REGULARITY AND RIGIDITY THEOREMS FOR A CLASS OF ANISOTROPIC NONLOCAL OPERATORS

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Abstract. We consider here operators which are sum of (possibly) fractional derivatives, with (possibly different) order. The main constructive assumption is that the operator is of order 2 in one variable. By constructing an explicit barrier, we prove a Lipschitz estimate which controls the oscillation of the solutions in such direction with respect to the oscillation of the nonlinearity in the same direction.

As a consequence, we obtain a rigidity result that, roughly speaking, states that if the nonlinearity is independent of a coordinate direction, then so is any global solution (provided that the solution does not grow too much at infinity). A Liouville type result then follows as a byproduct.

1. Introduction

Recently a good deal of research has been performed about nonlocal operators of fractional type, also in consideration of their probabilistic interpretation of Lévy processes. In this framework, it is natural to consider the superposition of different nonlocal operators in different directions, possibly with different (fractional) orders, in relation with the nonlocal diffusive equations in anisotropic media.

A first attempt to systematically study these anisotropic fractional operators was given in [5, 8, 6]. In particular, the regularity theory of these anisotropic operators is perhaps harder than expected, it still presents several open questions and some lack of regularity occurs in concrete examples (for instance, solutions of rather simple equations with smooth data and smooth domains may fail in this case to be smooth, see Theorem 1.2 in [8]). Roughly speaking, the lack of regularity may be caused by the combination of the nonlocal properties of the operator and the anisotropic structure of the operator. Namely, first the nonlocal feature may cause the solution to be only Hölder continuous at the boundary; then the anisotropic structure may relate the solution in the interior to values at (or close to) the boundary, and the nonlocal effect can somehow "propagate" the boundary singularity towards the interior, making a smooth interior regularity theory false in this case (see [8] for more details about it).

The goal of this paper is to provide a very simple approach to a Lipschitz-type regularity theory for a family of anisotropic integro-differential operators, obtained by the superposition of different operators in different coordinate directions, and possibly with different order of differentiation.

The main structural assumption that we take is that there is one "special" coordinate (say the last one) in which the operator is local and of second order. In this framework, we will control the derivative of the solution in this variable by uniform and universal quantities, depending on the data of the problem.

More precisely, the mathematical framework in which we work is the following. We denote by $\{e_1, \dots, e_n\}$ the Euclidean base of $\mathbb{R}^n$. Given a point $x \in \mathbb{R}^n$, we use the notation

$$x = (x_1, \dots, x_n) = x_1 e_1 + \dots + x_n e_n,$$

with $x_i \in \mathbb{R}$.

We divide the variables of $\mathbb{R}^n$ into m subgroups of variables, that is we consider $m \in \mathbb{N}$ and $N_1, \dots, N_m \in \mathbb{N}$, with $N_1 + \dots + N_{m-1} = n - 1$. For $i \in \{1, \dots, m\}$, we use the notation $N_i' := N_1 + \dots + N_i$, and we

2010 *Mathematics Subject Classification.* 35R11, 35B53, 35R09.

Key words and phrases. Nonlocal anisotropic integro-differential equations, regularity results.

The authors have been supported by the ERC grant 277749 "EPSILON Elliptic Pde's and Symmetry of Interfaces and Layers for Odd Nonlinearities".

take into account the set of coordinates

$$
\begin{aligned}
&X_1 := (x_1, \dots, x_{N_1}) \in \mathbb{R}^{N_1}\\
&X_2 := (x_{N_1+1}, \dots, x_{N_2'}) \in \mathbb{R}^{N_2}\\
&\vdots\\
&X_i := (x_{N_{i-1}'+1}, \dots, x_{N_i'}) \in \mathbb{R}^{N_i} \qquad (1)\\
&\vdots\\
&X_{m-1} := (x_{N_{m-2}'+1}, \dots, x_{N_{m-1}'}) \in \mathbb{R}^{N_{m-1}}\\
\text{and}\quad &X_m := x_n.
\end{aligned}
$$

Given $i \in \{1, \dots, m-1\}$ and $s_i \in (0, 1]$ we will consider the (possibly fractional) s_i-Laplacian in the ith set of coordinates X_i. For this scope, given $y = (y_1, \dots, y_{N_i}) \in \mathbb{R}^{N_i}$ it is useful to consider the increment induced by y with respect to the ith set of coordinates in $\mathbb{R}^n$, that is one defines

$$
y^{(i)} := y_1 e_{N_{i-1}'+1} + \cdots + y_{N_i} e_{N_i'} \in \mathbb{R}^n. \tag{2}
$$

With this notation, one can define the N_i-dimensional (possibly fractional) s_i-Laplacian in the ith set of coordinates X_i, for a (smooth) function $u : \mathbb{R}^n \to \mathbb{R}$ by

$$
(-\Delta_{X_i})^{s_i} u(x) := \begin{cases} -\partial^2_{x_{N_{i-1}'+1}} u(x) - \ \dots \ - \partial^2_{x_{N_i'}} u(x) & \text{if } s_i = 1, \\ c_{N_i,s_i} \displaystyle\int_{\mathbb{R}^{N_i}} \frac{2u(x) - u(x + y^{(i)}) - u(x - y^{(i)})}{|y^{(i)}|^{N_i+2s_i}} \, dy^{(i)} & \text{if } s_i \in (0, 1), \end{cases} \tag{3}
$$

The quantity c_{N_i,s_i} in (3) is just a positive normalization constant, whose explicit[1] value for $N \in \mathbb{N}$ and $s \in (0, 1)$ is taken to be

$$
c_{N,s} := \frac{2^{2s-1}\,\Gamma(s + \frac{N}{2})}{\pi^{\frac{N}{2}}\,|\Gamma(-s)|}, \tag{4}
$$

where Γ is the Euler's Gamma Function. We refer to [4, 9, 3] and to the references therein for further motivations about fractional operators.

In this paper we consider a pseudo-differential operator, which is the sum of (possibly) fractional Laplacians in the different coordinate directions X_i, with $i \in \{1, \dots, m-1\}$, plus a local second derivative in the direction x_n. The operators involved may have different orders and they may be multiplied by possibly different coefficients: that is, given $a_1, \dots, a_{m-1} \geqslant 0$ and $a > 0$, we define

$$
\begin{aligned}
L &:= \sum_{i=1}^{m-1} a_i (-\Delta_{X_i})^{s_i} - a\partial^2_{x_n} \\
&= \sum_{i=1}^{m} a_i (-\Delta_{X_i})^{s_i},
\end{aligned} \tag{5}
$$

where in the latter identity we used the convention that $a_m := a$, $s_1, \dots, s_{m-1} \in (0, 1]$ and $s_m := 1$.

Given the operator in (5), we stress that a very important structural difference with respect to the classical local case is that fractional objects are in general not reduced to the sum of their directional components[2]

[1]We wrote the value of $c_{N,s}$ as in (4) to be consistent with the literature, see e.g. notation of [2]. Of course, such value can be equivalently written in other forms, according to the different tastes. The explicit value of the normalization constant in (4) plays no major role in this paper, but it is useful for consistency properties as $s_i \to 1$.

[2]That is, if $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ the following formulas are false, unless $s = 1$:

$$
\begin{aligned}
&(-\partial^2_{x_1})^s + \cdots + (-\partial^2_{x_n})^s = (-\Delta_x)^s \\
\text{and}\quad &(-\Delta_{(x_1,\dots,x_N)})^s + (-\Delta_{(x_{N+1},\dots,x_{N+K})})^s = (-\Delta_{(x_1,\dots,x_{N+K})})^s.
\end{aligned}
$$

The main result that we prove in this paper is a Lipschitz regularity theory in the last coordinate variable that extends the one of [1] (which was obtained in the classical setting of local operators). To this goal, we denote by B_r^N the open ball of $\mathbb{R}^N$ centered at the origin and with radius R. Also, given $d_1, \dots, d_m > 0$, we set $d := (d_1, \dots, d_m)$ and

$$Q_d := B_{d_1}^{N_1} \times \cdots \times B_{d_{m-1}}^{N_{m-1}} \times (-d_m, d_m) = \prod_{i=1}^{m} B_{d_i}^{N_i},$$

where in the latter identity we used the convention that $N_m := 1$.

Also, given $\kappa > 0$ we denote by $Q_{d,\kappa}$ the dilation of factor κ in the last coordinate (leaving the others fixed), that is

$$Q_{d,\kappa} := B_{d_1}^{N_1} \times \cdots \times B_{d_{m-1}}^{N_{m-1}} \times (-\kappa d_m, \kappa d_m).$$

Of course, by construction $Q_{d,1} = Q_d$. In accordance with the constant fixed in (3), it is also convenient to introduce the following notation[3] for a suitable universal quantity, for any $i \in \{1, \dots, m\}$:

$$\eta_i := \frac{\Gamma\left(\frac{N_i}{2}\right)}{2^{2s_i}\,\Gamma(s_i+1)\,\Gamma\left(s_i + \frac{N_i}{2}\right)}. \tag{6}$$

With this notation, we have the following result:

Theorem 1.1. *Let $f : Q_{d,2} \to \mathbb{R}$ and $u : \mathbb{R}^n \to \mathbb{R}$ be a solution of $Lu = f$ in $Q_{d,2}$. Then, for any $t \in (-d_m, d_m)$,*

$$\frac{|u(te_n) - u(-te_n)|}{|t|} \leqslant \frac{d_m\,\mathfrak{S}}{a} + \frac{\tilde{C}\,d_m\,\|u\|_{L^\infty(\mathbb{R}^n)}}{\min\limits_{i\in\{1,\dots,m\}} (\eta_i d_i^{2s_i})}, \tag{7}$$

where

$$\mathfrak{S} := \sup_{(x,t)\in Q_d\times(0,d_m)} |f(x+te_n) - f(x-te_n)|,$$

$$\text{and} \quad \tilde{C} := \frac{2(a_1 + \cdots + a_m)}{a} + 1.$$

Higher regularity results (for different types of nonlocal anisotropic operators) have been obtained in [5, 8] (indeed, general anisotropic operators can be considered in [5, 8], but only the kernel with the same homogeneity were taken into account). Some advantages are offered by Theorem 1.1 with respect to the other results available in the literature. First of all, Theorem 1.1 comprises the case of operators of different orders (e.g. the s_i can be all different and both local and nonlocal operators can be superposed). Moreover, Theorem 1.1 may select the "local" coordinate direction independently on the others, in order to take into account the behavior of the nonlinearity in this single coordinate and detect its effect on the oscillation of the solution (notice in particular the term $\mathfrak{S}$ appearing in Theorem 1.1, which only depends on the oscillation of f in the last coordinate direction). As a matter of fact, the diffusive operators in the other variables can also degenerate (indeed a_i may vanish for some $i \in \{1, \dots, m-1\}$).

In addition, all the constants appearing in Theorem 1.1 can be computed explicitly without effort and the proof is rather simple and it makes use only of one explicit barrier (the barrier will be given in formula (18) and, as a matter of fact, this argument may be seen as the fractional counterpart of the regularity theory developed by [1] in the local framework).

As a technical remark, we point out that, for simplicity, the notion of solution in Theorem 1.1 is taken in the classical sense, i.e. the function u will be implicitly assumed to be smooth enough to compute the operator L pointwise (in this sense, formula (7) reads as an "a priori estimate"). Nevertheless, the same argument that we present goes through, for instance, by applying the operator to smooth functions that touch the solution from above/below, that is one can assume simply that the solution in Theorem 1.1 is taken in the viscosity sense (in this case, formula (7) reads as an "improvement of regularity").

[3] We observe that $\eta_i = 1/(2N_i)$ if $s_i = 1$, since $\Gamma\left(1 + \frac{N_i}{2}\right) = \frac{N_i}{2}\,\Gamma\left(\frac{N_i}{2}\right)$.

We point out that, as a simple consequence of Theorem 1.1, we obtain an interior Lipschitz estimate in the last variable:

Corollary 1.2. *Let $u:\mathbb{R}^n\to\mathbb{R}$ be a solution of $Lu=f$ in B_1. Then*

$$\|\partial_{x_n}u\|_{L^\infty(B_{1/2})}\leqslant C\left(\|f\|_{L^\infty(B_1)}+\|u\|_{L^\infty(\mathbb{R}^n)}\right), \tag{8}$$

for some $C>0$, depending on $a_1,\dots,a_m$, $s_1,\dots,s_{m-1}$, and $N_1,\dots,N_{m-1}$.

As a matter of fact, when $s_1=\dots=s_m=1$, Corollary 1.2 reduces to the classical Lipschitz regularity theory as presented in [1].

We observe that the regularity results obtained in this paper can be also combined efficiently with other results available in the literature, possibly leading to higher regularity results. To make a simple example of this feature, we give the following result:

Corollary 1.3. *Let $s\in(0,1)$, $a_1,\dots,a_{m-1}$, $a>0$ and*

$$L_*:=\sum_{i=1}^{m-1}a_i(-\Delta_{X_i})^s-a\partial^2_{x_n}. \tag{9}$$

Let $f\in L^\infty(\mathbb{R}^n)$ be Lipschitz continuous in B_1 with respect to the variable x_n. Let $u:\mathbb{R}^n\to\mathbb{R}$ be a solution of $L_*u=f$ in $B_1^{n-1}\times\mathbb{R}$. Then

$$\|u\|_{C^\gamma(B_{1/2})}\leqslant C\left(\|f\|_{L^\infty(\mathbb{R}^n)}+\|\partial_{x_n}f\|_{L^\infty(B_1)}+\|u\|_{L^\infty(\mathbb{R}^n)}\right),$$

where

$$\gamma:=\begin{cases}2s & \text{if } s\neq 1/2,\\ 1-\epsilon & \text{if } s=1/2\end{cases}$$

for some $C>0$, depending on $a_1,\dots,a_m$, s, and $N_1,\dots,N_{m-1}$ (with the caveat that when $s=1/2$, one can choose ϵ arbitrarily in $(0,1)$ and C will also depend on ϵ).

We observe that if $s>1/2$ in Corollary 1.3, then $\gamma=1+\eta$, with $\eta=2s-1\in(0,1)$, hence Corollary 1.3 gives that the solution lies in $C^{1+\eta}(B_{1/2})$.

Another interesting consequence of Theorem 1.1, is also the following rigidity result, valid when all the fractional exponents are larger than 1/2:

Theorem 1.4. *Let $f:\mathbb{R}^n\to\mathbb{R}$. Assume that*

$$\sigma:=2\min\{s_1,\dots,s_n\}-1>0.$$

Let $u:\mathbb{R}^n\to\mathbb{R}$ be a solution of $Lu=f$ in the whole of $\mathbb{R}^n$. Assume that f does not depend on the nth coordinate and that[4]

$$\|u\|_{L^\infty(B_R)}=o(R^\sigma) \tag{10}$$

as $R\to+\infty$.

Then u does not depend on the nth coordinate.

Remark 1.5. A simple, but interesting, consequence of Theorem 1.4 is that if (10) holds and f is identically zero, then

$$L_*u=0 \qquad \text{in } \mathbb{R}^{n-1},$$

where we used the notation in (9). Therefore, if L_* enjoys a Liouville property, then u is necessarily constant.

This feature holds, in particular, when $L_*=(-\partial^2_{x_1})^s+\dots+(-\partial^2_{x_{n-1}})^s$, see Theorem 2.1 in [5].

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[4]As customary, the notation in (10) simply means that

$$\lim_{R\to+\infty}\frac{\|u\|_{L^\infty(B_R)}}{R^\sigma}=0.$$

Remark 1.6. The observation in Remark 1.5 also says that, if (10) is satisfied and L_* enjoys a Liouville property, then the problem $Lu = f$ possesses a unique solution, up to an additive constant.

The rest of the paper is organized as follows. The proof of Theorem 1.1, based on the barrier method of [1], is contained in Section 2. Then, in Section 3, we combine our results with those of [5] and we prove Corollary 1.3. The proof of Theorem 1.4, which combines our result with a cutoff argument, is contained in Section 4.

2. Proof of Theorem 1.1

2.1. **A useful explicit barrier.** We recall here a useful barrier. Here and in what follows we use the standard "positive part" notation for any $t \in \mathbb{R}$, i.e.

$$t_+ := \max\{t, 0\}.$$

We will also exploit the notation in (1) and (6).

Lemma 2.1. *Let $s_i \in (0,1]$ and $d_i > 0$. For any $x = (X_1, \dots, X_{m-1}, x_n) \in \mathbb{R}^n$ let*

$$\Phi_{d_i}(z) := \eta_i \, (d_i^2 - |X_i|^2)_+^{s_i}.$$

Then, for any $x \in \mathbb{R}^n$ with $X_i \in B_{d_i}^{N_i}$, we have that

$$(-\Delta_{X_i})^s \Phi_{d_i}(x) = 1.$$

Proof. The result is obvious for $s_i = 1$ (recall the footnote on page 3), hence we suppose $s_i \in (0,1)$. We let $\Psi_{d_i} := \eta_i^{-1} \Phi_{d_i}(z) = (d_i^2 - |X_i|^2)_+^{s_i}$. By scaling variables $x_* := \frac{x}{d_i}$, $X_{*,i} := \frac{X_i}{d_i}$ and $\zeta_* := \frac{\zeta}{d_i}$, we obtain that

$$\begin{aligned}
(-\Delta)^{s_i} \Psi_{d_i}(x) &= c_{N_i,s_i} \int_{\mathbb{R}^{N_i}} \frac{2(d_i^2 - |X_i|^2)_+^{s_i} - (d_i^2 - |X_i + \zeta|^2)_+^{s_i} - (d_i^2 - |X_i - \zeta|^2)_+^{s_i}}{|\zeta|^{N_i + 2s_i}} \, d\zeta \\
&= c_{N_i,s_i} \, d_i^{2s_i} \, d_i^{N_i} \int_{\mathbb{R}^{N_i}} \frac{2(1 - |X_{*,i}|^2)_+^{s_i} - (1 - |X_{*,i} + \zeta_*|^2)_+^{s_i} - (1 - |X_{*,i} - \zeta_*|^2)_+^{s_i}}{d_i^{N_i + 2s_i} |X_{*,i}|^{N_i + 2s_i}} \, d\zeta_* \\
&= (-\Delta_{X_i})^{s_i} \Psi_1(x_*) \\
&= \frac{2^{2s_i} \, \Gamma(s_i + 1) \, \Gamma\left(s_i + \frac{N_i}{2}\right)}{\Gamma\left(\frac{N_i}{2}\right)},
\end{aligned}$$

see for instance Table 3 of [2] for the last identity (here, we used the notation Ψ_1 to denote Ψ_{d_i} when $d_i = 1$). □

2.2. **Completion of the proof of Theorem 1.1.** For any $t \in \mathbb{R}$, we define[5]

$$u^{\pm}(x,t) := u(x \pm t_+ e_n) = u(x_1, \dots, x_{n-1}, x_n \pm t_+). \tag{11}$$

Similarly, we define $f^{\pm}(x,t) := f(x \pm t_+ e_n)$. Let also

$$v(x,t) := u^+(x,t) - u^-(x,t) \quad \text{and} \quad g(x,t) := f^+(x,t) - f^-(x,t).$$

We fix $\nu \in (0,a)$ (to be taken as close to a as we wish in what fallows). Recalling (5), we introduce the operator

$$\begin{aligned}
L_* &:= L + \nu \partial_{x_n}^2 - \nu \partial_t^2 \\
&= \sum_{i=1}^{m} a_i (-\Delta_{X_i})^{s_i} - \nu \, (-\partial_{x_n}^2) - \nu \partial_t^2 \\
&= \sum_{i=1}^{m-1} a_i (-\Delta_{X_i})^{s_i} - (a - \nu) \partial_{x_n}^2 - \nu \partial_t^2.
\end{aligned} \tag{12}$$

[5] As a technical remark, we point out that the assumption that the operator is "local" in the last coordinate is used at this point, since if $t > 0$ we have that $u^{\pm}(x,t) = u(x \pm t e_n)$, and so, if we differentiate with respect to t in the domain $\{t \in (0, d_m)\}$, we have that $\partial_t u^{\pm}(x,t) = \pm \partial_{x_n} u(x \pm t e_n)$.

Notice that L_* is an operator with one variable more than L (namely, the new variable $t \in \mathbb{R}$). We claim that

$$L_* v = g \tag{13}$$

for any $(x,t) \in Q_d \times (0, d_m)$. To check (13), we first notice that if $(x,t) \in Q_d \times (0,d_m)$ then $x \pm t e_n \in Q_{d,n,2}$, and we know that $Lu = f$ in the latter set. Also, since the (fractional) Laplacian is translation invariant,

$$(-\Delta_{X_i})^{s_i} u^{\pm} = \Big((-\Delta_{X_i})^{s_i} u \Big)^{\pm} \tag{14}$$

for any $i \in \{1, \dots, m\}$ and any $(x,t) \in Q_d \times (0, d_m)$ (notice that the variable t plays the role of a fixed parameter here). Moreover

$$\partial_t^2 u^{\pm} = \Big(\partial_{x_n}^2 u \Big)^{\pm} \tag{15}$$

for any $(x,t) \in Q_d \times (0, d_m)$. In turn, we see that (12), (14) and (15) imply that

$$L_* u^{\pm} = (Lu)^{\pm}$$

and thus, by linearity,

$$L_* v = L_*(u^+ - u^-) = (Lu)^+ - (Lu)^- = f^+ - f^- = g,$$

which establishes (13). Now we set

$$\begin{aligned} c_o &:= \sum_{i=1}^m \eta_i d_i^{2s_i} + \frac{d_m^2}{2} = \sum_{i=1}^{m-1} d_i^{2s_i} + d_m^2 \\ \text{and} \qquad A_0 &:= \sum_{i=1}^m a_i. \end{aligned} \tag{16}$$

Let also

$$\begin{aligned} & A_1 := A_0 A_2 + \|g\|_{L^\infty(Q_d \times (0,d_m))} + (a - \nu), \\ \text{where} \qquad & A_2 := \frac{\|v\|_{L^\infty(\mathbb{R}^{n+1})}}{\min\limits_{i \in \{1,\dots,m\}} (\eta_i d_i^{2s_i})}. \end{aligned} \tag{17}$$

We consider the barrier

$$\begin{aligned} & \Phi(x,t) := \frac{A_1}{\nu} \Phi_1(t) + A_2 \Phi_2(x,t) \\ \text{with} \quad & \Phi_1(t) := \frac{t_+ \, (d_m - t)_+}{2} \\ \text{and} \quad & \Phi_2(x,t) = \Phi_2(X_1, \dots, X_{m-1}, X_m, t) := c_o - \sum_{i=1}^m \eta_i (d_i^2 - |X_i|^2)_+^{s_i} - \frac{(d_m^2 - t^2)_+}{2}. \end{aligned} \tag{18}$$

Notice that $\Phi_1 \geqslant 0$ and also, by (16),

$$\Phi_2 \geqslant c_o - \sum_{i=1}^m \eta_i d_i^{2s_i} - \frac{d_m}{2} = 0.$$

Consequently,

$$\Phi \geqslant 0 \tag{19}$$

Moreover, using the notation of Lemma 2.1, we see that

$$\Phi_2(x,t) = c_o - \sum_{i=1}^n \Psi_{d_i}(x_i) - \Psi_{d_m}(t).$$

Therefore, making use of Lemma 2.1, we conclude that, for any $(x,t)\in Q_d\times(0,d_m)$,

$$\begin{aligned} L_*\Phi(x,t) &= \sum_{i=1}^m a_i(-\Delta_{X_i})^{s_i}\Phi(x,t)+\nu\partial^2_{x_n}\Phi(x,t)-\nu\partial^2_t\Phi(x,t)\\ &= A_2\sum_{i=1}^m a_iA_2(-\Delta_{X_i})^{s_i}\Phi_2(x,t)+A_2\nu\partial^2_{x_n}\Phi_2(x,t)-\nu\partial^2_t\Phi(x,t)\\ &= -A_2\sum_{i=1}^m a_i - A_2\nu+\nu\left(\frac{A_1}{\nu}+A_2\right)\\ &= -A_2\sum_{i=1}^m a_i + A_1. \end{aligned}$$

That is, recalling (16) and (17),

$$\begin{aligned} L_*\Phi(x,t) = -A_2A_0+A_1 &= \|g\|_{L^\infty(Q_d\times(0,d_m))}+(a-\nu)\\ &\geqslant \pm g(x,t)+(a-\nu)=\pm L_*v(x,t)+(a-\nu), \end{aligned} \tag{20}$$

for any $(x,t)\in Q_d\times(0,d_m)$, where we used (13) in the last step.

Now we claim that

$$\Phi(x,t)\pm v(x,t)\geqslant 0 \text{ for any } (x,t)\in\mathbb{R}^{n+1}\setminus\big(Q_d\times(0,d_m)\big). \tag{21}$$

To check this, we take (x,t) outside $Q_d\times(0,d_m)$, and we distinguish three cases: either $t\leqslant 0$, or $t\geqslant d_m$, or $x\in\mathbb{R}^n\setminus Q_d$.

First, when $t\leqslant 0$, we have that $t_+=0$, so we use (11) to see that $u^+(x,0)=u(x)=u^-(x,0)$ and so $v(x,0)=0$. Then $\pm v(x,0)=0\leqslant\Phi(x,0)$ in this case, thanks to (19) and this establishes (21) when $t\leqslant 0$.

Now, let us deal with the case in which $t\geqslant d_m$. In this case $(d_m^2-t^2)_+=0$, hence, by (18),

$$\begin{aligned} \Phi(x,t) &\geqslant A_2\Phi_2(x,t)\\ &= A_2\left[c_o-\sum_{i=1}^m\eta_i\,(d_i^2-|X_i|^2)_+^{s_i}\right]\\ &\geqslant A_2\left[c_o-\sum_{i=1}^m\eta_i d_i^{2s_i}\right]\\ &= \frac{A_2d_m^2}{2}\\ &\geqslant \|v\|_{L^\infty(\mathbb{R}^{n+1})}\\ &\geqslant \pm v(x,t), \end{aligned}$$

as desired. It remains to consider the case $x\in\mathbb{R}^n\setminus Q_d$. Under this circumstance, we have that there exists $i_o\in\{1,\dots,m\}$ such that $|X_{i_o}|\geqslant d_{i_o}$. Accordingly

$$\sum_{i=1}^m(d_i^2-|X_i|^2)_+^{s_i}=\sum_{\substack{1\leqslant i\leqslant m\\ i\neq i_o}}(d_i^2-|X_i|^2)_+^{s_i}\leqslant\sum_{\substack{1\leqslant i\leqslant m\\ i\neq i_o}}d_i^{2s_i},$$

and so

$$\begin{aligned}
\Phi(x,t) &\geqslant A_2\Phi_2(x,t)\\
&\geqslant A_2\left[c_o-\sum_{\substack{1\leqslant i\leqslant m\\ i\neq i_o}}\eta_i d_i^{2s_i}-\frac{d_m^2}{2}\right]\\
&= A_2\eta_{i_o}d_{i_o}^{2s_{i_o}}\\
&\geqslant \|v\|_{L^\infty(\mathbb{R}^{n+1})}\\
&\geqslant \pm v(x,t),
\end{aligned}$$

which completes the proof of (21).

Now we show that the inequality in (21) propagates inside $Q_d\times(0,d_m)$, namely that

$$\Phi(x,t)\pm v(x,t)\geqslant 0 \text{ for any } (x,t)\in\mathbb{R}^{n+1}. \tag{22}$$

The proof of (22) is mostly Maximum Principle. The details are as follows. Suppose, by contradiction, that (22) were false. Then we set $h:=\Phi\pm v$. Notice that, since u is assumed to be continuous, so is h, due to (11), and then (21) would imply that

$$\min_{Q_d\times(0,d_m)} h =: \mu<0.$$

Let $\bar p:=(\bar x,\bar t)$ attaining the minimum of h, that is

$$(-\infty,0)\ni\mu=h(\bar p)\leqslant h(\xi),$$

for any $\xi\in\mathbb{R}^{n+1}$. By (21), we have that $\bar p$ lies in $Q_d\times(0,d_m)$, hence, by (20),

$$L_*h(\bar p)\geqslant a-\nu>0. \tag{23}$$

On the other hand, recalling the notation in (2), for any $i\in\{1,\dots,m-1\}$ and any $y\in\mathbb{R}^{N_i}$, we have that

$$2h(\bar p)-h(\bar p+y^{(i)})-h(\bar p-y^{(i)})\leqslant 0,$$

due to the minimality of $\bar p$. Similarly $(-\partial^2_{x_j})h(\bar p)\leqslant 0$ for any $j\in\{1,\dots,n\}$, as well as $(-\partial^2_t)h(\bar p)\leqslant 0$. Therefore $(-\Delta_{X_i})^{s_i}(\bar p)\leqslant 0$, for any $i\in\{1,\dots,m\}$. Consequently, by (12), we infer that $L_*h(\bar p)\leqslant 0$. The latter inequality is in contradiction with (23) and thus we have proved (22).

By choosing the sign in (22), we deduce that

$$|v(x,t)|\leqslant\Phi(x,t) \text{ for any } (x,t)\in\mathbb{R}^{n+1}. \tag{24}$$

Moreover, recalling (18) and (16), for any $t\in(0,d_m)$,

$$\begin{aligned}
\Phi_2(0,t) &= c_o-\sum_{i=1}^m\eta_i d_i^{2s_i}-\frac{(d_m^2-t^2)_+}{2}\\
&= \frac{d_m^2}{2}-\frac{(d_m^2-t^2)}{2}\\
&= \frac{t^2}{2}.
\end{aligned}$$

In addition,

$$\Phi_1(t)\leqslant d_m t_+,$$

therefore, by (18), for any $t\in(0,d_m)$,

$$\Phi(0,t)\leqslant\frac{A_1 d_m t}{\nu}+\frac{A_2t^2}{2}.$$

This and (24) imply that, for any $t \in (0, d_m)$,

$$\begin{aligned}\frac{|u(te_n) - u(-te_n)|}{|t|} &= \frac{|u^+(0,t) - u^-(0,t)|}{t} = \frac{|v(0,t)|}{t} \leqslant \frac{\Phi(0,t)}{t} \\ &\leqslant \frac{A_1\, d_m}{\nu} + \frac{A_2\, t}{2} \\ &= \frac{\left[A_0 A_2 + \|g\|_{L^\infty(Q_d \times (0,d_m))} + (a-\nu)\right] d_m}{\nu} + \frac{A_2\, t}{2}.\end{aligned}$$

Now we observe that the first term in the above inequality remains unchanged if we replace t with $-t$, and therefore the inequality is valid for any $t \in (-d_m, d_m)$. Furthermore, we can now take ν as close to a as we wish (recall that A_0 and A_2 do not depend on ν), hence we obtain that, for any $t \in (-d_m, d_m)$,

$$\begin{aligned}\frac{|u(te_n) - u(-te_n)|}{|t|^{s_n}} &\leqslant \frac{\left[A_0 A_2 + \|g\|_{L^\infty(Q_d \times (0,d_m))}\right] d_m}{a} + \frac{A_2\, t}{2} \\ &\leqslant \frac{\left[A_0 A_2 + \|g\|_{L^\infty(Q_d \times (0,d_m))}\right] d_m}{a} + \frac{A_2\, d_m}{2} \\ &= \frac{\|g\|_{L^\infty(Q_d \times (0,d_m))}\, d_m}{a} + A_2\, d_m \left(\frac{A_0}{a} + \frac{1}{2}\right) \\ &= \frac{\|g\|_{L^\infty(Q_d \times (0,d_m))}\, d_m}{a} + \frac{\|v\|_{L^\infty(\mathbb{R}^{n+1})}\, d_m}{\min\limits_{i \in \{1,\dots,m\}} (\eta_i d_i^{2s_i})} \left(\frac{a_1 + \dots + a_m}{a} + \frac{1}{2}\right).\end{aligned}$$

This completes the proof of Theorem 1.1.

3. Proof of Corollary 1.3

The proof combines Corollary 1.2 here with Theorem 1.1(a) in [5]. To this goal, fixed $t \in \left[-\frac{1}{1000}, \frac{1}{1000}\right]$ (to be taken arbitrarily small in the sequel) we define

$$u_\sharp(x) := \frac{u(x+te_n) - u(x)}{t} \quad \text{and} \quad f_\sharp(x) := \frac{f(x+te_n) - f(x)}{t}. \tag{25}$$

By formula (8) in Corollary 1.2, we already know that

$$\|u_\sharp\|_{L^\infty(\mathbb{R}^n)} \leqslant \|\partial_{x_n} u\|_{L^\infty(\mathbb{R}^n)} \leqslant C\left(\|f\|_{L^\infty(\mathbb{R}^n)} + \|u\|_{L^\infty(\mathbb{R}^n)}\right). \tag{26}$$

Also, we point out that $L_* u_\sharp = f_\sharp$ in $B_{99/100}$, and so, using again Corollary 1.2,

$$\|\partial_{x_n} u_\sharp\|_{L^\infty(B_{97/100})} \leqslant C\left(\|f_\sharp\|_{L^\infty(B_{99/100})} + \|u_\sharp\|_{L^\infty(\mathbb{R}^n)}\right).$$

This, combined with (26), gives that

$$\begin{aligned}\sup_{x \in B_{97/100}} \left|\frac{\partial_{x_n} u(x+te_n) - \partial_{x_n} u(x)}{t}\right| &= \|\partial_{x_n} u_\sharp\|_{L^\infty(B_{97/100})} \\ &\leqslant C\left(\|\partial_{x_n} f\|_{L^\infty(B_1)} + \|f\|_{L^\infty(\mathbb{R}^n)} + \|u\|_{L^\infty(\mathbb{R}^n)}\right).\end{aligned}$$

Hence, taking t to the limit,

$$\sup_{x \in B_{97/100}} |\partial^2_{x_n} u(x)| \leqslant C\left(\|\partial_{x_n} f\|_{L^\infty(B_1)} + \|f\|_{L^\infty(\mathbb{R}^n)} + \|u\|_{L^\infty(\mathbb{R}^n)}\right). \tag{27}$$

Now, given $i \in \{1, \dots, m-1\}$ we consider the sphere $S^{N_i - 1}$ in the Euclidean space $\mathbb{R}^{N_i}$ (of course, if $N_i = 1$, then S^{N_i-1} reduces to two points).

We also set $S^{n-2} := \{(x_1, \dots, x_{n-1}) \text{ s.t. } x_1^2 + \dots + x_{n-1}^2 = 1\}$ and we observe that each S^{N_i-1} is naturally immersed into S^{n-2} (in the same way as $\mathbb{R}^{N_i-1}$ is immersed into $\mathbb{R}^{n-1}$).

We denote by $\mathcal{H}_i$ the (N_i-1)-dimensional Hausdorff measure restricted to S^{N_i-1} (if $N_i=1$, we replace it by the Dirac's delta on the two points given by S^{N_i-1}). Then we consider the measure

$$\mu := \sum_{i=1}^{m-1} \frac{a_i\, c_{N_i,s_i}}{2} \mathcal{H}_i.$$

We fix $\tilde{x}_n \in \left[-\frac{1}{100}, \frac{1}{100}\right]$ and we set

$$\begin{aligned} &\tilde{u}(x_1,\dots,x_{n-1}) := u(x_1,\dots,x_{n-1},\tilde{x}_n) \\ \text{and} \quad &\tilde{f}(x_1,\dots,x_{n-1}) := f(x_1,\dots,x_{n-1},\tilde{x}_n) + a\partial^2_{x_n} u(x_1,\dots,x_{n-1},\tilde{x}_n). \end{aligned}$$

We use (3) and polar coordinates on $\mathbb{R}^{N_i}$ to see that, for any $\tilde{x}=(x_1,\dots,x_{n-1})\in B^{n-1}_{99/100}$,

$$\begin{aligned} \tilde{L}\tilde{u}(\tilde{x}) &:= \int_{S^{n-2}} \left[\int_{\mathbb{R}} \Big(\tilde{u}(\tilde{x}+\theta r)+\tilde{u}(\tilde{x}-\theta r)-2\tilde{u}(\tilde{x})\Big)\frac{dr}{|r|^{1+2s}}\right] d\mu(\theta) \\ &= \sum_{i=1}^{m-1} \frac{a_i\, c_{N_i,s_i}}{2} \int_{S^{N_i-1}} \left[\int_{\mathbb{R}} \Big(\tilde{u}(\tilde{x}+\theta r)+\tilde{u}(\tilde{x}-\theta r)-2\tilde{u}(\tilde{x})\Big)\frac{dr}{r^{1+2s}}\right] d\mathcal{H}_i(\theta) \\ &= \sum_{i=1}^{m-1} a_i\, c_{N_i,s_i} \int_{S^{N_i-1}} \left[\int_0^{+\infty} \Big(\tilde{u}(\tilde{x}+\theta r)+\tilde{u}(\tilde{x}-\theta r)-2\tilde{u}(\tilde{x})\Big)\frac{dr}{r^{1+2s}}\right] d\mathcal{H}_i(\theta) \\ &= \sum_{i=1}^{m-1} a_i\, c_{N_i,s_i} \int_{\mathbb{R}^{N_i}} \frac{\tilde{u}(\tilde{x}+y^{(i)})+\tilde{u}(\tilde{x}-y^{(i)})-2\tilde{u}(\tilde{x})}{|y^{(i)}|^{N_i+2s}}\, dy^{(i)} \\ &= \sum_{i=1}^{m-1} a_i\, (-\Delta_{X_i})^{s_i}\tilde{u}(\tilde{x}) \\ &= L_*\tilde{u}(\tilde{x}) + a\partial^2_{x_n} u(\tilde{x},\tilde{x}_n) \\ &= f(\tilde{x},\tilde{x}_n) + a\partial^2_{x_n} u(\tilde{x},\tilde{x}_n) \\ &= \tilde{f}(\tilde{x}). \end{aligned}$$

Notice that, with this setting, the operator $\tilde{L}$ satisfies formula (1.1) in [5].

Furthermore, we have that

$$\inf_{\nu\in S^{n-2}} \int_{S^{n-2}} |\nu\cdot\theta|\, d\mu(\theta) \geqslant \lambda, \tag{28}$$

for some $\lambda>0$. To prove it, we observe that if $\nu=(\nu_1,\dots,\nu_{n-1})\in S^{n-2}$, we have that $|\nu_j|\geqslant (n-1)^{-1/2}$, for at least one $j\in\{1,\dots,n-1\}$. Up to relabeling variables, we assume that $j=1$, and thus

$$\begin{aligned} \int_{S^{n-2}} |\nu\cdot\theta|^{2s}\, d\mu(\theta) &\geqslant \frac{a_1\, c_{N_1,s_1}}{2} \int_{S^{N_1-1}} |\nu_1\theta_1|^{2s}\, d\mu(\theta) \\ &\geqslant \frac{a_1\, c_{N_1,s_1}}{2\,(n-1)^s} \int_{S^{N_1-1}} |\theta_1|^{2s}\, d\mu(\theta), \end{aligned}$$

which proves (28).

In addition,

$$\mu(S^{n-2}) \leqslant \sum_{i=1}^{m-1} \frac{a_i\, c_{N_i,s_i}}{2} \mathcal{H}^{N_i-1}(S^{N_i-1}) < +\infty.$$

From this and (28), we conclude that condition (1.2) in [5] is satisfied. Accordingly, we can exploit Theorem 1.1(a) in [5] and conclude that

$$\begin{aligned} \|\tilde{u}\|_{C^\gamma(B^{n-1}_{3/4})} &\leqslant C\left(\|\tilde{u}\|_{L^\infty(\mathbb{R}^n)} + \|\tilde{f}\|_{L^\infty(B^{n-1}_{4/5})}\right) \\ &\leqslant C\left(\|u\|_{L^\infty(\mathbb{R}^n)} + \|f\|_{L^\infty(B_1)} + \|\partial^2_{x_n} u\|_{L^\infty(B_{97/100})}\right). \end{aligned}$$

This and (27) imply that

$$\|\tilde{u}\|_{C^\gamma(B^{n-1}_{3/4})} \leqslant C\left(\|\partial_{x_n} f\|_{L^\infty(B_1)} + \|f\|_{L^\infty(\mathbb{R}^n)} + \|u\|_{L^\infty(\mathbb{R}^n)}\right).$$

This gives the desired regularity in the set of variables $(x_1, \dots, x_{n-1})$. The regularity in the last variable follows from (27) and so the proof of Corollary 1.3 is complete.

4. Proof of Theorem 1.4

4.1. **A cutoff argument.** The purpose of this section is to localize the estimate of Theorem 1.1 by using a cutoff function. As customary in the fractional problems, regularity estimates cannot be completely localized, due to nonlocal effect, nevertheless our objective is to give quantitative bounds on the contribution "coming from infinity". For this scope, we use the notation $s_{\min} := \min\{s_1, \dots, s_n\}$ and $s_{\max} := \max\{s_1, \dots, s_n\}$ (a similar notation will also be exploited in the sequel for $a_{\min} := \min\{a_1, \dots, a_m\}$ and $a_{\max} := \max\{a_1, \dots, a_m\}$).

Lemma 4.1. *Let $R \geqslant 1$. If w vanishes identically in $(-3R, 3R)^n$, then*

$$\|Lw\|_{L^\infty((-R,R)^n)} \leqslant C_o \int_{2R}^{+\infty} \frac{\|w\|_{L^\infty(B_{2\rho}\setminus B_{\rho/2})}}{\rho^{1+2s_{\min}}}\, d\rho,$$

where

$$C_o := 2\sum_{i=1}^{n} a_i c_{N_i,s_i}\, \mathcal{H}^{N_i-1}(S^{N_i-1}). \tag{29}$$

Proof. Let $x \in (-R, R)$. We claim that

$$|(-\Delta_{X_i})^{s_i} w(x)| \leqslant 2c_{N_i,s_i}\; \mathcal{H}^{N_i-1}(S^{N_i-1}) \int_{2R}^{+\infty} \frac{\|w\|_{L^\infty(B_{2\rho}\setminus B_{\rho/2})}}{\rho^{1+2s_i}}\, d\rho \tag{30}$$

for each $i \in \{1, \dots, m\}$. To prove this, we notice that if $s_i = 1$ then the fact that w vanishes identically in a neighborhood of x implies that $-\Delta_{X_i} w(x) = 0$, and so (30) is obvious in this case. Thus, we can suppose that $s_i \in (0, 1)$, and we observe that, if $y^{(i)} \in [-2R, 2R]^{N_i}$ then $x+y^{(i)} \in (-3R, 3R)^n$ and so $w(x+y^{(i)}) = 0$. From this, it follows that

$$(-\Delta_{X_i})^{s_i} w(x) = c_{N_i,s_i} \int_{\mathbb{R}^{N_i}\cap\{|y^{(i)}|>2R\}} \frac{-w(x+y^{(i)}) - w(x-y^{(i)})}{|y^{(i)}|^{N_i+2s_i}}\, dy^{(i)}. \tag{31}$$

Also, if $|y^{(i)}| > 2R$ then $|y^{(i)}| > 2|x|$, thus

$$\begin{aligned} &|x \pm y^{(i)}| \geqslant |y^{(i)}| - |x| > \frac{|y^{(i)}|}{2} \\ \text{and} \qquad &|x \pm y^{(i)}| \leqslant |x| + |y^{(i)}| < 2|y^{(i)}|, \end{aligned}$$

and so $|w(x \pm y^{(i)})| \leqslant \|w\|_{L^\infty(B_{2|y^{(i)}|}\setminus B_{|y^{(i)}|/2})}$. As a consequence of this and of (31), we obtain

$$\begin{aligned} |(-\Delta_{X_i})^{s_i} w(x)| &\leqslant 2c_{N_i,s_i} \int_{\mathbb{R}^{N_i}\cap\{|y|>2R\}} \frac{\|w\|_{L^\infty(B_{2|y|}\setminus B_{|y|/2})}}{|y|^{N_i+2s_i}}\, dy \\ &= 2c_{s_i}\; \mathcal{H}^{N_i-1}(S^{N_i-1}) \int_{2R}^{+\infty} \frac{\|w\|_{L^\infty(B_{2\rho}\setminus B_{\rho/2})}}{\rho^{1+2s_i}}\, d\rho, \end{aligned}$$

which proves (30). The desired claim then follows recalling (5) and adding up the estimate in (30). □

Corollary 4.2. *Let $R \geqslant 1$. There exists $\eta_R \in C^\infty(\mathbb{R}^n)$ such that*

$$\eta_R = 1 \text{ in } (-3R, 3R)^n,\ \eta_R = 0 \text{ in } \mathbb{R}^n \setminus (-6R, 6R), \text{ and} \tag{32}$$

$$\|Lu - L(\eta_R u)\|_{L^\infty((-R,R)^n)} \leqslant C_o \int_{2R}^{+\infty} \frac{\|u\|_{L^\infty(B_{2\rho}\setminus B_{\rho/2})}}{\rho^{1+2s_{\min}}}\, d\rho, \tag{33}$$

with C_o as in (29).

Proof. Let $\eta_o \in C^\infty(\mathbb{R}, [0,1])$ with $\eta_o = 1$ in $(-1,1)$ and $\eta_o = 0$ outside $(-2,2)$. Let

$$\eta_R(x) := \prod_{i=1}^{n} \eta_o\left(\frac{x_i}{3R}\right).$$

Then η_R satisfies (32). Also, if we set $w := (1-\eta_R)u$, we have from (32) that $w = 0$ in $(-3R,3R)^n$. Thus, the estimate in (33) follows by writing $u - \eta_R u = w$, using the linearity of the operator L and Lemma 4.1. □

By combining Theorem 1.1 and Corollary 4.2, we can obtain a refined estimate in which the "contribution from infinity" in the right hand side of (7) is weighted "ring by ring":

Theorem 4.3. *Let $R \geqslant 1$ and $f : B_{6\sqrt{n}R} \to \mathbb{R}$. Let $u : \mathbb{R}^n \to \mathbb{R}$ be a solution of $Lu = f$ in $B_{6\sqrt{n}R}$. Then, for any $t \in (-\frac{R}{6\sqrt{n}}, \frac{R}{6\sqrt{n}})$,*

$$\begin{aligned}\frac{|u(te_n) - u(-te_n)|}{|t|} &\leqslant C \Bigg(R \sup_{(x,t)\in B_R\times(0,R)} |f(x+te_n) - f(x-te_n)| \\ &\quad + \frac{R\,\|u\|_{L^\infty(B_{6R})}}{R^{2s_{\min}}} + R\int_{2R}^{+\infty} \frac{\|u\|_{L^\infty(B_{2\rho}\setminus B_{\rho/2})}}{\rho^{1+2s_{\min}}}\,d\rho \Bigg),\end{aligned} \tag{34}$$

where $C > 0$ here only depends on n, $s_{\min}$, $s_{\max}$, $a_{\min}$ and $a_{\max}$.

Proof. In this argument, we will take the freedom of renaming constants as we please, line after line, by keeping the same name C. Using the notation of Corollary 4.2, we define $\tilde u := \eta_R u$ and, for any $x \in B_{6\sqrt{n}R}$, $\tilde f(x) := L\tilde u(x)$. Let also $\tilde g := L\tilde u - Lu$. By Corollary 4.2,

$$\|\tilde g\|_{L^\infty((-R,R)^n)} \leqslant C \int_{2R}^{+\infty} \frac{\|u\|_{L^\infty(B_{2\rho}\setminus B_{\rho/2})}}{\rho^{1+2s_{\min}}}\,d\rho.$$

By construction $\tilde f = f + \tilde g$, therefore

$$\begin{aligned}&\sup_{(x,t)\in(-\frac{R}{6\sqrt{n}},\frac{R}{6\sqrt{n}})^n\times(0,\frac{R}{6\sqrt{n}})} |\tilde f(x+te_n) - \tilde f(x-te_n)| \\ &\leqslant \sup_{(x,t)\in(-\frac{R}{6\sqrt{n}},\frac{R}{6\sqrt{n}})^n\times(0,\frac{R}{6\sqrt{n}})} |f(x+te_n) - f(x-te_n)| + 2\|\tilde g\|_{L^\infty((-\frac{R}{3\sqrt{n}},\frac{R}{3\sqrt{n}})^n)} \\ &\leqslant \sup_{(x,t)\in B_R\times(0,R)} |f(x+te_n) - f(x-te_n)| + C\int_{2R}^{+\infty} \frac{\|u\|_{L^\infty(B_{2\rho}\setminus B_{\rho/2})}}{\rho^{1+2s_{\min}}}\,d\rho.\end{aligned}$$

Notice also that $\tilde u = u$ in $(-R,R)^n$ and $\tilde u = 0$ outside B_{6R}. Thus, by applying Theorem 1.1 (here with $d_1 = \cdots = d_m = \frac{R}{3\sqrt{n}}$) to the function $\tilde u$, for any $t \in (-\frac{R}{6\sqrt{n}}, \frac{R}{6\sqrt{n}})$ we obtain that

$$\begin{aligned}&\frac{|u(te_n) - u(-te_n)|}{|t|} = \frac{|\tilde u(te_n) - \tilde u(-te_n)|}{|t|} \\ &\leqslant CR \sup_{(x,t)\in(-\frac{R}{6\sqrt{n}},\frac{R}{6\sqrt{n}})^n\times(0,\frac{R}{6\sqrt{n}})} |\tilde f(x+te_n) - \tilde f(x-te_n)| + \frac{CR\,\|\tilde u\|_{L^\infty(\mathbb{R}^n)}}{R^{2s_{\min}}} \\ &\leqslant CR \sup_{(x,t)\in B_R\times(0,R)} |f(x+te_n) - f(x-te_n)| + CR\int_{2R}^{+\infty} \frac{\|u\|_{L^\infty(B_{2\rho}\setminus B_{\rho/2})}}{\rho^{1+2s_{\min}}}\,d\rho + \frac{CR\,\|u\|_{L^\infty(B_{6R})}}{R^{2s_{\min}}},\end{aligned}$$

as desired. □

4.2. **Completion of the proof of Theorem 1.4.** Using L'Hôpital's Rule, we see that

$$\lim_{R\to+\infty} R \int_{2R}^{+\infty} \frac{\|u\|_{L^\infty(B_{2\rho}\setminus B_{\rho/2})}}{\rho^{1+2s_{\min}}}\,d\rho = \lim_{R\to+\infty} \frac{\displaystyle\int_{2R}^{+\infty} \frac{\|u\|_{L^\infty(B_{2\rho}\setminus B_{\rho/2})}}{\rho^{1+2s_{\min}}}\,d\rho}{R} = \lim_{R\to+\infty} \frac{\dfrac{\|u\|_{L^\infty(B_{4R}\setminus B_R)}}{(2R)^{1+2s_{\min}}}}{R^{-2}} = 0,$$

$$\text{and} \quad \lim_{R\to+\infty} \frac{R\,\|u\|_{L^\infty(B_{6R})}}{R^{2s_{\min}}} = 0,$$

thanks to (10). So, we can use Theorem 4.3 and pass formula (34) to the limit as $R \to +\infty$, and obtain that

$$\frac{|u(te_n) - u(-te_n)|}{|t|} = 0,$$

for any fixed $t \in \mathbb{R}$. This says that $u(te_n) = u(-te_n)$ for any $t \in \mathbb{R}$

Since the problem is translation invariant, we can apply the argument above in the neighborhood of any point, so we obtain that

$$u(p + te_n) = u(p - te_n) \tag{35}$$

for any $p \in \mathbb{R}^n$ and any $t \in \mathbb{R}$

Now take any point $x \in \mathbb{R}^n$ and any $\rho \in \mathbb{R}$. We take $p := x + \frac{\rho e_n}{2}$ and $t := \frac{\rho}{2}$. Notice that $p - te_n = x$ and $p + te_n = x + \rho e_n$, therefore (35) implies that $u(x) = u(x + \rho e_n)$, which completes the proof of Theorem 1.4.

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