EXISTENCE AND UNIQUENESS OF DYNAMIC EVOLUTIONS FOR A PEELING TEST IN DIMENSION ONE

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Abstract. In this paper we present a one-dimensional model of a dynamic peeling test for a thin film, where the wave equation is coupled with a Griffith criterion for the propagation of the debonding front. Our main results provide existence and uniqueness for the solution to this coupled problem under different assumptions on the data.

Keywords: Thin films; Dynamic debonding; Wave equation in time-dependent domains; Dynamic energy release rate; Energy-dissipation balance; Maximum dissipation principle; Griffith's criterion; Dynamic fracture.
2010 MSC: 35L05, 35Q74, 35R35, 74H20, 74K35.

Introduction

The study of crack growth based on Griffith's criterion has become of great interest in the mathematical community. The starting point was the seminal paper [15], where a precise variational scheme for the quasistatic evolution has been proposed. This strategy has been exploited under different hypotheses in [12, 6, 14, 7, 10, 23]. The approximation of brittle crack growth by means of phase-field models in the quasistatic regime has been studied in [18]. A comprehensive presentation of the variational approach to quasistatic fracture mechanics can be found in [5]. For the relationships between this approach and the general theory of rate-independent systems we refer to the recent book [29].

In the dynamic case no general formulation has been yet proposed and only preliminary results are available (see [30, 8, 11, 9]). A reasonable model for dynamic fracture should combine the equations of elasto-dynamics for the displacement u out of the crack with an evolution law which connects the crack growth with u. The only result in this direction, without strong geometrical assumptions on the cracks, has been obtained for a phase-field model [24], but the convergence of these solutions to a brittle crack evolution has not been proved in the dynamic case. In the latter model the equation of elasto-dynamics for u is coupled with a suitable minimality condition for the phase-field ζ at each time. Other models in materials science, dealing with damage or delamination, couple a second order hyperbolic equation for a function u with a first order flow rule for an internal variable ζ (see, e.g., [16, 4, 3, 31, 32, 21, 20] for viscous flow rules on ζ and [34, 36, 35, 33, 37, 2, 1, 38, 27, 28] for rate-independent evolutions of ζ).

In this work we contribute to the study of dynamic fracture by analysing a simpler one-dimensional model already considered in [17, Section 7.4]. This model exhibits some of the relevant mathematical difficulties due to the time dependence of the domain of the wave equation. More precisely, following [13, 26] we study a model of a dynamic peeling test for a thin film, initially attached to a planar rigid substrate; the process is assumed to depend only on one variable. This hypothesis is crucial for our analysis, since we frequently use d'Alembert's formula for the wave equation.

To describe the geometry of our problem, we fix an orthogonal coordinate system (x, y, z). We assume that the z-axis is vertical, the plane (x, y) coincides with the rigid substrate, and that the reference configuration of the film is the half plane $\{(x, y) : x \geq 0\}$. We also assume that the deformation of the film at time $t \geq 0$ is described by two functions h and u according to

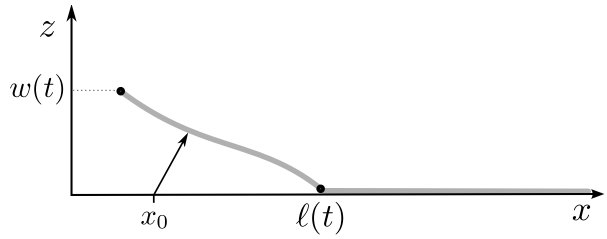

FIGURE 1. The curve $x \mapsto (x+h(t,x), u(t,x))$ describing the deformation of the film in the peeling test. The vector applied to the point x_0 is $(h(t,x_0), u(t,x_0))$

the formula $(x,y) \mapsto (x+h(t,x), y, u(t,x))$, i.e., the displacement is given by $(h(t,x), 0, u(t,x))$. Therefore the thin film at time t is uniquely determined by the parametric curve $x \mapsto (x+h(t,x), u(t,x))$ with $x \geq 0$, which represents its intersection with the vertical plane (x,z) (see Figure 1). To simplify our analysis we shall not consider the unilateral contact constraint $u(t,x) \geq 0$, thus neglecting the non-interpenetration of matter.

The film is assumed to be perfectly flexible, inextensible, and glued to the rigid substrate on the half line $\{(x,y,z) : x{\geq}\ell(t)\,,\, z{=}0\}$, where $\ell(t)$ is a nondecreasing function which represents the debonding front, with $\ell_0 := \ell(0) > 0$. This implies $h(t,x) = u(t,x) = 0$ for $x \geq \ell(t)$. At $x = 0$ we prescribe a vertical displacement $u(t,0) = w(t)$ depending on time $t \geq 0$, and a fixed tension so that the speed of sound in the film is constant. Using the linear approximation and the inextensibility it turns out that h can be expressed in terms of u as

$$h(t,x) = \frac{1}{2}\int_x^{+\infty} u_x(t,\xi)^2 \,\mathrm{d}\xi,$$

and u solves the problem

$$u_{tt}(t,x) - u_{xx}(t,x) = 0, \quad t > 0\,,\; 0 < x < \ell(t), \tag{0.1a}$$

$$u(t,0) = w(t), \quad t > 0, \tag{0.1b}$$

$$u(t,\ell(t)) = 0, \quad t > 0, \tag{0.1c}$$

where we normalised the speed of sound to one. The system is supplemented by the initial conditions

$$u(0,x) = u_0(x), \quad 0 < x < \ell_0, \tag{0.1d}$$

$$u_t(0,x) = u_1(x), \quad 0 < x < \ell_0, \tag{0.1e}$$

where u_0 and u_1 are prescribed functions.

The first result of the paper is that, under suitable assumptions on the functions u_0, u_1, ℓ, and w, problem (0.1) has a unique solution u (cf. Theorem 1.8). In particular, we always assume that $\dot\ell < 1$, which means that the debonding speed is less than the speed of sound.

In order to prove this theorem, we observe that, by d'Alembert's formula, u is a solution of (0.1a)&(0.1b) if and only if

$$u(t,x) = w(t{+}x) - f(t{+}x) + f(t{-}x), \tag{0.2}$$

for a suitable function $f\colon [-\ell_0, +\infty) \to \mathbb{R}$. Moreover, the boundary condition (0.1c) is satisfied if and only if

$$f(t{+}\ell(t)) = w(t{+}\ell(t)) + f(t{-}\ell(t)). \tag{0.3}$$

Using this formula, together with the monotonicity and continuity of ℓ, we can determine the values of $f(s)$ for $s \in [-\ell_0, t+\ell(t)]$ from the values of $f(s)$ for $s \in [-\ell_0, t-\ell(t)]$.

It is easy to see (cf. Proposition 1.6) that (0.2) implies that f is uniquely determined on $[-\ell_0, \ell_0]$ by the initial conditions u_0 and u_1 through an explicit formula (see (1.18)). If s_1 is the unique time such that $s_1 - \ell(s_1) = \ell_0$, formula (0.3) allows us to extend f to the interval $[-\ell_0, s_1+\ell(s_1)]$. Then, we consider the unique time s_2 such that $s_2-\ell(s_2) = s_1+\ell(s_1)$ and, using again formula (0.3), we are able to extend f to $[-\ell_0, s_2 + \ell(s_2)]$. In this way we can construct recursively a sequence s_n such that f is extended to $[-\ell_0, s_n + \ell(s_n)]$ and (0.3) holds for every $0 \le t \le s_n$. Since it is easy to see that $s_n \to +\infty$, we are able to extend f to $[-\ell_0, +\infty)$ in such a way that (0.3) holds for every $t > 0$. This contruction allows us also to obtain the expected regularity for u from our hypotheses on u_0, u_1, ℓ, and w.

In the second part of the paper only u_0, u_1, and w are given and the evolution of the debonding front ℓ has to be determined on the basis of an additional energy criterion. To formulate this criterion we fix once and for all the initial conditions u_0 and u_1 and we consider the internal energy of u as a functional depending on ℓ and w. More precisely,

$$\mathcal{E}(t; \ell, w) := \frac{1}{2}\int_0^{\ell(t)} u_x(t,x)^2 \,\mathrm{d}x + \frac{1}{2}\int_0^{\ell(t)} u_t(t,x)^2 \,\mathrm{d}x,$$

where u is the unique solution corresponding to u_0, u_1, ℓ, and w; the first term is the potential energy and the second one is the kinetic energy.

A crucial role is played by the dynamic energy release rate, which is defined as a (sort of) partial derivative of $\mathcal{E}$ with respect to the elongation of the debonded region. More precisely, to define the dynamic energy release rate $G_\alpha(t_0)$ at time t_0 corresponding to a speed $0 < \alpha < 1$ of the debonding front, we modify the debonding front ℓ and the vertical displacement w using the functions

$$\lambda(t) = \begin{cases} \ell(t), & t \le t_0, \\ (t-t_0)\alpha, & t > t_0, \end{cases} \qquad z(t) = \begin{cases} w(t), & t \le t_0, \\ w(t_0), & t > t_0, \end{cases}$$

and we set

$$G_\alpha(t_0) := \lim_{t \to t_0^+} \frac{\mathcal{E}(t_0; \lambda, z) - \mathcal{E}(t; \lambda, z)}{(t-t_0)\alpha}.$$

We prove in Proposition 2.1 that, given ℓ and w, the limit above exists for a.e. $t_0 > 0$ and for every $\alpha \in (0,1)$. Moreover, we prove that

$$G_\alpha(t_0) = 2\frac{1-\alpha}{1+\alpha}\dot{f}(t_0 - \ell(t_0))^2, \tag{0.4}$$

where f is the function which appears in (0.2). This formula shows, in particular, that $G_\alpha(t_0)$ depends only on α and on the values of $u(t,x)$ for $t \le t_0$ (see the discussion which leads to (2.3)).

In our model we describe a debonding process occurring between the glue and the film. Therefore it is natural to assume that the energy dissipated to debond a segment $[x_1, x_2]$, with $0 \le x_1 < x_2$, is given by

$$\int_{x_1}^{x_2} \kappa(x) \,\mathrm{d}x,$$

where $\kappa\colon [0, +\infty) \to (0, +\infty)$ represents the local toughness of the glue between the film and the substrate. This is the analogue of brittle behaviour in fracture mechanics. In this paper we do not consider cohesive debonding, which occurs when the adhesive properties of the glue undergo progressive deterioration. In this latter case κ should depend also on u.

In Section 2, starting from a maximum dissipation principle, we prove that the debonding front must satisfy the following energy criterion, called Griffith's criterion:

$$\begin{cases} \dot\ell(t) \geq 0, \\ G_{\dot\ell(t)}(t) \leq \kappa(\ell(t)), \\ \Big[G_{\dot\ell(t)}(t) - \kappa(\ell(t))\Big]\,\dot\ell(t) = 0, \end{cases} \tag{0.5}$$

for a.e. $t > 0$. The first condition asserts that the debonding can only grow (unidirectionality). The second condition states that the dynamic energy release rate is always bounded by the local toughness, while, accordingly to the third one, ℓ can increase with positive speed at t only when the dynamic energy release rate is critical at t, i.e., $G_{\dot\ell(t)}(t) = \kappa(\ell(t))$.

If u and ℓ are sufficiently regular, we shall see in (2.16) that $G_{\dot\ell(t)}(t) = \frac{1}{2}(1-\dot\ell(t)^2)u_x(t,\ell(t))^2$. Therefore (0.5) can be interpreted as a threshold condition on $u_x(t,\ell(t))^2$.

The main results of this paper are Theorems 3.1, 3.4, and 3.5, where we show existence and uniqueness of the solution (u,ℓ) to the coupled problem (0.1)&(0.5) under various assumptions on the data.

The strategy for the proof of these results is to write (0.5) as an ordinary differential equation for ℓ depending on the unknown function f. More precisely, starting from (0.4) we find that (0.5) is equivalent to

$$\begin{cases} \dot\ell(t) = \dfrac{2\dot f(t-\ell(t))^2 - \kappa(\ell(t))}{2\dot f(t-\ell(t))^2 + \kappa(\ell(t))} \vee 0, \quad \text{for a.e. } t > 0, \\ \ell(0) = \ell_0. \end{cases} \tag{0.6}$$

As observed above, f is uniquely determined on the interval $[-\ell_0, \ell_0]$ by the initial conditions u_0 and u_1. Therefore, we can solve (0.6) in a maximal interval $[0, s_1]$, where s_1 is the unique point such that $s_1 - \ell(s_1) = \ell_0$. We can now apply formula (0.3) to extend f to the interval $[-\ell_0, s_1+\ell(s_1)]$. Then we can extend the solution ℓ of (0.6) to a larger interval $[0, s_2]$, where s_2 is the only point such that $s_2-\ell(s_2) = s_1+\ell(s_1)$. Arguing recursively, we can find $f\colon [-\ell_0,+\infty) \to \mathbb{R}$ and $\ell\colon [0,+\infty) \to [0,+\infty)$ such that (0.6) is satisfied in $[0,+\infty)$. The three Theorems 3.1, 3.4, and 3.5 consider different assumptions on u_0, u_1, w, and κ, which require different techniques to solve the differential equation (0.6).

These results will be used in a forthcoming paper to study the limit of (a rescaled version of) the solutions, as the speed of external loading tends to zero. In particular we will examine the relationships between these limits and different notions of quasistatic evolution. An example in which the limit is not a solution of a quasistatic problem was given in [26], in the presence of discontinuities of κ.

1. The problem for prescribed debonding front

In this section we make precise the notion of solution of problem (0.1) when the evolution of the debonding front is prescribed. More precisely, we fix $\ell_0 > 0$ and $\ell\colon [0,+\infty) \to [\ell_0,+\infty)$ Lipschitz and such that

$$0 \leq \dot\ell(t) < 1, \text{ for a.e. } t > 0, \tag{1.1a}$$

$$\ell(0) = \ell_0. \tag{1.1b}$$

It will be convenient to introduce the following functions:

$$\varphi(t) := t - \ell(t) \text{ and } \psi(t) := t + \ell(t). \tag{1.2}$$

We observe that φ and ψ are strictly increasing, so we can define

$$\omega\colon [\ell_0,+\infty) \to [-\ell_0,+\infty), \quad \omega(t) := \varphi \circ \psi^{-1}(t). \tag{1.3}$$

Observe that

$$0 \le \omega(t_2) - \omega(t_1) \le t_2 - t_1, \text{ for every } 0 \le t_1 < t_2 \le T. \tag{1.4}$$

We use standard notations for the Sobolev spaces H^1 and H^1_{loc}. Moreover, for every $a \in \mathbb{R}$, we introduce the space

$$\widetilde{H}^1(a,+\infty) := \{u \in H^1_{\mathrm{loc}}(a,+\infty) : u \in H^1(a,b), \text{ for every } b > a\}.$$

We assume that

$$w \in \widetilde{H}^1(0,+\infty). \tag{1.5}$$

As for the initial data we require

$$u_0 \in H^1(0,\ell_0), \quad u_1 \in L^2(0,\ell_0), \tag{1.6a}$$

and the compatibility conditions

$$u_0(0) = w(0), \quad u_0(\ell_0) = 0. \tag{1.6b}$$

We set

$$\Omega := \{(t,x) : t > 0\,,\, 0 < x < \ell(t)\},$$

and

$$\Omega_T := \{(t,x) : 0 < t < T\,,\, 0 < x < \ell(t)\}.$$

We will look for solutions in the space

$$\widetilde{H}^1(\Omega) := \{u \in H^1_{\mathrm{loc}}(\Omega) : u \in H^1(\Omega_T), \text{ for every } T > 0\}.$$

Moreover, we set for $k \ge 0$

$$\widetilde{C}^{k,1}(0,+\infty) := \{f \in C^k([0,+\infty)) : f \in C^{k,1}([0,T]) \text{ for every } T > 0\}$$

and

$$\widetilde{C}^{k,1}(\Omega) := \{u \in C^k(\overline{\Omega}) : u \in C^{k,1}(\overline{\Omega}_T), \text{ for every } T > 0\}.$$

Definition 1.1. *We say that $u \in \widetilde{H}^1(\Omega)$ (resp. in $H^1(\Omega_T)$) is a solution of* (0.1a)–(0.1c) *if $u_{tt} - u_{xx} = 0$ holds in the sense of distributions in Ω (resp. in Ω_T) and the boundary conditions are intended in the sense of traces.*

Given a solution $u \in \widetilde{H}^1(\Omega)$ in the sense of Definition 1.1, we extend u to $(0,+\infty)^2$ (still denoting it by u), by setting $u = 0$ in $(0,+\infty)^2 \setminus \Omega$. Note that this agrees with the interpretation of u as vertical displacement of the film which is still glued to the substrate for $(t,x) \notin \Omega$. For a fixed $T > 0$, we define $Q_T := (0,T)\times(0,\ell(T))$ and we observe that $u \in H^1(Q_T)$ because of the boundary conditions (0.1b)&(0.1c). Further, we need to impose the initial position and velocity of u. While condition in (0.1d) can be formulated in the sense of traces, we have to give a precise meaning to the second condition. Since $H^1((0,T)\times(0,\ell_0)) = H^1(0,T;L^2(0,\ell_0)) \cap L^2(0,T;H^1(0,\ell_0))$, we have $u_t, u_x \in L^2(0,T;L^2(0,\ell_0))$. This implies that $u_t, u_{xx} \in L^2(0,T;H^{-1}(0,\ell_0))$ and, by the wave equation, $u_{tt} \in L^2(0,T;H^{-1}(0,\ell_0))$. Therefore $u_t \in H^1(0,T;H^{-1}(0,\ell_0)) \subset C^0([0,T];H^{-1}(0,\ell_0))$ and we can impose condition (0.1e) as an equality between elements of $H^{-1}(0,\ell_0)$. This discussion shows that the following definition makes sense.

Definition 1.2. *We say that $u \in \widetilde{H}^1(\Omega)$ (resp. $H^1(\Omega_T)$) is a solution of* (0.1) *if Definition 1.1 holds and the initial conditions* (0.1d)&(0.1e) *are satisfied in the sense of $L^2(0,\ell_0)$ and $H^{-1}(0,\ell_0)$, respectively.*

In the following discussion T and u are fixed as above. We consider the change coordinates

$$\begin{cases} \xi = t - x, \\ \eta = t + x, \end{cases} \tag{1.7}$$

which maps the set Ω_T into $\widetilde{\Omega}$. In terms of the new function

$$v(\xi,\eta) := u(\frac{\xi+\eta}{2}, \frac{\eta-\xi}{2}), \tag{1.8}$$

the wave equation (0.1a) (weakly formulated) reads as

$$v_{\eta\xi} = 0 \quad \text{in } \mathcal{D}'(\widetilde{\Omega}). \tag{1.9}$$

This means that for every test function $\alpha \in \mathcal{C}_c^\infty(\widetilde{\Omega})$ we have

$$0 = \langle v_{\eta\xi}, \varphi\rangle = -\int_{\widetilde{\Omega}} v_\eta(\xi,\eta)\, \alpha_\xi(\xi,\eta)\, \mathrm{d}\xi\, \mathrm{d}\eta. \tag{1.10}$$

For every $\xi \in \mathbb{R}$ let

$$\widetilde{\Omega}^\xi := \{\eta \in \mathbb{R} : (\xi,\eta) \in \widetilde{\Omega}\},$$

and, similarly, for every $\eta \in \mathbb{R}$ let

$$\widetilde{\Omega}_\eta := \{\xi \in \mathbb{R} : (\xi,\eta) \in \widetilde{\Omega}\}.$$

Notice that, thanks to (1.1a), $\widetilde{\Omega}^\xi$ and $\widetilde{\Omega}_\eta$ are intervals. Moreover $\widetilde{\Omega}^\xi \neq \emptyset$ if and only if $\xi \in (-\ell_0, T)$ and similarly $\widetilde{\Omega}_\eta \neq \emptyset$ if and only if $\eta \in (0, T+\ell(T))$.

Lemma 1.3. *A function $v \in H^1(\widetilde{\Omega})$ is a solution to* (1.9) *if and only if there exist functions $f \in H^1_{\mathrm{loc}}(-\ell_0, T)$ and $g \in H^1_{\mathrm{loc}}(0, T+\ell(T))$ such that*

$$\int_{-\ell_0}^{T} \dot f(\xi)^2 |\widetilde{\Omega}^\xi|\, \mathrm{d}\xi < +\infty, \tag{1.11a}$$

$$\int_0^{T+\ell(t)} \dot g(\eta)^2 |\widetilde{\Omega}_\eta|\, \mathrm{d}\eta < +\infty, \tag{1.11b}$$

and

$$v(\xi,\eta) = f(\xi) + g(\eta), \qquad \textit{for a.e. } (\xi,\eta) \in \widetilde{\Omega}. \tag{1.12}$$

Proof. Let $v \in H^1(\widetilde{\Omega})$ be a solution to (1.9). Using a standard argument for the slicing of H^1 functions, we deduce from (1.10) that for a.e. $\eta \in (0, T+\ell(T))$ we have $v_\eta(\cdot,\eta) \in L^2(\widetilde{\Omega}_\eta)$ and

$$\int_{\widetilde{\Omega}_\eta} v_\eta(\xi,\eta) \dot\beta(\xi)\, \mathrm{d}\xi = 0, \quad \text{for every } \beta \in C_c^\infty(\widetilde{\Omega}_\eta).$$

This implies that v_η is in $H^1(\widetilde{\Omega}_\eta)$ and its derivative in the sense of distributions vanishes in $\widetilde{\Omega}_\eta$. Therefore for a.e. $\eta \in (0, T+\ell(T))$ there exists $\Phi(\eta) \in \mathbb{R}$ such that

$$v_\eta(\xi,\eta) = \Phi(\eta), \quad \text{for a.e. } \xi \in \widetilde{\Omega}_\eta. \tag{1.13}$$

Let us prove that $\Phi \in L^2_{\mathrm{loc}}(0, T+\ell(T))$. First, by applying the Fubini Theorem to v_η, we deduce that the function Φ belongs to $L^2(\widetilde{\Omega}^\xi)$ for a.e. $\xi \in (-\ell_0, T)$. On the other hand, for every $\eta_0 \in (0, T+\ell(T))$ there exists $\xi_0 \in (-\ell_0, T)$ such that $\eta_0 \in \widetilde{\Omega}^\xi$ for every ξ in a suitable neighbourhood of ξ_0. Together with the previous result this gives $\Phi \in L^2_{\mathrm{loc}}(0, T+\ell(T))$.

Let now g be a primitive of Φ, which clearly belongs to $H^1_{\mathrm{loc}}(0, T+\ell(T))$. By (1.13) and the Fubini Theorem, for a.e. $\xi \in (-\ell_0, T)$ we have $v_\eta(\xi,\eta) = \dot g(\eta)$ for a.e. $\eta \in \widetilde{\Omega}^\xi$; therefore for a.e. $\xi \in (-\ell_0, T)$ there exists $f(\xi) \in \mathbb{R}$ such that $v(\xi,\eta) = f(\xi) + g(\eta)$ for a.e. $\eta \in \widetilde{\Omega}^\xi$. Using again the Fubini Theorem, for a.e. $\eta \in (0, T+\ell(T))$ we obtain $v(\xi,\eta) = f(\xi) + g(\eta)$ for a.e. $\xi \in \widetilde{\Omega}_\eta$. This implies that for a.e. $\eta \in (0, T+\ell(T))$ the function f belongs to $H^1(\widetilde{\Omega}_\eta)$. Arguing as above we deduce that $f \in H^1_{\mathrm{loc}}(-\ell_0, T)$. In conclusion, for every solution v to (1.9), with $v \in H^1(\widetilde{\Omega})$, there exist $f \in H^1_{\mathrm{loc}}(-\ell_0, T)$ and $g \in H^1_{\mathrm{loc}}(0, T+\ell(T))$ such that (1.12) is satisfied.

Moreover, taking the derivative with respect to ξ we find that for a.e. $\eta \in (0, T+\ell(T))$, $v_\xi(\xi,\eta) = \dot{f}(\xi)$ for a.e. $\xi \in \widetilde{\Omega}_\eta$. By the Fubini Theorem

$$\int_{-\ell_0}^{T} \dot{f}(\xi)^2 |\widetilde{\Omega}^\xi| \,\mathrm{d}\xi = \int_{\widetilde{\Omega}} v_\xi(\xi,\eta)^2 \,\mathrm{d}\xi\,\mathrm{d}\eta < +\infty.$$

Similarly we prove that

$$\int_{0}^{T+\ell(T)} \dot{g}(\xi)^2 |\widetilde{\Omega}_\eta| \,\mathrm{d}\xi = \int_{\widetilde{\Omega}} v_\eta(\xi,\eta)^2 \,\mathrm{d}\xi\,\mathrm{d}\eta < +\infty.$$

Conversely, assume that $f \in H^1_{\mathrm{loc}}(-\ell_0, T)$ and $g \in H^1_{\mathrm{loc}}(0, T+\ell(T))$ satisfy (1.11) and define v as in (1.12). Then, by the Fubini Theorem, f and g belong to $H^1(\widetilde{\Omega})$. Moreover, $v \in H^1(\widetilde{\Omega})$ and (1.9) is satisfied. □

In the next proposition we return to the variables (t,x) and use Lemma 1.3 to characterise the solutions of problem (0.1a)–(0.1c) according to Definition 1.1. Notice that the boundary conditions imply a relationship between the functions f and g of the previous lemma, so that the solution can be written using either of them.

In this characterisation we use the functions φ and ψ defined in (1.2). We extend ψ^{-1} to $[0,+\infty)$ by setting $\psi^{-1}(s) := 0$ for $s \in [0, \ell_0)$. Notice that all integrands in (1.14) are nonnegative and recall that $a \vee b = \max\{a,b\}$ and $a \wedge b = \min\{a,b\}$.

Proposition 1.4. *Let $T > 0$ and assume* (1.1) *and* (1.5)*. There exists a weak solution $u \in H^1(\Omega_T)$ to problem* (0.1a)–(0.1c) *(in the sense of Definition 1.1) if and only if there exists a function $f \in H^1_{\mathrm{loc}}(-\ell_0, T+\ell(T))$ with*

$$\int_{-\ell_0}^{T-\ell(T)} \dot{f}(s)^2 (\varphi^{-1}(s) - (s \vee 0)) \,\mathrm{d}s + \int_{T-\ell(T)}^{T} \dot{f}(s)^2 (T - (s \vee 0)) \,\mathrm{d}s < +\infty, \tag{1.14a}$$

$$\int_{0}^{T+\ell(T)} (\dot{w}(s) - \dot{f}(s))^2 ((s \wedge T) - \psi^{-1}(s)) \,\mathrm{d}s < +\infty, \tag{1.14b}$$

whose continuous representative satisfies $f(0) = 0$ and

$$f(t+\ell(t)) = w(t+\ell(t)) + f(t-\ell(t)), \quad \textit{for every } t \in (0,T). \tag{1.15}$$

In this case u is given by

$$u(t,x) = w(t+x) - f(t+x) + f(t-x), \quad \textit{for a.e. } (t,x) \in \Omega_T. \tag{1.16}$$

Proof. Using (1.7), (1.8), and (1.12), we can assert that every weak solution $u \in H^1(\Omega_T)$ of problem (0.1) has the form

$$u(t,x) = f(t-x) + g(t+x), \quad \text{for a.e. } (t,x) \in \Omega_T, \tag{1.17}$$

for some functions $f \in H^1_{\mathrm{loc}}(-\ell_0, T)$ and $g \in H^1_{\mathrm{loc}}(0, T+\ell(t))$ satisfying (1.11). Then, by the boundary condition (0.1b) and by the continuity of f, g, and w in $(0,T)$,

$$u(t,0) = w(t) = f(t) + g(t), \quad \text{for a.e. } t \in (0,T).$$

From now on we consider the consider the continuous representatives of f, g, and w. We observe that $g = w - f$ everywhere in $(0,T)$ and $w - f$ is continuous in $[0,T)$ (indeed, w is continuous in $[0,T]$ because $w \in H^1(0,T)$, while f is continuous in $(-\ell_0, T)$ because $f \in H^1_{\mathrm{loc}}(-\ell_0, T)$).

Therefore we can extend g at zero by continuity. Analogously, f can be extended at T by continuity, so that $w(t) = f(t) + g(t)$ for every t in $[0,T]$. We can also extend f by setting $f = w - g$ in $(T, T + \ell(T))$, so that $f \in H^1_{\mathrm{loc}}(-\ell_0, T+\ell(T))$. In particular, (1.17) reduces to (1.16). Moreover, by the second boundary condition $u(t, \ell(t)) = 0$ we obtain (1.15). By a direct computation of $|\widetilde{\Omega}^\xi|$ and of $|\widetilde{\Omega}_\eta|$, one sees that conditions (1.11) are equivalent to (1.14). The condition $f(0) = 0$ can be obtained adding a suitable constant. □

Remark 1.5. The results obtained up to now hold also in the case $\ell_0 = 0$, provided that $\ell(t) > 0$ for every $t > 0$ and that $w(0) = 0$.

In the remaining part of the section we focus on the case $\ell_0 > 0$. We begin with a Proposition which gives the connection between f and the initial conditions (0.1d)&(0.1e).

Proposition 1.6. *Let $T > 0$ and assume* (1.1), (1.5), *and* (1.6a). *Let $f \in H^1_{\mathrm{loc}}(-\ell_0, T+\ell(T))$ satisfy* (1.14), (1.15), *and $f(0) = 0$, and let u be defined by* (1.16). *Then, u is solution to problem* (0.1) *in $H^1(\Omega_T)$, according to Definition 1.2, if and only if*

$$f(s) = w(s) - \frac{u_0(s)}{2} - \frac{1}{2}\int_0^s u_1(x)\,\mathrm{d}x - w(0) + \frac{u_0(0)}{2}, \quad \textit{for every } s \in [0, \ell_0], \tag{1.18a}$$

$$f(s) = \frac{u_0(-s)}{2} - \frac{1}{2}\int_0^{-s} u_1(x)\,\mathrm{d}x - \frac{u_0(0)}{2}, \qquad \textit{for every } s \in (-\ell_0, 0]. \tag{1.18b}$$

Proof. We already know, by Proposition 1.4, that u is a solution to problem (0.1a)–(0.1c). We compute the time derivative of u using (1.16) and we obtain

$$u_t(t,x) = \dot{w}(t+x) - \dot{f}(t+x) + \dot{f}(t-x), \quad \text{for a.e. } (t,x) \in \Omega_T.$$

Assume that (0.1d)&(0.1e) holds. By (1.16) and (1.24a), taking $(t,x) = (0,s)$, we deduce that

$$u_0(s) = w(s) - f(s) + f(-s), \quad \text{for every } s \in [0, \ell_0), \tag{1.19a}$$

$$u_1(s) = \dot{w}(s) - \dot{f}(s) + \dot{f}(-s), \quad \text{for a.e. } s \in (0, \ell_0), \tag{1.19b}$$

where we have used the continuity property of f and the initial conditions according to Definition 1.2. By adding (1.19b) to the derivative of (1.19a), we find that

$$\dot{f}(s) = \dot{w}(s) - \frac{\dot{u}_0(s) + u_1(s)}{2}, \quad \text{for a.e. } s \in (0, \ell_0). \tag{1.20}$$

Therefore, integrating (1.20), we obtain (1.18). Equality (1.19a) enables us to determine f in the interval $[-\ell_0, 0]$, leading to (1.18b).

Conversely, assume that (1.18) holds. Then (1.19a) follows easily and, taking the derivative of (1.18), we obtain also (1.19b). Finally, (1.19), together with (1.16) and (1.24a), gives (0.1d)&(0.1e) in the sense of Definition 1.2. □

Remark 1.7. Conditions (1.5) and (1.6a), together with (1.18), show that $f \in H^1(-\ell_0, \ell_0)$ and (1.18b) holds for every $s \in [-\ell_0, 0]$.

We are now in a position to give the main result of this section, which gives existence and uniqueness of a solution to problem (0.1), according to Definition 1.2.

Theorem 1.8. *Assume* (1.1), (1.5), *and* (1.6). *Then there is a unique solution $u \in \widetilde{H}^1(\Omega)$ to problem* (0.1), *according to Definition 1.2. Moreover, there is a unique function $f\colon [-\ell_0, +\infty) \to \mathbb{R}$, with $f(0) = 0$ and $f \in \widetilde{H}^1(-\ell_0, +\infty)$, such that* (1.16) *holds.*

Proof. By Propositions 1.4 and 1.6, it is enough to construct a function $f\colon [-\ell_0, +\infty) \to \mathbb{R}$, with $f \in \widetilde{H}^1(-\ell_0, +\infty)$, such that (1.18) holds and

$$f(t+\ell(t)) = w(t+\ell(t)) + f(t-\ell(t)), \tag{1.21}$$

for every $t \in [0, +\infty)$. We use (1.18) and Remark 1.7 to define f in $[-\ell_0, \ell_0]$. To conclude the proof we now have to extend it to $(\ell_0, +\infty)$ in such a way that (1.21) is satisfied.

We set $t_0 := \ell_0$ and $t_1 := \omega^{-1}(t_0)$ and we define f in $(t_0, t_1]$ by

$$f(t) = w(t) + f(\omega(t)), \tag{1.22}$$

for every $t \in (t_0, t_1]$. Since $w,\ f \in H^1(-\ell_0, t_0)$ (see Remark 1.7) and ω is Lipschitz between (t_0, t_1) and $(-\ell_0, \ell_0)$ by (1.4), we have $f \in H^1(t_0, t_1)$. Using the compatibility conditions (1.6b) we deduce from (1.18) and (1.22) that $f(t_0^-) = f(t_0^+)$, hence $f \in H^1(-\ell_0, t_1)$. Moreover, by (1.22), we obtain that (1.21) is satisfied in $[0, \psi^{-1}(t_1))$.

We now define inductively a sequence t_i by setting $t_{i+1} := \omega^{-1}(t_i)$. Let us prove that $t_i \to +\infty$. From the definition of φ and ψ and from the inequality $\ell(t) \geq \ell_0$ we deduce that $\varphi^{-1}(t) \geq t + \ell_0$ and $\psi(t) \geq t + \ell_0$. By the monotonicity of ψ we thus find that

$$\omega^{-1}(t) \geq \psi(t + \ell_0) \geq t + 2\ell_0,$$

which implies $t_{i+1} - t_i \geq 2\ell_0$ and therefore $t_i \to +\infty$.

Assume that for some i the function f has already been defined in $[-\ell_0, t_i]$ so that $f \in H^1(-\ell_0, t_i)$ and (1.21) holds for every $t \in [0, \psi^{-1}(t_i))$. We define f in $[t_i, t_{i+1}]$ by (1.22) for every $t \in [t_i, t_{i+1}]$. With this construction (1.22) holds for every $t \in [t_0, t_i)$, hence (1.21) holds for every $t \in [0, \psi^{-1}(t_{i+1}))$. Since f is continuous at $t_{i-1} \in (-\ell_0, t_i)$, we deduce from (1.22) that f is continuous at t_i, which implies $f \in H^1(-\ell_0, t_{i+1})$.

Since $t_i \to +\infty$, this construction leads to $f \in H^1_{\mathrm{loc}}(-\ell_0, +\infty)$ satisfying (1.21) for every $t \in [0, +\infty)$. Condition (1.18) is obviously satisfied.

This construction shows that the function $f\colon [-\ell_0, +\infty) \to \mathbb{R}$ satisfying (1.18) in $[-\ell_0, \ell_0]$ and (1.21) for every $t \in [0, +\infty)$ is uniquely determined. Thanks to Propositions 1.4 and 1.6 this gives the uniqueness of the solution u. □

Remark 1.9. Theorem 1.8 implies that the solution of problem (0.1) according to Definition 1.2 has a continuous representative which satisfies

$$u(t, x) = w(t{+}x) - f(t{+}x) + f(t{-}x), \quad \text{for every } (t, x) \in \Omega, \tag{1.23}$$

for a suitable function $f \in \widetilde{H}^1(-\ell_0, +\infty)$ such that

$$f(t) = w(t) + f(\omega(t)), \qquad \text{for every } t \geq \ell_0.$$

From now on we shall identify u with its continuous representative. Equality (1.23) implies that, for every $t > 0$, the function $u(t, \cdot)$ belongs to $H^1(0, \ell(t))$. Moreover, for every $t > 0$, the partial derivatives, defined as the limits of the corresponding difference quotients, exist for a.e. $(t, x) \in \Omega$ and satisfy the equalities

$$u_t(t, x) = \dot{w}(t{+}x) - \dot{f}(t{+}x) + \dot{f}(t{-}x), \tag{1.24a}$$

$$u_x(t, x) = \dot{w}(t{+}x) - \dot{f}(t{+}x) - \dot{f}(t{-}x). \tag{1.24b}$$

Therefore, if we set $u = 0$ on $(0, +\infty)^2 \setminus \Omega$, taking into account the boundary condition $u(t, \ell(t)) = 0$, we obtain that $u(t, \cdot) \in H^1(0, +\infty)$ for every $t > 0$. Moreover $t \mapsto u(t, \cdot)$ belongs to $C^0([0, +\infty); H^1(0, +\infty))$, while $t \mapsto u_t(t, \cdot)$ and $t \mapsto u_x(t, \cdot)$ belong to $C^0([0, +\infty); L^2(0, +\infty))$.

Remark 1.10. We denote by ω^k the composition of ω with itself k times. The construction of f in the proof of the previous theorem shows that for every $s \in [\ell_0, +\infty)$ there exists a nonnegative integer n, depending on s and with $n \leq \frac{s+\ell_0}{2\ell_0}$, such that $\omega^n(s) \in [-\ell_0, \ell_0)$ and

$$f(s) = \sum_{k=0}^{n-1} w(\omega^k(s)) + f(\omega^n(s)). \tag{1.25}$$

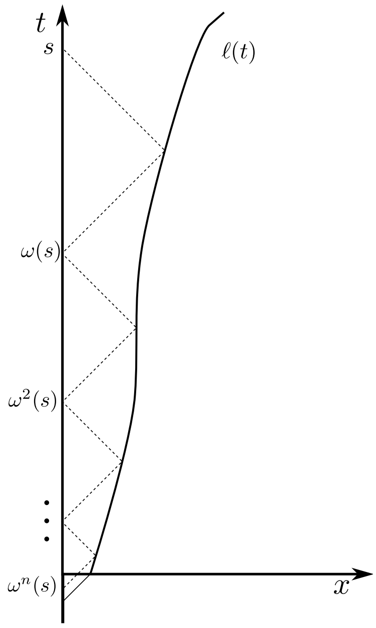

FIGURE 2. Construction of the sequence in Remark 1.10

Since $f(\omega^n(s))$ can be computed using (1.18), this provides an alternative formula of f in $[-\ell_0,+\infty)$, whose geometrical meaning is described in Figure 2.

Remark 1.11 (Causality)**.** In order to prove Theorem 1.8, we needed formula (1.16), which expresses $u(t,x)$ using $w(t{+}x)-f(t{+}x)$. Hence, $u(t,x)$ seems to depend on the value of the prescribed vertical displacement at a time larger than t. However, one can see that $u(t,x)$ can be alternatively written using the data of the problem (the initial conditions, the boundary condition w, and the prescribed debonding front ℓ) evaluated only at times smaller than t.

Indeed, if $t{+}x\le\ell_0$, formula (1.18) shows that $w(t{+}x)-f(t{+}x)$ only depends on the initial conditions. On the other hand, for every (t,x) such that $t{+}x>\ell_0$ there exists $s>0$ such that $t{+}x=s{+}\ell(s)=\psi(s)$, because ψ is invertible. Therefore, using (1.15) we get

$$w(t{+}x)-f(t{+}x)=f(\omega(t{+}x)). \tag{1.26}$$

Notice that $\omega(t{+}x)\le\omega(t{+}\ell(t))=t{-}\ell(t)<t$.

If the vertical displacement w is prescribed only in a time interval $[0,T]$, we can extend it to any $\tilde w\in\widetilde H^1(0,+\infty)$ such that $\tilde w=w$ in $[0,T]$ in order to apply Theorem 1.8. Then, by (1.26), the solution u will not depend on the chosen extension.

Remark 1.12 (Regularity)**.** The regularity of the solution to problem (0.1) depends on the data. If we assume that the debonding front ℓ is of class $C^{1,1}(0,+\infty)$, the loading w belongs to $\widetilde C^{1,1}(0,+\infty)$, and the initial conditions satisfy $u_0\in C^{1,1}([0,\ell_0])$, $u_1\in C^{0,1}([0,\ell_0])$, and

$$u_1(0)=\dot w(0), \tag{1.27a}$$

$$\dot u_0(\ell_0)\dot\ell(0)+u_1(\ell_0)=0, \tag{1.27b}$$

then the solution u is of class $\widetilde C^{1,1}(\Omega)$, as one can see using the construction introduced in the proof of Theorem 1.8. Indeed, the function f constructed in Theorem 1.8 belongs to $C^{1,1}([-\ell_0,\ell_0])$ by (1.6b), (1.18), and (1.27a), while $f\in C^{1,1}([t_i,t_{i+1}])$ by (1.21). We already know that f is continuous at t_i by (1.6b); the continuity of $\dot f$ at t_i is a consequence of (1.18), (1.21), and (1.27b). This implies that $f\in\widetilde C^{1,1}(-\ell_0,+\infty)$ and guarantees the $C^{1,1}$-regularity of the solution u in the whole of Ω. If condition (1.27b) does not hold, we still have $f \in C^{1,1}([-\ell_0, \ell_0])$ and $f \in C^{1,1}([t_i, t_{i+1}])$ for every $i \geq 0$, but the function $\dot{f}$ may be discontinuous at the points t_i; in this case u is only piecewise regular in Ω. Similarly, if condition (1.27a) does not hold, we may have discontinuities of $\dot{f}$ at 0 and, by the "bounce formula" (1.15), at times $\omega^{-1}(0)$, $\omega^{-2}(0)$, ...

We conclude this section with some results on the energy balance for a solution to problem (0.1). For a solution $u \in \widetilde{H}^1(\Omega)$ to problem (0.1) the derivatives $u_x(t,x)$ and $u_t(t,x)$ are defined for every $t > 0$ and almost every $x > 0$ by Remark 1.9. The energy of u is defined for every $t \in [0, +\infty)$ by

$$\mathcal{E}(t) := \frac{1}{2}\int_0^{\ell(t)} u_x(t,x)^2 \,\mathrm{d}x + \frac{1}{2}\int_0^{\ell(t)} u_t(t,x)^2 \,\mathrm{d}x, \tag{1.28}$$

where the first term is the potential energy and the second one is the kinetic energy.

Proposition 1.13. *Let $u \in \widetilde{H}^1(\Omega)$ be a solution to problem* (0.1)*. Then $\mathcal{E}\colon [0,+\infty) \to \mathbb{R}$ is absolutely continuous in $[0,T]$ for every $T > 0$. Moreover we have*

$$\mathcal{E}(t) = \int_{t-\ell(t)}^{t} \dot{f}(s)^2 \,\mathrm{d}s + \int_t^{t+\ell(t)} [\dot{w}(s) - \dot{f}(s)]^2 \,\mathrm{d}s \tag{1.29}$$

for every $t \in [0,+\infty)$, where f is as in Proposition 1.4.

Proof. Using (1.23) and (1.24), we can write

$$\begin{aligned}
&\frac{1}{2}\int_0^{\ell(t)} u_x(t,x)^2 \,\mathrm{d}x + \frac{1}{2}\int_0^{\ell(t)} u_t(t,x)^2 \,\mathrm{d}x \\
&= \frac{1}{2}\int_0^{\ell(t)} \Big[\big(\dot{w}(t{+}x) - \dot{f}(t{+}x) + \dot{f}(t{-}x)\big)^2 + \big(\dot{w}(t{+}x) - \dot{f}(t{+}x) - \dot{f}(t{-}x)\big)^2\Big] \,\mathrm{d}x \\
&= \int_{t-\ell(t)}^{t} \dot{f}(s)^2 \,\mathrm{d}s + \int_t^{t+\ell(t)} [\dot{w}(s) - \dot{f}(s)]^2 \,\mathrm{d}s,
\end{aligned}$$

where in the last equality we have used obvious changes of variables. Since the expression in last line of the last formula is absolutely continuous on $[0,T]$ for every $T > 0$, the proof is complete. □

Proposition 1.14. *Let u and $\mathcal{E}$ be as in Proposition 1.13. Then $\mathcal{E}$ satisfies the energy balance*

$$\mathcal{E}(t) = \mathcal{E}(0) - 2\int_0^t \dot{\ell}(s)\frac{1{-}\dot{\ell}(s)}{1{+}\dot{\ell}(s)}\dot{f}(s{-}\ell(s))^2 \,\mathrm{d}s - \int_0^t [\dot{w}(s) - 2\dot{f}(s)]\dot{w}(s) \,\mathrm{d}s, \tag{1.30}$$

for every $t \in [0,+\infty)$.

The second integral in (1.30) can be interpreted as the work corresponding to the prescribed displacement. The first integral is related to the notion of dynamic energy release rate as explained in Section 2.

Proof. Thanks to (1.29), for a.e. $t \in [0,+\infty)$ we have

$$\dot{\mathcal{E}}(t) = [\dot{w}(t{+}\ell(t)) - \dot{f}(t{+}\ell(t))]^2(1{+}\dot{\ell}(t)) - [\dot{w}(t) - \dot{f}(t)]^2 + \dot{f}(t)^2 - \dot{f}(t{-}\ell(t))^2(1{-}\dot{\ell}(t)). \tag{1.31}$$

The boundary condition $u(t,\ell(t)) = 0$ together with (1.23) gives $w(t{+}\ell(t)) - f(t{+}\ell(t)) + f(t{-}\ell(t)) = 0$ for every $t \geq 0$. By differentiating we obtain

$$\dot{w}(t{+}\ell(t))(1{+}\dot{\ell}(t)) - \dot{f}(t{+}\ell(t))(1{+}\dot{\ell}(t)) + \dot{f}(t{-}\ell(t))(1{-}\dot{\ell}(t)) = 0$$

for a.e. $t \in [0, +\infty)$. From this equality and from (1.31) we obtain, with easy algebraic manipulations,

$$\dot{\mathcal{E}}(t) = -2\dot{\ell}(t)\frac{1-\dot{\ell}(t)}{1+\dot{\ell}(t)}\dot{f}(t-\ell(t))^2 - [\dot{w}(t) - 2\dot{f}(t)]\dot{w}(t),$$

for a.e. $t \in [0, +\infty)$. This proves (1.30), since $\mathcal{E}$ is absolutely continuous on $[0, T]$ for every $T > 0$. $\square$

2. Dynamic energy release rate and Griffith's criterion

In this section we introduce in a rigorous way the dynamic energy release rate in our context; such a notion will be used to formulate Griffith's criterion throughout the paper. To this end we assume that the debonding front $t \mapsto \ell(t)$ satisfies (1.1a). Let u be the solution to (0.1) in Ω, with $w \in \widetilde{H}^1(0, +\infty)$, $u_0 \in H^1(0, \ell_0)$, and $u_1 \in L^2(0, \ell_0)$, satisfying the compatibility conditions (1.6b). (See Remark 1.11.)

2.1. **Dynamic energy release rate.** To define the dynamic energy release rate we fix $\bar{t} > 0$ and consider virtual modifications z and λ of the functions w and ℓ after $\bar{t}$. We then consider the corresponding solution v to problem (0.1) and we study the dependence of its energy on z and λ. More precisely, we consider a function $z \in \widetilde{H}^1(0, +\infty)$ and a function $\lambda\colon [0, +\infty) \to [\ell_0, +\infty)$ satisfying condition (1.1a), with

$$z(t) = w(t) \quad \text{and} \quad \lambda(t) = \ell(t) \quad \text{for every } t \le \bar{t}. \tag{2.1}$$

We consider the problem

$$\begin{cases} v_{tt}(t,x) - v_{xx}(t,x) = 0, & t > 0,\ 0 < x < \lambda(t), \\ v(t,0) = z(t), & t > 0, \\ v(t,\lambda(t)) = 0, & t > 0, \\ v(0,x) = u_0(x), & 0 \le x \le \ell_0, \\ v_t(0,x) = u_1(x), & 0 \le x \le \ell_0, \end{cases} \tag{2.2}$$

whose solution has to be interpreted in the sense of Definition 1.2 and of Remark 1.9. We recall that by Remark 1.11 $v(t,x) = u(t,x)$ for every $(t,x) \in \Omega_{\bar{t}}$. By the previous results, there exists a unique function $g \in \widetilde{H}^1(-\ell_0, +\infty)$ with $g(0) = 0$ such that

$$v(t,x) = z(t+x) - g(t+x) + g(t-x).$$

By Remark 1.11 we have

$$g = f \quad \text{ in } [-\ell_0, \bar{t}]. \tag{2.3}$$

Recalling (1.28), we now define

$$\mathcal{E}(t; \lambda, z) := \frac{1}{2}\int_0^{\lambda(t)} \left[v_x(t,x)^2 + v_t(t,x)^2\right] \mathrm{d}x. \tag{2.4}$$

By Proposition 1.14 we have

$$\dot{\mathcal{E}}(t; \lambda, z) = -2\dot{\lambda}(t)\frac{1-\dot{\lambda}(t)}{1+\dot{\lambda}(t)}\dot{g}(t-\lambda(t))^2 - \dot{z}(t)[\dot{z}(t) - 2\dot{g}(t)] \quad \text{for a.e. } t > 0. \tag{2.5}$$

This is not enough for our purposes, since we want to compute the right derivative $\dot{\mathcal{E}}_r(\bar{t}; \lambda, z)$ at $t = \bar{t}$. This will be done in the next proposition. We recall that, by definition, $\bar{t} \in [0, +\infty)$ is a right Lebesgue point of $\dot{\lambda}$ if there exists $\alpha \in \mathbb{R}$ such that

$$\frac{1}{h}\int_{\bar{t}}^{\bar{t}+h} \left|\dot{\lambda}(t) - \alpha\right| \mathrm{d}t \to 0, \qquad \text{as } h \to 0^+. \tag{2.6}$$

We say that $\bar t$ is a right L^2-Lebesgue point for $\dot z$ and $\dot g$, respectively, if there exist β and γ in $\mathbb{R}$ such that

$$\frac{1}{h}\int_{\bar t}^{\bar t+h} |\dot z(t)-\beta|^2 \,\mathrm{d}t \to 0 \quad \text{ and } \quad \frac{1}{h}\int_{\bar t}^{\bar t+h} |\dot g(t)-\gamma|^2 \,\mathrm{d}t \to 0, \qquad \text{as } h\to 0^+. \tag{2.7}$$

It is easy to see that, in this case, we also have

$$\frac{1}{h}\int_{\bar t}^{\bar t+h} |\dot z(t)(\dot z(t)-2\dot g(t))-\beta(\beta-2\gamma)| \,\mathrm{d}t \to 0, \qquad \text{as } h\to 0^+. \tag{2.8}$$

Proposition 2.1. *Assume* (1.1) *(with* $\ell_0>0$*),* (1.5)*, and* (1.6)*. Then there exists a set* $N\subset[0,+\infty)$*, with measure zero, depending only on* ℓ*,* w*,* u_0*, and* u_1*, such that the following property holds for every* $\bar t\in[0,+\infty)\setminus N$*: if* λ *and* z *are as above, if* v*,* g*, and* $\mathcal{E}(\cdot;\lambda,z)$ *are defined by* (2.2)*&*(2.4)*, if* $\dot\lambda$ *has a right Lebesgue point at* $\bar t$*, and if* $\dot z$ *has a right* L^2*-Lebesgue point at* $\bar t$*, then* $\bar t$ *is a right* L^2*-Lebesgue point for* $\dot g$ *and*

$$\dot{\mathcal{E}}_r(\bar t;\lambda,z)=-2\alpha\frac{1-\alpha}{1+\alpha}\dot f(\bar t-\ell(\bar t))^2-\beta(\beta-2\gamma), \tag{2.9}$$

where α*,* β*, and* γ *are as in* (2.6) *and* (2.7)*.*

Proof. We consider the points $\bar t$ with the following properties:

a) $\dot f$ exists at $\bar t-\ell(\bar t)$ and $\displaystyle\lim_{h\to0^+}\frac{1}{h}\int_{\bar t-\ell(\bar t)}^{\bar t-\ell(\bar t)+h}\Big|\dot f(s)^2-\dot f(\bar t-\ell(\bar t))^2\Big|\,\mathrm{d}s=0$;

b$_1$) if $\bar t\le\ell_0$, $\bar t$ is an L^2-Lebesgue point for $\dot u_0$ and u_1;

b$_2$) if $\bar t\ge\ell_0$, $\bar t$ is a Lebesgue point for $\dot\omega$ and $\omega(\bar t)$ is an L^2-Lebesgue point for $\dot f$.

We call E the set of the points satisfying all the properties above. It is well known that $N:=[0,+\infty)\setminus E$ has measure zero. Let us fix $\bar t\in E$.

Let us prove that $\dot g$ has a right L^2-Lebesgue point at $\bar t$. This is clear if $\bar t<\ell_0$. Assume $\bar t\ge\ell_0$; using (1.22) and (2.3), we have

$$g(t)=z(t)+g(\omega(t))=z(t)+f(\omega(t)),\ \text{ for every } t\in[\ell_0,\bar t+\ell(\bar t)].$$

Then we have

$$\dot g(t)=\dot z(t)+\dot f(\omega(t))\dot\omega(t), \qquad \text{for a.e. } t\in(\ell_0,\bar t+\ell(\bar t)].$$

Since $\dot z$ has a right L^2-Lebesgue point at $\bar t$, it is enough to prove that $\dot f(\omega(t))\dot\omega(t)$ has a right L^2-Lebesgue point at $t=\bar t$. Since $\bar t\in E$, there exist a and b in $\mathbb{R}$ such that

$$\frac{1}{h}\int_{\omega(\bar t)}^{\omega(\bar t)+h}\Big|\dot f(s)-a\Big|^2\,\mathrm{d}s\to0 \quad \text{ and } \quad \frac{1}{h}\int_{\bar t}^{\bar t+h}|\dot\omega(s)-b|^2\,\mathrm{d}s\to0, \tag{2.10}$$

where in the last formula we used the fact that $\dot\omega$ is bounded. We now have

$$\begin{aligned}&\frac{1}{h}\int_{\bar t}^{\bar t+h}\Big|\dot f(\omega(t))\dot\omega(t)-ab\Big|^2\,\mathrm{d}t\\ \le&\frac{2}{h}\int_{\bar t}^{\bar t+h}\Big|\dot f(\omega(t))-a\Big|^2\dot\omega(t)^2\,\mathrm{d}t+\frac{2}{h}\int_{\bar t}^{\bar t+h}a^2\,|\dot\omega(t)-b|^2\,\mathrm{d}t.\end{aligned} \tag{2.11}$$

Using the change of variables $s=\omega(t)$ and the inequalities $0\le\dot\omega\le1$, we deduce from (2.10) that the right hand side in (2.11) tends to zero. This proves that $\bar t$ is a right L^2-Lebesgue point for $\dot g$.

We now prove the formula for the right derivative of the energy at $\bar t \in E$. By (2.5), we have

$$
\begin{aligned}
&\left|\frac{\mathcal{E}(\bar t{+}h;\lambda,z)-\mathcal{E}(\bar t;\lambda,z)}{h}-\left(-2\alpha\frac{1-\alpha}{1+\alpha}\dot f(\bar t{-}\ell(\bar t))^2-\beta(\beta-2\gamma)\right)\right|\\
&\le\left|\frac{2}{h}\int_{\bar t}^{\bar t+h}\left(\dot\lambda(t)\frac{1-\dot\lambda(t)}{1+\dot\lambda(t)}\dot g(t{-}\lambda(t))^2-\alpha\frac{1-\alpha}{1+\alpha}\dot f(\bar t{-}\ell(\bar t))^2\right)\mathrm{d}t\right|\\
&\quad+\left|\frac{1}{h}\int_{\bar t}^{\bar t+h}(\dot z(t)(\dot z(t)-2\dot g(t))-\beta(\beta-2\gamma))\,\mathrm{d}t\right|\\
&\le\frac{2}{h}\int_{\bar t}^{\bar t+h}\dot\lambda(t)\frac{1-\dot\lambda(t)}{1+\dot\lambda(t)}\left|\dot g(t{-}\lambda(t))^2-\dot f(\bar t{-}\ell(\bar t))^2\right|\mathrm{d}t\\
&\quad+\frac{2}{h}\dot f(\bar t{-}\ell(\bar t))^2\int_{\bar t}^{\bar t+h}\left|\dot\lambda(t)\frac{1-\dot\lambda(t)}{1+\dot\lambda(t)}-\alpha\frac{1-\alpha}{1+\alpha}\right|\mathrm{d}t\\
&\quad+\frac{1}{h}\int_{\bar t}^{\bar t+h}|\dot z(t)(\dot z(t)-2\dot g(t))-\beta(\beta-2\gamma)|\,\mathrm{d}t=:I_h^1+I_h^2+I_h^3.
\end{aligned}\tag{2.12}
$$

By (2.3) we can replace $\dot g(\cdot)$ by $\dot f(\cdot)$ in I_h^1. Hence

$$
\begin{aligned}
I_h^1&\le\frac{2}{h}\int_{\bar t}^{\bar t+h}(1-\dot\lambda(t))\left|\dot f(t{-}\lambda(t))^2-\dot f(\bar t{-}\ell(\bar t))^2\right|\mathrm{d}t\\
&\le\frac{2}{h}\int_{\bar t-\ell(\bar t)}^{\bar t-\ell(\bar t)+h}\left|\dot f(s)^2-\dot f(\bar t{-}\ell(\bar t))^2\right|\mathrm{d}s\to 0,\qquad\text{as } h\to 0^+,
\end{aligned}\tag{2.13}
$$

where we have used the change of variables $s=t{-}\lambda(t)$ and the fact that $\ell(\bar t)=\lambda(\bar t)\le\lambda(\bar t+h)$. Moreover, since the function $x\mapsto x\frac{1-x}{1+x}$ is Lipschitz and since $\bar t$ is a right Lebesgue point for $\dot\lambda$, we conclude that

$$
I_h^2\to 0,\qquad\text{as } h\to 0^+.\tag{2.14}
$$

Equations (2.12)–(2.14), together with (2.8), prove (2.9). □

Remark 2.2. The set N introduced in Proposition 2.1 can be chosen in such a way that $N\cap[0,t]$ depends only on the restriction of ℓ and w to $[0,t]$, cf. also (1.3). Moreover, (1.25) shows that (2.7) does not depend on the choice of λ but only on z.

We are now in a position to introduce the notion of dynamic energy release rate, which measures the amount of energy spent during the debonding evolution. It is defined as a sort of partial derivative of $\mathcal{E}$ with respect to the elongation of the debonded region. More precisely, we fix $\bar t>0$, we consider an arbitrary virtual extension λ of $\ell|_{[0,\bar t]}$ with right speed α at $\bar t$ in the sense of (2.6), and we freeze the loading after time $\bar t$ at the level $w(\bar t)$. The derivative of the energy $\mathcal{E}$ with respect to the elongation is obtained by taking the time derivative and dividing it by the velocity α.

Definition 2.3. *For a.e. $\bar t>0$ and every $\alpha\in(0,1)$ the dynamic energy release rate corresponding to the velocity α of the debonding front is defined as*

$$
G_\alpha(\bar t):=-\frac{1}{\alpha}\dot{\mathcal{E}}_r(\bar t;\lambda,\bar z),
$$

where $\lambda\colon[0,+\infty)\to[\ell_0,+\infty)$ is an arbitrary extension of $\ell|_{[0,\bar t]}$ satisfying conditions (1.1a), (2.1), *and* (2.6), *while $\bar z(t)=w(t)$ for every $t\le\bar t$ and $\bar z(t)=w(\bar t)$ for every $t>\bar t$.*

Proposition 2.1 implies that

$$G_\alpha(\bar t) = 2\frac{1-\alpha}{1+\alpha}\dot f(\bar t - \ell(\bar t))^2 \quad \text{for a.e. } \bar t > 0. \tag{2.15}$$

In particular, $G_\alpha(\bar t)$ depends on λ only through α, so the definition is well posed.

Straightforward computations based on (1.24b) show that, when the solution is regular enough so that $u_x(\bar t, \ell(\bar t))$ is well defined for a.e. $\bar t > 0$, the dynamic energy release rate can also be expressed as

$$G_\alpha(\bar t) = \frac{1}{2}(1-\alpha^2)u_x(\bar t, \ell(\bar t))^2. \tag{2.16}$$

This is consistent with the formulas given in [13].

The dynamic energy release rate can be extended to the case $\alpha = 0$, by continuity, as

$$G_0(\bar t) := 2\dot f(\bar t - \ell(\bar t))^2. \tag{2.17}$$

We observe that, by (2.15), $G_\alpha(\bar t)$ is continuous and strictly monotone with respect to α and

$$G_\alpha(\bar t) < G_0(\bar t), \text{ for every } \alpha \in (0,1), \quad G_\alpha(\bar t) \to 0 \text{ for } \alpha \to 1^-, \tag{2.18}$$

for a.e. $\bar t > 0$.

2.2. **Griffith's criterion.** To introduce Griffith's criterion for the debonding model we consider the notion of local toughness of the glue between the substrate and the film. This is a measurable function $\kappa\colon [0,+\infty) \to [c_1, c_2]$, with $0 < c_1 < c_2$, with the following mechanical interpretation: the energy dissipated to debond a segment $[x_1, x_2]$, with $0 \le x_1 < x_2$ is given by

$$\int_{x_1}^{x_2} \kappa(x)\,\mathrm{d}x.$$

This implies that, for every $t > 0$, the energy dissipated in the debonding process in the time interval $[0,t]$ is

$$\int_{\ell_0}^{\ell(t)} \kappa(x)\,\mathrm{d}x.$$

In our model we postulate the following energy-dissipation balance: for every $t > 0$ we have

$$\mathcal{E}(t;\ell,w) + \int_{\ell_0}^{\ell(t)} \kappa(x)\,\mathrm{d}x = \mathcal{E}(0;\ell,w) - \int_0^t \dot w(s)[\dot w(s) - 2\dot f(s)]\,\mathrm{d}s, \tag{2.19}$$

where the last term is the work of the external loading. By (1.30), (2.15), and (2.17) we obtain that (2.19) is equivalent to

$$\int_{\ell_0}^{\ell(t)} \kappa(x)\,\mathrm{d}x = \int_0^t G_{\dot\ell(s)}(s)\dot\ell(s)\,\mathrm{d}s,$$

which, in turn, is equivalent to

$$\kappa(\ell(t))\dot\ell(t) = G_{\dot\ell(t)}(t)\dot\ell(t), \quad \text{for a.e. } t > 0. \tag{2.20}$$

In addition to the energy-dissipation balance we postulate the following maximum dissipation principle, as proposed in [22]: for a.e. $t > 0$

$$\dot\ell(t) = \max\{\alpha \in [0,1) : \kappa(\ell(t))\alpha = G_\alpha(t)\alpha\}. \tag{2.21}$$

This means that the debonding front must move as fast as possible, consistent with the energy-dissipation balance (2.19). We observe that the set $\{\alpha \in [0,1) : \kappa(\ell(t))\alpha = G_\alpha(t)\alpha\}$ has at most one element different from zero, by the strict monotonicity of $\alpha \mapsto G_\alpha(t)$. Therefore the maximum dissipation principle (2.21) simply states that the debonding front must move when this is possible.

Our postulates imply the following properties.

- For a.e. $t > 0$, if $\dot\ell(t) > 0$, then $\kappa(\ell(t)) = G_{\dot\ell(t)}(t)$.
- For a.e. $t > 0$, if $\dot\ell(t) = 0$ then $\kappa(\ell(t)) \geq G_{\dot\ell(t)}(t) = G_0(t)$. Indeed, if the opposite inequality holds, by continuity and by (2.18) then there exists $\alpha > 0$ such that $\kappa(\ell(t)) = G_\alpha(t)$, which contradicts (2.18).

This amounts to the following system, which will be called Griffith's criterion in analogy to the corresponding criterion in Fracture Mechanics: for a.e. $t > 0$

$$\dot\ell(t) \geq 0, \tag{2.22a}$$

$$G_{\dot\ell(t)}(t) \leq \kappa(\ell(t)), \tag{2.22b}$$

$$\Big[G_{\dot\ell(t)}(t) - \kappa(\ell(t))\Big]\,\dot\ell(t) = 0. \tag{2.22c}$$

Conversely, we now show that Griffith's criterion implies both the energy-dissipation balance and the maximum dissipation. Indeed, the third condition in Griffith's criterion implies (2.20) which is equivalent to the energy-dissipation balance. As for the maximum dissipation, (2.20) implies that $\dot\ell(t) \in \{\alpha \in [0,1) : \kappa(\ell(t))\alpha = G_\alpha(t)\alpha\}$. Recalling that this set has at most one positive element, we only need to prove that if $\dot\ell(t) = 0$, then there is no positive $\alpha > 0$ such that $G_\alpha(t) = \kappa(\ell(t))$. This is a consequence of the inequality in (2.18) and of (2.22b).

We conclude this section by proving that Griffith's criterion is equivalent to the following ordinary differential equation:

$$\dot\ell(t) = \frac{2\dot f(t-\ell(t))^2 - \kappa(\ell(t))}{2\dot f(t-\ell(t))^2 + \kappa(\ell(t))} \vee 0 \quad \text{for a.e. } t \in [0,+\infty). \tag{2.23}$$

We recall that $G_0(t) = 2\dot f(t-\ell(t))^2$, by (2.17). If $G_0(t) \leq \kappa(\ell(t))$, then the right hand side of (2.23) is zero. Moreover, by the strict monotonicity of $\alpha \mapsto G_\alpha(t)$ we have $G_\alpha(t) < \kappa(\ell(t))$ for every $\alpha > 0$, hence (2.22c) gives $\dot\ell(t) = 0$. Therefore (2.23) is satisfied in this case. Conversely, if $G_0(t) > \kappa(\ell(t))$, then the right hand side of (2.23) is strictly positive and $\dot\ell(t)$ is the unique $\alpha \in (0,1)$ such that $G_\alpha(t) = \kappa(\ell(t))$. Using (2.15), one sees that (2.23) holds.

3. Evolution of the debonding front

In this section we prove existence and uniqueness of a pair $(u(t,x), \ell(t))$ where u solves problem (0.1) (in the sense of Definitions 1.1 and 1.2) and ℓ satisfies Griffith's criterion (2.22) as formulated in the discussion above. By (2.23) we look for functions $t \mapsto f(t)$, $t \mapsto \ell(t)$ satisfying

$$\begin{cases} \dot\ell(t) = \dfrac{2\dot f(t-\ell(t))^2 - \kappa(\ell(t))}{2\dot f(t-\ell(t))^2 + \kappa(\ell(t))} \vee 0, & \text{for a.e. } t > 0, \\ \ell(0) = \ell_0. \end{cases} \tag{3.1}$$

We recall that, in order to solve system (0.1) in Ω_T for some $T > 0$, it is sufficient to apply Proposition 1.4 and find the related function f defined in $[-\ell_0, T+\ell(T)]$; the solution u is then given by (1.16). The pair (f,ℓ) is found by recursively applying an alternate scheme where the two systems (0.1) and (3.1) are solved separately and iteratively. More precisely, one starts from the definition of f in $[-\ell_0, \ell_0]$, given by Proposition 1.6. Thus (3.1) can be solved in a time interval $[0, s_1]$ such that the right-hand side of the differential equation is defined; this is illustrated in the proof of the theorem below. The debonding front $\ell\colon [0,s_1] \to [\ell_0, +\infty)$ turns out to be as in the assumptions of Section 1, hence f can be defined in a subsequent interval $[\ell_0, t_1]$ thanks to the "bounce formula" (1.15). This alternate scheme is then iterated in order to find the solution in the whole domain.

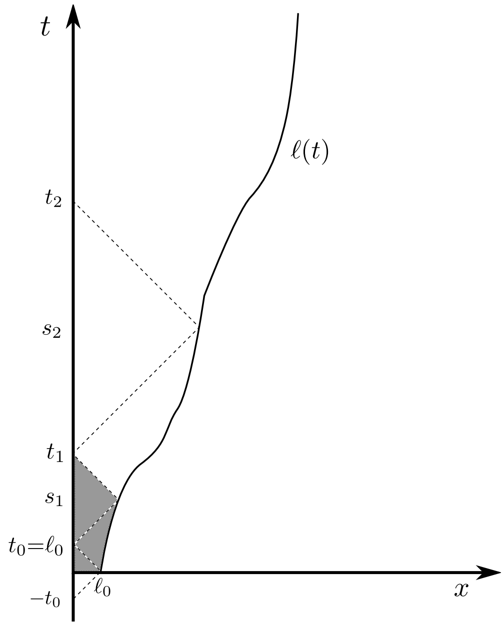

FIGURE 3. Construction of the solution $(\ell(t), u(t,x))$

We are now in a position to state the first existence result under regularity assumptions on the data. The main point is to solve (3.1) in the first time interval $[0, s_1]$.

Theorem 3.1. *Let $u_0 \in C^{1,1}([0, \ell_0])$, $u_1 \in C^{0,1}([0, \ell_0])$, and $w \in \widetilde{C}^{1,1}(0, +\infty)$ be such that* (1.6b) *and* (1.27a) *hold. Assume that the local toughness $\kappa\colon [0, +\infty) \to [c_1, c_2]$ belongs to $\widetilde{C}^{0,1}(0, +\infty)$. Assume in addition that*

$$u_1(\ell_0) + \dot{u}_0(\ell_0) \left\{ \frac{2\left[-\frac{\dot{u}_0(\ell_0)}{2} + \frac{u_1(\ell_0)}{2}\right]^2 - \kappa(\ell_0)}{2\left[-\frac{\dot{u}_0(\ell_0)}{2} + \frac{u_1(\ell_0)}{2}\right]^2 + \kappa(\ell_0)} \vee 0 \right\} = 0. \tag{3.2}$$

Then, there exists a unique pair $(u, \ell) \in \widetilde{H}^1(\Omega) \times \widetilde{C}^{0,1}(0, +\infty)$ satisfying (0.1)*&*(3.1)*. Moreover, one has $(u, \ell) \in \widetilde{C}^{1,1}(\Omega) \times \widetilde{C}^{1,1}(0, +\infty)$ and $0 \le \dot{\ell}(t) < 1$ for every $t \in [0, +\infty)$.*

Proof. We define f in the interval $[-\ell_0, \ell_0]$ by (1.18). Our regularity assumptions and the condition (1.27a) guarantee that $f \in C^{1,1}([-\ell_0, \ell_0])$. Therefore the right hand side of the differential equation in (3.1) is Lipschitz and bounded by a constant strictly smaller than one. We now set $t_0 := \ell_0$. We can thus find a unique solution to (3.1) defined up to the unique time s_1 with $s_1 - \ell(s_1) = t_0$. Notice that $\ell \in C^{1,1}([0, s_1])$. Moreover, by (1.18), (1.27a), and (3.1), $\dot{\ell}(0)$ coincides with the term in curly brackets in (3.2), hence condition (1.27b) is satisfied. With the aid of the "bounce formula" (1.15), we can now find the value of f in the interval $[t_0, t_1]$ where $t_1 = s_1 + \ell(s_1)$. By Remark 1.12, f and $\dot{f}$ are continuous at t_0. By now, the problem is uniquely solved with a pair (u, ℓ), with ℓ defined in $[0, s_1]$ and u defined (through formula (1.23)) in $\overline{\Omega}_{s_1} \cup \{(t, x) : t \in [s_1, t_1],\ 0 \le x \le t_1 - t\}$, that is the grey part in Figure 3. We also notice that $f \in C^{1,1}([t_0, t_1])$, so that we can repeat the previous argument in order to find a unique solution to the differential equation in (3.1), with initial conditions given by $\ell(s_1)$, in the time interval $[s_1, s_2]$, where $s_2 - \ell(s_2) = t_1$. Applying again (1.15) we can define f on the interval $[t_1, t_2]$, where $t_2 = s_2 + \ell(s_2)$. Arguing as in Remark 1.12, we can deduce that $f \in C^{1,1}([t_1, t_2])$ and f, $\dot{f}$ are continuous at t_1. Formula (1.23) leads to a unique solution u of problem (0.1) defined in $\overline{\Omega}_{s_2} \cup \{(t,x) : t \in [s_2, t_2],\ 0 \leq x \leq t_2 - t\}$. By iterating this argument we construct two sequences $\{s_i\}$ and $\{t_i\}$, with $t_i < s_{i+1} < t_{i+1}$ and $t_{i+1} = s_{i+1} + \ell(s_{i+1}) \geq t_i + \ell_0$ and we extend progressively the definitions of ℓ and f to the intervals $[0, s_i]$ and $[-\ell_0, t_i]$ respectively. Since $t_i \to +\infty$, we are able to find a unique solution (u, ℓ) to the coupled problem defined in $\Omega \times [0, +\infty)$. The inequality $0 \leq \dot{\ell}(t) < 1$ follows easily from the equation (3.1). $\square$

Remark 3.2. We make some remarks on the role of conditions (1.27) in Theorem 3.1. (Recall that (1.27b) follows from (3.2).) When they are not satisfied, arguing as in the previous proof we see that $f \in \widetilde{C}^{0,1}(-\ell_0, +\infty)$ and $\ell \in C^{0,1}(0, +\infty)$, and they are only piecewise $C^{1,1}$. Indeed, $\dot{f}$ may have discontinuities at times 0 and ℓ_0 (and their subsequent times $\omega^{-1}(0)$, $\omega^{-1}(\ell_0)$, etc., according to the previous construction). Such discontinuities generate forward and backward shock waves travelling with speed 1 and -1, respectively, and represented by lines $R_1^+ := \{(t,t) : t \in [0, \varphi^{-1}(0)]\}$ and $S_1^- := \{(t, \ell_0 - t) : t \in [0, \ell_0]\}$. At time $t = \varphi^{-1}(0)$, R_1^+ intersects the front of debonding, causing a discontinuity for $\dot{\ell}$; the forward shock wave is then reflected into a backward shock wave $R_2^- := \{(t, \omega^{-1}(0) - t) : t \in [\varphi^{-1}(0), \omega^{-1}(0)]\}$. Analogously, the backward shock wave S_1^- intersects the axis $x = 0$ and it is transformed into a forward shock wave $S_2^+ := \{(t, t - \ell_0) : t \in [\ell_0, \varphi^{-1}(\ell_0)]\}$. By iterating this argument we construct lines where the following Rankine-Hugoniot conditions for the derivatives of u hold:

$$[\![u_x]\!] + [\![u_t]\!] = 0 \text{ on } \bigcup_{i=1}^{\infty} \left(R_{2i-1}^+ \cup S_{2i}^+\right) \text{ and } [\![u_x]\!] - [\![u_t]\!] = 0 \text{ on } \bigcup_{i=0}^{\infty} \left(R_{2i}^- \cup S_{2i-1}^-\right),$$

where $[\![\cdot]\!]$ denotes the difference between the values of the functions across the discontinuity line.

Remark 3.3. Under the assumptions of Theorem 3.1 we have the equality

$$\kappa(\ell_0) = G_{\dot{\ell}(0)}(0). \tag{3.3}$$

Indeed, the formula for $\dot{\ell}(0)$ in the proof implies that, if $\dot{\ell}(0) > 0$, we have

$$u_1(\ell_0) + \dot{u}_0(\ell_0) \frac{2\left[-\frac{\dot{u}_0(\ell_0)}{2} + \frac{u_1(\ell_0)}{2}\right]^2 - \kappa(\ell_0)}{2\left[-\frac{\dot{u}_0(\ell_0)}{2} + \frac{u_1(\ell_0)}{2}\right]^2 + \kappa(\ell_0)} = 0,$$

which implies

$$\kappa(\ell_0) = \frac{1}{2}\left[\dot{u}_0(\ell_0)^2 - u_1(\ell_0)^2\right] = G_{\dot{\ell}(0)}(0),$$

where the last equality follows from (2.16) and (1.27b). If instead $\dot{\ell}(0) = 0$, by analogous computations we find that

$$\kappa(\ell_0) = \frac{1}{2}\dot{u}_0(\ell_0)^2 = G_0(0),$$

which concludes the proof of (3.3).

We now prove existence and uniqueness for the coupled system (0.1)&(3.1) under weaker regularity assumptions on the data. More precisely, we assume

$$u_0 \in C^{0,1}([0, \ell_0]), \quad u_1 \in L^\infty(0, \ell_0), \text{ and } w \in \widetilde{C}^{0,1}(0, +\infty). \tag{3.4}$$

In Theorem 3.4 we assume that the local toughness κ is constant, while in Theorem 3.5 we consider a nonconstant toughness. Since the arguments in the proof are different, we prefer to present both cases separately.

Theorem 3.4. *Let u_0, u_1, and w satisfy* (1.6b) *and* (3.4) *and let the local toughness κ be a positive constant. Then, there exists a unique pair $(u,\ell) \in \widetilde{H}^1(\Omega)\times\widetilde{C}^{0,1}(0,+\infty)$ satisfying* (0.1)*&*(3.1)*. Moreover, one has $u \in \widetilde{C}^{0,1}(\Omega)$ and for every $T > 0$ there exists $L_T < 1$ such that*

$$0 \le \dot{\ell}(t) \le L_T \quad \text{for a.e. } t \in (0,T). \tag{3.5}$$

Proof. We define f in $[-\ell_0,\ell_0]$ by (1.18). Since our regularity assumptions imply only that $f \in C^{0,1}([-\ell_0,\ell_0])$, we now have to justify existence and uniqueness of a local solution to (3.1). This is done by reducing the problem to an autonomous equation, using the fact that κ is constant. Set $z(t) := t-\ell(t)$. Then the Cauchy problem (3.1) reduces to

$$\begin{cases} \dot{z}(t) = F(z), \\ z(0) = -\ell_0, \end{cases} \tag{3.6}$$

where

$$F(z) := 1 - \frac{\left(2\dot{f}(z)^2 - \kappa\right) \vee 0}{2\dot{f}(z)^2 + \kappa}.$$

Since $\dot{f}$ is bounded on $[-\ell_0,\ell_0]$, there exists a constant $c_0 \in (0,1)$ such that $F(z) \ge c_0$ for a.e. $z \in [-\ell_0,\ell_0]$. The standard formula for the solution of autonomous Cauchy problems implies that, setting

$$s_1 = \int_{-\ell_0}^{\ell_0} \frac{\mathrm{d}z}{F(z)},$$

problem (3.6) has a unique solution $z \in C^{0,1}([0,s_1])$ and that this solution satisfies

$$\int_{-\ell_0}^{z(t)} \frac{\mathrm{d}z}{F(z)} = t, \qquad \text{for every } t \in [0,s_1].$$

Notice that s_1 is the unique point such that $s_1-\ell(s_1) = \ell_0$. Since $\dot{\ell}(t) = 1-\dot{z}(t) < 1-c_0$, we have that $\omega(t)$ (see (1.3)) is bi-Lipschitz and thus, by the bounce formula (1.22), $f \in C^{0,1}([t_0,t_1])$, where $t_0 = \ell_0$ and $t_1 = \omega^{-1}(t_0) = s_1 + \ell(s_1)$. Then one can argue iteratively imitating the proof of Theorem 3.1, without the part concerning the continuity of $\dot{f}$. We thus find a unique solution (u,ℓ) on $\Omega\times[0,+\infty)$ which now belongs to $\widetilde{C}^{0,1}(\Omega)\times\widetilde{C}^{0,1}(0,+\infty)$. ☐

We extend this result to a wider class of local toughnesses.

Theorem 3.5. *Let u_0, u_1, and w satisfy* (1.6b) *and* (3.4) *and let $\kappa \in \widetilde{C}^{0,1}(\ell_0,+\infty)$ with $c_1 \le \kappa \le c_2$. Then, there exists a unique pair $(u,\ell) \in \widetilde{H}^1(\Omega)\times\widetilde{C}^{0,1}(0,+\infty)$ satisfying* (0.1)*&*(3.1)*. Moreover, $u \in \widetilde{C}^{0,1}(\Omega)$ and for every $T > 0$ there exists $L_T < 1$ such that* (3.5) *is satisfied.*

Proof. As in the proof of Theorem 3.4, we only have to study (3.1) in a first time interval $[0,s_1]$. Set $z(t) = t-\ell(t)$. We look for solutions to the system

$$\begin{cases} \dot{z}(t) = \dfrac{2\kappa(t-z)}{2\dot{f}(z)^2 + \kappa(t-z)} \wedge 1, \\ z(0) = -\ell_0. \end{cases}$$

Any solution must satisfy $\dot{z} > 0$ a.e. and therefore $t \mapsto z(t)$ is invertible. The equation solved by $t(z)$ is

$$\frac{\mathrm{d}t}{\mathrm{d}z} = \left(\frac{1}{2} + \frac{\dot{f}(z)^2}{\kappa(t-z)}\right) \vee 1 =: \Phi(z,t), \tag{3.7}$$

with initial condition $t(-\ell_0) = 0$. Recalling that $\dot{f}$ is bounded in $[-\ell_0,\ell_0]$, it is easy to prove that Φ is locally Lipschitz in t, uniformly with respect to z.

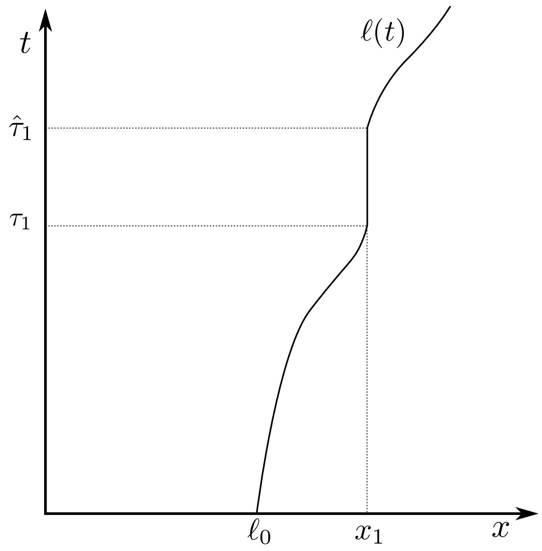

FIGURE 4. A jump of the local toughness at x_1 may lead to a solution lingering at x_1 in a time interval $[\tau_1, \hat{\tau}_1]$

We can thus apply classical results of ordinary differential equations (see, e.g., [19, Theorem 5.3]) and get a unique solution $z \mapsto t(z)$ to (3.7). Then z is found by inverting the function $t(z)$ and finally $\ell(t) = t - z(t)$ is the unique solution to (3.1) up to time $s_1 = t(\ell_0)$, which is the unique point such that $s_1 - \ell(s_1) = \ell_0$. Property (3.5) follows from the differential equation. The proof is concluded by an iterative argument based on the "bounce formula" (1.15) as for the previous theorems. $\square$

Remark 3.6. The previous result can be adapted to the case where κ is piecewise Lipschitz. More precisely, we assume that there exist a finite or infinite sequence $\ell_0 = x_0 < x_1 < x_2 < \dots,$ without accumulation points, and a sequence κ_n of Lipschitz functions on $[x_{n-1}, x_n]$ such that $\kappa(x) = \kappa_n(x)$ for $x \in [x_{n-1}, x_n)$. Using the arguments of Theorem 3.5, we can solve the coupled system for (u, ℓ) with κ replaced by κ_1. It may happen that $\ell(t) < x_1$ for every t. In this case the problem is solved and the discontinuities play no role. Assume, in contrast, that there exists τ_1 such that $\ell(\tau_1) = x_1$. To extend ℓ after this time, we solve the equation in (3.1) with κ replaced by κ_2 and initial condition $\ell(\tau_1) = x_1$ and then we apply the iterative procedure of Theorem 3.5 with κ replaced by κ_2 as long as $\ell(t) < x_2$. If there exists τ_2 such that $\ell(\tau_2) = x_2$, then we iterate this argument using as local toughness κ_3.

Note that the equation may lead to a solution satisfying $\ell(t) = x_1$ for every $t \in [\tau_1, \hat{\tau}_1]$, for some $\hat{\tau}_1 > \tau_1$. This happens if and only if $2\dot{f}(t - \ell(t))^2 - \kappa_2(\ell(t)) \le 0$ for a.e. $t \in [\tau_1, \hat{\tau}_1]$, that is $G_0(t) \le \kappa_2(x_1)$ for a.e. $t \in [\tau_1, \hat{\tau}_1]$.

Particular cases of piecewise constant local toughnesses κ have been studied in detail in [13, 26]. Our analysis proves the uniqueness of the solution obtained in those papers.

Acknowledgements. This material is based on work supported by the European Research Council through the Advanced Grant 290888 "Quasistatic and Dynamic Evolution Problems in Plasticity and Fracture" and by the Italian Ministry of Education, University, and Research through the Project "Calculus of Variations" (PRIN 2010-11). The authors are members of the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM).

References

[1] J. F. Babadjan and M. G. Mora, *Approximation of dynamic and quasi-static evolution problems in elasto-plasticity by cap models*, Quart. Applied Math., 73 (2015), pp. 265–316.

[2] S. Bartels and T. Roubíček, *Thermo-visco-elasticity with rate-independent plasticity in isotropic materials undergoing thermal expansion*, ESAIM Math. Model. Numer. Anal., 45 (2011), pp. 477–504.

[3] E. Bonetti and G. Bonfanti, *Well-posedness results for a model of damage in thermoviscoelastic materials*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 25 (2008), pp. 1187–1208.

[4] E. Bonetti, G. Schimperna, and A. Segatti, *On a doubly nonlinear model for the evolution of damaging in viscoelastic materials*, J. Differential Equations, 218 (2005), pp. 91–116.

[5] B. Bourdin, G. A. Francfort, and J.-J. Marigo, *The variational approach to fracture*, J. Elasticity 91 (2008), no. 1-3, pp. 5–148.

[6] A. Chambolle, *A density result in two-dimensional linearized elasticity, and applications*, Arch. Ration. Mech. Anal., 167 (2003), pp. 211–233.

[7] G. Dal Maso, G. A. Francfort, and R. Toader, *Quasistatic crack growth in nonlinear elasticity*, Arch. Ration. Mech. Anal., 176 (2005), pp. 165–225.

[8] G. Dal Maso and C. J. Larsen, *Existence for wave equations on domains with arbitrary growing cracks*, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl., 22 (2011), pp. 387–408.

[9] G. Dal Maso, C. J. Larsen, and R. Toader, *Existence for constrained dynamic Griffith fracture with a weak maximal dissipation condition*, J. Mech. Phys. Solids (2016), http://dx.doi.org/10.1016/j.jmps.2016.04.033, in press.

[10] G. Dal Maso and G. Lazzaroni, *Quasistatic crack growth in finite elasticity with non-interpenetration*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 27 (2010), pp. 257–290.

[11] G. Dal Maso and I. Lucardesi, *The wave equation on domains with cracks growing on a prescribed path: existence, uniqueness, and continuous dependence on the data*, Preprint SISSA 47/2015/MATE.

[12] G. Dal Maso and R. Toader, *A model for the quasi-static growth of brittle fractures: existence and approximation results*, Arch. Rational Mech. Anal., 162 (2002), pp. 101–135.

[13] P.-E. Dumouchel, J.-J. Marigo, and M. Charlotte, *Dynamic fracture: an example of convergence towards a discontinuous quasistatic solution*, Contin. Mech. Thermodyn., 20 (2008), pp. 1–19.

[14] G. A. Francfort and C. J. Larsen, *Existence and convergence for quasi-static evolution in brittle fracture*, Comm. Pure Appl. Math., 56 (2003), pp. 1465–1500.

[15] G. A. Francfort and J.-J. Marigo, *Revisiting brittle fracture as an energy minimization problem*, J. Mech. Phys. Solids, 46 (1998), pp. 1319–1342.

[16] M. Frémond and B. Nedjar, *Damage, gradient of damage and principle of virtual power*, Internat. J. Solids Structures, 33 (1996), pp. 1083–1103.

[17] L. B. Freund, *Dynamic fracture mechanics*, Cambridge Monographs on Mechanics and Applied Mathematics, Cambridge University Press, Cambridge, 1990.

[18] A. Giacomini, *Ambrosio-Tortorelli approximation of quasi-static evolution of brittle fractures*, Calc. Var. Partial Differential Equations, 22 (2005), pp. 129–172.

[19] J. K. Hale, *Ordinary differential equations*, Robert E. Krieger Publishing Co., Inc., Huntington, N.Y., second ed., 1980.

[20] C. Heinemann and C. Kraus, *Existence of weak solutions for a hyperbolic-parabolic phase field system with mixed boundary conditions on nonsmooth domains*, SIAM J. Math. Anal., 47 (2015), pp. 2044–2073.

[21] ———, *Existence of weak solutions for a PDE system describing phase separation and damage processes including inertial effects*, Discrete Contin. Dyn. Syst., 35 (2015), pp. 2565–2590.

[22] C. J. Larsen, *Models for dynamic fracture based on Griffith's criterion*, in IUTAM Symp. Variational Concepts with Applications to the Mechanics of Materials, ed. K. Hackl, Springer, 2010, pp. 131–140.

[23] ———, *Epsilon-stable quasi-static brittle fracture evolution*, Comm. Pure Appl. Math., 63 (2010), pp. 630–654.

[24] C. J. Larsen, C. Ortner, and E. Süli, *Existence of solutions to a regularized model of dynamic fracture*, Math. Models Methods Appl. Sci., 20 (2010), pp. 1021–1048.

[25] G. Lazzaroni, *Quasistatic crack growth in finite elasticity with Lipschitz data*, Ann. Mat. Pura Appl. (4) 190 (2011), pp. 165–194.

[26] G. Lazzaroni, R. Bargellini, P.-E. Dumouchel, and J.-J. Marigo, *On the role of kinetic energy during unstable propagation in a heterogeneous peeling test*, Int. J. Fract., 175 (2012), pp. 127–150.

[27] G. Lazzaroni, R. Rossi, M. Thomas, and R. Toader, *Rate-independent damage in thermo-viscoelastic materials with inertia*, Preprint SISSA 52/2014/MATE.

[28] G. B. Maggiani and M. G. Mora, *A dynamic evolution model for perfectly plastic plates*, Preprint 2015.

[29] A. Mielke and T. Roubíček, *Rate-independent systems*, vol. 193 of Applied Mathematical Sciences, Springer, New York, 2015.
[30] S. Nicaise and A.-M. Sändig, *Dynamic crack propagation in a 2D elastic body: the out-of-plane case*, J. Math. Anal. Appl., 329 (2007), pp. 1–30.
[31] E. Rocca and R. Rossi, *A degenerating PDE system for phase transitions and damage*, Math. Models Methods Appl. Sci., 24 (2014), pp. 1265–1341.
[32] ———, *"Entropic" solutions to a thermodynamically consistent PDE system for phase transitions and damage*, SIAM J. Math. Anal., 47 (2015), pp. 2519–2586.
[33] R. Rossi and T. Roubíček, *Thermodynamics and analysis of rate-independent adhesive contact at small strains*, Nonlinear Anal., 74 (2011), pp. 3159–3190.
[34] T. Roubíček, *Rate-independent processes in viscous solids at small strains*, Math. Methods Appl. Sci., 32 (2009), pp. 825–862.
[35] ———, *Thermodynamics of rate-independent processes in viscous solids at small strains*, SIAM J. Math. Anal., 42 (2010), pp. 256–297.
[36] ———, *Adhesive contact of visco-elastic bodies and defect measures arising by vanishing viscosity*, SIAM J. Math. Anal., 45 (2013), pp. 101–126.
[37] ———, *Nonlinearly coupled thermo-visco-elasticity*, NoDEA Nonlinear Differential Equations Appl., 20 (2013), pp. 1243–1275.
[38] T. Roubíček and G. Tomassetti, *Thermomechanics of damageable materials under diffusion: modelling and analysis*, Z. Angew. Math. Phys., 66 (2015), pp. 3535–3572.

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