Conformal transformation on metric measure spaces

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Abstract

We intrinsically study the conformal transformations on metric measure spaces, including the Sobolev space, the differential structure and the curvature-dimension condition under conformal transformations. As an application, we will show how the conformal transformations change the curvature-dimension condition.

Keywords: curvature-dimension condition, Bakry-Émery theory, conformal transformation, Ricci tensor, metric measure space.

Contents

1 Introduction

Let (M, g, Vol_g) be a Riemannian manifold with dimension n, $\mathrm{Ricci}(\cdot,\cdot)$ be the Ricci tensor on it. Let w be a smooth function on M, the corresponding Riemannian manifold under conformal transformation be defined as $(M, e^{2w}g, e^{nw}\mathrm{Vol}_g)$. Then we have the following formula (see Theorem 1.159, [6]) which builds a link between the Ricci tensor of the new manifold and the old one,

$$\mathrm{Ricci}' = \mathrm{Ricci} - (n-2)(\mathrm{D}\mathrm{d}w - \mathrm{d}w \otimes \mathrm{d}w) + (\Delta w - (n-2)|\mathrm{d}w|^2)g \tag{1.1}$$

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where $\mathrm{D}\mathrm{d}w$ is the Hessian of w and Δ is the Laplace-Beltrami operator on (M, g).

The conformal transformation as above plays an important role in the study of differential geometry, also has potential applications in non-smooth case since it can be obtained in an intrinsic way: Let w be a bounded continuous function on a metric measure space $(X, \mathrm{d}, \mathfrak{m})$, we can build a weighted metric measure space $M' := (X, \mathrm{d}', \mathfrak{m}')$ where

- we replace $\mathfrak{m}$ by the weighted measure with density e^{Nw}:

$$\mathfrak{m}' = e^{Nw}\,\mathfrak{m};$$

- we replace d by $\mathrm{d}' = e^w \mathrm{d}$, i.e. the weighted metric with conformal factor e^w:

$$\mathrm{d}'(x, y) = \inf_{\gamma}\{\int_0^1 |\dot{\gamma}_t| e^{w(\gamma_t)}\,\mathrm{d}t : \gamma \in AC([0,1]; X), \gamma_0 = x, \gamma_1 = y\}.$$

Then we have a natural question, can we prove the formula (1.1) in a non-smooth framework? A reasonable restriction on the metric measure space is the so-called curvature-dimension condition ($\mathrm{CD}(K,\infty)$, $\mathrm{CD}(K,N)$, etc).

The notion of synthetic Ricci curvature bounds, or curvature-dimension conditions of metric measure space were founded by Lott-Sturm-Villani (see [18] and [16] for $\mathrm{CD}(K,\infty)$ and $\mathrm{CD}(K,N)$ conditions) and Bacher-Sturm (see [5] for $\mathrm{CD}^*(K,N)$ condition) about ten years ago. More recently, based on some new results on the Sobolev spaces on metric measure space (see [3]), $\mathrm{RCD}(K,\infty)$ and $\mathrm{RCD}^*(K,N)$ conditions which are refinements of curvature-dimension conditions were proposed by Ambrosio-Gigli-Savaré (see [4] and [1]). Moreover, the non-smooth Bakry-Émery theory, which offers an equivalent description of $\mathrm{RCD}(K,\infty)$ (and $\mathrm{RCD}^*(K,N)$) conditions, was studied by Ambrosio-Gigli-Savaré (see [4] and [2]) and Erbar-Kuwada-Sturm (see [8]). These Riemannian curvature-dimension conditions are stable with respect to the measured Gromov-Hausdorff convergence, and cover the cases of Riemannian manifolds, smooth metric measure spaces, Alexandrov spaces and their limits.

In the case of $\mathrm{RCD}(K,\infty)$ space, we have a natural Dirichlet form and in particular, there is a well-posed measure-valued Laplacian (see [9] and [15]). Then, Bakry-Émery's Γ_2 calculus and Bochner-type inequality are known to be valid in non-smooth sense (see [4], [8] and [15]). In [10], Gigli builds a differential structure on metric measure spaces which is suitable to deal with the non-smooth (co)tangent fields (see the preliminary section). In this framework, the Hessian and the Ricci curvature tensor on $\mathrm{RCD}(K,\infty)$ metric measure spaces is well defined. Furthermore, in [12], the N-Ricci tensor on $\mathrm{RCD}^*(K,N)$ spaces is defined by the author.

In [17], Sturm proves the formula (1.1) under some smoothness assumptions. The current work is to apply the tools and results on the differential structure of $\mathrm{RCD}(K,\infty)$ spaces which are developed in [10] (and [12]) to prove Sturm's result in the case of $\mathrm{RCD}^*(K,N)$ which is a lower-regular situation (see Theorem 4.4). As an application, we will obtain an estimate of the lower Ricci curvature under conformal transformation.

The organization of this paper is the following. In Section 2 we will introduce the notions of Sobolev space, non-smooth Bakry-Émery theory, the tangent/cotangent module and analytic dimension of metric measure spaces. In Section 3, we will study the conformal transformation on metric measure spaces, the Sobolev space as well as the differential structures under conformal transformation. All these objects will be considered in pure intrinsic ways. In Section 4, we will prove Theorem 4.4 which is a generalization of the formula (1.1) on $\mathrm{RCD}^*(K,N)$ spaces. As a corollary, we obtain a precise N-Ricci curvature bound estimate under conformal transformation in Corollary 4.5.

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2 Preliminaries

The main object we studied in this paper are metric measure spaces. Basic assumptions on metric measure spaces are the following. Let $M = (X, \mathrm{d}, \mathfrak{m})$, we assume that (X, d) is a geodesic space and $\mathfrak{m}$ is a d-Borel measure satisfying the following property

$$\operatorname{supp}\mathfrak{m} = X, \qquad \mathfrak{m}(B_r(x)) < c_1 \exp\left(c_2 r^2\right) \quad \text{for every} \quad r > 0,$$

for some constants $c_1, c_2 \geq 0$ and a point $x \in X$. Our results are mainly concentrated on $\mathrm{RCD}^*(K,N)$ metric measure spaces, where $K \in \mathbb{R}$ and $N \in [1,\infty]$ (when $N = \infty$ it is $\mathrm{RCD}(K,\infty)$ space). $\mathrm{RCD}(K,\infty)$ and $\mathrm{RCD}^*(K,N)$ conditions are refinements of the curvature-dimensions proposed by Lott-Sturm-Villani (see [18] and [16] for $\mathrm{CD}(K,\infty)$) and Bacher-Sturm (see [5] for $\mathrm{CD}^*(K,N)$). The general inclusion of these curvature dimension conditions are

$$\mathrm{RCD}^*(K,N) \subset \mathrm{CD}^*(K,N) \quad \text{and} \quad \mathrm{RCD}(K,\infty) \subset \mathrm{CD}(K,\infty),$$

and

$$\mathrm{RCD}^*(K,N) \subset \mathrm{RCD}(K,\infty) \quad \text{and} \quad \mathrm{CD}^*(K,N) \subset \mathrm{CD}(K,\infty).$$

More details about the curvature dimension condition $\mathrm{RCD}^*(K,N)$ can be found in and [1, 4].

The space of finite Borel measures on M, equipped with the total variation norm $\|\cdot\|_{\mathrm{TV}}$, is denoted by $\mathrm{Meas}(M)$.

For $f : X \mapsto \mathbb{R}$, the local Lipschitz constant $\mathrm{lip}(f) : X \mapsto [0,\infty]$ is defined as

$$\mathrm{lip}(f)(x) := \begin{cases} \overline{\lim}_{y\to x} \frac{|f(y)-f(x)|}{\mathrm{d}(x,y)} & \text{if } x \text{ is not isolated,} \\ 0, & \text{otherwise.} \end{cases}$$

The (global) Lipschitz constant is defined in the usual way as

$$\mathrm{Lip}(f) := \sup_{x\neq y} \frac{|f(y)-f(x)|}{\mathrm{d}(x,y)}.$$

Since (X, d) is a geodesic space, we know $\mathrm{Lip}(f) = \sup_x \mathrm{lip}(f)(x)$.

The Sobolev space $W^{1,2}(M)$ is defined as in [3]. We say that $f \in L^2(X, \mathfrak{m})$ is a Sobolev function in $W^{1,2}(M)$ if there exists a sequence of Lipschitz functions functions $\{f_n\} \to f$ in L^2 such that $\mathrm{lip}(f_n) \to G$ in L^2 for some $G \in L^2(X, \mathfrak{m})$. It is known that there exists a minimal function G in $\mathfrak{m}$-a.e. sense. We call the minimal G the weak gradient of the function f, and denote it by $|\mathrm{D}f|$ or $|\mathrm{D}f|_M$ to indicate which space we are considering.

Then we equip $W^{1,2}(X, \mathrm{d}, \mathfrak{m})$ with the norm

$$\|f\|^2_{W^{1,2}(X,\mathrm{d},\mathfrak{m})} := \|f\|^2_{L^2(X,\mathfrak{m})} + \||\mathrm{D}f|\|^2_{L^2(X,\mathfrak{m})}.$$

It is part of the definition of $\mathrm{RCD}^*(K, N)$ space that $W^{1,2}(X, \mathrm{d}, \mathfrak{m})$ is a Hilbert space, in which case $(X, \mathrm{d}, \mathfrak{m})$ is called infinitesimally Hilbertian. In this article, we will assume that all the metric measure spaces are infinitesimally Hilbertian.

As a consequence of the definition above, we have the lower semi-continuity: if $\{f_n\}_n \subset W^{1,2}(X, \mathrm{d}, \mathfrak{m})$ is a sequence converging to some f in $\mathfrak{m}$-a.e. sense and such that $\{|\mathrm{D}f_n|\}_n$ is bounded in $L^2(X, \mathfrak{m})$, then $f \in W^{1,2}(X, \mathrm{d}, \mathfrak{m})$ and

$$|\mathrm{D}f| \leq G, \qquad \mathfrak{m} - a.e.,$$

for every L^2-weak limit G of some subsequence of $\{|\mathrm{D}f_n|\}_n$. Furthermore, we have the following proposition.

Proposition 2.1 (see [3])**.** *Let $(X, \mathrm{d}, \mathfrak{m})$ be a metric measure space. Then the Lipschitz functions are dense in energy in $W^{1,2}(M)$ in the sense that: for any $f \in W^{1,2}(M)$ there exists a sequence of Lipschitz functions $\{f_n\}_n \subset L^2(X, \mathfrak{m})$ such that $f_n \to f$ and $\mathrm{lip}(f_n) \to |\mathrm{D}f|$ in L^2.*

Then we discuss a bit the notion of 'tangent/cotangent vector field' in non-smooth framework, more details can be found in [10].

Definition 2.2 (L^2-normed L^∞-module)**.** Let $M = (X, \mathrm{d}, \mathfrak{m})$ be a metric measure space. A L^2-normed $L^\infty(X, \mathfrak{m})$-module is a Banach space $(\mathbf{B}, \|\cdot\|_{\mathbf{B}})$ equipped with a bilinear map

$$\begin{aligned} L^\infty(X, \mathfrak{m}) \times \mathbf{B} &\mapsto \mathbf{B}, \\ (f, v) &\mapsto f \cdot v \end{aligned}$$

such that

$$\begin{aligned} (fg) \cdot v &= f \cdot (g \cdot v), \\ \mathbf{1} \cdot v &= v \end{aligned}$$

for every $v \in \mathbf{B}$ and $f, g \in L^\infty(M)$, where $\mathbf{1} \in L^\infty(X, \mathfrak{m})$ is identically equal to 1 on X, and a 'pointwise norm' $|\cdot| : \mathbf{B} \mapsto L^2(X, \mathfrak{m})$ which maps $v \in \mathbf{B}$ to a non-negative L^2-function such that

$$\begin{aligned} \|v\|_{\mathbf{B}} &= \||v|\|_{L^2} \\ |f \cdot v| &= |f||v|, \quad \mathfrak{m} - \text{a.e.} \end{aligned}$$

for every $f \in L^\infty(X, \mathfrak{m})$ and $v \in \mathbf{B}$.

Now we can define the tangent/cotangent modules of M as typical L^2-normed modules. We define the 'Pre-Cotangent Module' $\mathcal{PCM}$ as the set consisting of the elements of the from $\{(B_i, f_i)\}_{i\in\mathbb{N}}$, where $\{B_i\}_{i\in\mathbb{N}}$ is a Borel partition of X, and $\{f_i\}_i$ are Sobolev functions such that $\sum_i \int_{B_i} |\mathrm{D}f_i|^2 < \infty$.

We define an equivalence relation on $\mathcal{PCM}$ via

$$\{(A_i, f_i)\}_{i\in\mathbb{N}} \sim \{(B_j, g_j)\}_{j\in\mathbb{N}} \quad \text{if} \quad |\mathrm{D}(g_j - f_i)| = 0, \quad \mathfrak{m} - \text{a.e. on } A_i \cap B_j.$$

We denote the equivalence class of $\{(B_i, f_i)\}_{i\in\mathbb{N}}$ by $[(B_i, f_i)]$. In particular, we call $[(X, f)]$ the differential of a Sobolev function f and denote it by $\mathrm{d}f$.

Then we define the following operations:

1) $[(A_i, f_i)] + [(B_i, g_i)] := [(A_i \cap B_j, f_i + g_j)]$;

2) Multiplication by scalars: $\lambda[(A_i, f_i)] := [(A_i, \lambda f_i)]$;

3) Multiplication by simple functions: $(\sum_j \lambda_j \chi_{B_j})[(A_i, f_i)] := [(A_i \cap B_j, \lambda_j f_i)]$;

4) Pointwise norm: $|[(A_i, f_i)]| := \sum_i \chi_{A_i} |\mathrm{D}f_i|$,

where χ_A denotes the characteristic function on the set A.

It can be seen that all the operations above are continuous on $\mathcal{PCM}/\sim$ with respect to the norm $\|[(A_i, f_i)]\| := \sqrt{\int |[(A_i, f_i)]|^2 \,\mathfrak{m}}$ and the $L^\infty(M)$-norm on the space of simple functions. Therefore we can extend them to the completion of $(\mathcal{PCM}/\sim, \|\cdot\|)$ and we denote this completion by $L^2(T^*M)$. As a consequence of our definition, we can see that $L^2(T^*M)$ is the $\|\cdot\|$ closure of $\{\sum_{i\in I} a_i \mathrm{d}f_i : |I| < \infty, a_i \in L^\infty(M), f_i \in W^{1,2}\}$ (see Proposition 2.2.5 in [10] for a proof). It can also be seen from the definition and the infinitesimal Hilbertianity assumption on M that $L^2(T^*M)$ is a Hilbert space equipped with the inner product induced by $\|\cdot\|$. Moreover, $(L^2(T^*M), \|\cdot\|, |\cdot|)$ is a L^2-normed module according to the Definition 2.2, which we shall call cotangent module of M.

We then define the tangent module $L^2(TM)$ as $\mathrm{Hom}_{L^\infty(M)}(L^2(T^*M), L^1(M))$, i.e. $T \in L^2(T^*M)$ if it is a continuous linear map from $L^2(T^*M)$ to $L^1(M)$ viewed as Banach spaces satisfying the homogeneity:

$$T(fv) = fT(v), \quad \forall v \in L^2(T^*M), \quad f \in L^\infty(M).$$

It can be seen that $L^2(TM)$ has a natural L^2-normed $L^\infty(M)$-module structure and is isometric to $L^2(T^*M)$ both as a module and as a Hilbert space. We denote the corresponding element of $\mathrm{d}f$ in $L^2(TM)$ by ∇f and call it the gradient of f. It can be seen that the Riesz theorem for Hilbert modules (see Chapter 1 of [10]) that $\mathrm{d}f(\nabla f) := \nabla f(\mathrm{d}f) = |\mathrm{D}f|^2$. The natural pointwise norm on $L^2(TM)$ (we also denote it by $|\cdot|$) satisfies $|\nabla f| = |\mathrm{d}f| = |\mathrm{D}f|$. It can be seen that $\{\sum_{i\in I} a_i \nabla f_i : |I| < \infty, a_i \in L^\infty(M), f_i \in W^{1,2}\}$ is a dense subset in $L^2(TM)$.

Since we assume that the space is infinitesimally Hilbertian, we have a natural carré du champ operator $\Gamma(\cdot,\cdot) : [W^{1,2}(M)]^2 \mapsto L^1(M)$ defined by

$$\Gamma(f,g) := \frac{1}{4}\Big(|\mathrm{D}(f+g)|^2 - |\mathrm{D}(f-g)|^2\Big).$$

We denote $\Gamma(f,f)$ by $\Gamma(f)$.

Then we have a pointwise inner product $\langle\cdot,\cdot\rangle : [L^2(T^*M)]^2 \mapsto L^1(M)$ satisfying

$$\langle \mathrm{d}f, \mathrm{d}g\rangle := \Gamma(f,g)$$

for $f,g \in W^{1,2}(M)$. We know (also from Riesz theorem) that the gradient ∇g is exactly the element in $L^2(TM)$ such that $\nabla g(\mathrm{d}f) = \langle \mathrm{d}f, \mathrm{d}g\rangle, \mathfrak{m}$-a.e. for every $f \in W^{1,2}(M)$. Therefore, $L^2(TM)$ inherits a pointwise inner product from $L^2(T^*M)$ and we still use $\langle\cdot,\cdot\rangle$ to denote it.

It is known from [3] and [10] that the following basic calculus rules hold in $\mathfrak{m}$-a.e. sense.

We have

- $\mathrm{d}(fg) = f\mathrm{d}g + g\mathrm{d}f$,
- $\mathrm{d}(\varphi\circ f) = \varphi'\circ f\mathrm{d}f$,

for every $f,g \in W^{1,2}(M)$, and $\varphi : \mathbb{R} \mapsto \mathbb{R}$ smooth.

We then define the Laplacian by duality (integration by part).

Definition 2.3 (Measure valued Laplacian, [9,10])**.** The space $\mathrm{D}(\mathbf{\Delta}) \subset W^{1,2}(M)$ is the space of $f \in W^{1,2}(M)$ such that there is a measure $\mu \in \mathrm{Meas}(M)$ satisfying

$$\int \varphi\,\mu = -\int \Gamma(\varphi, f)\,\mathfrak{m}, \forall\varphi : M \mapsto \mathbb{R}, \text{ Lipschitz with bounded support.}$$

In this case the measure μ is unique and we denote it by $\mathbf{\Delta} f$. If $\mathbf{\Delta} f \ll m$, we denote its density with respect to $\mathfrak{m}$ by Δf.

It is proved in [9] that the following rules hold for the Laplacian:

- $\mathbf{\Delta}(fg) = f\mathbf{\Delta}g + g\mathbf{\Delta}f + 2\Gamma(f,g)\,\mathfrak{m}$,
- $\mathbf{\Delta}(\varphi\circ f) = \varphi'\circ f\mathbf{\Delta}f + \varphi''\circ f\Gamma(f)\,\mathfrak{m}$,

for every $f,g \in \mathrm{D}(\mathbf{\Delta}) \cap L^\infty(M)$, and $\varphi : \mathbb{R} \mapsto \mathbb{R}$ smooth.

We define $\mathrm{TestF}(M) \subset W^{1,2}(M)$, the space of test functions as

$$\mathrm{TestF}(M) := \{f \in \mathrm{D}(\mathbf{\Delta}) \cap L^\infty : |\mathrm{D}f| \in L^\infty \quad\text{and}\quad \Delta f \in W^{1,2}(M)\}.$$

It is known from [15] and [4] that $\mathrm{TestF}(M)$ is an algebra and it is dense in $W^{1,2}(M)$ when M is $\mathrm{RCD}(K,\infty)$. In particular, we know $\{\sum_{i\in I} a_i\nabla f_i : |I| < \infty, a_i \in L^\infty(M), f_i \in \mathrm{TestF}(M)\}$ is dense in $L^2(TM)$.

We also have the following lemma.

Lemma 2.4 ([15]). *Let $M = (X, \mathrm{d}, \mathfrak{m})$ be a $\mathrm{RCD}(K, \infty)$ space, $f \in \mathrm{TestF}(M)$ and $\Phi \in C^\infty(\mathbb{R})$ be with $\Phi(0) = 0$. Then $\Phi \circ f \in \mathrm{TestF}(M)$.*

It is proved in [15] that $\Gamma(f, g) \in \mathrm{D}(\mathbf{\Delta}) \subset W^{1,2}(M)$ for any $f, g \in \mathrm{TestF}(M)$. Therefore we can define the Hessian and Γ_2 operator as follows.

Let $f \in \mathrm{TestF}(M)$. We define the Hessian $\mathrm{H}_f : \{\nabla g : g \in \mathrm{TestF}(M)\}^2 \mapsto L^0(M)$ by

$$2\mathrm{H}_f(\nabla g, \nabla h) = \Gamma(g, \Gamma(f, h)) + \Gamma(h, \Gamma(f, g)) - \Gamma(f, \Gamma(g, h))$$

for any $g, h \in \mathrm{TestF}(M)$. It can be seen that H_f can be extended to a symmetric $L^\infty(M)$-bilinear map on $L^2(TM)$ and continuous with values in $L^0(M)$.

Let $f, g \in \mathrm{TestF}(M)$. We define the measure valued operator $\mathbf{\Gamma}_2(\cdot, \cdot)$ by

$$\mathbf{\Gamma}_2(f, g) := \frac{1}{2}\mathbf{\Delta}\Gamma(f, g) - \frac{1}{2}\big(\Gamma(f, \Delta g) + \Gamma(g, \Delta f)\big)\,\mathfrak{m},$$

and we put $\mathbf{\Gamma}_2(f) := \mathbf{\Gamma}_2(f, f)$.

Then we can characterize the curvature-dimension condition using non-smooth Bakry-Émery theory. The following proposition is proved in [2] ($N = \infty$) and [8] ($N < \infty$), we rewrite it according to the results in [15] (see Lemma 3.2 and Theorem 4.1 there). We say that a metric measure space $M = (X, \mathrm{d}, \mathfrak{m})$ has Sobolev-to-Lipschitz property if: for any function $f \in W^{1,2}(X)$ such that $|\mathrm{D}f| \in L^\infty$, we can find a Lipschitz continuous function $\bar{f}$ such that $f = \bar{f}$ $\mathfrak{m}$-a.e. and $\mathrm{Lip}(\bar{f}) = \text{ess sup } |\mathrm{D}f|$.

Proposition 2.5 (Bakry-Émery condition, [2], [8], [15]). *Let $M = (X, \mathrm{d}, \mathfrak{m})$ be an infinitesimal Hilbert space satisfying the Sobolev-to-Lipschitz property,* $\mathrm{TestF}(M)$ *is dense in $W^{1,2}(M)$. Then it is a $\mathrm{RCD}^*(K, N)$ space with $K \in \mathbb{R}$ and $N \in [1, \infty]$ if and only if*

$$\mathbf{\Gamma}_2(f) \geq \Big(K|\mathrm{D}f|^2 + \frac{1}{N}(\Delta f)^2\Big)\,\mathfrak{m}$$

for any $f \in \mathrm{TestF}(M)$.

Remark 2.6. In some results in the references, the Sobolev-to-Lipschitz property here is replaced by the following condition:

$$\mathrm{d}(x, y) = \sup\{f(x) - f(y) : f \in W^{1,2}(M) \cap C_b(M), |\mathrm{D}f| \leq 1, \quad \mathfrak{m} - \text{a.e.}\}.$$

It can be seen that we can obtain this property from Sobolev-to-Lipschitz property by considering the functions $\{\mathrm{d}(z, \cdot) : z \in X\}$.

Now we turn to the dimension of M which is understood as the dimension of $L^2(TM)$ as a L^∞-module. Let B be a Borel set. We denote the subset of $L^2(TM)$ consisting of those v such that $\chi_{B^c} v = 0$ by $L^2(TM)_{|_B}$.

Definition 2.7 (Local independence). Let B be a Borel set with positive measure. We say that $\{v_i\}_1^n \subset L^2(TM)$ is independent on B if

$$\sum_i f_i v_i = 0, \quad \mathfrak{m} - \text{a.e. on } B$$

holds if and only if $f_i = 0$ $\mathfrak{m}$-a.e. on B for each i.

Definition 2.8 (Local span and generators)**.** Let B be a Borel set in X and $V := \{v_i\}_{i\in I} \subset L^2(TM)$. The span of V on B, denoted by $\mathrm{Span}_B(V)$, is the subset of $L^2(TM)_{|B}$ with the following property: there exist a Borel decomposition $\{B_n\}_{n\in\mathbb{N}}$ of B and families of vectors $\{v_{i,n}\}_{i=1}^{m_n} \subset L^2(TM)$ and functions $\{f_{i,n}\}_{i=1}^{m_n} \subset L^\infty(M)$, $n = 1, 2, ...$, such that

$$\chi_{B_n} v = \sum_{i=1}^{m_n} f_{i,n} v_{i,n}$$

for each n. We call the closure of $\mathrm{Span}_B(V)$ the space generated by V on B.

We say that $L^2(TM)$ is finitely generated if there is a finite family $v_1, ..., v_n$ spanning $L^2(TM)$ on X, and locally finitely generated if there is a partition $\{E_i\}$ of X such that $L^2(TM)_{|E_i}$ is finitely generated for every $i \in \mathbb{N}$.

Definition 2.9 (Local basis and dimension)**.** We say that a finite set $v_1, ..., v_n$ is a basis on a Borel set B if it is independent on B and $\mathrm{Span}_B\{v_1, ..., v_n\} = L^2(TM)_{|B}$. If $L^2(TM)$ has a basis of cardinality n on B, we say that it has dimension n on B, or that its local dimension on B is n. If $L^2(TM)$ does not admit any local basis of finite cardinality on any subset of B with positive measure, we say that the local dimension of $L^2(TM)$ on B is infinity.

It can be proved (see Proposition 1.4.4 in [10] for example) that the definitions of basis and dimension are well posed. As a consequence of this definition, we can prove the existence of a unique decomposition $\{E_n\}_{n\in\mathbb{N}\cup\{\infty\}}$ of X such that for each E_n with positive measure, $n \in \mathbb{N}\cup\{\infty\}$, $L^2(TM)$ has dimension n on E_n. Furthermore, thanks to the infinitesimal Hilbertianity we have the following proposition.

Proposition 2.10 (Theorem 1.4.11, [10])**.** *Let $(X, \mathrm{d}, \mathfrak{m})$ be a $\mathrm{RCD}(K, \infty)$ metric measure space. Then there exists a unique decomposition $\{E_n\}_{n\in\mathbb{N}\cup\{\infty\}}$ of X such that*

- *For any $n \in \mathbb{N}$ and any $B \subset E_n$ with finite positive measure, $L^2(TM)$ has a unit orthogonal basis $\{e_{i,n}\}_{i=1}^{n}$ on B,*
- *For every subset B of E_∞ with finite positive measure, there exists a unit orthogonal set $\{e_{i,B}\}_{i\in\mathbb{N}\cup\{\infty\}} \subset L^2(TM)_{|B}$ which generates $L^2(TM)_{|B}$,*

where unit orthogonal of a countable set $\{v_i\}_i \subset L^2(TM)$ on B means $\langle v_i, v_j\rangle = \delta_{ij}$ $\mathfrak{m}$-a.e. on B.

Definition 2.11 (Analytic Dimension)**.** Let $\{E_n\}_{n\in\mathbb{N}\cup\{\infty\}}$ be the decomposition given in Proposition 2.10. We define the local dimension $\dim_{\mathrm{loc}} : M \mapsto \mathbb{N}$ by $\dim_{\mathrm{loc}}(x) = n$ on E_n. We say that the dimension of $L^2(TM)$ is k if $k = \sup\{n : \mathfrak{m}(E_n) > 0\}$. We define the analytic dimension of M as the dimension of $L^2(TM)$ and denote it by $\dim_{\max} M$.

We have the following proposition about the analytic dimension of $\mathrm{RCD}^*(K, N)$ spaces.

Proposition 2.12 (See [12]). *Let $M = (X, \mathrm{d}, \mathfrak{m})$ be a* $\mathrm{RCD}^*(K, N)$ *metric measure space. Then* $\dim_{\max} M \le N$. *Furthermore, if the local dimension on a Borel set E is N, we have* $\mathrm{trH}_f(x) = \Delta f(x)$ $\mathfrak{m}$*-a.e.* $x \in E$ *for every* $f \in \mathrm{TestF}$.

Combining the results in Proposition 2.10 and Proposition 2.12, we know there is a canonical coordinate system, i.e. there exists a partition of X: $\{E_n\}_{n \le N}$, such that $\dim_{\mathrm{loc}}(x) = n$ on E_n and $\{e_{i,n}\}_i, n = 1, ..., \lfloor N \rfloor$ are the unit orthogonal basis on corresponding E_n. Then we can do computations on $\mathrm{RCD}^*(K, N)$ spaces in a similar way as on manifolds. For example, the pointwise Hilbert-Schmidt norm $|S|_{\mathrm{HS}}$ of a L^∞-bilinear map $S : [L^2(TM)]^2 \mapsto L^0(M)$ can be defined in the following way. Letting $S_1, S_2 : [L^2(TM)]^2 \mapsto L^0(M)$ be symmetric bilinear maps, we define $\langle S_1, S_2 \rangle_{\mathrm{HS}}$ as a function such that $\langle S_1, S_2 \rangle_{\mathrm{HS}} := \sum_{i,j} S_1(e_{i,n}, e_{j,n}) S_2(e_{i,n}, e_{j,n})$, $\mathfrak{m}$-a.e. on E_n. Clearly, this definition is well posed. In particular, we can define the Hilbert-Schmidt norm of S as $\sqrt{\langle S, S \rangle_{\mathrm{HS}}}$ and denote it by $|S|_{\mathrm{HS}}$, it can be seen that it is compatible with the classical definition of Hilbert-Schmidt norm. The trace of S can be written in the way that $\mathrm{tr} S = \langle S, \mathrm{Id}_{\dim_{\mathrm{loc}}} \rangle_{\mathrm{HS}}$ where $\mathrm{Id}_{\dim_{\mathrm{loc}}}$ is the unique map satisfying $\mathrm{Id}_{\dim_{\mathrm{loc}}}(e_{i,n}, e_{j,n}) = \delta_{ij}$, $\mathfrak{m}$-a.e. on E_n.

3 Conformal transformation

In this section we will study the conformal transformation on metric measure space, some basic definitions and facts are the following. It can be seen that all the objects about the conformal transformation are intrinsically defined.

Let $w \in \mathrm{TestF}(M)$ be a test function on the metric measure space $(X, \mathrm{d}, \mathfrak{m})$, we construct a weighted metric measure space $M' := (X, \mathrm{d}', \mathfrak{m}')$ where

- we replace $\mathfrak{m}$ by the weighted measure with density e^{Nw}:

$$\mathfrak{m}' = e^{Nw}\, \mathfrak{m};$$

- we replace d by $\mathrm{d}' = e^w \mathrm{d}$, i.e. the weighted metric with conformal factor e^w:

$$\mathrm{d}'(x, y) = \inf_\gamma \{ \int_0^1 |\dot{\gamma}_t| e^{w(\gamma_t)}\, \mathrm{d}t : \gamma \in AC([0, 1]; X), \gamma_0 = x, \gamma_1 = y \}$$

Since any test function in a $\mathrm{RCD}(K, \infty)$ space is bounded and Lipschitz continuous, we know the coefficient/weight e^w is also bounded and continuous. Therefore, the topology of the new weighted space (X, d') coincides with the topology of (X, d), (X, d') is still a complete length space, the measure $\mathfrak{m}'$ is a Borel measure with respect to d'. In case M is $\mathrm{RCD}^*(K, N)$, it is locally compact by Bishop-Gromov inequality, then (X, d') is also geodesic. It can be seen that the conformal transformation is reversible, i.e. the space M can be obtained from M' through conformal transformation, in which case the function w should be replaced by $-w$.

In the case of smooth metric measure space, we have the following assertions. The gradient and Laplace operator on M' (which are defined in the same way) are denoted by ∇' and Δ' respectively.

- The Sobolev spaces $W^{1,2}(M)$ and $W^{1,2}(M')$ coincide as sets;
- Let $f \in W^{1,2}(M) = W^{1,2}(M')$, $|\mathrm{d}f|_{M'} = e^{-w}|\mathrm{d}f|_M$, $\Gamma'(f) = e^{-2w}\Gamma(f)$ and $\nabla' f = e^{-2w}\nabla f$;
- Let $X, Y \in TM = TM'$, $\langle X, Y\rangle_{M'} = e^{2w}\langle X, Y\rangle$;
- For any $u \in C^\infty(M) = C^\infty(M')$, we have $\Delta' u = e^{-2w}\big(\Delta u + (N-2)\Gamma(w,u)\big)$.

Now we will prove the non-smooth counterparts of these properties. First of all, we have simple Lemma concerning the identification of Sobolev spaces.

Lemma 3.1 (see [11])**.** *Let $M = (X, \mathrm{d}, \mathfrak{m})$ and $M' = (X, \mathrm{d}', \mathfrak{m}')$ be metric measure spaces where $\mathrm{d} \geq \mathrm{d}'$, d' induces the same topology of d, and $c\mathfrak{m} \leq \mathfrak{m}' \leq C\mathfrak{m}$ for some $c, C > 0$. Then $W^{1,2}(M') \subset W^{1,2}(M)$ and for any function $f \in W^{1,2}(M')$, we have: $|\mathrm{D}f|_M \leq |\mathrm{D}f|_{M'}$ $\mathfrak{m}$-a.e. .*

Lemma 3.2. *Let $M = (X, \mathrm{d}, \mathfrak{m})$ be a metric measure space, M' be the space constructed as above. Then $W^{1,2}(M)$ and $W^{1,2}(M')$ coincide as sets and*

$$|\mathrm{D}f|_{M'} = e^{-w}|\mathrm{D}f|_M, \quad \mathfrak{m} - a.e.$$

for any $f \in W^{1,2}(M)$. In particular, we know M' is infinitesimally Hilbertian.

Proof. Let $\epsilon > 0$ be arbitrary positive number. For $x \in X$, pick $r > 0$ such that

$$\sup_{y \in B_r(x)} \max\left\{\frac{e^{w(x)}}{e^{w(y)}}, \frac{e^{w(y)}}{e^{w(x)}}\right\} < 1 + \epsilon,$$

where $B_r(x)$ is the open ball in (X, d) with radius r

Then for any Lipschitz function g, we have

$$\begin{aligned}
\mathrm{lip}_{M'}(g)(x) &= \varlimsup_{y \to x} \frac{|g(y) - g(x)|}{\mathrm{d}'(y, x)} = \varlimsup_{B_r(x) \ni y \to x} \frac{|g(y) - g(x)|}{\mathrm{d}'(y, x)} \\
&\leq (1+\epsilon)e^{-w(x)} \varlimsup_{B_r(x) \ni y \to x} \frac{|g(y) - g(x)|}{\mathrm{d}(y, x)} \\
&= (1+\epsilon)e^{-w(x)}\mathrm{lip}_M(g)(x).
\end{aligned}$$

Similarly, we have $\mathrm{lip}_{M'}(g)(x) \geq (1+\epsilon)^{-1}e^{-w(x)}\mathrm{lip}_M(g)(x)$. Since the choice of ϵ is arbitrary, we know

$$\mathrm{lip}_{M'}(g)(x) = e^{-w(x)}\mathrm{lip}_M(g)(x).$$

Recall the fact that e^w is also bounded and continuous, by Lemma 3.1 we know that $W^{1,2}(M)$ and $W^{1,2}(M')$ coincide as sets. Let $f \in W^{1,2}(M) = W^{1,2}(M')$, and $\{f_n\}_n$ be the Lipschitz functions as in Proposition 2.1 such that $f_n \to f$ and $\mathrm{lip}_M(f_n) \to |\mathrm{D}f|_M$ in $L^2(X, \mathfrak{m})$. Then we know $f_n \to f$ and $\mathrm{lip}_{M'}(f_n) \to e^{-w}|\mathrm{D}f|_M$ in $L^2(X, \mathfrak{m}')$. Therefore, by the lower semi-continuity we know $|\mathrm{D}f|_{M'} \leq e^{-w}|\mathrm{D}f|_M$, $\mathfrak{m}$-a.e..

Conversely, we can exchange the roles of M and M', i.e. M is the weighted space based on M' with respect to the weight e^{-w}. Hence by the same argument we can prove $|\mathrm{D}f|_{M'} \geq e^{-w}|\mathrm{D}f|_M$, $\mathfrak{m}$-a.e., then we prove this lemma.

□

According to the lemma above, $\Gamma'(\cdot,\cdot) = e^{-2w}\Gamma(\cdot,\cdot)$ and the natural energy form on M' is defined by

$$W^{1,2}(M') \ni f \mapsto \int \Gamma'(f)\,\mathfrak{m}' = \int |\mathrm{D}f|^2\, e^{(N-2)w}\,\mathfrak{m}.$$

It can be checked that the Laplacian on M' can be represented in the following way.

Lemma 3.3. *Let M and M' be metric measure spaces as discussed above. Then* $\mathrm{D}(\mathbf{\Delta}') = \mathrm{D}(\mathbf{\Delta})$ *and* $\mathrm{TestF}(M') = \mathrm{TestF}(M)$. *For any $f \in \mathrm{D}(\mathbf{\Delta}')$, we have*

$$\mathbf{\Delta}' f = e^{(N-2)w}\big(\mathbf{\Delta} f + (N-2)\Gamma(w,f)\,\mathfrak{m}\big).$$

Furthermore, we have the formula:

$$\Delta' f = e^{-2w}\big(\Delta f + (N-2)\Gamma(w,f)\big), \quad \mathfrak{m} - a.e.$$

where $f \in \mathrm{TestF}(M')$.

Proof. Let $f \in \mathrm{D}(\mathbf{\Delta})$. Then there exists a measure $\mathbf{\Delta} f$ such that

$$\int \phi\,\mathbf{\Delta} f = -\int \Gamma(\phi, f)\,\mathfrak{m}$$

for any Lipschitz function ϕ with bounded support.

Thus for any Lipschitz function φ with bounded support, we have

$$\begin{aligned}
& \int \varphi e^{(N-2)w}\big(\mathbf{\Delta} f + (N-2)\Gamma(w,f)\,\mathfrak{m}\big) \\
= & \int e^{(N-2)w}\varphi\mathbf{\Delta} f + \int (N-2)\varphi\Gamma(w,f)\,e^{(N-2)w}\,\mathfrak{m} \\
= & -\int \Gamma(e^{(N-2)w}\varphi, f)\,\mathfrak{m} + \int (N-2)\varphi\Gamma(w,f)\,e^{(N-2)w}\,\mathfrak{m} \\
= & -\int e^{(N-2)w}\Gamma(\varphi,f)\,\mathfrak{m} + \int (2-N)\varphi e^{(N-2)w}\Gamma(w,f)\,\mathfrak{m} + \int (N-2)\varphi\Gamma(w,f)\,e^{(N-2)w}\,\mathfrak{m} \\
= & -\int e^{(N-2)w}\Gamma(\varphi,f)\,\mathfrak{m} \\
= & -\int \Gamma'(\varphi,f)\,\mathfrak{m}'.
\end{aligned}$$

Therefore, we know $f \in \mathrm{D}(\mathbf{\Delta}')$ and by uniqueness we know

$$\mathbf{\Delta}' f = e^{(N-2)w}(\mathbf{\Delta} f + (N-2)\Gamma(w,f)\,\mathfrak{m}), \quad \mathfrak{m} - \text{a.e.}.$$

Conversely, we can prove $\mathrm{D}(\mathbf{\Delta}') \subset \mathrm{D}(\mathbf{\Delta})$. Combining the result of Lemma 3.2 we know $\mathrm{TestF}(M') = \mathrm{TestF}(M)$. In particular, when $\mathbf{\Delta}' f \ll \mathfrak{m}'$, we know $\mathbf{\Delta} f \ll \mathfrak{m}$ and

$$\Delta' f = e^{-2w}(\Delta f + (N-2)\Gamma(w,f)), \quad \mathfrak{m} - \text{a.e.}.$$

$\square$

Lemma 3.4. *Let M and M' be metric measure spaces as discussed above. Then $\nabla' f = e^{-2w}\nabla f$ and $\langle X, Y\rangle_{M'} = e^{2w}\langle X, Y\rangle$, $\mathfrak{m}$-a.e.*

Proof. Let $f, g \in W^{1,2}(M) = W^{1,2}(M')$. Then by definition we know

$$\mathrm{d}f(\nabla' g) = \langle \nabla f, \nabla g\rangle_{M'}$$

and

$$\langle \nabla f, \nabla g\rangle_{M'} = \Gamma'(f, g) = e^{-2w}\Gamma(f, g) = e^{-2w}\langle \nabla f, \nabla g\rangle_M = e^{-2w}\mathrm{d}f(\nabla g).$$

Then we know $\mathrm{d}f(\nabla' g) = e^{-2w}\mathrm{d}f(\nabla g)$, therefore $\nabla' g = e^{-2w}\nabla g$.

Furthermore, we know

$$\langle X, Y\rangle_{M'} = e^{-2w}\langle X, Y\rangle_M,$$

for any $X = \nabla f, Y = \nabla g$, $f, g \in W^{1,2}(M)$. Hence by linearity and the density we $\langle X, Y\rangle_{M'} = e^{-2w}\langle X, Y\rangle_M$ holds for any $X, Y \in L^2(TM)$. □

Lemma 3.5. *Let M be a $\mathrm{RCD}^*(K, N)$ metric measure space where $N < \infty$, and M' be the weighted space defined as before. Then M' satisfies the Sobolev-to-Lipschitz property.*

Proof. Let $f \in W^{1,2}(M')$ with $|\mathrm{D}f|_{M'} \in L^\infty(M')$. By Lemma 3.2 we know $|\mathrm{D}f|_{M'} = e^{-w}|\mathrm{D}f| \in L^\infty(M')$, then $|\mathrm{D}f| \in L^\infty(M)$. Since M has the Sobolev-to-Lipschitz property (see [4]), we may assume that f is Lipschitz on both M and M'. Then it is rest to prove that $\mathrm{Lip}_{M'}(f) = \text{ess sup } |\mathrm{D}f|_{M'} = \text{ess sup } e^{-w}|\mathrm{D}f|$. As the inequality $\mathrm{Lip}_{M'}(f) \geq \text{ess sup } |\mathrm{D}f|_{M'}$ is trivial, we just need to prove the opposite one.

In the proof of Lemma 3.2 we know

$$\mathrm{lip}_{M'}(f)(x) = e^{-w(x)}\mathrm{lip}_M(f)(x).$$

It is known that $\mathrm{RCD}^*(K, N)$ space is doubling (see [16]) and supports a 1-1 weak local Poincaré inequality (see [13, 14]). According to Theorem 6.1 in [7] we know $\mathrm{lip}_M(f)(x) = |\mathrm{D}f|$. Therefore we have

$$\mathrm{lip}_{M'}(f)(x) = e^{-w(x)}|\mathrm{D}f|(x) \leq \text{ess sup } e^{-w}|\mathrm{D}f|.$$

Furthermore, we know $\mathrm{Lip}_{M'}(f) \leq \text{ess sup } |\mathrm{D}f|_{M'}$ and we complete the proof. □

4 Ricci curvature tensor under conformal transformation

In this section we will prove the formula (1.1) in $\mathrm{RCD}^*(K, N)$ case.

First of all, it can be seen (see [12]) that the N-Ricci tensor $\mathbf{Ricci}_N$ can be defined in the following way.

Definition 4.1 (N-Ricci tensor)**.** Let $f \in \mathrm{TestF}(M)$, $\mathbf{Ricci}_N(\nabla f, \nabla f) \in \mathrm{Meas}(M)$ be defined by

$$\mathbf{Ricci}_N(\nabla f, \nabla f) := \mathbf{\Gamma}_2(f) - \big(|\mathrm{H}_f|_{\mathrm{HS}}^2 + \frac{1}{N - \dim_{\mathrm{loc}}}(\mathrm{tr}\mathrm{H}_f - \Delta f)^2\big)\,\mathfrak{m}.$$

We now recall the following result which is proved in [12], Theorem 4.4. Here we modify the statement a bit according to Proposition 2.5.

Theorem 4.2. *Let M be a* $\mathrm{RCD}^*(K,N)$ *space. Then*

$$\mathbf{Ricci}_N(\nabla f, \nabla f) \geq K|\mathrm{D}f|^2\,\mathfrak{m},$$

and

$$\mathbf{\Gamma}_2(f) \quad \geq \quad \Big(\frac{(\Delta f)^2}{N} + \mathbf{Ricci}_N(\nabla f, \nabla f)\Big)\,\mathfrak{m} \tag{4.1}$$

holds for any $f \in \mathrm{TestF}(M)$. Conversely, let M be an infinitesimal Hilbert space satisfying the Sobolev-to-Lipschitz property, $\mathrm{TestF}(M)$ *be dense in $W^{1,2}(M)$. Assume that*

(1) $\dim_{\max} M \leq N$

(2) $\mathrm{tr}\mathrm{H}_f = \Delta f$ $\mathfrak{m} -$ *a.e. on* $\{\dim_{\mathrm{loc}} = N\}, \forall f \in \mathrm{TestF}(M)$

(3) $\mathbf{Ricci}_N \geq K$

for some $K \in \mathbb{R}$, $N \in [1, +\infty]$, then it is $\mathrm{RCD}^*(K,N)$.

According to the definition 4.1, we need to compute the Hilbert-Schmidt norm of the Hessian under conformal transformation. We have the following lemma. Notice that the Hessian has nothing to do with the weighted measure.

Lemma 4.3. *Let $M = (X, \mathrm{d}, \mathfrak{m})$ be a* $\mathrm{RCD}^*(K,N)$ *metric measure space, e^w be the weight where $w \in \mathrm{TestF}(M)$. Then for any $f \in \mathrm{TestF}(M)$, the following formulas*

$$\begin{aligned}|\mathrm{H}'_f|_{\mathrm{HS}}^2 = e^{-4w}\big(|\mathrm{H}_f|_{\mathrm{HS}}^2 \quad &+ \quad 2\Gamma(f)\Gamma(w) + (\dim_{\mathrm{loc}} - 2)\Gamma(f,w)^2 \\ &- \quad 2\Gamma(w, \Gamma(f)) + 2\Gamma(f,w)\mathrm{tr}\mathrm{H}_f\big)\end{aligned}$$

and

$$\mathrm{tr}\mathrm{H}'_f = e^{-2w}\big(\mathrm{tr}\mathrm{H}_f - 2\Gamma(f,w) + \dim_{\mathrm{loc}}\Gamma(f,w)\big)$$

hold $\mathfrak{m}$-a.e. .

Proof. Letting g, h be arbitrary test functions, we know

$$\begin{aligned}
\mathrm{H}'_f(\nabla' g, \nabla' h) \quad &= \quad \frac{1}{2}\big(\Gamma'(g, \Gamma'(f,h)) + \Gamma'(h, \Gamma'(f,g)) - \Gamma'(f, \Gamma'(g,h))\big) \\
&= \quad \frac{e^{-2w}}{2}\big(\Gamma(g, e^{-2w}\Gamma(f,h)) + \Gamma(h, e^{-2w}\Gamma(f,g)) - \Gamma(f, e^{-2w}\Gamma'(g,h))\big) \\
&= \quad \frac{e^{-4w}}{2}\big(\Gamma(g, \Gamma(f,h)) + \Gamma(h, \Gamma(f,g)) - \Gamma(f, \Gamma'(g,h)) \\
&\qquad - 2\Gamma(g,w)\Gamma(f,h) - 2\Gamma(h,w)\Gamma(f,g) + 2\Gamma(f,w)\Gamma(g,h)\big) \\
&= \quad e^{-4w}\big(\mathrm{H}_f(\nabla g, \nabla h) - \Gamma(g,w)\Gamma(f,h) - \Gamma(h,w)\Gamma(f,g) + \Gamma(f,w)\Gamma(g,h)\big) \\
&= \quad e^{-4w}\big(\mathrm{H}_f(\nabla g, \nabla h) - \langle \nabla g, \nabla w\rangle\langle \nabla f, \nabla h\rangle - \langle \nabla h, \nabla w\rangle\langle \nabla f, \nabla g\rangle \\
&\qquad + \langle \nabla f, \nabla w\rangle\langle \nabla g, \nabla h\rangle\big)
\end{aligned}$$

holds $\mathfrak{m}$-a.e..

Then we replace g, h by linear combinations of test function in the equalities above. First of all, we can replace $\nabla' g$ by $\sum_i \nabla' g_i$ and replace $\nabla' h$ by $\sum_j \nabla' h_j$ in $\mathrm{H}'_f(\nabla' g, \nabla' h)$. Then by approximation and the continuity of Hessian as a bilinear map on $L^2(TM)$, we can replace $\nabla' g, \nabla' h$ by e'_i, e'_j where $\{e'_i\}_i$ is a unit orthogonal base on M with respect to $\Gamma'(\cdot,\cdot)$. It can be seen from Lemma 3.4 that ∇g and ∇h should be simultaneously replaced by $e^w e_i$ and $e^w e_j$ where $\{e_i\}_i$ is the corresponding unit orthogonal base with respect to $\Gamma(\cdot,\cdot)$. Hence we obtain

$$(\mathrm{H}'_f)_{ij} = e^{-2w}\big((\mathrm{H}_f)_{ij} - w_i f_j - w_j f_i + \Gamma(f, w)\delta_{ij}\big),$$

$\mathfrak{m}$-a.e., where we keep the notion $(T)_{ij} = T(e_i, e_j)$ for a bilinear map T and $f_i = \langle \nabla f, e_i\rangle$ for a function f. Then we know

$$\begin{aligned}
|\mathrm{H}'_f|^2_{\mathrm{HS}} &= \sum_{i,j} (\mathrm{H}'_f)^2_{ij} \\
&= e^{-4w} \sum_{i,j} \big((\mathrm{H}_f)_{ij} - w_i f_j - w_j f_i + \Gamma(f, w)\delta_{ij}\big)^2 \\
&= e^{-4w}\big(|\mathrm{H}_f|^2_{\mathrm{HS}} + 2\Gamma(f)\Gamma(w) + (\dim_{\mathrm{loc}} - 2)\Gamma(f, w)^2 \\
&\quad - 2\Gamma(w, \Gamma(f)) + 2\Gamma(f, w)\mathrm{tr}\mathrm{H}_f\big)
\end{aligned}$$

holds $\mathfrak{m}$-a.e., which is the thesis.

In the same way, we know

$$\mathrm{tr}\mathrm{H}'_f(x) = \sum_{i=j} (\mathrm{H}'_f)_{ij} = e^{-2w}\big(\mathrm{tr}\mathrm{H}_f - 2\Gamma(f, w) + \dim_{\mathrm{loc}}(x)\Gamma(f, w)\big)$$

for $\mathfrak{m}$-a.e. $x \in X$. □

Theorem 4.4. *Let $M = (X, \mathrm{d}, \mathfrak{m})$ be a $\mathrm{RCD}^*(K, N)$ metric measure space, e^w be the weight where $w \in \mathrm{TestF}(M)$. The corresponding metric measure space under conformal transformation is $M' = (X, \mathrm{d}', \mathfrak{m}')$ where $\mathrm{d}' = e^w\mathrm{d}$ and $\mathfrak{m}' = e^{Nw}\mathfrak{m}$. Then the N-Ricci tensor of M' can be computed in the following way:*

$$\begin{aligned}
\mathbf{Ricci}'_N(\nabla' f, \nabla' f) = e^{-4w}\big(\mathbf{Ricci}_N(\nabla f, \nabla f) &+ [-\Delta w - (N-2)\Gamma(w)]\Gamma(f)\,\mathfrak{m} \\
&- (N-2)[\mathrm{H}_w(\nabla f, \nabla f) - \Gamma(w, f)^2]\,\mathfrak{m}\big).
\end{aligned}$$

Proof. According to the Definition 4.1, we just need to compute $\mathbf{\Gamma}'_2$. Letting $f \in \mathrm{TestF}(M)$, by definition we know

$$\mathbf{\Gamma}'_2(f) = \frac{1}{2}\mathbf{\Delta}'(\Gamma'(f)) - \Gamma'(\Delta' f, f)\,\mathfrak{m}.$$

$$\begin{aligned}
\mathbf{\Delta}'(\Gamma'(f)) &= e^{-2w}\big(\mathbf{\Delta}(e^{-2w}\Gamma(f)) + (N-2)\Gamma(w, e^{-2w}\Gamma(f))\,\mathfrak{m}\big) \\
&= e^{-2w}\big(\mathbf{\Delta}(e^{-2w})\Gamma(f) + 2\Gamma(e^{-2w}, \Gamma(f))\,\mathfrak{m} + e^{-2w}\mathbf{\Delta}(\Gamma(f)) \\
&\quad + (N-2)e^{-2w}\Gamma(w, \Gamma(f))\,\mathfrak{m} - 2(N-2)e^{-2w}\Gamma(f)\Gamma(w)\,\mathfrak{m}\big) \\
&= e^{-4w}\big(4\Gamma(f)\Gamma(w) - 2\Delta w\Gamma(f) - 4\Gamma(w, \Gamma(f)) \\
&\quad + (N-2)\Gamma(w, \Gamma(f)) - 2(N-2)\Gamma(f)\Gamma(w)\big)\,\mathfrak{m} + e^{-4w}\mathbf{\Delta}(\Gamma(f)),
\end{aligned}$$

and

$$\begin{aligned}
\Gamma'(\Delta' f, f) &= e^{-2w}\big(\Gamma(f, e^{-2w}(\Delta f + (N-2)\Gamma(w,f)))\big)\\
&= e^{-4w}\big(\Gamma(f, (\Delta f + (N-2)\Gamma(w,f)) - 2\Gamma(w,f)(\Delta f + (N-2)\Gamma(w,f))\big)\\
&= e^{-4w}\big(\Gamma(f,\Delta f) - 2\Gamma(w,f)\Delta f - 2(N-2)\Gamma(w,f)^2\\
&\qquad + (N-2)\Gamma(f,\Gamma(f,w))\big).
\end{aligned}$$

Therefore, we know

$$\begin{aligned}
\mathbf{\Gamma}_2'(f) &= e^{-4w}\big(\mathbf{\Gamma}_2(f)\big) + e^{-4w}\Big((4-N)\Gamma(f)\Gamma(w) - \Delta w\Gamma(f) + \frac{N-6}{2}\Gamma(w,\Gamma(f))\\
&\quad - (N-2)\Gamma(f,\Gamma(f,w)) + 2\Gamma(f,w)\Delta f + 2(N-2)\Gamma(w,f)^2\Big)\,\mathfrak{m}. \quad (4.2)
\end{aligned}$$

By definition, Lemma 4.3 and the formula (4.2) above we have

$$\begin{aligned}
\mathbf{Ricci}_N'(\nabla' f, \nabla' f) :&= \mathbf{\Gamma}_2'(f) - |\mathrm{H}_f'|_{\mathrm{HS}}^2 - \frac{1}{N-\dim_{\mathrm{loc}}}(\mathrm{trH}_f' - \Delta' f)^2\big)\,\mathfrak{m}\\
&= e^{-4w}\big(\mathbf{\Gamma}_2(f)\big) + e^{-4w}\Big((4-N)\Gamma(f)\Gamma(w) - \Delta w\Gamma(f) + \frac{N-6}{2}\Gamma(w,\Gamma(f))\\
&\quad - (N-2)\Gamma(f,\Gamma(f,w)) + 2\Gamma(f,w)\Delta f + 2(N-2)\Gamma(w,f)^2\Big)\,\mathfrak{m}\\
&\quad -e^{-4w}\big(|\mathrm{H}_f|_{\mathrm{HS}}^2 + 2\Gamma(f)\Gamma(w) + (\dim_{\mathrm{loc}}-2)\Gamma(f,w)^2\\
&\quad - 2\Gamma(w,\Gamma(f)) + 2\Gamma(f,w)\mathrm{trH}_f\big)\,\mathfrak{m}\\
&\quad -\frac{e^{-4w}}{N-\dim_{\mathrm{loc}}}\big(\Delta f - \mathrm{trH}_f + (N-\dim_{\mathrm{loc}})\Gamma(f,w)\big)^2\,\mathfrak{m}\\
&= e^{-4w}\Big(\mathbf{\Gamma}_2(f) - |\mathrm{H}_f|_{\mathrm{HS}}^2\,\mathfrak{m} - \frac{1}{N-\dim_{\mathrm{loc}}}(\Delta f - \mathrm{trH}_f)^2\,\mathfrak{m}\Big)\\
&\quad +e^{-4w}\Big((N-2)\big(\Gamma(w,f)^2 - \mathrm{H}_w(\nabla f,\nabla f)\big) - \Gamma(f)\big((N-2)\Gamma(w) + \Delta w\big)\Big)\,\mathfrak{m}\\
&= e^{-4w}\big(\mathbf{Ricci}_N(\nabla f,\nabla f) + [-\Delta w - (N-2)\Gamma(w)]\Gamma(f)\,\mathfrak{m}\\
&\quad - (N-2)[\mathrm{H}_f(\nabla f,\nabla f) - \Gamma(w,f)^2]\,\mathfrak{m}\big),
\end{aligned}$$

which is the result we need. □

At the end, we introduce a direct application of this main theorem which offers us a precise estimate of the lower Ricci curvature bound.

Corollary 4.5. *Let M be a* $\mathrm{RCD}^*(K,N)$ *space, M' be the conformal transformed spaces. Then M' satisfies the* $\mathrm{RCD}^*(K',N)$ *condition in case*

$$K' := \inf_{x\in X} e^{-4w}\Big[K - \Delta w + (N-2)\Gamma(w) - \sup_{f\in \mathrm{TestF}(M)} \frac{N-2}{\Gamma(f)}\big(\mathrm{H}_w(\nabla f,\nabla f) - \Gamma(w,f)^2\big)\Big]$$

is a real number.

Proof. We know M' is infinitesimally Hilbertian from Lemma 3.2, M' has Sobolev-to-Lipschitz property from Lemma 3.5 and $\mathrm{TestF}(M')$ is dense in $W^{1,2}(M')$ from Lemma 3.3. It is sufficient to check the conditions (1),(2) in the Theorem 4.2.

(1) By definition and Lemma 3.2 we know that the conformal transformation will not change the local/analytic dimension. Hence by Proposition 2.12 we know $\dim_{\max} M' \leq N$.

(2) Let $f \in \mathrm{TestF}(M) = \mathrm{TestF}(M')$. It is proved in Lemma 3.3 and Lemma 4.3 that

$$\mathrm{trH}'_f - \Delta' f = e^{-2w}\big(\mathrm{trH}_f - \Delta f + (\dim_{\mathrm{loc}} - N)\Gamma(f, w)\big).$$

On the set $\{\dim_{\mathrm{loc}} M = N\} = \{\dim_{\mathrm{loc}} M' = N\}$, we know $\mathrm{trH}_f = \Delta f$ by Proposition 2.12. Therefore $\mathrm{trH}'_f = \Delta' f$ $\mathfrak{m}$-a.e. on $\{\dim_{\mathrm{loc}} M' = N\}$.

□

References

[1] L. Ambrosio, N. Gigli, A. Mondino, and T. Rajala, *Riemannian Ricci curvature lower bounds in metric measure spaces with $\sum$-finite measure.* Trans. Amer. Math. Soc., arXiv:1207.4924, 2012.

[2] L. Ambrosio, N. Gigli, and G. Savaré, *Bakry-Émery curvature-dimension condition and Riemannian Ricci curvature bounds.* Annals of Prob., (2014)

[3] ——, *Calculus and heat flow in metric measure spaces and applications to spaces with Ricci bounds from below*, Invent. Math., 195 (2014), pp. 289–391.

[4] ——, *Metric measure spaces with Riemannian Ricci curvature bounded from below*, Duke Math. J., 163 (2014), pp. 1405–1490.

[5] K. Bacher, K. -T. Sturm, *Localization and tensorization properties of the curvature-dimension condition for metric measure spaces*, J. Funct. Anal., 259 (2010), pp. 28-56.

[6] A.L. Besse, *Einstein manifolds.* Springer 1987.

[7] J. Cheeger, *Differentiability of Lipschitz functions on metric measure spaces*, Geom. Funct. Anal., 9 (1999), pp. 428–517.

[8] M. Erbar, K. Kuwada, and K. -T. Sturm, *On the equivalence of the entropic curvature-dimension condition and Bochners inequality on metric measure spaces.* Preprint, arXiv:1303.4382, 2013.

[9] N. Gigli, *On the differential structure of metric measure spaces and applications*, Memoirs of the AMS, (2014).

[10] ——, *Non-smooth differential geometry.* Preprint, arXiv:1407.0809, 2014.

[11] N. Gigli, B. Han, *Sobolev Space on Warped Product.* Preprint.

[12] B. Han, *Ricci tensor on* $\mathrm{RCD}^*(K, N)$ *spaces.* Preprint, arXiv: 1412.0441, 2015.

[13] T. Rajala, *Improved geodesics for the reduced curvature-dimension condition in branching metric spaces*, Disc. Cont. Dyn. Sist., 33 (2013), pp. 3043-3056.

[14] T. Rajala, *Interpolated measures with bounded density in metric spaces satisfying the curvature-dimension conditions of Sturm*, J. Funct. Anal., 263 (2012), no. 4, pp. 896–924.

[15] G. Savaré, *Self-improvement of the Bakry-Émery condition and Wasserstein contraction of the heat flow in* $\mathrm{RCD}(K,\infty)$ *metric measure spaces*, Disc. Cont. Dyn. Sist., 34 (2014), pp. 1641–1661.

[16] K. -T. Sturm, *On the geometry of metric measure spaces*, I, II Acta Math., 196 (2006), pp. 65–137.

[17] ——, *Ricci tensor for diffusion operators and curvature-dimension inequalities under conformal transformations and time changes*. Preprint, arXiv:1401.0687, 2014.

[18] J. Lott and C. Villani, *Ricci curvature for metric-measure spaces via optimal transport*, Ann. of Math. (2), 169 (2009), pp. 903–991.