\psi^2}) V_2 \,\mathrm{d}\eta\,\mathrm{d}\tau \\
&= \int_{\mathbb{R}^2} \nabla^f\nabla^f \left(\frac{\psi}{\sqrt{1+\psi^2}} \right) V_2 + \partial_\tau f \nabla^f \left(\frac{\psi}{\sqrt{1+\psi^2}} \right) V_2 + \\
&\quad + \nabla^f(\partial_\tau f) \frac{\psi}{\sqrt{1+\psi^2}} V_2 + \partial_\tau f \nabla^f \left(\frac{\psi}{\sqrt{1+\psi^2}} \right) V_2 + (\partial_\tau f)^2 \frac{\psi}{\sqrt{1+\psi^2}} V_2 + \\
&\qquad\qquad - \frac{\psi}{\sqrt{1+\psi^2}} \partial_\tau \psi V_2 \,\mathrm{d}\eta\,\mathrm{d}\tau.
\end{aligned}
$$

Therefore, using the fact that $\partial_\tau\psi = \nabla^f(\partial_\tau f) + (\partial_\tau f)^2$, we get that (9) is equivalent to

$$
\nabla^f\nabla^f \left(\frac{\psi}{\sqrt{1+\psi^2}} \right) + 2\partial_\tau f \cdot \nabla^f \left(\frac{\psi}{\sqrt{1+\psi^2}} \right) = 0.
$$

□

7. Second Contact Variation

Similarly to the previous sections, we set $\psi := \nabla^f f$.

Proposition 7.1. *If the intrinsic graph of $f \in \mathscr{C}^1_{\mathbb{W}}$ is an area-minimizing surface, then, for all $V_1, V_2 \in \mathscr{C}^\infty_c(\mathbb{R}^2)$, we have:*

$$
\begin{aligned}
(14) \quad 0 \le I\!I_f(V_1,V_2) := \int_{\mathbb{R}^2} & \frac{(\Delta^f V_2 - 2\psi\nabla^f V_1 - f\Delta^f V_1)^2}{(1+\psi^2)^{\frac{3}{2}}} + \\
& + \frac{\psi}{(1+\psi^2)^{\frac12}} \Big(-4\Delta^f V_2 \cdot \nabla^f V_1 - 2\nabla^f V_2 \cdot \Delta^f V_1 + \\
& \qquad + 6f \cdot \nabla^f V_1 \cdot \Delta^f V_1 + 6\psi \cdot (\nabla^f V_1)^2 \Big) + \\
& + 2\frac{\psi}{(1+\psi^2)^{\frac12}} (\Delta^f V_2 - 2\psi\nabla^f V_1 - f\Delta^f V_1)(\partial_\eta V_1 + \partial_\tau V_2) + \\
& + 2(1+\psi^2)^{\frac12} (\partial_\eta V_1 \partial_\tau V_2 - \partial_\tau V_1 \partial_\eta V_2) \, \mathrm{d}\eta \, \mathrm{d}\tau.
\end{aligned}
$$

7.1. **Proof of Proposition 7.1.** Let $\omega \subset \mathbb{R}^2$ be an open and bounded set and $V = (V_1, V_2) : \mathbb{R}^2 \to \mathbb{R}^2$ a smooth vector field with $\mathrm{spt} V \subset\subset \omega$. Let $\phi^\epsilon = (\phi^\epsilon_1, \phi^\epsilon_2) : \mathbb{R}^2 \to \mathbb{R}^2$ be a smooth one-parameter family of diffeomorphism such that $\{\phi^\epsilon \ne \mathrm{Id}\} \subset \mathrm{spt} V$ for all $\epsilon > 0$ and, for all $p \in \mathbb{R}^2$,

$$
\begin{cases} \phi^0(p) = p \\ \partial_\epsilon \phi^\epsilon(p)|_{\epsilon=0} = V(p). \end{cases}
$$

Define $W_i(p) := \partial^2_\epsilon \phi^\epsilon_i(p)|_{\epsilon=0}$. Then $W = (W_1, W_2) : \mathbb{R}^2 \to \mathbb{R}^2$ is a smooth vector field with $\mathrm{spt} W \subset\subset \omega$.

As for the first variation, see Section 6.1, define

$$
\gamma(\epsilon) := \int_\omega \sqrt{1 + (\nabla^{f_\epsilon} f_\epsilon)^2} \, \mathrm{d}\eta \, \mathrm{d}\tau.
$$

Lemma 7.2. *The function $\gamma : I \to \mathbb{R}$ is twice continuously differentiable and*

$$
\begin{aligned}
(15) \quad \gamma''(\epsilon) = \int_\omega & \frac{A_f(\epsilon)^2}{(1 + (\nabla^{f_\epsilon} f_\epsilon \circ \phi^\epsilon)^2)^{\frac32}} J_{\phi^\epsilon} + \frac{(\nabla^{f_\epsilon} f_\epsilon \circ \phi^\epsilon) B_f(\epsilon)}{(1 + (\nabla^{f_\epsilon} f_\epsilon \circ \phi^\epsilon)^2)^{\frac12}} J_{\phi^\epsilon} + \\
& + 2\frac{(\nabla^{f_\epsilon} f_\epsilon \circ \phi^\epsilon) A_f(\epsilon)}{(1 + (\nabla^{f_\epsilon} f_\epsilon \circ \phi^\epsilon)^2)^{\frac12}} \partial_\epsilon J_{\phi^\epsilon} + (1 + (\nabla^{f_\epsilon} f_\epsilon \circ \phi^\epsilon)^2)^{\frac12} \partial^2_\epsilon J_{\phi^\epsilon} \, \mathrm{d}y \, \mathrm{d}z,
\end{aligned}
$$

where $A_f(\epsilon)$ is defined as in (13) *and*

$$
\begin{aligned}
B_f(\epsilon) := & \frac{\Delta^f \partial^2_\epsilon \phi^\epsilon_2}{(\nabla^f \phi^\epsilon_1)^2} - 2\frac{\Delta^f \partial_\epsilon \phi^\epsilon_2 \cdot \nabla^f \partial_\epsilon \phi^\epsilon_1}{(\nabla^f \phi^\epsilon_1)^3} + \\
& - 2\frac{\Delta^f \partial_\epsilon \phi^\epsilon_2 \cdot \nabla^f \partial_\epsilon \phi^\epsilon_1}{(\nabla^f \phi^\epsilon_1)^3} - 2\frac{\Delta^f \phi^\epsilon_2 \cdot \nabla^f \partial^2_\epsilon \phi^\epsilon_1}{(\nabla^f \phi^\epsilon_1)^3} + 6\frac{\Delta^f \phi^\epsilon_2 \cdot (\nabla^f \partial_\epsilon \phi^\epsilon_1)^2}{(\nabla^f \phi^\epsilon_1)^4} + \\
& - \frac{\nabla^f \partial^2_\epsilon \phi^\epsilon_2 \cdot \Delta^f \phi^\epsilon_1}{(\nabla^f \phi^\epsilon_1)^3} - \frac{\nabla^f \partial_\epsilon \phi^\epsilon_2 \cdot \Delta^f \partial_\epsilon \phi^\epsilon_1}{(\nabla^f \phi^\epsilon_1)^3} + 3\frac{\nabla^f \partial_\epsilon \phi^\epsilon_2 \cdot \Delta^f \phi^\epsilon_1 \cdot \nabla^f \partial_\epsilon \phi^\epsilon_1}{(\nabla^f \phi^\epsilon_1)^4} + \\
& + 3\frac{\nabla^f \partial_\epsilon \phi^\epsilon_2 \cdot \nabla^f \partial_\epsilon \phi^\epsilon_1 \cdot \Delta^f \phi^\epsilon_1}{(\nabla^f \phi^\epsilon_1)^4} + 3\frac{\nabla^f \phi^\epsilon_2 \cdot \nabla^f \partial^2_\epsilon \phi^\epsilon_1 \cdot \Delta^f \phi^\epsilon_1}{(\nabla^f \phi^\epsilon_1)^4} + \\
& + 3\frac{\nabla^f \phi^\epsilon_2 \cdot \nabla^f \partial_\epsilon \phi^\epsilon_1 \cdot \Delta^f \partial_\epsilon \phi^\epsilon_1}{(\nabla^f \phi^\epsilon_1)^4} - 12\frac{\nabla^f \phi^\epsilon_2 \cdot (\nabla^f \partial_\epsilon \phi^\epsilon_1)^2 \cdot \Delta^f \phi^\epsilon_1}{(\nabla^f \phi^\epsilon_1)^5} +
\end{aligned}
$$

$$- \frac{\nabla^f \partial_\epsilon \phi_2^\epsilon \cdot \Delta^f \partial_\epsilon \phi_1^\epsilon}{(\nabla^f \phi_1^\epsilon)^3} - \frac{\nabla^f \phi_2^\epsilon \cdot \Delta^f \partial_\epsilon^2 \phi_1^\epsilon}{(\nabla^f \phi_1^\epsilon)^3} + 3 \frac{\nabla^f \phi_2^\epsilon \cdot \Delta^f \partial_\epsilon \phi_1^\epsilon \cdot \nabla^f \partial_\epsilon \phi_1^\epsilon}{(\nabla^f \phi_1^\epsilon)^4}.$$

Proof. This lemma is a continuation of Lemma 6.4.

First, suppose $f \in \mathscr{C}^\infty(\mathbb{R}^2)$. Then, the function γ is smooth and its second derivative is

$$\begin{aligned}\gamma''(\epsilon) = \int_\omega & \frac{(\partial_\epsilon(\nabla^{f_\epsilon} f_\epsilon \circ \phi^\epsilon))^2}{(1 + (\nabla^{f_\epsilon} f_\epsilon \circ \phi^\epsilon)^2)^{\frac{3}{2}}} J_{\phi^\epsilon} + \\ & + \frac{(\nabla^{f_\epsilon} f_\epsilon \circ \phi^\epsilon) \partial_\epsilon^2 (\nabla^{f_\epsilon} f_\epsilon \circ \phi^\epsilon)}{(1 + (\nabla^{f_\epsilon} f_\epsilon \circ \phi^\epsilon)^2)^{\frac{1}{2}}} J_{\phi^\epsilon} + \\ & + 2 \frac{(\nabla^{f_\epsilon} f_\epsilon \circ \phi^\epsilon) \partial_\epsilon (\nabla^{f_\epsilon} f_\epsilon \circ \phi^\epsilon)}{(1 + (\nabla^{f_\epsilon} f_\epsilon \circ \phi^\epsilon)^2)^{\frac{1}{2}}} \partial_\epsilon J_{\phi^\epsilon} + \\ & + (1 + (\nabla^{f_\epsilon} f_\epsilon \circ \phi^\epsilon)^2)^{\frac{1}{2}} \partial_\epsilon^2 J_{\phi^\epsilon} \, \mathrm{d}y \, \mathrm{d}z.\end{aligned}$$

One checks by direct computation that

$$\begin{aligned}\partial_\epsilon(\nabla^{f_\epsilon} f_\epsilon \circ \phi^\epsilon) &= A_f(\epsilon), \\ \partial_\epsilon^2(\nabla^{f_\epsilon} f_\epsilon \circ \phi^\epsilon) &= B_f(\epsilon),\end{aligned}$$

thus (15) is proven in the smooth case.

Next, suppose $f = f_\infty$ is the limit in $\mathscr{C}^1_{\mathbb{W}}$ of a sequence $f_k \in \mathscr{C}^\infty(\mathbb{R}^2)$, as in Lemma 2.4. Define $f_{k,\epsilon}$ and $I \subset \mathbb{R}$ and $\gamma_k : I \to \mathbb{R}$ as in the proof of Lemma 6.4. Define also $\eta : I \to \mathbb{R}$ as the right-hand side of (15). By Lemma 3.1, $\{A_{f_k}\}_{k \in \mathbb{N}}$ and $\{B_{f_k}\}_{k \in \mathbb{N}}$ converge to A_f and B_f uniformly on I. Therefore, we have that the convergences $\gamma_k \to \gamma$ and $\gamma_k' \to \gamma'$ and $\gamma_k'' \to \eta$ are uniform on I. We conclude that $\gamma \in \mathscr{C}^2(I)$ and $\gamma'' = \eta$. □

Next, one can directly check that

$$\begin{aligned}\gamma''(0) = \int_\omega & \frac{(\Delta^f V_2 - 2\psi \nabla^f V_1 - f \Delta^f V_1)^2}{(1 + \psi^2)^{\frac{3}{2}}} + \\ + \frac{\psi}{(1+\psi^2)^{\frac{1}{2}}} & (\Delta^f W_2 - f \cdot \Delta^f W_1 - 2\psi \cdot \nabla^f W_1 - 4 \Delta^f V_2 \cdot \nabla^f V_1 - 2 \nabla^f V_2 \cdot \Delta^f V_1 + \\ & + 6 f \cdot \nabla^f V_1 \cdot \Delta^f V_1 + 6\psi \cdot (\nabla^f V_1)^2) + \\ & + 2 \frac{\psi}{(1+\psi^2)^{\frac{1}{2}}} (\Delta^f V_2 - 2\psi \nabla^f V_1 - f \Delta^f V_1)(\partial_\eta V_1 + \partial_\tau V_2) + \\ & + (1+\psi^2)^{\frac{1}{2}} (\partial_\eta W_1 + \partial_\tau W_2 + 2(\partial_\eta V_1 \partial_\tau V_2 - \partial_\tau V_1 \partial_\eta V_2)) \, \mathrm{d}\eta \, \mathrm{d}\tau.\end{aligned}$$

Finally, if Γ_f is a minimal surface, then $\gamma'(0) = 0$ and $\gamma''(0) \geq 0$. Notice that the terms containing W_1 and W_2 in the expression of $\gamma''(0)$ are zero because $\gamma'(0) = 0$. So, the second variation formula (14) is proven. □

8. Contact variations in the case $\Delta^f f = 0$

In this final section we prove our main result. We show that there is a quite large class of functions in $\mathscr{C}^1_{\mathbb{W}}$ that satisfy both conditions on the first and second contact variation. Since we know that the only intrinsic graphs of smooth functions that are area minimizers are the vertical planes, our result shows that variations along contact diffeomorphisms are not selective enough.

As usual, we set $\psi := \nabla^f f$.

Lemma 8.1. *Let $f \in \mathscr{C}^\infty(\mathbb{R}^2)$ be such that $\Delta^f f = 0$. Then*

$$II_f(V_1, V_2) = \int_{\mathbb{R}^2} \frac{(\Delta^f V_2 - 2\psi \nabla^f V_1 - f \Delta^f V_1)^2}{(1+\psi^2)^{\frac{3}{2}}} + \\ + \partial_\tau \left(\frac{\psi}{(1+\psi^2)^{\frac{1}{2}}} \right) \left(\nabla^f V_2 - \nabla^f (f V_1) \right)^2 \, \mathrm{d}\eta \, \mathrm{d}\tau.$$

The proof is very technical and it is postponed to the last section below.

Theorem 8.2. *Let $f \in \mathscr{C}^1_{\mathbb{W}}$ be such that $\Delta^f f = 0$. Then both equalities* (8) *and* (9) *and also the inequality* (14) *are satisfied for all $V_1, V_2 \in \mathscr{C}^\infty_c(\mathbb{R}^2)$.*

Proof. We first prove that both equalities (8) and (9) are satisfied. Let $\{f_k\}_{k\in\mathbb{N}} \subset \mathscr{C}^\infty(\mathbb{R}^2)$ be a sequence converging to f in $\mathscr{C}^1_{\mathbb{W}}$ and such that $\Delta^{f_k} f_k = 0$, as in Lemma 3.6. Fix $V_1, V_2 \in \mathscr{C}^\infty_c(\mathbb{R}^2)$. Then (10) and (8) are satisfied by all f_k thanks to Propositions 6.2 and 6.3. Passing to the limit $k \to \infty$, we prove that f satisfies them too.

Now, we prove that the inequality (14) holds true. If $f \in \mathscr{C}^\infty(\mathbb{R}^2)$, then we can apply Lemma 8.1, where $\partial_\tau \left(\frac{\psi}{(1+\psi^2)^{\frac{1}{2}}} \right) = \frac{\partial_\tau \psi}{(1+\psi^2)^{\frac{3}{2}}} \ge 0$ because of Lemma 3.3. So, (14) is proven for f smooth. For $f \in \mathscr{C}^1_{\mathbb{W}}$, let $\{f_k\}_{k\in\mathbb{N}} \subset \mathscr{C}^\infty(\mathbb{R}^2)$ as in Lemma 3.6. From Lemma 3.1 follows that, for fixed $V_1, V_2 \in \mathscr{C}^\infty_c(\mathbb{R}^2)$, it holds

$$\lim_{k\to\infty} II_{f_k}(V_1, V_2) = II_f(V_1, V_2),$$

thus $II_f(V_1, V_2) \ge 0$. □

8.1. **Proof of Lemma 8.1.** The proof of this lemma is just a computation, but quite elaborate. For making the formulas more readable, we decided to drop the sign of integral along the proof. In other words, all equalities in this section are meant as equalities of integrals on $\mathbb{R}^2$. We will constantly use the formulas listed in Appendix A together with $\nabla^f \psi = 0$.

Before of all, we reorganise the integral in (14):

$$\text{(ⓐ)} \qquad \frac{(\Delta^f V_2 - 2\psi \nabla^f V_1 - f \Delta^f V_1)^2}{(1+\psi^2)^{\frac{3}{2}}} +$$

$$\text{(ⓑ)} \qquad + \frac{\psi}{(1+\psi^2)^{\frac{1}{2}}} \left(+6f \cdot \nabla^f V_1 \cdot \Delta^f V_1 + 6\psi \cdot (\nabla^f V_1)^2 \right)$$

$$\text{(ⓒ)} \qquad + 2\frac{\psi}{(1+\psi^2)^{\frac{1}{2}}} \left(-2\psi \nabla^f V_1 - f \Delta^f V_1 \right) \partial_\eta V_1$$

$$\text{(ⓓ)} \qquad + 2\frac{\psi}{(1+\psi^2)^{\frac{1}{2}}} \Delta^f V_2 \partial_\tau V_2$$

$$\text{(ⓔ)} \qquad + \frac{\psi}{(1+\psi^2)^{\frac{1}{2}}} \left(-4\Delta^f V_2 \cdot \nabla^f V_1 - 2\nabla^f V_2 \cdot \Delta^f V_1 \right)$$

$$(\text{f}) \qquad +2\frac{\psi}{(1+\psi^2)^{\frac{1}{2}}}\Big(\Delta^f V_2 \partial_\eta V_1 + (-2\psi\nabla^f V_1 - f\Delta^f V_1)\partial_\tau V_2\Big)$$

$$(\text{g}) \qquad +2(1+\psi^2)^{\frac{1}{2}}(\partial_\eta V_1 \partial_\tau V_2 - \partial_\tau V_1 \partial_\eta V_2).$$

In the following lemmas we will study b+c, d and e + f + g separately in order to obtain the expansion of the square in the second term of the integral in Lemma 8.1.

Lemma 8.3.

$$\text{b} + \text{c} = \partial_\tau\left(\frac{\psi}{(1+\psi^2)^{\frac{1}{2}}}\right)\Big(\nabla^f(fV_1)\Big)^2.$$

Proof.

$$\begin{aligned}
\text{b} &= \frac{\psi}{(1+\psi^2)^{\frac{1}{2}}}\Big(6f\nabla^f V_1 \Delta^f V_1 + 6\psi(\nabla^f V_1)^2\Big)\\
&= \frac{\psi}{(1+\psi^2)^{\frac{1}{2}}}\Big(3f\nabla^f(\nabla^f V_1)^2 + 6\psi(\nabla^f V_1)^2\Big)\\
&= 3\frac{\psi}{(1+\psi^2)^{\frac{1}{2}}}\Big(\nabla^f(f(\nabla^f V_1)^2) + \psi(\nabla^f V_1)^2\Big)\\
&= -3\frac{\psi}{(1+\psi^2)^{\frac{1}{2}}} f\partial_\tau f(\nabla^f V_1)^2 + 3\frac{\psi^2}{(1+\psi^2)^{\frac{1}{2}}}(\nabla^f V_1)^2\\
&= -\frac{3}{2}\frac{\psi}{(1+\psi^2)^{\frac{1}{2}}}\partial_\tau(f^2)(\nabla^f V_1)^2 + 3\frac{\psi^2}{(1+\psi^2)^{\frac{1}{2}}}(\nabla^f V_1)^2.
\end{aligned}$$

$$\begin{aligned}
\text{c} &= -2\frac{\psi}{(1+\psi^2)^{\frac{1}{2}}}(2\psi\nabla^f V_1 + f\Delta^f V_1)\partial_\eta V_1\\
&= -2\frac{\psi}{(1+\psi^2)^{\frac{1}{2}}}(2\psi\nabla^f V_1 + f\Delta^f V_1)(\nabla^f V_1 - f\partial_\tau V_1)\\
&= -4\frac{\psi^2}{(1+\psi^2)^{\frac{1}{2}}}(\nabla^f V_1)^2 + 4\frac{\psi^2}{(1+\psi^2)^{\frac{1}{2}}} f\nabla^f V_1 \partial_\tau V_1 +\\
&\qquad - 2\frac{\psi}{(1+\psi^2)^{\frac{1}{2}}} f\Delta^f V_1 \nabla^f V_1 + 2\frac{\psi}{(1+\psi^2)^{\frac{1}{2}}} f^2\Delta^f V_1 \partial_\tau V_1.
\end{aligned}$$

We have two particular terms in this expression:

$$\begin{aligned}
\text{@} &:= 4\frac{\psi^2}{(1+\psi^2)^{\frac{1}{2}}} f\nabla^f V_1 \partial_\tau V_1 + 2\frac{\psi}{(1+\psi^2)^{\frac{1}{2}}} f^2\Delta^f V_1 \partial_\tau V_1\\
&= 2\frac{\psi}{(1+\psi^2)^{\frac{1}{2}}}\nabla^f(f^2\nabla^f V_1)\partial_\tau V_1\\
&= -2\frac{\psi}{(1+\psi^2)^{\frac{1}{2}}}(f^2\nabla^f V_1)\nabla^f\partial_\tau V_1 - 2\partial_\tau f\frac{\psi}{(1+\psi^2)^{\frac{1}{2}}}(f^2\nabla^f V_1)\partial_\tau V_1\\
&= -2\frac{\psi}{(1+\psi^2)^{\frac{1}{2}}} f^2\nabla^f V_1(\nabla^f\partial_\tau V_1 + \partial_\tau f\partial_\tau V_1)
\end{aligned}$$

$$
\begin{aligned}
&= -2\frac{\psi}{(1+\psi^2)^{\frac{1}{2}}} f^2 \nabla^f V_1 \partial_\tau \nabla^f V_1 \\
&= -\frac{\psi}{(1+\psi^2)^{\frac{1}{2}}} f^2 \partial_\tau (\nabla^f V_1)^2 \\
&= \partial_\tau \left(\frac{\psi}{(1+\psi^2)^{\frac{1}{2}}} \right) f^2 (\nabla^f V_1)^2 + \frac{\psi}{(1+\psi^2)^{\frac{1}{2}}} \partial_\tau (f^2) (\nabla^f V_1)^2
\end{aligned}
$$

and

$$
\begin{aligned}
\beta &:= -2\frac{\psi}{(1+\psi^2)^{\frac{1}{2}}} f \Delta^f V_1 \nabla^f V_1 \\
&= -\frac{\psi}{(1+\psi^2)^{\frac{1}{2}}} f \nabla^f (\nabla^f V_1)^2 \\
&= \frac{\psi^2}{(1+\psi^2)^{\frac{1}{2}}} (\nabla^f V_1)^2 + \frac{\psi}{(1+\psi^2)^{\frac{1}{2}}} f \partial_\tau f (\nabla^f V_1)^2 \\
&= \frac{\psi^2}{(1+\psi^2)^{\frac{1}{2}}} (\nabla^f V_1)^2 + \frac{\psi}{(1+\psi^2)^{\frac{1}{2}}} \frac{\partial_\tau (f^2)}{2} (\nabla^f V_1)^2.
\end{aligned}
$$

Therefore:

$$
\begin{aligned}
c &= -4\frac{\psi^2}{(1+\psi^2)^{\frac{1}{2}}} (\nabla^f V_1)^2 + \alpha + \beta \\
&= -3\frac{\psi^2}{(1+\psi^2)^{\frac{1}{2}}} (\nabla^f V_1)^2 + \partial_\tau \left(\frac{\psi}{(1+\psi^2)^{\frac{1}{2}}} \right) f^2 (\nabla^f V_1)^2 + \\
&\qquad + \frac{3}{2} \frac{\psi}{(1+\psi^2)^{\frac{1}{2}}} \partial_\tau (f^2) (\nabla^f V_1)^2.
\end{aligned}
$$

Putting this together,

$$
\begin{aligned}
b + c &= \partial_\tau \left(\frac{\psi}{(1+\psi^2)^{\frac{1}{2}}} \right) f^2 (\nabla^f V_1)^2 \\
&= \partial_\tau \left(\frac{\psi}{(1+\psi^2)^{\frac{1}{2}}} \right) (\nabla^f (f V_1) - \psi V_1)^2 \\
&= \partial_\tau \left(\frac{\psi}{(1+\psi^2)^{\frac{1}{2}}} \right) ((\nabla^f (f V_1))^2 + (\psi V_1)^2 - 2\nabla^f (f V_1) \psi V_1) \\
&= \partial_\tau \left(\frac{\psi}{(1+\psi^2)^{\frac{1}{2}}} \right) ((\nabla^f (f V_1))^2 - (\psi V_1)^2 - f \nabla^f (\psi V_1^2)) \\
&\overset{(*)}{=} \partial_\tau \left(\frac{\psi}{(1+\psi^2)^{\frac{1}{2}}} \right) ((\nabla^f (f V_1))^2 - (\psi V_1)^2 + \nabla^f f \psi V_1^2) \\
&= \partial_\tau \left(\frac{\psi}{(1+\psi^2)^{\frac{1}{2}}} \right) (\nabla^f (f V_1))^2.
\end{aligned}
$$

In $(*)$ we used formula (16). $\square$

Lemma 8.4.

$$ⓓ = \partial_\tau \left(\frac{\psi}{(1+\psi^2)^{\frac{1}{2}}} \right) (\nabla^f V_2)^2.$$

Proof.

$$\begin{aligned} ⓓ &= 2\frac{\psi}{(1+\psi^2)^{\frac{1}{2}}} \Delta^f V_2 \partial_\tau V_2 \\ &\overset{(*)}{=} -2\frac{\psi}{(1+\psi^2)^{\frac{1}{2}}} \nabla^f V_2 \partial_\tau \nabla^f V_2 \\ &= -\frac{\psi}{(1+\psi^2)^{\frac{1}{2}}} \partial_\tau (\nabla^f V_2)^2 \\ &= \partial_\tau \left(\frac{\psi}{(1+\psi^2)^{\frac{1}{2}}} \right) (\nabla^f V_2)^2. \end{aligned}$$

In $(*)$ we used formula (16). □

Lemma 8.5.

$$ⓔ + ⓕ + ⓖ = -2\partial_\tau \left(\frac{\psi}{(1+\psi^2)^{\frac{1}{2}}} \right) \nabla^f (fV_1) \nabla^f V_2.$$

Proof.

$$\begin{aligned} ⓖ &= 2(1+\psi^2)^{\frac{1}{2}} (\partial_\eta V_1 \partial_\tau V_2 - \partial_\tau V_1 \partial_\eta V_2) \\ &= 2(1+\psi^2)^{\frac{1}{2}} ((\nabla^f V_1 - f\partial_\tau V_1)\partial_\tau V_2 - \partial_\tau V_1 (\nabla^f V_2 - f\partial_\tau V_2)) \\ &= 2(1+\psi^2)^{\frac{1}{2}} (\nabla^f V_1 \partial_\tau V_2 - \partial_\tau V_1 \nabla^f V_2). \end{aligned}$$

$$\begin{aligned} ⓕ &= 2\frac{\psi}{(1+\psi^2)^{\frac{1}{2}}} \Big(\Delta^f V_2 \partial_\eta V_1 + (-2\psi \nabla^f V_1 - f\Delta^f V_1)\partial_\tau V_2 \Big) \\ &= 2\frac{\psi}{(1+\psi^2)^{\frac{1}{2}}} \Big(\Delta^f V_2 (\nabla^f V_1 - f\partial_\tau V_1) - 2\psi \nabla^f V_1 \partial_\tau V_2 - f\Delta^f V_1 \partial_\tau V_2 \Big) \\ &= 2\frac{\psi}{(1+\psi^2)^{\frac{1}{2}}} \Big(-f\partial_\tau V_1 \Delta^f V_2 - f\partial_\tau V_2 \Delta^f V_1 + \Delta^f V_2 \nabla^f V_1 - 2\psi \nabla^f V_1 \partial_\tau V_2 \Big) \\ &\overset{(*)}{=} 2\frac{\psi}{(1+\psi^2)^{\frac{1}{2}}} \Big(\psi \partial_\tau V_1 \nabla^f V_2 + f\partial_\tau (\nabla^f V_1) \nabla^f V_2 + \psi \partial_\tau V_2 \nabla^f V_1 + \\ &\qquad + f\partial_\tau (\nabla^f V_2) \nabla^f V_1 + \Delta^f V_2 \nabla^f V_1 - 2\psi \nabla^f V_1 \partial_\tau V_2 \Big) \\ &= 2\frac{\psi}{(1+\psi^2)^{\frac{1}{2}}} \Big(\psi \partial_\tau V_1 \nabla^f V_2 + f\partial_\tau (\nabla^f V_1) \nabla^f V_2 + \\ &\qquad + f\partial_\tau (\nabla^f V_2) \nabla^f V_1 + \Delta^f V_2 \nabla^f V_1 - \psi \nabla^f V_1 \partial_\tau V_2 \Big). \end{aligned}$$

In $(*)$ we used formula (16).

$$ⓕ + ⓖ = 2\frac{(1+\psi^2)}{(1+\psi^2)^{\frac{1}{2}}} (\nabla^f V_1 \partial_\tau V_2 - \partial_\tau V_1 \nabla^f V_2) +$$

$$
\begin{aligned}
&+2\frac{\psi}{(1+\psi^2)^{\frac12}}\left(\psi\partial_\tau V_1\nabla^f V_2 + f\partial_\tau(\nabla^f V_1)\nabla^f V_2 + \right.\\
&\qquad \left. + f\partial_\tau(\nabla^f V_2)\nabla^f V_1 + \Delta^f V_2\nabla^f V_1 - \psi\nabla^f V_1\partial_\tau V_2\right)\\
&= 2\frac{1}{(1+\psi^2)^{\frac12}}(\nabla^f V_1\partial_\tau V_2 - \partial_\tau V_1\nabla^f V_2) +\\
&\qquad + 2\frac{\psi}{(1+\psi^2)^{\frac12}}\left(f\partial_\tau(\nabla^f V_1\nabla^f V_2) + \Delta^f V_2\nabla^f V_1\right).
\end{aligned}
$$

$$
\begin{aligned}
&ⓔ + ⓕ + ⓖ = \frac{\psi}{(1+\psi^2)^{\frac12}}\left(-4\Delta^f V_2\nabla^f V_1 - 2\nabla^f V_2\Delta^f V_1\right) + ⓕ + ⓖ\\
&= 2\frac{\psi}{(1+\psi^2)^{\frac12}}\left(-\Delta^f V_2\nabla^f V_1 - \nabla^f V_2\Delta^f V_1 + f\partial_\tau(\nabla^f V_1\nabla^f V_2)\right) +\\
&\qquad + 2\frac{1}{(1+\psi^2)^{\frac12}}(\nabla^f V_1\partial_\tau V_2 - \partial_\tau V_1\nabla^f V_2)\\
&= 2\frac{\psi}{(1+\psi^2)^{\frac12}}\left(-\nabla^f(\nabla^f V_2\nabla^f V_1) + f\partial_\tau(\nabla^f V_1\nabla^f V_2)\right) +\\
&\qquad + 2\frac{1}{(1+\psi^2)^{\frac12}}(\nabla^f V_1\partial_\tau V_2 - \partial_\tau V_1\nabla^f V_2)\\
&= 2\frac{\psi}{(1+\psi^2)^{\frac12}}\partial_\tau f\nabla^f V_2\nabla^f V_1 - 2\partial_\tau\left(\frac{\psi}{(1+\psi^2)^{\frac12}}\right) f\nabla^f V_1\nabla^f V_2 +\\
&\quad - 2\frac{\psi}{(1+\psi^2)^{\frac12}}\partial_\tau f\nabla^f V_1\nabla^f V_2 + 2\frac{1}{(1+\psi^2)^{\frac12}}(\nabla^f V_1\partial_\tau V_2 - \partial_\tau V_1\nabla^f V_2)\\
&= -2\partial_\tau\left(\frac{\psi}{(1+\psi^2)^{\frac12}}\right) f\nabla^f V_1\nabla^f V_2 + 2\frac{1}{(1+\psi^2)^{\frac12}}(\nabla^f V_1\partial_\tau V_2 - \partial_\tau V_1\nabla^f V_2).
\end{aligned}
$$

In particular, we have

$$
\begin{aligned}
\frac{1}{(1+\psi^2)^{\frac12}}(\nabla^f V_1\partial_\tau V_2 - \partial_\tau V_1\nabla^f V_2) &= -\frac{1}{(1+\psi^2)^{\frac12}}\partial_\tau\nabla^f V_2 V_1 +\\
&\quad + \partial_\tau\left(\frac{1}{(1+\psi^2)^{\frac12}}\right)\nabla^f V_2 V_1 + \frac{1}{(1+\psi^2)^{\frac12}}\partial_\tau\nabla^f V_2 V_1\\
&= \partial_\tau\left(\frac{1}{(1+\psi^2)^{\frac12}}\right)\nabla^f V_2 V_1
\end{aligned}
$$

and

$$
\partial_\tau\left(\frac{1}{(1+\psi^2)^{\frac12}}\right) = -\frac{1}{(1+\psi^2)^{\frac32}}\psi\partial_\tau\psi = -\psi\partial_\tau\left(\frac{\psi}{(1+\psi^2)^{\frac12}}\right).
$$

Therefore

$$
ⓔ + ⓕ + ⓖ =
$$

$$= -2\partial_\tau\left(\frac{\psi}{(1+\psi^2)^{\frac{1}{2}}}\right)f\nabla^f V_1\nabla^f V_2 + 2\frac{1}{(1+\psi^2)^{\frac{1}{2}}}(\nabla^f V_1\partial_\tau V_2 - \partial_\tau V_1\nabla^f V_2)$$

$$= -2\partial_\tau\left(\frac{\psi}{(1+\psi^2)^{\frac{1}{2}}}\right)f\nabla^f V_1\nabla^f V_2 - 2\psi\partial_\tau\left(\frac{\psi}{(1+\psi^2)^{\frac{1}{2}}}\right)\nabla^f V_2 V_1$$

$$= -2\partial_\tau\left(\frac{\psi}{(1+\psi^2)^{\frac{1}{2}}}\right)\nabla^f(fV_1)\nabla^f V_2.$$

□

Appendix A. Useful formulas

In the case $f \in \mathscr{C}^\infty(\mathbb{R}^2)$, the adjoint operator of ∇^f is

$$(\nabla^f)^* = -\nabla^f - \partial_\tau f,$$

i.e., if $A, B \in \mathscr{C}^\infty(\mathbb{R}^2)$ and one of them has compact support, then

$$\int_{\mathbb{R}^2} A \cdot \nabla^f B \,\mathrm{d}\eta\,\mathrm{d}\tau = -\int_{\mathbb{R}^2} \nabla^f A \cdot B + \partial_\tau f \cdot A \cdot B \,\mathrm{d}\eta\,\mathrm{d}\tau.$$

Notice that, if f is smooth, the following holds:

$$\partial_\eta = \nabla^f - f\partial_\tau,$$
$$\partial_\tau\nabla^f = \nabla^f\partial_\tau + \partial_\tau f\partial_\tau.$$

If $A, B, C \in \mathscr{C}^\infty(\mathbb{R}^2)$ and one of them has compact support, then

(16)
$$\int_{\mathbb{R}^2} A \cdot \partial_\tau B \cdot \nabla^f C \,\mathrm{d}\eta\,\mathrm{d}\tau = -\int_{\mathbb{R}^2} \Big(\nabla^f A \cdot \partial_\tau B \cdot C + A \cdot \partial_\tau\nabla^f B \cdot C\Big)\,\mathrm{d}\eta\,\mathrm{d}\tau.$$

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