LONG-TERM PLANNING VERSUS SHORT-TERM PLANNING IN THE ASYMPTOTICAL LOCATION PROBLEM

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Abstract. Given the probability measure ν over the given region $\Omega \subset \mathbb{R}^n$, we consider the optimal location of a set Σ composed by n points in Ω in order to minimize the average distance $\Sigma \mapsto \int_\Omega \mathrm{dist}\,(x, \Sigma)\, d\nu$ (the classical optimal facility location problem). The paper compares two strategies to find optimal configurations: the long-term one which consists in placing all n points at once in an optimal position, and the short-term one which consists in placing the points one by one adding at each step at most one point and preserving the configuration built at previous steps. We show that the respective optimization problems exhibit qualitatively different asymptotic behavior as $n \to \infty$, although the optimization costs in both cases have the same asymptotic orders of vanishing.

1. Introduction

Planning an economic activity is in general an extremely complex problem, where a high number of parameters often intervene. In addition, the attitude of the planner has to be taken into account: long-term planners take their decisions through an optimization process over a large time horizon, while short-term planners behave by optimizing day-by-day their strategies. Usually, the first kind of behavior is perceived as more virtuous and efficient, while the second is seen as easier to implement.

In this paper we analyze the long-term and short-term strategies in a very simple model problem, and we give a way to measure the efficiency of the first versus the second. The problem we consider is the so-called *location problem* which can be roughly described as follows. Suppose one has to open a certain number $n \in \mathbb{N}$ of identical facilities (e.g. plants, shops, distribution centers, cinemas etc.) in the given urban region $\Omega \subset \mathbb{R}^d$ which will be modeled by a compact convex set (this convexity assumption is made for simplicity in order to avoid ambiguities between euclidean and geodesic distances). If the density of population in Ω is given by a known Borel probability measure ν, then the simplest way to measure the optimality of the chosen configuration of facilities modeled by a set $\Sigma \subset \Omega$ consisting of at most n points (i.e. $\#\Sigma \leq n$), is clearly to calculate the average distance the people have to cover to reach the nearest facility

$$F(\Sigma) := \int_\Omega \mathrm{dist}\,(x, \Sigma) d\nu(x). \tag{1.1}$$

Hence the owner of the facilities is interested in locating them in such a way as to minimize F. That is why, if he is able to open all the facilities at once, he would choose a configuration $\Sigma = \Sigma_n$ solving the following problem.

Problem 1.1. *Minimize the functional $\Sigma \mapsto F(\Sigma)$ subject to the constraints*

$$\Sigma \subset \Omega,\ \#\Sigma \leq n.$$

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We will further refer to such problem as a long-term planning problem since it models the optimal choice of facility location so as to satisfy the global (long-term) needs of the facility owner. If $\Sigma = \Sigma_n$ solves Problem 1.1, we will denote $l_n := F(\Sigma_n)$ the respective optimal cost. It is worth mentioning that such a problem has been extensively studied (see e.g. [16, 17, 10] for recent surveys on the subject). Nevertheless a lot of interesting and important questions regarding this problem still remain open.

If, however, the owner of the facilities is unable to open all the facilities immediately (say, if he does not have enough financial resources to do that), he will try to open them one by one, trying to minimize the average distance functional F at each step (i.e. when opening each facility), but, of course, taking into consideration the location of facilities already opened at previous steps. Such a short-term optimization strategy amounts to solving the following problem (in the sequel referred to as short-term planning problem).

Problem 1.2. *For each $n \in \mathbb{N}$, $n \geq 1$, find a set $\Sigma = \Sigma'_n$ minimizing the functional $\Sigma \mapsto F(\Sigma)$ subject to the constraints*

$$\Sigma'_{n-1} \subset \Sigma \subset \Omega,\ \#\Sigma \leq n,$$

where $\Sigma'_0 := \emptyset$.

The above model, to the best of our knowledge, can be traced back to [15], where the short-term facility allocation strategy is called a *myopic allocation policy* (see also references therein for similar formulations). In the same paper the authors propose also several other allocation strategies which can be considered as intermediate between the short-term and the long-term ones.

We will further denote $s_n := F(\Sigma'_n)$ the optimal cost of the above short-term optimization problem.

In this paper we discuss the asymptotic behavior of solutions to the above two problems as $n \to \infty$, by studying the weak limits (in a suitable sense) of optimal configurations Σ_n and Σ'_n as well as the optimal costs l_n and s_n. Namely, we will be interested in finding answers to the following questions.

(A) Find the asymptotic order of l_n (resp. s_n) as $n \to \infty$, i.e. find an exponent $\alpha > 0$ such that

$$C_1 n^{-\alpha} \leq l_n \leq C_2 n^{-\alpha} \qquad (\text{resp. } C_1 n^{-\alpha} \leq s_n \leq C_2 n^{-\alpha})$$

for some positive constants C_1 and C_2 and for n sufficiently large. To simplify the notation, in the sequel we will write in this case $l_n \sim n^{-\alpha}$ (resp. $s_n \sim n^{-\alpha}$).

(B) Find precise asymptotic estimates for l_n (resp. s_n), i.e. find $\lim_n n^\alpha l_n$ (resp. $\lim_n n^\alpha s_n$), or just $\liminf$ and $\limsup$, should the limit not exist.

(C) Describe the asymptotic behavior of the minimizers, i.e. find all the weak limits, in a suitable sense, of subsequences of minimizers.

In the pioneering paper [15] some accurate numerical calculations of optimal short-term configurations for the case of the uniform density on a line and on a two-dimensional square have been provided for rather small values of n. Moreover, some deep insights have been formulated regarding the behavior of solutions. Nevertheless, the above questions have not been explicitly addressed nor even formally posed. In this paper we provide rigorous results concerning the nature of the problem which partially confirm the insights of [15], and also go further. In Section 2 we summarize all the known results on the asymptotic behavior of solutions to the long-term optimization Problem 1.1 that we need for the purpose of comparison with the short-term optimization Problem 1.2. In Sections 3 and 4 we show that, although the optimal cost of the short-term optimization Problem 1.2 has the same order of asymptotic expansion as $n \to \infty$ (i.e. the answer to question (A) is the same for both problems), the two problems are qualitatively different even in the simplest one-dimensional case $d = 1$, with $\nu = \mathcal{L}^1$. This case is actually treated in details in Section 4, where we describe explicitly the algorithm that gives the sequence of points composing Σ'_n. To answer question (B) we prove that the above lim inf is different from the corresponding lim sup and strictly greater than 1 and, for question (C), we consider the weak limits in the sense of measures of the distributions of the points and we prove that there is an infinity of cluster points and none of them is the uniform density. Finally, we conclude the paper by some remarks and open questions supported by numerical evidence, as well as the description of a similar problem we feel important for applications.

2. The long-term problem

The asymptotic behavior of the long-term optimal location Problem 1.1 has been intensively studied both using geometric (see e.g. [9] and [10]) and variational methods [3, 11], in the latter case mainly by means of $\Gamma-$convergence tools (see [8] for details on the theory). In order to later make a comparison with the respective properties of the short-term problem, we summarize here the most important properties of this problem, Notice first that

$$l_n \leq Cn^{-1/d}. \tag{2.1}$$

This estimate is straightforward, if one considers a set Σ composed by n points placed on a uniform grid of size approximately equal to $n^{-1/d}$.

We use $\Gamma-$convergence theory to find answers to the questions (A)–(C) posed in the introduction. In order to apply it, we need to work with functionals defined on a common space, which we choose to be the space of all Borel probability measures $\mathcal{P}(\Omega)$. To this aim we identify each set $\Sigma \subset \Omega$ having $\#\Sigma < +\infty$ with the measure $\mu_\Sigma \in \mathcal{P}(\Omega)$ defined by

$$\mu_\Sigma := \frac{1}{\#\Sigma} \sum_{x\in\Sigma} \delta_x.$$

We define now a sequence of functionals on the space $\mathcal{P}(\Omega)$ by setting

$$\mathfrak{F}_n(\mu) := \begin{cases} n^{1/d} F(\Sigma), & \text{if } \mu = \mu_\Sigma,\ \#\Sigma \leq n, \\ +\infty, & \text{otherwise.} \end{cases} \tag{2.2}$$

The coefficient $n^{1/d}$ in the above formula prevents the minimization from degenerating and is chosen according to (2.1). It is straightforward to recognize that minimizing $\mathfrak{F}_n$ is equivalent to solving Problem 1.1 up to the above identification of sets with probability measures.

Here we give the $\Gamma-$convergence result only for the case $\nu = f \cdot \mathcal{L}^d$. This result could easily be generalized to a generic measure ν by inserting in the $\Gamma-$limit only the absolutely continuous part of ν with respect to $\mathcal{L}^d$. The proof of this theorem, for the case when f is a lower semicontinuous function, can be found in [3]. Anyway we state such a result under the more general assumption $f \in L^1(\Omega)$: the proof of this extension can be deduced from the techniques developed by Mosconi and Tilli in [11] for the study of a similar problem (the so-called *irrigation problem*).

Theorem 2.1. *The sequence of functionals* $\{\mathfrak{F}_n\}_n$ Γ*-converges with respect to the weak* convergence of measures to the functional* $\mathfrak{F}_\infty$*:* $\mathcal{P}(\Omega) \to \bar{\mathbb{R}}$ *defined by the formula*

$$\mathfrak{F}_\infty(\mu) := \theta_d \int_\Omega \frac{f}{\rho^{1/d}}\, d\mathcal{L}^d, \quad \text{where } \mu = \rho \cdot \mathcal{L}^d + \mu_{sing},$$

where $\rho \in L^1(\Omega)$*,* μ_{sing} *stands for the singular part of the measure* μ *with respect to* $\mathcal{L}^d$*, and* θ_d *is a constant depending only on the dimension* d *satisfying* $0 < \theta_d < \infty$

and given by

$$\theta_d := \inf\left\{\liminf_n n^{1/d}\int_{[0,1]^d} \operatorname{dist}(x,\Sigma_n)dx \;:\; \Sigma_n\subset[0,1]^d,\ \#\Sigma_n\le n\right\}.$$

The most important consequences of this theorem are summarized in the following corollary and answer the questions we are interested in. For the reader's convenience, we will simply provide a short proof for the optimality of the measure $\bar\mu$ provided by Corollary 2.2 below, where a complete answer to questions (A), (B) and (C) is given.

Corollary 2.2. *The following assertions hold.*

(A) *One has* $l_n \sim n^{-1/d}$.

(B) *More precisely, one has*

$$\lim_n n^{1/d} l_n = \min\left\{\mathfrak{F}_\infty(\mu) \;:\; \mu\in\mathcal{P}(\Omega)\right\} = \theta_d \|f\|_{d/(d+1)} > 0,$$

(C) *Denoting by* $\mu_n := \mu_{\Sigma_n}$ *the measures associated to a sequence of minimizers for Problem 1.1, we have that* $\mu_n \overset{*}{\rightharpoonup} \bar\mu$ *as* $n\to\infty$ *in the weak* sense of measures, where* $\bar\mu$ *is the unique minimizer of* $\mathfrak{F}_\infty$ *and is given by the formula*

$$\bar\mu = c f^{d/(d+1)}\cdot\mathcal{L}^d \text{ with } c := \left(\int_\Omega f^{d/(d+1)}\,d\mathcal{L}^d\right)^{-1}.$$

In particular, if ν *has constant density, i.e.* $\nu = c\cdot\mathcal{L}^d$, *then* $\bar\mu$ *has constant density as well, namely,* $\bar\mu=\nu$.

Proof. This statement comes from well-known properties of $\Gamma-$convergence (i.e. convergence of minima and of minimizers), once we find the unique minimizer of $\mathfrak{F}_\infty$. Due to the strictly decreasing nature of $\mathfrak{F}_\infty$ with respect to ρ on $\{f>0\}$ and to the fact that μ_{sing} and $\rho 1_{\{f=0\}}$ do not affect the value of $\mathfrak{F}_\infty$, it is straightforward that the minimizers should be absolutely continuous and concentrated on $\{f>0\}$. To identify the density of the absolutely continuous part, set $\lambda := f^{d/(d+1)}\cdot\mathcal{L}^d\llcorner\Omega$, and $w := \rho f^{-d/(d+1)}$. With this notation, finding the minimizers to $\mathfrak{F}_\infty$ is equivalent to minimizing the functional $w\mapsto \int_\Omega w^{-1/d}\,d\lambda$ over the set

$$\left\{w\in L^1(\Omega,\lambda),\ w\ge 0,\ \int_\Omega w\,d\lambda = 1\right\}.$$

By convexity of the map $w\mapsto w^{-1/d}$, it immediately follows from Jensen inequality that the minimum of the latter functional is attained at a constant function w. This shows $\bar\mu = cf^{d/(d+1)}\cdot\mathcal{L}^d$ and the computation of the constant c follows from the constraint $\bar\mu\in\mathcal{P}(\Omega)$. At last, the value of $\min\mathfrak{F}_\infty$ is obtained by plugging $\bar\mu$ into the expression for $\mathfrak{F}_\infty$. $\square$

The exact values of the constants θ_d are known in the one-dimensional and two-dimensional cases. Namely, if $\Omega=[0,1]$ and $\nu=\mathcal{L}^1\llcorner\Omega$ it is actually easy to compute explicitly the unique minimizer of Problem 1.1 which is given by the set of n points located at the centers of n equal disjoint intervals forming a partition of Ω. In other words, one has

$$\Sigma_n = \bigcup_{i=1}^n\left\{\frac{2i-1}{2n}\right\},\quad F(\Sigma_n)=\frac{1}{4n},\quad \text{so that}\quad \theta_1=\frac14.$$

In the two-dimensional case when $\Omega=[0,1]^2$ and $\nu=\mathcal{L}^2\llcorner\Omega$ it is known that the configuration of n points placed in centers of regular hexagons, is asymptotically optimal as $n \to \infty$ (see [9] or [10]), which gives possibility to compute explicitly the constant θ_2. Namely one gets

$$\theta_2 = \int_\sigma |x|\,dx = \frac{3\log 3 + 4}{6\sqrt{2}\,3^{3/4}} \approx 0.377$$

where $\sigma \subset \mathbb{R}^2$ stands for the regular hexagon of unit area centered at the origin.

3. The short-term problem

In this section we answer question (A) posed in the Introduction regarding the short-term optimal location Problem 1.2. Namely, we will show that

$$s_n \sim n^{-1/d} \tag{3.1}$$

whenever $\nu \ll \mathcal{L}^d$, similarly to the asymptotic estimate $l_n \sim n^{-1/d}$ proved in Section 2.

Before proving (3.1) we find it important to remark that the values s_n may actually depend not only on n, but also on the chosen sequence of solutions $\{\Sigma'_n\}$ of the short-term Problem 1.2. This is due to the fact that at each minimization step both the position of the next point and the new minimum value may depend on the history, i.e. on the configuration chosen on the previous steps. In other words, the choice of the optimal set at each step may affect the minimal values of all the following steps as the following example shows.

Example 3.1. Consider the one-dimensional situation $d = 1$ with $\Omega := [0, 4]$ and

$$\nu := \left(\frac{1}{2}\cdot 1_{[0,1]} + \frac{1}{4}1_{[2,4]}\right)\cdot \mathcal{L}^1.$$

Then one clearly has that both singletons $\{1\}$ and $\{2\}$ are solutions to the short-term location problem at the first step $n = 1$, and both give the value $s_1 = 5/4$. Now, if one takes $\Sigma'_1 := \{1\}$, then at the second optimization step $n = 2$ we get the unique minimizer $\Sigma'_2 := \{1, 3\}$, which gives the value $s_2 = 1/2$. On the other hand, if at the first step one takes $\Sigma'_1 := \{2\}$, then at the second step one gets the unique minimizer $\Sigma'_2 := \{1/2, 2\}$ which gives a different value $s_2 = 5/8$.

Coming back to proving (3.1), we observe that it is impossible to use the Γ-convergence theory for this purpose. Namely, the constraint $\Sigma'_{n+1} \supset \Sigma'_n$ which is imposed at each minimization step, actually gives raise to a sequence of problems that, once rescaled and expressed in terms of probability measures as in Section 2, involve the functionals $\mathfrak{F}'_n$ given by

$$\mathfrak{F}'_n(\mu) = \begin{cases} \mathfrak{F}_n(\mu), & \text{if either} \quad \mu = \frac{n-1}{n}\mu_{\Sigma'_{n-1}} + \frac{1}{n}\delta_x,\ x \in \Omega, \\ & \qquad\quad \text{or} \quad \mu = \mu_{\Sigma'_{n-1}}, \\ +\infty, & \text{otherwise.} \end{cases}$$

It is therefore not difficult to see that, whenever $\mu_{\Sigma'_n} \overset{*}{\rightharpoonup} \bar\mu$ in the weak* sense of measures as $n \to \infty$, then the $\Gamma-$limit functional $\mathfrak{F}'_\infty$ would be finite only on $\bar\mu$ itself. This means that taking the limit of a sequence μ_{Σ_n} and using the fact that it minimizes $\mathfrak{F}'_\infty$ gives no additional information on the limit itself. Hence it is not possible to find $\bar\mu$ in this way, contrary to the long-term case. We will therefore analyze the short-term Problem 1.2 directly, without using $\Gamma-$convergence tools.

The following assertion is valid.

Theorem 3.2. *For any probability measure ν there exists a constant C_2 such that*

$$s_n \le C_2 n^{-1/d};$$

moreover, if $\nu << \mathcal{L}^d$, there also exists a positive constant C_1 such that

$$C_1 n^{-1/d} \le s_n.$$

The two constants C_1 and C_2 do not depend on the choice of the sequence of solutions to the short-term optimal location Problem 1.2. In particular, (3.1) *holds.*

Proof. Let $\{\Sigma'_n\}$ (resp. $\{\Sigma_n\}$) be a sequence of minimizers for the short-term optimal location Problem 1.2 (resp. long-term optimal location Problem 1.1), where $\nu = f \cdot \mathcal{L}^d$, $f \in L^1(\Omega)$. Of course, the long-term cost is lower than the short-term one, namely,

$$F(\Sigma_n) \leq F(\Sigma'_n).$$

This provides the required estimate from below as a consequence of Corollary 2.2, under the same assumption of such a corollary.

Now we come back to the case of an arbitrary ν and we want to prove an estimate of the form

$$F(\Sigma'_n) \leq Bn^{-1/d}$$

for a suitable constant B independent of the choice of the sequence of solutions to the short-term optimal location Problem 1.2.

Once we have a set Σ'_n, we want to estimate by how much the functional F decreases when we add a point $x_0 \in \Omega$. For the sake of brevity denote $\delta(x) := \operatorname{dist}(x, \Sigma'_n)$. If we set $\Sigma := \Sigma'_n \cup \{x_0\}$, it is clear that for $x \in B(x_0, \delta(x_0)/4)$ we have $\operatorname{dist}(x, \Sigma) < \delta(x_0)/4$, $\delta(x) > \frac{3}{4}\delta(x_0)$ and hence $\operatorname{dist}(x, \Sigma) < \delta(x) - \delta(x_0)/2$. Thus, if we set

$$g(x_0) := \nu\left(B\left(x_0, \frac{\delta(x_0)}{4}\right)\right)\frac{\delta(x_0)}{2},$$

we get

$$F(\Sigma) \leq F(\Sigma'_n) - g(x_0) \quad \text{and} \quad F(\Sigma'_{n+1}) \leq F(\Sigma'_n) - \sup_{x_0 \in \Omega} g(x_0).$$

To estimate $\sup_{x_0 \in \Omega} g(x_0)$ we use the inequalities

$$\begin{aligned}\int_\Omega g(x_0)dx_0 &= \int_\Omega dx_0 \int_\Omega d\nu(x) \frac{\delta(x_0)}{2} 1_{\{|x-x_0|<\delta(x_0)/4\}}(x)\\ &= \int_\Omega d\nu(x) \int_\Omega \frac{\delta(x_0)}{2} 1_{\{|x-x_0|<\delta(x_0)/4\}}(x_0)dx_0.\end{aligned}$$

We may estimate this last quantity by using the fact that, for $x_0 \in B(x, \delta(x)/5)$ the condition $|x - x_0| < \delta(x_0)/4$ is always satisfied as well as the inequality $\delta(x_0) \geq \frac{4}{5}\delta(x)$. This implies

$$\begin{aligned}\int_\Omega g(x_0)dx_0 &\geq \int_\Omega d\nu(x) \int_\Omega \frac{\delta(x_0)}{2} 1_{\{|x-x_0|<\delta(x)/5\}}(x_0)dx_0\\ &\geq \int_\Omega d\nu(x) \frac{4\omega_d}{2 \cdot 5^{d+1}} \delta(x)^{d+1} \geq \frac{2\omega_d}{5^{d+1}} \left(\int_\Omega d\nu(x)\delta(x)\right)^{d+1},\end{aligned}$$

where ω_d stands for the volume of the unit ball in $\mathbb{R}^d$. The last inequality in the above chain is an application of Jensen inequality since ν is a probability measure.

We have therefore obtained

$$\sup_{x_0 \in \Omega} g(x_0) \geq \frac{1}{|\Omega|} \int_\Omega g(x_0)dx_0 \geq C\left(\int_\Omega d\nu(x)\delta(x)\right)^{d+1} = CF(\Sigma'_n)^{d+1},$$

which implies

$$F(\Sigma'_{n+1}) \leq F(\Sigma'_n) - CF(\Sigma'_n)^{d+1},$$

where $C > 0$ depends only on the dimension d of the underlying space. The conclusion follows now from Lemma 3.4 below (minding that the value s_1 is independent of the choice of the sequence of solutions to the short-term optimal location Problem 1.2). □

Remark 3.3. The lower estimate of Theorem 3.2 is valid whenever ν is such that the same estimate holds true for the long-term problem (i.e. if ν is such that the infimum of the $\Gamma-$limit of the sequence $\mathfrak{F}_n$ is positive). This happens, for instance, whenever ν is absolutely continuous with respect to the Lebesgue measure, though it suffices that the absolutely continuous part of ν with respect to the Lebesgue measure be nonvanishing. In fact, it can be quite easily proven that the expression of the $\Gamma-$limit contains only the latter absolutely continuous part.

Lemma 3.4. *Let $\{a_n\}$ be a sequence of nonnegative numbers satisfying $a_{n+1} \leq a_n - Ca_n^{d+1}$ for all $n \in \mathbb{N}$, where $C > 0$. Then there exists a number $B > 0$ (depending only on a_1, C and d), such that $a_n \leq Bn^{-1/d}$ for all $n \in \mathbb{N}$.*

Proof. The proof will be performed by induction, simultaneously with the choice of B.

The step $n = 1$ is satisfied choosing $B \geq a_1$. Now we look for a condition on B such that the following statement is satisfied: if for some $n \in \mathbb{N}$ one has $a_n \leq Bn^{-1/d}$, then $a_{n+1} \leq B(n+1)^{-1/d}$. Since

$$a_{n+1} \leq a_n - Ca_n^{d+1} \leq Bn^{-1/d} - Ca_n^{d+1},$$

then we must impose that

$$Bn^{-1/d} - Ca_n^{d+1} \leq B(n+1)^{-1/d}.$$

Using again the fact that $a_n \leq Bn^{-1/d}$, we get

$$Bn^{-1/d} \leq B(n+1)^{-1/d} + CB^{d+1}n^{-(d+1)/d},$$

or equivalently

$$\frac{\left(1+\frac{1}{n}\right)^{1/d} - 1}{1/n} \leq CB^d \left(1 + \frac{1}{n}\right)^{1/d}.$$

Minding that

$$\frac{\left(1+\frac{1}{n}\right)^{1/d} - 1}{1/n} \nearrow \frac{\mathrm{d}(x^{1/d})}{\mathrm{d}x}|_{x=1} = 1/d,$$

it is sufficient to impose $1/d \leq CB^d\left(1 + 1/n\right)^{1/d}$, or even $1/d \leq 2CB^d$. Therefore, the choice $B := a_1 \vee (1/2dC)^{1/d}$ suffices for the proof to be concluded by induction. □

4. The one-dimensional case with uniform measure

In this section we address questions (B) and (C) posed in the Introduction regarding the short-term optimal location Problem 1.2. In particular, studying question (B), we will show that, even in a very simple one-dimensional situation with ν being the uniform measure on an interval, the ratio s_n/l_n (which is never smaller than one) does not have a limit as $n \to \infty$, and hence, neither does $n^{1/d}s_n$ (although by Theorem 3.2 one has $n^{1/d}s_n \sim 1$), since by Corollary 2.2 $\lim_n n^{1/d}l_n$ exists. Moreover, we will show that in this case even $\liminf_n s_n/l_n > 1$. We further show that in the same situation there are infinitely many limit measures (in the weak* sense) of the sequences of $\mu_{\Sigma'_n}$ as $n \to \infty$, where Σ'_n solves Problem 1.2, and what is more, neither of such limit measures is equal to the unique limit measure of the sequence of solutions to the long-term Problem 1.1.

In this section we restrict ourselves to the case $\Omega = [0, 1]$ and $\nu = \mathcal{L}^1$. We will further identify each set $\Sigma \subset \Omega$ having finite number of points with the partition of Ω by the points of Σ. Namely, if $\Sigma = \{x_1, \ldots, x_k\}$, we will always order the elements of Σ in such a way that

$$x_1 \leq x_2 \leq \ldots \leq x_k,$$

and identify Σ with the partition of Ω given by the intervals $(\Delta_i)_{i=1}^{k+1}$, where

$$\Delta_1 := [0, x_1], \Delta_2 := [x_1, x_2], \ldots, \Delta_k = [x_{k-1}, x_k], \Delta_{k+1} := [x_k, 1].$$

The intervals Δ_1 and Δ_{k+1} will be further called *external intervals*, while all the other intervals of this partition will be called *internal intervals*. We will also identify the same set with the $(k+1)$-dimensional vector whose entries are the lengths of the intervals of the respective partition

$$(|\Delta_1|, \ldots, |\Delta_{k+1}|) = (x_1, x_2 - x_1, x_3 - x_2, \ldots, x_k - x_{k-1}, 1 - x_k),$$

and we write

$$\Sigma \simeq (x_1, x_2 - x_1, x_3 - x_2, \ldots, x_k - x_{k-1}, 1 - x_k).$$

For instance, the set $\{1/3, 2/3\}$ is identified with the respective partition of $[0,1]$, and with the vector $(1/3, 1/3, 1/3)$, i.e. $\{1/3, 2/3\} \simeq (1/3, 1/3, 1/3)$. The intervals $[0, 1/3]$ and $[2/3, 1]$ are external intervals while the interval $[1/3, 2/3]$ is internal.

Proposition 4.1. *Given a $\Sigma_k \subset \Omega$, assume that $\Sigma_k \simeq (\lambda_1, \ldots, \lambda_{k+1})$, and let Σ_{k+1} be a minimizer of F over all sets $\Sigma \subset \Omega$ such that $\Sigma \supset \Sigma_k$ and $\#\Sigma_{k+1} = k+1$. Then $\Sigma_{k+1} = \Sigma_k \cup \{x\}$, while either of the following two conditions hold.*

(1) *Either x is the center of some internal interval Δ_i, so that*

$$\Sigma_{k+1} \simeq (\lambda_1, \ldots, \lambda_{i-1}, \lambda_i/2, \lambda_i/2, \lambda_{i+1}, \ldots, \lambda_{k+1}),$$

while

$$F(\Sigma_{k+1}) = F(\Sigma_k) - \lambda_i^2/8,$$

(2) *or x divides some external interval Δ_1 or Δ_{k+1} with the length ratio $1:2$ closer to the boundary of $[0,1]$, so that*

$$\Sigma_{k+1} \simeq (\lambda_1/3, 2\lambda_1/3, \lambda_2, \ldots, \lambda_{k+1}) \text{ or } \Sigma_{k+1} \simeq (\lambda_1 \ldots, \lambda_k, 2\lambda_{k+1}/3, \lambda_{k+1}/3)$$

while

$$F(\Sigma_{k+1}) = F(\Sigma_k) - \lambda_i^2/3,$$

where $i = 1$ (if $x \in \Delta_1$) or $i = k+1$ (if $x \in \Delta_{k+1}$).

Proof. If $x \in \Delta_i$, $i \notin \{1, k+1\}$ (i.e. Δ_i is internal), then

$$\begin{aligned} F(\Sigma_{k+1}) &= \int_{[0,1]\setminus\Delta_i} \operatorname{dist}(z, \Sigma_k)\, dz + \int_{\Delta_i} \operatorname{dist}(z, \{x_{i-1}, x, x_i\})\, dz \\ &= \int_0^1 \operatorname{dist}(z, \Sigma_k)\, dz - \frac{(x_i - x)(x - x_{i-1})}{2}, \end{aligned}$$

and to conclude, it is enough to maximize the last quadratic function of x. The case when Δ_i is external, say, $i = 1$ (the case $i = k+1$ is completely symmetric), is absolutely analogous, once one notes that

$$\begin{aligned} F(\Sigma_{k+1}) &= \int_{[0,1]\setminus\Delta_1} \operatorname{dist}(z, \Sigma_k)\, dz + \int_{\Delta_i} \operatorname{dist}(z, \{x, x_1\})\, dz \\ &= \int_0^1 \operatorname{dist}(z, \Sigma_k)\, dz - \frac{x_1^2}{2} - \left(\frac{x^2}{2} + \frac{(x_1 - x)^2}{4}\right) \\ &= F(\Sigma_k) - \frac{3x^2 - 2x_1 x + x_1^2}{4}. \end{aligned}$$

Here as well it is sufficient to maximize the quadratic part. $\square$

We will further say that the point $x \in \Delta_i$ is in optimal position, if either Δ_i is an internal interval and x is its center, or Δ_i is an external interval, and x divides it with the length ratio $1:2$ closer to the boundary of $[0,1]$. The above proposition says that whenever $\Sigma_k \subset \Omega$, $\#\Sigma_k = k$ and $\Sigma_{k+1} \subset \Omega$, $\#\Sigma_{k+1} = k+1$ solve Problem 1.2, then $\Sigma_{k+1} = \Sigma_k \cup \{x\}$ with x in optimal position.

We set now

$$\Omega_i := \left[\frac{1}{2\cdot 3^{i+1}}, \frac{1}{2\cdot 3^{i}}\right] \cup \left[1-\frac{1}{2\cdot 3^{i}}, 1-\frac{1}{2\cdot 3^{i+1}}\right], \qquad i\in\mathbb{N},$$

so that, clearly $\{\Omega_i\}_i$ gives a partition of Ω, while $|\Omega_i| = 2/3^{i+1}$. This allows us to formulate the following corollary to the above Proposition 4.1.

Corollary 4.2. *Let $\Sigma_k' \subset \Omega$, $\#\Sigma_k' = k$ be a solution to Problem 1.2. Then, for the corresponding partition of Ω one has that for each interval Δ_i there exists a unique couple of numbers $(j,h)\in\mathbb{N}^2$, such that $|\Delta_i| = 3^{-j}2^{-h}$, while*

(1) *if Δ_i is internal, then $j\ge 1$ and $\Delta_i\subset\Omega_j$.*
(2) *if Δ_i is external (i.e. $i=1$ or $i=k+1$), then $h=1$ and*

$$j = \begin{cases} 0, & k=1,\\ j_0 \text{ or } j_0-1, & \text{otherwise},\end{cases}$$

where $j_0 := \sup\{j : \Omega_j \supset \Delta_m \text{ for some } m=2,\dots,k\}$.

Proof. The proof is easily obtained by induction on k. □

Consider an arbitrary $\Sigma_k' \simeq (\lambda_1,\dots,\lambda_{k+1})$ solving Problem 1.2. Thanks to the corollary 4.2 we will identify each number λ_i corresponding to the internal interval Δ_i with the respective couple $(j,h)\in\mathbb{N}^2$, $j\ge 1$, such that $\lambda_i = 3^{-j}2^{-h}$. We will further identify every external interval Δ_i with the couple $(j+1/2,-1/2)$, where $j\in\mathbb{N}$ is such that $\lambda_i = 3^{-j}2^{-1}$ (the reason for the latter identification will be explained in a moment).

Consider now the set D of couples $(j,h)\in\mathbb{R}^2$, where $(j,h)\in\mathbb{N}^2$, $j\ge 1$, or $j\in\mathbb{N}+1/2$, $h=-1/2$. We have that each interval Δ_i, and each λ_i is identified with a unique point $d_{\lambda_i}\in D$. We introduce the ordering on D according to the following definition.

Definition 4.3. *We denote $(j_2,h_2)\succ(j_1,h_1)$, if*

$$h_2 - h_1 + (j_2-j_1)\log 3/\log 2 < 0.$$

Observe that the above relation well orders the set D. We have now the following simple assertion.

Proposition 4.4. *Let $\Sigma_k'\subset\Omega$, $\#\Sigma_k' = k$ be a solution to Problem 1.2. Let $p,q\in\{1,\dots,k+1\}$, and consider $d_{\lambda_p} = (j_{\lambda_p}, h_{\lambda_p}), d_{\lambda_q} = (j_{\lambda_q}, h_{\lambda_q})\in D$. Consider $\Sigma_{k+1}^p := \Sigma_k'\cup\{x^p\}$ and $\Sigma_{k+1}^q := \Sigma_k'\cup\{x^q\}$, where $x^p\in\Delta_p$ and $x^q\in\Delta_q$ are in optimal positions. If $d_{\lambda_q}\succ d_{\lambda_p}$, then*

$$F(\Sigma_{k+1}^p) > F(\Sigma_{k+1}^q).$$

Proof. One considers separately three cases.

Case 1. Both Δ_p and Δ_q are internal intervals. Then

$$\begin{aligned} F(\Sigma_{k+1}^p) &= F(\Sigma_k') - \lambda_p^2/8 \\ &= F(\Sigma_k') - 3^{-2j_{\lambda_p}}2^{-2h_{\lambda_p}}/8 \\ &> F(\Sigma_k') - 3^{-2j_{\lambda_q}}2^{-2h_{\lambda_q}}/8 = F(\Sigma_{k+1}^q), \end{aligned}$$

because $h_{\lambda_q} - h_{\lambda_p} + (j_{\lambda_q} - j_{\lambda_p})\log 3/\log 2 < 0$ implies $3^{-2j_{\lambda_p}}2^{-2h_{\lambda_p}} < 3^{-2j_{\lambda_q}}2^{-2h_{\lambda_q}}$.

Case 2. One of the intervals (say, Δ_p) is internal, another one (Δ_q) is external. Then, minding that $h_{\lambda_p} = -1/2$, and hence $\lambda_p = 3^{-(j_{\lambda_p}-1/2)}2^{-1} = 3^{-j_{\lambda_p}+1/2}2^{2h_{\lambda_p}}$, we get

$$\begin{aligned} F(\Sigma_{k+1}^p) &= F(\Sigma_k') - \lambda_p^2/3 \\ &= F(\Sigma_k') - 3^{-2j_{\lambda_p}}2^{4h_{\lambda_p}} \\ &> F(\Sigma_k') - 3^{-2j_{\lambda_q}}2^{-2h_{\lambda_q}}/8 = F(\Sigma_{k+1}^q), \end{aligned}$$

because $h_{\lambda_q} - h_{\lambda_p} + (j_{\lambda_q} - j_{\lambda_p}) \log 3/\log 2 < 0$ and $h_{\lambda_p} = -1/2$ implies $3^{-2j_{\lambda_p}} 2^{4h_{\lambda_p}} < 3^{-2j_{\lambda_q}} 2^{-2h_{\lambda_q}}/8$.

CASE 3. Both intervals Δ_p and Δ_q are external, so that $\lambda_p = 3^{-j_{\lambda_p}+1/2} 2^{-1}$ and $\lambda_q = 3^{-j_{\lambda_q}+1/2} 2^{-1}$, while $j_{\lambda_q} < j_{\lambda_p}$. Then

$$\begin{aligned} F(\Sigma^p_{k+1}) &= F(\Sigma'_k) - \lambda^2_{\lambda_p}/3 \\ &= F(\Sigma'_k) - 3^{-2j_{\lambda_p}+1} 2^{-2}/3 \\ &> F(\Sigma'_k) - 3^{-2j_{\lambda_q}+1} 2^{-2}/3 = F(\Sigma^q_{k+1}), \end{aligned}$$ □

As an immediate consequence of Proposition 4.4 we get the following theorem.

Theorem 4.5. *Let $\Sigma'_k \subset \Omega$, $\#\Sigma'_k = k$ be a solution to Problem 1.2. Let $d = (j,h) \in D$ be maximal (with respect to the order $\succ$) among the elements $d_{\lambda_q} \in D$ corresponding to $q \in \{1,\ldots,k+1\}$. If $\Sigma'_{k+1} \subset \Omega$, $\#\Sigma'_{k+1} = k+1$ solves Problem 1.2, then $\Sigma'_{k+1} = \Sigma'_k \cup \{x\}$ where $x \in \Delta_p$ is in optimal position and $p \in \{1,\ldots,k+1\}$ is such that $d = d_{\lambda_p}$ (such an element may be nonunique).*

We are now able to formulate the following statement which says exactly what the sequence $\{\Sigma'_k\}$ of solutions to Problem 1.2 looks like in the particular case we are considering.

Corollary 4.6. *Enumerate D in the order given by the relation $\succ$, so that $D = \{d(i)\}_{i=1}^\infty$, $d(i+1) \succ d(i)$. The sequence of solutions $\Sigma'_k \subset \Omega$, $\#\Sigma'_k = k$ to Problem 1.2 can be described in the following way by induction on $(D,\succ)$. Each $d(i) = (j(i),h(i)) \in D$ gives rise to a part $\mathcal{S}^i$ of the sequence of optimal sets.*

- *$\mathcal{S}^0$ consists of the unique set $\Sigma'_1 \simeq (1/2,1/2)$.*
- *The set $\mathcal{S}^{i+1}$ consists of the sets $\Sigma'_{k+1},\ldots,\Sigma'_h$, where Σ'_k is the last element of $\mathcal{S}^i$,*

  $$\Sigma'_{j+1} = \Sigma'_j \cup \{x_j\}, \qquad j = k,\ldots,h-1,$$

  $h := k + 2^{h(i)} - 1$, $x_j \in \Delta_p \subset \Omega_{j(i)}$ in optimal position, and $p \in \{1,\ldots,j\}$ is an arbitrary index satisfying $d(i) = d_{\lambda_p}$.

Proof. The proof is easily obtained by induction on the well-ordered set $(D,\succ)$. □

Remark 4.7. An easy consequence of what has been proven so far, is that in the case we are considering (that is, when ν is the uniform measure over the interval) the value of s_n *does not* depend on the particular sequence of solutions $\{\Sigma'_n\}$ to short-term Problem 1.2, i.e. it depends only on the index n.

Example 4.8. We consider the first elements of the possible sequences of solutions $\Sigma'_k \subset \Omega$, $\#\Sigma'_k = k$ to Problem 1.2.

(1) $i = 0$, $\mathcal{S}^0 = \{\Sigma'_1\}$, where

$$\Sigma'_1 \simeq (1/2,1/2).$$

(2) $i = 1$, $d(1) := (j(1),h(1)) = (1/2,-1/2) \in D$. From $d(1) = d_{\lambda_p}$ we get $\lambda_p = 3^{-j(1)+1/2} 2^{-1} = 1/2$, which corresponds to the two external intervals of length 1/2. Hence, $\mathcal{S}^1 := \{\Sigma'_2,\Sigma'_3\}$, where

$$\begin{aligned} &\text{either } \Sigma'_2 \simeq (1/6,1/3,1/2) \text{ or } \Sigma'_2 \simeq (1/2,1/3,1/6), \\ &\Sigma'_3 \simeq (1/6,1/3,1/3,1/6). \end{aligned}$$

(3) $i = 2$, $d(2) := (j(2),h(2)) = (1,0) \in D$. From $d(2) = d_{\lambda_p}$ we get $\lambda_p = 3^{-j(2)} 2^{-h(2)} = 1/3$, which corresponds to the two internal intervals of length 1/3, both belonging to $\Omega_{j(2)} = \Omega_1$. Hence, $\mathcal{S}^2 := \{\Sigma'_4,\Sigma'_5\}$, where

$$\begin{aligned} &\text{either } \Sigma'_4 \simeq (1/6,1/6,1/6,1/3,1/6) \text{ or } \Sigma'_4 \simeq (1/6,1/3,1/6,1/6,1/6), \\ &\Sigma'_5 \simeq (1/6,1/6,1/6,1/6,1/6,1/6). \end{aligned}$$

(4) $i = 3$, $d(3) := (j(3), h(3)) = (3/2, -1/2) \in D$. From $d(3) = d_{\lambda_p}$ we get $\lambda_p = 3^{-j(3)+1/2}2^{-1} = 1/6$, which corresponds to the two external intervals of length $1/6$. Hence, $\mathcal{S}^3 := \{\Sigma'_6, \Sigma'_7\}$, where

$$\begin{aligned} \text{either } \Sigma'_4 &\simeq (1/18, 1/9, 1/6, 1/6, 1/6, 1/6, 1/6) \\ \text{or } \Sigma'_4 &\simeq (1/6, 1/6, 1/6, 1/6, 1/6, 1/9, 1/18), \\ \Sigma'_5 &\simeq (1/18, 1/9, 1/6, 1/6, 1/6, 1/6, 1/9, 1/18). \end{aligned}$$

(5) $i = 4$, $d(4) := (j(4), h(4)) = (1, 1) \in D$. From $d(4) = d_{\lambda_p}$ we get $\lambda_p = 3^{-j(4)}2^{-h(4)} = 1/6$, which corresponds to the four internal intervals of length $1/6$, all belonging to $\Omega_{j(4)} = \Omega_1$. Hence, $\mathcal{S}^3 := \{\Sigma'_8, \Sigma'_9, \Sigma'_{10}, \Sigma'_{11}\}$, where

$$\begin{aligned} \text{either } \Sigma'_8 &\simeq (1/18, 1/9, 1/12, 1/12, 1/6, 1/6, 1/6, 1/9, 1/18) \\ \text{or } \Sigma'_8 &\simeq (1/18, 1/9, 1/6, 1/12, 1/12, 1/6, 1/6, 1/9, 1/18) \\ \text{or } \Sigma'_8 &\simeq (1/18, 1/9, 1/6, 1/6, 1/12, 1/12, 1/6, 1/9, 1/18) \\ \text{or } \Sigma'_8 &\simeq (1/18, 1/9, 1/6, 1/6, 1/6, 1/12, 1/12, 1/9, 1/18), \\ \text{either } \Sigma'_9 &\simeq (1/18, 1/9, 1/12, 1/12, 1/12, 1/12, 1/6, 1/6, 1/9, 1/18) \\ \text{or } \Sigma'_9 &\simeq (1/18, 1/9, 1/12, 1/12, 1/6, 1/12, 1/12, 1/6, 1/9, 1/18) \\ \text{or } \Sigma'_9 &\simeq (1/18, 1/9, 1/12, 1/12, 1/6, 1/6, 1/12, 1/12, 1/9, 1/18) \\ \text{or } \Sigma'_9 &\simeq (1/18, 1/9, 1/6, 1/12, 1/12, 1/12, 1/12, 1/6, 1/9, 1/18) \\ \text{or } \Sigma'_9 &\simeq (1/18, 1/9, 1/6, 1/12, 1/12, 1/6, 1/12, 1/12, 1/9, 1/18) \\ \text{or } \Sigma'_9 &\simeq (1/18, 1/9, 1/6, 1/6, 1/12, 1/12, 1/12, 1/12, 1/9, 1/18), \\ &\cdots\cdots \end{aligned}$$

We are now able to claim the following assertions answering questions (B) and (C) for the case we are considering.

Theorem 4.9. *Assume $\nu = \mathcal{L}^1 \llcorner [0, 1]$. Then the bounded sequence $\{ns_n\}$ does not converge as $n \to \infty$. Further, for any sequence $\{\Sigma'_n\}$ of solutions to Problem 1.2, the sequence of probability measures $\mu_{\Sigma'_n}$ has infinitely many limit measures in the weak* sense as $n \to \infty$.*

Proof. Fixed a $j \in \mathbb{N}$, consider the sequence of indices k_n such that Σ'_{k_n} induces a subdivision of the set Ω_j into 2^n equal subintervals while the partition of Ω corresponding to Σ'_{k_n+1} divides further one of the latter subintervals into two equal intervals. This means that the sequence of partitions corresponding to optimal sets Σ'_k, starting from $k = k_n$ and up to $k = k_n + 2^n$, will be obtained by dividing into two equal parts at each step one of the 2^n subintervals in Ω_j. Set now

$$a_n := \int_{[0,1]\setminus\Omega_j} \operatorname{dist}(x, \Sigma'_{k_n})dx \quad \text{and} \quad b_n := \int_{\Omega_j} \operatorname{dist}(x, \Sigma'_{k_n})dx.$$

Fix an arbitrary dyadic number $\lambda \in (0, 1)$ and consider the new sequence of indices $k'_n := k_n + \lambda 2^n$, so that the partition of Ω corresponding to $\Sigma_{k'_n}$ coincides with that corresponding to Σ_{k_n} up to the fact that some of the subintervals in Ω_0, namely, a fraction λ of the total, have been split in two parts. This implies that

$$k'_n l_{k'_n} = k'_n F(\Sigma'_{k'_n}) = (k_n + \lambda 2^n)(a_n + (1 - \lambda/2)b_n),$$

splitting an interval in two equal parts reduces the average distance from its points to Σ by a factor two. Mind that

$$k_n l_{k_n} = k_n F(\Sigma'_{k_n}) = k_n(a_n + b_n),$$

and assume by contradiction that, $\lim_k kl_k$ exists. Then $\lim_n k_n' l_{k_n'} = \lim_n k_n l_{k_n}$, and hence,

$$\begin{aligned}(k_n+\lambda 2^n)\Big(a_n+\Big(1-\frac{\lambda}{2}\Big)b_n\Big)-k_n(a_n+b_n)&\\ &=\lambda\Big(2^n a_n-\frac{1}{2}k_n b_n+\left(1-\frac{\lambda}{2}\right)2^n b_n\Big)\\ &=\lambda\Big(2^n a_n-\frac{1}{2}k_n b_n+\frac{1}{9}\Big(1-\frac{\lambda}{2}\Big)\Big)\to 0\end{aligned}$$

as $n\to\infty$. Therefore, $2^n a_n - k_n b_n/2 \to \lambda/18 - 1/9$ as $n\to\infty$, which means that, taking two different dyadic values of $\lambda\in(0,1)$, the sequence $\{2^n a_n - k_n b_n/2\}$ has two different limits. This contradiction proves the first claim.

To prove the second claim, we first show that the limit measures of the sequence $\{\mu_{\Sigma_l'}\}$ are not unique, where $l\in[k_n, k_n+2^n]$. To this aim, suppose by contradiction that all the subsequences of the above sequence converge in the weak* sense to the same limit measure μ as $n\to\infty$. In particular, this is the case of $\{\mu_{\Sigma_{k_n}'}\}$, which clearly then converges to a uniform measure over Ω_j since the points of Σ_{k_n}' are uniformly distributed over Ω_j. Observing that $k_n\ge 2^n$, and hence

$$1/(1+\lambda)\le k_n/k_n'\le 1,$$

we may assume without loss of generality that up to extracting a subsequence of k_n (not relabeled), the sequence $\{k_n/k_n'\}$ converges to some finite limit. Hence

$$1=\lim_{n\to\infty}\frac{\mu_{\Sigma_{k_n'}'}(\Omega_j)}{\mu_{\Sigma_{k_n}'}(\Omega_j)}=\lim_{n\to\infty}\frac{(1+\lambda)2^n/k_n'}{2^n/k_n}=(1+\lambda)\lim_{n\to\infty}\frac{k_n}{k_n'},$$

which gives $\lim_n k_n/k_n'\neq 1$. But since

$$\mu_{\Sigma_{k_n'}'}(\Omega_j)=\frac{k_n}{k_n'}\mu_{\Sigma_{k_n}'}(\Omega_j)+\frac{\lambda 2^n}{k_n'},$$

then passing to a limit in the above relationship as $n\to\infty$, we get

$$\begin{aligned}\mu(\Omega_j)&=\lim_{n\to\infty}\frac{k_n}{k_n'}\mu(\Omega_j)+\lim_{n\to\infty}\frac{\lambda 2^n}{k_n'}\\ &=\mu(\Omega_j)\lim_{n\to\infty}\frac{k_n}{k_n'}+\lim_{n\to\infty}\left(1-\frac{k_n}{k_n'}\right).\end{aligned}$$

Minding that $\lim_n k_n/k_n'\neq 1$, the above relationship is only possible when $\mu(\Omega_j)=1$, i.e. when the unique limit measure μ is concentrated on Ω_j, which is clearly not the case because

$$\mu(\Omega_{j+1})\ge\frac{\mu(\Omega_j)}{3}. \tag{4.1}$$

To verify the latter inequality, it is enough to notice that

$$\frac{\mu_{\Sigma_{k_n}'}(\Omega_{j+1})}{\mu_{\Sigma_{k_n}'}(\Omega_j)}=\frac{2^h+1}{2^n+1},\ \text{with } (j,n)\succ(j+1,h),$$

so that $h\ge n-\log 3/\log 2$, which gives (4.1) in the limit as $n\to\infty$.

At last, to show that the limit measures of the sequence $\{\mu_{\Sigma_n'}\}$ are infinite, it is enough to vary $j\in\mathbb{N}$. □

In Figure 1 below we provide a graph of s_n/l_n for the case we are considering calculated according to the algorithm given by Theorem 4.5 and Corollary 4.6.

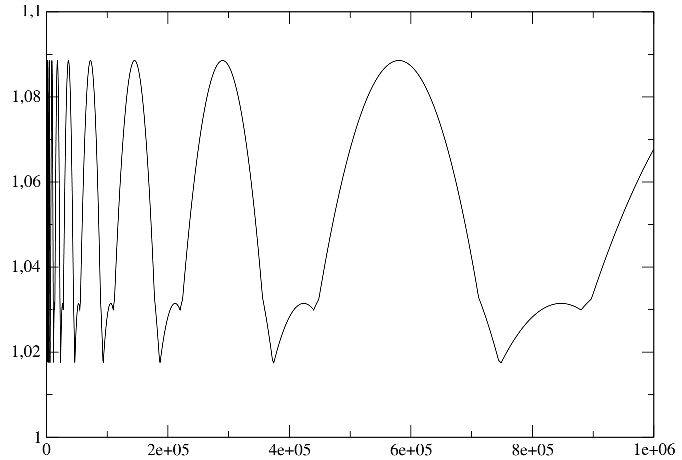

FIGURE 1. Plot of s_n/l_n for the case of uniform density over $[0,1]$.

Theorem 4.10. *No sequence $\{\Sigma'_n\}$ of solutions to Problem 1.2 (with ν uniform measure on $[0,1]$) has the Lebesgue measure $\mathcal{L}^1 \llcorner [0,1]$ as a limit measure of some subsequence of $\mu_{\Sigma'_n}$ in the weak* sense as $n \to \infty$.*

Proof. By using Corollary 4.6, to any k we can associate a set Ω_j in the following way. Let $i \in \mathbb{N}$ be such that $\Sigma'_k \in \mathcal{S}^i$. Consider now the pair $(j(i), h(i)) \in D$ corresponding to the index i, and set $j := j(i)$. The index $j = j[k]$ associated to a set Σ'_k represents the set Ω_j where we last put a point in building Σ'_k and also the set Ω_j which is being divided in a dyadic way by the optimal sequence at step k. Namely, if we look at the points composing Σ'_k, we see the following picture.

- For every $j > \max_{k' \leq k} j[k']$ no point of Σ_k belongs to Ω_j.
- Every Ω_j with $j \leq \max_{k' \leq k} j[k']$ and $j \neq j[k]$ contains a certain number 2^h of points of Σ'_k (precisely we have $h = \max\{h : (j,h) \prec (j[k], k)\}$).
- At last, in $\Omega_{j[k]}$ we cannot exactly predict the number of points of Σ'_k.

Up to choosing a subsequence of n (not relabeled), we can suppose the existence of two distinct indices j_1 and j_2 such that for any n (from the chosen subsequence) we have $j[n] \neq j_1$, and $j[n] \neq j_2$. Extracting a further subsequence of n (again not relabeled), we may assume that $\max_{k' \leq n} j[k'] \geq (j_1 \vee j_2)$, and hence the number of points of Σ'_n in the sets Ω_{j_1} and Ω_{j_2} is a power of two. This means that, if we set $\mu_n := \mu_{\Sigma'_{k_n}}$ and we suppose $\mu_n \overset{*}{\rightharpoonup} \mu = \mathcal{L}^1 \llcorner [0,1]$ in the weak* sense as $n \to \infty$, we get

$$2^{\lambda(n,j_1,j_2)} = \frac{\mu_n(\Omega_{j_1})}{\mu_n(\Omega_{j_2})} \to \frac{\mu(\Omega_{j_1})}{\mu(\Omega_{j_2})} = 3^{j_2 - j_1},$$

where $\lambda(n, j_1, j_2)$ is an integer exponent depending only on n, j_1 and j_2. Yet this is a contradiction as a sequence of powers of two can converge only to 0, $+\infty$ or a power of two. $\square$

The above theorem gives a rigorous formulation of the insight of the authors of [15] who noticed that a short-term strategy in the above one-dimensional situation, compared to other allocation policies, induces “a more uniformly spaced allocation yet not *completely uniform*”.

Corollary 4.11. *One has*

$$1 < \liminf_{n\to\infty} s_n/l_n < \limsup_{n\to\infty} s_n/l_n,$$

if ν is the uniform measure over $[0,1]$.

Proof. The second inequality is just the reformulation of the first claim of Theorem 4.9 minding Corollary 2.2. To prove the first inequality, note that $s_n \geq l_n$, while, should the above $\liminf$ be equal to one, we could build a subsequence $\{\Sigma'_{k_n}\}$ of solutions to short-term Problem 1.2 such that the respective sequence of measures $\{\mu_{\Sigma'_{k_n}}\}$ is asymptotically optimal for the sequence of functionals $\{\mathfrak{F}_{k_n}\}_n$ defined by (2.2). From general results in $\Gamma-$convergence theory we would obtain $\mu_{\Sigma_{k_n}} \stackrel{*}{\rightharpoonup} \bar{\mu}$ in the weak* sense as $n \to \infty$, where $\bar{\mu}$ is the limit measure for the long-term Problem 1.1, i.e. the Lebesgue measure over $[0,1]$, which contradicts Theorem 4.10. □

5. Concluding remarks and open problems

We conclude the paper by a list of remarks and open problems regarding the model studied in this paper, which we consider to be interesting for further study.

Question I. It seems interesting to understand whether the phenomena studied in Section 4 (e.g. non existence of the limit of s_n/l_n, the fact that the respective $\liminf$ is strictly greater than one, non-uniqueness of the limit measures for the short-term location problem etc.) for the case $\nu = \mathcal{L}^1 \llcorner [0,1]$ occur in the case of generic measure ν in one-dimensional case $d = 1$. Further, it seems to be important for applications to characterize completely the limit measures of the short-term problem and to obtain sharp estimates on $\liminf_{n\to\infty} s_n/l_n$ and $\limsup_{n\to\infty} s_n/l_n$ even for the uniform measure ν. In the latter case, the numerical computation provided in this paper (Figure 1) suggests that both limits are close to 1. In practice, one could assume that asymptotically the value of the functional in the short-term problem is almost the same as in the long-term one, which would give a rigorous statement of the idea first suggested in [15].

Question II. In the case of the generic space dimension $d > 1$ we are able to prove only the asymptotic order estimate of the value of the functional in the short-term problem. It seems therefore important to study the above posed problems for generic space dimension even in the simplest case when, say, ν is the uniform measure on the unit ball or on the unit square. Figure 2 shows what one can expect about the behavior of s_n in the case of a uniform density on a unit square (for convenience, the normalized values s_n/l_n^∞ are provided, where $l_n^\infty := n^{-1/d}\theta_2\|f\|_{d/(d+1)}$ stands for the asymptotical value of the long-term minima according to Theorem 2.1), while Figure 3 shows how the distribution of points in this case looks like. Both figures are obtained by a numerical calculation on a uniform 200×200 grid. We just point out that the plot of $n \mapsto s_n/l_n^\infty$ in principle may depend on the sequence Σ'_n, i.e. on the history. It seems important therefore to understand whether $\lim_n s_n/l_n$ depends on the sequence Σ'_n. More results of numerical computations for the short-term optimal location problem can be found on the web page [18].

Question III. At last, it is natural to mention here a similar problem introduced in [5] and sometimes called *irrigation problem* (see [4, 6, 7, 12, 13, 14]) on minimization of the average distance functional, but over compact connected sets of finite length, rather than discrete sets of points (as in this paper). The statement of such a problem is obtained by replacing the constraint on cardinality $\#\Sigma$ of the unknown minimizer Σ in the location problem by the similar constraint on the one-dimensional Hausdorff measure $\mathcal{H}^1(\Sigma)$. The problem is therefore that of finding a minimizer of the cost $F\colon \Sigma \mapsto \int_\Omega \operatorname{dist}(x,\Sigma)\, d\nu(x)$ over all compact and connected

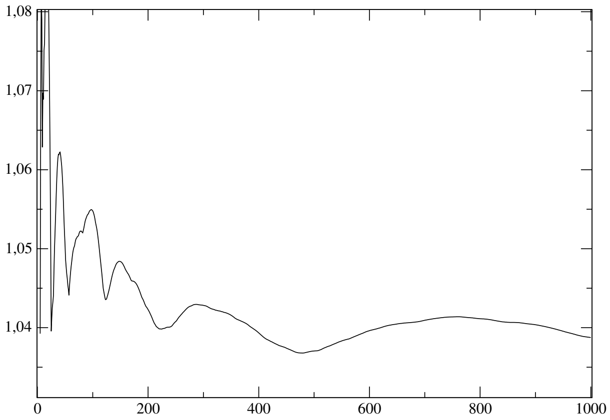

Figure 2. Plot of s_n/l_n^∞ for the case of a uniform density on a unit square

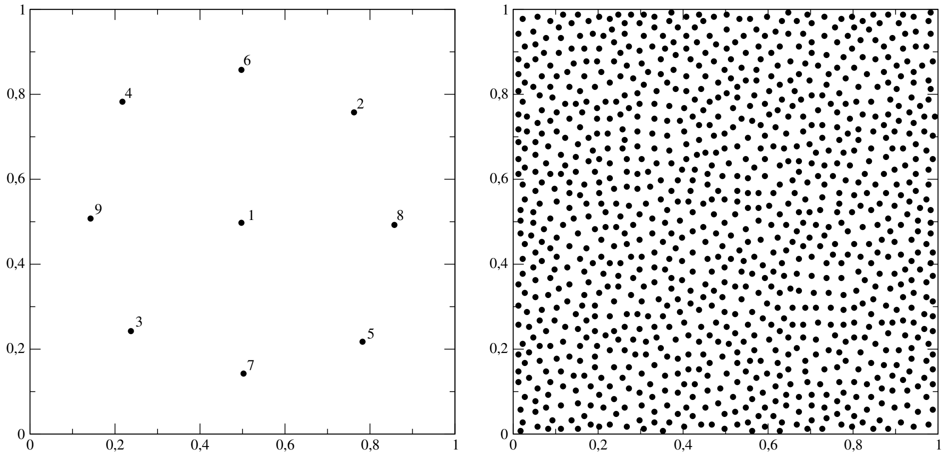

Figure 3. Short-term location for the case of a uniform density on a unit square of (left) the first 9 points (right) 1000 points.

sets $\Sigma \subset \bar{\Omega}$ satisfying $\mathcal{H}^1(\Sigma) \leq l$ with given $l > 0$. This problem, which again can be interpreted as the long-term one, admits also a natural short-term approach. Namely, by letting the parameter l increase, we would like to find an irrigation set Σ which increases continuously with l, in such a way that the functional F decreases as fast as possible. This construction may be made rigorous through a slight modification of the well-known method of *minimizing movements* (see for instance [1, 2] for the presentation of the theory). Namely, fixed an arbitrary time step $\tau > 0$, we minimize the functional $\Sigma \mapsto F(\Sigma) + \mathcal{H}^1(\Sigma \setminus \Sigma_k^\tau)^2/2\tau$ over the set of all connected compact subsets of $\overline{\Omega}$ satisfying the constraint $\Sigma_{k+1}^\tau \supset \Sigma_k^\tau$, where $\Sigma_0^\tau := \emptyset$ (alternatively, one could drop the $\mathcal{H}^1-$penalization term in the above functional and add the additional constraint $\mathcal{H}^1(\Sigma_{k+1}) \leq \mathcal{H}^1(\Sigma_k^\tau) + \tau$ instead). Let then $t \mapsto \Sigma^\tau(t)$ be the piecewise constant map defined by

$$\Sigma^\tau(t) := \Sigma_k^\tau \qquad \text{for } t \in [k\tau, (k+1)\tau[.$$

One should first study whether in this way one can find a well-defined increasing evolution $\Sigma(t)$ as a limit (in the Hausdorff topology) of $\Sigma^\tau(t)$ as $\tau \to 0$. Further, several questions then arise about the behavior of $\Sigma(t)$. For instance, it has been proven in [6, 7, 13, 14] that the solutions of the long-term irrigation problem under suitable conditions on problem data do not contain loops, have a finite number of endpoints and may have only regular tripods (i.e. triple junctions with the branches with infinitesimal angles 120° between each other) as branching points, which are at most finite in number. It is interesting to verify whether the same or similar properties hold also for solutions of the short-term problem $\Sigma(t)$ for all (or for some) $t \in \mathbb{R}^+$, or whether such solutions are just simple curves without branching points. Besides, one is also interested in the asymptotic density of the curve $\Sigma(t)$ (i.e. the weak* limits of the measures $\mathcal{H}^1 \llcorner \Sigma(t)/\mathcal{H}^1(\Sigma(t))$ as $t \to \infty$), which for the long-term irrigation problem has been studied in [11].

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