On a fourth order Stekloff eigenvalue problem

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Abstract

We study a biharmonic Stekloff eigenvalue problem. We prove some new results and we collect and refine a number of known results. Moreover, we highlight the main open problems still to be solved.

1 Introduction and results

Let \( \Omega \subset \mathbb{R}^n \) \((n \geq 2)\) be a bounded domain with \( \partial \Omega \in C^2 \), let \( d \in \mathbb{R} \) and consider the boundary eigenvalue problem

\[
\begin{aligned}
\Delta^2 u &= 0 \quad \text{in } \Omega \\
u &= 0 \quad \text{on } \partial \Omega \\
\Delta u &= du_\nu \quad \text{on } \partial \Omega ,
\end{aligned}
\]

(1.1)

where \( u_\nu \) denotes the outer normal derivative of \( u \) on \( \partial \Omega \). We are interested in studying the eigenvalues of (1.1), namely those values of \( d \) for which the problem admits nontrivial solutions, the corresponding eigenfunctions. The purpose of the present paper is to collect a number of known (and old) results, to prove some new results and to suggest some open questions.

Elliptic problems with eigenvalues in the boundary conditions are usually called Stekloff problems from their first appearance in [20]. By solution of (1.1) we mean a function \( u \in H^2 \cap H^1_0(\Omega) \) such that

\[
\int_{\Omega} \Delta u \Delta v \, dx = d \int_{\partial \Omega} u_\nu v_\nu \, dS \quad \text{for all } v \in H^2 \cap H^1_0(\Omega).
\]

(1.2)

By taking \( v = u \) in (1.2), it is clear that all the eigenvalues of (1.1) are strictly positive. Let \( \mathcal{H}(\Omega) := [H^2 \cap H^1_0(\Omega)] \setminus H^2_0(\Omega) \) and

\[
d_1 = d_1(\Omega) := \min_{u \in \mathcal{H}(\Omega)} \frac{\int_{\Omega} |\Delta u|^2}{\int_{\partial \Omega} u_\nu^2} .
\]

(1.3)

*Financial support by the Vigoni programme of CRUI (Rome) and DAAD (Bonn) is gratefully acknowledged.
The number $d_1$ represents the least positive eigenvalue and $d_1^{-1/2}$ is the norm of the compact linear operator $H^2 \cap H_0^1(\Omega) \to L^2(\partial \Omega)$, defined by $u \mapsto u_\nu|_{\partial \Omega}$. Moreover, as pointed out by Kuttler [10], $d_1$ is the sharp constant for a priori estimates for the Laplace equation

\begin{equation}
\begin{cases}
\Delta v = 0 & \text{in } \Omega \\
v = g & \text{on } \partial \Omega
\end{cases}
\end{equation}

where $g \in L^2(\partial \Omega)$. Indeed, using Fichera’s principle of duality (see [6] and also (1.11) below), for the solution $v$ of (1.4) one has

$$d_1(\Omega) \cdot \|v\|_{L^2(\Omega)}^2 \leq \|g\|_{L^2(\partial \Omega)}^2$$

and $d_1(\Omega)$ is the largest possible constant for this inequality.

The first eigenvalue $d_1$ also plays a crucial role in the positivity preserving property for the biharmonic operator $\Delta^2$ under the boundary conditions $u = \Delta u - du_\nu = 0$ on $\partial \Omega$, see [3, 7]. It is shown there that if $d \geq d_1$, then the positivity preserving property fails, whereas it holds when $d$ is in a left neighborhood of $d_1$ (possibly $d \in (-\infty, d_1)$). We refer to [3, 7] for further details. We also refer to [11] for several inequalities between the eigenvalues of (1.1) and other eigenvalue problems.

The boundary condition in (1.1) has an interesting interpretation in theory of elasticity. Consider the model problem

\begin{equation}
\begin{cases}
\Delta^2 u = f & \text{in } \Omega \\
u = \Delta u - (1 - \sigma)\kappa u_\nu = 0 & \text{on } \partial \Omega
\end{cases}
\end{equation}

where $\Omega \subset \mathbb{R}^2$ is an open bounded domain with smooth boundary, $\sigma \in (-1, 1/2)$ is the Poisson ratio and $\kappa$ is the mean curvature of the boundary. Problem (1.5) describes the deformation $u$ of the linear elastic supported plate $\Omega$ under the action of the transversal exterior force $f = f(x)$, $x \in \Omega$. The Poisson ratio $\sigma$ of an elastic material is the negative transverse strain divided by the axial strain in the direction of the stretching force. In other words, this parameter measures the transverse expansion (resp. contraction) if $\sigma \geq 0$ (resp. $\sigma < 0$) when the material is compressed by an external force. We refer to [13, 21] for more details. The restriction on the Poisson ratio is due to thermodynamic considerations of strain energy in the theory of elasticity. As shown in [13], there exist materials for which the Poisson ratio is negative and the limit case $\sigma = -1$ corresponds to materials with an infinite flexural rigidity, see [21, p.456]. This limit value for $\sigma$ is strictly related to the eigenvalue problem (1.1). Indeed, if we assume that $\Omega$ is the unit disk, then $d = (1 - \sigma)\kappa = 1 - \sigma$. Moreover, by Theorem 3 below, the first eigenvalue $d$ of (1.1) is equal to 2, which implies $\sigma = -1$. Hence, the limit value $\sigma = -1$, which is not allowed from a physical point of view, also changes the structure of the stationary problem (1.5): when $\Omega$ is the unit disk and $d = (1 - \sigma)\kappa = 2$, (1.5) either admits an infinite number of solutions or it admits no solutions at all, depending on $f$.

We are firstly interested in the description of the spectrum of (1.1). Throughout this paper we endow the Hilbert space $H^2 \cap H_0^1(\Omega)$ with the scalar product

$$\langle u, v \rangle = \int_\Omega \Delta u \Delta v \, dx .$$

Consider the subspace

$$Z = \{ v \in C^\infty(\overline{\Omega}) : \Delta^2 u = 0, \; u = 0 \; \text{on } \partial \Omega \}$$

and denote by $V$ the completion of $Z$ with respect to the scalar product in (1.6). Then, we prove
Theorem 1. Assume that $\Omega \subset \mathbb{R}^n$ ($n \geq 2$) is an open bounded domain with $C^2$ boundary. Then, problem (1.1) admits infinitely many (countable) eigenvalues. The only eigenfunction of one sign is the one corresponding to the first eigenvalue. The set of eigenfunctions forms a complete orthonormal system in $V$.

The vector space $V$ also has a different interesting characterization:

Theorem 2. Assume that $\Omega \subset \mathbb{R}^n$ ($n \geq 2$) is an open bounded domain with $C^2$ boundary. Then, the space $H^2 \cap H^1_0(\Omega)$ admits the following orthogonal decomposition with respect to the scalar product (1.6)

$$H^2 \cap H^1_0(\Omega) = V \oplus H^2_0(\Omega).$$

Moreover, if $v \in H^2 \cap H^1_0(\Omega)$ and if $v = v_1 + v_2$ is the corresponding orthogonal decomposition, then $v_1 \in V$ and $v_2 \in H^2_0(\Omega)$ are weak solutions of

$$\begin{cases}
\Delta^2 v_1 = 0 & \text{in } \Omega \\
v_1 = 0 & \text{on } \partial \Omega \\
(v_1)_\nu = v_\nu & \text{on } \partial \Omega
\end{cases} \quad \text{and} \quad \begin{cases}
\Delta^2 v_2 = \Delta^2 v & \text{in } \Omega \\
v_2 = 0 & \text{on } \partial \Omega \\
(v_2)_\nu = 0 & \text{on } \partial \Omega
\end{cases} \quad (1.8)$$

When $\Omega = B$ (the unit ball) we may determine explicitly all the eigenvalues of (1.1). To this end, consider the spaces of harmonic homogeneous polynomials:

$$D_k := \{ P \in C^\infty(\mathbb{R}^n); \Delta P = 0 \text{ in } \mathbb{R}^n, \ P \text{ is an homogeneous polynomial of degree } k - 1 \}. \quad (1.9)$$

Also, denote by $\mu_k$ the dimension of $D_k$. In particular, we have

$$D_1 = \text{span}\{1\}, \quad \mu_1 = 1,$$

$$D_2 = \text{span}\{x_i; (i = 1, ..., n)\}, \quad \mu_2 = n,$$

$$D_3 = \text{span}\{x_ix_j; x_1^2 - x_h^2; (i, j = 1, ..., n, i \neq j, h = 2, ..., n)\}, \quad \mu_3 = \frac{n^2 + n - 2}{2}.$$

Then, we prove

Theorem 3. If $n \geq 2$ and $\Omega = B$, then for all $k = 1, 2, 3, ...$

(i) the eigenvalues of (1.1) are $d_k = n + 2(k - 1)$;

(ii) the multiplicity of $d_k$ equals $\mu_k$;

(iii) for all $\psi_k \in D_k$, the function $\varphi_k(x) := (1 - |x|^2)\psi_k(x)$ is an eigenfunction corresponding to $d_k$.

Remark 1. Theorems 1 and 3 become false if $n = 1$. It is shown in [3] that the one dimensional problem

$$u'' = 0 \quad \text{in } (-1, 1), \quad u(\pm 1) = u''(-1) + du'(-1) = u''(1) - du'(1) = 0, \quad (1.9)$$

admits only two eigenvalues, $d_1 = 1$ and $d_2 = 3$, each one of multiplicity 1. The reason of this striking difference is that the “boundary space” of (1.9) has precisely dimension 2, one for each endpoint of the interval $(-1, 1)$. This result is consistent with Theorem 3 since $\mu_1 = \mu_2 = 1$ and $\mu_3 = 0$ whenever $n = 1$. 

3
By combining Theorems 1 and 3 we obtain

**Corollary 1.** Assume that \( n \geq 2 \) and that \( \Omega = B \). Assume moreover that for all \( k \in \mathbb{N} \) the set \( \{\psi^k_\ell : \ell = 1, \ldots, \mu_k\} \) is a basis of \( D_k \) chosen in such a way that the corresponding functions \( \varphi^k_\ell \) are orthonormal with respect to the scalar product (1.6). Then, for any \( u \in V \) there exists a sequence \( \{\alpha^k_\ell\} \in \ell^2 \) (\( k \in \mathbb{N}; \ell = 1, \ldots, \mu_k \)) such that

\[
  u(x) = (1 - |x|^2) \sum_{k=1}^{\infty} \sum_{\ell=1}^{\mu_k} \alpha^k_\ell \psi^k_\ell(x) \quad \text{for a.e. } x \in B .
\]

We now restrict our attention to the *first* eigenvalue \( d_1(\Omega) \). To this end, we recall a consequence of Fichera’s principle of duality [6]. Assume that \( \partial \Omega \in C^2 \) and let

\[
  C^2_H(\Omega) := \{ v \in C^2(\overline{\Omega}); \Delta v = 0 \text{ in } \Omega \} .
\]

We consider the norm defined by \( \|v\|_H := \|v\|_{L^2(\partial \Omega)} \) for all \( v \in C^2_H(\Omega) \). Then, we consider

\[
  H := \text{the closure of } C^2_H(\Omega) \text{ with respect to the norm } \| \cdot \|_H .
\]

Finally, we define

\[
  \delta_1 = \delta_1(\Omega) := \min_{h \in H \setminus \{0\}} \frac{\int_{\partial \Omega} h^2}{\int_{\Omega} h^2} . \tag{1.10}
\]

The minimum in (1.10) is achieved. To see this, combine the continuous embedding (for weakly harmonic functions) \( H^{-1/2}(\partial \Omega) \subset L^2(\Omega) \) (see Théorème 6.6 in Ch. 2 in [14]) with the compact embedding \( L^2(\partial \Omega) \subset H^{-1/2}(\partial \Omega) \).

Fichera’s principle [6] states that

\[
  \delta_1(\Omega) = d_1(\Omega) \quad \text{for all } \Omega \text{ such that } \partial \Omega \in C^2 . \tag{1.11}
\]

For the reader’s convenience we quote a new proof of this statement in Section 5.

**Problem 1.** Does equality (1.11) hold also if \( \partial \Omega \notin C^2 \)? This question is strictly related with the following: are the maps \( \Omega \mapsto d_1(\Omega) \) and \( \Omega \mapsto \delta_1(\Omega) \) continuous with respect to Hausdorff convergence of domains?

In view of the important applications explained in the introduction, one is interested in finding both lower and upper bounds for \( d_1(\Omega) \). First, we extend to any space dimension \( n \geq 2 \) a lower bound for \( d_1(\Omega) \) obtained for planar domains \( (n = 2) \) by Payne [15], see also Kuttler [9] for a different proof. This estimate is useful for convex domains since it involves the mean curvature of the boundary as stated in the following

**Theorem 4.** Let \( \Omega \subset \mathbb{R}^n \) \( (n \geq 2) \) a bounded convex open domain with \( C^2 \) boundary. For all \( x \in \partial \Omega \), let \( K(x) \) denote the mean curvature at \( x \) and let

\[
  \overline{K} := \min_{x \in \partial \Omega} K(x) .
\]

Then \( d_1(\Omega) \geq n \overline{K} \) and equality holds if and only if \( \Omega \) is a ball.
By rescaling, it is not difficult to see that the map \( \Omega \mapsto d_1(\Omega) \) is homogeneous of degree \(-1\), namely, for any \( \Omega \) and \( k > 0 \) we have \( d_1(\Omega) = kd_1(k\Omega) \). This suggests that \( d_1 \) should somehow be related to the “isoperimetric ratio” \( \frac{|\partial \Omega|}{|\Omega|} \). As noticed by Kuttler [9], this is indeed the case: by taking \( h \equiv 1 \) in (1.10) and using (1.11) one readily gets

\[
d_1(\Omega) \leq \frac{|\partial \Omega|}{|\Omega|} \quad \text{for all } \Omega.
\] (1.12)

In Section 7, we determine extremal sets for the isoperimetric inequality (1.12):

**Theorem 5.** Let \( \Omega \subset \mathbb{R}^n \) (\( n \geq 2 \)) be a bounded connected domain with \( C^2 \) boundary. Then

\[
d_1(\Omega) = \frac{|\partial \Omega|}{|\Omega|}
\]

if and only if \( \Omega \) is a ball.

**Problem 2.** The above mentioned homogeneity also suggests that the map \( \Omega \mapsto d_1(\Omega) \) could be monotone decreasing with respect to domain inclusions. Is this true?

We now deal with a particular class of nonsmooth domains. It is clear that (planar) rectangles are slightly easier to handle than general domains. We consider the two families of rectangles

\[
\forall a \in \left(0, \frac{\pi}{2}\right], \quad R_a := (0, \pi - a) \times (0, a), \quad \forall \alpha \in (0, \sqrt{\pi}], \quad Q_\alpha := \left(0, \frac{\pi}{\alpha}\right) \times (0, \alpha). \quad (1.13)
\]

We note that the \( R_a \)'s have the same perimeter \( 2\pi \) as the unit disk, whereas the \( Q_\alpha \)'s have the same area \( \pi \) as the unit disk. Moreover, \( R_{\pi/2} \) and \( Q_{\sqrt{\pi}} \) are squares.

Smith [18, 19] conjectured that for any domain \( \Omega \), one has \( d_1(\Omega) \geq d_1(\Omega^*) \), where \( \Omega^* \) is the ball having the same measure as \( \Omega \). In particular, for planar domains \( \Omega \) of measure \( \pi \) (as the unit disk), this would mean that \( d_1(\Omega) \geq 2 \). This conjecture was disproved by Kuttler [9] which shows that

\[
d_1(Q_{\sqrt{\pi}}) < 1.9889... \quad (1.14)
\]

In fact, Kuttler’s estimate [9, p.3] is given for the square \( Q' := (0,1)^2 \) for which \( d_1(Q') < 3.5254... \), so that (1.14) is obtained by rescaling this inequality. In order to prove (1.14), Kuttler uses directly the characterization (1.3) of \( d_1(Q_{\sqrt{\pi}}) \) and finds a suitable linear combination of a fourth order polynomial with the stress function of \( Q_{\sqrt{\pi}} \) to get an upper bound for \( d_1(Q_{\sqrt{\pi}}) \). In Section 8, by using Fichera’s principle (1.11) and a much simpler trial function, we improve (1.14) with the following

\[
d_1(Q_{\sqrt{\pi}}) < 1.96256. \quad (1.15)
\]

We now consider rectangles with sizes of different length. In Section 9 we show that if they maintain the same perimeter and become “thin” then \( d_1 \) tends to infinity:

**Theorem 6.** For all \( 0 < a \leq \frac{\pi}{2} \) let \( R_a \) be as in (1.13). Then,

\[
\frac{\pi}{2} \leq \liminf_{a \to 0} \left[ a \cdot d_1(R_a) \right] \leq \limsup_{a \to 0} \left[ a \cdot d_1(R_a) \right] \leq \pi d_1(R_{\pi/2}) < (2.2146)\pi.
\]
Theorem 6 complements some numerical approximations of $d_1$ for rectangles obtained by Kuttler [10] using a posteriori / a priori inequalities, see also [12]. We recall here his results: for our convenience, we scale [10, Table 1] to the case of the rectangles $R_a$ defined in (1.13) and we add the last column, according to Theorem 6.

<table>
<thead>
<tr>
<th>$\frac{\pi}{2}$</th>
<th>$\frac{5\pi}{11}$</th>
<th>$\frac{4\pi}{9}$</th>
<th>$\frac{3\pi}{7}$</th>
<th>$\frac{2\pi}{5}$</th>
<th>$\frac{\pi}{3}$</th>
<th>$\frac{2\pi}{7}$</th>
<th>$\rightarrow 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d_1(R_a)$</td>
<td>2.2118</td>
<td>2.2261</td>
<td>2.2331</td>
<td>2.2459</td>
<td>2.4493</td>
<td>2.666</td>
<td>$\rightarrow \infty$</td>
</tr>
<tr>
<td>$d_1(R_a)$</td>
<td>2.2133</td>
<td>2.2304</td>
<td>2.2359</td>
<td>2.2498</td>
<td>2.2878</td>
<td>2.4542</td>
<td>2.6839</td>
</tr>
</tbody>
</table>

In view of these results, Kuttler suggests a new and weaker conjecture, which we state for any space dimension $n$:

**Conjecture 1.** [9]

Let $B \subset \mathbb{R}^n$ denote the unit ball. Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain whose surface measure satisfies $|\partial \Omega| = |\partial B|$. Then, $n = d_1(B) \leq d_1(\Omega)$.

In other words, the first eigenvalue $d_1$ is expected to be minimal on the ball, among all domains having the same surface measure. In connection with this conjecture, we also make the following

**Remark 2.** For the second order Stekloff problem

$$\left\{ \begin{array}{ll} \Delta u = 0 & \text{in } \Omega \\ u_\nu = \lambda u & \text{on } \partial \Omega , \end{array} \right. \quad (1.16)$$

it is known (see [5, Theorem 3]) that the first (nontrivial) eigenvalue $\lambda_1 = \lambda_1(\Omega)$ satisfies $\lambda_1(\Omega) \leq \lambda_1(\Omega^*)$ where

$$\lambda_1 = \lambda_1(\Omega) := \inf_{u \in \mathcal{H}(\Omega)} \frac{\int_{\Omega} |\nabla u|^2}{\int_{\partial \Omega} u^2} \quad \text{and} \quad \mathcal{H}(\Omega) := \left\{ u \in H^1(\Omega) \setminus H^1_0(\Omega); \int_{\partial \Omega} u = 0 \right\} .$$

Therefore, (1.15) and the possible validity of Conjecture 1 would show that the fourth order problem (1.1) and the second order problem (1.16) are completely different.

Consider now the rectangles $Q_\alpha$ defined in (1.13) and which have fixed area. By Theorem 6 and by rescaling (it suffices to put $\alpha = \sqrt{a\pi/(\pi - a)}$) we obtain

**Corollary 2.** For all $0 < \alpha \leq \sqrt{\pi}$ let $Q_\alpha$ be as in (1.13). Then,

$$\frac{\pi}{2} \leq \liminf_{\alpha \to 0} \left[ \alpha \cdot d_1(Q_\alpha) \right] \leq \limsup_{\alpha \to 0} \left[ \alpha \cdot d_1(Q_\alpha) \right] \leq \pi d_1(R_{\pi/2}) < (2.2146)\pi .$$

Therefore, also for thinning rectangles of fixed area, the first eigenvalue $d_1$ tends to infinity, although at a lower rate. Then, we may also rescale Table 1 above and obtain for the rectangles $Q_\alpha$ in (1.13):
Table 2. Numerical estimates for \( d_1(Q_\alpha) \) of rectangles \( Q_\alpha \) such that \( |Q_\alpha| = \pi \).

<table>
<thead>
<tr>
<th>smallest side ( \alpha )</th>
<th>( \sqrt{\pi} )</th>
<th>( \sqrt{5\pi}/6 )</th>
<th>( \sqrt{4\pi}/5 )</th>
<th>( \sqrt{3\pi}/4 )</th>
<th>( \sqrt{2\pi}/3 )</th>
<th>( \sqrt{\pi}/2 )</th>
<th>( \sqrt{2\pi}/5 )</th>
<th>( \rightarrow 0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( d_1(Q_\alpha) &gt; )</td>
<td>1.9601</td>
<td>1.9647</td>
<td>1.9667</td>
<td>1.97</td>
<td>1.9838</td>
<td>2.0465</td>
<td>2.1347</td>
<td>( \rightarrow \infty )</td>
</tr>
<tr>
<td>( d_1(Q_\alpha) &lt; )</td>
<td>1.9615</td>
<td>1.9685</td>
<td>1.9693</td>
<td>1.9734</td>
<td>1.9866</td>
<td>2.0506</td>
<td>2.149</td>
<td></td>
</tr>
</tbody>
</table>

The numerical estimate in the first column of Table 2 seems to show that (1.15) is almost optimal.

**Problem 3.** From (1.15) we know that the ball (at least if \( n = 2 \)) is not the minimizer for \( d_1 \) among domains having the same measure. Does there exist an optimal shape for this minimization problem? At least in the class of convex domains, we feel that the answer could be affirmative.

2 Proof of Theorem 1

Let \( Z \) be as in (1.7) and define on \( Z \) the scalar product given by

\[
(u, v)_W := \int_{\partial \Omega} u \nu v \nu dS \quad \text{for all } u, v \in Z
\]

(2.1)

and we denote by \( W \) the completion of \( Z \) with respect to this scalar product. Then, we prove:

**Lemma 1.** The (Hilbert) space \( V \) is compactly embedded into the (Hilbert) space \( W \).

**Proof.** By definition of \( d_1 \) we have

\[
\|u\|_W = \|u\|_{L^2(\partial \Omega)} \leq d_1^{-1/2} \|\Delta u\|_{L^2(\Omega)} = d_1^{-1/2} \|u\|_V \quad \forall u \in Z
\]

(2.2)

and hence any Cauchy sequence in \( Z \) with respect to the norm of \( V \) is a Cauchy sequence with respect to the norm of \( W \). Since \( V \) is the completion of \( Z \) with respect to (1.6), it follows immediately that \( V \subset W \). The continuity of this inclusion can be obtained by density from (2.2).

It remains to prove that this embedding is compact. To this purpose, let \( u_m \rightharpoonup u \) in \( V \), so that also \( u_m \rightarrow u \) in \( H^1_0(\Omega) \). Then by trace embedding and compact embedding \( H^{1/2}(\partial \Omega) \subset L^2(\partial \Omega) \) we obtain immediately \( u_m \rightarrow u \) in \( W \). \( \square \)

Denote by \( I_1 : V \rightarrow W \) the embedding \( V \subset W \) and by \( I_2 : W \rightarrow V' \) the continuous linear operator defined by

\[
\langle I_2 u, v \rangle = (u, v)_W \quad \forall u \in W, \forall v \in V.
\]

Moreover, let \( L : V \rightarrow V' \) be the linear operator given by

\[
\langle Lu, v \rangle = \int_{\Omega} \Delta u \Delta v \, dx \quad \forall u, v \in V.
\]

Then \( L \) is an isomorphism and in view of Lemma 1, the linear operator \( K = L^{-1}I_2I_1 : V \rightarrow V \) is compact. Since for \( n \geq 2 \), \( V \) is an infinite dimensional Hilbert space and \( K \) is a compact self-adjoint operator with strictly positive eigenvalues then \( V \) admits an orthonormal base of eigenfunctions of \( K \) and the set of the eigenvalues of \( K \) can be ordered in a strictly decreasing sequence \( \{\mu_i\} \) which converges to zero.

Therefore problem (1.2) admits an infinite set of eigenvalues given by \( d_i = \frac{1}{\mu_i} \) and the eigenfunctions of (1.2) coincide with the eigenfunctions of \( K \). The demonstration of Theorem 1 will be complete once we prove
Lemma 2. If $d_k$ is an eigenvalue of (1.1) corresponding to a positive eigenfunction $\varphi_k$ then $d_k = d_1$.

Proof. Since $\varphi_k > 0$ in $\Omega$ and $\varphi_k = 0$ on $\partial \Omega$, then $(\varphi_k)_\nu \leq 0$ on $\partial \Omega$ and in turn $\Delta \varphi_k = d_k (\varphi_k)_\nu \leq 0$ on $\partial \Omega$. Therefore by $\Delta^2 \varphi_k = 0$ in $\Omega$ and the weak comparison principle, we infer $\Delta \varphi_k \leq 0$ in $\Omega$. Moreover, since $\varphi_k > 0$ in $\Omega$ and $\varphi_k = 0$ on $\partial \Omega$, the Hopf boundary lemma implies that $(\varphi_k)_\nu < 0$ on $\partial \Omega$.

Let $\varphi_1$ be a positive eigenfunction corresponding to the first eigenvalue $d_1$ (see Theorem 1 in [3]). Then $\varphi_1$ also satisfies $(\varphi_1)_\nu < 0$ on $\partial \Omega$ and hence from

$$d_k \int_{\partial \Omega} (\varphi_k)_\nu (\varphi_1)_\nu dS = \int_{\Omega} \Delta \varphi_k \Delta \varphi_1 dx = d_1 \int_{\partial \Omega} (\varphi_k)_\nu (\varphi_1)_\nu dS > 0$$

we obtain $d_k = d_1$. □

3 Proof of Theorem 2

We start by proving that $Z^\perp = H^2_0(\Omega)$. Let $v \in Z$ and $w \in H^2 \cap H^1_0(\Omega)$. After two integrations by parts we obtain

$$\int_{\Omega} \Delta^2 vw \, dx = \int_{\Omega} \Delta^2 wv \, dx + \int_{\partial \Omega} (w_\nu \Delta v - w (\Delta v)_\nu) \, dS = \int_{\partial \Omega} w_\nu \Delta v \, dS$$

for all $v \in Z$ and $w \in H^2 \cap H^1_0(\Omega)$. This proves that $w_\nu = 0$ on $\partial \Omega$ if and only if $w \in Z^\perp$ and hence $V^\perp = Z^\perp = H^2_0(\Omega)$.

Let $v \in H^2 \cap H^1_0(\Omega)$ and consider the first Dirichlet problem in (1.8):

$$\begin{cases}
\Delta^2 v_1 = 0 & \text{in } \Omega \\
v_1 = 0 & \text{on } \partial \Omega \\
(\nu v)_\nu = v_\nu & \text{on } \partial \Omega.
\end{cases} \quad (3.1)$$

Since $v_\nu \in H^{1/2}(\partial \Omega)$, by Lax-Milgram Theorem and Théorème 8.3 in Ch. 1 in [14], we deduce that (3.1) admits a unique solution $v_1 \in H^2 \cap H^1_0(\Omega)$ such that

$$\|\Delta v_1\|_{L^2(\Omega)} \leq C \|v_\nu\|_{H^{1/2}(\partial \Omega)} \quad (3.2)$$

This proves that $v_1 \in V$. Let $v_2 = v - v_1$, then $(v_2)_\nu = 0$ on $\partial \Omega$ and, in turn, $v_2 \in H^2_0(\Omega)$. Moreover, by (3.1) we infer

$$\int_{\Omega} \Delta v_2 \Delta w \, dx = \int_{\Omega} \Delta v \Delta w \, dx - \int_{\Omega} \Delta v_1 \Delta w \, dx = \int_{\Omega} \Delta v \Delta w \, dx \quad \forall w \in H^2_0(\Omega) \quad (3.3)$$

which proves that $v_2$ is a weak solution of the second problem in (1.8).

4 Proof of Theorem 3

We start with the following technical result:
Lemma 3. Let $k \in \mathbb{N}$, let $\Omega \subset \mathbb{R}^n$ ($n \geq 2$) be a bounded domain with boundary $\partial \Omega$ of class $C^{k+1}$. Let $u \in C^{k+1}(\overline{\Omega})$ be such that $u \equiv 0$ on $\partial \Omega$. Consider the function $\varphi : \Omega \to \mathbb{R}$ defined by

$$
\varphi(x) := \frac{u(x)}{\text{dist}(x, \partial \Omega)}.
$$

Then, there exists $\delta > 0$ such that $\varphi \in C^k(\Omega_\delta)$, with $\Omega_\delta = \{x \in \overline{\Omega} : \text{dist}(x, \partial \Omega) < \delta\}$.

Proof. It is well-known (see e.g. [2]) that there exists $\delta > 0$ such that $x \mapsto \text{dist}(x, \partial \Omega)$ is of class $C^{k+1}(\Omega_\delta)$; this will give the “size” of $\delta$. Therefore, by local $C^{k+1}$-charts, we may restrict our attention to the case where

$$
\Omega = \mathbb{R}^n_+ , \quad \partial \Omega = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n ; x_1 = 0\} , \quad \text{dist}(x, \partial \Omega) = x_1 , \quad \varphi(x) = \frac{u(x)}{x_1},
$$

with $u \in C^{k+1}(\{x_1 \geq 0\})$ and $u(x) = 0$ whenever $x \in \partial \Omega$. We denote $x = (x_1, x')$; then, by the mean value Theorem, we have

$$
\varphi(x) = \frac{u(x) - u(0, x')}{x_1} = \frac{1}{x_1} \int_0^1 \frac{\partial u}{\partial x_1}(tx_1, x') x_1 dt = \int_0^1 \frac{\partial u}{\partial x_1}(tx_1, x') dt.
$$

Since $\frac{\partial u}{\partial x_1} \in C^k(\{x_1 \geq 0\})$, we conclude that also $\varphi \in C^k(\{x_1 \geq 0\})$. \qed

Consider now an eigenfunction $u$ of (1.1). Then $u \in C^\infty(\overline{B})$ and by Lemma 3 we can write

$$
u(x) = (1 - |x|^2)\varphi(x) \quad \quad (4.1)
$$

with $\varphi \in C^\infty(\overline{B})$. We have

$$
u_{x_i} = -2x_i \varphi + (1 - |x|^2)\varphi_{x_i},
$$

and on $\partial B$,

$$
u = x \cdot \nabla u = x \cdot (-2x \varphi + (1 - |x|^2)\nabla \varphi) = -2\varphi. \quad \quad (4.2)
$$

Moreover,

$$\Delta u = -2n\varphi - 4x \cdot \nabla \varphi + (1 - |x|^2)\Delta \varphi. \quad \quad (4.3)
$$

Hence

$$\Delta u = -2n\varphi - 4\varphi_{x_i} \quad \text{on} \quad \partial B. \quad \quad (4.4)
$$

From (4.3) we get for $i = 1, \ldots, n$,

$$(\Delta u)_{x_i} = -(2n + 4)\varphi_{x_i} - 4 \sum_{j=1}^n x_j \varphi_{x_j x_i} - 2x_i \Delta \varphi + (1 - |x|^2)\Delta \varphi_{x_i},
$$

and therefore

$$(\Delta u)_{x_i x_i} = -2(n + 4)\varphi_{x_i x_i} - 4x \cdot \nabla (\varphi_{x_i x_i}) - 2\Delta \varphi - 4x_i (\Delta \varphi)_{x_i} + (1 - |x|^2)\Delta^2 \varphi_{x_i x_i}.
$$

Summing with respect to $i$ and recalling that $u$ is biharmonic in $B$, we obtain

$$0 = \Delta^2 u = -2(n + 4)\Delta \varphi - 4x \cdot \nabla \Delta \varphi - 2n\Delta \varphi - 4x \cdot \nabla \Delta \varphi + (1 - |x|^2)\Delta^2 \varphi
$$

$$(1 - |x|^2)\Delta^2 \varphi - 8x \cdot \nabla \Delta \varphi - 4(n + 2)\Delta \varphi. \quad \quad (4.5)$$
Writing (4.5) as an equation in \( w = \Delta \varphi \), we get
\[
(1 - |x|^2)\Delta w - 8x \cdot \nabla w - 4(n + 2)w = 0 \quad \text{in } B,
\]
so that
\[
0 = -(1 - |x|^2)4\Delta w + 8(1 - |x|^2)x \cdot \nabla w + 4(n + 2)(1 - |x|^2)^3w. \tag{4.6}
\]
Multiplying the right hand side of (4.6) by \( w \) and integrating by parts over \( B \), we obtain
\[
\int_B (1 - |x|^2)^4|\nabla w|^2 + 4(n + 2) \int_B (1 - |x|^2)^3w^2 = \int_{\partial B} (1 - |x|^2)^4ww_{\nu} = 0.
\]
Hence \( \Delta \varphi = w \equiv 0 \) in \( B \). Now from (1.1), (4.2) and (4.4) we get
\[
\varphi_{\nu} = \frac{d - n}{2}\varphi \quad \text{on } \partial B. \tag{4.7}
\]
Therefore, we obtained the following result:

**Lemma 4.** The number \( d \) is an eigenvalue of (1.1) with corresponding eigenfunction \( u \) if and only if \( \varphi \) defined by (4.1) is an eigenfunction of the boundary eigenvalue problem
\[
\left\{ \begin{array}{l}
\Delta \varphi = 0 \quad \text{in } B \\
\varphi_{\nu} = a\varphi \quad \text{on } \partial B,
\end{array} \right. \tag{4.8}
\]
where \( a = \frac{d - n}{2} \).

So we are led to study the eigenvalues of the second order Stekloff problem (4.8). Since we were unable to find an explicit reference, we quickly explain how to obtain them. In radial and angular coordinates \((r, \theta)\), the equation in (4.8) reads
\[
\frac{\partial^2 \varphi}{\partial r^2} + \frac{n - 1}{r} \frac{\partial \varphi}{\partial r} + \frac{1}{r^2} \Delta_\theta \varphi = 0, \tag{4.9}
\]
where \(-\Delta_\theta\) denotes the Laplace-Beltrami operator on \( \partial B \). From [4, p.160] we quote

**Lemma 5.** The Laplace-Beltrami operator \(-\Delta_\theta\) admits a sequence of eigenvalues \( \{\lambda_k\} \) having multiplicity \( \mu_k \) equal to the number of independent harmonic homogeneous polynomials of degree \( k - 1 \). Moreover, \( \lambda_k = (k - 1)(n + k - 3) \).

In the sequel, we denote by \( e^\ell_k \) \((\ell = 1, \ldots, \mu_k)\) the independent normalized eigenfunctions corresponding to \( \lambda_k \). Then, one seeks functions \( \varphi = \varphi(r, \theta) \) of the kind
\[
\varphi(r, \theta) = \sum_{k=1}^{\infty} \sum_{\ell=1}^{\mu_k} \varphi_k^\ell(r)e^\ell_k(\theta).
\]
Hence, by differentiating the series, we obtain
\[
\Delta \varphi(r, \theta) = \sum_{k=1}^{\infty} \sum_{\ell=1}^{\mu_k} \left( \frac{d^2}{dr^2} \varphi_k^\ell(r) + \frac{n - 1}{r} \frac{d}{dr} \varphi_k^\ell(r) - \frac{\lambda_k}{r^2} \varphi_k^\ell(r) \right) e^\ell_k(\theta) = 0.
\]
Therefore, we must solve the equations

\[
\frac{d^2}{dr^2} \varphi_k^\ell(r) + \frac{n-1}{r} \frac{d}{dr} \varphi_k^\ell(r) - \frac{\lambda_k}{r^2} \varphi_k^\ell(r) = 0 \quad k = 1, 2, \ldots \ell = 1, \ldots, \mu_k.
\]  

(4.10)

With the change of variables \( r = e^t \) \((t \leq 0)\), equation (4.10) becomes a linear constant coefficients equation. It has two linearly independent solutions, but one is singular. Hence, up to multiples, the only regular solution of (4.10) is given by \( \varphi_k^\ell(r) = r^{k-1} \) because

\[
\frac{2 - n + \sqrt{(n-2)^2 + 4\lambda_k}}{2} = k - 1.
\]

Since the boundary condition in (4.8) reads \( \frac{d}{dr} \varphi_k^\ell(1) = a \varphi_k^\ell(1) \) we immediately infer that \( a = \bar{k} - 1 \) for some \( \bar{k} \). In turn, Lemma 4 tells us that

\[
d_k = n + 2(\bar{k} - 1).
\]

The proof of Theorem 3 is so complete.

5 Proof of Fichera’s principle of duality (1.11)

We say that \( \delta \) is an eigenvalue relative to problem (1.10) if there exists \( g \in H \) such that

\[
\delta \int_\Omega gv = \int_{\partial \Omega} gv \quad \text{for all } v \in H.
\]

Clearly, \( \delta_1 \) is the least eigenvalue. We prove (1.11) by showing that both \( \delta_1 \geq d_1 \) and \( \delta_1 \leq d_1 \).

Proof of \( \delta_1 \geq d_1 \). Let \( h \) be a minimizer for \( \delta_1 \), then

\[
\delta_1 \int_\Omega hv = \int_{\partial \Omega} hv \quad \text{for all } v \in H.
\]

(5.1)

Let \( u \in H(\Omega) \) be the unique solution of

\[
\begin{cases}
\Delta u = h & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega.
\end{cases}
\]

(5.2)

Integrating by parts we have

\[
\int_\Omega hv = \int_\Omega v\Delta u = \int_{\partial \Omega} vu_\nu \quad \text{for all } v \in H \cap C^2(\Omega).
\]

By a density argument, the latter follows for all \( v \in H \). Inserting this into (5.1) gives

\[
\delta_1 \int_{\partial \Omega} vu_\nu = \int_{\partial \Omega} v\Delta u \quad \text{for all } v \in H.
\]
This yields $\Delta u = \delta_1 u_\nu$ on $\partial \Omega$. Therefore,

$$\delta_1 = \frac{\int_{\partial \Omega} h^2}{\int_{\Omega} h^2} = \frac{\int_{\Omega} |\Delta u|^2}{\int_{\Omega} |u_\nu|^2} = \delta_1^2 \frac{\int_{\partial \Omega} u_\nu^2}{\int_{\Omega} |\Delta u|^2}.$$

In turn, this implies that

$$\delta_1 = \frac{\int_{\partial \Omega} |\Delta u|^2}{\int_{\partial \Omega} u_\nu^2} \geq \min_{v \in H(\Omega)} \frac{\int_{\partial \Omega} |\Delta v|^2}{\int_{\partial \Omega} v_\nu^2} = d_1.$$  

**Proof of $\delta_1 \leq d_1$.** Let $u$ be a minimizer for $d_1$ in (1.3), then $\Delta u = d_1 u_\nu$ on $\partial \Omega$ so that $\Delta u \in H^{1/2}(\partial \Omega) \subset L^2(\partial \Omega)$ and

$$\int_{\Omega} v \Delta u = d_1 \int_{\partial \Omega} v u_\nu$$  

for all $v \in H$. (5.3)

Let $h := \Delta u$ so that $h \in L^2(\Omega) \cap L^2(\partial \Omega)$. Moreover, $\Delta h = \Delta^2 u = 0$ (in distributional sense) and hence $h \in H$. Two integrations by parts (and a density argument) yield

$$\int_{\Omega} h v = \int_{\partial \Omega} v u_\nu$$  

for all $v \in H$.

Replacing this into (5.3) gives

$$\int_{\partial \Omega} h v = d_1 \int_{\Omega} h v$$  

for all $v \in H$.

This proves that $h$ is an eigenfunction relative to problem (1.10) with corresponding eigenvalue $d_1$. Since $\delta_1$ is the least eigenvalue, we obtain $d_1 \geq \delta_1$.

### 6 Proof of Theorem 4

Let $\varphi$ be a first eigenfunction of (1.1) such that $\varphi > 0$ in $\Omega$ and $\varphi_\nu < 0$ on $\partial \Omega$ (see Lemma 2). The boundary condition $\Delta \varphi = d_1 \varphi_\nu$ on $\partial \Omega$ also reads

$$\varphi_{\nu\nu} + (n - 1) K \varphi_\nu = d_1 \varphi_\nu$$  

on $\partial \Omega$  

(6.1)  

(see e.g. (4.68) p.62 in [16]). Therefore

$$(\varphi_\nu^2)_\nu = 2 \varphi_{\nu\nu} \varphi_\nu = 2 [d_1 - (n - 1) K] \varphi_\nu^2$$

so that if we put $D^2 \varphi D^2 \varphi = \sum_{i,j=1}^n (\partial_{ij} \varphi)^2$, by (1.1) and integration by parts, we obtain

$$2 \int_{\partial \Omega} [d_1 - (n - 1) K] \varphi_\nu^2 dS = \int_{\partial \Omega} (\varphi_\nu^2)_\nu dS = \int_{\partial \Omega} (|\nabla \varphi|^2)_\nu dS = \int_{\Omega} \Delta (|\nabla \varphi|^2) dx$$
\[2 \int_{\Omega} \nabla (\Delta \varphi) \nabla \varphi \, dx + 2 \int_{\Omega} D^2 \varphi D^2 \varphi \, dx = -2 \int_{\Omega} \varphi \Delta^2 \varphi \, dx + 2 \int_{\partial \Omega} \varphi (\Delta \varphi)_{\nu} \, dS + 2 \int_{\Omega} D^2 \varphi D^2 \varphi \, dx\]
\[= 2 \int_{\Omega} D^2 \varphi D^2 \varphi \, dx \geq \frac{2}{n} \int_{\Omega} |\Delta \varphi|^2 \, dx.\]

Finally, by (1.2) we have
\[2 \int_{\partial \Omega} [d_1 - (n-1) K] \varphi^2_{\nu} \, dS \geq \frac{2d_1}{n} \int_{\partial \Omega} \varphi^2_{\nu} dS\]
from which we obtain
\[\int_{\partial \Omega} \left( \frac{d_1}{n} - K \right) \varphi^2_{\nu} dS \geq 0, \quad (6.2)\]
which implies at once that \(d_1 \geq nK\).

It remains to prove that equality holds if and only if \(\Omega\) is a ball. If \(d_1 = nK\), then \(d_1 \leq nK(x)\) for \(x \in \partial \Omega\) and since \(\varphi_{\nu} < 0\) on \(\partial \Omega\), by (6.2) we infer that \(K(x) \equiv \frac{d_1}{n}\). This proves that \(\Omega\) is a ball in view of Alexandrov’s characterization of spheres [1].

7 Proof of Theorem 5

Assume that equality holds in (1.12). Then, \(h \equiv 1\) is a minimizer for (1.10) and, according to Fichera’s principle (see (5.2)), the minimizer \(u\) of (1.3) is the stress function for \(\Omega\) (the solution of the torsion problem), namely
\[\begin{cases} -\Delta u = 1 & \text{in } \Omega \\ u = 0 & \text{on } \partial \Omega. \end{cases}\]

Since \(u\) also solves the Euler equation (1.1) with \(d = d_1\), we have a solution to the problem
\[-\Delta u = 1 \quad \text{in } \Omega, \quad u_{\nu} = -d_1^{-1} \quad \text{on } \partial \Omega, \quad u = 0 \quad \text{on } \partial \Omega.\]

By a result of Serrin [17], this shows that \(\Omega\) is a ball and completes the proof.

8 Proof of (1.15)

In view of the results in [8], we may argue as for (5.2) in order to show that \(d_1 \leq \delta_1\).

For our convenience, we translate the square \(Q_{\sqrt{\pi}}\) and consider instead
\[Q := \left( -\frac{\sqrt{\pi}}{2}, \frac{\sqrt{\pi}}{2} \right)^2.\]

For all \(k \in \mathbb{R}\), consider the harmonic function \(h_k(x, y) := x^4 + y^4 - 6x^2y^2 + k\). Then by (1.10)-(1.11) we have
\[d_1(Q) \leq \delta_1(Q) \leq \frac{\int_{\partial Q} h_k^2 dS}{\int_Q h_k^2 \, dx dy} \quad \text{for all } k \in \mathbb{R}. \quad (8.1)\]
Via direct computation we obtain
\[
\int_{\partial Q} h_k^2 dS = \sqrt{\pi} \left( 4k^2 - \frac{2}{5} \pi^2 k + \frac{59}{1200} \pi^4 \right) \quad \text{and} \quad \int_Q h_k^2 dxdy = \pi k^2 - \frac{1}{30} \pi^3 k + \frac{59}{25200} \pi^5.
\]
We recall that \(3.141592 < \pi < 3.141593\). Hence, if we choose \(k = 2.69\) we obtain
\[
\int_{\partial Q} h_k^2 dS < \sqrt{3.141593} \left( 4 \cdot (2.69)^2 - \frac{2}{5} \cdot (2.69) \cdot (3.141592)^2 + \frac{59}{1200} \cdot (3.141593)^4 \right) < 40.56426
\]
and
\[
\int_Q h_k^2 dxdy > (3.141592) \cdot (2.69)^2 - \frac{(3.141593)^3}{30} \cdot (2.69) + \frac{59}{25200} \cdot (3.141592)^5 > 20.66911.
\]
By inserting these estimates into (8.1) we obtain (1.15).

9 Proof of Theorem 6

Let \(u \in H^2 \cap H^1_0(R_{\pi/2})\) be a (positive) minimizer for (1.3) when \(\Omega = R_{\pi/2}\):
\[
d_1(R_{\pi/2}) = \frac{\int_{R_{\pi/2}} |\Delta u|^2}{\int_{\partial R_{\pi/2}} u^2}. 
\]
By uniqueness of the minimizer, \(u\) is symmetric in \(R_{\pi/2}\) so that
\[
\int_0^{\pi/2} u_x^2(0, y) dy = \int_0^{\pi/2} u_x^2 \left( \frac{\pi}{2}, y \right) dy = \int_0^{\pi/2} u_y^2(x, 0) dx = \int_0^{\pi/2} u_y^2 \left( x, \frac{\pi}{2} \right) dx. \quad (9.1)
\]
Fix \(a \in (0, \pi/2)\), let \(R_a := (0, \pi - a) \times (0, a)\) and consider the function \(v \in H^2 \cap H^1_0(R_a)\) defined by
\[
v(x, y) = u \left( \frac{\pi x}{2(\pi - a)}, \frac{\pi y}{2a} \right).
\]
Then,
\[
d_1(R_a) \leq \frac{\int_{R_a} |\Delta v|^2}{\int_{\partial R_a} v^2}. \quad (9.2)
\]
and we estimate the two integrals in the right hand side of (9.2). We have
\[
v_x(x, y) = \frac{\pi}{2(\pi - a)} \cdot u_x \left( \frac{\pi x}{2(\pi - a)}, \frac{\pi y}{2a} \right) \quad \text{and} \quad v_{xx}(x, y) = \left( \frac{\pi}{2(\pi - a)} \right)^2 \cdot u_{xx} \left( \frac{\pi x}{2(\pi - a)}, \frac{\pi y}{2a} \right),
\]
\[
v_y(x, y) = \frac{\pi}{2a} \cdot u_y \left( \frac{\pi x}{2(\pi - a)}, \frac{\pi y}{2a} \right) \quad \text{and} \quad v_{yy}(x, y) = \left( \frac{\pi}{2a} \right)^2 \cdot u_{yy} \left( \frac{\pi x}{2(\pi - a)}, \frac{\pi y}{2a} \right).
\]
Hence, applying (9.1) and with obvious changes of variables, we obtain

\[
\int_0^a v_x^2(\pi - a, y) \, dy = \int_0^a v_x^2(0, y) \, dy = \frac{\pi^2}{4(\pi - a)^2} \int_0^a u_x^2 \left(0, \frac{\pi y}{2a}\right) \, dy = \\
= \frac{a \pi}{2(\pi - a)^2} \int_0^{\pi/2} u_x^2(0, y) \, dy = \frac{a \pi}{8(\pi - a)^2} \int_{\partial R_{\pi/2}} u_y^2 ,
\]

\[
\int_0^{\pi-a} v_y^2(x, a) \, dx = \int_0^{\pi-a} v_y^2(x, 0) \, dx = \frac{\pi^2}{4a^2} \int_0^{\pi-a} u_y^2 \left(\frac{\pi x}{2(\pi - a)}, 0\right) \, dx = \\
= \frac{\pi(\pi - a)}{2a^2} \int_0^{\pi/2} u_y^2(x, 0) \, dx = \frac{\pi(\pi - a)}{8a^2} \int_{\partial R_{\pi/2}} u_y^2 .
\]

Therefore, we infer

\[
\int_{\partial R_a} v^2 = \frac{\pi}{4} \left(\frac{a}{(\pi - a)^2} + \frac{\pi - a}{a^2}\right) \int_{\partial R_{\pi/2}} u^2 .
\] (9.3)

Moreover, with a change of variables, we also obtain

\[
\int_{R_a} |\Delta v|^2 = \int_{R_a} \left(\frac{\pi}{2(\pi - a)}\right)^2 \cdot u_{xx} \left(\frac{\pi x}{2(\pi - a)}, \frac{\pi y}{2a}\right) + \left(\frac{\pi}{2a}\right)^2 \cdot u_{yy} \left(\frac{\pi x}{2(\pi - a)}, \frac{\pi y}{2a}\right) \right)^2 \, dx \, dy = \\
= 4a(\pi - a)\pi^2 \int_{R_{\pi/2}} \frac{u_{xx}(x, y)^2 + u_{xx}(x, y) \cdot u_{yy}(x, y) + u_{yy}(x, y)^2}{16(\pi - a)^4 + 8a^2(\pi - a)^2 + 16a^4} \, dx \, dy .
\] (9.4)

Next, as noticed by Kuttler [10, p.334], we recall that two integration by parts yield

\[
\int_{R_{\pi/2}} u_{xx}(x, y) \cdot u_{yy}(x, y) \, dx \, dy = \int_{R_{\pi/2}} u_y^2(x, y) \, dx \, dy > 0 .
\]

Hence, we may estimate (9.4) as follows

\[
\int_{R_a} |\Delta v|^2 \leq \frac{\pi^2(\pi - a)}{4a^3} \int_{R_{\pi/2}} |\Delta u|^2 .
\] (9.5)

Inserting (9.3) and (9.5) into (9.2) yields

\[
d_1(R_a) \leq \frac{(\pi - a)^3}{a(\pi^2 - 3\pi a + 3a^2)} d_1(R_{\pi/2}) .
\]

Letting \(a \to 0\), shows that

\[
\limsup_{a \to 0} [a \cdot d_1(R_a)] \leq \pi d_1(R_{\pi/2}) ,
\]

which is precisely the upper bound in the statement of Theorem 6.

In order to prove the lower bound, we rewrite [10, (15)] as

\[
d_1(R_a) \geq \frac{\pi}{2} \sqrt{\frac{1}{(\pi - a)^2} + \frac{1}{a^2}}
\]

and we let \(a \to 0\).
References


