VARIATION FORMULAS FOR H-RECTIFIABLE SETS

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Abstract. We compute a first- and second-variation formula for the area of H-rectifiable sets in the Heisenberg group along a contact flow. In particular, the formula holds for sets with locally finite H-perimeter, with no further regularity.

1. Introduction

In this paper, we compute the first- and second-variation formula for H-perimeter of sets in the sub-Riemannian Heisenberg group $\mathbb{H}^n$, with minimal regularity assumptions. Let $A \subset \mathbb{H}^n$ be a bounded open set in the n-th Heisenberg group and let $E \subset \mathbb{H}^n$ be a set with finite H-perimeter in A. This perimeter is defined starting from a scalar product $\langle \cdot, \cdot \rangle_H$ on the horizontal bundle H of $\mathbb{H}^n$. We denote by μ_E the H-perimeter measure of E, by $P(E, A) = \mu_E(A)$ the H-perimeter of E in A, and by $\nu_E \in H$ the measure theoretic horizontal inner normal of E. Let $\Psi_s : A \to \mathbb{H}^n$, $s \in [-\delta, \delta]$, be the flow of a contact vector field V, defined for some $\delta = \delta(A, V) > 0$.

The main result of the paper is the following

Theorem 1.1. *There exists a positive constant $C = C(A, V)$ independent of E such that, letting $E_s = \Psi_s(E)$ and $A_s = \Psi_s(A)$, we have*

$$\left| P(E_s, A_s) - P(E, A) - s \int_A \mathscr{F}_V(\nu_E)\, d\mu_E - \frac{s^2}{2} \int_A \mathscr{S}_V(\nu_E)\, d\mu_E \right| \leq C P(E, A) s^3, \quad (1.1)$$

for any $s \in [-\delta, \delta]$.

The functions $\mathscr{F}_V, \mathscr{S}_V : H \to \mathbb{R}$ are the first- and second-variation kernels. The first-variation kernel is

$$\mathscr{F}_V(\nu_E) = \mathrm{div} V + \mathscr{Q}_V(\nu_E),$$

where $\mathscr{Q}_V$ is the following quadratic form on H

$$\mathscr{Q}_V(\nu_E) = \langle [V, \nu_E], \nu_E \rangle_H.$$

The bracket $[V, \nu_E]$ is computed pointwise freezing ν_E at one point and extending the vector in a left-invariant way. Notice that $[V, \nu_E]$ is a horizontal vector field because V is contact. The divergence appearing in $\mathscr{F}_V$ is the divergence associated with the Haar measure of $\mathbb{H}^n$, that is the Lebesgue measure. The kernel $\mathscr{F}_V$ is well defined μ_E-a.e. for any set with locally finite H-perimeter.

Date: December 21, 2015.

2010 *Mathematics Subject Classification.* 49Q05, 53C17.

Key words and phrases. Heisenberg group, variation formulas, contact vector fields.

The second-variation kernel is

$$\mathscr{S}_V(\nu_E) = \operatorname{div}(J_V V) + \mathscr{F}_V(\nu_E)^2 - 2\mathscr{Q}_V(\nu_E)^2 + \mathscr{R}_V(\nu_E),$$

where J_V is the Jacobian (differential) of V and $\mathscr{R}_V : H \to \mathbb{R}$ is the quadratic form

$$\mathscr{R}_V(\nu_E) = \langle [V, [V, \nu_E]], \nu_E\rangle_H + \|\mathscr{L}_V^*(\nu_E)\|_H^2.$$

The mapping $\mathscr{L}_V : H \to H$ is the Lie derivative $\mathscr{L}_V(X) = [V, X]$, that is $\mathbb{R}$-linear, and $\mathscr{L}_V^* : H \to H$ is the adjoint mapping, $\langle \mathscr{L}_V X, Y\rangle_H = \langle X, \mathscr{L}_V^* Y\rangle_H$ for all $X, Y \in H$. Also $\mathscr{S}_V$ is well defined for sets with locally finite H-perimeter.

In Section 5, we give explicit formulas for $\mathscr{F}_V$ and $\mathscr{S}_V$ in terms of the generating function of the contact vector field V. The first variation kernel is

$$\mathscr{F}_V(\nu_E) = -4(n+1)T\psi - \nabla_H^2\psi(\nu_E, J\nu_E)$$

where $\nabla_H^2\psi$ is the horizontal Hessian of the generating function ψ, $J : H \to H$ is the standard complex structure, and T is the Reeb vector field. For $\mathscr{S}_V$ in terms of ψ see Proposition 5.2.

Formula (1.1) is interesting because it holds for the most general class of sets. It is nontrivial since it also applies to nonrectifiable hypersurfaces with fractional dimension, see [6]. In fact, we deduce the formula from an analogous variation formula for H-rectifiable sets, see Theorem 4.4. Variation formulas for smooth sets were already obtained, e.g., in [3, Section 8], [9, Section 4], [5]. We think that Theorem 1.1 could be useful in the study of regularity of H-minimal surfaces and of stability of critical sets for H-perimeter, in the study of the isoperimetric problem in $\mathbb{H}^n$ and of other problems under low regularity assumptions, as the Bernstein problem (see [2]). In fact, if E locally minimizes H-perimeter in an open set A, then for any contact vector field V with compact support in A we have

$$\int_A \mathscr{F}_V(\nu_E)d\mu_E = 0 \qquad \text{and} \qquad \int_A \mathscr{S}_V(\nu_E)d\mu_E \geq 0. \tag{1.2}$$

The first order necessary condition was used in [11] to study the harmonic approximation of H-perimeter minimizing boundaries, and here we are giving a detailed proof of the tools used in that paper. The regularity problem for H-perimeter minimizing sets is still completely open.

Formula (1.1) has, however, some drawbacks. First, in contrast with the usual Riemannian second-variation formula and its relation with the Ricci curvature and the second fundamental form, it is not easy to catch any clear geometric meaning of the kernels $\mathscr{F}_V$ and $\mathscr{S}_V$. Secondly, restricting variations to contact vector fields seems to cause a loss of information. This is already evident in the use of the first order necessary condition in (1.2) made in [11] and the reason is that a contact vector field depends on the first derivatives of the generating function.

The proof of Theorem 1.1 is divided into several steps. First, we prove it for smooth sets. Here, the key point is to get a constant C in the right hand side of (1.1) independent of the set. Then, we extend the formula to H-regular hypersurfaces and finally to H-rectifiable sets.

The computations of Section 3 make transparent the role of contact flows and have a general character, independent of the specific structure of $\mathbb{H}^n$. The approximation techniques of Section 4 also have a general character. The extension of the results from H-rectifiable sets to sets with finite H-perimeter is based on the structure theorem for the reduced boundary proved in [4].

2. Preliminary definitions

As customary, we denote points $p \in \mathbb{H}^n$ by

$$\mathbb{R}^{2n+1} \ni (x_1, \dots, x_n, y_1, \dots, y_n, t) = (z_1, \dots, z_n, t) \in \mathbb{C}^n \times \mathbb{R},$$

where $z_j = x_j + iy_j$ and $j = 1, \dots, n$. The Lie algebra of left-invariant vector fields in $\mathbb{H}^n$ is spanned by the vector fields

$$X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t}, \quad Y_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t}, \quad \text{and} \quad T = \frac{\partial}{\partial t}, \tag{2.3}$$

for $j = 1, \dots, n$. In the sequel, we shall frequently use the alternative notation $X_j = Y_{j-n}$ for $j = n+1, \dots, 2n$. We denote by H the horizontal bundle of $T\mathbb{H}^n$. Namely, for any $p = (z, t) \in \mathbb{H}^n$ we let

$$H_p = \mathrm{span}\big\{X_1(p), \dots, X_{2n}(p)\big\}.$$

We fix on H the scalar product $\langle \cdot, \cdot \rangle_H$ that makes $X_1, \dots, X_{2n}$ orthonormal.

A horizontal section $\varphi \in C^1_c(A, H)$, where $A \subset \mathbb{H}^n$ is an open set, is a vector field of the form

$$\varphi = \sum_{j=1}^{2n} \varphi_j X_j,$$

where $\varphi_j \in C^1_c(A)$. We identify φ with its horizontal coordinates $(\varphi_1, \dots, \varphi_{2n}) \in \mathbb{R}^{2n}$. With abuse of notation, for horizontal vectors φ and ν we let $\langle \varphi, \nu \rangle_H = \langle \varphi, \nu \rangle$ and $\|\nu\|_H = |\nu|$, where $\langle \cdot, \cdot \rangle$ and $|\cdot|$ are the standard scalar product and norm in $\mathbb{R}^{2n}$.

The *H-perimeter* of a $\mathscr{L}^{2n+1}$-measurable set $E \subset \mathbb{H}^n$ in an open set $A \subset \mathbb{H}^n$ is

$$P(E, A) = \sup\left\{ \int_E \mathrm{div}\varphi \, d\mathscr{L}^{2n+1} : \varphi \in C^1_c(A, H), \|\varphi\|_\infty \leq 1 \right\}.$$

If $P(E, A) < \infty$ we say that E has finite H-perimeter in A. If $P(E, A') < \infty$ for any open set $A' \subset\subset A$, we say that E has locally finite H-perimeter in A. In this case, the open sets mapping $A \mapsto P(E, A)$ extends to a Radon measure μ_E on A that is called *H-perimeter measure* induced by E. Moreover, there exists a μ_E-measurable function $\nu_E : A \to H$ such that $|\nu_E| = 1$ μ_E-a.e. and the Gauss-Green integration by parts formula

$$\int_A \langle \varphi, \nu_E \rangle \, d\mu_E = -\int_E \mathrm{div}\varphi \, d\mathscr{L}^{2n+1}$$

holds for any $\varphi \in C^1_c(A, H)$. The vector ν_E is called *horizontal inner normal* of E in A.

Let $\varrho(p,q) = \|q^{-1} * p\|_\infty$ be the box-distance between the points $p, q \in \mathbb{H}^n$, where $\|(z,t)\|_\infty = \max\{|z|, |t|^{1/2}\}$ is the box norm and $*$ is the Heisenberg product associated with the basis of left-invariant vector fields (2.3).

For any set $M \subset \mathbb{H}^n$, $s \geq 0$ and $\delta > 0$ we define

$$\mathscr{S}^s_\delta(M) = \inf\Big\{\omega_s \sum_{j\in\mathbb{N}} (\operatorname{diam} B_j)^s : M \subset \bigcup_{j\in\mathbb{N}} B_j, B_j \subset \mathbb{H}^n\, \varrho\text{-balls with} \operatorname{diam} B_j < \delta\Big\},$$

where $\omega_s > 0$ is a suitable normalization constant. The s-dimensional spherical Hausdorff measure of M is

$$\mathscr{S}^s(M) = \lim_{\delta\to 0} \mathscr{S}^s_\delta(M).$$

The relevant ϱ-Hausdorff dimension for us is $s = 2n+1$. For this reason, we use the short notation $\mathscr{S} = \mathscr{S}^{2n+1}$. By the representation theorem in [4], for any set $E \subset \mathbb{H}^n$ with locally finite H-perimeter we have

$$\mu_E = \mathscr{S} \llcorner \partial^* E, \tag{2.4}$$

where $\llcorner$ denotes restriction, and $\partial^* E$ is the reduced boundary of E, i.e., the set of points $p \in \mathbb{H}^n$ such that: i) $\mu_E(B(p,r)) > 0$ for all $r > 0$; ii) there holds $|\nu_E(p)| = 1$, and iii)

$$\lim_{r\to 0} ⨍_{B(p,r)} \nu_E d\mu_E = \nu_E(p).$$

In particular, the representation formula (2.4) holds when E is a set such that $M = \partial E = \partial^* E$ is an H-regular surface (see Section 4). For sets with smooth boundary a representation formula for μ_E will be presented and used later in Section 3, see (3.30) and (3.31).

3. Variation formula in the smooth case

In this section, we compute the second order Taylor formula with Lagrange remainder for the variation of H-perimeter of smooth hypersurfaces along a contact flow. The term "smooth" will always mean C^∞-smooth and vector fields will always be smooth vector fields in $\mathbb{H}^n$. By $\langle\cdot,\cdot\rangle$ we denote the standard scalar product of vectors in $\mathbb{R}^{2n+1}$.

3.1. **Preliminary computations along flows.** Let $\{\Psi_s\}_{s\in\mathbb{R}}$ be the flow of diffeomorphisms in $\mathbb{H}^n$ generated by a vector field V. Given a smooth hypersurface $M \subset \mathbb{H}^n$, we denote by N_s the Euclidean normal to $\Psi_s(M)$. We fix a base point $p \in M$ and we let $p_s = \Psi_s(p)$. Reference to the base point will often be omitted. In our computations, there will appear the function

$$\vartheta(s) = \langle J_V N_s, N_s\rangle, \quad s \in \mathbb{R}, \tag{3.5}$$

where the vectors are evaluated at p_s and J_V is the Jacobian of the vector field V generating the flow Ψ_s. However, our final formulas are independent of ϑ and, in particular, they do not depend on the Euclidean normal but only on the horizontal normal. This is of crucial importance for the extension of the formulas to sets with finite H-perimeter of Section 4.

For any vector field W and for any fixed base point, we define the function $s \mapsto F_W(s)$

$$F_W(s) = \langle W, N_s \rangle (p_s), \quad s \in \mathbb{R}. \tag{3.6}$$

The definition depends on the starting hypersurface M that we are considering.

Lemma 3.1. *The function $s \mapsto F_W(s)$ satisfies the differential equation*

$$F'_W(s) = F_{[V,W]}(s) + \vartheta(s) F_W(s), \qquad s \in \mathbb{R}. \tag{3.7}$$

Proof. Fix an orthonormal frame of vector fields $V_1, \dots, V_{2n}$ that are tangent to M. Thus, $J_{\Psi_s} V_1, \dots, J_{\Psi_s} V_{2n}$ are tangent to $\Psi_s(M)$, and so

$$\langle J_{\Psi_s} V_i, N_s \rangle = 0 \quad i = 1, \dots, n.$$

Differentiating the above identity with respect to s yields

$$\langle J_V J_{\Psi_s} V_i, N_s \rangle + \langle J_{\Psi_s} V_i, N'_s \rangle = 0, \quad i = 1 \dots, 2n, \tag{3.8}$$

where N'_s is the derivative of $s \mapsto N_s(p_s)$ with respect to s. On the other hand, differentiating the identity $|N_s|^2 = 1$, we get $\langle N'_s, N_s \rangle = 0$, that is, N'_s is tangent to $\Psi_s(M)$. Using the fact that $J_{\Psi_s}|_{s=0}$ is the identity, we deduce that the derivative of N_s at $s = 0$ is

$$\begin{aligned} N'_0 = \sum_{i=1}^{2n} \langle V_i, N'_0 \rangle V_i &= - \sum_{i=1}^{2n} \langle J_V V_i, N \rangle V_i \\ &= - \sum_{i=1}^{2n} \langle V_i, J_V^* N \rangle V_i \\ &= \langle J_V^* N, N \rangle N - J_V^* N, \end{aligned} \tag{3.9}$$

where the second identity is justified by (3.8) computed at $s = 0$. An analogous argument shows that (3.9) holds for all $s \in \mathbb{R}$, that is

$$N'_s = \langle J_V^* N_s, N_s \rangle N_s - J_V^* N_s. \tag{3.10}$$

The derivative of F_W is

$$F'_W(s) = \langle J_W V, N_s \rangle + \langle W, N'_s \rangle,$$

that, by (3.10) and by the definition of adjoint map, becomes

$$\begin{aligned} F'_W(s) &= \langle J_W V - J_V W, N_s \rangle + \langle J_V N_s, N_s \rangle \langle W, N_s \rangle \\ &= \langle [V, W], N_s \rangle + \langle J_V N_s, N_s \rangle \langle W, N_s \rangle \\ &= F_{[V,W]}(s) + \vartheta(s) F_W(s). \end{aligned}$$

This ends the proof. □

Now consider the function $s \mapsto K_s$ defined in the following way

$$K_s = \Big(\sum_{j=1}^{2n} F_{X_j}(s)^2\Big)^{1/2}. \tag{3.11}$$

The function K_s is the integral kernel for the H-area of the hypersurface $\Psi_s(M)$, introduced in (3.30) below. The derivative of K_s is related to the function $s \mapsto G_s$

$$G_s = \frac{1}{K_s}\sum_{j=1}^{2n} F_{X_j}(s) F_{[V,X_j]}(s), \tag{3.12}$$

that is defined when $K_s \neq 0$. On the other hand, the derivative of G_s is related to the function $s \mapsto H_s$

$$H_s = \frac{1}{K_s}\sum_{j=1}^{2n} F_{[V,X_j]}(s)^2 + F_{X_j}(s) F_{[V,[V,X_j]]}(s). \tag{3.13}$$

Finally, the derivative of H_s is related to the function $s \mapsto L_s$

$$L_s = \frac{1}{K_s}\sum_{j=1}^{2n} 3F_{[V,X_j]}(s) F_{[V,[V,X_j]]}(s) + F_{X_j}(s) F_{[V,[V,[V,X_j]]]}(s). \tag{3.14}$$

The functions G_s, H_s, and L_s appear in the first, second, and third derivatives of H-perimeter.

Lemma 3.2. *As long as $K_s \neq 0$, the functions K_s, G_s, and H_s satisfy the following differential equations*

$$K_s' = \vartheta(s) K_s + G_s, \tag{3.15}$$

$$G_s' = \vartheta(s) G_s - \frac{G_s^2}{K_s} + H_s, \tag{3.16}$$

$$H_s' = \vartheta(s) H_s - \frac{G_s H_s}{K_s} + L_s, \tag{3.17}$$

where L_s is as in (3.14)*. Moreover, we have*

$$\Big(\frac{G_s}{K_s}\Big)' = \frac{H_s}{K_s} - 2\frac{G_s^2}{K_s^2}, \tag{3.18}$$

$$\Big(\frac{G_s}{K_s}\Big)'' = -4\frac{G_s}{K_s}\Big(\frac{G_s}{K_s}\Big)' + \frac{L_s}{K_s} - 2\frac{G_s H_s}{K_s^2}, \tag{3.19}$$

$$\Big(\frac{H_s}{K_s}\Big)' = \frac{L_s}{K_s} - 2\frac{G_s H_s}{K_s^2}. \tag{3.20}$$

Proof. We have

$$K_s' = \frac{1}{K_s}\sum_{j=1}^{2n} F_{X_j} F_{X_j}',$$

and, by (3.7) and the definition (3.12) of G_s, this formula gives identity (3.15).

Differentiating (3.12) we obtain

$$\begin{aligned} G_s' &= -\frac{K_s'}{K_s^2}\sum_{j=1}^{2n} F_{X_j}F_{[V,X_j]} \\ &\quad + \frac{1}{K_s}\sum_{j=1}^{2n}(F_{[V,X_j]}+\vartheta F_{X_j})F_{[V,X_j]} + F_{X_j}(F_{[V,[V,X_j]]}+\vartheta F_{[V,X_j]}). \end{aligned}$$

Inserting K_s' into this formula, we obtain

$$G_s' = -\frac{G_s}{K_s}(\vartheta(s)K_s + G_s) + 2\vartheta(s)G_s + H_s = \vartheta(s)G_s - \frac{G_s^2}{K_s} + H_s.$$

This is formula (3.16). Differentiating (3.13), using (3.7) and (3.15), we find

$$\begin{aligned} H_s' &= -\frac{K_s'}{K_s}H_s + \frac{1}{K_s}\sum_{j=1}^{2n} 2F_{[V,X_j]}(F_{[V,[V,X_j]]}+\vartheta F_{[V,X_j]}) \\ &\qquad + (F_{[V,X_j]}+\vartheta F_{X_j})F_{[V,[V,X_j]]} + F_{X_j}(F_{[V,[V,[V,X_j]]]}+\vartheta F_{[V,[V,X_j]]}) \\ &= -\frac{\vartheta(s)K_s+G_s}{K_s}H_s + 2\vartheta(s)H_s + L_s \\ &= \vartheta(s)H_s - \frac{G_sH_s}{K_s} + L_s. \end{aligned}$$

This is formula (3.17).

The formulas (3.18), (3.19), and (3.20) follow from (3.15), (3.16), and (3.17). In fact, we have

$$\Big(\frac{G_s}{K_s}\Big)' = \frac{1}{K_s}\Big(\vartheta(s)G_s - \frac{G_s^2}{K_s} + H_s\Big) - \frac{G_s}{K_s}\frac{\vartheta(s)K_s+G_s}{K_s} = \frac{H_s}{K_s} - 2\frac{G_s^2}{K_s^2},$$

and

$$\begin{aligned} \Big(\frac{G_s}{K_s}\Big)'' &= -4\frac{G_s}{K_s}\Big(\frac{G_s}{K_s}\Big)' + \frac{H_s'}{K_s} - \frac{H_s}{K_s}\frac{K_s'}{K_s} \\ &= -4\frac{G_s}{K_s}\Big(\frac{G_s}{K_s}\Big)' + \frac{H_s' - \vartheta H_s}{K_s} - \frac{H_sG_s}{K_s^2} \\ &= -4\frac{G_s}{K_s}\Big(\frac{G_s}{K_s}\Big)' + \frac{L_s - G_sH_s/K_s}{K_s} - \frac{H_sG_s}{K_s^2} \\ &= -4\frac{G_s}{K_s}\Big(\frac{G_s}{K_s}\Big)' + \frac{L_s}{K_s} - 2\frac{G_sH_s}{K_s^2}. \end{aligned}$$

The computations for $(H_s/K_s)'$ are analogous and are omitted. □

The (Euclidean) *tangential Jacobian determinant* of a smooth mapping $\Psi : \mathbb{H}^n \to \mathbb{H}^n$ restricted to a hypersurface $M \subset \mathbb{H}^n$ is the mapping $\mathscr{J}\Psi : M \to \mathbb{R}$

$$\mathscr{J}\Psi(p) = \sqrt{\det(J_\Psi|_M^* \circ J_\Psi|_M)(p)}, \qquad p \in M, \tag{3.21}$$

where $J_\Psi|_M$ at $p \in M$ is the restriction of the Jacobian J_Ψ to T_pM and * denotes the adjoint mapping.

The (Euclidean) *tangential divergence* of a vector field V on a hypersurface M is

$$\mathrm{div}_M V = \mathrm{div} V - \langle J_V N, N\rangle, \tag{3.22}$$

where N is the Euclidean normal to M and J_V is the Jacobian of V.

Lemma 3.3. *Let $M \subset \mathbb{H}^n$ be a smooth hypersurface and $\{\Psi_s\}_{s\in\mathbb{R}}$ be the flow of diffemorphisms generated by a vector field V. For any $s \in \mathbb{R}$ we have*

$$(\mathscr{J}\Psi_s)' = \mathscr{J}\Psi_s \,\mathrm{div}_{\Psi_s(M)} V(p_s). \tag{3.23}$$

The proof of (3.23) is well known (see e.g. [13]) and it is omitted. In the next step, we compute the derivatives of the product $K_s \mathscr{J}\Psi_s$.

Lemma 3.4. *Let $M \subset \mathbb{H}^n$ be a smooth hypersurface and $\{\Psi_s\}_{s\in\mathbb{R}}$ be the flow of diffemorphisms generated by a vector field V. For any $s \in \mathbb{R}$ such that $K_s \neq 0$ we have*

$$(K_s \mathscr{J}\Psi_s)' = K_s \mathscr{J}\Psi_s \Big(\mathrm{div} V(p_s) + \frac{G_s}{K_s}\Big), \tag{3.24}$$

$$(K_s \mathscr{J}\Psi_s)'' = K_s \mathscr{J}\Psi_s \Big[\Big(\mathrm{div} V(p_s) + \frac{G_s}{K_s}\Big)^2 + \mathrm{div}(J_V V)(p_s) - 2\frac{G_s^2}{K_s^2} + \frac{H_s}{K_s}\Big], \tag{3.25}$$

$$(K_s \mathscr{J}\Psi_s)''' = K_s \mathscr{J}\Psi_s \big(A_s + B_s\big), \tag{3.26}$$

where

$$\begin{aligned}
A_s &= \Big(\mathrm{div} V(p_s) + \frac{G_s}{K_s}\Big)\Big[\Big(\mathrm{div} V(p_s) + \frac{G_s}{K_s}\Big)^2 + \mathrm{div}(J_V V)(p_s) - 2\frac{G_s^2}{K_s^2} + \frac{H_s}{K_s}\Big],\\
B_s &= 2\Big(\mathrm{div} V(p_s) + \frac{G_s}{K_s}\Big)\Big(\mathrm{div}(J_V V)(p_s) + \Big(\frac{G_s}{K_s}\Big)'\Big) + \mathrm{div}(J_{J_V V} V)(p_s)\\
&\quad - 4\frac{G_s}{K_s}\Big(\frac{G_s}{K_s}\Big)' + \Big(\frac{H_s}{K_s}\Big)'.
\end{aligned}$$

Proof. From (3.15), (3.23), and from the definition (3.22) of tangential divergence, we obtain

$$\begin{aligned}
(K_s \mathscr{J}\Psi_s)' &= K_s' \mathscr{J}\Psi_s + K_s (\mathscr{J}\Psi_s)'\\
&= (\vartheta(s) K_s + G_s) \mathscr{J}\Psi_s + K_s \mathscr{J}\Psi_S \mathrm{div}_{\Psi_s(S)} V(p_s)\\
&= K_s \mathscr{J}\Psi_s \Big(\vartheta(s) + \frac{G_s}{K_s} + \mathrm{div}_{\Psi_s(S)} V(p_s)\Big)\\
&= K_s \mathscr{J}\Psi_s \Big(\frac{G_s}{K_s} + \mathrm{div} V(p_s)\Big).
\end{aligned}$$

In order to compute the second derivative, first observe that

$$\frac{\partial}{\partial s} \mathrm{div} V(p_s) = \mathrm{div}\Big(\frac{\partial}{\partial s}(V \circ \Psi_s)\Big) = \mathrm{div}(J_V V)(p_s).$$

Thus, we obtain

$$\begin{aligned}(K_s\mathscr{J}\Psi_s)'' &= (K_s\mathscr{J}\Psi_s)'\Big(\mathrm{div}V(p_s)+\frac{G_s}{K_s}\Big)+K_s\mathscr{J}\Psi_s\Big(\mathrm{div}V(p_s)+\frac{G_s}{K_s}\Big)'\\ &= K_s\mathscr{J}\Psi_s\Big[\Big(\mathrm{div}V(p_s)+\frac{G_s}{K_s}\Big)^2+\mathrm{div}(J_VV)(p_s)+\Big(\frac{G_s}{K_s}\Big)'\Big].\end{aligned}$$

The formula for the third derivative is obtained in a similar way and we omit the computations. □

Remark 3.5. The function ϑ introduced in (3.5) appears in the formulas (3.15)–(3.17). However, in (3.18)–(3.20) the function does not appear and thus the derivatives of $K_s\mathscr{J}\Psi_s$ in (3.24)–(3.26) are independent of ϑ.

3.2. **Contact flows.** Let $A\subset\mathbb{H}^n$ be an open set. A C^∞-diffeomorphism $\Psi : A\to\Psi(A)\subset\mathbb{H}^n$ is a *contact diffeomorphism* if for any $p\in A$ the differential J_Ψ satisfies

$$J_\Psi(H_p)=H_{\Psi(p)},$$

where H is the horizontal bundle of $\mathbb{H}^n$. Contact diffeomorphisms play a central role in geometric, conformal, and metric analysis in the Heisenberg group. From our point of view, they are important for the following reason. If Ψ is a contact diffeomorphism and $E\subset\mathbb{H}^n$ is a (bounded) set with finite H-perimeter then also $\Psi(E)$ is a set with finite H-perimeter. If Ψ is not contact then this property may fail even for a linear mapping Ψ.

A vector field V in $\mathbb{H}^n$ is a *contact vector field* if it generates a flow of contact diffeomorphisms. The following proposition is well-known and lists some characterizations of contact vector fields. We refer the reader to [7, Section 5] for a derivation of formula (3.27) below.

Proposition 3.6. *Let V be a smooth vector field in $\mathbb{H}^n$. The following statements are equivalent:*

i) *V is a contact vector field.*
ii) *For any $p\in\mathbb{H}^n$ and for any $j=1,\dots,n$ we have*

$$[V,X_j](p)\in H_p \quad \textit{and} \quad [V,Y_j](p)\in H_p.$$

iii) *There exists a function $\psi\in C^\infty(\mathbb{H}^n)$ such that $V=V_\psi$ with*

$$V_\psi=-4\psi T+\sum_{j=1}^n(Y_j\psi)X_j-(X_j\psi)Y_j. \tag{3.27}$$

Let $M\subset\mathbb{H}^n$ be a smooth hypersurface, let N_s be the Euclidean normal to $\Psi_s(M)$, and let $F_W(s)=\langle W,N_s\rangle(p_s)$ be the function in (3.6), for some base point $p\in M$. The functions G_s, H_s, and L_s in (3.12)–(3.14) depend on M. However, we have the following result.

Lemma 3.7. *Let Ψ_s be the flow of a contact vector fields V in $\mathbb{H}^n$. For any compact set $Q \subset \mathbb{H}^n$ there exists a constant $C = C(Q, V) > 0$ independent of the initial surface M such that*

$$|G_s| + |H_s| + |L_s| \leq C K_s, \tag{3.28}$$

and

$$\Big|\Big(\frac{G_s}{K_s}\Big)'\Big| + \Big|\Big(\frac{G_s}{K_s}\Big)''\Big| + \Big|\Big(\frac{H_s}{K_s}\Big)'\Big| \leq C, \tag{3.29}$$

provided $K_s \neq 0$ and $p_s \in Q$.

Proof. By the characterization ii) of Proposition 3.6, the vector fields $[V, X_j]$, $[V, [V, X_j]]$, and $[V[V, [V, X_j]]]$ are horizontal. The expressions (3.12)–(3.14) for G_s, H_s, and L_s are thus homogeneous of degree 1 in X_j, $j = 1, \ldots, 2n$. By a continuity argument, it follows that there exists a constant $C > 0$ depending on V and Q but independent of M such that $|G_s| + |H_s| + |L_s| \leq C K_s$ for all $p_s \in Q$.

The estimates (3.29) follow from (3.28) and from the formulas (3.18)–(3.20). □

From Lemma 3.7, we deduce the following corollary.

Lemma 3.8. *Let Ψ_s be the flow of a smooth contact vector fields V in $\mathbb{H}^n$. For any compact set $Q \subset \mathbb{H}^n$ there exists a constant $C = C(Q, V) > 0$ independent of the initial surface M such that*

$$|(K_s \mathscr{J}\Psi_s)'| + |(K_s \mathscr{J}\Psi_s)''| + (K_s \mathscr{J}\Psi_s)'''| \leq C K_s,$$

provided $K_s \neq 0$ and $p_s \in Q$.

In fact, also $\mathscr{J}\Psi_s$ is locally bounded independently from the initial surface M.

3.3. **Variation formulas in the smooth case.** Let UH be the unit horizontal bundle of $\mathbb{H}^n$. Namely, $\nu \in UH$ if $\nu \in H$ and for any $p \in H_p$ we have

$$\nu(p) = \sum_{j=1}^{2n} \nu_j X_j(p) \quad \text{with} \quad \sum_{j=1}^{2n} \nu_j^2 = 1.$$

We identify $\nu(p)$ with its horizontal coordinates $(\nu_1, \ldots, \nu_{2n}) \in \mathbb{R}^{2n}$.

For a smooth hypersurface $M \subset \mathbb{H}^n$, we define its horizontal normal at $p \in M$ as the vector $\nu_M \in UH_p$ that, in horizontal coordinates, is defined as

$$\nu_M = \frac{1}{K}\big(\langle X_1, N\rangle, \ldots, \langle X_{2n}, N\rangle\big), \qquad \text{at the point } p,$$

where N is the Euclidean normal and $K = (\sum_{j=1}^{2n} \langle X_j, N\rangle^2)^{1/2}$. The definition of ν_M depends on a choice of sign for N and it is possible when $K \neq 0$. The H-area of M in an open set $A \subset \mathbb{H}^n$ is

$$\mathscr{A}_H(M, A) = \int_{M \cap A} K \, d\mathscr{H}^{2n}, \tag{3.30}$$

where $\mathscr{H}^{2n}$ is the standard $2n$-dimensional Hausdorff measure in $\mathbb{H}^n = \mathbb{R}^{2n+1}$. The H-area measure of M is the measure

$$\mu_M = K\mathscr{H}^{2n} \llcorner M. \tag{3.31}$$

When $M = \partial E$ is the boundary of a smooth set E we have $\mu_E = \mu_M$.

For any contact vector field V, we define quadratic forms $\mathscr{Q}_V, \mathscr{R}_V : UH \to \mathbb{R}$ in the following way. Fix a point $p \in \mathbb{H}^n$ and a vector $\nu \in UH_p$. Take a smooth hypersurface $M \subset \mathbb{H}^n$ such that $p \in M$ is noncharacteristic (i.e., $T_pM \neq H_p$) and $\nu_M(p) = \nu$. Let Ψ_s be the contact flow generated by V, with base point p. We define

$$\mathscr{Q}_V(\nu) = \frac{G_s}{K_s}\Big|_{s=0} \quad \mathscr{R}_V(\nu) = \frac{H_s}{K_s}\Big|_{s=0}, \tag{3.32}$$

where G_s, K_s, and H_s are defined in (3.11)–(3.13) starting from the functions $F_W(s) = \langle W, N_s\rangle$ where N_s is the Euclidean normal to the hypersurface $\Psi_s(M)$ at the point $p_s = \Psi_s(p)$. The quantities in (3.32) do not depend on the choice of M, see Lemma 3.9.

Recalling that the bracket $[V,\nu]$ is computed pointwise by extending $\nu = \nu(p)$ in a left-invariant way, we have the following

Lemma 3.9. *For any contact vector field V and any $\nu \in UH$, we have:*

i) $\mathscr{Q}_V(\nu) = \langle [V,\nu], \nu\rangle_H$;
ii) $\mathscr{R}_V(\nu) = \langle [V,[V,\nu]], \nu\rangle_H + \|\mathscr{L}_V^*(\nu)\|_H^2$.

Proof. The relation between the Euclidean normal N to the hypersurface M and the horizontal normal $\nu \in UH$ is

$$\nu_j = \frac{\langle X_j, N\rangle}{K}, \quad j = 1, \dots, 2n. \tag{3.33}$$

By the formula (3.12) and by standard linear algebra, we have

$$\mathscr{Q}_V(\nu) = \frac{1}{K^2}\sum_{j=1}^{2n} \langle X_j, N\rangle \langle [V, X_j], N\rangle = \langle [V,\nu],\nu\rangle_H.$$

By (3.13) and (3.33), the quadratic form $\mathscr{R}_V : UH_p \to \mathbb{R}$ in (3.32) is $\mathscr{R}_V = \mathscr{R}_V^1 + \mathscr{R}_V^2$, with

$$\mathscr{R}_V^1(\nu) = \frac{1}{K^2}\sum_{j=1}^{2n} \langle [V,[V,X_j]], N\rangle \langle X_j, N\rangle = \langle [V,[V,\nu]],\nu\rangle_H,$$

and

$$\mathscr{R}_V^2(\nu) = \frac{1}{K^2}\sum_{j=1}^{2n} \langle [V,X_j], N\rangle^2 = \sum_{j=1}^{2n} \langle [V,X_j],\nu\rangle_H^2 = \sum_{j=1}^{2n} \langle X_j, \mathscr{L}_V^*(\nu)\rangle_H^2 = \|\mathscr{L}_V^*(\nu)\|_H^2.$$

□

Motivated by the formulas in (3.24) and (3.25), let us introduce the short notation

$$\mathscr{F}_V(p,\nu) = \mathrm{div} V + \mathscr{Q}_V(\nu) \tag{3.34}$$

$$\mathscr{S}_V(p,\nu) = \mathrm{div}(J_V V) + (\mathscr{F}_V(\nu))^2 - 2\mathscr{Q}_V(\nu)^2 + \mathscr{R}_V(\nu), \tag{3.35}$$

where functions are evaluated at a noncharacteristic point $p \in M$ and $\nu = \nu_M(p)$ is the horizontal normal of M at p. We call $\mathscr{F}_V$ the *first-variation kernel* and $\mathscr{S}_V$ the *second-variation kernel* along V. We omit dependence on p and we let $\mathscr{F}_V(\nu) = \mathscr{F}_V(p,\nu)$ and $\mathscr{S}_V(p,\nu) = \mathscr{S}_V(\nu)$. We shall compute explicit formulas for the kernels $\mathscr{F}_V$ and $\mathscr{S}_V$ in Section 5.

In the next theorem, we compute the second variation formula for the H-area of smooth hypersurfaces. A sketch of the proof of the formula up to the first order appeared in the lecture notes [10].

Theorem 3.10. *Let $A \subset \mathbb{H}^n$ be a bounded open set and let $M \subset \mathbb{H}^n$ be a smooth hypersurface with finite H-area in A. Let $\Psi : [-\delta,\delta] \times A \to \mathbb{H}^n$, $\delta = \delta(A,V) > 0$, be the flow of a contact vector field V. Then there exists a positive constant $C = C(A,V)$ independent of M such that*

$$\Big|\mathscr{A}_H(M_s,A_s) - \mathscr{A}_H(M,A) - s\int_A \mathscr{F}_V(\nu_M)\,d\mu_M - \frac{s^2}{2}\int_A \mathscr{S}_V(\nu_M)\,d\mu_M\Big| \le C\mathscr{A}_H(M,A)s^3, \tag{3.36}$$

for any $s \in [-\delta,\delta]$, where $M_s = \Psi_s(M)$ and $A_s = \Psi_s(A)$.

Proof. Let N, N_s be the Euclidean unit normals to $M \cap A$ and $M_s \cap A_s$, respectively. We choose a coherent orientation. Let K_s be the H-area kernel of M_s introduced in (3.11). By the definition of a contact diffeomorphism, we have $K_0(p) \neq 0$ if and only if $K_s(p_s) \neq 0$. The set where $K_0 = 0$ is contained in a smooth hypersurface of M and is $\mathscr{H}^{2n}$-negligible.

By the change of variable formula for surface integrals, we have

$$\begin{aligned}\mathscr{A}_H(M_s,A_s) &= \int_{M_s\cap A_s} \Big(\sum_{i=1}^{2n}\langle X_j, N_s\rangle^2\Big)^{1/2} d\mathscr{H}^{2n} \\ &= \int_{M\cap A} K_s \mathscr{J}\Psi_s d\mathscr{H}^{2n} = P(s),\end{aligned}$$

where $\mathscr{J}\Psi_s$ is the tangential Jacobian defined in (3.21). The function $P(s)$ defined in the last line has the following Taylor expansion

$$P(s) = P(0) + sP'(0) + \frac{s^2}{2}P''(0) + \frac{s^3}{6}P'''(\bar{s}), \tag{3.37}$$

for some $\bar{s} \in [0,s]$.

The exchange of integral and derivative in s,

$$\frac{d}{ds}\int_{M\cap A} K_s \mathscr{J}\Psi_s d\mathscr{H}^{2n} = \int_{M\cap A} \big(K_s \mathscr{J}\Psi_s\big)' d\mathscr{H}^{2n},$$

is justified by (3.24). In fact, the right hand side in (3.24) is bounded by the estimates of Lemma 3.8. The same holds for the second and third derivatives in s. Then, we have

$$P(0) = \int_{m\cap A} K_0 \, d\mathscr{H}^{2n} = \mathscr{A}_H(M,A),$$

and, by (3.32)–(3.35),

$$\begin{aligned} P'(0) &= \int_{M\cap A} \big(K_s \mathscr{J}\Psi_s\big)'\big|_{s=0} \, d\mathscr{H}^{2n} = \int_{M\cap A} \big(\mathrm{div} V + \mathscr{Q}_V(\nu_M)\big) K_0 \, d\mathscr{H}^{2n} \\ &= \int_A \mathscr{F}_V(\nu_M)\, d\mu_M, \\ P''(0) &= \int_{M\cap A} \big(K_s \mathscr{J}\Psi_s\big)''\big|_{s=0} \, d\mathscr{H}^{2n} = \int_A \mathscr{S}_V(\nu_M)\, d\mu_M. \end{aligned}$$

The third derivative satisfies the bound

$$\begin{aligned} \Big|\int_{M\cap A} \big(K_s \mathscr{J}\Psi_s\big)''' d\mathscr{H}^{2n}\Big| &\leq C_1 \int_{M\cap A} K_s \, d\mathscr{H}^{2n} \\ &\leq C_2 \int_{M\cap A} K_0 \, d\mathscr{H}^{2n} = C_2 \mathscr{A}_H(M,A). \end{aligned}$$

The estimate $K_s \leq C_3 K_0$ follows from $K_s' \leq C_4 K_s$, that is a consequence of (3.15). The constants C_1, C_2, C_3, C_4 are independent of M.

Now formula (3.36) follows from (3.37). □

4. Variation formulas for H-rectifiable sets

In this section we extend Theorem 3.10 to H-rectifiable sets, and in particular to sets with finite H-perimeter. In a first step, we extend the theorem to H-regular hypersurfaces, and in a second step to H-rectifiable sets.

4.1. **Variation of the area of H-regular surfaces.** A function $g : A \to \mathbb{R}$, $A \subset \mathbb{H}^n$ open set, is of class $C^1_H(A)$ if g is continuous and the derivatives $X_1 g, \dots, X_{2n} g$ in the sense of distributions are (represented by) continuous functions in A. The horizontal gradient of $g \in C^1_H(A)$ is the vector valued mapping $\nabla_H g \in C(A;\mathbb{R}^{2n})$, $\nabla_H g = \big(X_1 g, \dots, X_{2n} g\big)$.

A set $M \subset \mathbb{H}^n$ is an H-regular hypersurface if for all $p \in M$ there exists an open neighbourhood A of p and a function $g \in C^1_H(A)$ such that $M \cap A = \big\{q \in A : g(q) = 0\big\}$ and $|\nabla_H g(p)| \neq 0$.

The main result of this section is the following

Theorem 4.1. *Let $A \subset \mathbb{H}^n$ be a bounded open set and $E \subset \mathbb{H}^n$ a set with finite H-perimeter in A such that $\partial E \cap A$ is an H-regular hypersurface. Let $\Psi : [-\delta,\delta] \times A \to \mathbb{H}^n$, $\delta = \delta(A,V) > 0$, be the flow generated by a contact vector field V. There exists a positive constant $C = C(A,V)$ independent of E such that, letting $E_s = \Psi_s(E)$ and $A_s = \Psi_s(A)$, we have*

$$\Big| P(E_s, A_s) - P(E,A) - s\int_A \mathscr{F}_V(\nu_E)\, d\mu_E - \frac{s^2}{2}\int_A \mathscr{S}_V(\nu_E)\, d\mu_E \Big| \leq C P(E,A) s^3, \quad (4.38)$$

for any $s \in [-\delta, \delta]$.

The starting point of the proof is the following technical lemma, that is an easy adaptation of Lemma 4.4 in [14].

Lemma 4.2. *Let* M *be an* H*-regular hypersurface such that* $M = \{p \in U : g(p) = 0\}$ *with* $U \subset \mathbb{H}^n$ *open and bounded set and* $g \in C^1_H(U)$ *such that* $\nabla_H g \neq 0$ *on* M*. Then, for any* $p \in M$ *there exist an open neighbourhood* $A \subset U$ *and a function* $f \in C^1_H(A)$ *such that* $M = \{q \in A : f(q) = 0\}$, $\nabla_H f \neq 0$ *on* A, *and* $f \in C^\infty(A \setminus M)$.

Proof of Theorem 4.1. We can assume that $M = \partial E \cap A$ is given by the zero set of a function $f \in C^1_H(A) \cap C^\infty(A \setminus M)$ as in Lemma 4.2; we can also assume that $f < 0$ on E. Let Ψ_s be the flow generated by a contact vector field V. For $s \in [-\delta, \delta]$, we define

$$A_s = \Psi_s(A), \qquad f_s = f \circ \Psi_s, \qquad M_s = \{q \in A_s : f_s(q) = 0\} = \Psi_s(M).$$

Using the property of a contact flow it is easy to check that each M_s is an H-regular hypersurface with defining function $f_s \in C^1_H(A_s) \cap C^\infty(A_s \setminus M_s)$.

For any $r \in \mathbb{R}$ we define the sets

$$E^r = \{p \in A : f(p) < r\}, \quad E = E^0,$$
$$E^r_s = \{q \in A_s : f_s(q) < r\} = \Psi_s(E^r), \quad E_s = E^0_s = \Psi_s(E^0).$$

We have $\partial E \cap A = M$ and $\partial E_s \cap A_s = M_s$. Since $\nabla_H f \neq 0$ on A, $\partial E^r \cap A$ and $\partial E^r_s \cap A_s$ are smooth hypersurfaces.

Let $s \in [-\delta, \delta]$ be fixed. By Theorem 3.10 and by the standard representation of H-perimeter for smooth surfaces, we have

$$\left| P(E^r_s, A_s) - P(E^r, A) - s \int_A \mathscr{F}_V(\nu_{E^r}) d\mu_{E^r} - \frac{s^2}{2} \int_A \mathscr{S}_V(\nu_{E^r}) d\mu_{E^r} \right| \leq C P(E^r, A) s^3, \tag{4.39}$$

where C is a constant independent of r and s. It is easy to see that as $r \to 0$ we have

$$\chi_{E^r} \to \chi_E \quad \text{in } L^1(A). \tag{4.40}$$

We claim that we also have

$$\lim_{r \to 0} P(E^r, A) = P(E, A), \qquad \lim_{r \to 0} P(E^r_s, A_s) = P(E_s, A_s). \tag{4.41}$$

We prove the claim in the left hand side of (4.41). Since $|\nabla_H f| \neq 0$ in A, we can assume, up to a rotation and a localization argument, that $X_1(f) \geq \varepsilon_0 > 0$ on A. By the implicit function theorem of [4], there exist an open set $I \subset \mathbb{R}^{2n}$ and continuous functions $\Phi^r : I \to \mathbb{H}^n$ such that

$$P(E^r, A) = \int_I \frac{|\nabla_H f|}{X_1 f} (\Phi^r(x)) d\mathscr{L}^{2n}(x). \tag{4.42}$$

We may assume that I does not depend on r, via choosing a cylindrical structure of A along X_1. As $r \to 0$, the function $\Phi^r(x)$ converge to $\Phi(x)$ uniformly in x (see the proof of [14, Proposition 4.5]) and, by the (uniform) continuity of $\nabla_H f$ and $X_1 f$, we conclude that

$$\lim_{r\to 0}\frac{|\nabla_H f|}{X_1 f}(\Phi^r(x)) = \frac{|\nabla_H f|}{X_1 f}(\Phi^0(x)),$$

uniformly in $x \in I$. Exchanging integral and limit in (4.42), this proves our claim.

From (4.40) and (4.41), we deduce by Lemma 2.5 in [14] that for $r \to 0$ we have the weak* convergence of measures

$$\nu_{E^r}\mu_{E^r} \rightharpoonup \nu_E \mu_E.$$

Thus, by Reshetnyak continuity theorem, see e.g. [1, Theorem 2.39], we deduce that

$$\begin{aligned}\lim_{r\to 0}\int_A \mathscr{F}_V(\nu_{E^r})d\mu_{E^r} &= \int_A \mathscr{F}_V(\nu_E)d\mu_E \quad \text{and}\\ \lim_{r\to 0}\int_A \mathscr{S}_V(\nu_{E^r})d\mu_{E^r} &= \int_A \mathscr{S}_V(\nu_E)d\mu_E.\end{aligned} \tag{4.43}$$

Now, using (4.43) and (4.41), and passing to the limit in the formula (4.39), we obtain (4.38). □

4.2. **Variation formulas for H-rectifiable sets.** In this section, we prove the variation formula for H-rectifiable sets. In this formula, the H-perimeter measure μ_E associated with a set E of locally finite perimeter is replaced by the spherical Hausdorff measure $\mathscr{S} = \mathscr{S}^{2n+1}$.

We introduce the notion of H-rectifiable set for the dimension $2n+1$. The notion of rectifiability for a generic dimension is studied in [8].

Definition 4.3. A Borel set $R \subset \mathbb{H}^n$ is H-rectifiable if there exists a sequence of H-regular hypersurfaces $\{M_j\}_{j\in\mathbb{N}}$ with $\mathscr{S}(M_j) < \infty$ such that

$$\mathscr{S}\Big(R \setminus \bigcup_{j\in\mathbb{N}} M_j\Big) = 0.$$

The importance of the notion of R-rectifiability is due to the fact that the reduced boundary of sets with locally finite H-perimeter is rectifiable in the sense of Definition 4.3, see [4]. So Theorem 1.1 for sets with finite H-perimeter in the Introduction follows directly from Theorem 4.4 below for H-rectifiable sets.

We define a mapping $\nu_R : R \to \mathbb{R}^{2n}$ letting $\nu_R(p) = \nu_{M_j}(p)$ where $j \in \mathbb{N}$ is the unique integer such that $p \in M_j \setminus \cup_{i<j} M_i$ and letting $\nu_R = 0$ if there is no such j. The function ν_R is Borel measurable and it is well-defined up to a sign. Namely, if $\{M_j^1\}_{j\in\mathbb{N}}$ and $\{M_j^2\}_{j\in\mathbb{N}}$ are two sequences of H-regular hypersurfaces such that

$$\mathscr{S}\Big(R \setminus \bigcup_{j\in\mathbb{N}} M_j^1\Big) = 0, \quad \mathscr{S}\Big(R \setminus \bigcup_{j\in\mathbb{N}} M_j^2\Big) = 0,$$

then for $\mathscr{S}$-a.e. $p \in R$ we have $\nu_R^1(p) = \nu_R^2(p)$ or $\nu_R^1(p) = -\nu_R^2(p)$, where ν_R^1 and ν_R^2 are defined as above by means of $\{M_j^1\}_{j\in\mathbb{N}}$ and $\{M_j^2\}_{j\in\mathbb{N}}$, respectively. The proof of these claims can be found in [12, Appendix B].

We call ν_R the *horizontal normal* of the H-rectifiable set R. The fact that ν_R is unique only up to the sign does not affect our results, because the first- and second-variation kernels are symmetric, in the sense that

$$\mathscr{F}_V(\nu) = \mathscr{F}_V(-\nu) \quad \text{and} \quad \mathscr{S}_V(\nu) = \mathscr{S}_V(-\nu)$$

for any $\nu \in UH$.

Theorem 4.4. *Let $A \subset \mathbb{H}^n$ be a bounded open set and $R \subset A$ an H-rectifiable set with $\mathscr{S}(R) < \infty$ and with horizontal normal ν_R. Let $\Psi : [-\delta, \delta] \times A \to \mathbb{H}^n$, $\delta = \delta(A, V) > 0$, be the flow generated by a contact vector field V. Then there exists a positive constant $C = C(A, V)$ such that*

$$\left| \mathscr{S}(\Psi_s(R)) - \mathscr{S}(R) - s \int_R \mathscr{F}_V(\nu_R) d\mathscr{S} - \frac{s^2}{2} \int_R \mathscr{S}_V(\nu_R) d\mathscr{S} \right| \leq C \mathscr{S}(R) s^3, \quad (4.44)$$

for all $s \in [-\delta, \delta]$.

Proof. When $R = \partial E \cap A$ is an H-regular hypersurface bounding a set E, formula (4.44) is formula (4.38) with $\nu_R = \nu_E$ and $\mu_E = \mathscr{S} \llcorner \partial R$, which holds because of (2.4).

Step 1. We prove formula (4.44) in the case that $R = \Sigma \subset M$ is a Borel subset of an H-regular hypersurface M.

Since $\mathscr{S} \llcorner M$ is a Radon measure, there exists a sequence of open sets $A_j \subset \mathbb{H}^n$ such that $\Sigma \subset A_j$ and

$$\lim_{j \to \infty} \mathscr{S}(M \cap A_j) = \mathscr{S}(\Sigma).$$

For the same reason, for any fixed $s \in [-\delta, \delta]$, there exists a sequence B_j of open sets such that $\Psi_s(\Sigma) \subset B_j$ and

$$\lim_{j \to \infty} \mathscr{S}(\Psi_s(M) \cap B_j) = \mathscr{S}(\Psi_s(\Sigma)).$$

Letting, $U_j = A_j \cap \Psi_s^{-1}(B_j)$, we have $\Sigma \subset U_j \subset A_j$ and $\Psi_s(\Sigma) \subset \Psi_s(U_j) \subset B_j$, and thus

$$\lim_{j \to \infty} \mathscr{S}(M \cap U_j) = \mathscr{S}(\Sigma), \qquad \lim_{j \to \infty} \mathscr{S}(\Psi_s(M \cap U_j)) = \mathscr{S}(\Psi_s(\Sigma)). \quad (4.45)$$

The sets $\Sigma_j = M \cap U_j$ are H-regular hypersurfaces and thus formula (4.44) holds for them with $\nu_{\Sigma_j} = \nu_M$ on Σ_j. By dominated convergence, we have

$$\begin{aligned} \lim_{j \to \infty} \int_{\Sigma_j} \mathscr{F}_V(\nu_M)\, d\mathscr{S} &= \int_\Sigma \mathscr{F}_V(\nu_M)\, d\mathscr{S}, \\ \lim_{j \to \infty} \int_{\Sigma_j} \mathscr{S}_V(\nu_M)\, d\mathscr{S} &= \int_\Sigma \mathscr{S}_V(\nu_M)\, d\mathscr{S}. \end{aligned} \quad (4.46)$$

From (4.45) and (4.46), we deduce (4.44).

Step 2. We prove formula (4.44) for a general H-rectifiable set R such that $\mathscr{S}(R) < \infty$. Then we have

$$R = N \cup \bigcup_{j=1}^{\infty} \Sigma_j \subset A,$$

with $\mathscr{S}(N) = 0$ and for pairwise disjoint Borel sets $\Sigma_j \subset M_j$ with M_j H-regular and $\mathscr{S}(\Sigma_j) < \infty$. By Step 1, we have for any $j \in \mathbb{N}$

$$\Big| \mathscr{S}(\Psi_s(\Sigma_j)) - \mathscr{S}(\Sigma_j) - s \int_{\Sigma_j} \mathscr{F}_V(\nu_R) d\mathscr{S} - \frac{s^2}{2} \int_{\Sigma_j} \mathscr{S}_V(\nu_R) d\mathscr{S} \Big| \le C\mathscr{S}(\Sigma_j) s^3,$$

for all $s \in [-\delta, \delta]$. Taking into account the fact that $\mathscr{S}(\Psi_s(N)) = 0$, formula (4.44) then follows by summation on j. □

5. Variation kernels and generating function

Let $V = V_\psi$ be a contact vector field of the form (3.27) for some $\psi \in C^\infty(\mathbb{H}^n)$. We compute the first- and second-variation kernels in terms of the generating function ψ.

5.1. **Formula for $\mathscr{F}_V$.** By the formula (3.27) for a contact vector field, we have

$$V = -4\psi T - J\nabla_H \psi, \tag{5.47}$$

where $J : H \to H$ is the standard complex structure on H, i.e., the linear mapping such that $JX_j = Y_j$ and $JY_j = -X_j$, and

$$\nabla_H \psi = \sum_{j=1}^{2n} (X_j \psi) X_j \tag{5.48}$$

is the horizontal gradient of ψ. The horizontal Hessian of ψ is the bilinear form $\nabla_H^2 \psi : H \times H \to \mathbb{R}$ such that

$$\nabla_H^2 \psi(X_j, X_i) = X_j X_i \psi, \quad i, j = 1, \dots, 2n.$$

We can think of the horizontal Hessian also as the linear mapping $\nabla_H^2 \psi : H \to H$ such that $\nabla_H^2 \psi(X_j, X_i) = \langle \nabla_H^2 \psi(X_j), X_i \rangle_H$, i.e.,

$$\nabla_H^2 \psi(X_j) = \sum_{i=1}^{2n} (X_j X_i \psi) X_i, \quad j = 1, \dots, 2n.$$

Proposition 5.1. *For any contact vector field V and $\nu \in UH$, the first-variation kernel is*

$$\mathscr{F}_V(\nu) = -4(n+1)T\psi - \nabla_H^2 \psi(\nu, J\nu).$$

Proof. The first-variation kernel is $\mathscr{F}_V(\nu) = \mathrm{div}V + \mathscr{Q}_V(\nu)$, see (3.34). In order to compute the divergence $\mathrm{div}V$, we first observe that

$$\mathrm{div}(J\nabla_H \psi) = \sum_{j=1}^{n} Y_j X_j \psi - X_j Y_j \psi = 4nT\psi. \tag{5.49}$$

Then we have

$$\mathrm{div}V = \mathrm{div}(-4\psi T - J\nabla_H \psi) = -4T\psi - \mathrm{div}(J\nabla_H \psi) = -4(n+1)T\psi.$$

We compute the quadratic form $\mathscr{Q}_V$. From the commutation relations $[X_i, Y_j] = -4\delta_{ij}T$ and $[X_j, T] = [Y_j, T] = 0$ for $i, j = 1, \dots, n$, and from (5.47) we have, for any $j = 1, \dots, 2n$,

$$\begin{aligned}
[V, X_j] &= [-4\psi T - J\nabla_H\psi, X_j] \\
&= 4(X_j\psi)T - \sum_{j=1}^{2n}(X_i\psi)[JX_i, X_j] - (X_jX_i\psi)JX_i \\
&= X_jJ\nabla_H\psi,
\end{aligned} \tag{5.50}$$

where the vector field X_j acts on the horizontal coordinates of $J\nabla_H\psi$. By formula i) of Lemma 3.9 for $\mathscr{Q}_V$, by (5.50), by the isometric property of J, and by $J^2 = -\mathrm{Id}$, we have

$$\begin{aligned}
\mathscr{Q}_V(\nu) = \langle [V, \nu], \nu\rangle_H &= \sum_{j=1}^{2n} \nu_j \langle [V, X_j], \nu\rangle_H \\
&= \sum_{j=1}^{2n} \nu_j \langle X_jJ\nabla_H\psi, \nu\rangle_H = -\sum_{j=1}^{2n} \nu_j\langle X_j\nabla_H\psi, J\nu\rangle_H \\
&= -\nabla_H^2\psi(\nu, J\nu).
\end{aligned}$$

The claim follows. □

5.2. **Formula for $\mathscr{S}_V$.** The formula for $\mathscr{S}_V(\nu)$ in terms of V is in (3.35). We need to compute the quadratic form $\mathscr{R}_V$ and the divergence $\mathrm{div}(J_VV)$.

We define the bilinear form $\nabla_{JH}^2\psi : H \times H \to \mathbb{R}$ such that

$$\nabla_{JH}^2\psi(X_j, X_i) = (JX_j)(JX_i)\psi, \quad j = 1, \dots, 2n,$$

that also induces a linear mapping $\nabla_{JH}^2\psi : H \to H$. We denote by $\nabla_H^2\psi\nabla_{JH}^2\psi$ the bilinear form associated with the composition of the linear operators $\nabla_H^2\psi$ and $\nabla_{JH}^2\psi$, while $(\nabla_H^2\psi)^*$ is the adjoint of $\nabla_H^2\psi$.

Proposition 5.2. *For any contact vector field V and for any $\nu \in UH$ we have*

$$\mathrm{div}(J_VV) = 8(n+1)T^2\psi^2 + 4n\langle\nabla_H\psi, J\nabla_HT\psi\rangle_H - \mathrm{trace}\big(\nabla_H^2\psi\nabla_{JH}^2\psi\big)$$

and

$$\mathscr{R}_V(\nu) = -(V\nabla_H^2\psi)(\nu, J\nu) - \nabla_H^2\psi\nabla_{JH}^2\psi(\nu,\nu) + \|(\nabla_H^2\psi)^*(J\nu)\|_H^2.$$

Proof. By formula ii) in Lemma 3.9, we have $\mathscr{R}_V(\nu) = \langle [V, [V, \nu]], \nu\rangle_H + \|\mathscr{L}_V^*(\nu)\|_H^2$. By formulas (5.50) and (5.48), one gets

$$\begin{aligned}
[V, [V, \nu]] &= \sum_{j=1}^{2n} \nu_j [V, X_jJ\nabla_H\psi] = \sum_{i,j=1}^{2n} \nu_j[V, (X_jX_i\psi)JX_i] \\
&= \sum_{i,j=1}^{2n} \nu_j\big\{(VX_jX_i\psi)JX_i + (X_jX_i\psi)(JX_i)J\nabla_H\psi\big\},
\end{aligned}$$

and the scalar product with ν is

$$
\begin{aligned}
\langle [V,[V,\nu]],\nu\rangle_H &= \sum_{i,j=1}^{2n} \nu_j\big\{ -(VX_jX_i\psi)\langle X_i, J\nu\rangle_H + (X_jX_i\psi)\langle (JX_i)J\nabla_H\psi,\nu\rangle_H\big\}\\
&= -(V\nabla_H^2\psi)(\nu,J\nu) - \sum_{i,j,k=1}^{2n} \nu_j\nu_k (X_jX_i\psi)(JX_i)(JX_k)\psi\\
&= -(V\nabla_H^2\psi)(\nu,J\nu) - (\nabla_H^2\psi\nabla_{JH}^2\psi)(\nu,\nu).
\end{aligned}
$$

Moreover, by (5.50) we have

$$
\begin{aligned}
\|\mathscr{L}_V^*(\nu)\|_H^2 &= \sum_{j=1}^{2n}\langle X_j,\mathscr{L}_V^*(\nu)\rangle_H^2 = \sum_{j=1}^{2n}\langle \mathscr{L}_V(X_j),\nu\rangle_H^2 = \sum_{j=1}^{2n}\langle [V,X_j],\nu\rangle_H^2\\
&= \sum_{j=1}^{2n}\langle X_jJ\nabla_H\psi,\nu\rangle_H^2 = \sum_{j=1}^{2n}\langle \nabla_H^2\psi(X_j),J\nu\rangle_H^2 = \|(\nabla_H^2\psi)^*(J\nu)\|_H^2.
\end{aligned}
$$

Now we compute the divergence $\operatorname{div}(J_VV)$. Using the relations $X_j(JX_i) = -2\delta_{ij}T$, we obtain

$$
\begin{aligned}
J_VX_j &= -2(X_j\psi)T - J(X_j\nabla_H\psi), \quad j=1,\dots,2n,\\
J_VT &= -4(T\psi)T - J(T\nabla_H\psi),
\end{aligned}
$$

and thus the vector field J_VV is

$$
\begin{aligned}
J_VV &= 4\psi\{4(T\psi)T + J(T\nabla_H\psi)\} + \sum_{j=1}^{2n}(X_j\psi)\{2(JX_j\psi)T + (JX_j)J\nabla_H\psi\}\\
&= 8(T\psi^2)T + 4\psi J(\nabla_H T\psi)) + J\Big(\sum_{j=1}^{2n}(X_j\psi)(JX_j)\nabla_H\psi\Big).
\end{aligned}
$$

We used the identity $\langle\nabla_H\psi, J\nabla_H\psi\rangle_H = 0$. Since $(JX_j)X_i = X_i(JX_j) + [JX_j, X_i]$, we have

$$
\sum_{j=1}^{2n}(X_j\psi)(JX_j)\nabla_H\psi = 4(T\psi)\nabla_H\psi + \sum_{j=1}^{2n}(X_j\psi)\nabla_H(JX_j)\psi
$$

and, since T commutes with ∇_H, we finally obtain the following formula for J_VV

$$
J_VV = 8T\psi^2T + 4T(\psi J\nabla_H\psi) + \sum_{j=1}^{2n}(X_j\psi)J\nabla_H(JX_j)\psi.
$$

Using $T\langle\nabla_H\psi, J\nabla_H\psi\rangle_H = 0$ and formula (5.49), one gets

$$
\begin{aligned}
\operatorname{div}(J_VV) &= 8T^2\psi^2 + 4T\operatorname{div}(\psi J\nabla_H\psi) + \sum_{j=1}^{2n}\operatorname{div}\big((X_j\psi)J\nabla_H(JX_j)\psi\big)\\
&= 8(n+1)T^2\psi^2 + 4n\langle\nabla_H\psi, J\nabla_HT\psi\rangle_H - \sum_{i,j=1}^{2n} X_jX_i\psi(JX_i)(JX_j)\psi,
\end{aligned}
$$

and the claim follows. □

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