AN ELEMENTARY PROOF OF THE RANK-ONE THEOREM FOR BV FUNCTIONS

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Abstract. We provide a simple proof of a result, due to G. Alberti, concerning a rank-one property for the singular part of the derivative of vector-valued functions of bounded variation.

In this paper we provide a short, elementary proof of the following result by G. Alberti [1] concerning a rank-one property for the derivative of a function with bounded variation.

Theorem. Let \( \Omega \subset \mathbb{R}^n \) be an open set, \( u : \Omega \rightarrow \mathbb{R}^m \) a function of bounded variation and let \( D_s u \) be the singular part of \( Du \) with respect to the Lebesgue measure \( \mathcal{L}^n \). Then \( D_s u \) is a rank-one measure, i.e., the (matrix-valued) function \( \frac{D_s u}{|D_s u|}(x) \) has rank one \( |D_s u| \)-a.e. \( x \in \Omega \).

We recall that a function \( u \in L^1(\Omega, \mathbb{R}^m) \) has bounded variation in \( \Omega \) \((u \in BV(\Omega, \mathbb{R}^m))\) if the derivatives \( Du \) of \( u \) in the sense of distributions are represented by a (matrix-valued) measure with finite total variation. The measure \( Du \) can then be decomposed as the sum \( Du = D_a u + D_s u \) of a measure \( D_a u \), that is absolutely continuous with respect to \( \mathcal{L}^n \), and a measure \( D_s u \) that is singular with respect to \( \mathcal{L}^n \). The Radon-Nikodym derivative \( \frac{D_a u}{|Du|} \) of \( D_a u \) with respect to its total variation \( |D_a u| \) is a \( |D_a u| \)-measurable map from \( \Omega \) to \( \mathbb{R}^{m \times n} \). The Theorem states that this map takes values in the space of rank-one matrices. See [2] for more details on BV functions.

The Theorem above was conjectured by L. Ambrosio and E. De Giorgi in [3]. It was first proved by G. Alberti in [1] by introducing new tools and using sophisticated techniques in Geometric Measure Theory. A new proof has been announced by G. De Philippis and F. Rindler as one of the consequences of the forthcoming, profound PDE result [4].

On the contrary, our proof of the Theorem above is elementary: it stems from well-known geometric properties relating the derivative of a BV function and the perimeter of its subgraph. The main new tool is the following lemma, where we denote by \( \pi : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^n \) the canonical projection \( \pi(x_1, \ldots, x_{n+1}) := (x_1, \ldots, x_n) \).

Lemma. Let \( \Sigma_1, \Sigma_2 \) be \( C^1 \) hypersurfaces in \( \mathbb{R}^{n+1} \) with unit normals \( \nu_{\Sigma_1}, \nu_{\Sigma_2} \). Then, the set
\[
T := \{ p \in \Sigma_1 : \exists q \in \Sigma_2 \land \pi^{-1}(\pi(p)) = (\nu_{\Sigma_1}(p))_{n+1} = 0 \text{ and } \nu_{\Sigma_1}(p) \neq \pm \nu_{\Sigma_2}(q) \}
\]
is \( \mathcal{H}^n \)-negligible.

We postpone the proof of the Lemma in order to directly address the proof of the main result.

Proof of the Theorem. Let \( u = (u_1, \ldots, u_m) \in BV(\Omega, \mathbb{R}^m) \). It is not restrictive to assume that \( \Omega \) is bounded. For any \( i = 1, \ldots, m \) we write \( D_s u_i = \sigma_i |D_s u_i| \) for a \( |D_s u_i| \)-measurable map \( \sigma_i : \Omega \rightarrow \mathbb{S}^{n-1} \). We also let \( E_i := \{(x, t) \in \Omega \times \mathbb{R} : t < u_i(x)\} \) be the subgraph of \( u_i \); it is well known that \( E_i \) has finite perimeter in \( \Omega \times \mathbb{R} \). Denoting by \( \partial^* E_i \) the reduced boundary of \( E_i \) and by \( \nu_i \) the measure theoretic inner normal to \( E_i \), we have (see e.g. [5, Section 4.1.5])
\[
|D_s u_i| = \pi_\#(\mathcal{H}^n \res S_i) \quad \text{for } S_i := \{ p \in \partial^* E_i : (\nu_i(p))_{n+1} = 0 \},
\]
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where $\pi_\#$ denotes push-forward of measures. The set $S_i$ is $n$-rectifiable and we can assume that it is contained in the union $\cup_{h \in \mathbb{N}} \Sigma_i^h$ of $C^1$ hypersurfaces $\Sigma_i^h$ in $\mathbb{R}^{n+1}$.

By [5, Section 4.1.5], the Lemma above and well-known properties of rectifiable sets, the following properties hold for $H^n$-a.e. $p \in S_i$:

\begin{equation}
\nu_{\partial^* E_i}(p) = (\sigma_i(\pi(p)),0)
\end{equation}

(1)

if $p \in \Sigma_i^h$, then $\nu_i(p) = \pm \nu_{\Sigma_i^h}(p)$

(2)

if $p \in \Sigma_i^h$ and $q \in S_j \cap \Sigma_k^j \cap \pi^{-1}(\pi(p))$, then $\nu_{\Sigma_i^h}(p) = \pm \nu_{\Sigma_k^j}(q)$.

(3)

Up to modifying $S_i$ on a $H^n$-negligible set and $\sigma_i$ on a $|D_s u_i|$-negligible set, we can assume that (1),(2) and (3) hold everywhere on $S_i$ and that $\sigma_i = 0$ on $\Omega \setminus \pi(S_i)$.

Since $D_s u = (\sigma_1|D_s u_1|, \ldots, \sigma_m|D_s u_m|)$ and $|D_s u|$ is concentrated on $\pi(S_1) \cup \cdots \cup \pi(S_m)$, it is enough to prove that the matrix-valued function $(\sigma_1, \ldots, \sigma_m)$ has rank 1 on $\pi(S_1) \cup \cdots \cup \pi(S_m)$. This will follow if we prove that the implication

\[ i,j \in \{1, \ldots, m\}, i \neq j, \ x \in \pi(S_i) \implies \sigma_j(x) \in \{0, \sigma_i(x), -\sigma_i(x)\} \]

holds. If $i,j,x$ are as above and $x \notin \pi(S_j)$, then $\sigma_j(x) = 0$. Otherwise, $x \in \pi(S_i) \cap \pi(S_j)$, i.e., there exist $p \in S_i$ and $h \in \mathbb{N}$ such that $\pi(p) = x$ and $\sigma_i(x) = \pm \nu_{\Sigma_i^h}(p)$ and there exist $q \in S_j$ and $k \in \mathbb{N}$ such that $\pi(q) = x$ and $\sigma_j(x) = \pm \nu_{\Sigma_k^j}(p)$. By (3) we obtain $\sigma_j(x) = \pm \sigma_i(x)$, as wished.

\[ \square \]

**Proof of the Lemma.** Consider the sets $\Sigma := \Sigma_1 \times \Sigma_2 \subset \mathbb{R}^{2n+2}$ and

\[ \Delta := \{ \xi = (x,t,y,s) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} = \mathbb{R}^{2n+2} : x = y \} \]

Then $\Sigma$ is a $2n$-dimensional manifold of class $C^1$ and $\Delta$ is a smooth $(n+2)$-dimensional manifold in $\mathbb{R}^{2n+2}$. Let us consider the set

\[ R := \{ \xi = (x,t,y,s) \in \Delta \cap \Sigma : (\nu_{\Sigma_1}(x,t))_{n+1} = (\nu_{\Sigma_2}(x,s))_{n+1} = 0 \text{ and } \nu_{\Sigma_1}(x,t) \neq \pm \nu_{\Sigma_2}(x,s) \} \]

By construction, the intersection between $\Delta$ and $\Sigma$ is transversal at every point of $R$, thus $R$ is contained in a $n$-dimensional submanifold $\tilde{R} \subset \Delta \cap \Sigma$ of class $C^1$. Let $\phi$ be the projection $\phi(x,t,y,s) := (x,t)$; notice that $\phi(\tilde{R}) \subset \Sigma_1$ and $T = \phi(R)$. Moreover, for every $\xi \in R$ the differential $d\phi_\xi : T_{\xi} \tilde{R} \to T_{\phi(\xi)} \Sigma_1$ is not surjective, because the vector $(0, \ldots, 0, 1)$ is in the kernel of $d\phi_\xi$. By the area formula we deduce that $H^n(T) = H^n(\phi(R)) = 0$, as desired.

\[ \square \]

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**References**


