INTRINSIC REGULAR GRAPHS IN HEISENBERG GROUPS VS. WEAK SOLUTIONS OF NON LINEAR FIRST-ORDER PDEs

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Abstract. We continue to study $\mathbb{H}$- regular graphs, a class of intrinsic regular hypersurfaces in the Heisenberg group $\mathbb{H}^n = \mathbb{C}^n \times \mathbb{R} \equiv \mathbb{R}^{2n+1}$ endowed with a left- invariant metric d_∞ equivalent to its Carnot- Carathéodory metric. Here we investigate their relationships with suitable weak solutions of nonlinear first- order PDEs. As a consequence this implies some of their geometric properties: a uniqueness result for $\mathbb{H}$- regular graphs of prescribed horizontal normal as well as their (Euclidean) regularity as long as there is regularity on the horizontal normal.

1. Introduction and statement of the main results

A fundamental problem of geometric analysis is the investigation of the interplay between a surface of a given manifold and its normal. Typically this investigation consists of the study of suitable PDEs once a system of coordinates for the surface has been fixed. Following this strategy, the present paper deals with relationships between weak solutions of nonlinear first order PDEs and $\mathbb{H}$- regular intrinsic graphs. The $\mathbb{H}$- regular intrinsic graphs are a class of *intrinsic regular* hypersurfaces in the setting of the Heisenberg group $\mathbb{H}^n = \mathbb{C}^n \times \mathbb{R} \equiv \mathbb{R}^{2n+1}$, endowed with a left- invariant metric d_∞ equivalent to its Carnot-Carathéodory (CC) metric.

Given an intrinsic graph $S = \Phi(\omega) \subset \mathbb{H}^n$ (see Definition 2.6 and (1.8)) where $\phi : \omega \subset \mathbb{R}^{2n} \to \mathbb{R}$, we will study the relationships between S and ϕ so that S is an $\mathbb{H}$- regular surface (see Definition 2.5) and ϕ is a suitable solution of the system

$$\nabla^\phi \phi = w \quad \text{in}\, \omega\,, \tag{1.1}$$

being ∇^ϕ the family of the first order differential operators defined in (1.12) and (1.13), $w \in C^0(\omega; \mathbb{R}^{2n-1})$ prescribed. In the first Heisenberg group $\mathbb{H}^1$ (1.1) reduces to the classical Burgers' equation. The system (1.1) geometrically is a prescribed normal vector field PDE for the intrinsic graph S. In [1] $\nabla^\phi \phi$ has been recognized as intrinsic gradient of ϕ in a suitable differential structure as we will define later. The notion of intrinsic graph has been introduced in [18] in the setting of a Carnot group and deeply studied in the setting of $\mathbb{H}^n$ in [1], although it was already implicitly used in [15].

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The intrinsic graphs in Carnot groups had two main applications so far. The former has been in the theory of rectifiability in Carnot groups. In fact, in [17] classical De Giorgi's rectifiability and divergence theorem for sets of finite perimeter have been fully extended to a Carnot group of step 2 (see also [23]). The latter has been in the framework of minimal surfaces in $\mathbb{H}^n$ (see [24],[10], [3],[11], [6] and [7]).

We shall denote the points of $\mathbb{H}^n$ by $P = [z,t] = [x+iy,t]$, $z \in \mathbb{C}^n$, $x,y \in \mathbb{R}^n$, $t \in \mathbb{R}$, and also by $P = (x_1,\dots,x_n,y_1,\dots,y_n,t) = (x_1,\dots,x_n,x_{n+1},\dots,x_{2n},t)$. If $P = [z,t]$, $Q = [\zeta,\tau] \in \mathbb{H}^n$ and $r > 0$, following the notations of [8], where the reader can find an exhaustive introduction to the Heisenberg group, we define the group operation

$$P \cdot Q := \left[z+\zeta, t+\tau - \frac{1}{2}\Im m(z \cdot \bar{\zeta}) \right] \tag{1.2}$$

and the family of non isotropic dilations

$$\delta_r(P) := [rz, r^2 t], \text{ for } r > 0. \tag{1.3}$$

We denote as $P^{-1} := [-z,-t]$ the inverse of P and as 0 the origin of $\mathbb{R}^{2n+1}$.

Moreover $\mathbb{H}^n$ can be endowed with the homogeneous norm

$$\|P\|_\infty := \max\{|z|, |t|^{1/2}\} \tag{1.4}$$

and the distance d_∞ we shall deal with is defined as

$$d_\infty(P,Q) := \|P^{-1} \cdot Q\|_\infty. \tag{1.5}$$

It is well-known that $\mathbb{H}^n$ is a Lie group of topological dimension $2n+1$, whereas the Hausdorff dimension of $(\mathbb{H}^n, d_\infty)$ is $Q := 2n+2$ (see Proposition 2.1).

$\mathbb{H}^n$ is a Carnot group of step 2. Indeed, its Lie algebra $\mathfrak{h}_n$ is (linearly) generated by

$$X_j = \frac{\partial}{\partial x_j} - \frac{y_j}{2}\frac{\partial}{\partial t}, \qquad Y_j = \frac{\partial}{\partial y_j} + \frac{x_j}{2}\frac{\partial}{\partial t}, \qquad \text{for } j = 1,\dots,n; \qquad T = \frac{\partial}{\partial t}, \tag{1.6}$$

and the only non-trivial commutator relations are $[X_j, Y_j] = T$, for $j = 1,\dots,n$. We shall identify vector fields and associated first order differential operators; thus the vector fields $X_1,\dots,X_n$, $Y_1,\dots,Y_n$ generate a vector bundle on $\mathbb{H}^n$, the so called *horizontal* vector bundle $\mathrm{H}\mathbb{H}^n$ according to the notation of Gromov (see [19]), that is a vector subbundle of $\mathrm{T}\mathbb{H}^n$, the tangent vector bundle of $\mathbb{H}^n$.

The two key points we want to stress now are the notions of intrinsic regular hypersurface and graph in $\mathbb{H}^n$. A general and more complete discussion of these topics in Carnot groups can be found in [18].

Let us recall that in the Euclidean setting $\mathbb{R}^n$, a C^1-hypersurface can be equivalently viewed as the (local) set of zeros of a function $f : \mathbb{R}^n \to \mathbb{R}$ with non-vanishing gradient. Such a notion was easily transposed in [15] to the Heisenberg group, since an intrinsic notion of $C^1_{\mathbb{H}}$-functions has been introduced by Folland and Stein (see [14]): we can state that a continuous real function f on $\mathbb{H}^n$ belongs to $C^1_{\mathbb{H}}(\mathbb{H}^n)$ if $\nabla_{\mathbb{H}} f$ (in the sense of distributions) is a continuous vector-valued function. We shall say that $S \subset \mathbb{H}^n$ is an $\mathbb{H}$- *regular surface* if it is locally defined as the set of points $P \in \mathbb{H}$ such that $f(P) = 0$, provided that $\nabla_{\mathbb{H}} f \neq 0$ on S (see Definition 2.5). Due to the fact it is not restrictive, we will deal in the following with $\mathbb{H}$- regular surfaces S which are locally zero level sets of function $f \in C^1_{\mathbb{H}}$ with $X_1 f \neq 0$. A few comments are now in order to point out similar geometric properties (in the measure theoretical sense) of the $\mathbb{H}$- regular surfaces and classical (Euclidean) regular surfaces and to also mention some of their applications.

First of all, we point out that the class of $\mathbb{H}$- regular surfaces is deeply different from the class of Euclidean regular surfaces, in the sense that there are $\mathbb{H}$- regular surfaces in $\mathbb{H}^1 \equiv \mathbb{R}^3$ that are (Euclidean) fractal sets (see [20]), and conversely there are differentiable 2-submanifolds in $\mathbb{R}^3$ that are not $\mathbb{H}$- regular hypersurfaces (see [15], Remark 6.2). We notice that Euclidean differentiable $2n$-manifolds are $\mathbb{H}$- regular surfaces provided they do not contain characteristic points, i.e. points P such that the Euclidean tangent space at P coincides with the horizontal fiber $\mathrm{H}\mathbb{H}^n_P$ at P. According to Frobenius' theorem, for a general smooth manifold, the set of characteristic points has empty interior; in fact there are few characteristic points (see, for instance, [8], sections 4.5 and 4.6).

The important point supporting the choice of the notion is the fact that this definition yields an Implicit Function Theorem, proved in [15] for the Heisenberg group and in [16] for a general Carnot group (see also [9] for an extension to a CC metric space), so that a $\mathbb{H}$- regular surface locally is a X_1-*graph*, namely (see Definition 2.6) there is a continuous parameterization of S

$$\Phi : \omega \subset (\mathbb{V}_1, |\cdot|) \to (S, d_\infty) \tag{1.7}$$

$$\Phi(A) := A \cdot (\phi(A) e_1) \tag{1.8}$$

where $\phi : \omega \to \mathbb{R}$ is continuous, $\mathbb{V}_1 := \{(x,y,t) \in \mathbb{H}^n : x_1 = 0\}$, $\omega \subset \mathbb{V}_1$, $\{e_j : j = 1, \dots, 2n+1\}$ and $|\cdot|$ denote respectively the standard basis in $\mathbb{R}^{2n+1} \equiv \mathbb{H}^n$ and the Euclidean distance in $\mathbb{V}_1 \equiv \mathbb{R}^{2n}$. In particular every smooth hypersurface is locally an intrinsic graph outside its characteristic points. In general, such a parameterization is not continuously differentiable or even Lipschitz continuous. Indeed, its best Hölder continuous regularity turns out to be of order 1/2 with respect to the distances given in (1.7) ([20]).

A natural question arising is the characterization of the functions $\phi : \omega \to \mathbb{R}$ such that $S = \Phi(\omega)$ is $\mathbb{H}$- regular. A characterization has been proposed in [1]. More precisely there is a natural identification between $\mathbb{V}_1$ and $\mathbb{R}^{2n}$ given by the diffeomorphism

$$\iota : \mathbb{R}^{2n} \longrightarrow \mathbb{V}_1 \subset \mathbb{H}^n \tag{1.9}$$

defined as

$$\iota(\eta, \tau) = (0, \eta, \tau), \tag{1.10}$$

when $n = 1$; while when $n \geq 2$ and $(\eta, v, \tau) \in \mathbb{R}^{2n} \equiv \mathbb{R}_\eta \times \mathbb{R}^{2n-2}_v \times \mathbb{R}_\tau$, ι is defined as

$$\iota((\eta, v, \tau)) = (0, v_2, \dots, v_n, \eta, v_{n+2}, \dots, v_{2n}, \tau), \tag{1.11}$$

where $v = (v_2, \dots, v_n, v_{n+2}, \dots, v_{2n})$. The tangent space of $\mathbb{V}_1$ is linearly generated by the vector fields which are the restrictions of $X_2, \dots, X_n, Y_1, \dots, Y_n, T$ to $\mathbb{V}_1$, and so we can define the vector fields $\widetilde{X}_2 \dots, \widetilde{X}_n, \widetilde{Y}_1, \dots, \widetilde{Y}_n$ and $\widetilde{T}$ on $\mathbb{R}^{2n}$ given by $\widetilde{X}_j := (\iota^{-1})_* X_j$ and $\widetilde{Y}_j := (\iota^{-1})_* Y_j, \widetilde{T} := (\iota^{-1})_* T$, where $(\iota^{-1})_*$ is the usual push- forward of vector fields after the diffeomorphism ι^{-1}. For $n+1 \leq j \leq 2n$ we will also use the notation $\widetilde{X}_j := \widetilde{Y}_{j-n}$.

Let $\phi : \omega \to \mathbb{R}$ be a given continuous function; we will denote by $\nabla^\phi := (\nabla_2^\phi, \dots, \nabla_{2n}^\phi)$ the family of $(2n-1)$ first-order differential operators defined by

$$\nabla_j^\phi := \begin{cases} \widetilde{X}_j = \dfrac{\partial}{\partial v_j} - \dfrac{v_{j+n}}{2}\dfrac{\partial}{\partial \tau} & \text{if } 2 \le j \le n \\ \widetilde{Y}_1 + \phi \widetilde{T} = \dfrac{\partial}{\partial \eta} + \phi \dfrac{\partial}{\partial \tau} & \text{if } j = n+1 \\ \widetilde{Y}_{j-n} = \dfrac{\partial}{\partial v_j} + \dfrac{v_{j-n}}{2}\dfrac{\partial}{\partial \tau} & \text{if } n+2 \le j \le 2n \end{cases}, \tag{1.12}$$

when $n \ge 2$ while, when $n = 1$, we put

$$\nabla^\phi = \nabla_2^\phi := \widetilde{Y}_1 + \phi \widetilde{T} = \frac{\partial}{\partial \eta} + \phi \frac{\partial}{\partial \tau}\,. \tag{1.13}$$

We also put $\nabla_{n+1}^\phi = W^\phi$. The (nonlinear) differential operator

$$C^1(\omega) \ni \phi \to \mathfrak{B}\phi := W^\phi \phi \tag{1.14}$$

is a Burgers' type operator which can be represented in distributional form as

$$\mathfrak{B}\phi = \frac{\partial \phi}{\partial \eta} + \frac{1}{2}\frac{\partial \phi^2}{\partial \tau}\,. \tag{1.15}$$

In [1] it has been proved that each $\mathbb{H}$- regular graph $\Phi(\omega)$ admits an *intrinsic gradient* $\nabla^\phi \phi \in C^0(\omega; \mathbb{R}^{2n})$, in the sense of distributions, which shares a lot of properties with the Euclidean gradient.

Let us recall that the same problem was studied in [9] in the general setting of a CC space. A study similar to the one in [1] has been recently carried out in [2] for $\mathbb{H}$- regular intrinsic graphs in $\mathbb{H}^n$ with arbitrary codimension.

Now we are ready to state the main results of this article. In section 3 we establish the relationships between $\mathbb{H}$- regular graphs and the notion of *broad* solution* for (1.1).

Let $n \ge 2$ and $A_0 = (\eta_0, v_0, \tau_0) \in \mathbb{R}^{2n} = \mathbb{R}_\eta \times \mathbb{R}_v^{2n-2} \times \mathbb{R}_\tau$, let us define

$$I_r(A_0) := \{(\eta, v, \tau) \in \mathbb{R}^{2n} : |\eta - \eta_0| < r,\, |v - v_0| < r,\, |\tau - \tau_0| < r\} =$$

$$= (\eta_0 - r, \eta_0 + r) \times B(v_0, r) \times (\tau_0 - r, \tau_0 + r)$$

where $B(v_0, r)$ denotes the Euclidean open ball in $\mathbb{R}^{2n-2}$ centered at v_0, with radius $r > 0$. When $n = 1$ and $A_0 = (\eta_0, \tau_0) \in \mathbb{R}^2 = \mathbb{R}_\eta \times \mathbb{R}_\tau$

$$I_r(A_0) := \{(\eta, \tau) \in \mathbb{R}^2 : |\eta - \eta_0| < r,\, |\tau - \tau_0| < r\} = (\eta_0 - r, \eta_0 + r) \times (\tau_0 - r, \tau_0 + r).$$

Definition 1.1. *Let $\omega \subset \mathbb{R}^{2n}$ be an open set and let $\phi : \omega \to \mathbb{R}$ and $w = (w_2, ..., w_{2n}) : \omega \to \mathbb{R}^{2n-1}$ be continuous functions. We say that ϕ is a broad* solution of the system* (1.1) *if, for every $A \in \omega$, $\forall\, j = 2, ..., 2n$, there exists a map, we will call exponential map,*

$$\exp_A(\cdot \nabla_j^\phi)(\cdot) : [-\delta_2, \delta_2] \times \overline{I_{\delta_2}(A)} \to \overline{I_{\delta_1}(A)} \Subset \omega\,, \tag{1.16}$$

where $0 < \delta_2 < \delta_1$, such that, if $\gamma_j^B(s) = \exp_A(s\nabla_j^\phi)(B)$,

(E.1): $\gamma_j^B \in C^1([-\delta_2, \delta_2])$

(E.2): $\begin{cases} \dot\gamma_j^B = \nabla_j^\phi \circ \gamma_j^B \\ \gamma_j^B(0) = B \end{cases}$

(E.3): $\phi\left(\gamma_j^B(s)\right) - \phi\left(\gamma_j^B(0)\right) = \int_0^s w_j\left(\gamma_j^B(\sigma)\right)\, d\sigma \quad \forall s \in [-\delta_2, \delta_2]$

$\forall\, B \in\, I_{\delta_2}(A)$, $\forall\, j = 2, ..., 2n$.

The notion of broad* solution extends, when $n = 1$, the classical notion of broad solution for Burgers' equation, provided that ϕ and w are locally Lipschitz continuous (see [5]). In our case ϕ and w are supposed to be only continuous, then the classical theory of solutions for ODEs collapses and the notion of broad solution does not apply (see [12] for an interesting account of this subject and its recent developments). Let us explicitly stress that both the uniqueness and the global continuity of the exponential maps (1.16) are not guaranteed (see, for instance, [25], Remark 4.34).

In our context the notion of broad* solution has to be understood as a notion of C^1-differentiability with respect to the vector fields ∇^ϕ. In fact, we prove that the notion of $\mathbb{H}$-regular intrinsic graph and the one of broad* solution of the system (1.1) are equivalent.

Theorem 1.2. *Let $\omega \subset \mathbb{R}^{2n}$ be an open set and let $\phi : \omega \to \mathbb{R}$ and $w = (w_2, ..., w_{2n}) : \omega \to \mathbb{R}^{2n-1}$ be continuous functions. Then the following conditions are equivalent:*

i:

$$\phi \text{ is a broad* solution of the system } \nabla^\phi \phi = w \text{ in } \omega\,; \tag{1.17}$$

ii: *$S = \Phi(\omega)$ is $\mathbb{H}$- regular and $\nu_S^{(1)}(P) < 0$ for all $P \in S$, where $\nu_S(P) = \left(\nu_S^{(1)}(P), ..., \nu_S^{(2n)}(P)\right)$ denotes the horizontal normal to S at a point $P \in S$. Moreover*

$$\nu_S(P) = \left(-\frac{1}{\sqrt{1+|\nabla^\phi\phi|^2}}, \frac{\nabla^\phi\phi}{\sqrt{1+|\nabla^\phi\phi|^2}}\right)\left(\Phi^{-1}(P)\right)$$

$\forall\, P \in S$ where $\nabla^\phi\phi$ denotes the intrinsic gradient of ϕ.

Let us explicitly point out that the characterization of $\mathbb{H}$-regular intrinsic graphs in Theorem 1.2 is not contained in [1] (see Theorem 2.7). Indeed, those results yield the conclusion of Theorem 1.2 provided the additional assumption that ϕ is little Hölder continuous of order 1/2 (see Lemma 3.1). Here the key step to the proof of Theorem 1.2 will be to achieve 1/2-little Hölder continuity when ϕ is supposed to be only a (continuous) broad* solution of the system (1.1) (see Theorem 3.2). Theorem 1.2 also yields that each Lipschitz continuous solution ϕ of the system (1.1), with continuous datum w, induces a $\mathbb{H}$- regular graph (see Corollary 3.5). Moreover a broad* solution of (1.1) turns out to be a distributional solution (see Corollary 3.6).

A local uniqueness result for broad* solutions of (1.1) is also proved (see Theorem 3.8).

In the section 4 we will study the Euclidean regularity of the $\mathbb{H}$- regular graph $S = \Phi(\omega)$, through the regularity of its intrinsic gradient $\nabla^\phi\phi$. With $Lip(\omega)$ and $Lip_{loc}(\omega)$ we denote, respectively, the set of Lipschitz and locally Lipschitz continuous functions in ω.

Theorem 1.3. *Let $\omega \subset \mathbb{R}^{2n}$ be an open set, let $\Phi(\omega)$ be $\mathbb{H}$- regular in $\mathbb{H}^n$. Assume that $W^\phi\phi \in Lip_{loc}(\omega)$. Then $\phi \in Lip_{loc}(\omega)$.*

Let us point out that Theorem 1.3 is optimal. Indeed, Example 2.8 in [3] assures a function $\phi : \omega := (-1,1)\times\mathbb{R}\to\mathbb{R}$, $\phi(\eta,\tau) := \dfrac{\tau}{\eta+\frac{\tau}{|\tau|}}$, which induces a $\mathbb{H}$- regular graph $\Phi(\omega)\subset\mathbb{H}^1$, and its intrinsic gradient $\nabla^{\phi}\phi\equiv 0$ in ω. Moreover $\phi\in Lip_{loc}(\omega)\setminus C^1(\omega)$.

Weakening the assumption $W^{\phi}\phi\in Lip(\omega)$ with $W^{\phi}\phi\in C^{0,\alpha}(\omega)$, Theorem 1.3 falls down. For instance, we can construct a function $\phi\in C^{0,\alpha}(\omega)$, for each $\alpha\in\left(\frac{1}{2},1\right)$, such that $\Phi(\omega)$ is $\mathbb{H}$- regular in $\mathbb{H}^1$ and $W^{\phi}\phi\in C^{0,2\alpha-1}(\omega)$ (see [1], Corollary 5.11).

Eventually a regularizing effect is stressed when $n\geq 2$ (see also Theorem 4.3, Corollary 4.4 and Remark 4.5).

Theorem 1.4. *Let $n\geq 2$, $\omega\subseteq\mathbb{R}^{2n}$ be an open set and let $\phi\in Lip(\omega)$ and $w=(w_2,\dots,w_{2n})\in Lip(\omega;\mathbb{R}^{2n-1})$ be such that $\nabla^{\phi}\phi=w$ a.e. in ω. Then $\phi\in C^1(\omega)$.*

Corollary 1.5. *Let $n\geq 2$, $\omega\subset\mathbb{R}^n$ and let $\Phi(\omega)$ be $\mathbb{H}$- regular.*

- **i:** *Suppose that $\nabla^{\phi}\phi\in Lip_{loc}(\omega;\mathbb{R}^{2n-1})$, then $\phi\in C^1(\omega)$.*
- **ii:** *Suppose that $\nabla^{\phi}\phi\in C^k(\omega;\mathbb{R}^{2n-1})$, then $\phi\in C^k(\omega)$.*

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2. Notations and preliminary results

We will denote by $\tau_P:\mathbb{H}^n\to\mathbb{H}^n$ the group of left- translations defined as $Q\mapsto\tau_P(Q):=P\cdot Q$ for any fixed $P\in\mathbb{H}^n$.

Proposition 2.1 ([15])**.** *The function d_∞ defined by (1.5) is a distance in $\mathbb{H}^n$ and for any bounded subset Ω of $\mathbb{H}^n$ there exist positive constants $c_1(\Omega)$, $c_2(\Omega)$ such that*

$$c_1(\Omega)|P-Q|_{\mathbb{R}^{2n+1}}\leq d_\infty(P,Q)\leq c_2(\Omega)|P-Q|^{1/2}_{\mathbb{R}^{2n+1}} \tag{2.1}$$

for $P,Q\in\Omega$. In particular, the topologies induced by d_∞ and by the Euclidean distance coincide on $\mathbb{H}^n$. Finally the distance d_∞ is equivalent to the Carnot- Carathéory distance d_C associated with the horizontal bundle $H\mathbb{H}^n$.

From now on, $U(P,r)$ will be the open ball with center P and radius r with respect to the distance d_∞.

If Ω is an open subset of $\mathbb{H}^n$ and $f\in C^1(\Omega)$, we will define as *horizontal gradient* of f the vector $\nabla_{\mathbb{H}}f:=(X_1f,\dots,X_nf,Y_1f,\dots,Y_nf)$. It is well-know that $\nabla_{\mathbb{H}}$ acts as a gradient operator in $\mathbb{H}^n$.

Lemma 2.2 ([14], theorem 1.41)**.** *Let $\Omega\subseteq\mathbb{H}^n$ be a connected open set and let $f\in L^1_{loc}(\Omega)$ be such that $\nabla_{\mathbb{H}}f=0$ in the sense of distributions. Then $f\equiv cost$ in Ω.*

Let $Lip_{\mathbb{H}}(\Omega)$ denote the set of functions $f:\Omega\to\mathbb{R}$ such that there exists $L>0$ for which

$$|f(P)-f(Q)|\leq L\,d_\infty(P,Q)\quad\forall P,\,Q\in\Omega\,. \tag{2.2}$$

Remark 2.3. Because of (2.1), $Lip_{\mathbb{H}}(\Omega)\subset C^0(\Omega)$.

The following characterization of $Lip_{\mathbb{H}}(\mathbb{H}^n)$ holds (see, for instance, [8], section 6.2).

Theorem 2.4. *The following are equivalent:*

i: $f \in Lip_{\mathbb{H}}(\mathbb{H}^n)$;

ii: $f \in L^\infty_{loc}(\mathbb{H}^n)$ *and there exists* $\nabla_{\mathbb{H}} f \in (L^\infty(\mathbb{H}^n))^{2n}$ *in the sense of distributions. Moreover the constant* L *in* (2.2) *can be chosen as* $L = c(n)\|\nabla_{\mathbb{H}} f\|_{(L^\infty(\mathbb{H}^n))^{2n}}$ *and* $c(n)$ *is a positive constant depending only on* n.

Definition 2.5. *We shall say that* $S \subset \mathbb{H}^n$ *is a* $\mathbb{H}$- *regular hypersurface if for every* $P \in S$ *there exist an open ball* $U(P,r)$ *and a function* $f \in C^1_{\mathbb{H}}(U(P,r))$ *such that*

i: $S \cap U(P,r) = \{Q \in U(P,r) : f(Q) = 0\}$;

ii: $\nabla_{\mathbb{H}} f(P) \neq 0$.

We will denote by $\nu_S(P)$ *the horizontal normal to* S *at a point* $P \in S$, *i.e. the unit vector*

$$\nu_S(P) := -\frac{\nabla_{\mathbb{H}} f(P)}{|\nabla_{\mathbb{H}} f(P)|_P}.$$

Observe that the parameterization $\Phi : \omega \to \mathbb{H}^n$ in (1.8) reads as follows

$$\begin{array}{ll} \Phi(\eta, v, \tau) = (\phi(\eta, v, \tau), v_2, \dots, v_n, \eta, v_{n+2}, \dots, v_{2n}, \tau - \frac{\eta}{2}\phi(\eta, v, \tau)) & \text{if } n \geq 2 \\ \Phi(\eta, \tau) = (\phi(\eta, \tau), \eta, \tau - \frac{\eta}{2}\phi(\eta, \tau)) & \text{if } n = 1 \end{array} . \tag{2.3}$$

Definition 2.6. *A set* $S \subset \mathbb{H}^n$ *is an* X_1*-graph if there is a function* $\phi : \omega \subset \mathbb{R}^{2n} \to \mathbb{R}$ *such that* $S = \Phi(\omega) = \{\iota(A) \cdot \phi(A) e_1 : A \in \omega\}$.

Let us summarize one of the main results contained in [1] (see Theorems 1.2 and 1.3).

Theorem 2.7. *Let* $\omega \subset \mathbb{R}^{2n}$ *be an open set, let* $\phi : \omega \to \mathbb{R}$ *be a continuous function and let* $\Phi : \omega \to \mathbb{H}^n$ *be the parameterization in* (1.8). *Then the following conditions are equivalent:*

i: $S = \Phi(\omega)$ *is an* $\mathbb{H}$- *regular surface and* $\nu_S^{(1)}(P) < 0$ *for all* $P \in S$, *where* $\nu_S(P) = \left(\nu_S^{(1)}(P), \dots, \nu_S^{(2n)}(P)\right)$ *is the horizontal normal to* S *at a point* $P \in S$.

ii: *the distribution* $\nabla^\phi \phi$ *is represented by a function* $w = (w_2, ..., w_{2n}) \in C^0(\omega; \mathbb{R}^{2n-1})$ *and there exists a family* $(\phi_\epsilon)_{\epsilon > 0} \subset C^1(\omega)$ *such that, for any open set* $\omega' \Subset \omega$, *we have*

$$\phi_\epsilon \to \phi \ \text{ and } \ \nabla^{\phi_\epsilon} \phi_\epsilon \to w \ \text{ uniformly in } \omega'. \tag{2.4}$$

Moreover, for every open set $\omega' \Subset \omega$

$$\lim_{r \to 0^+} \sup \left\{ \frac{|\phi(A) - \phi(B)|}{\sqrt{|A - B|}} : A, B \in \omega', 0 < |A - B| < r \right\} = 0\,, \tag{2.5}$$

$$\nu_S(P) = \left(-\frac{1}{\sqrt{1 + |\nabla^\phi \phi|^2}}, \frac{\nabla^\phi \phi}{\sqrt{1 + |\nabla^\phi \phi|^2}} \right) (\Phi^{-1}(P)) \qquad \text{for every } P \in S, \tag{2.6}$$

and

$$\mathcal{S}^{Q-1}_\infty(S) = c(n) \int_\omega \sqrt{1 + |\nabla^\phi \phi|^2}\, d\mathcal{L}^{2n} \tag{2.7}$$

where $\mathcal{L}^{2n}$ *denotes the Lebesgue* $2n$*-dimensional measure on* $\mathbb{R}^{2n}$*,* $\mathcal{S}_\infty^{Q-1}$ *denotes the* $(Q-1)$*-dimensional spherical Hausdorff measure induced in* $(\mathbb{H}^n, d_\infty)$ *and* $c(n)$ *a positive constant depending only on* n*.*

Because of Theorem 2.7, we will call $\nabla^\phi \phi = \left(\widetilde{X}_2\phi, ..., \widetilde{X}_n\phi, \mathfrak{B}\phi, \widetilde{Y}_2\phi, ..., \widetilde{Y}_n\phi\right)$ the *intrinsic gradient* of ϕ in ω, provided $\Phi(\omega)$ is $\mathbb{H}$- regular. Let $n \geq 2$, we will denote by $\widetilde{\nabla}_{\mathbb{H}}$ the family of $2n-2$ vector fields on $\mathbb{R}^{2n}$

$$\widetilde{\nabla}_{\mathbb{H}} := \left(\widetilde{X}_2, \ldots, \widetilde{X}_n, \widetilde{Y}_2, \ldots \widetilde{Y}_n\right) \tag{2.8}$$

where $\widetilde{X}_j$ and $\widetilde{Y}_j$ $(j = 2, \ldots n)$ are defined in (1.12).

Definition 2.8. *Let* $\Omega \subseteq \mathbb{R}^n$ *be a bounded open set.*

i: *Given* $\alpha \in (0,1)$*, let* $h^\alpha(\overline{\Omega})$ *denote the set of functions* $f \in C^0(\overline{\Omega})$ *such that*

$$\lim_{r\to 0} L_\alpha(\bar{\Omega}, f, r) = 0\,,$$

where

$$L_\alpha(f, \overline{\Omega}, r) := \sup\left\{\frac{|f(x) - f(y)|}{|x-y|^\alpha} \,:\, x, y \in \overline{\Omega},\, 0 < |x-y| < r\right\}. \tag{2.9}$$

We will denote by $L_0(f, \overline{\Omega}, r)$ *the modulus of continuity of a function* $f \in C^0(\bar{\Omega})$*, i.e. the quantity in* (2.9) *with* $\alpha = 0$*.*

ii: *Let* $h^\alpha_{loc}(\Omega)$ *denote the set of functions* $f \in C^0(\Omega)$ *such that* $f \in h^\alpha(\overline{\Omega'})$*, for each open set* $\Omega' \Subset \Omega$*.*

iii: *Given* $f \in Lip(\Omega)$*, let*

$$L_1(f, \overline{\Omega}) := \sup\left\{\frac{|f(x) - f(y)|}{|x-y|} \,:\, x, y \in \overline{\Omega},\, x \neq y\right\}. \tag{2.10}$$

In this second part of the section we shall recall some notions and results about entropy solutions for scalar conservation laws introduced in [21] (see, also [5], chapter 4 and [13], section 11.4.3).

Definition 2.9. *Let* $f \in Lip_{loc}(\mathbb{R})$*. Two smooth functions* $e, d : \mathbb{R} \to \mathbb{R}$ *comprise an entropy/entropy-flux pair for the conservation law* $u_t + f(u)_x = g(t,x)$ *provided*

i: e *is convex*

ii: $e' \cdot f' = d'$

In the following let $I = (-r_0, r_0)$, $T > 0$, $\omega = (0,T) \times (-r_0, r_0)$.

Definition 2.10. *Let* $f \in Lip_{loc}(\mathbb{R})$*,* $g \in L^1(\omega)$*,* $u_0 \in L^\infty(I)$*. We call* $u \in C^0([0,T]; L^1(I)) \cap L^\infty(\omega)$ *an entropy solution of*

$$\begin{cases} u_t + f(u)_x = g(t,x) & \text{in}\,\omega \\ u = u_0 & \text{on}\,\{0\} \times I \end{cases} \tag{2.11}$$

provided that u *satisfies*

i: $\forall\, v \in C_c^\infty(\omega)$ *with* $v \geq 0$*, for each smooth entropy/entropy flux* $e, d : \mathbb{R} \to \mathbb{R}$

$$\int_\omega \left[e(u)v_t + d(u)v_x + e'(u)gv\right]\; dt\, dx \geq 0,$$

ii: $\lim_{t\to 0^+} \| u(t,\cdot) - u_0 \|_{L^1(I)} = 0$.

A well-known method in constructing an entropy solution u is the approximation of u by suitable regular solutions (see for instance [5] section 4.4 and [13] section 11.4.2, Theorem 2). In particular the following result will be crucial for our purposes.

Proposition 2.11. *Let* $(u_\epsilon)_\epsilon \subset Lip([0,T]\times[-r_0,r_0])$, $(g_\epsilon)_\epsilon \subset L^1([0,T]\times[-r_0,r_0])$, $f \in Lip_{loc}(\mathbb{R})$ *be such that*

$$u_{\epsilon,t} + f'(u_\epsilon)u_{\epsilon,x} = g_\epsilon \qquad \mathcal{L}^2 - \text{a.e. in}\,(0,T)\times(-r_0,r_0)\,. \tag{2.12}$$

Let us assume that

$$u_\epsilon \to u \qquad \text{uniformly in}\,[0,T]\times[-r_0,r_0]\,, \tag{2.13}$$

$$g_\epsilon \to g \qquad \text{in}\, L^1([0,T]\times[-r_0,r_0])\,. \tag{2.14}$$

Then u is an entropy solution of (2.11) *with* $u_0(x) = u(0,x)$.

We shall introduce now a slight refinement of the well-known uniqueness result due to Kružhkov in order to obtain a local uniqueness result for entropy solutions of (2.11).

Theorem 2.12. *Let* $g \in L^1(\omega)$ *and let* $u, \tilde{u} \in C^0([0,T];L^1(I)) \cap L^\infty(\omega)$ *be two entropy solutions of the problem* (2.11). *Let M, L be constants such that*

$$|u(t,x)| \le M, \quad |\tilde{u}(t,x)| \le M \qquad \forall (t,x) \in \omega\,, \tag{2.15}$$

$$|f(u_1) - f(u_2)| \le L|u_1 - u_2| \qquad \forall\, u_1, u_2 \in [-M,M]\,. \tag{2.16}$$

Then, $\forall\, r \in (0,r_0)$ *such that* $r + LT < r_0$, $\forall\, 0 \le \tau_0 \le \tau \le T$, *one has*

$$\int_{|x|\le r} |u(\tau,x) - \tilde{u}(\tau,x)|\,dx \le \int_{|x|\le r+L(\tau-\tau_0)} |u(\tau_0,x) - \tilde{u}(\tau_0,x)|\,dx\,. \tag{2.17}$$

In particular when $\tau_0 = 0$ *and* $u(0,\cdot) = \tilde{u}(0,\cdot)$ *a.e. in I then*

$$u(t,x) = \tilde{u}(t,x) \qquad \mathcal{L}^2 - \text{a.e.}\ (t,x) \in (0,T)\times(-r,r).$$

The classical proof of Theorem 2.12 is contained in [21], section 3 Theorem 1, when $r_0 = +\infty$, $f \in C^1(\mathbb{R}^2)$ $g \in C^1(\mathbb{R}^2)$. A detailed proof can be found in [4].

By Theorem 2.12, we easily obtain the following local uniqueness result for entropy solutions of Burgers' equation that will be needed later.

Corollary 2.13. *Let* $g \in L^1((0,T)\times(-r_0,r_0))$, $u_0 \in L^\infty(-r_0,r_0)$, $M > 0$. *Let* $\mathcal{E}_M(T,r_0)$ *denote the class of functions* $u \in C^0([0,T];L^1(-r_0,r_0))$ *such that*

$$|u(t,x)| \le M \quad \mathcal{L}^2 - a.e.(t,x) \in (0,T)\times(-r_0,r_0)\,.$$

Let u, $\tilde{u} \in \mathcal{E}_M(T,r_0)$ *be entropy solutions of the initial value problem*

$$\begin{cases} u_t + \left(\frac{u^2}{2}\right)_x = g & \text{in}\,(0,T)\times(-r_0,r_0) \\ u(0,x) = u_0(x) & \forall\, x \in (-r_0,r_0) \end{cases}.$$

Then, if $r + M\,T < r_0$, $u(t,x) = \tilde{u}(t,x)$ $\mathcal{L}^2 - \text{a.e.}\ (t,x) \in (0,T)\times(-r,r)$.

Finally let us recall the following link between entropy solutions and $\mathbb{H}$- regular intrinsic graphs, already pointed out in [1], Remark 5.2.

Proposition 2.14. *Let $\omega = (-r_0, r_0) \times (-r_0, r_0)$. Assume that $S = \Phi(\omega) \subset \mathbb{H}^1$ is $\mathbb{H}$-regular and let $w := W^\phi \phi \in C^0(\omega)$. Then ϕ is an entropy solution of the initial value problem*

$$\begin{cases} u_\eta + \left(\frac{u^2}{2}\right)_\tau = w & \text{in } (0, r_0) \times (-r_0, r_0) \\ u(0,\tau) = \phi(0,\tau) & \forall\, \tau \in [-r_0, r_0] \end{cases}.$$

3. $\mathbb{H}$- Regularity and Weak Solutions of Non Linear First-Order PDEs

In this section we are going to prove Theorem 1.2. Its proof relies on two preliminary results. The former is the following one given in [1], though not explicitly stated.

Lemma 3.1. *The conclusion of Theorem* 1.2 *holds provided that the assumption*

$$\phi \in h^{\frac{1}{2}}_{loc}(\omega) \tag{3.1}$$

is also required in the statement **i**.

Proof. **i** $\Rightarrow$ **ii** The implication follows at once using Theorems 1.2 and 5.7 contained in [1]. **ii** $\Rightarrow$ **i**: By Theorems 1.2 and 1.3 in [1], we obtain that (3.1) holds and there is a family $(\phi_\epsilon)_\epsilon \subset C^1(\omega)$ such that

$$\phi_\epsilon \to \phi, \qquad \nabla^{\phi_\epsilon}\phi_\epsilon \to \nabla^\phi \phi \tag{3.2}$$

uniformly on the compact sets contained in ω. Finally by (3.2) and Lemma 5.6 in [1], we obtain (1.17). □

In order to obtain Theorem 1.2 we need only to show that the assumption (3.1) can be omitted. More precisely we prove the following regularity result for broad* solutions (see also [1], Theorem 5.8).

Theorem 3.2. *Let $\phi : \omega \to \mathbb{R}$ and $w = (w_2, ..., w_{2n}) : \omega \to \mathbb{R}^{2n-1}$ be continuous functions. Assume that ϕ is a broad* solution of* (1.1). *Then for each $A_0 \in \omega$ there exist $0 < r_2 < r_1$ and a function $\alpha : (0, +\infty) \to [0, +\infty)$, which depends only on A_0, $\|\phi\|_{L^\infty(I_{r_1}(A_0))}$, $\|w\|_{L^\infty(I_{r_1}(A_0))}$ and on the modulus of continuity of w_{n+1} on $I_{r_1}(A_0)$, such that $\lim_{r\to 0}\alpha(r) = 0$ and*

$$L_{\frac{1}{2}}(\phi, \overline{I_{r_2}(A_0)}, r) = \sup\left\{\frac{|\phi(A)-\phi(B)|}{|A-B|^{1/2}} : A, B \in \overline{I_{r_2}(A_0)},\ 0 < |A-B| \le r\right\} \le \alpha(r) \tag{3.3}$$

for all $r \in (0, r_2)$.

Before the proof of Theorem 3.2, we shall introduce a key preliminary result which will be needed in section 4 too.

Lemma 3.3. *Let $Q_1 := [-\delta_2, \delta_2] \times [\tau_0 - \delta_1, \tau_0 + \delta_1]$ and $Q_2 := [-\delta_2, \delta_2] \times [\tau_0 - \delta_2, \tau_0 + \delta_2]$ with $0 < \delta_2 < \delta_1$. Let $f_i \in C^0(Q_1)$ $(i = 1, 2)$ and $x : Q_2 \to [\tau_0 - \delta_1, \tau_0 + \delta_1]$ be given such that*

i: $x(\cdot, \tau) \in C^2([-\delta_2, \delta_2]) \quad \forall \tau \in [\tau_0 - \delta_2, \tau_0 + \delta_2]$*;*

ii:

$$\begin{cases} \dfrac{d^i}{ds^i}x(s,\tau) = f_i(s,x(s,\tau)) \quad (i=1,2) \\ x(0,\tau)=\tau \end{cases} \quad \forall s\in[-\delta_2,\delta_2],\ \tau\in[\tau_0-\delta_2,\tau_0+\delta_2]\,.$$

Then

$$L_{\frac{1}{2}}(g,[\tau_0-\delta_2,\tau_0+\delta_2],r) \le \max\left\{r^{1/4}, 2\sqrt{2\,L_0(f_2,Q_1,r+2\,c_0\,r^{1/4})}\right\} \tag{3.4}$$

for each $r\in(0,r_0)$*, where* $g(\tau):=f_1(0,\tau)$*,* $c_0:=2\,\|f_1\|_{L^\infty(Q_1)}$*,* $0<r_0<\dfrac{\delta_2^4}{16}$.

Moreover, if $f_2\in Lip(Q_1)$ *and* $L_1:=L_1(f_2,Q_1)$*, then*

$$L_1(g,[\tau_0-\delta_2,\tau_0+\delta_2]) \le \frac{2}{\delta_2}\,. \tag{3.5}$$

Proof. Let

$$\beta(r):=L_0(f_2,Q_1,r), \quad \alpha(r):=\max\left\{r^{1/4}, 2\sqrt{2\,\beta(r+2\,c_0\,r^{1/4})}\right\}$$

if $r\ge 0$ and observe that

$$\frac{\beta\left(r+\frac{2\,c_0\sqrt{r}}{\alpha(r)}\right)}{\alpha(r)^2} \le \frac{1}{8} \quad \forall r>0\,. \tag{3.6}$$

Firstly, we shall prove (3.4). We argue by contradiction. Assume there exist $\tau_0-\delta_2\le\tau_2<\tau_1\le\tau_0+\delta_2$, $0<\bar r<r_0$ such that

$$0<|\tau_1-\tau_2|\le\bar r\,, \tag{3.7}$$

$$\frac{|g(\tau_1)-g(\tau_2)|}{\sqrt{\tau_1-\tau_2}} > \alpha(\bar r)\,, \tag{3.8}$$

and let us prove there exists $s^*\in[-\delta_2,\delta_2]$ such that

$$x(s^*,\tau_1)=x(s^*,\tau_2) \tag{3.9}$$

and

$$f_1((s^*,x(s^*,\tau_1))\ne f_1((s^*,x(s^*,\tau_2))\,. \tag{3.10}$$

This is a contradiction and (3.4) will be proved.

We shall introduce the curves $\gamma_\tau(s):=(s,x(s,\tau))$ if $s\in[-\delta_2,\delta_2]$. Assuming **i** and **ii** we can represent each $x(\cdot,\tau)$ for each $\tau\in[\tau_0-\delta_2,\tau_0+\delta_2]$ as

$$\begin{aligned} x(s,\tau) &= \tau+\textstyle\int_0^s f_1(\gamma_\tau(\sigma))\,d\sigma \\ &= \tau+f_1(0,\tau)\,s+\displaystyle\int_0^s(s-\sigma)f_2(\gamma_\tau(\sigma))\,d\sigma \quad \forall s\in[-\delta_2,\delta_2]\,. \end{aligned} \tag{3.11}$$

By the first equality in (3.11) we obtain

$$|x(s,\tau)-x(s,\tau')|\le|\tau-\tau'|+c_0\,|s| \quad \forall s\in[-\delta_2,\delta_2],\ \tau,\tau'\in[\tau_0-\delta_2,\tau_0+\delta_2]$$

and then, being β increasing,

$$|f_2(\gamma_\tau(\sigma))-f_2(\gamma_{\tau'}(\sigma))|\le\beta(|\gamma_\tau(\sigma)-\gamma_{\tau'}(\sigma)|)\le\beta(|\tau-\tau'|+c_0\,|s|) \tag{3.12}$$

for each $|\sigma| \le |s|$ and $\tau, \tau' \in [\tau_0 - \delta_2, \tau_0 + \delta_2]$.

In particular, by the second equality in (3.11) and (3.12),

$$x(s,\tau) - x(s,\tau') \le \tau - \tau' + (g(\tau) - g(\tau'))s + \beta(|\tau - \tau'| + c_0\,|s|)\,s^2 \tag{3.13}$$

for $0 \le s \le \delta_2$, for each $\tau, \tau' \in [\tau_0 - \delta_2, \tau_0 + \delta_2]$.

By (3.8) we obtain

$$g(\tau_1) - g(\tau_2) < -\alpha(\bar r)\sqrt{\tau_1 - \tau_2} \tag{3.14}$$

or

$$g(\tau_1) - g(\tau_2) > \alpha(\bar r)\sqrt{\tau_1 - \tau_2} \tag{3.15}$$

Let $\bar s := 2\,\dfrac{\sqrt{\tau_1 - \tau_2}}{\alpha(\bar r)}$ then

$$\bar s \in [0, \delta_2], \quad x(\bar s, \tau_1) - x(\bar s, \tau_2) < 0\,. \tag{3.16}$$

Indeed, by (3.7) and the definition of α, $\bar s \le 2\,\dfrac{\sqrt{\tau_1 - \tau_2}}{\alpha(|\tau_1 - \tau_2|)} \le 2\,(\tau_1 - \tau_2)^{1/4} \le 2\,\bar r^{1/4} \le 2\,r_0^{1/4} \le \delta_2$. On the other hand by (3.13) (with $s = \bar s$, $\tau = \tau_1$, $\tau' = \tau_2$), (3.14) and (3.6)

$$x(\bar s, \tau_1) - x(\bar s, \tau_2) \le \tau_1 - \tau_2 - 2\,(\tau_1 - \tau_2) + 4\,\frac{\beta(|\tau_1 - \tau_2| + c_0\,\bar s)}{\alpha(\bar r)^2}\,(\tau_1 - \tau_2) =$$

$$= (\tau_1 - \tau_2)\left(-1 + 4\,\frac{\beta(\bar r + 2\,c_0\sqrt{\bar r}/\alpha(\bar r))}{\alpha(\bar r)^2}\right) \le -\frac{1}{2}\,(\tau_1 - \tau_2) < 0\,.$$

Then (3.16) follows. Let

$$s^* := \sup\{s \in [0, \delta_2] : x(s, \tau_1) > x(s, \tau_2)\} \tag{3.17}$$

then by (3.16) $0 < s^* < \bar s \le \delta_2$ and it satisfies (3.9).

If (3.15) holds, let us consider

$$f_1^*(\eta, \tau) = -f_1(-\eta, \tau), \quad f_2^*(\eta, \tau) = f_2(-\eta, \tau) \quad (\eta, \tau) \in Q_1$$
$$x^*(s, \tau) = x(-s, \tau), (s, \tau) \in Q_2\,,$$
$$g^*(\tau) = -f_1(0, \tau) \quad \tau \in [\tau_0 - \delta_1, \tau_0 + \delta_1]\,.$$

Then, since in this case

$$\frac{d^i}{ds^i} x^*(s, \tau) = f_i^*(s, x^*(s, \tau)) \ \text{ if } |s| \le \delta_2,\ \tau \in [\tau_0 - \delta_1, \tau_0 + \delta_1], \quad (i = 1, 2)$$
$$g^*(\tau_1) - g^*(\tau_2) < -\alpha(\bar r)\sqrt{\tau_1 - \tau_2}\,,$$

we can repeat the argument above, obtaining that there exists $-\delta_2 < s* < 0$ such that (3.9) still holds. Let us prove now (3.10). For instance, assume (3.14). From (3.11) and (3.12),

$$f_1(\gamma_{\tau_1}(s*)) - f_1(\gamma_{\tau_2}(s*)) = g(\tau_1) - g(\tau_2) + \int_0^{s*} (f_2(\gamma_{\tau_1}(\sigma)) - f_2(\gamma_{\tau_2}(\sigma)))\,d\sigma \le$$

$$\le g(\tau_1) - g(\tau_2) + \beta(|\tau_1 - \tau_2| + c_0\,s*)\,s* \le g(\tau_1) - g(\tau_2) + \beta(|\tau_1 - \tau_2| + c_0\,\bar s)\,\bar s$$

$$\le -\alpha(\bar r)\sqrt{\tau_1 - \tau_2} + 2\,\frac{\beta(|\tau_1 - \tau_2| + c_0\,\bar s)}{\alpha(\bar r)}\sqrt{\tau_1 - \tau_2} \le$$

$$\leq -\alpha(\bar r)\sqrt{\tau_1-\tau_2} + 2\,\frac{\beta(\bar r + 2\,c_0\sqrt{\bar r}/\alpha(\bar r))}{\alpha(\bar r)}\sqrt{\tau_1-\tau_2} =$$

$$= 2\,\alpha(\bar r)\sqrt{\tau_1-\tau_2}\left[-\frac{1}{2} + \frac{\beta(\bar r + 2\,c_0\sqrt{\bar r}/\alpha(\bar r))}{\alpha(\bar r)^2}\right].$$

From (3.6), $f_1(\gamma_{\tau_1}(s*)) - f_1(\gamma_{\tau_2}(s*))) < 0$ and (3.10) follows.

Let us prove now (3.5). The proof scheme partially follows the previous one. By contradiction, assume, for instance, there exist $\tau_0-\delta_2 \leq \tau_2 < \tau_1 \leq \tau_0+\delta_2$ such that

$$K := \frac{g(\tau_1)-g(\tau_2)}{\tau_1-\tau_2} < -\frac{2}{\delta_2}\,. \tag{3.18}$$

Otherwise we can argue as before to reduce to this case. Then we need only to prove there exists $0 < s^* < \delta_2$ such that (3.9) holds. In fact, we can apply now the classical uniqueness result for ODE solutions with Lipschitz continuous data to the Cauchy problem

$$\begin{cases} \dfrac{d^2}{ds^2}y(s) = f_2(s,y(s)) \\ y(s^*) = \tau^*,\ \dfrac{d}{ds}y(s^*) = f_1(s^*,\tau^*) \end{cases},$$

where $\tau^* = x(s^*,\tau_1) = x(s^*,\tau_2)$ and thereby a contradiction.

Let s^* be as in (3.17), then $0 < s^* \leq \delta_2$. Since $f_2 \in Lip(Q_1)$, by the second equality in (3.11) and (ii), for $0 \leq s \leq \delta_2$,

$$x(s,\tau_1) - x(s,\tau_2) \leq \tau_1-\tau_2 + (g(\tau_1)-g(\tau_2))s + L_1 s\int_0^s |x(\sigma,\tau_1)-x(\sigma,\tau_2)|d\sigma. \tag{3.19}$$

We shall prove (3.9). Let $u(s) := \int_0^s (x(\sigma,\tau_1)-x(\sigma,\tau_2))d\sigma$ if $0 \leq s \leq s^*$, then by (3.19)

$$\frac{d}{ds}u(s) \leq a(s) + b(s)u(s) \quad 0 \leq s \leq s^*$$

with $a(s) := \tau_1-\tau_2 + (g(\tau_1)-g(\tau_2))s$, $b(s) = L_1\,s$. By applying Gronwall's inequality (see, for instance, [13], appendix B.2 j), if $0 \leq s \leq s^*$,

$$0 \leq \int_0^s (x(\sigma,\tau_1)-x(\sigma,\tau_2))d\sigma = u(s) \leq \exp\left(\int_0^s b(\sigma)d\sigma\right)\cdot\left[u(0) + \int_0^s a(\sigma)d\sigma\right] = \tag{3.20}$$

$$= \exp\left(L_1\,\frac{s^2}{2}\right)\left[(\tau_1-\tau_2)s + \frac{g(\tau_1)-g(\tau_2)}{2}s^2\right] = \exp\left(L_1\,\frac{s^2}{2}\right)(\tau_1-\tau_2)\,s\left(1+\frac{K}{2}\,s\right).$$

Let $\bar s := -2/K$ and notice that $0 < \bar s < \delta_2$ by (3.18). Then we imply $0 < s^* \leq \bar s < \delta_2$ from (3.20) and (3.9) holds. □

Remark 3.4. To obtain (3.5) we have actually exploited the weaker assumption

$$|f_2(\eta,\tau) - f_2(\eta,\tau')| \leq L_1\,|\tau-\tau'| \quad \forall \eta \in [-\delta_2,\delta_2],\ \tau \in [\tau_0-\delta_1,\tau_0+\delta_1]\,,$$

instead of $f_2 \in Lip(Q_1)$.

Proof of Theorem 3.2. Let $A_0 = (\eta_0, \tau_0) \in \omega$ if $n = 1$ and $A_0 = (\eta_0, v_0, \tau_0) \in \omega$ if $n \geq 2$. As ϕ is a broad* solution of (1.1), there exists a family of exponential maps at A_0

$$\exp_{A_0}(\cdot \nabla_j^{\phi})(\cdot) : [-\delta_2, \delta_2] \times \overline{I_{\delta_2}(A_0)} \to \overline{I_{\delta_1}(A_0)} \Subset \omega\,, \tag{3.21}$$

where $0 < \delta_2 < \delta_1$ and $j = 2, \ldots, 2n$, satisfying (E_1), (E_2) and (E_3).

Let us denote $I_1 := \overline{I_{\delta_1}(A_0)}$, $I_2 := \overline{I_{\delta_2}(A_0)}$, $K := \sup_{A\in I_1} |A|, M := \|\phi\|_{L^\infty(I_1)}, N := \|\nabla^\phi \phi\|_{L^\infty(I_1)}$; let $\beta(r) := L_0(w_{n+1}, I_1, r)$ be the modulus of continuity of w_{n+1} on I_1.

Let $A = (\eta, \tau) \in I_2$ if $n = 1$ and $A = (\eta, v, \tau) \in I_2$ if $n \geq 2$. Denote with $\gamma_A(s) = \gamma_{n+1}^A(s) = \exp_{A_0}(sW^\phi)(A)$ if $s \in [-\delta_2, \delta_2]$ and let $\gamma_A(s) = (\eta + s, \tau_A(s))$ if $n = 1$ and $\gamma_A(s) = (\eta + s, v, \tau_A(s))$ if $n \geq 2$. Then τ_A satisfies

$$\begin{cases} \dfrac{d^2}{ds^2}\tau_A(s) = \dfrac{d}{ds}[\phi(\gamma_A(s))] = w_{n+1}(\gamma_A(s)) \quad \forall s \in [-\delta_2, \delta_2] \\ \tau_A(0) = \tau, \quad \dfrac{d}{ds}\tau_A(0) = \phi(A) \end{cases} . \tag{3.22}$$

Let us observe that

$$\exp_{A_0}(\cdot W^\phi)(\cdot) : [-r_2, r_2] \times \overline{I_{r_2}(A_0)} \to \overline{I_{\delta_2}(A_0)} = I_2 \tag{3.23}$$

provided that

$$r_2 < \frac{\delta_2}{M + 2}\,. \tag{3.24}$$

Indeed, if $(s, A) \in [-r_2, r_2] \times \overline{I_{r_2}(A_0)}$, then by (3.21),(3.24) and (E_2)

$$\gamma_A(s) - A_0 = \begin{cases} (\eta - \eta_0 + s, \tau_A(s) - \tau_0) & \text{if}\, n = 1 \\ (\eta - \eta_0 + s, v - v_0, \tau_A(s) - \tau_0) & \text{if}\, n \geq 2 \end{cases} \in \overline{I_{\delta_2}(0)}\,.$$

Firstly, let us consider the case $n = 1$ and divide the proof in three steps.

Step 1. Let us prove that

$$\sup\left\{\frac{|\phi(A) - \phi(B)|}{|A - B|^{1/2}} : A = (\eta, \tau), B = (\eta, \tau') \in I_2, 0 < |A - B| \leq r\right\} \leq \alpha_1(r). \tag{3.25}$$

for every $r \in (0, r_0)$ where

$$\alpha_1(r) := \max\left\{r^{1/4}, \sqrt{L_0(w_{n+1}, I_1, r + 2\, M\, r^{1/4})}\right\}, \quad 0 < r_0 < \frac{\delta_2^4}{16}\,. \tag{3.26}$$

Let $A = (\eta, \tau) \in I_2 = [\eta_0 - \delta_2, \eta_0 + \delta_2] \times [\tau_0 - \delta_2, \tau_0 + \delta_2]$ and let $x(s, \tau) := \tau_A(s)$ if $|s| \leq \delta_2$ and $\tau \in [\tau_0 - \delta_2, \tau_0 + \delta_2]$, $f_{1,\eta}(s, \tau) := \phi(\eta + s, \tau)$, $f_{2,\eta}(s, \tau) := w_2(\eta + s, \tau)$, $g_\eta(\tau) = \phi(\eta, \tau)$ if $(s, \tau) \in Q_1 := [-\delta_2, \delta_2] \times [\tau_0 - \delta_1, \tau_0 + \delta_1]$ and $\eta \in [\eta_0 - \delta_2, \eta_0 + \delta_2]$ is fixed. By (3.22) and since

$$\|f_{1,\eta}\|_{L^\infty(Q_1)} \leq M, \; L_0(f_{2,\eta}, Q_1, r) \leq L_0(w_2, I_1, r) \quad \forall \eta \in [\eta_0 - \delta_2, \eta_0 + \delta_2],$$

we can apply (3.4) of Lemma 3.3 and (3.25) follows.

Step 2. We shall prove that

$$\sup\left\{\frac{|\phi(A) - \phi(B)|}{|A - B|^{1/2}} : A = (\eta, \tau), B = (\eta', \tau) \in \overline{I_{r_2}(A_0)}, 0 < |A - B| \leq r\right\} \leq \alpha_2(r) \tag{3.27}$$

for every $r \in (0, r_2)$ where

$$\alpha_2(r) := \sqrt{M}\,\alpha_1(M\,r) + N\sqrt{r}, \quad 0 < r_2 < \min\left\{\frac{\delta_2}{M+2}, \frac{r_0}{M}\right\} \tag{3.28}$$

and $\alpha_1(r)$ and r_0 are as in (3.26). We proceed by contradiction. Suppose there exist $\bar{A} = (\bar{\eta}', \bar{\tau}), \bar{B} = (\bar{\eta}, \bar{\tau}) \in \overline{I_{r_2}(A_0)}$, $0 < \bar{r} \le r_2$ such that $0 < |\bar{A} - \bar{B}| \le \bar{r}$ and

$$\frac{|\phi(\bar{A}) - \phi(\bar{B})|}{|\bar{A} - \bar{B}|^{1/2}} > \sqrt{M}\,\alpha_1(M\,\bar{r}) + N\,\sqrt{\bar{r}}\,. \tag{3.29}$$

Let $\bar{C} := \gamma_{\bar{A}}(\bar{\eta} - \bar{\eta}') = (\bar{\eta}, \bar{\tau}')$ and notice that $\bar{C} \in I_2$ by (3.23) and (3.24). Moreover

$$|\bar{\tau}' - \bar{\tau}| = \left| \int_0^{\bar{\eta} - \bar{\eta}'} \phi(\gamma_{\bar{A}}(\sigma))\,d\sigma \right| \le M\,|\bar{\eta} - \bar{\eta}'|\,. \tag{3.30}$$

On the other hand, by (3.29) and (E_3),
(3.31)

$$\begin{aligned} |\phi(\bar{B}) - \phi(\bar{C})| \ &\ge |\phi(\bar{B}) - \phi(\bar{A})| - |\phi(\bar{A}) - \phi(\bar{C})| \\ \ge \ &\left[\sqrt{M}\,\alpha_1(M\,\bar{r}) + N\,\sqrt{\bar{r}} - N\,\sqrt{|\bar{\eta} - \bar{\eta}'|}\,\right] \sqrt{|\bar{\eta} - \bar{\eta}'|} \ge \sqrt{M}\,\alpha_1(M\,\bar{r})\,\sqrt{|\bar{\eta} - \bar{\eta}'|} \end{aligned}$$

Let us notice that $\bar{\tau} \ne \bar{\tau}'$. Otherwise $\bar{C} = (\bar{\eta}, \bar{\tau}') = (\bar{\eta}, \bar{\tau}) = \bar{B}$ and, since $\alpha_1(r) > 0\,\forall r > 0$, by (3.31), $M = 0$. Therefore $\phi \equiv 0$ in I_1 and we reach a contradiction because of (3.29).

By (3.31) and (3.30), $\bar{B} = (\bar{\eta}, \bar{\tau})$, $\bar{C} = (\bar{\eta}, \bar{\tau}') \in I_2$ and

$$\frac{|\phi(\bar{B}) - \phi(\bar{C})|}{\sqrt{|\bar{B} - \bar{C}|}} \ge \alpha_1(M\,\bar{r})$$

with $0 < |\bar{B} - \bar{C}| = |\bar{\tau} - \bar{\tau}'| \le M\,\bar{r} \le M\,r_2 \le r_0$ and thereby a contradiction for step 1.

Step 3. Let $A = (\eta, \tau)$, $B = (\eta', \tau') \in \overline{I_{r_2}(A_0)}$ with $0 < |A - B| \le r$, then

$$\frac{|\phi(A) - \phi(B)|}{|A - B|^{1/2}} \le \frac{|\phi(\eta, \tau) - \phi(\eta', \tau)|}{|\eta - \eta'|^{1/2}} + \frac{|\phi(\eta, \tau) - \phi(\eta, \tau')|}{|\tau - \tau'|^{1/2}}\,.$$

Steps 1, 2 and 3 conclude the proof when $n = 1$, choosing $r_1 = \delta_1$, r_2 as in (3.28) and $\alpha(r) = \alpha_1(r) + \alpha_2(r)$ where $\alpha_1(r)$ and $\alpha_2(r)$ are respectively defined in (3.26) and (3.28).

Let us consider now the case $n \ge 2$. Let $\widehat{\cdot} : \mathbb{R}^{2n} = \mathbb{R}_\eta \times \mathbb{R}_v^{2n-2} \times \mathbb{R}_\tau \to \mathbb{R}^2 = \mathbb{R}_\eta \times \mathbb{R}_\tau$ be the projection defined as $\widehat{(\eta, v, \tau)} = (\eta, \tau)$. Let us notice that $\widehat{I_r(A)} = I_r(\hat{A})$ for each $A \in \mathbb{R}^{2n}$. For fixed $v \in \overline{B(v_0, \delta_1)}$ let us define

$$\phi_v(\eta, \tau) := \phi(\eta, v, \tau), \quad w_v(\eta, \tau) := w_{n+1}(\eta, v, \tau) \quad \text{if} \quad (\eta, \tau) \in \overline{I_{\delta_1}(\hat{A}_0)}$$

and notice that

$$\widehat{\exp_{A_0}}(sW^\phi)(A) = \exp_{\hat{A}_0}(sW^{\phi_v})(\hat{A}) \quad s \in [-\delta_2, \delta_2]$$

for each $A \in \overline{I_{\delta_2}(A_0)}$ where $\exp_{A_0}(\cdot W^\phi)(\cdot)$ is the exponential map in (3.21) with $j = n+1$. In particular

$$\exp_{\hat{A}_0}(\cdot W^{\phi_v})(\cdot) : [-\delta_2, \delta_2] \times \overline{I_{\delta_2}(\hat{A}_0)} \to \overline{I_{\delta_1}(\hat{A}_0)}$$

and it satisfies (E_1), (E_2) and (E_3) in the case n=1 with $w_2 = w_v$. Moreover

$$\begin{aligned} M_v := \|\phi_v\|_{L^\infty(\overline{I_{\delta_1}(\hat{A}_0)})} &\leq M,\ N_v := \|w_v\|_{L^\infty(\overline{I_{\delta_1}(\hat{A}_0)})} \leq N, \\ L_0(w_v, \overline{I_{\delta_1}(\hat{A}_0)}, r) &\leq L_0(w_{n+1}, \overline{I_{\delta_1}(A_0)}, r) \end{aligned} \tag{3.32}$$

for each $v \in \overline{B(v_0, \delta_1)}$ and $r > 0$. We can apply the previous case $n = 1$ and, by (3.32),

$$\sup\left\{\frac{|\phi(A) - \phi(B)|}{|A - B|^{1/2}} : A = (\eta, v, \tau), B = (\eta', v, \tau') \in \overline{I_{r_2}(A_0)}, 0 < |A - B| \leq r\right\} \leq \alpha_3(r) \tag{3.33}$$

for each $r \in (0, r_2)$, where $\alpha_3(r) = \alpha_1(r) + \alpha_2(r)$ and $\alpha_1(r)$ is defined in (3.26), $\alpha_2(r)$ and r_2 are defined in (3.28). In order to achieve the proof we can follow the argument in step 5 of the proof of Theorem 5.8 in [1]. Then we can carry out the same estimates and we obtain

$$\frac{|\phi(A) - \phi(B)|}{|A - B|^{1/2}} \leq N\,|A - B|^{1/2} + \left(\frac{K}{2} + 2\right)\alpha_3(|A - B|)$$

for each A, $B \in \overline{I_{r_2}(A_0)}$ and $0 < |A - B| \leq r_2$. □

Corollary 3.5. *Let $\phi \in Lip_{loc}(\omega)$, $w \in C^0(\omega; \mathbb{R}^{2n-1})$ be such that $\nabla^\phi \phi = w$ a.e. in ω. Then $\Phi(\omega)$ is $\mathbb{H}$- regular. In particular $\Phi(\omega)$ turns out to be $\mathbb{H}$- regular when $\phi \in C^1(\omega)$.*

Proof. By Theorem 1.2, we need only to show (1.17). Let $A \in \omega$, then, by the classical ODE theory, there exists $0 < \delta_2 < \delta_1$ such that for each $B \in I_{\delta_2}(A)$, $\forall\, j = 2, ..., n$ there is a unique solution $\gamma_j^B : [-\delta_2, \delta_2] \to \overline{I_{\delta_1}(A)} \Subset \omega$ of the Cauchy problem

$$\begin{cases} \dot{\gamma}_j^B(s) = \nabla_j^\phi(\gamma_j^B(s)) & \forall s \in [-\delta_2, \delta_2] \\ \gamma_j^B(0) = B. \end{cases}.$$

Thus (E_1) and (E_2) in Definition 1.1 follow. Since $\phi \in Lip_{loc}(\omega)$, $[-\delta_2, \delta_2] \ni s \to \phi(\gamma_j^B(s))$ is differentiable a.e. and $\dfrac{d}{ds}\phi(\gamma_j^B(s)) = w_j(\gamma_j^B(s))$ a.e. $s \in [-\delta_2, \delta_2]$. Then (E_3) holds too. □

Corollary 3.6. *Let $\phi \in C^0(\omega)$ be a broad* solution of (1.1) with $w = (w_2, ..., w_{2n}) \in C^0(\omega; \mathbb{R}^{2n-1})$. Then ϕ is also a distributional solution, i.e. for each $\varphi \in C_c^\infty(\omega)$*

$$\int_\omega \phi \widetilde{X}_i \varphi \, d\mathcal{L}^{2n} = -\int_\omega w_i \varphi \, d\mathcal{L}^{2n} \qquad \forall\, i \neq n+1\,, \tag{3.34}$$

$$\int_\omega \left(\phi \frac{\partial \varphi}{\partial \eta} + \frac{1}{2}\phi^2 \frac{\partial \varphi}{\partial \tau}\right) d\mathcal{L}^{2n} = -\int_\omega w_{n+1} \varphi \, d\mathcal{L}^{2n}\,. \tag{3.35}$$

Proof. By Theorems 1.2 and 2.7 there exists a family $(\phi_\epsilon)_\epsilon \subset C^1(\omega)$ such that $\phi_\epsilon \to \phi$, $\nabla^{\phi_\epsilon}\phi_\epsilon \to w$ uniformly in ω' for each open set $\omega' \Subset \omega$. Integrating by parts we obtain (3.34) and (3.35). □

Remark 3.7. Corollary 3.5 yields that the $\mathbb{H}$- regular graphs need not be C^1 Euclidean regular. Actually there are examples of $\mathbb{H}$- regular graphs $S = \Phi(\omega)$ in $\mathbb{H}^1 \equiv \mathbb{R}^3$ such that $\mathcal{H}^{2+\epsilon}(S) > 0$ $\forall\, 0 < \epsilon < \frac{1}{2}$ (see [20]), i.e. S looks like a fractal set in $\mathbb{R}^3$ from the Euclidean metric point of view. By Theorem 1.2, the defining function $\phi : \omega \to \mathbb{R}$ of the graph is a broad* solution of the system $\nabla^\phi\phi = w$ in ω, for a suitable continuous function $w : \omega \to \mathbb{R}$. As S is not a 2-rectifiable set from the Euclidean metric point of view, $\phi \notin BV_{loc}(\omega)$, where $BV_{loc}(\omega)$ denotes the space of functions with locally bounded variation in ω (see also [1], Corollary 5.10). A similar $\mathbb{H}$- regular graph can be constructed in $\mathbb{H}^n$ with $n \geq 2$, arguing as in [20].

We are now going to study the local uniqueness of broad* solution of the system (1.1).

Theorem 3.8. *Let $M > 0$, $A_0 = (\eta_0, \tau_0) \in \mathbb{R}^2 = \mathbb{R}_\eta \times \mathbb{R}_\tau$ if $n = 1$, $A_0 = (\eta_0, v_0, \tau_0) \in \mathbb{R}^{2n} = \mathbb{R}_\eta \times \mathbb{R}_v^{2(n-1)} \times \mathbb{R}_\tau$ if $n \geq 2$, $r_0 > 0$, $w = (w_2, ..., w_{2n}) \in C^0(\overline{I_{r_0}(A_0)}; \mathbb{R}^{2n-1})$ be given. Let $\phi_i \in C^0(\overline{I_{r_0}(A_0)})$ (i=1,2) verifying $|\phi_i(A)| \leq M \quad \forall A \in \overline{I_{r_0}(A_0)}$.*

- **i:** *Let $n = 1$, $\phi_0 \in C^0([\tau_0 - r_0, \tau_0 + r_0])$, let ϕ_i $(i = 1, 2)$ be broad* solutions of the initial value problem*

$$\begin{cases} W^\phi\phi = w & \text{in}\, I_{r_0}(A_0) \\ \phi(\eta_0, \tau) = \phi_0(\tau) & \forall \tau \in [\tau_0 - r_0, \tau_0 + r_0] \end{cases} . \tag{3.36}$$

  Then $\phi_1 = \phi_2$ in $I_r(A_0)$, if $0 < r < \frac{r_0}{1+M}$.
- **ii:** *Let $n \geq 2$, $\alpha \in \mathbb{R}$ let ϕ_i $(i = 1, 2)$ be broad* solutions of the initial value problem*

$$\begin{cases} \nabla^\phi\phi = w & \text{in}\, I_{r_0}(A_0) \\ \phi(A_0) = \alpha \end{cases} . \tag{3.37}$$

  Then $\phi_1 = \phi_2$ in $I_r(A_0)$, if $0 < r < \frac{r_0}{1+M}$.

Remark 3.9. It is well-known that the uniqueness falls down for the problem

$$\begin{cases} W^\phi\phi = 0 & \text{in}\, I_{r_0}((0,0)) \\ \phi(\eta, 0) = 0 & \forall \tau \in [-r_0, r_0] \end{cases} . \tag{3.38}$$

For instance, the functions $\phi_1 := 0$ and $\phi_2(\eta, \tau) := \dfrac{\tau}{\eta + c}$ with $c \in \mathbb{R}$ are broad* solutions of (3.38) for r_0 small enough.

Proof. **i** Firstly, without loss of generality, we can assume that $A_0 = (0, 0)$. Otherwise, let us consider $\phi^*(\eta, \tau) = \phi(\eta - \eta_0, \tau - \tau_0)$ and the associated initial value problem

$$\begin{cases} W^{\phi^*}\phi^* = w^* & \text{in}\, I_{r_0}((0,0)) \\ \phi^*(0, \tau) = \phi_0^*(\tau) & \forall \tau \in [-r_0, +r_0] \end{cases} , \tag{3.39}$$

where $w^*(\eta, \tau) = w(\eta - \eta_0, \tau - \tau_0)$, $\phi_0^*(\tau) = \phi_0(\tau - \tau_0)$, $(\eta, \tau) \in I_{r_0}((0, 0))$, $\tau \in [-r_0, r_0]$. By definition, it is easy to see that ϕ is a broad* solution of (3.36) if and only if ϕ^* is a broad* solution of (3.39).

Let ϕ_i, $i = 1, 2$, be broad* solutions of the problem (3.36). Then $S_i = \Phi_i\left(\overline{I_{r_0}((0, 0))}\right)$ are $\mathbb{H}$- regular with $\omega = I_{r_0}((0, 0))$, because of Theorem 1.2. Moreover ϕ_i are entropy solutions of the problem

$$\begin{cases} u_\eta + uu_\tau = g & \text{in}\, (0, r_0) \times (-r_0, r_0) \\ u(0, \tau) = \phi_0(\tau) & \forall \tau \in [-r_0, r_0] \end{cases} \tag{3.40}$$

with $g(\eta,\tau) = w(\eta,\tau)$, because of Proposition 2.14. Thus Corollary 2.13 yields that $\phi_1 = \phi_2$, $\mathcal{L}^2$ − a.e. in $(0,r)\times(-r,r)$, when $r < \frac{r_0}{1+M}$ and, by the continuity of ϕ_i,

$$\phi_1 = \phi_2 \qquad \text{in } (0,r)\times(-r,r). \tag{3.41}$$

On the other hand, arguing as before, the functions defined by $u_i(\eta,\tau) := -\phi_i(-\eta,\tau)$ $(\eta,\tau)\in[0,r_0]\times[-r_0,r_0]$ still turn out to be entropy solutions of the problem (3.40), with $g(\eta,\tau) = w(-\eta,\tau)$ $(\eta,\tau)\in(0,r_0)\times(-r_0,r_0)$. Therefore

$$\phi_1 = \phi_2 \qquad \text{in } (-r,0)\times(-r,r). \tag{3.42}$$

Thus we complete the proof by (3.41) and (3.42) .

ii As before, we can assume that $A_0 = (0, v_0, 0)$. Let ϕ_i, $i = 1, 2$, be broad* solution of (3.37), with $n \geq 2$. Fix $\eta \in (-r_0, r_0)$ and define

$$f_i^{(\eta)}(v,\tau) = \phi_i(\eta, v,\tau) \qquad (v,\tau)\in B(v_0,r_0)\times(-r_0,r_0).$$

Using Theorems 1.2 and 2.7, there exist two families $(\phi_{i,\epsilon})_\epsilon \subset C^1(I_r(A_0))$ such that

$$\phi_{i,\epsilon}\to\phi_i, \qquad \nabla^{\phi_{i,\epsilon}}\phi_{i,\epsilon}\to w \quad \text{uniformly in } \overline{I_r(A_0)}\,, \tag{3.43}$$

for every $0 < r < r_0$. From (3.43) and for a fixed $\eta\in(-r,r)$, it follows that

$$\widetilde{\nabla}_{\mathbb{H}} f_i^{(\eta)} = \hat{w}_{n+1}(\eta,\cdot,\cdot) \quad \text{in } B(v_0,r)\times(-r,r)\,, \tag{3.44}$$

in the sense of distributions, where $\widetilde{\nabla}_{\mathbb{H}}$ is the family of vectors fields in (2.8) and $\hat{w}_{n+1} := (w_2,\dots,w_n,w_{n+2},\dots,w_{2n})$. Define $f^{(\eta)}(v,\tau) := f_1^{(\eta)}(v,\tau) - f_2^{(\eta)}(v,\tau)$, then it holds that

$$\widetilde{\nabla}_{\mathbb{H}} f^{(\eta)} = 0 \quad \text{in } B(v_0,r)\times(-r,r)\,, \tag{3.45}$$

in the sense of distributions. By (3.45) and Lemma 2.2, there exists a function $\psi = \psi(\eta)$: $(-r,r)\to\mathbb{R}$ such that

$$\phi_2(\eta,v,\tau) = \psi(\eta) + \phi_1(\eta,v,\tau) \quad \forall(\eta,v,\tau)\in I_r(A_0). \tag{3.46}$$

From $\phi_i(A_0) = \alpha$, $i = 1, 2$, and (3.46), we obtain $\psi(0) = 0$. Then

$$\phi_0^{(v)} := \phi_1(0,v,\tau) = \phi_2(0,v,\tau) \quad \forall\,(v,\tau)\in B(v_0,r)\times(-r,r). \tag{3.47}$$

Fix now $v\in B(v_0,r)$ and define $u_i \equiv u_i^{(v)}(\eta,\tau) = \phi_i(\eta,v,\tau)$ if $(\eta,\tau)\in(0,r)\times(-r,r)$. In order to achieve the proof, we need only to show that u_i, $i = 1, 2$, are entropy solutions of the initial value problem

$$\begin{cases} u_\eta + uu_\tau = g & \text{in } (0,\tau)\times(-r_0,+r_0) \\ u(0,\tau) = \phi_0^{(v)}(\tau) & \forall\tau\in[-r_0,+r_0] \end{cases}$$

where $g(\eta,\tau) := w_{n+1}(\eta,v,\tau)$. Indeed, by Corollary 2.13, as before we can conclude that $\phi_1 = \phi_2$ in $I_r(A_0)$. For fixed $v\in B(v_0,r)$, let

$$u_{i,\epsilon}(\eta,\tau) := \phi_{i,\epsilon}(\eta,v,\tau),\ g_{i,\epsilon}(\eta,\tau) := W^{\phi_{i,\epsilon}}\phi_{i,\epsilon}(\eta,v,\tau) \quad (\eta,\tau)\in[0,r]\times[-r,r]\,.$$

Proposition 2.11 and (3.43) imply that u_i, $i = 1, 2$, are entropy solutions. □

4. Euclidean Regularity of $\mathbb{H}$- Regular Graphs

In this section we are going to prove Theorems 1.3 and 1.4 and some of their consequences as well. Before the proof of Theorem 1.3 we will need some preliminary results.

Lemma 4.1. *Let $A_0 = (\eta_0, \tau_0) \in \mathbb{R}^2$ if $n = 1$, $A_0 = (\eta_0, v_0, \tau_0) \in \mathbb{R}^{2n}$ if $n \geq 2$, $r_0 > 0$, let $\phi : I_{r_0}(A_0) \to \mathbb{R}$ and $w = (w_2, \dots, w_{2n}) : I_{r_0}(A_0) \to \mathbb{R}^{2n-1}$ be given continuous functions. Assume that*

i: *ϕ is a broad* solution of $\nabla^\phi \phi = w$ in $I_{r_0}(A_0)$;*

ii: *$w_{n+1} \in Lip(\overline{I_{r_0}(A_0)})$.*

Then, for some $0 < r < r_0$, if $n = 1$

$$\sup\left\{\frac{|\phi(A)-\phi(B)|}{|A-B|} : A = (\eta,\tau), B = (\eta,\tau') \in \overline{I_r(A_0)},\ A \neq B\right\} < \infty\,; \tag{4.1}$$

if $n \geq 2$

$$\sup\left\{\frac{|\phi(A)-\phi(B)|}{|A-B|} : A = (\eta,v,\tau), B = (\eta,v,\tau') \in \overline{I_r(A_0)},\ A \neq B\right\} < \infty\,. \tag{4.2}$$

Proof. We are going to follow here the same strategy of the proof of Theorem 3.2.

Being ϕ a broad* solution, there exists a family of exponential maps at A_0

$$\exp_{A_0}(\cdot\nabla_j^\phi)(\cdot) : [-\delta_2, \delta_2] \times \overline{I_{\delta_2}(A_0)} \to \overline{I_{\delta_1}(A_0)} \Subset I_{r_0}(A_0) \tag{4.3}$$

where $0 < \delta_2 < \delta_1$ and $j = 2, \dots, 2n$ satisfying (E_1), (E_2) and (E_3).

We shall denote $I_1 := \overline{I_{\delta_1}(A_0)}$, $I_2 := \overline{I_{\delta_2}(A_0)}$. Let $A = (\eta, \tau) \in I_2$ if $n = 1$ and $A = (\eta, v, \tau) \in I_2$ if $n \geq 2$. Denote $\gamma_A(s) = \gamma_{n+1}^A(s) = \exp_{A_0}(sW^\phi)(A)$ if $s \in [-\delta_2, \delta_2]$ and let $\gamma_A(s) = (\eta + s, \tau_A(s))$ if $n = 1$ and $\gamma_A(s) = (\eta + s, v, \tau_A(s))$ if $n \geq 2$. Then τ_A satisfies

$$\begin{cases} \dfrac{d^2}{ds^2}\tau_A(s) = \dfrac{d}{ds}[\phi(\gamma_A(s))] = w_{n+1}(\gamma_A(s)). \\ \tau_A(0) = \tau, \quad \dfrac{d}{ds}\tau_A(0) = \phi(A) \end{cases} \tag{4.4}$$

Firstly, let us consider the case $n = 1$. Let $A = (\eta, \tau) \in I_2 = [\eta_0 - \delta_2, \eta_0 + \delta_2] \times [\tau_0 - \delta_2, \tau_0 + \delta_2]$ and let $x(s, \tau) := \tau_A(s)$ if $|s| \leq \delta_2$ and $\tau \in [\tau_0 - \delta_2, \tau_0 + \delta_2]$, $f_{1,\eta}(s, \tau) := \phi(\eta + s, \tau)$, $f_{2,\eta}(s, \tau) := w_2(\eta + s, \tau)$, $g_\eta(\tau) = \phi(\eta, \tau)$ if $(s, \tau) \in Q_1 := [-\delta_2, \delta_2] \times [\tau_0 - \delta_1, \tau_0 + \delta_1]$ and $\eta \in [\eta_0 - \delta_2, \eta_0 + \delta_2]$ is fixed. By (4.4) and since

$$L_1(f_{2,\eta}, [\tau_0 - \delta_1, \tau_0 + \delta_1]) \leq L_1(f_2, \overline{I_1}) < \infty \quad \forall \eta \in [\eta_0 - \delta_2, \eta_0 + \delta_2]$$

we can apply (3.5) of Lemma 3.3 and (4.1) follows with $r = \delta_2$. In the case $n \geq 2$ and $A = (\eta, v, \tau) \in I_2 = [\eta_0 - \delta_2, \eta_0 + \delta_2] \times \overline{B(v_0, \delta_2)} \times [\tau_0 - \delta_2, \tau_0 + \delta_2]$ let $x(s, \tau) := \tau_A(s)$ if $|s| \leq \delta_2$ and $\tau \in [\tau_0 - \delta_2, \tau_0 + \delta_2]$, $f_{1,\eta,v}(s, \tau) := \phi(\eta + s, v, \tau)$, $f_{2,\eta,v}(s, v, \tau) := w_{n+1}(\eta + s, v, \tau)$, $g_{\eta,v}(\tau) = \phi(\eta, v, \tau)$ if $(s, \tau) \in Q_1 := [-\delta_2, \delta_2] \times \overline{B(v_0, \delta_1)} \times [\tau_0 - \delta_1, \tau_0 + \delta_1]$ and $\eta \in [\eta_0 - \delta_2, \eta_0 + \delta_2]$, $v \in \overline{B(v_0, \delta_2)}$ are fixed. By (4.4) and since

$$L_1(f_{2,\eta,v}, [\tau_0 - \delta_1, \tau_0 + \delta_1]) \leq L_1(f_2, \overline{I_1}) < \infty \quad \forall \eta \in [\eta_0 - \delta_2, \eta_0 + \delta_2],\ v \in \overline{B(v_0, \delta_1)}\,,$$

we can argue as before to obtain (4.2). □

Remark 4.2. In order to obtain (4.1) and (4.2), by Remark 3.4, we can actually weaken the assumption $w_{n+1} \in Lip(\overline{I_{r_0}(A_0)})$ with

$$\sup\left\{\frac{|w_{n+1}(A)-w_{n+1}(B)|}{|A-B|} : A=(\eta,\tau), B=(\eta,\tau')\in\overline{I_{r_0}(A_0)},\ A\neq B\right\}<\infty \qquad \text{if } n=1 \text{ and}$$

$$\sup\left\{\frac{|w_{n+1}(A)-w_{n+1}(B)|}{|A-B|} : A=(\eta,v,\tau), B=(\eta,v,\tau')\in\overline{I_{r_0}(A_0)},\ A\neq B\right\}<\infty \qquad \text{if } n\geq 2.$$

Proof of Theorem 1.3. : Let $A_0 \in \omega$ and $r_0 > 0$ be such that $I_{r_0}(A_0) \Subset \omega$. We need only to prove that $\phi \in Lip(I_r(A_0))$ for some $0 < r < r_0$.

Let $A_0 = (\eta_0, \tau_0) \in \mathbb{R}^2$ if $n = 1$, $A_0 = (\eta_0, v_0, \tau_0) \in \mathbb{R}^{2n}$ if $n \geq 2$. Observe that, by Theorem 1.2, ϕ is a broad* solution of the system

$$\nabla^{\phi}\phi = w \quad \text{in}\, \omega := I_{r_0}(A_0)\,. \tag{4.5}$$

Then we can apply Lemma 4.1 and, for some $0 < r < r_0$, we obtain that

$$|\phi(\eta,\tau)-\phi(\eta,\tau')|\leq L\,|\tau-\tau'| \quad \forall\eta\in[\eta_0-r,\eta_0+r],\ \tau\,,\tau'\in[\tau_0-r,\tau_0+r]$$

if $n = 1$ and

$$|\phi(\eta,v,\tau)-\phi(\eta,v,\tau')|\leq L\,|\tau-\tau'| \quad \forall\eta\in[\eta_0-r,\eta_0+r],\ v\in\overline{B(v_0,r)},\ \tau\,,\tau'\in[\tau_0-r,\tau_0+r]$$

if $n \geq 2$. Notice also that in both cases there exists

$$\frac{\partial\phi}{\partial\tau}\in L^{\infty}(\omega) \tag{4.6}$$

in the sense of distributions. Moreover, through a standard approximation argument by convolution,

$$\frac{\partial\phi^2}{\partial\tau}=2\phi\frac{\partial\phi}{\partial\tau}\ \in L^{\infty}(\omega) \tag{4.7}$$

in the sense of distributions. Let us recall now that by Corollary 3.6 ϕ is also a distributional solution of (4.5). By (3.35) and (4.7) there exists

$$\frac{\partial\phi}{\partial\eta}=w_{n+1}-\frac{1}{2}\frac{\partial\phi^2}{\partial\tau}\ \in L^{\infty}(\omega)\,.$$

Meanwhile, by (4.6) and (3.34), there exist

$$\frac{\partial\phi}{\partial v_j}=w_j+\frac{v_{j+n}}{2}\frac{\partial\phi}{\partial\tau}\in L^{\infty}_{loc}(\omega), \qquad \frac{\partial\phi}{\partial v_{j+n}}=w_{j+n}-\frac{v_j}{2}\frac{\partial\phi}{\partial\tau}\in L^{\infty}_{loc}(\omega).$$

□

Let us deal now with the case $n \geq 2$.

Theorem 4.3. *Let $\omega \subseteq \mathbb{R}^{2n}$ be an open set with $n \geq 2$, let $\phi : \omega \to \mathbb{R}$, $w = (w_2, ..., w_{n+1}, ..., w_{2n}) : \omega \to \mathbb{R}^{2n-1}$. Let us assume*

i: *$\phi \in L^{\infty}_{loc}(\omega)$, $w_i \in L^{\infty}_{loc}(\omega)$ $\forall\, i = 2, ..., 2n$ and, for some $i_0 = 2, ..., n$, there exists*

$$\widetilde{X}_{i_0}w_{i_0+n}-\widetilde{Y}_{i_0}w_{i_0}\ \in L^{\infty}_{loc}(\omega) \tag{4.8}$$

in the sense of distributions;

ii: *ϕ is a distributional solution of the system* (1.1).

Then $\phi \in Lip_{loc}(\omega)$.

Proof. Because of the commutator relation $[\widetilde{X}_{i_0}, \widetilde{Y}_{i_0}] = \widetilde{T}$, there exists

$$\frac{\partial \phi}{\partial \tau} = \widetilde{X}_{i_0} w_{i_0+n} - \widetilde{Y}_{i_0} w_{i_0} \quad \in L^\infty_{loc}(\omega) \tag{4.9}$$

in the sense of distributions. By (4.9), there exist, for $j = 2, ..., n$,

$$\frac{\partial \phi}{\partial v_j} = w_j + \frac{v_{j+n}}{2}\frac{\partial \phi}{\partial \tau} \in L^\infty_{loc}(\omega), \qquad \frac{\partial \phi}{\partial v_{j+n}} = w_{j+n} - \frac{v_j}{2}\frac{\partial \phi}{\partial \tau} \in L^\infty_{loc}(\omega).$$

in the sense of distributions. Arguing now as in the proof of (4.7), there exists

$$\frac{\partial \phi^2}{\partial \tau} = 2\phi \frac{\partial \phi}{\partial \tau} \quad \in L^\infty_{loc}(\omega)\,. \tag{4.10}$$

in sense of distributions. Then $\dfrac{\partial \phi}{\partial \eta} = w_{n+1} - \dfrac{1}{2}\dfrac{\partial \phi^2}{\partial \tau} \quad \in L^\infty_{loc}(\omega)$. □

Corollary 4.4. *Following the same assumptions of Theorem* 4.3*, let us replace* (4.8) *with*

$$w_j \in C^k(\omega) \tag{4.11}$$

for $j = 2, \dots, 2n$, and some integer $k \geq 1$. Then $\phi \in C^k(\omega)$.

Proof. By Theorem 4.3, (4.11) and (4.9) $\phi \in Lip_{loc}(\omega)$ and there exists

$$\frac{\partial \phi}{\partial \tau} = \widetilde{X}_{i_0} w_{i_0+n} - \widetilde{Y}_{i_0} w_{i_0} \quad \in C^{k-1}(\omega) \tag{4.12}$$

As in the proof of Theorem 4.3, there exists for $j = 2. \dots, n$

$$\frac{\partial \phi}{\partial v_j} = w_j + \frac{v_{j+n}}{2}\frac{\partial \phi}{\partial \tau} \in C^{k-1}(\omega), \qquad \frac{\partial \phi}{\partial v_{j+n}} = w_{j+n} - \frac{v_j}{2}\frac{\partial \phi}{\partial \tau} \in C^{k-1}(\omega) \tag{4.13}$$

in the sense of distributions. In order to complete the proof we need to show there exists

$$\frac{\partial \phi}{\partial \eta} \in C^{k-1}(\omega) \tag{4.14}$$

in the sense of distributions. In fact, from (4.12), (4.13) and (4.14), through a standard approximation argument by convolution, it follows that $\phi \in C^k(\omega)$. Let us prove (4.14). As in (4.10), we need only to prove there exists

$$\frac{\partial \phi^2}{\partial \tau} = 2\phi \frac{\partial \phi}{\partial \tau} \quad \in C^{k-1}(\omega)$$

This, for instance, follows by induction with respect to k. □

Remark 4.5. The example given in the introduction shows that Corollary 4.4 falls down when $n = 1$.

Proof of Theorem 1.4. : We need only to prove that

$$\frac{\partial \phi}{\partial \tau} \quad \in C^0(\omega)\,. \tag{4.15}$$

Indeed, by (4.15) and arguing as in the proof of Corollary 4.4, we obtain $\phi \in C^1(\omega)$. We restrict to deal with the linear system $\widetilde{\nabla}_{\mathbb{H}}\phi = \hat{w}_{n+1}$ in ω, where $\widetilde{\nabla}_{\mathbb{H}}$ is the family of vector fields defined in (2.8) and $\hat{w}_{n+1} := (w_2, \dots, w_n, w_{n+2}, \dots, w_{2n})$. Without loss of generality, we can suppose that $\omega = \mathbb{R}^{2n}$. Otherwise, for a fixed open set $\omega' \Subset \omega$, let $\chi \in C_c^\infty(\omega)$ be a cut- off function such that $\chi \equiv 1$ in ω'. Then we can replace ϕ and $\hat{w}_{n+1}$ by $\phi^* := \chi\,\phi \in Lip(\mathbb{R}^{2n})$ and $\hat{w}^*_{n+1} := (w^*_2, \dots, w^*_n, w^*_{n+2}, \dots, w^*_{2n})$ where $w^*_j := \chi\, w_j + \widetilde{X}_j \chi\, \phi \in Lip(\mathbb{R}^{2n})$ if $j = 2, \dots n$ and $w^*_j := \chi\, w_j + \widetilde{Y}_j \chi\, \phi \in Lip(\mathbb{R}^{2n})$. Moreover we can suppose that $\widetilde{\nabla}_{\mathbb{H}}\phi(A) = \hat{w}_{n+1}(A)$ for all $A \in \mathbb{R}^{2n}$ since w is continuous. We split the proof in four steps.

Step 1: We observe that there exist

$$\left(\widetilde{X}_j \frac{\partial \phi}{\partial \tau}, \widetilde{Y}_j \frac{\partial \phi}{\partial \tau}\right) = \left(\frac{\partial w_j}{\partial \tau}, \frac{\partial w_{j+n}}{\partial \tau}\right) \in \left(L^\infty(\mathbb{R}^{2n})\right)^2 \tag{4.16}$$

in the sense of distributions, for $j = 2, ..., n$.

Step 2: Fix $\eta \in \mathbb{R}$ and define $u_\eta(v, \tau) := \dfrac{\partial \phi}{\partial \tau}(\eta, v, \tau)$ for $(v, \tau) \in \mathbb{R}^{2n-1}$. By (4.16) and Theorem 2.4, we obtain that

$$u_\eta \in Lip_{\mathbb{H}}\left(\mathbb{H}^{n-1}\right) \quad \forall \eta \in \mathbb{R}\,, \tag{4.17}$$

where $Lip_{\mathbb{H}}(\mathbb{H}^{n-1})$ denotes the space of intrinsic locally Lipschitz functions in $\mathbb{H}^{n-1}$, with respect to the distance (1.5) d_∞ in $\mathbb{H}^{n-1} \equiv \mathbb{R}^{2n-1}_{(v,\tau)}$ and

$$\left\|\left(\widetilde{X}_j u_\eta, \widetilde{Y}_j u_\eta\right)\right\|_{(L^\infty(\mathbb{H}^{n-1}))^2} \le \left\|\left(\frac{\partial w_j}{\partial \tau}, \frac{\partial w_{j+n}}{\partial \tau}\right)\right\|_{(L^\infty(\mathbb{R}^{2n}))^2} < \infty \qquad \forall \eta \in \mathbb{R}. \tag{4.18}$$

Observe also that $\dfrac{\partial \phi}{\partial \tau}(\eta, \cdot, \cdot) \in C^0(\mathbb{H}^{n-1})$ $\forall \eta \in \mathbb{R}$. In fact, by (4.17) and Remark 2.3, it follows that $u_\eta \in Lip_{\mathbb{H}}(\mathbb{H}^{n-1}) \subseteq C^0(\mathbb{H}^{n-1})$.

Step 3: Let us prove that, for every fixed $(v, \tau) \in \mathbb{H}^{n-1}$, $\dfrac{\partial \phi}{\partial \tau}(\cdot, v, \tau) \in C^0(\mathbb{R})$. We need only to show that if $\eta_h \to \eta_0$ then $\dfrac{\partial \phi}{\partial \tau}(\eta_h, v, \tau) \to \dfrac{\partial \phi}{\partial \tau}(\eta_0, v, \tau)$. Because of $\widetilde{X}_j\phi(\eta_h, v, \tau) = w_j(\eta_h, v, \tau)$ and $\widetilde{Y}_j\phi(\eta_h, v, \tau) = w_{j+n}(\eta_h, v, \tau)$, then, $\mathcal{L}^{2n-1} - a.e.\, (v, \tau) \in \mathbb{H}^{n-1}$,

$$\frac{\partial \phi}{\partial \tau}(\eta_h, v, \tau) = \left(\widetilde{X}_j\widetilde{Y}_j\phi - \widetilde{Y}_j\widetilde{X}_j\phi\right)(\eta_h, v, \tau) = \widetilde{X}_j w_{j+n}(\eta_h, v, \tau) - \widetilde{Y}_j w_j(\eta_h, v, \tau). \tag{4.19}$$

Let us define, for $(v, \tau) \in \mathbb{H}^{n-1}$ and a fixed $j \in \{2, ..., n\}$, $w_h(v, \tau) = \widetilde{X}_j w_{j+n}(\eta_h, v, \tau) - \widetilde{Y}_j w_j(\eta_h, v, \tau)$. The sequence $(w_h)_h \subseteq L^\infty(\mathbb{H}^{n-1})$ and $\sup_{h\in\mathbb{N}} ||w_h||_{L^\infty(\mathbb{H}^{n-1})} < \infty$, then there exists $w^* \in L^\infty(\mathbb{H}^{n-1})$ such that, up to a subsequence, $w_h \to w^*$ in $L^\infty(\mathbb{H}^{n-1})$ -weak*. We show now that, $\mathcal{L}^{2n-1} - a.e.\, (v, \tau) \in \mathbb{H}^{n-1}$,

$$w^*(v, \tau) = \widetilde{X}_j w_j(\eta_0, v, \tau) - \widetilde{Y}_j w_{j+n}(\eta_0, v, \tau) = \frac{\partial \phi}{\partial \tau}(\eta_0, v, \tau)\,. \tag{4.20}$$

Using the definition of weak*- convergence, $\forall\, \varphi \in C^1_c(\mathbb{H}^{n-1})$

$$\int_{\mathbb{H}^{n-1}} w^*(v, \tau)\varphi(v, \tau)\, dv\, d\tau = \lim_h \int_{\mathbb{H}^{n-1}} w_h(v, \tau)\varphi(v, \tau)\, dv\, d\tau =$$

$$= \lim_h \int_{\mathbb{H}^{n-1}} \left[\left(\widetilde{X}_j w_{j+n}\right)(\eta_h, v, \tau) - \left(\widetilde{Y}_j w_j\right)(\eta_h, v, \tau)\right] \varphi(v, \tau)\, dv\, d\tau =$$

$$= -\lim_h \int_{\mathbb{H}^{n-1}} \Big[w_{j+n}(\eta_h, v, \tau)\widetilde{X}_j\varphi(v,\tau) - w_j(\eta_h, v, \tau)\widetilde{Y}_j\varphi(v,\tau)\Big]\, dv\, d\tau =$$

$$= -\int_{\mathbb{H}^{n-1}} \Big[w_{j+n}(\eta_0, v, \tau)\widetilde{X}_j\varphi(v,\tau) - w_j(\eta_0, v, \tau)\widetilde{Y}_j\varphi(v,\tau)\Big]\, dv\, d\tau =$$

$$= \int_{\mathbb{H}^{n-1}} \Big[\Big(\widetilde{X}_j w_{j+n}\Big)(\eta_0, v, \tau) - \Big(\widetilde{Y}_j w_j\Big)(\eta_0, v, \tau)\Big] \varphi(v,\tau)\, dv\, d\tau =$$

$$= \int_{\mathbb{H}^{n-1}} \frac{\partial \phi}{\partial \tau}(\eta_0, v, \tau)\varphi(v,\tau)\, dv\, d\tau$$

and so we obtain (4.20). Define $u_h(v,\tau) := u_{\eta_h}(v,\tau) = \dfrac{\partial\phi}{\partial\tau}(\eta_h, v, \tau)\ (v,\tau)\in\mathbb{H}^{n-1}$. By (4.19) and (4.20)

$$u_h \to u_{\eta_0} \qquad \text{in}\, L^\infty(\mathbb{H}^{n-1})\text{−weak}\ast\,. \tag{4.21}$$

Moreover with step 1 we understand that the sequence $(u_h)_h \subseteq Lip_{\mathbb{H}}(\mathbb{H}^{n-1})$ and

$$\sup_{\mathbb{H}^{n-1}} |u_h| \le \sup_{\mathbb{R}^{2n}} \left|\frac{\partial\phi}{\partial\tau}\right|,$$

$\exists\, L>0:\ \big|u_h(v,\tau) - u_h(v',\tau')\big| \le L d_\infty\big((v,\tau),(v',\tau')\big) \quad \forall\,(v,\tau),(v',\tau')\in\mathbb{H}^{n-1},\ \forall\, h\in\mathbb{N}.$
Referring to Arzelá- Ascoli's Theorem, up to a subsequence, there exists $u^*\in Lip_{\mathbb{H}}(\mathbb{H}^{n-1})$ such that

$$u_h \to u^* \qquad \text{uniformly on the compact sets of} \quad \mathbb{H}^{n-1}\,. \tag{4.22}$$

Using the uniqueness, (4.21) and (4.22) $u_{\eta_0} = u^*$ $\mathcal{L}^{2n-1}$-a.e. in $\mathbb{H}^{n-1}$. Moreover, because of $u_{\eta_0}, u^* \in C^0(\mathbb{H}^{n-1})$,

$$\frac{\partial\phi}{\partial\tau}(\eta_0, v, \tau) = u^*(v,\tau) \qquad \forall\,(v,\tau)\in\mathbb{H}^{n-1}\,. \tag{4.23}$$

From (4.22) and (4.23) we have the desired result.
Step 4: Let us show (4.15). We shall prove that for each sequence $((\eta_h, v_h, \tau_h))_h \subset \mathbb{R}^{2n}$ with $(\eta_h, v_h,\tau_h)\to(\eta_0, v_0,\tau_0)$, then $\lim_{h\to\infty} \dfrac{\partial\phi}{\partial\tau}(\eta_h, v_h,\tau_h) = \dfrac{\partial\phi}{\partial\tau}(\eta_0, v_0,\tau_0)$. Observe that

$$\frac{\partial\phi}{\partial\tau}(\eta_h, v_h,\tau_h) - \frac{\partial\phi}{\partial\tau}(\eta_0, v_0,\tau_0) =$$

$$= \left(\frac{\partial\phi}{\partial\tau}(\eta_h, v_h,\tau_h) - \frac{\partial\phi}{\partial\tau}(\eta_h, v_0,\tau_0)\right) + \left(\frac{\partial\phi}{\partial\tau}(\eta_h, v_0,\tau_0) - \frac{\partial\phi}{\partial\tau}(\eta_0, v_0,\tau_0)\right) = I_h^{(1)} + I_h^{(2)}.$$

By step 2, there exists $L>0$ such that $\forall\,(v,\tau),(v',\tau')\in\mathbb{H}^{n-1}$, $\forall\eta\in\mathbb{R}$

$$\left|\frac{\partial\phi}{\partial\tau}(\eta, v,\tau) - \frac{\partial\phi}{\partial\tau}(\eta, v',\tau')\right| \le\ L\, d_\infty\big((v,\tau),(v',\tau')\big)\,.$$

Thus $\lim_{h\to 0} I_h^{(1)} = 0$ and step 3 implies $\lim_{h\to 0} I_h^{(2)} = 0$ as well. □

Proof of Corollary 1.5: This follows by applying, respectively, Theorems 1.2, 1.4 and 4.3 and Corollaries 3.6 and 4.11. □

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