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Density constraints in Optimal Transport, PDEs and Mean Field Games

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To Tímea, for her endless love

Abstract

Motivated by some questions raised by F. Santambrogio (see [San12b]), this thesis is devoted to the study of *Mean Field Games* and models involving *optimal transport* with *density constraints*.

To study second order MFG models in the spirit of [San12b], as a possible first step we introduce and show the well-posedness of a diffusive crowd motion model with density constraints (generalizing in some sense the works by B. Maury *et al.*, see [MRCS10, RC11]). The model is described by the evolution of the people's density, that can be seen as a curve in the Wasserstein space. From the PDE point of view, this corresponds to a modified Fokker-Planck equation, with an additional gradient of a pressure (only living in the saturated zone $\{\rho = 1\}$) in the drift. We provide a uniqueness result for the pair density and pressure (ρ, p) by passing through the dual equation and using some well-known parabolic estimates.

Initially motivated by the *splitting algorithm* (used for the above existence result), we study some fine properties of the Wasserstein projection below a given threshold. Embedding this question into a larger class of variational problems involving optimal transport, we show BV estimates for the optimizers. Other possible applications (for partial optimal transport, shape optimization and degenerate parabolic problems) of these BV estimates are also discussed.

Changing the point of view, we also study *variational* Mean Field Game models with density constraints. In this sense, the MFG systems are obtained as first order optimality conditions of two convex problems in duality. In these systems an additional term appears, interpreted as a price to be paid when agents pass through saturated zones. Firstly, profiting from the regularity results of elliptic PDEs, we give the existence and characterization of the solutions of stationary second order MFGs with density constraints. As a byproduct we characterize the subdifferential of a convex functional introduced initially by Benamou-Brenier (see [BB00]) to give a dynamic formulation of the optimal transport problem. Secondly, (based on a penalization technique) we prove the well-posedness of a class of first order evolutive MFG systems with density constraints. An unexpected connection with the incompressible Euler's equations *à la* Brenier is also given.

Résumé

Movité par des questions posées par F. Santambrogio (voir [San12b]), cette thèse est dédiée à l'étude de *jeux à champ moyen* et des modèles impliquant le *transport optimal* avec *contraintes de densité*.

A fin d'étudier des modèles de MFG d'ordre deux dans l'esprit de [San12b], on introduit en tant que brique élementaire un modèle diffusif de mouvement de foule avec contraintes de densité (en généralisant dans une sense les travaux de Maury *et al.*, voir [MRCS10, RC11]). Le modèle est décrit par l'évolutions de la densité de la foule, qui peut être vu comme une courbe dans l'espace de Wasserstein. Du point de vu EDP, ça correspond à une équation de Fokker-Planck modifiée, avec un terme supplémentaire, le gradient d'une pression (seulement dans la zone saturée $\{\rho = 1\}$) dans le drift. En passant par l'équation duale et en utilisant des estimations paraboliques bien connues, on démontre l'unicité du pair densité et pression (ρ, p).

Motivé initialement par *l'algorithm de splitting* (utilisé dans le résultat d'existence ci-dessus), on étudie des propriétés fines de la projection de Wasserstein en dessous d'un seuil donné. Intégrant cette question dans une classe plus grande de problèmes impliquant le transport optimal, on démontre des estimations BV pour les optimiseurs. D'autres applications possibles (en transport partiel, optimisation de forme et problèmes paraboliques dégénérés) de ces estimations BV sont également discutées.

En changeant de point de vu, on étudie également des modèles de MFG *variationnels* avec contraintes de densité. Dans ce sens, les systèmes de MFG sont obtenus comme conditions d'optimalité de premier ordre pour deux problèmes convexes en dualité. Dans ces systèmes un terme additionnel apparaît, interpreté comme un prix à payer quand les agents passent dans des zones saturées. Premièrement, en profitant des résultats de régularité elliptique, on montre l'existence et la caractérisation de solutions des MFG de deuxième ordre stationnaires avec contraintes de densité. Comme résultat additionnel, on caractérise le sous-différentiel d'une fonctionnelle introduite par Benamou-Brenier (voir [BB00]) pour donner une formulation dynamique du problème de transport optimal. Deuxièmement, (basé sur une technique de pénalisation) on montre qu'une classe de systèmes de MFG de premier ordre avec contraintes de densité est bien posée. Une connexion inattendu avec les équations d'Euler incompressible *à la* Brenier est égalment donnée.

Preface

My adventure in mathematics at Orsay started with a *modest proposal* of Filippo Santambrogio. On that day, the enthusiasm and brightness of Filippo definitely convinced me to pursue a PhD under his guidance in the fascinating subjects of optimal transport, mean field games and partial differential equations. Et voilà, it is almost the end of this chapter of my adventure. It has been a wonderful three years. I believe I improved a lot both mathematically and personally. I hope that I also learned how to be more rigorous and to take care of all the little details. Still, there will be always room for improvements. There will be always interesting mathematical problems and theories to study and a new language to learn.

The problems that I studied have been (and still are) challenging and the steps towards to the understanding have been difficult sometimes. However, I am very happy that I could understand them a bit and that I can see now some portions of the big picture. I also thank Filippo for this very nice subject and especially for his trust in me (that motivated a lot, even in the moments when I had less self-confidence).

What I learned besides the theories is, that mathematics beyond its beauty, it is not just about calculations on any piece of paper or formulas on blackboards, but it is also about people. During these three years I had the chance to meet and discuss with many amazing people. I was always excited during these discussion anywhere next to a coffee, in a bar, on a train or during a Skype conversation. I am grateful for the ideas I have been shared, and advises I have been given.

I am very thankful for being part of this mathematical family.

Orsay, *Alpár Richárd Mészáros*
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I am deeply thankful to Guillaume Carlier and Wilfrid Gangbo for accepting to be referees of this thesis, reading carefully the manuscript in just a couple of weeks. I thank them all the discussions we had and all the advices they gave concerning the manuscript and other problems as well. I also thank Guillaume for the (large amount of) recommendation letters and Prof. Gangbo for his encouragement and trust in me, which led me to apply for postdoc positions in the US.

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The day to day life in Orsay was very pleasant in the Mathematics Department. I am especially thankful for the PDE team and my office mates in 256-258. I thank the seniors, Jean, Nicolas, Nina, Pierre and Tony for helping me to integrate in this community. I thank Anthony, Clémentine, Cristèle, Fatima, Loïc, Paul and Xia Bo for the good humor, nice atmosphere and the recent cookie-tasting sessions (for which I still owe you one). Special thanks go to Antonin and Yueyuan. Thank you Antonin (and Céline) for your trip to Transylvania for our wedding and also your availability and help with my administrative questions. Thank you Yueyuan for your friendship, for the two football matches watched in the two Parisian stadiums and for the walks towards home after some longer stays in the office. I am also thankful for the 'question du midi' semi-math sessions to Pierre, Antonin and Paul. I also thank Catherine Poupon and Valérie Lavigne for their help with administrative issues.

My mathematical studies started back in Cluj Napoca, where I met some people to whom today I am thankful for choosing this path. I am especially thankful to Szilárd András who led my very first steps in mathematics and to Sándor Kristály who encouraged, tutored and gave advises several times when I needed.

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Meg szeretném köszönni szüleimnek, hogy felneveltek, hogy minden döntésemben támogattak és távolról is mindig mellettem álltak. Öcsémnek, Mózesnek is köszönetet mondok, mert mindig testvérként állt mellettem. Köszönöm apósomnak, anyósomnak és Hunornak is a támogatásukat.

At last, but not least I would like to thank my wife, Tímea her endless and unconditioned love and constant support. I am thankful also for her patience even when I was telling her my mathematical problems at home.

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Contents

Introduction Générale

Dans cette thèse on étudie des modèles différens venant du transport optimal, des équations aux dérivées partielles décrivant le mouvement des foules et des jeux à champ moyen. Dans tous ces modèles — comme le titre le souligne — le dénominateur commun est la notion de *contrainte de densité*. Les contraintes de densité apparaissent naturellement lorsqu'on veut modéliser les effets de congestion. On peut imaginer la situation suivante : nous venons de construire un nouveau département de mathématiques (comme il sera le cas à Orsay). Pour des raisons de sécurité on veut concevoir un dispositif qui dit aux gens comment évacuer le bâtiment de façon 'optimale' en cas d'une urgence. Dans ce contexte 'optimale' veut dire pas seulement de la manière la plus rapide/plus courte possible, mais on prend en considération les effets possibles de congestion aussi. Ceci est tout à fait une question importante, parce que dans des endroits étroits (par exemple à côté des portes) nous pouvons habituellement s'attendre à avoir une concentration plus élevée de personnes. Par conséquent, si notre dispositif pouvait prendre aussi en cosidération la contrainte que à chaque instant et chaque endroit du bâtiment la densité de la population reste en dessous d'un seuil donné (par exemple 5,4 personnes sur chaque mètre carré, une valeur qui est généralement utilisée dans les applications), la procédure d'évacuation serait parfaite.

La réalisation possible d'un dispositif comme ça serait sûrement une tâche difficile. L'une des principales raisons est que dans un cas d'une situation d'urgence, les gens ont la tendance à oublier de réfléchir rationnellement. Néanmoins, du point de vue mathématique cela crée des questions très intéressantes et non triviales. Décrit mathématiquement, le dispositif ci-dessus pourrait fonctionner de la façon suivante : comme entrée il reçoit à chaque instant la densité et le *champ de vitesse souhaité* de la population, et retourne un nouveau champ de vitesse (donc il dit aux gens à quelle vitesse et dans quelle direction aller). Ce nouveau champ de vitesse est construit de façon à ce que personne ne soit autorisé à se déplacer de manière telle que la densité dépasse le seuil de saturation. La vitesse souhaitée est supposée connue (elle peut dépendre de la distance de la porte la plus proche, etc.) et elle est la même pour tous.

À ce point là, on remarque que dans tous nos modèles à venir, nous allons décrire le mouvement d'une foule/population d'agents par l'évolution de leur densité. Donc, on considère des modèles macroscopiques.

Au cours des dernières années, de nombreux modèles différents ont été proposés pour étudier les mouvements de foule avec effets de congestion. En plus, ces modèles peuvent parfois servir de base pour comprendre certains phénomènes venant de la biologie (tels que la migration cellulaire, la croissance tumorale, la formation de structures et textures), la physique des particules ou de l'économie. Pour une liste non exhaustive de bibliographie dans ce cadre nous nous référons à [Cha07, CR05, CPT14b, Dog08, Hel92, HM95, Hug02, Hug03, MV07, MRCS10, MRCSV11, MRCS14, RC11, PQV14, AKY14].

Dans ce contexte, la situation décrite précédemment modélise un soi-disant effet de *congestion forte* (nous nous référons par exemple à [MRCS10, MRCSV11, MV07, RC11]). Dans ce sens, le champ de vitesse souhaité doit être modifié pour éviter les zones très concentrées. Des modèles similaires, avec des effets de *congestion douce* existent également. Dans ceux-ci, les gens ralentissent dès que la densité de la zone se rapproche du seuil (au lieu d'être affectés seulement quand ils sont dans une zone totalement saturée).

Les modèles étudiés dans cette thèse sont motivés par le premier type de considération, par les modèles macroscopiques de mouvement de foule avec des effets de congestion forte. Le pilier central de notre analyse est la théorie du *transport optimal*. Cette théorie est très puissante et elle nous permet d'étudier et de comprendre plusieurs phénomènes liés à des contraintes de densité dans différents modèles de manière unifiée.

La genèse de cette thèse a commencé avec le *modest proposal* de F. Santambrogio (voir [San12b]). Il a proposé un modèle de *jeux à champ moyen* (voir [LL06a, LL06b, LL07, Lio08]), où on impose une contrainte de densité. Nous allons présenter la théorie de MFG plus en détail en début de la Partie ii. Néanmoins, nous soulignons que dans les modèles de MFG *les agents* jouent un jeu différentiel non coopératif, où chacun doit choisir une stratégie. Par conséquent, dans ces modèles on veut pas seulement comprendre l'évolution de la densité de la population, mais on veut décrire la fonction valeur et la stratégie optimale de chaque agent aussi. Les modèles de [San12b] visent à généraliser ceux du mouvement de foule (discutés auparavant) dans le cas où les gens sont *stratégiques*. Dans [San12b], le modèle a seulement été construit, et aucun résultat rigoureux n'a pas été fourni. Étant également le sujet du mémoire de M2 (voir [Més12]), il s'est avéré que les questions soulevées dans [San12b] sont loin d'être triviales. Une des raisons est la faible régularité que l'on pouvait espérer pour la fonction valeur qui résout une équation de Hamilton-Jacobi-Bellman de premier ordre. Cela nous a empêché de construire un schéma de point fixe raisonnable (une technique utilisée avec succès dans de nombreux autres modèles de MFG), en tenant compte également de la *pression*, la nouvelle variable venant en dualité avec la contrainte de densité.

Une première tentative pour résoudre ce problème, était l'étude d'un modèle diffusif, où une diffusion non dégénérée est incluse dans l'équation de HJB et dans l'équations de continuité (transformant ce dernier en une équation de Fokker-Planck). Comme première étape, cela nécessitait l'étude de l'existence et de l'unicité d'une solution de l'équation de Fokker-Planck avec contrainte de densité. Cet objectif a été atteint avec succès et il a conduit à un nouveau modèle diffusif macroscopique de mouvement de foule avec contrainte de densité (ce sera l'objet du Chapitre 2 et le Chapitre 3 ; il a également fait l'objet de deux papiers, voir [MS15a, DMM15]). À un certain point dans l'analyse effectuée dans [MS15a], nous avions besoin de certaines estimations plus fines sur les mesures projetées en dessous d'un certain seuil dans le sens Wasserstein. Plus précisément, une estimation BV sur les mesures projetées nous permettrait d'obtenir des résultats de compacité pour certaines courbes dans l'espace Wasserstein, construits par un schéma de type *splitting*. Par ceci, on pouvait démontrer la convergence de l'algorithme et donc le résultat d'existence. Les estimations BV ont été réalisées non seulement pour les mesures projetées, mais pour les optimiseurs d'une grande classe de problèmes variationnels impliquant le transport optimal (ce qui est le sujet du Chapitre 1 et de [DPMSV15]).

En parallèle de la direction présentée ci-dessus, nous avons étudié les questions soulevées dans [San12b] aussi d'un point de vue différent. Nous avons étudié deux types de modèles de MFG possédant une *structure variationnelle*. Dans ces deux modèles, les approches utilisées sont liés par leur formulation variationnelle. Les approches utilisées dans les deux modèles rappellent celle étudiée par J.-D. Benamou et Y. Brenier (voir [BB00]) pour donner une formulation dynamique du problème de transport optimal de Monge-Kantorovich.

Premièrement, nous avons montré le caractère bien-posé et on a caractérisé les solutions de certains systèmes de MFG diffusifs stationnaires sous contraintes de densité. L'effet régularisant de l'équation de Fokker-Planck stationnaire et la structure elliptique nous ont permis d'imposer la contrainte de densité (que nous avons montré être qualifiée) directement au niveau du problème d'optimisation. Cela fait l'objet du Chapitre 4 (voir aussi [MS15b]).

Deuxièmement, nous avons montré le caractère bien-posé des systèmes de MFG évolutifs de premier ordre avec contraintes de densité. Ici, nous avons obtenu la contrainte de densité par la limite de certaines pénalisations (une procédure initialement proposée également dans [San12b]). De plus, nous avons obtenu un lien surprenant entre notre modèle de MFG avec des contraintes de densité et les équations d'Euler décrivant le mouvement des fluides parfaits incompressibles (voir [Bre99, AF09]). Ces modèles font l'objet du Chapitre 5 (voir aussi [CMS15]). Notons que les systèmes de MFG (obtenus comme conditions d'optimalité des problèmes variationnels correspondants) présentés dans la Partie ii montrent quelques différences par rapport aux originaux dérivés formellement dans [San12b]. Cela est dû à la différence de l'interprétation du champ de pression, considéré comme multiplicateur de Lagrange pour la contrainte de densité.

Maintenant, nous allons décrire en détail les principaux résultats mathématiques inclus dans cette thèse. Nous allons voir comment ils sont présentés par rapport aux chapitres aussi. Chaque chapitre se concentre essentiellement sur un papier. Ceux-ci sont soit acceptés pour publication, soit soumis ou en préparation.

DESCRIPTION MATHÉMATIQUE DES RÉSULTATS

Chapitres 1, 2 et 3 construisent la Partie i de la thèse et ils contiennent les résultats sur les modèles venant purement de transport optimal et les mouvements de foule macroscopiques avec des contraintes de densité.

Le Chapitre 1 est basé sur un travail commun avec G. De Philippis, F. Santambrogio et B. Velichkov (voir [DPMSV15]). Ici, notre objectif principal était d'étudier certaines propriétés fines de l'opérateur de projection dans l'espace Wasserstein $\mathbb{W}_2(\Omega), (\Omega \subseteq \mathbb{R}^d)$. En fait, nous incorporons cette question dans un ensemble plus large de problèmes. Notamment, nous étudions certaines propriétés quantitatives et la régularité des minimiseurs du problème d'optimisation

$$\min_{\varrho \in \mathscr{P}_2(\Omega)} \frac{1}{2} W_2^2(\varrho, g) + \tau F(\varrho),$$

où W_2 indique la distance 2-Wasserstein sur $\mathscr{P}_2(\Omega)$, $F : \mathscr{P}_2(\Omega) \to \mathbb{R}$ est une fonctionnelle donnée, $\tau > 0$ est un paramètre qui peut éventuellement être petit, et g est une probabilité donnée dans $\mathcal{P}_2(\Omega)$ (l'espace de mesures de probabilité sur $\Omega \subseteq \mathbb{R}^d$ avec deuxième moment fini $\int_\Omega |x|^2 \, d\varrho(x) < +\infty$). Le problème ci-dessus peut être reconnu comme une étape dans la discrétisation en temps (τ étant le paramètre de discrétisation, dans ce cas) du flot-gradient de la fonctionnelle F, où $g = \varrho_k^\tau$ est une mesure précédemment construite et le ϱ optimal est en effet la suivante. Les algorithmes, où chaque pas de temps a la forme du problème d'optimisation ci-dessus, sont généralement appelés des schémas de JKO dans la communauté du transport optimal (voir [JKO98]). Sous des hypothèses appropriées, à la limite lorsque $\tau \to 0$, la suite de mesures optimales converge vers une courbe de mesures qui est le flot-gradient de F. Notons ici que pour les flots-gradient, le paramètre de discrétisation τ tend vers zéro, donc, en ce qui concerne les estimations sur les optimiseurs, on peut vouloir les obtenir de manière indépendante de τ. Nous pouvons imaginer d'autres modèles qui rentrent dans le cadre du problème d'optimisation ci-dessus: g pourrait représenter certaines ressources, et ϱ la répartition des usines autour d'eux; g la distribution de certains magasins/banques/écoles, etc., et ϱ la répartition des personnes. D'autres modèles sophistiqués sur la planification urbaine, le traitement d'images, etc. existent aussi. Ici en général $\tau > 0$ est fixé.

Notre premier objectif était d'étudier le comportement de l'opérateur de projection, plus précisément les mesures projetées en-dessous d'un certain seuil. C'est bien le cas du problème ci-dessus, si on prend formellement F la fonction d'indicatrice de l'ensemble $\mathcal{K}_f := \{\rho \in \mathscr{P}_2(\Omega) : \rho \leq f\,\mathrm{d}x\}$, où f est une fonction positive avec $\int_\Omega f(x)\,\mathrm{d}x \geq 1$. Dans les applications, en général, il est raisonnable de choisir f constant. Puisque ce type de problème ne nécessite pas une dépendance en $\tau > 0$, nous choisisons simplement $\tau = 1$.

Nos principaux résultats dans ce chapitre sont :

Théorème 0.0.1. *Soit $\Omega \subset \mathbb{R}^d$ un ensemble convexe (éventuellement non-borné), soit $h : \mathbb{R}_+ \to \mathbb{R} \cup \{+\infty\}$ une fonction convexe et s.c.i. et $g \in \mathscr{P}_2(\Omega) \cap BV(\Omega)$. Si $\bar{\varrho}$ est l'optimiseur du problème variationnel suivant*

$$\min_{\varrho \in \mathscr{P}_2(\Omega)} \frac{1}{2} W_2^2(\varrho, g) + \int_\Omega h(\varrho(x))\,\mathrm{d}x\,,$$

alors

$$\int_\Omega |\nabla \bar{\varrho}|\,\mathrm{d}x \leq \int_\Omega |\nabla g|\,\mathrm{d}x\,. \tag{0.0.1}$$

Par un argument d'approximation (approximation de la fonction d'indicatrice de l'ensemble $\mathcal{K}_f$ par des fonctionnelles convexes s.c.i.), le résultat ci-dessus vaut en particulier pour le problème de projection sous la forme suivante:

Théorème 0.0.2. *Soit $\Omega \subset \mathbb{R}^d$ un ensemble convexe (possiblement non-borné), $g \in \mathscr{P}_2(\Omega) \cap BV(\Omega)$ et soit $f \in BV_{\mathrm{loc}}(\Omega)$ une fonction avec $\int_\Omega f\,\mathrm{d}x \geq 1$. Si*

$$\bar{\varrho} = \operatorname{argmin}\left\{ W_2^2(\varrho, g) \;:\; \varrho \in \mathscr{P}_2(\Omega),\ \varrho \in \mathcal{K}_f \right\}, \tag{0.0.2}$$

alors

$$\int_\Omega |\nabla \bar{\varrho}|\,\mathrm{d}x \leq \int_\Omega |\nabla g|\,\mathrm{d}x + 2\int_\Omega |\nabla f|\,\mathrm{d}x. \tag{0.0.3}$$

Dans le cas où $f \equiv 1$, on obtient un résultat sur la décroissance de la variation totale par l'opérateur de projection. Nous remarquons que la constante 2 dans l'inégalité (0.0.3) est sharp. Les estimations BV sont utiles lorsque la projection est traitée comme une étape d'une procedure d'évolution discrétisée. Par exemple, une borne BV permet de transformer la convergence faible en convergence forte dans L^1. Aussi, si nous considérons une EDP mélangeant une évolution lisse, comme l'évolution de Fokker-Planck, et certaines étapes de projection (pour imposer une contrainte de densité, comme dans les problèmes de mouvement de foule : nous allons décrire cela plus tard, et ce sera traité en détail dans le Chapitre 2), on pourrait se demander quelles bornes sur la régularité de la solution sont conservées en temps. Du fait que les discontinuités dans la mesure projetée détruisent tout type de norme $W^{1,p}$, c'est naturel de chercher des bornes BV.

Le cœur de la preuve des estimations BV précédentes est l'inégalité suivante. Prenons $\varrho, g \in \mathscr{P}_2(\Omega)$ et (φ, ψ) un couple de potentiels de Kantorovich dans le transport optimal de ϱ à g et $H : \mathbb{R}^d \to \mathbb{R}$ une fonction paire et convexe (nous ne nous soucions pas des hypothèses de régularité dans cette description heuristique). Alors, on a

$$\int_\Omega \nabla H(\nabla \varphi) \cdot \nabla \varrho + \nabla H(\nabla \psi) \cdot \nabla g \, \mathrm{d}x \geq 0.$$

Il semble que cette inégalité décrit certaines caractéristiques géométriques non triviales du problème de transport optimal entre ϱ et g, qui ne sont pas complètement comprises, sauf dans certains cas particuliers. Le cas $H(z) = |z|^2/2$, par exemple, est une conséquence de la convexité géodésique de la fonctionnelle entropie. Pour démontrer les estimations BV décrites ci-dessus, on utilise l'inégalité pour $H(z) = |z|$. Dans le même contexte, nous avons également donné une nouvelle preuve rigoureuse du fait que la mesure projetée sature la contrainte (ce qui est également connu dans le cadre du transport partiel). Plus précisément, il existe un optimiseur unique $\bar{\varrho}$ dans (0.0.2) et il existe un ensemble mesurable $B \subseteq \Omega$ tel que

$$\bar{\varrho} = g^{\mathrm{ac}} \mathbb{1}_B + f \mathbb{1}_{B^c}.$$

Notons que pour cette propriété, il n'y a pas besoin d'imposer la régularité BV sur g et f, qui est juste nécessaire pour l'estimation (0.0.3).

À la fin du chapitre, nous discutons des applications possibles (également sous forme de questions ouvertes) des estimations BV établies antérieurement. Premièrement, nous observons que certaines question de la théorie de *transport optimal partiel*, étudiée récemment par L.A. Caffarelli-R. McCann et A. Figalli (voir [CM10, Fig10], où l'objectif est de transporter de manière optimale seulement une partie donnée d'une mesure à l'autre) peuvent être formulées dans notre cadre (tels que la régularité de la frontière libre résultant quand on projette une mesure). Nos estimations BV peuvent être utiles dans l'étude du problème de transport partiel lui-même, qui peut être considéré en fait comme un problème de double projection. Nous discutons également d'autres applications possibles pour l'optimisation de forme et des problèmes d'évolution d'ensembles. Nous fournissons une nouvelle preuve, basée sur le transport optimal, de la décroissance de la variation totale pour les équations de diffusion dégénérés (comme l'équation des milieux poreux).

Dans le Chapitre 2 — basé sur un travail commun avec F. Santambrogio (voir [MS15a]) — nous proposons un nouveau modèle macroscopique de mouvement de foule avec contraintes de densité, soit avec congestion forte. Motivé par les modèles de premier ordre étudiés récemment, due à *Maury et al.* (voir

[MRCS10, MRCS14, MRCSV11]), on analyse un modèle de deuxième ordre. Du point de vue de la modélisation nous imposons un caractère aléatoire dans le mouvement des individus. Mathématiquement, cela peut être vu au niveau macroscopique comme une diffusion non dégénérée engendrée par un mouvement brownien et le modèle tout entier peut être décrit à l'aide d'une équation de Fokker-Planck 'modifiée'. Ici, le mot 'modifiée' fait référence au fait que l'on doit modifier le champ de vitesse des gens sur les zones saturées.

Nous décrivons notre modèle par l'évolution de la densité de la foule $[0,T] \ni t \mapsto \rho_t$, qui est une famille de mesures de probabilité dépendant du temps sur $\Omega \subset \mathbb{R}^d$ (un domaine borné et convexe avec bord lipschitzien). Il est donné un champ de vitesse spontané $\mathbf{u} : [0,T] \times \Omega \to \mathbb{R}^d$, qui représente la vitesse souhaitée que chaque personne suivrait en l'absence des autres. On impose seulement une regularité L^∞ sur ce champ. Pour équiper le modèle avec des contraintes de densité — $\rho \leq 1$ p.p. dans $[0,T] \times \Omega$, ce qui implique que nous devons imposer $\mathscr{L}^d(\Omega) > 1$ —, nous introduisons l'ensemble des vitesses admissibles. Ce sont les champs qui n'augmentent pas la densité sur les zones déjà saturées, donc formellement nous posons

$$\mathrm{adm}(\rho) := \left\{ \mathbf{v} : \Omega \to \mathbb{R}^d : \nabla \cdot \mathbf{v} \geq 0 \text{ sur } \{\rho = 1\} \text{ et } \mathbf{v} \cdot \mathbf{n} \leq 0 \text{ sur } \partial\Omega \right\}.$$

Maintenant, nous nous intéressons à la résolution de l'équation de Fokker-Planck modifiée

$$\begin{cases} \partial_t \rho_t - \Delta \rho_t + \nabla \cdot \left(\rho_t P_{\mathrm{adm}(\rho_t)}[\mathbf{u}_t] \right) = 0, \\ \rho(0,x) = \rho_0(x), \text{ in } \Omega, \end{cases} \tag{0.0.4}$$

où $P_{\mathrm{adm}(\rho)} : L^2(\Omega;\mathbb{R}^d) \to L^2(\Omega;\mathbb{R}^d)$ représente la projection L^2 sur l'ensemble convexe fermé $\mathrm{adm}(\rho)$ et ρ_0 est la densité initiale donnée de la foule. Observons que l'on pourrait se demander si nous devrions projeter le 'champ de vitesse tout entier' $-\nabla\rho_t/\rho_t + \mathbf{u}_t$. En fait, cela est la même que projeter seulement $\mathbf{u}_t$, parce que dans la région $\{\rho_t = 1\}$ on a $-\nabla\rho_t/\rho_t = 0$. Ainsi, le point principal est que ρ est advecté par un champ de vecteur, compatible avec les contraintes, le plus proche possible de celui spontané.

Malgré le fait que nous avons ajouté une diffusion non dégénérée au modèle, ce qui a un effet de régularisation, en raison de l'opérateur de projection, le nouveau champ de vitesses est très irrégulier (seulement L^2) et il dépend de manière non-locale de la densité elle-même. Ainsi, la théorie classique échouera dans l'analyse du problème (0.0.4). Pour traiter cette question, nous devons redéfinir l'ensemble des vitesses admissibles par dualité (comme cela a été fait pour les modèles du premier ordre, voir [MRCS10, RC11]):

$$\mathrm{adm}(\rho) = \left\{ \mathbf{v} \in L^2(\rho) : \int_\Omega \mathbf{v} \cdot \nabla p \leq 0,\ \forall p \in H^1(\Omega), p \geq 0, p(1-\rho) = 0 \text{ p.p.} \right\}.$$

À l'aide de cette formulation, nous avons toujours la décomposition orthogonale

$$\mathbf{u} = P_{\mathrm{adm}(\rho)}[\mathbf{u}] + \nabla p,$$

où

$$p \in \text{press}(\rho) := \left\{ p \in H^1(\Omega) : p \geq 0,\ p(1-\rho) = 0 \text{ p.p.} \right\}.$$

En effet, les cônes $\text{adm}(\rho)$ et $\nabla\text{press}(\rho)$ sont en dualité. Via cette approche le système (0.0.4) peut être réécrit comme un système pour (ρ, p) :

$$\begin{cases} \partial_t \rho_t - \Delta\rho_t + \nabla \cdot (\rho_t(\mathbf{u}_t - \nabla p_t)) = 0 \\ p \geq 0,\ \rho \leq 1,\ p(1-\rho) = 0,\ \rho(0,x) = \rho_0(x), \text{ in } \Omega. \end{cases} \tag{0.0.5}$$

Nous pouvons naturellement équiper ce système de ses conditions de Neumann naturelles sur le bord.

L'une des principales contributions du Chapitre 2 est le résultat d'existence pour le système (0.0.5). Ceci est réalisé par un algorithm bien choisi, discret en temps, de type *splitting*. Il procède comme suit : pour un pas de temps $\tau > 0$ nous construisons de manière récursive les mesures ρ_k^τ ($k \in \{0, \ldots, N\}$, où $N := [T/\tau]$) par notre *schéma principal.* Ce schéma est le suivant : on suit l'équation de Fokker-Planck sans contrainte pendant un temps τ avec donné initiale ρ_k^τ ; notons cette solution au temps τ par ρ_τ ; la nouvelle densité est ensuite construite comme $\rho_{k+1}^\tau := P_{\mathcal{K}_1}[\rho_\tau]$, où $P_{\mathcal{K}_1}$ désigne maintenant l'opérateur de projection 2-Wasserstein sur l'ensemble $\mathcal{K}_1 := \{\rho \in \mathscr{P}(\Omega) : \rho \leq 1 \text{ p.p.}\}$. Ensuite, il faut itérer ces deux étapes.

Nous continuons notre analyse en construisant des interpolations appropriées $\rho_t^\tau, t \in [0,T]$ entre les ρ_k^τ. Ces interpolations peuvent être considérées comme des courbes dans $\mathbb{W}_2(\Omega)$. Nous devons construire des vitesses $\mathbf{v}_k^\tau$ et des moments $\mathbf{E}_k^\tau$ discrets aussi. Pour démontrer la convergence quand $\tau \downarrow 0$, il faut des résultats de compacité pour les courbes ρ^τ. Ceux-ci reposent sur une comparaison standard entre le dérivé métrique en $\mathbb{W}_2(\Omega)$ et la dissipation de l'entropie le long des courbes ρ^τ. Par cela, on obtient la compacité dans l'espace $H^1([0,T]; \mathbb{W}_2(\Omega))$. Pour identifier l'équation limite quand $\tau \downarrow 0$, en fait, nous utilisons plusieurs interpolations entre les $\rho_k^\tau, \mathbf{v}_k^\tau$ et $\mathbf{E}_k^\tau$. Enfin, par cette procédure, on obtient l'existence d'un couple (ρ, p) qui satisfait le système (0.0.5) dans le sens des distributions.

Comme manière alternative d'obtenir la compacité pour les courbes ρ^τ, nous montrons des bornes uniformes en $\tau > 0$ dans l'espace $\text{Lip}([0,T]; \mathbb{W}_1(\Omega))$. Ce résultat est obtenu en combinant certaines estimations "sharp" BV pour l'équation de Fokker-Planck sans la contrainte de densité d'une part avec des estimations BV pour les mesures projetées (fournies au Chapitre 1) d'autre part. Puisque notre *schéma principal* consiste à suivre l'équation de Fokker-Planck sans contrainte, puis projeter sur l'ensemble $\mathcal{K}_1$, et comme la dérivé métrique en $\mathbb{W}_1(\Omega)$ le long de la solution de l'équation de Fokker-Planck sans contrainte est de l'ordre de

$$\int_\Omega |\nabla\rho_t| + |\mathbf{u}_t|\rho_t \,\mathrm{d}x,$$

c'est facile de deviner pourquoi nous recherchons des estimations BV à la fois pour l'équation de Fokker-Planck et pour les mesures projetées. Ceci donne une motivation supplémentaire aux résultats établis dans le Chapitre 1. Notons que nous fournissons une courte section sur les variantes possibles de notre *schéma principal*. Ici, nous discutons les similitudes et les difficultés possibles sur les autres schémas, qui contiennent quelques étapes de flot-gradient aussi. Pour les champs de vecteurs purement gradient une approche de flot-gradient peut également être utilisé, de la même manière que dans [MRCS10].

Enfin, nous remarquons que l'estimation BV pour l'équation de Fokker-Planck (sans contrainte de densité) semble une question délicate et elle a son propre intérêt. Comme une sorte d'annexe dans la dernière section de ce chapitre, nous fournissons les estimations que nous avons pu trouver. Certains d'entre elles sont valables pour les champs de vecteur lipschitziens, certains ont seulement été prouvé pour des champs $C^{1,1}$ et leur validité pour les champs de vecteur lipschitziens en général est ouverte.

Le but du Chapitre 3 est de compléter le Chapitre 2 avec des résultats d'unicité. L'unicité des solutions est essentielle si l'on veut inclure un système comme (0.0.5) décrivant le mouvement de foules sous contraintes de densité dans un modèle plus grand, comme les *jeux à champ moyen*, et l'on vise à étudier la question de l'existence pour le système MFG avec un schéma de point fixe. Par ailleurs, la question de l'unicité pour les modèles diffusifs de mouvement de foule avec des contraintes de densité était une pièce manquante dans toute sa généralité. Pour les systèmes de premier ordre (voir [MRCS10, MRCSV11]), l'unicitié était bien connue (au moins parmi les spécialistes) dans certains cas (comme pour les champs de vecteurs monotones) et elle a été écrit d'abord dans [Més12]. Néanmoins, par souci de complétude, nous fournissons une preuve rigoureuse (et simplifiée) pour les modèles de premier ordre aussi.

Les deux stratégies utilisées dans les deux types de modèles sont cependant très différentes. Pour les systèmes du premier ordre, nous supposerons que le champ de vitesse souhaité, $\mathbf{u} : [0,T] \times \Omega \to \mathbb{R}^d$ ($\Omega \subset \mathbb{R}^d$ est borné, convexe et avec bord lipschitzien) satisfait une propriété de monotonie : il existe $\lambda \in \mathbb{R}$ tel que

$$[\mathbf{u}_t(x) - \mathbf{u}_t(y)] \cdot (x-y) \leq \lambda |x-y|^2, \text{ p.p. } x, y \in \Omega, \ \forall t \in [0,T].$$

Puis l'idée est de prouver une propriété de contraction de la distance de Wasserstein W_2 le long de solutions. En utilisant la propriété de monotonie pour le champ de vecteur $\mathbf{u}$, avec la formule du dérivé en temps de $W_2^2(\rho_t^1, \rho_t^2)/2$ (voir [AGS08]) le long deux solutions (ρ^1, p^1) et (ρ^2, p^2) on obtient

$$W_2^2(\rho_t^1, \rho_t^2) \leq e^{2\lambda t} W_2^2(\rho_0^1, \rho_0^2), \text{ pour } \mathscr{L}^1 - \text{p.p. } t \in [0,T],$$

ce qui implique l'unicité pour ρ. Ici nous avons également utilisé la propriété qui dit que si φ^t est un potentiel de Kantorovich dans le transport optimal de ρ_t^1 à ρ_t^2, alors

$$\int_\Omega \nabla \varphi^t \cdot \nabla p_t \, dx \geq 0, \text{ pour } \mathscr{L}^1 - \text{p.p. } t \in [0,T].$$

L'unicité pour p résulte de l'observation que $p_t^1 - p_t^2$ est harmonique et p_t^1 et p_t^2 (pour $\mathscr{L}^1 -$ p.p. $t \in [0,T]$) s'annulent sur le même ensemble de mesure de Lebesgue positive. Notons que pour obtenir une propriété de contraction pour W_2^2 le long de deux solutions, l'hypothèse de monotonie sur le champ de vitesse est naturelle. La même hypothèse a été imposée dans [NPS11] pour étudier la propriété de contraction le long les solutions de l'équation de Fokker-Planck pour une classe générale des distances de transport.

La stratégie pour le cas de deuxième ordre (0.0.5) repose fortement sur la propriété régularisante du laplacien. Cela nous permet de ne pas imposer de régularité supplémentaire sur le champ $\mathbf{u}$, et nous demandons seulement (comme pour l'existence) $\mathbf{u} \in L^\infty$. En utilisant la formulation faible de (0.0.5) pour deux solutions (ρ^1, p^1) et (ρ^2, p^2), on introduit le problème adjoint

$$\begin{cases} A\partial_t\phi + (A+B)\Delta\phi + A\mathbf{u}\cdot\nabla\phi = AG, & \text{dans } [0,T[\times\Omega, \\ \nabla\phi\cdot\mathbf{n} = 0 \text{ sur } [0,T]\times\partial\Omega, & \phi(T,\cdot) = 0 \text{ p.p dans } \Omega, \end{cases} \tag{0.0.6}$$

où

$$A := \frac{\rho^1-\rho^2}{(\rho^1-\rho^2)+(p^1-p^2)}, \quad B := \frac{p^1-p^2}{(\rho^1-\rho^2)+(p^1-p^2)}$$

et G est une function lisse arbitraire. Après la régularisation de A et B, nous obtenons une famille d'équations uniformément paraboliques. En utilisant certaines estimations paraboliques de base pour ces problèmes et la formulation faible pour la différence des deux solutions d'une manière appropriée, on obtient

$$\int_0^T\int_\Omega(\rho^1-\rho^2)G\,\mathrm{d}x\,\mathrm{d}t = 0,$$

ce qui, par le caractère arbitraire de G, donne l'unicité de ρ. L'unicité de p suit par le même argument que dans le cas de premier ordre.

Ce chapitre est basé sur un travail commun avec S. Di Marino (voir [DMM15]).

Composé du Chapitre 4 et du Chapitre 5, la Partie ii est dédiée à l'étude de certains systèmes de *jeux à champ moyen* sous contraintes de densité. Motivé par les questions soulevées par F. Santambrogio dans [San12b], en fait cette partie est considérée comme le cœur de la thèse.

Introduits il y a une dizaine d'années par J.-M. Lasry et P.-L. Lions (voir [LL06a, LL06b, LL07] et aussi [HMC06]), les jeux à champ moyen visent à modéliser des limites d'équilibres de Nash des jeux différentiels (stochastiques), lorsque le nombre de joueurs tend vers l'infini. Ainsi, les systèmes MFG sont liés au problème de commande optimale d'un agent typique, où la densité de la population intervient comme paramètre, plus précisément

$$u(t,x) := \inf_\gamma\left\{\int_t^T L(\gamma(s),\dot\gamma(s)) + f(\gamma(s), m(s,\gamma(s)))\,\mathrm{d}s + g(\gamma(T))\right\}, \tag{0.0.7}$$

où la minimisation est prise parmi les courbes (suffisamment régulières) $\gamma : [0,T] \to \mathbb{R}^d$ avec $\gamma(t) = x$; $L : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ est une fonction lagrangienne donnée, $f : \mathbb{R}^d \times \mathscr{P}_1(\mathbb{R}^d) \to \mathbb{R}$ et $g : \mathbb{R}^d \to \mathbb{R}$ represent le coût courant et le coût final du système, respectivement. Par des méthodes classiques de la théorie de la commande optimale, la fonction valeur résout formellement une équation de Hamilton-Jacobi-Bellman. La densité de population est transporté par le champ de vitesse donné par la commande optimale $\alpha^* := -D_pH(\cdot, Du)$ dans le problème ci-dessus, donc formellement on obtient un système d'EDP couplé, que nous allons appeler d'après Lasry et Lions un système de MFG:

$$\begin{cases} \text{(i)} & -\partial_t u + H(x, Du) = f(x,m) & \text{in } (0,T) \times \mathbb{R}^d \\ \text{(ii)} & \partial_t m - \nabla \cdot \left(m D_p H(x, Du)\right) = 0 & \text{in } (0,T) \times \mathbb{R}^d \\ \text{(iii)} & u(T,x) = g(x), \quad m(0,x) = m_0(x) & \text{in } \mathbb{R}^d. \end{cases} \tag{0.0.8}$$

Ici $H : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ est la transformée de Legendre-Fenchel p.r. à la deuxième variable du lagrangien L et $m_0 \in \mathscr{P}_1(\mathbb{R}^d)$ est la densité initiale de la population. Une solution (u, m) du système ci-dessus représente une configuration d'équilibre aussi. Notons que l'agent typique doit "prédire" en quelque sorte l'évolution de l'ensemble de la population des autres agents afin d'être en mesure de résoudre son problème de commande optimal. Après l'obtention de la commande optimale et le calcul de l'évolution de la "vraie" densité, si cela correspond à la prédiction on dit que m est une *équilibre de Nash*. En d'autres termes (u, m) est une solution du système de MFG (0.0.8).

F. Santambrogio dans [San12b] se demandait si un système de MFG du type (0.0.8) peut être obtenu de façon rigoureuse avec la contrainte supplémentaire que $m(t,x) \le 1$ pour p.p. $(t,x) \in (0,T) \times \mathbb{R}^d$. L'auteur a discuté deux façons possibles pour attaquer cette question. La première est au niveau du problème de contrôle optimal (0.0.7), où les vitesses des courbes γ devraient être affectées par le gradient du champ de pression introduit (de même que dans les modèles de mouvement de foule), plus précisément les compétiteurs γ et α satisfont $\gamma(t) = x$ et $\dot\gamma(s) = \alpha(s) - \nabla p_s(\gamma(s)), s > t$. Cela conduit formellement à un système comme (0.0.8), où la pression $p : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ intervient comme nouvelle variable. D'après nos connaissances, l'analyse rigoureuse de cette approche est encore ouverte.

La deuxième alternative proposée par F. Santambrogio était d'essayer d'obtenir un système comme (0.0.8) avec la contrainte de densité comme limites des conditions d'optimalité de certains problèmes variationnels pénalisés.

Il est bien connu déjà depuis les travaux de J.-M. Lasry et P.-L Lions, que le système de MFG correspond formellement aux conditions d'optimalité de certains problèmes de contrôle optimal d'EDP. Plus précisément, la fonction valeur u est (formellement) donnée comme un minimiseur de la fonctionnelle

$$\mathcal{A}(u) := \int_0^T \int_{\mathbb{R}^d} F^*(x, -\partial_t u + H(x, Du))\,\mathrm{d}x\,\mathrm{d}t - \int_{\mathbb{R}^d} u(0,x)\,\mathrm{d}m_0(x), \tag{0.0.9}$$

sous la contrainte $u(T,x) = g(x)$, où $F = F(x,m)$ est une primitive de $f = f(x,m)$ p.r. à m et F^* est la transformée de Legendre-Fenchel p.r. à la deuxième variable. De la même manière, m est (formellement) donné comme le minimum du problème

$$\mathcal{B}(m,\mathbf{w}) := \int_{\mathbb{R}^d} g(x)m(T,x)\,\mathrm{d}x + \int_0^T\int_{\mathbb{R}^d} m(t,x)L\left(x, -\frac{\mathbf{w}}{m}\right) + F(x,m(t,x))\,\mathrm{d}x\,\mathrm{d}t \tag{0.0.10}$$

sous la contrainte

$$\partial_t m + \nabla\cdot\mathbf{w} = 0 \quad \text{in } (0,T)\times\mathbb{T}^d, \qquad m(0) = m_0.$$

La proposition de F. Santambrogio dans [San12b] était d'utiliser $F(x,m) := m^n/n$ et de prendre la limite lorsque $n\to\infty$. Par cette méthode, formellement, à la limite, la fonction F disparaît et la contrainte supplémentaire $m \le 1$ p.p. apparaît. En fait, ceci est l'un des résultats que nous allons prouver rigoureusement au Chapitre 5. Une autre idée, similaire, est utilisée dans le Chapitre 4 pour montrer le caractère bien-posé des modèles de MFG stationnaires du second ordre avec des contraintes de densité. On va décrire ces résultats maintenant en détails.

Basé sur un travail commun avec F.J. Silva (voir [MS15b]), dans le Chapitre 4 nous étudions une classe de modèles de MFG stationnaires de deuxième ordre avec contraintes de densité. Les systèmes stationnaires ont déjà été introduits dans les travaux originaux de J.-M Lasry et P.-L. Lions (et plus tard étudiés dans [CLLP13, CLLP12]). Ils peuvent être considérés comme la limite moyenne en temps long/limite ergodique des systèmes dépendant du temps.

Dans ce chapitre, nous utilisons une technique variationnelle (similaire à celle présentée avant) et nous obtenons le système de MFG avec des contraintes de densité comme conditions d'optimalité pour ce problème. Pour décrire cela, soit $\Omega\subset\mathbb{R}^d$ ($d\ge 2$) un ouvert borné non vide avec une frontère lisse, tel que $\mathscr{L}^d(\Omega) > 1$. De plus, soit $f:\Omega\times\mathbb{R}\to\mathbb{R}$ une fonction continue, croissante dans la deuxème variable, et on définie $\ell_q:\mathbb{R}\times\mathbb{R}^d\to\overline{\mathbb{R}}$ et $\mathcal{B}_q : W^{1,q}(\Omega)\times L^q(\Omega)^d\to\overline{\mathbb{R}}$ comme

$$\ell_q(a,b) := \begin{cases} \frac{1}{q}\frac{|b|^q}{a^{q-1}}, & \text{si } a>0,\\ 0, & \text{si } (a,b)=(0,0),\\ +\infty, & \text{sinon.}\end{cases} \qquad \mathcal{B}_q(m,\mathbf{w}) := \int_\Omega \ell_q(m(x),\mathbf{w}(x))\,\mathrm{d}x. \tag{0.0.11}$$

La fonctionnelle convexe, s.c.i. $\mathcal{B}_q$ est précisément celle introduite par J.-D. Benamou et Y. Brenier pour etudier une reformulation dynamique du problème de transport optimal de Monge-Kantorovich. Dans notre contexte, nous travaillons en effet avec une restriction de cette dernière à l'espace $W^{1,q}(\Omega)\times L^q(\Omega)^d$.

On considère le probeème

$$\begin{aligned} \min \quad & \mathcal{B}_q(m,\mathbf{w}) + \int_\Omega F(x,m(x))\,\mathrm{d}x, \\ \text{s.c.} \quad & -\Delta m + \nabla\cdot\mathbf{w} = 0 \ \text{ in } \Omega, \quad (\nabla m - \mathbf{w})\cdot\mathbf{n} = 0 \quad \text{sur } \partial\Omega, \\ & \int_\Omega m(x)\,\mathrm{d}x = 1, \quad m \le 1, \end{aligned} \tag{P_q}$$

où, comme avant, $F(x,m)$ est une primitive de $f(x,m)$ p.r. à la seconde variable. Remarquons que, au lieu d'utiliser une pénalisation de la contrainte de densité, nous l'incluons directement dans le problème. Pour travailler de cette façon, il faut assurer une *condition de qualification/condition de point intérieur* pour la contrainte. Comme pour un $\mathbf{w}$ donné l'équation de Fokker-Planck stationnaire fournit la régularité pour m, nous avons besoin de bien choisir nos espaces fonctionnels.

Notre premier résultat préparatoire (avant d'écrire les conditions d'optimalité pour le problème ci-dessus) est une description précise du sous-différentiel de la fonctionnelle $\mathcal{B}_q$. Il semble que ce type de résultat est nouveau dans la littérature. Nous divisons maintenant nos principaux résultats en deux catégories, en fonction de la valeur de q.

Premier cas : $q > d$. Dans ce cas, en utilisant la méthode directe classique du calcul des variations, nous démontrons l'existence d'une solution $(m,\mathbf{w})$ de (P_q). En utilisant que $m \in W^{1,q} \hookrightarrow C(\overline{\Omega})$, nous sommes en mesure de calculer le sous-différentiel de $\mathcal{B}_r(m,\mathbf{w})$ pour tout $1 < r \le q$. En plus, l'injection dans L^∞ de m nous permet de démontrer que les contraintes dans (P_q) sont qualifiées. En utilisant la formule pour le sous-différentiel avec $r = q$ et des arguments classiques en analyse convexe, nous dérivons l'existence de $u \in W^{1,s}(\Omega)$ ($s \in [1, d/(d-1)[$), $\lambda \in \mathbb{R}$ et de deux mesures μ et p positives régulières telles que

$$\begin{cases} -\Delta u + \frac{1}{q'}|\nabla u|^{q'} + \mu - p - \lambda = f(x,m), & \text{dans } \Omega, \\ -\Delta m - \nabla\cdot\left(m|\nabla u|^{\frac{2-q}{q-1}}\nabla u\right) = 0, & \text{dans } \Omega, \\ \nabla m\cdot\mathbf{n} = \nabla u\cdot\mathbf{n} = 0, & \text{sur } \partial\Omega, \\ \int_\Omega m\,\mathrm{d}x = 1, \ \ 0 \le m \le 1, & \text{dans } \Omega, \\ \operatorname{spt}(\mu) \subseteq \{m = 0\}, \quad \operatorname{spt}(p) \subseteq \{m = 1\}, \end{cases} \tag{MFG_q}$$

où le système d'EDP est satisfait au sens faible et $q' := q/(q-1)$. Dans le système ci-dessus, p apparaît comme un multiplicateur de Lagrange associé à la contrainte $m \le 1$ et peut être interprété comme une sorte de "pression". Notons que cette pression diffère de celle présentée dans les modèles précédents de mouvements de foule congestionnés. En effet, dans ces EPD la pression est apparue à travers son gradient (parce qu'elle était une variable duale pour une contrainte de divergence), ici, elle apparaît comme une variable duale pour la contrainte $m \leq 1$, donc il est naturel de l'avoir dans cette forme dans l'équation duale. Notons aussi le fait que la contrainte de densité implique $m \in L^\infty$ de façon naturelle et, dans cet espace, la condition de qualification de contrainte tient automatiquement. On peut se demander pourquoi nous avons besoin de passer à travers les espaces de Sobolev et de la théorie de la régularité elliptique. La raison en est très simple: si l'on a choisi la topologie L^∞ pour m, la pression p comme variable duale vivrait a priori dans $(L^\infty)^*$, un espace qui est difficile à manier. Contrairement, si $q > d$ on a $W^{1,q}(\Omega) \hookrightarrow C(\overline{\Omega})$, l'espace dual de ce dernier étant l'espace des mesures de Radon signées.

Nous calculons aussi le problème dual associé à (P_q) en récupérant (MFG_q) par dualité. Enfin, dans l'ensemble ouvert $\{0 < m < 1\}$ nous prouvons certains résultats de régularité locaux pour le couple (m, u) en utilisant un argument de *bootstrap*.

Deuxième cas : $1 < q \leq d$. Dans ce cas, même si l'existence d'une solution est toujours vraie, m est en général discontinu, ce qui implique que les arguments utilisés dans le calcul du sous-différentiel de $\mathcal{B}_q(m, \mathbf{w})$ ne sont plus valables. En plus, dans la topologie $W^{1,q}$ pour m la contrainte $0 \leq m \leq 1$ n'est pas en général qualifiée. Pour surmonter ces problèmes techniques, nous utilisons un argument d'approximation. En ajoutant le terme $\varepsilon\mathcal{B}_r(m, \mathbf{w})$ avec $r > d$ à la fonction de coût et en utilisant les arguments du *Premier cas*, on obtient un système similaire à (MFG_q) en fonction de $\varepsilon > 0$. Ensuite, par le biais de quelques limites uniformes par rapport à ε et de résultats connus sur les estimations du gradient pour les solutions d'équations elliptiques avec des données mesures (voir par exemple [Min07] pour des résultats récents de régularité qui traitent également des problèmes non linéaires), lorsque $\varepsilon \downarrow 0$, nous pouvons prouver l'existence de points limites satisfaisant (MFG_q), où les propriétés de concentration pour p et μ doivent être interprétées au sens faible.

Puisque nous avons obtenu les systèmes de MFG dans ce chapitre par une manière directe, à l'aide de la caractérisation du sous-différentiel de la fonctionnelle $\mathcal{B}_q$, nous avons observé que l'on peut retirer l'hypothèse de convexité sur la fonction F. Par conséquent, il est possible de considerer une plus grande classe de problèmes non-convexes et l'analyse effectuée avant s'applique toujours. De plus, certains problèmes sans contraintes de densité peuvent être traités de cette façon. Nous avons également inclus ces remarques dans ce chapitre. Ces extensions possibles sont l'objet d'une collaboration en cours avec F.J. Silva (voir [MS]).

Le Chapitre 5 est le denier chapitre de cette thèse. Il est basé sur un travail commun avec P. Cardaliaguet et F. Santambrogio (voir [CMS15]). Comme nous l'avons souligné quelques pages avant, ici, nous étudions le caractère bien-posé des systèmes de MFG évolutifs de premier ordre sous des contraintes de densité. Notre stratégie est variationnelle, et repose sur une technique de pénalisation, mentionnée aussi dans [San12b]. Ainsi nous obtenons un système de MFG avec la contrainte supplémentaire sur la densité comme conditions d'optimalité de deux problèmes de contrôle optimal en dualité. Ces deux problèmes sont ceux pour les fonctionnelles $\mathcal{A}$ et $\mathcal{B}$ décrites dans (0.0.9) et (0.0.10). Pour éviter des technicités sur la frontière, dans l'ensemble de ce chapitre nous travaillons avec des conditions périodiques au bord, donc nous avons mis $\Omega := \mathbb{T}^d$, le tôre plat $\mathbb{R}^d/\mathbb{Z}^d$. La contrainte de densité est donnée par une constante positive $\overline{m} > 1 = 1/\mathscr{L}^d(\mathbb{T}^d)$ et elle peut être modélisée au moyen d'un prix infini à payer si un agent passe par une zone où $m > \overline{m}$.

Néanmoins, il y a plusieurs problèmes avec l'interprétation d'un système MFG quand on a affaire à une contrainte de densité. En effet, le problème de contrôle optimal pour les agents dans l'interprétation ci-dessus n'a pas plus de sens pour la raison suivante : si, d'une part, la contrainte $m \leq \overline{m}$ est satisfaite, alors le problème de minimisation pour les agents (en raison du fait qu'ils sont considérés négligeables par rapport aux autres) ne voit pas cette contrainte et le couple (u, m) est la solution d'un système de MFG standard ; mais cette solution n'a aucune raison de satisfaire la contrainte, et il y a une contradiction. D'autre part, s'il y a des endroits où $m(t, x) > \overline{m}$, alors les joueurs ne passent pas par ces endroits parce que leur coût est infini : mais alors la densité aux ces endroits est zéro, et il y a à nouveau une contradiction. Ainsi, afin de comprendre le système de MFG avec une contrainte de densité, il faut changer notre point de vue. Nous verrons qu'il existe plusieurs façons de comprendre plus profondément les phénomènes derrière cette question.

Une manière de résoudre la question ci-dessus est de travailler au niveau des deux problèmes d'optimisation. La contrainte de densité est incluse dans la fonction F, plus précisément nous posons $F = +\infty$ sur l'ensemble où la deuxième variable est dans $(-\infty, 0) \cup (\overline{m}, +\infty)$. En utilisent une relaxation similaire à celle du problème pour $\mathcal{A}$, comme dans [Car13b], nous montrons l'existence d'une solution. L'existence pour le problème dual, qui porte sur $\mathcal{B}$, est une simple conséquence du théorème de dualité de Fenchel-Rockafellar. Le système de conditions d'optimalité pour ces deux problèmes est le système de MFG suivant :

$$\begin{cases} \text{(i)} & -\partial_t u + H(x, Du) = f(x, m) + \beta & \text{dans } (0, T) \times \mathbb{T}^d \\ \text{(ii)} & \partial_t m - \nabla \cdot (m D_p H(x, Du)) = 0 & \text{dans } (0, T) \times \mathbb{T}^d \\ \text{(iii)} & u(T, x) = g(x) + \beta_T, \quad m(0, x) = m_0(x) & \text{dans } \mathbb{T}^d \\ \text{(iv)} & 0 \leq m \leq \overline{m} & \text{dans } [0, T] \times \mathbb{T}^d \end{cases} \tag{0.0.12}$$

À côté de la contrainte de densité attendue (iv), deux termes supplémentaires apparaissent : β en (i) et β_T end (iii). On peut voir que ces deux quantités sont non négatives et se concentrent sur l'ensemble $\{m = \overline{m}\}$. Elles correspondent formellement à un prix supplémentaire payé par les joueurs s'ils passent par des zones où la concentration est saturée, plus précisément où $m = \overline{m}$. En d'autres termes, le nouveau problème de contrôle optimal pour les joueurs est maintenant (formellement)

$$u(t,x) = \inf_{\substack{\gamma \\ \gamma(t)=x}} \int_t^T L(\gamma(s),\dot{\gamma}(s)) + f(\gamma(s),m(s,\gamma(s)) + \beta(s,\gamma(s))\,\mathrm{d}s + g(\gamma(T)) + \beta_T(\gamma(T)), \tag{0.0.13}$$

et donc (encore formellement) u satisfait le principe de programmation dynamique : pour tout $0 \leq t_1 \leq t_2 < T$,

$$u(t_1,x) = \inf_{\substack{\gamma \\ \gamma(t_1)=x}} \int_{t_1}^{t_2} L(\gamma(s),\dot{\gamma}(s)) + f(\gamma(s),m(s,\gamma(s)) + \beta(s,\gamma(s))\,\mathrm{d}s + u(t_2,\gamma(t_2)). \tag{0.0.14}$$

Les "prix supplémentaires" β et β_T découragent un nombre trop élevé de joueurs à être attirés par la région où la contrainte est saturée, assurant ainsi que la contrainte de densité (iv) soit satisfaite.

Une autre façon d'interpréter la contrainte de densité, est par un argument d'approximation. On peut utiliser une approximation $f^\varepsilon(x,m)$ qui tend vers $f(x,m)$ uniformément si $m \in [0,\overline{m}]$ et $+\infty$ si $m \in (\overline{m},+\infty)$. Pour ces couplages la théorie classique s'applique, et nous montrons que (u,m) correspond à la configuration limite (à des sous-suites près) des $(u^\varepsilon,m^\varepsilon)$ correspondants, lorsque $\varepsilon \downarrow 0$.

L'une des principales contributions de ce chapitre est la détermination de certains liens forts entre notre modèle de MFG avec contrainte de densité et les équations d'Euler incompressibles étudiés par Y. Brenier (voir [Bre99]) et aussi par L. Ambrosio et A. Figalli (voir [AF09]). En fait, cette connexion n'est pas surprenante. Tout d'abord, la contrainte d'incompressibilité dans le modèle de Brenier pour étudier des fluides parfaits introduit bien un champ de pression. Moralement le même effet se produit si on impose la contrainte de densité pour des MFG (avec la seule différence qu'on impose une contrainte de densité unilatérale, et donc la pression a un signe). Deuxièmement, à la fois le modèle de Brenier et le nôtre ont une structure variationnelle, similaire aussi à celle introduite par Benamou et Brenier dans [BB00]. Par conséquent, les termes β et β_T, que nous appelons "prix supplémentaires" pour les agents (qui apparaissent uniquement lorsqu'ils traversent des zones saturées) dans (0.0.12) correspondent à une sorte de pression de la mécanique des fluides. Cette observation motive le titre de ce chapitre aussi.

En utilisant des techniques similaires à celles de [Bre99] et [AF09, AF08], nous montrons que β est une fonction $L^2_{\mathrm{loc}}((0,T);BV(\mathbb{T}^d)) \hookrightarrow L^{d/(d-1)}_{\mathrm{loc}}((0,T)\times\mathbb{T}^d)$ (a priori, on ne pouvait que supposer qu'il s'agisse seulement d'une mesure) et β_T est $L^1(\mathbb{T}^d)$. À l'aide d'un exemple, nous montrons que cette intégrabilité peut échouer au voisinage du temps final $t = T$, montrant également une sorte d'*optimalité* du résultat. Cette propriété de régularité nous permettra de donner une meilleure (bien que faible) signification au problème du contrôle (0.0.13) et d'obtenir des conditions d'optimalité le long de trajectoires individuelles de chauque agent. Nos techniques pour procéder avec l'analyse reposent sur les propriétés des mesures définies sur des chemins, que nous appellerons des *flot sous contrainte de densité* dans notre contexte, et on exploite certaines propriétés d'une fonctionnelle maximale de type Hardy-Littlewood. Nous décrivons ces résultats plus en détail. Les preuves sont fortement inspirées de [AF09].

Soit Γ l'ensemble des courbes $\gamma : [0, T] \to \mathbb{T}^d$ absolument continues et $\mathscr{P}_2(\Gamma)$ l'ensemble des mesures de probabilité Borelliennes $\tilde{\boldsymbol{\eta}}$ définies sur Γ telles que

$$\int_\Gamma \int_0^T |\dot\gamma(s)|^2 \, \mathrm{d}s \, \mathrm{d}\tilde{\boldsymbol{\eta}}(\gamma) < +\infty.$$

On appelle $\tilde{\boldsymbol{\eta}}$ un *flot sous contrainte de densité* si $0 \le \tilde m_t \le \overline{m}$ p.p. dans $\mathbb{T}^d$ pour tout $t \in [0, T]$, où $\tilde m_t := (e_t)_\# \tilde{\boldsymbol{\eta}}$. Ici $e_t : \Gamma \to \mathbb{T}^d$ désigne l'application d'évaluation au temps $t \in [0, T]$, $e_t(\gamma) := \gamma(t)$. Soit (u, m, β, β_T) une solution du système de MFG (0.0.12).

On dit que $\boldsymbol{\eta} \in \mathscr{P}_2(\Gamma)$ est un *flot sous contrainte de densité optimal* associé à la solution (u, m, β, β_T) si $m(t, \cdot) = (e_t)_\# \boldsymbol{\eta}$, pour tout $t \in [0, T]$ et l'égalité d'énergie suivante est satisfaite

$$\begin{aligned}\int_{\mathbb{T}^d} u(0^+, x) m_0(x) \, \mathrm{d}x = &\int_{\mathbb{T}^d} g(x) m(T, x) \, \mathrm{d}x + \overline{m} \int_{\mathbb{T}^d} \beta_T \, \mathrm{d}x \\ &+ \int_\Gamma \int_0^T L(\gamma(t), \dot\gamma(t)) \, \mathrm{d}t \, \mathrm{d}\boldsymbol{\eta}(\gamma) \\ &+ \int_0^T \int_{\mathbb{T}^d} (f(x, m(t, x)) + \beta(t, x)) \, m(t, x) \, \mathrm{d}x \, \mathrm{d}t.\end{aligned}$$

Soit $\alpha := f(\cdot, m) + \beta$. Pour décrire notre résultat final, nous allons définir la notion suivante. Pour $0 < t_1 < t_2 < T$, nous disons qu'un chemin $\gamma \in H^1([0, T]; \mathbb{T}^d)$ avec $M\hat\alpha(\cdot, \gamma) \in L^1_{\mathrm{loc}}((0, T))$ est *minimisant* sur l'interval de temps $[t_1, t_2]$ pour le problème (0.0.14) si on a

$$\begin{aligned}\hat u(t_2^+, \gamma(t_2)) + \int_{t_1}^{t_2} L(\gamma(t), \dot\gamma(t)) + \hat\alpha(t, \gamma(t)) \, \mathrm{d}t \le \hat u(t_2^-, \gamma(t_2) + \omega(t_2)) + \\ + \int_{t_1}^{t_2} L(\gamma(t) + \omega(t), \dot\gamma(t) + \dot\omega(t)) + \hat\alpha(t, \gamma(t) + \omega(t)) \, \mathrm{d}t,\end{aligned}$$

pour tout $\omega \in H^1([t_1, t_2]; \mathbb{T}^d)$ tel que $\omega(t_1) = 0$ et $M\hat\alpha(\cdot, \gamma + \omega) \in L^1([t_1, t_2])$. Ici $\hat\alpha$ désigne un représentant spécial de $\alpha \in L^{d/(d-1)}_{\mathrm{loc}}((0, T) \times \mathbb{T}^d)$ défini comme étant

$$\hat\alpha(t, x) := \limsup_{\varepsilon \downarrow 0} (\alpha(t, \cdot) \star \rho_\varepsilon)(x),$$

pour p.t. $t \in (0, T)$, où la mollification est réalisée avec le noyau de la chaleur ρ_ε. Cette hypothèse pour la classe des compétiteurs, plus précisément $M\hat\alpha(\cdot, \gamma) \in$

$L^1_{\text{loc}}((0,T))$, est imposée pour être capable de gérer les passages à la limite (dans la régularisation, lorsque $\varepsilon \downarrow 0$). Nous aurons besoin de certaines bornes uniformes ponctuelles sur $\alpha \star \rho_\varepsilon$, donc nous allons utiliser les propriétés de la fonctionnelle maximale de type Hardy-Littlewood définie à l'aide du noyau de la chaleur. Ainsi, pour tout $h \in L^1(\mathbb{T}^d)$ nous posons

$$(Mh)(x) := \sup_{\varepsilon>0} \int_{\mathbb{R}^d} |h(x+\varepsilon y)|\rho(y)\,\mathrm{d}y.$$

Notons que, en particulier, il existe une constante $C > 0$ telle que $\|Mh\|_{L^r(\mathbb{T}^d)} \leq C\|h\|_{L^r}$ pour tout $h \in L^r(\mathbb{T}^d)$ et $r > 1$. En plus $M(h \star \rho_\varepsilon) \leq Mh$. Notre résultat principal dans ce contexte est le suivant.

Théorème 0.0.3. *Il existe au moins un flot sous contrainte de densité optimal $\boldsymbol{\eta}$. De plus, pour tout $0 < t_1 < t_2 < T$, $\boldsymbol{\eta}$ se concentre sur des chemins minimisants sur l'intervalle de temps $[t_1, t_2]$ pour le problème (0.0.14) au sens décrit auparavant.*

Enfin, ce théorème fournit l'existence d'une *faible équilibre de Nash locale* dans notre modèle.

Soit (u, m, β, β_T) une solutions du système de MFG avec des contraintes de densité sur $[0,T] \times \mathbb{T}^d$. On dit que (m, β, β_T) est un *équilibre de Nash local faible* s'il existe un flot sous contrainte de densité optimal $\boldsymbol{\eta} \in \mathscr{P}_2(\Gamma)$ (construit à l'aide de (m, β, β_T)) qui est concentré sur les chemins localement minimisants du problème (0.0.14). En particulier, on a que $m_t = (e_t)_\#\boldsymbol{\eta}$ et $0 \leq m_t \leq \overline{m}$ p.p. dans $\mathbb{T}^d$ pour tout $t \in [0,T]$.

Structurellement, à côté des cinq chapitres détaillés précédemment, la thèse contient deux petits chapitres non numérotés supplémentaires. Dans la Partie i nous avons inclus une "Boîte à outils de transport optimal" (*Optimal Transport toolbox*), où nous avons recueilli tous les résultats et les références classiques sur la théorie du transport optimal qui sont nécessaires par la suite. Comme la Partie ii est entièrement dédiée aux systèmes de MFG avec des contraintes de densité, nous avons inclus un court chapitre sur l'histoire de MFG, juste avant les deux principaux chapitres de cette partie. Enfin, nous terminons la thèse avec un appendice, où nous avons recueilli quelques résultats bien connus d'analyse convexe, de théorie de $\Gamma-$convergence et des résultats bien connus sur l'existence et la régularité des solutions des équations elliptiques avec des données mesures.

General Introduction

In this thesis we study different models coming from optimal transport, partial differential equations describing crowd motion and Mean Field Games. In all these models – as the title already highlights – the common denominator is the notion of *density constraint*. Density constraints arise naturally when one wants to model congestion effects. Imagine the following situation: we have just built a new mathematics department (as it shall be the case in Orsay). For security reasons, we want to design a device that tells the people how to evacuate 'optimally' the building in case of an emergency. In this context 'optimally' means not only a fastest/shortest possible way, but which takes in consideration the possible congestion effects as well. This is quite an important issue, because at narrower places (for instance next to the doors) we can usually expect to have higher concentration of people. Hence, if our device could take in consideration also the constraint that at each moment and each place of the building the density of the people remains below a given threshold (for instance 5.4 people on each square meter, a value which is usually used in applications), the evacuation procedure would be perfect.

The possible realization of such a device would be for sure a hard task. One of the main reasons is that in the case of an emergency, people tend to forget to think rationally. Nevertheless, from the mathematical point of view this creates some very interesting and non-trivial questions. Mathematically described, the above device could work in the following way: as input it receives at each time the density and the *desired velocity field* of the people, and returns a new velocity (hence it tells to the people at which speed and in which direction to go). This new velocity is constructed in a manner that no people are allowed to move in such a way that the density goes beyond the saturation threshold. The desired velocity is supposed to be known (it can depend on the distance to the nearest door, etc.) and it is the same for everybody.

At this point we remark that in all of our forthcoming models we shall describe the motion of a crowd/population of agents through the evolution of their density. Hence we consider macroscopic models.

In the past few years many different models have been proposed to study crowd motions with congestions effects. Actually these models sometimes can serve as basis to understand some phenomena coming from biology (such as cell migration, tumor growth, pattern formation), particle physics or economics. For a non-exhaustive list of bibliography in this setting we refer to [Cha07, CR05, CPT14b, Dog08, Hel92, HM95, Hug02, Hug03, MV07, MRCS10, MRCSV11, MRCS14, RC11, PQV14, AKY14].

In this context, the previously described situation is modeling a so-called *hard congestion* effect (we refer for example to [MRCS10, MRCSV11, MV07, RC11]). In this sense, the desired velocity field of the people has to be modified in order to avoid highly concentrated zones. Similar models, with so-called *soft congestion* effects also exist. In these ones people will slow down as soon as the density of the zone they are in approaches the limit threshold (instead of being unaffected except when they are in a fully saturated zone).

The models studied in this thesis are motivated by the first type of consideration, i.e. by macroscopic crowd motion models with hard congestion effects. The central pillar of our analysis is the theory of *optimal transport*. This theory is very powerful and it allows us to study and understand several phenomena related to density constraints in different models by a unified manner.

The genesis of this thesis started with the *modest proposal* of F. Santambrogio (see [San12b]). He proposed a model of *Mean Field Games* (see [LL06a, LL06b, LL07, Lio08]), where one imposes a density constraint. We shall present the theory of MFG more in details at the beginning of Part ii. Nevertheless, let us point out that in MFG models *agents* are playing a non-cooperative differential game, where everyone has to choose a strategy. Hence, in these models one wants not only to understand the evolution of the density of the population, but to describe the value function and the optimal strategy of each agent. The models in [San12b] aim to generalize the ones on crowd motion (discussed before), in the sense that people are *strategic*. In [San12b] only the model has been built, but no rigorous well-posedness results were provided. Being also the subject of the MSc mémoire (see [Més12]), it turned out that the questions raised in [San12b] are far from being trivial. One of the reasons is the low regularity that one could expect for the value function solving a first order Hamilton-Jacobi-Bellman equation. This prevented us to build a reasonable fixed point scheme (a technique successfully used in many other MFG models), taking into account also the *pressure field*, the new variable arising in duality with the density constraint.

A first attempt to solve this issue, was the study a diffusive model, where a non-degenerate diffusion is included both in HJB and the continuity equations (transforming the latter one into a Fokker-Planck equation). As a first step, this required the study of the well-posedness of the Fokker-Planck equation with density constraint. This objective has been successfully achieved and it resulted in a new diffusive model of macroscopic crowd motion with density constraint (this shall be the subject of Chapter 2 and Chapter 3 and it has also been the subject of two papers, see [MS15a, DMM15]). At some point in the analysis performed in [MS15a], we needed some finer estimates on projected measures below a certain threshold in the Wasserstein sense. More precisely, a BV estimate on the projected measures would allow us to obtain some compactness results for some curves in the Wasserstein space, constructed by a *splitting-type procedure*. By this, one could prove the convergence of the algorithm and thus the existence result. The required BV estimates have been achieved not only for projected measures, but for the optimizers of a larger class of variational problems involving optimal transport (this is the subject of Chapter 1 and [DPMSV15]).

In parallel to the above presented direction, we investigated the questions raised in [San12b] also from a different point of view. We studied two type of MFG models possessing a *variational structure*. In the two different models the used approaches are related through their variational formulation. The approaches used in both models recall the one studied by J.-D. Benamou and Y. Brenier (see [BB00]) to give a dynamical formulation of the Monge-Kantorovich optimal transport problem.

Firstly, we showed the well-posedness and characterized the solutions of some diffusive stationary MFG systems under density constraints. The regularizing effect of the stationary Fokker-Planck equation and the elliptic structure allowed us to impose the density constraint (which we showed to be qualified) directly on the level of the optimization problem. This is the subject of Chapter 4 (see also [MS15b]).

Secondly, we showed the well-posedness of first order evolutive MFG systems with density constraints. Here we obtained the density constraint by the limit of some penalizations (a method suggested initially also in [San12b]). Moreover, we obtained a surprising link between our model of MFG with density constraints and the Euler's equations describing the movement of perfect incompressible fluids (see [Bre99, AF09]). These models are the subject of Chapter 5 (see also [CMS15]). Let us notice that the MFG systems (obtained as optimality conditions of the corresponding variational problems) presented in Part ii show some differences from the original ones derived formally in [San12b]. This is due to the different interpretation of the pressure field, seen as Lagrange multiplier for the density constraint.

Now we shall describe in details the main mathematical results included in the present thesis. We shall see how they are presented with respect to the chapters as well. Each chapter is based roughly on a paper. These are either accepted for publication, submitted or in preparation.

MATHEMATICAL DESCRIPTION OF THE RESULTS

Chapters 1, 2 and 3 build Part i of the thesis and they contain the results on the models arising purely from optimal transport and macroscopic crowd movements with density constraints.

Chapter 1 is based on a joint work with G. De Philippis, F. Santambrogio and B. Velichkov (see [DPMSV15]). Here our main objective was to study some fine properties of the projection operator in the Wasserstein space $\mathbb{W}_2(\Omega), (\Omega \subseteq \mathbb{R}^d)$. In fact we embed this question into a larger set of problems. Namely, we study some quantitative properties and regularity of the minimizers of the optimization problem

$$\min_{\varrho\in\mathscr{P}_2(\Omega)} \frac{1}{2}W_2^2(\varrho, g) + \tau F(\varrho),$$

where W_2 stands for the 2-Wasserstein distance on $\mathscr{P}_2(\Omega)$, $F : \mathscr{P}_2(\Omega) \to \mathbb{R}$ is a given functional, $\tau > 0$ is a parameter which can possibly be small, and g is a given probability in $\mathcal{P}_2(\Omega)$ (the space of probability measures on $\Omega \subseteq \mathbb{R}^d$ with finite second moment $\int_\Omega |x|^2 \, d\varrho(x) < +\infty$). The above problem can be recognized as one step in the time-discretization (τ being the discretization parameter in this case) of the gradient flow of the functional F, where $g = \varrho_k^\tau$ is a previously constructed measure and the optimal ϱ is in fact the next one. The algorithms, where each time step has the form of the above optimization problem, are usually called JKO schemes in the optimal transport community (see [JKO98]). Under suitable assumptions, at the limit when $\tau \to 0$, the sequence of the optimal measures converges to a curve of measures which is the gradient flow of F. Let us remark here that for gradient flows, the discretization parameter τ is sent to zero, hence regarding estimates on the optimizers, one may want to obtain some which are independent of τ. We can imagine other models entering in the above optimization problem: g could represent some resources, and ϱ the distribution of factories around them; g the distribution of some stores/banks/schools, etc. and ϱ the distribution of people. Other sophisticated models in urban planning, image processing, etc. exist as well. Here in general $\tau > 0$ is fixed.

Our very first objective was to study the behavior of the projection operator, i.e. projected measures below a certain threshold. In the above problem this is the case if we formally set F to be the indicator function of the set $\mathcal{K}_f := \{\rho \in \mathscr{P}_2(\Omega) : \rho \le f \, dx\}$, where f is a positive function with $\int_\Omega f(x) \, dx \ge 1$. In applications, in general it is reasonable to choose f to be constant. Since this type of problem does not require a dependence on $\tau > 0$, we simply set it to be $\tau = 1$.

Our main results in this chapter read as follows:

Theorem 0.0.1. *Let $\Omega \subset \mathbb{R}^d$ be a (possibly unbounded) convex set, $h : \mathbb{R}_+ \to \mathbb{R} \cup \{+\infty\}$ be a convex and l.s.c. function and $g \in \mathscr{P}_2(\Omega) \cap BV(\Omega)$. If $\bar{\varrho}$ is a minimizer of the following variational problem*

$$\min_{\varrho \in \mathscr{P}_2(\Omega)} \frac{1}{2} W_2^2(\varrho, g) + \int_\Omega h(\varrho(x))\, dx\,,$$

then

$$\int_\Omega |\nabla \bar{\varrho}|\, dx \leq \int_\Omega |\nabla g|\, dx\,. \tag{0.0.15}$$

By an approximation argument (approximating the indicator function of the set $\mathcal{K}_f$ by convex l.s.c. functionals), the above result applies in particular for the projection problem in the following form:

Theorem 0.0.2. *Let $\Omega \subset \mathbb{R}^d$ be a (possibly unbounded) convex set, $g \in \mathscr{P}_2(\Omega) \cap BV(\Omega)$ and let $f \in BV_{\text{loc}}(\Omega)$ be a function with $\int_\Omega f\, dx \geq 1$. If*

$$\bar{\varrho} = \operatorname{argmin}\left\{ W_2^2(\varrho, g) \ :\ \varrho \in \mathscr{P}_2(\Omega),\ \varrho \in \mathcal{K}_f \right\}, \tag{0.0.16}$$

then

$$\int_\Omega |\nabla \bar{\varrho}|\, dx \leq \int_\Omega |\nabla g|\, dx + 2\int_\Omega |\nabla f|\, dx. \tag{0.0.17}$$

In the case when $f \equiv 1$, one obtains a total variation decay result for the projection operator. We remark that the constant 2 in the inequality (0.0.17) is sharp. The BV estimates are useful when the projection is treated as one time-step of a discretized evolution process. For instance, a BV bound allows to transform weak convergence into strong L^1 convergence. Also, if we consider a PDE mixing a smooth evolution, such as the Fokker-Planck evolution, and some projection steps (in order to impose a density constraint, as in crowd motion issues: we describe this later, and it is treated in details in Chapter 2), one could wonder which bounds on the regularity of the solution are preserved in time. From the fact that the discontinuities in the projected measure destroy any kind of $W^{1,p}$ norm, it is natural to look for BV bounds.

The heart of the proof of the above BV estimates is the following inequality. Let us consider $\varrho, g \in \mathscr{P}_2(\Omega)$, (φ, ψ) a pair of Kantorovich potentials in the optimal transport of ϱ onto g and $H : \mathbb{R}^d \to \mathbb{R}$ an even, convex function (we skip the regularity assumptions in this heuristic description). Then one has

$$\int_\Omega \nabla H(\nabla \varphi) \cdot \nabla \varrho + \nabla H(\nabla \psi) \cdot \nabla g\, dx \geq 0.$$

It seems that this inequality encodes some non-trivial geometric features of the optimal transport problem between ϱ and g, which are not completely understood, except for some particular cases. The case of $H(z) = |z|^2/2$ for instance is a consequence of the geodesic convexity of the entropy functional.

To prove the above described BV estimates, we use the inequality for $H(z) = |z|$. In the same context, we also gave a new rigorous proof of the fact that the projected measure saturates the constraint (which is also known in the framework of partial transport). More precisely, there exists a unique optimizer $\bar{\varrho}$ in (0.0.16) and there exists a measurable set $B \subseteq \Omega$ such that

$$\bar{\varrho} = g^{\mathrm{ac}} \mathbb{1}_B + f \mathbb{1}_{B^c}.$$

Let us remark that for this property we do not have to impose the BV regularity on g and f. These ones are only needed for the estimate (0.0.17).

At the end of the chapter we discuss possible applications (also in the form of open questions) of the previously established BV estimates. First, we observe that some question from the so-called *optimal partial transport* theory, investigated recently by L.A. Caffarelli-R. McCann and A. Figalli (see [CM10, Fig10], where the objective is to transport optimally only a given portion of a measure onto another one) can be formulated in our framework (such as the regularity of the free boundary arising when one projects a measure). Our BV estimates could be useful in the study of the partial transport problem itself, which can be seen actually as a double projection problem. We also discuss some other possible applications for shape optimization and set evolution problems and we provide a new, transport-based, proof for the total variation estimates for degenerate diffusion equations (such as the porous media equation).

In Chapter 2 — based on a joint work with F. Santambrogio (see [MS15a]) — we propose a new model on macroscopic crowd motion with density constraints, i.e. with hard congestion. Motivated by the recently studied first order models, due to *Maury et al.* (see [MRCS10, MRCS14, MRCSV11]), we analyze a second order system. From the modeling point of view we impose a randomness in the movement of the individuals. Mathematically, this can be seen at the macroscopic level as a non-degenerate diffusion driven by a Brownian motion and the whole model can be described with the help of a 'modified' Fokker-Planck equation. Here the word 'modified' refers to the fact that one has to modify the velocity field of the people on the saturated zones.

We describe our model through the evolution of the density of the crowd $[0, T] \ni t \mapsto \rho_t$, which is a time-dependent family of probability measures on $\Omega \subset \mathbb{R}^d$ (a bounded and convex domain with Lipschitz boundary). There is given a spontaneous velocity field $\mathbf{u} : [0, T] \times \Omega \to \mathbb{R}^d$, which represents the desired velocity that each individual would follow in the absence of the others. We impose only L^∞ regularity on this field. To equip the model with density constraints — $\rho \leq 1$ a.e. in $[0, T] \times \Omega$, which implies that we have to impose $\mathscr{L}^d(\Omega) > 1$ —, we introduce the set of admissible velocities. These are the fields which do not increase the density on the already saturated zones, i.e. formally we set

$$\mathrm{adm}(\rho) := \left\{ \mathbf{v} : \Omega \to \mathbb{R}^d : \nabla \cdot \mathbf{v} \geq 0 \text{ on } \{\rho = 1\} \text{ and } \mathbf{v} \cdot \mathbf{n} \leq 0 \text{ on } \partial\Omega \right\}.$$

Now we are interested in solving the modified Fokker-Planck equation

$$\begin{cases} \partial_t\rho_t - \Delta\rho_t + \nabla\cdot\left(\rho_t P_{\mathrm{adm}(\rho_t)}[\mathbf{u}_t]\right) = 0, \\ \rho(0,x) = \rho_0(x), \text{ in } \Omega, \end{cases} \tag{0.0.18}$$

where $P_{\mathrm{adm}(\rho)} : L^2(\Omega;\mathbb{R}^d) \to L^2(\Omega;\mathbb{R}^d)$ represent the L^2 projection onto the convex closed set $\mathrm{adm}(\rho)$ and ρ_0 is the given initial density of the crowd. Observe that one could wonder if we should project instead the 'full velocity field' $-\nabla\rho_t/\rho_t + \mathbf{u}_t$. Actually this is the same as projecting only $\mathbf{u}_t$, because in the region $\{\rho_t = 1\}$ one has $-\nabla\rho_t/\rho_t = 0$. Thus the main point is that ρ is advected by a vector field, compatible with the constraints, which is the closest to the spontaneous one.

Despite the fact that we added a non-degenerate diffusion to the model, which has a regularization effect, because of the projection operator, the new velocity field is highly irregular (only L^2) and it depends on a nonlocal way on the density itself. Hence any classical theory will fail in the analysis of Problem (0.0.18). To handle this issue we need to redefine the set of admissible velocities by duality (as it has been done for first order models, see [MRCS10, RC11]):

$$\mathrm{adm}(\rho) = \left\{\mathbf{v}\in L^2(\rho) : \int_\Omega \mathbf{v}\cdot\nabla p \le 0,\ \forall p\in H^1(\Omega), p\ge 0, p(1-\rho) = 0 \text{ a.e}\right\}.$$

With the help of this formulation we always have the orthogonal decomposition

$$\mathbf{u} = P_{\mathrm{adm}(\rho)}[\mathbf{u}] + \nabla p,$$

where

$$p \in \mathrm{press}(\rho) := \left\{p\in H^1(\Omega) : p\ge 0,\ p(1-\rho) = 0 \text{ a.e.}\right\}.$$

Indeed, the cones $\mathrm{adm}(\rho)$ and $\nabla\mathrm{press}(\rho)$ are dual to each other. Via this approach the system (0.0.18) can be rewritten as a system for (ρ, p) which is

$$\begin{cases} \partial_t\rho_t - \Delta\rho_t + \nabla\cdot(\rho_t(\mathbf{u}_t - \nabla p_t)) = 0 \\ p\ge 0,\ \rho\le 1,\ p(1-\rho) = 0,\ \rho(0,x) = \rho_0(x), \text{ in } \Omega. \end{cases} \tag{0.0.19}$$

We can naturally endow this system with natural Neumann boundary conditions.

One of the main contributions of Chapter 2 is the existence result for system (0.0.19). This is achieved by a well-chosen discrete in time *splitting algorithm*. It goes as follows: for a time step $\tau > 0$ we build recursively the measures ρ_k^τ ($k\in\{0,\dots,N\}$, where $N := [T/\tau]$) via our *main scheme*. This scheme is the following: follow the unconstrained Fokker-Planck equation during a time τ with initial data ρ_k^τ. Let us denote this solution at time τ by ρ_τ. The new density is then constructed as $\rho_{k+1}^\tau := P_{\mathcal{K}_1}[\rho_\tau]$, where $P_{\mathcal{K}_1}$ denotes now the 2-Wasserstein projection operator onto the set $\mathcal{K}_1 := \{\rho\in\mathscr{P}(\Omega) : \rho\le 1 \text{ a.e}\}$. Now one has to iterate these two steps.

We continue our analysis by constructing suitable interpolations $\rho_t^\tau, t \in [0,T]$ between the ρ_k^τ's. These interpolations can be seen as curves in $\mathbb{W}_2(\Omega)$. We have to build discrete velocities $\mathbf{v}_k^\tau$ and momentums $\mathbf{E}_k^\tau$ as well. To prove the convergence as $\tau \downarrow 0$, one needs some compactness results for the curves ρ^τ. These rely on some standard comparison between the metric derivative in $\mathbb{W}_2(\Omega)$ and the dissipation of the entropy along the curves ρ^τ. By this, one obtains compactness in the space $H^1([0,T];\mathbb{W}_2(\Omega))$. In order to identify the limit equation as $\tau \downarrow 0$, in fact we use several interpolations between $\rho_k^\tau, \mathbf{v}_k^\tau$ and $\mathbf{E}_k^\tau$. Finally, by this procedure one obtains the existence of a pair (ρ, p) which satisfies the system (0.0.19) in the sense of distribution.

As an alternative way to get compactness for the curves ρ^τ, we show uniform bounds in $\tau > 0$ in the space $\mathrm{Lip}([0,T];\mathbb{W}_1(\Omega))$. This is achieved by combining some sharp BV estimates for the unconstrained Fokker-Planck equation on the one hand and by the use of the BV estimates for the projected measures (provided in Chapter 1) on the other hand. Since our *main scheme* consists in following the unconstrained Fokker-Planck equation and then projecting onto the set $\mathcal{K}_1$, and since the metric derivative in $\mathbb{W}_1(\Omega)$ along the solution of the unconstrained Fokker-Planck equation is of the order of

$$\int_\Omega |\nabla \rho_t| + |\mathbf{u}_t|\rho_t \, \mathrm{d}x,$$

it is easy to guess why we look for BV estimates both for the Fokker-Planck equation and the projected measures. By this, the results established in Chapter 1 gain another motivation. Notice that we provide a short section on possible variants of our *main scheme*. Here we discuss the similarities and the difficulties on possible other schemes where we include some gradient flow steps as well. For gradient vector fields a pure gradient flow approach can be also used, similarly as in [MRCS10].

Finally let us remark that the BV estimates for the (unconstrained) Fokker-Planck equation seem to be a delicate matter and they have their own interest. As sort of an appendix, we provide the estimates that we were able to find, in the last section of this chapter. Some of them are valid for Lipschitz vector field, some have only been proven for $C^{1,1}$ and their validity for Lipschitz vector fields in general is open.

The purpose of Chapter 3 is to complete Chapter 2 with uniqueness results. Uniqueness of solutions is essential if one wants to include a system like (0.0.19) describing crowd motion with density constraints into a larger model, as *Mean Field Games* and one aims to study the existence question for the larger system via a fixed point scheme. This is the case for some models of MFG. Moreover, the question of uniqueness for diffusive crowd motion models with density constraints was a missing puzzle in its full generality. For first order systems (see [MRCS10, MRCSV11]) it was well known (among specialists) in some cases (such as for monotone vector fields) and it was first written in

[Més12]. Nevertheless for the sake of completeness, we provide a rigorous (and simplified) proof for the first order models as well.

The two strategies used in the two type of models however are very different. For first order systems, we shall assume that the desired velocity field $\mathbf{u} : [0,T]\times\Omega\to\mathbb{R}^d$ ($\Omega\subset\mathbb{R}^d$ is bounded, convex and with Lipschitz boundary) satisfies a monotonicity property, i.e. there exists $\lambda\in\mathbb{R}$ such that

$$[\mathbf{u}_t(x)-\mathbf{u}_t(y)]\cdot(x-y)\le\lambda|x-y|^2,\ \text{a.e. } x,y\in\Omega,\ \forall t\in[0,T].$$

Then the idea is to prove a contraction property of the Wasserstein distance W_2 along two solutions. Using the monotonicity property for the vector field $\mathbf{u}$, together with the formula for the time derivative of $W_2^2(\rho_t^1,\rho_t^2)/2$ (see [AGS08]) along two solutions (ρ^1,p^1) and (ρ^2,p^2) one obtains

$$W_2^2(\rho_t^1,\rho_t^2)\le e^{2\lambda t}W_2^2(\rho_0^1,\rho_0^2),\ \text{for } \mathscr{L}^1-\text{a.e. } t\in[0,T],$$

which implies the uniqueness for ρ. Here we also used a fine property saying that if φ^t is a Kantorovich potential in the optimal transport of ρ_t^1 onto ρ_t^2, then

$$\int_\Omega\nabla\varphi^t\cdot\nabla p_t\,\mathrm{d}x\ge0,\ \text{for } \mathscr{L}^1-\text{a.e. } t\in[0,T].$$

The uniqueness for p follows from the observation that $p_t^1-p_t^2$ is harmonic and p_t^1 and p_t^2 (for $\mathscr{L}^1-$a.e. $t\in[0,T]$) vanish on the same set of positive Lebesgue measure. Let us mention that to obtain a contraction property for W_2^2 along two solutions, the monotonicity assumption on the velocity field is natural. The same assumption has been imposed in [NPS11] to study the contraction property along the solutions of the Fokker-Planck equation for a general class of transport distances.

The strategy for the second order case (0.0.19) highly relies on the regularizing property of the Laplacian. Because of this, we do not need extra regularity on the velocity $\mathbf{u}$ and we only require (as for existence) $\mathbf{u}\in L^\infty$. Using the weak formulation of (0.0.19) for two solutions (ρ^1,p^1) and (ρ^2,p^2), we introduce the dual problem

$$\begin{cases} A\partial_t\phi+(A+B)\Delta\phi+A\mathbf{u}\cdot\nabla\phi=AG, & \text{in } [0,T[\times\Omega,\\ \nabla\phi\cdot\mathbf{n}=0 \text{ on } [0,T]\times\partial\Omega, & \phi(T,\cdot)=0 \text{ a.e. in } \Omega,\end{cases}\tag{0.0.20}$$

where

$$A:=\frac{\rho^1-\rho^2}{(\rho^1-\rho^2)+(p^1-p^2)},\quad B:=\frac{p^1-p^2}{(\rho^1-\rho^2)+(p^1-p^2)}$$

and G is an arbitrary smooth function. After regularizing A and B we obtain a family of uniformly parabolic equations. Using some basic parabolic estimates for these problems and writing the weak formulation for the difference of the two solutions in a proper way, one obtains that

$$\int_0^T\int_\Omega(\rho^1-\rho^2)G\,\mathrm{d}x\,\mathrm{d}t=0,$$

which by the arbitrariness of G gives the uniqueness of ρ. The uniqueness of p follows by the same argument as in the first order case.

This chapter is based on a joint work with S. Di Marino (see [DMM15]).

Composed by Chapter 4 and Chapter 5, Part ii is dedicated to the study of some *Mean Field Game* systems under density constraints. Motivated by the questions raised by F. Santambrogio in [San12b], actually this part is considered as the core of the thesis.

Introduced roughly ten years ago by J.-M. Lasry and P.-L. Lions (see [LL06a, LL06b, LL07] and also [HMC06]), Mean Field Games aim to model limits of Nash equilibria of (stochastic) differential games, when the number of players tends to infinity. Thus MFG systems are linked to the optimal control problem of a typical agent, where the density of the whole population enters as a parameter, i.e.

$$u(t,x) := \inf_{\gamma} \left\{ \int_t^T L(\gamma(s), \dot\gamma(s)) + f(\gamma(s), m(s,\gamma(s)))\,\mathrm{d}s + g(\gamma(T)) \right\}, \quad (0.0.21)$$

where the minimization is taken among (sufficiently regular) curves $\gamma : [0,T] \to \mathbb{R}^d$ with $\gamma(t) = x$; $L : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is a given Lagrangian function, $f : \mathbb{R}^d \times \mathscr{P}_1(\mathbb{R}^d) \to \mathbb{R}$ and $g : \mathbb{R}^d \to \mathbb{R}$ represent the running- and the final cost of the system, respectively. By standard methods from optimal control theory, the value function formally solves a Hamilton-Jacobi-Bellman equation. The density of the population is transported by the velocity field given by the optimal control $\alpha^* := -D_pH(\cdot, Du)$ in the above problem, hence formally one obtains a coupled PDE system, that we shall call after Lasry and Lions an MFG system:

$$\begin{cases} \text{(i)} & -\partial_t u + H(x,Du) = f(x,m) & \text{in } (0,T)\times\mathbb{R}^d \\ \text{(ii)} & \partial_t m - \nabla\cdot\big(mD_pH(x,Du)\big) = 0 & \text{in } (0,T)\times\mathbb{R}^d \\ \text{(iii)} & u(T,x) = g(x), \quad m(0,x) = m_0(x) & \text{in } \mathbb{R}^d. \end{cases} \quad (0.0.22)$$

Here $H : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is the Legendre-Fenchel transform w.r.t. the second variable of the Lagrangian L and $m_0 \in \mathscr{P}_1(\mathbb{R}^d)$ is the initial density of the population. A solution (u,m) of the above system encodes an equilibrium configuration as well. Note that the typical agent has to "predict" somehow the evolution of the whole agent population in order to be able to solve his/her optimal control problem. After obtaining the optimal control and computing the evolution of the "true" density, if this corresponds to the prediction one says that m is a *Nash equilibrium*. In other words (u,m) is a solution of the MFG system (0.0.22).

F. Santambrogio in [San12b] asked whether an MFG system, similar to (0.0.22) can be rigorously obtained together with the additional constraint that $m(t,x) \leq$

1 for a.e. $(t, x) \in (0, T) \times \mathbb{R}^d$. The author discussed two possible ways to attack this question. The first one is at the level of the optimal control problem (0.0.21), where the velocities of the curves γ should be affected by the gradient of the introduced pressure field (similarly as in the models for crowd motion), i.e. the competitors γ and α satisfy $\gamma(t) = x$ and $\dot{\gamma}(s) = \alpha(s) - \nabla p_s(\gamma(s)), s > t$. This leads formally to a system like (0.0.22), where the pressure $p : [0, T] \times \mathbb{R}^d \to \mathbb{R}$ enters as a new variable. To the best of our knowledge, the rigorous analysis in this approach is still open.

The second alternative suggested by F. Santambrogio was to try to obtain a system like (0.0.22) together with the density constraint as limits of optimality conditions of some penalized variational problems.

It is well known already from the works of J.-M. Lasry and P.-L Lions that the MFG system formally corresponds to the optimality conditions of some optimal control problems with PDE constraints. More precisely, the value function u is (formally) given as a minimizer of the functional

$$\mathcal{A}(u) := \int_0^T \int_{\mathbb{R}^d} F^*(x, -\partial_t u + H(x, Du))\, \mathrm{d}x\, \mathrm{d}t - \int_{\mathbb{R}^d} u(0, x)\, \mathrm{d}m_0(x), \quad (0.0.23)$$

subject to the constraint that $u(T, x) = g(x)$, where $F = F(x, m)$ is an antiderivative of $f = f(x, m)$ with respect to m and F^* is its Legendre-Fenchel conjugate w.r.t. the second variable. In the same way m is (formally) given as a minimum of the problem

$$\mathcal{B}(m, \mathbf{w}) := \int_{\mathbb{R}^d} g(x) m(T, x)\, \mathrm{d}x + \int_0^T \int_{\mathbb{R}^d} m(t, x) L\left(x, -\frac{\mathbf{w}}{m}\right) + F(x, m(t, x))\, \mathrm{d}x\, \mathrm{d}t \quad (0.0.24)$$

subject to the constraint

$$\partial_t m + \nabla \cdot \mathbf{w} = 0 \quad \text{in } (0, T) \times \mathbb{T}^d, \qquad m(0) = m_0.$$

The proposition of F. Santambrogio in [San12b] was to use $F(x, m) := m^n/n$ and take the limit as $n \to \infty$. By this, formally at the limit the function F disappears and the additional constraint $m \leq 1$ a.e. appears. Actually this is one of the results that we shall prove rigorously in Chapter 5. A different, but similar idea is used in Chapter 4 to show the well-posedness of second order stationary MFG models with density constraints. Let us describe these results now in details.

Based on a joint work with F.J. Silva (see [MS15b]), in Chapter 4 we study a class of stationary second order MFG models with density constraints. Stationary systems have been introduced already in the original works of J.-M. Lasry and P.-L. Lions (and later studied in [CLLP13, CLLP12]). They can be seen as long time average/ergodic limit of time dependent systems.

In this chapter we use a variational technique (similar to the one presented before) and obtain the MFG system with density constraints as optimality conditions for this problem. To describe this, let $\Omega \subset \mathbb{R}^d$ ($d \geq 2$) be a non-empty bounded open set with smooth boundary, such that $\mathscr{L}^d(\Omega) > 1$. Moreover, let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a continuous function which is non-decreasing in the second variable and define $\ell_q : \mathbb{R} \times \mathbb{R}^d \to \overline{\mathbb{R}}$ and $\mathcal{B}_q : W^{1,q}(\Omega) \times L^q(\Omega)^d \to \overline{\mathbb{R}}$ as

$$\ell_q(a,b) := \begin{cases} \frac{1}{q}\frac{|b|^q}{a^{q-1}}, & \text{if } a > 0, \\ 0, & \text{if } (a,b) = (0,0), \\ +\infty, & \text{otherwise.} \end{cases} \qquad \mathcal{B}_q(m, \mathbf{w}) := \int_\Omega \ell_q(m(x), \mathbf{w}(x))\, dx. \tag{0.0.25}$$

The convex, l.s.c. functional $\mathcal{B}_q$ is precisely the one introduced by J.-D. Benamou and Y. Brenier to study a dynamical reformulation of the Monge-Kantorovich optimal transport problem. In our setting we work actually with a restriction of it to the space $W^{1,q}(\Omega) \times L^q(\Omega)^d$.

We consider the problem

$$\begin{aligned} \min \quad & \mathcal{B}_q(m, \mathbf{w}) + \int_\Omega F(x, m(x))\, dx, \\ \text{s.t.} \quad & -\Delta m + \nabla \cdot \mathbf{w} = 0 \ \text{ in } \Omega, \quad (\nabla m - \mathbf{w}) \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega, \\ & \int_\Omega m(x)\, dx = 1, \quad m \leq 1, \end{aligned} \tag{P_q}$$

where, as before, $F(x, m)$ is an antiderivative of $f(x, m)$ with respect to the second variable. Let us remark that instead of using a penalization for the density constraint, we include it directly into the problem. Working this way, one has to ensure an *interior point/constraint qualification* condition for the constraint. Since for a given $\mathbf{w}$ the stationary Fokker-Planck equation provides regularity for m, we need to chose wisely the functional spaces.

Our first preparatory result (before writing down the optimality conditions for the above problem) is a precise description of the subdifferential of the functional $\mathcal{B}_q$. It seems that this type of result is new in the literature. We divide now our main results in two classes, depending on the value of q.

Case 1: $q > d$. In this case, using the classical direct method of the calculus of variations, we prove the existence of a solution $(m, \mathbf{w})$ of (P_q). Using that $m \in W^{1,q} \hookrightarrow C(\overline{\Omega})$, we are able to compute the subdifferential of $\mathcal{B}_r(m, \mathbf{w})$ for any $1 < r \leq q$. Moreover, the injection in L^∞ of m allows us to prove that the constraints in (P_q) are qualified. Using the computation of the subdifferential with $r = q$ and classical arguments in convex analysis, we derive the existence of $u \in W^{1,s}(\Omega)$ $(s \in [1, d/(d-1)[)$, $\lambda \in \mathbb{R}$ and two nonnegative regular measures μ and p such that

$$\begin{cases} -\Delta u + \frac{1}{q'}|\nabla u|^{q'} + \mu - p - \lambda = f(x,m), & \text{in } \Omega, \\ -\Delta m - \nabla \cdot \left(m|\nabla u|^{\frac{2-q}{q-1}} \nabla u \right) = 0, & \text{in } \Omega, \\ \nabla m \cdot \mathbf{n} = \nabla u \cdot \mathbf{n} = 0, & \text{on } \partial\Omega, \\ \int_\Omega m \, dx = 1, \ \ 0 \le m \le 1, & \text{in } \Omega, \\ \text{spt}(\mu) \subseteq \{m = 0\}, \ \ \text{spt}(p) \subseteq \{m = 1\}, \end{cases} \tag{MFG_q}$$

where the system of PDEs is satisfied in the weak sense and $q' := q/(q-1)$. In the above system, p appears as a Lagrange multiplier associated to the constraint $m \le 1$ and can be interpreted as a sort of a "pressure" term. Let us remark that this pressure differs from the one introduced in the previous congested crowd movement models. Indeed, in those PDEs the pressure appeared through its gradient (because it was a dual variable for a divergence constraint), here it appears as a dual variable for the constraint $m \le 1$, hence it is natural to have it in this form in the dual equation. Let us remark the fact that the density constraint implies $m \in L^\infty$ in a natural way and, in this space the constraint qualification condition automatically holds. One could wonder why do we need to pass through Sobolev spaces and elliptic regularity theory. The reason is very straightforward: if one would chose the L^∞ topology for m, the pressure p as dual variable would live a priori in $(L^\infty)^*$, a space that is hard to work with. Contrary, if $q > d$ one has $W^{1,q}(\Omega) \hookrightarrow C(\overline{\Omega})$, the dual space of which being the space of signed Radon measures.

We also compute the dual problem associated to (P_q) recovering (MFG_q) by duality. Finally, in the open set $\{0 < m < 1\}$ we prove some local regularity results for the pair (m, u) using a bootstrap argument.

Case 2: $1 < q \le d$. In this case, even if the existence of a solution still holds true, m is in general discontinuous, which implies that the arguments employed in the computation of the subdifferential of $\mathcal{B}_q(m, \mathbf{w})$ are no longer valid. Moreover, in the topology $W^{1,q}$ for m the constraint $0 \le m \le 1$ is in general not qualified. In order to overcome these issues, we use an approximation argument. By adding the term $\varepsilon\mathcal{B}_r(m, \mathbf{w})$ with $r > d$ to the cost function and using the arguments in *Case 1* we obtain a system similar to (MFG_q) depending on $\varepsilon > 0$. Then, by means of some uniform bounds with respect to ε and well-known results on estimates on the gradients for solutions of elliptic equations with measure data (see for instance [Min07] for recent regularity results treating also nonlinear problems), as $\varepsilon \downarrow 0$ we can prove the existence of limit points satisfying (MFG_q) where the concentration properties for p and μ have to be understood in a weak sense.

Since we obtained the MFG systems in this chapter by a direct way, with the help of the characterization of the subdifferential of the functional $\mathcal{B}_q$, we observed that one can remove the convexity assumption on the function F. Hence it is possible to consider a larger class of non-convex problems and the analysis performed before still applies. Moreover, some problems without density constraints can interestingly be treated in this way. We also included these remarks into this chapter. These possible extensions are the subject of an ongoing collaboration with F.J. Silva (see [MS]).

Chapter 5 is the last chapter of this thesis. It is based on a joint work with P. Cardaliaguet and F. Santambrogio (see [CMS15]). As it was highlighted couple of pages before, here we study the well-posedness of first order evolutive MFG systems under density constraints. Our strategy is variational, which relies on a penalization technique, mentioned also in [San12b]. Thus we obtain an MFG system with the additional constraint on the density as optimality conditions of two optimal control problems in duality. These two problems are the ones for the functionals $\mathcal{A}$ and $\mathcal{B}$ described in (0.0.23) and (0.0.24). To avoid boundary issues, in the whole of this chapter we work with periodic boundary conditions, hence we set $\Omega := \mathbb{T}^d$, the $d-$dimensional flat torus $\mathbb{R}^d/\mathbb{Z}^d$. The density constraint is given by a positive constant $\overline{m} > 1 = 1/\mathscr{L}^d(\mathbb{T}^d)$ and it can be modeled by means of an infinite price to pay if an agent goes through a saturated zone.

Nevertheless, there are several issues in the interpretation of an MFG system when there is a density constraint. Indeed, the optimal control problem for the typical agents in the above interpretation does not make sense anymore for the following reason: if, on the one hand, the constraint $m \le \overline{m}$ is fulfilled, then the minimization problem of the agents (due to the fact that they are considered negligible against the others) does not see this constraint and the pair (u, m) is the solution of a standard MFG system; but this solution has no reason to satisfy the constraint, and there is a contradiction. On the other hand, if there are places where $m(t, x) > \overline{m}$, then the players do not go through these places because their cost is infinite there: but then the density at such places is zero, and there is again a contradiction. So, in order to understand the MFG system with a density constraint, one has to change our point of view. We shall see that there are several ways to understand more deeply the phenomena behind this question.

One way to solve the above issue is to work at the level of the two optimization problems. The density constraint is included into the function F, i.e. we set it to be $+\infty$ on the set $(-\infty, 0) \cup (\overline{m}, +\infty)$ for its second variable. Using a similar relaxation for the problem for $\mathcal{A}$ as in [Car13b], we show the existence of a solution. The existence for the dual problem, the one for $\mathcal{B}$, is a simple consequence of the Fenchel-Rockafellar duality theorem. The system of optimality conditions for these two problems is the following system of MFG:

$$\begin{cases} \text{(i)} & -\partial_t u + H(x, Du) = f(x,m) + \beta & \text{in } (0,T)\times\mathbb{T}^d \\ \text{(ii)} & \partial_t m - \nabla\cdot\big(mD_pH(x,Du)\big) = 0 & \text{in } (0,T)\times\mathbb{T}^d \\ \text{(iii)} & u(T,x) = g(x)+\beta_T, \quad m(0,x) = m_0(x) & \text{in } \mathbb{T}^d \\ \text{(iv)} & 0 \le m \le \overline{m} & \text{in } [0,T]\times\mathbb{T}^d \end{cases} \tag{0.0.26}$$

Beside the expected density constraint (iv), two extra terms appear: β in (i) and β_T in (iii). These two quantities turn out to be nonnegative and concentrated on the set $\{m = \overline{m}\}$. They formally correspond to an extra price payed by the players to go through zones where the concentration is saturated, i.e., where $m = \overline{m}$. In other words, the new optimal control problem for the players is now (formally)

$$u(t,x) = \inf_{\substack{\gamma \\ \gamma(t)=x}} \int_t^T L(\gamma(s),\dot\gamma(s)) + f(\gamma(s), m(s,\gamma(s))) + \beta(s,\gamma(s))\,\mathrm{d}s + g(\gamma(T)) + \beta_T(\gamma(T)), \tag{0.0.27}$$

and thus (still formally) u satisfies the dynamic programming principle: for any $0 \le t_1 \le t_2 < T$,

$$u(t_1,x) = \inf_{\substack{\gamma \\ \gamma(t_1)=x}} \int_{t_1}^{t_2} L(\gamma(s),\dot\gamma(s)) + f(\gamma(s), m(s,\gamma(s))) + \beta(s,\gamma(s))\,\mathrm{d}s + u(t_2,\gamma(t_2)). \tag{0.0.28}$$

The "extra prices" β and β_T discourage too many players to be attracted by the area where the constraint is saturated, thus ensuring the density constraints (iv) to be fulfilled.

Another way to interpret the density constraint, is by an approximation argument. One can use an approximation $f^\varepsilon(x,m)$ which tends to $f(x,m)$ uniformly if $m \in [0,\overline{m}]$ and to $+\infty$ if $m \in (\overline{m},+\infty)$. For these couplings the classical theory applies, and we show that (u,m) corresponds to the limit configuration (up to subsequences) as $\varepsilon \downarrow 0$ of the corresponding $(u^\varepsilon, m^\varepsilon)$.

One of the main contributions of this chapter is the determination of some strong connections between our model of MFG with density constraint and the incompressible Euler's equations studied by Y. Brenier (see [Bre99]) and also by L. Ambrosio and A. Figalli (see [AF09]). Actually this connection is not that surprising. Firstly, the incompressibility constraint in the model of Brenier to study perfect fluids will introduce the pressure field. Morally the same effect happens imposing density constraint for MFG (with the only difference that we impose a one-sided density constraint, and hence the pressure has a sign).

Secondly, both the model by Brenier and ours have a variational structure, similar also to the one introduced by Benamou and Brenier in [BB00]. Therefore, the terms β and β_T, that we call "additional prices/costs" for the agents (appearing only if they pass through saturated zones) in (0.0.26) correspond to a sort of pressure field from fluid mechanics. This observation motivates the title of this chapter as well.

Using similar techniques as in [Bre99] and [AF09, AF08] we show that β is an $L^2_{\rm loc}((0,T);BV(\mathbb{T}^d)) \hookrightarrow L^{d/(d-1)}_{\rm loc}((0,T)\times\mathbb{T}^d)$ function (while a priori it was only supposed to be a measure) and β_T is $L^1(\mathbb{T}^d)$. With the help of an example we show that this local integrability up to the final time $t = T$ may fail, showing also some sort of sharpness of the result. This regularity property will allow us to give a clearer (weak) meaning to the control problem (0.0.27), obtaining optimality conditions along single agent trajectories. Our techniques to proceed with the analysis rely on the properties of measures defined on paths, that we shall call *density-constrained flows* in our context, and we are exploiting some properties of a Hardy-Littlewood type maximal functional as well. Let us describe these results more in details. The proofs are highly inspired from [AF09].

Let Γ denote the set of absolutely continuous curves $\gamma : [0,T] \to \mathbb{T}^d$ and $\mathscr{P}_2(\Gamma)$ the set of Borel probability measures $\tilde{\boldsymbol{\eta}}$ defined on Γ such that

$$\int_\Gamma \int_0^T |\dot\gamma(s)|^2 \,\mathrm{d}s \,\mathrm{d}\tilde{\boldsymbol{\eta}}(\gamma) < +\infty.$$

We call $\tilde{\boldsymbol{\eta}}$ a *density-constrained flow* if $0 \le \tilde m_t \le \overline{m}$ a.e. in $\mathbb{T}^d$ for all $t \in [0,T]$, where $\tilde m_t := (e_t)_\# \tilde{\boldsymbol{\eta}}$. Here $e_t : \Gamma \to \mathbb{T}^d$ denotes the evaluation map at time $t \in [0,T]$, i.e. $e_t(\gamma) := \gamma(t)$. Let us consider a solution (u, m, β, β_T) of the MFG system (0.0.26).

We say that an $\boldsymbol{\eta} \in \mathscr{P}_2(\Gamma)$ is an *optimal density-constrained flow* associated to the solution (u, m, β, β_T) if $m(t,\cdot) = (e_t)_\# \boldsymbol{\eta}$, for all $t \in [0,T]$ and the following energy equality holds

$$\begin{aligned}\int_{\mathbb{T}^d} u(0^+,x)m_0(x)\,\mathrm{d}x = &\int_{\mathbb{T}^d} g(x)m(T,x)\,\mathrm{d}x + \overline{m}\int_{\mathbb{T}^d}\beta_T\,\mathrm{d}x\\ &+ \int_\Gamma\int_0^T L(\gamma(t),\dot\gamma(t))\,\mathrm{d}t\,\mathrm{d}\boldsymbol{\eta}(\gamma)\\ &+ \int_0^T\int_{\mathbb{T}^d} (f(x,m(t,x)) + \beta(t,x))\,m(t,x)\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

Let us set $\alpha := f(\cdot, m) + \beta$. To describe our final result, we shall define the following notion. Given $0 < t_1 < t_2 < T$, we say that a path $\gamma \in H^1([0,T];\mathbb{T}^d)$

with $M\hat{\alpha}(\cdot,\gamma) \in L^1_{\rm loc}((0,T))$ is *minimizing* on the time interval $[t_1,t_2]$ in the problem (0.0.28) if one has

$$\hat{u}(t_2^+,\gamma(t_2)) + \int_{t_1}^{t_2} L(\gamma(t),\dot{\gamma}(t)) + \hat{\alpha}(t,\gamma(t))\,{\rm d}t \le \hat{u}(t_2^-,\gamma(t_2)+\omega(t_2)) + \\ + \int_{t_1}^{t_2} L(\gamma(t)+\omega(t),\dot{\gamma}(t)+\dot{\omega}(t)) + \hat{\alpha}(t,\gamma(t)+\omega(t))\,{\rm d}t,$$

for all $\omega \in H^1([t_1,t_2];\mathbb{T}^d)$ such that $\omega(t_1)=0$ and $M\hat{\alpha}(\cdot,\gamma+\omega) \in L^1([t_1,t_2])$. Here $\hat{\alpha}$ denotes a special representative of $\alpha \in L^{d/(d-1)}_{\rm loc}((0,T)\times\mathbb{T}^d)$ defined as

$$\hat{\alpha}(t,x) := \limsup_{\varepsilon\downarrow 0}(\alpha(t,\cdot)\star\rho_\varepsilon)(x),$$

for a.e. $t \in (0,T)$, where the mollification is performed with the heat kernel ρ_ε. The hypothesis for the class of competitors, i.e. $M\hat{\alpha}(\cdot,\gamma) \in L^1_{\rm loc}((0,T))$ is imposed to be able to handle the passages to limit (in the regularization, as $\varepsilon \downarrow 0$). We will need some uniform point-wise bounds on $\alpha \star \rho_\varepsilon$, hence we shall use the properties of the Hardy-Littlewood-type maximal function defined with the help of the heat kernel. Thus for any $h \in L^1(\mathbb{T}^d)$ we set

$$(Mh)(x) := \sup_{\varepsilon>0}\int_{\mathbb{R}^d}|h(x+\varepsilon y)|\rho(y)\,{\rm d}y.$$

Note that in particular there exists a constant $C>0$ such that $\|Mh\|_{L^r(\mathbb{T}^d)} \le C\|h\|_{L^r}$ for all $h \in L^r(\mathbb{T}^d)$ and $r>1$. Moreover $M(h\star\rho_\varepsilon) \le Mh$. Our main result in this setting is the following.

Theorem 0.0.3. *There exists at least one optimal density-constrained flow $\boldsymbol{\eta}$. Moreover, for any $0<t_1<t_2<T$, $\boldsymbol{\eta}$ is concentrated on minimizing paths on the time interval $[t_1,t_2]$ for the problem (0.0.28) in the sense described before.*

Finally, this theorem provides the existence of *local weak Nash equilibria* in our model.

Let (u,m,β,β_T) be a solution of the MFG system with density constraints on $[0,T]\times\mathbb{T}^d$. We say that (m,β,β_T) is a local weak Nash equilibrium, if there exists an optimal density-constrained flow $\boldsymbol{\eta} \in \mathscr{P}_2(\Gamma)$ (constructed with the help of (m,β,β_T)) which is concentrated on locally minimizing paths for Problem (0.0.28). In particular one has that $m_t = (e_t)_\#\boldsymbol{\eta}$ and $0 \le m_t \le \overline{m}$ a.e. in $\mathbb{T}^d$ for all $t \in [0,T]$.

Structurally, besides the five chapters detailed previously, the thesis contains two extra small unnumbered chapters. In Part i we included an *Optimal Transport toolbox*, where we collected all the classical results and references on the theory of optimal transport, that are needed afterwards. Since Part ii is fully dedicated to MFG systems with density constraints, we included a short chapter on the history of MFG, just before the two main chapters in this part. Finally, we end the thesis with an Appendix, where we collected some well-known results from convex analysis, the theory of $\Gamma-$convergence and some well-known results on the existence and regularity of solutions of elliptic equations with measure data.

Notations

Let us set the basic notions which shall be used on the forthcoming pages. Our analysis is performed in the $d-$dimensional euclidean space $\mathbb{R}^d$ where always $d \geq 2$. $\Omega \subseteq \mathbb{R}^d$ denotes a non-empty, bounded or unbounded open (or sometimes compact, which will be specified in the context) set with a smooth boundary and we denote by $\mathbf{n}$ the outward normal to $\partial\Omega$. Let us set $|\cdot|$ for the usual euclidean norm on $\mathbb{R}^d$ and, given a Lebesgue measurable set $A \subseteq \mathbb{R}^d$, if it is not ambiguous, we also use $|A|$ for its d-dimensional Lebesgue measure.

We denote by $\mathscr{M}(\Omega)$ the space of (signed) Radon measures defined on Ω. We set $\mathscr{M}_+(\Omega)$ (respectively $\mathscr{M}_-(\Omega)$) for the subset of $\mathscr{M}(\Omega)$ of non-negative (respectively non-positive) Radon measures. Given the Hahn-Jordan decomposition $\mu = \mu^+ - \mu^-$, with $\mu^+, \mu^- \in \mathscr{M}_+(\Omega)$, we set $|\mu| := \mu^+(\Omega) + \mu^-(\Omega)$ for the total variation of μ. We also denote by $\mathscr{M}^{ac}(\Omega)$ and $\mathscr{M}^s(\Omega)$ the spaces of absolutely continuous and singular measures w.r.t. the Lebesgue measure, respectively. For $\mu \in \mathscr{M}(\Omega)$ one uses the decomposition $\mu = \mu^{\mathrm{ac}} + \mu^{\mathrm{s}}$, where $\mu^{\mathrm{ac}} \in \mathscr{M}^{ac}(\Omega)$ and $\mu^{\mathrm{s}} \in \mathscr{M}^s(\Omega)$. For notational convenience, if $\mu \in \mathscr{M}^{ac}(\Omega)$ we will also denote by μ its density w.r.t. the Lebesgue measure. Given $\mu \in \mathscr{M}(\Omega)$ we set $\mu \llcorner A$ for its restriction to $A \in \mathscr{B}(\Omega)$, defined as $\mu \llcorner A(B) := \mu(A \cap B)$ for all $B \in \mathscr{B}(\Omega)$ (where $\mathscr{B}(\Omega)$ denotes the Borel σ-algebra on Ω).

$\mathscr{P}(\Omega)$ denotes the space of Borel probability measures on Ω and $\mathscr{P}_p(\Omega) \subseteq \mathscr{P}(\Omega)$ (for $p \geq 1$) the space of probability measures with finite pth order moment, i.e. $\mu \in \mathscr{P}_p(\Omega)$ iff $\int_\Omega |x|^p \,\mathrm{d}\mu < +\infty$. If Ω is compact, the two spaces actually coincide and we do not make any difference between them.

We say that a family of probability measures $(\mu_n)_{n\geq 0}$ *narrowly converges* to $\mu \in \mathscr{P}(\Omega)$ if $\int_\Omega \phi \,\mathrm{d}\mu_n \to \int_\Omega \phi \,\mathrm{d}\mu$ for all $\phi \in C^0_b(\Omega)$, where $C^0_b(\Omega)$ is the space of continuous and bounded functions on Ω. We call this convergence for simplicity by *weak convergence of probability measures* and denote by $\mu_n \rightharpoonup \mu$. If Ω is compact, actually this notion is the same as the weak-$\star$ convergence in $\mathscr{M}(\Omega)$.

We say that $\mu \in \mathscr{P}(\Omega)$ is *tight* if for all $\varepsilon > 0$ there exists a compact set $K_\varepsilon \subset \Omega$ such that $\mu(\Omega \setminus K_\varepsilon) < \varepsilon$. *Prokhorov's theorem* is a classical result which gives a necessary and sufficient characterization of the sequential weak compactness of families of probability measures: if a sequence $(\mu_n)_{n\geq 0} \subset \mathscr{P}(\Omega)$ is uniformly tight, then there exists $\mu \in \mathscr{P}(\Omega)$ and a subsequence $(\mu_{n_k})_{n_k \geq 0} \subset (\mu_n)_{n\geq 0}$ such that $\mu_{n_k} \rightharpoonup \mu$ as $k \to \infty$. Conversely, if $\mu_n \rightharpoonup \mu$, then the sequence is uniformly tight. This theorem can be stated for any complete separable metric (Polish) space as ambient space. An equivalent characterization of the tightness of a probability measure $\mu \in \mathscr{P}(\Omega)$ is the following: there exists $\Psi : \Omega \to [0, +\infty]$ a coercive function with compact sub-level sets $\{x \in \Omega : \Psi(x) \leq c\}$ such that $\int_\Omega \Psi(x)\, \mathrm{d}\mu(x) < +\infty$.

Finally, let us collect below the basic notations used in this thesis:

$\overline{\mathbb{R}}$	$\mathbb{R} \cup \{+\infty\}$;
$\mathbb{1}_A$	characteristic function of the set A, i.e. $\mathbb{1}_A(x) = 1$ if $x \in A$ and $\mathbb{1}_A(x) = 0$ if $x \notin A$;
χ_A	indicator function of the set A, i.e. $\chi_A(x) = 0$ if $x \in A$ and $\chi_A(x) = +\infty$ if $x \notin A$;
I_d	identity matrix in $\mathbb{R}^{d\times d}$;
$x \cdot y$	scalar product of $x, y \in \mathbb{R}^d$;
$\mathbf{u}, \mathbf{v}, \mathbf{w}, \ldots$	vector fields, i.e. functions on X with values in $\mathbb{R}^d$;
$\mathbf{n}$	outer normal to a set with Lipschitz boundary;
$\lvert \cdot \rvert$	euclidean norm on $\mathbb{R}^d$;
$\mathscr{L}^d$	$d-$dimensional Lebesgue measure (if it is not ambiguous, we also use the notation $\lvert \cdot \rvert$);
$\mathrm{d}x$	integration w.r.t. $\mathscr{L}^d$, i.e. $\mathrm{d}\mathscr{L}^d$;
$\mathscr{H}^d$	$d-$dimensional Hausdorff measure;
$\mathscr{B}(X)$	Borel $\sigma-$algebra on X;
$\mathscr{P}(X)$	Borel probability measures on X;
$\mathscr{P}_p(X)$	subspace of $\mathscr{P}(X)$ with finite pth order moment ($p \geq 1$);
$\mathscr{M}(X)$	finite signed Radon measures on X;
$\mathscr{M}_+(X); \mathscr{M}_-(X)$	non-negative and non-positive elements from $\mathscr{M}(X)$;
$\mathscr{M}^{ac}(X); \mathscr{M}^s(X)$	absolutely continuous and singular measures w.r.t. $\mathscr{L}^d$ on X;
$\mathscr{M}(X)^d$	vector valued finite Radon measures on X;
$\mu \ll \nu$	the measure μ is absolutely continuous w.r.t. the measure ν;

$\frac{\mathrm{d}\mu}{\mathrm{d}\nu} = f$; $\mathrm{d}\mu = f \cdot \mathrm{d}\nu$	Radon-Nikodým derivative of μ w.r.t. ν or f is the density of μ w.r.t. ν;
$\mathrm{spt}(\mu)$	support of the measure μ, i.e. the set $\{x \in X : \mu(B(x,r)) > 0\ \forall r > 0\}$;
$C^0(X); C(X)$	continuous functions on X;
$C^0_b(X)$	bounded and continuous functions on X;
$C^{k,\alpha}(\mathbb{R}^d)$	Hölder continuous functions with k Hölder continuous derivatives;
$C^\infty_c(\mathbb{R}^d)$	smooth functions with compact support in $\mathbb{R}^d$;
$\mathrm{Lip}(X)$	the space of Lipschitz continuous functions on X with values in $\mathbb{R}$;
$\mathrm{Lip}(f)$	Lipschitz constant of a function $f \in \mathrm{Lip}(X)$;
$L^p(X)$	standard Lebesgue space of order $p \in [1, +\infty]$ on X w.r.t. $\mathscr{L}^d$;
$L^p_\mu(X)$	Lebesgue space on X w.r.t. the measure μ;
$W^{k,p}(X)$	standard Sobolev space on X;
$W^{s,p}(X)$	fractional Sobolev space if $s \notin \mathbb{Z}$;
$W^{k,p}_\diamond(X)$	Sobolev functions with zero mean;
$W^{k,p}(X)^d; W^{k,p}(X;\mathbb{R}^d)$	vector valued Sobolev functions;
$H^k(X)$	the space $W^{k,2}(X)$;
X^*	topological dual of the (Banach) space X;
$\langle\cdot,\cdot\rangle_{X^*,X}; \langle\langle\cdot,\cdot\rangle\rangle$	duality brackets between X^* and X;

Part I

DENSITY CONSTRAINTS IN OPTIMAL TRANSPORT AND CROWD MOTION

Optimal Transport toolbox

The theory of Optimal Transport plays a central role throughout the present thesis. Thus we devote this short chapter to collect all the necessary tools and results which are used on the forthcoming pages. For more details on this topic we refer to the by now standard references, to the two monographs of C. Villani (see [Vil09, Vil03]). We use also the forthcoming book of F. Santambrogio (see [San15]) and the monograph of L. Ambrosio, N. Gigli and G. Savaré ([AGS08]) as basic bibliography in the sequel.

The history of OT goes back to 1781, when the French mathematician G. Monge asked the following question (see [Mon81]): "which is the best way to transport a sand pile into a hole with the same volume?"

This question can be formulated mathematically as follows: given two positive Borel measures μ and ν defined on two (compact) subsets $X \subseteq \mathbb{R}^d$ and $Y \subseteq \mathbb{R}^d$ respectively, with the same mass (without loss of generality we set the mass equals to 1, hence μ and ν are probability measures), find the map $T : X \to Y$ that *transports (pushes forward)* μ onto ν, i.e. $T_\sharp\mu = \nu$ (meaning $\nu(A) = \mu(T^{-1}(A))$ for all $A \subset \mathbb{R}^d$ Borel set) and minimizes the transport cost

$$\int_X |x - T(x)| \, \mathrm{d}\mu(x).$$

In this setting Monge's problem reads as follows:

$$\inf_{T_\sharp\mu=\nu} \int_X |x - T(x)| \, \mathrm{d}\mu(x) \tag{MP}$$

A competitor T for (MP) is called a *transport map*, while to an optimizer we refer as *optimal transport map*.

From Monge to Brenier via Kantorovich

One of the difficulties in solving the above problem is coming from the high nonlinearity of the constraint $T_\sharp\mu = \nu$. To see this, let us assume that both measures are absolutely continuous w.r.t. the Lebesque measure with densities f and g respectively. Then the constraint can be written in the following way: for all $\phi \in C_b^0(Y)$

$$\int_X \phi(T(x))f(x)\,\mathrm{d}x = \int_Y \phi(y)g(y)\,\mathrm{d}y = \int_X \phi(T(x))|\det DT(x)|g(T(x))\,\mathrm{d}x,$$

and by density arguments the constraint can be transformed formally into the Jacobian (Monge-Ampère type) equation

$$|\det DT| = \frac{f}{g\circ T},$$

which is a fully nonlinear PDE. Note that the meaning of the above equation should be clarified, but let us skip the rigorous interpretation in this heuristic presentation.

Solving Problem (MP) some other obstructions may arise: the set of maps with the constraint $T_\# \mu = \nu$ can be empty. Indeed, for example if $\mu = \delta_x$ and $\nu = \frac{1}{4}\delta_{y_1} + \frac{3}{4}\delta_{y_2}$, with $x \in X$ and $y_1, y_2 \in Y, y_1 \neq y_2$, any transport map should split the mass concentrated in the point x, which is impossible. Moreover, it also could happen that no map realizes the minimum even if the set of competitors is not empty.

Because of these issues Monge's problem in its full generality remained unsolved over more than one and a half century. In 1942 the Soviet Nobel Memorial prize laureate in Economics, L. Kantorovich came up with the solution (see [Kan42]). He relaxed Monge's problem in the following way: instead of looking for optimal transport maps, he redefined the problem over a larger class of competitors, which we call *transport plans*. Kantorovich's problem reads as follows

$$\inf_{\gamma\in\Pi(\mu,\nu)} \int_{X\times Y} |x-y|\,\mathrm{d}\gamma(x,y) \tag{KP}$$

where we denote the set of plans by

$$\Pi(\mu,\nu) := \{\gamma \in \mathscr{P}(X\times Y) \;:\; (\pi^x)_\#\gamma = \mu \;\text{ and }\; (\pi^y)_\#\gamma = \nu\},$$

and π^x and π^y stand for the two canonical projections from $X \times Y$ onto X and Y respectively. The set $\Pi_o(\mu,\nu) \subseteq \Pi(\mu,\nu)$ stands for the optimal plans in (KP).

We observe that (KP) became a linear problem with linear constraints (and $\Pi(\mu,\nu) \neq \emptyset$, since $\mu \otimes \nu$ is always an element of it), thus by the direct method of calculus of variations one can easily show the existence of an optimal transport plan γ.

In Problem (KP) one can exchange the cost function $|x-y|$ to any $c : X \times Y \to [0,+\infty]$ proper l.s.c. function and the existence result still holds.

Another feature that Problem (KP) has is, that it admits a dual problem. For a general cost function, this can be written as

$$\sup\left\{\int_X \varphi(x)\,\mathrm{d}\mu(x) + \int_Y \psi(y)\,\mathrm{d}\nu(y) \;:\; \varphi(x)+\psi(y) \le c(x,y)\right\} \tag{DP}$$

where the supremum is taken over functions $\varphi \in C_b(X), \psi \in C_b(Y)$. The competitors in Problem (DP) are called *Kantorovich potentials* in the transport of μ onto ν. By the means of a Rockafellar type theorem and standard techniques from convex analysis one can show that Problem (KP) and Problem (DP) are in duality, moreover the dual problem also has a solution.

For a function $u : X \to \overline{\mathbb{R}}$ one can introduce its *c-transform* $u^c : Y \to \overline{\mathbb{R}}$ (analogously to the well-known Legendre-Fenchel transform from convex analysis) by

$$u^c(y) := \inf_{x \in X} \{c(x,y) - u(x)\} .$$

This notion turned to be very useful in the study of Problem (DP). Indeed, one can show that the optimal potentials always have the form (φ, φ^c).

After the solution provided by L. Kantorovich, OT fell again into oblivion for almost five decades. Its true renaissance started in the late 1980's with a seminal work of Y. Brenier (see [Bre87] and also [Bre91]). During the same period M. Cullen had the observation that some techniques from OT can be used in the study of the so-called semi-geostrophic equation, which has applications in meteorology. J. Mather also realized that there are fundamental connections between OT and Lagrangian dynamics. All these works were crucial in the understanding that OT actually connects PDEs, geometry, economics and physics.

By the work of Y. Brenier we can relate also the optimizers in Problem (MP) and Problem (KP) with the help of the optimal Kantorovich potential φ from Problem (DP). Under the additional assumption that $\mu \ll \mathscr{L}^d$, Y. Brenier showed for the quadratic cost $c(x,y) := \frac{1}{2}|x-y|^2$ that there exists a unique optimal transport map in Problem (MP) which is a gradient of a convex function, in addition the relation

$$T(x) = \nabla \left(\frac{1}{2}|x|^2 - \varphi(x) \right) = x - \nabla \varphi(x)$$

holds, for any φ optimal Kantorovich potential from the transport of μ onto ν in Problem (DP). Moreover $\gamma := (\mathrm{id}, T)_\# \mu$ is the optimal plan in Problem (KP). Nowadays we usually refer to this result as Brenier's Theorem.

Brenier's Theorem had been generalized in the following years in several directions. First, the condition $\mu \ll \mathscr{L}^d$ can be replaced with "$\mu(A) = 0$ for every $A \subset \mathbb{R}^d$ such that $\mathscr{H}^{d-1}(A) < +\infty$", which implies the existence of an optimal transport map as well. Moreover, a Brenier-type theorem still holds if one replaces the quadratic cost by $h(x-y)$, where h is strictly convex. To these, and to some other fine geometrical results we refer to the work of W. Gangbo and R.J. McCann (see [GM96]). Secondly, R.J. McCann obtained similar results on Riemannian manifolds (see [McC01]). In [Gig11], N. Gigli characterized the class of 'good' measures, for which an optimal transport map exists.

WASSERSTEIN SPACES AND SOME PREPARATORY RESULTS

The costs of the form $c(x,y) = |x-y|^p$ for $1 \le p \le +\infty$ play a special role in this setting. With the help of them one can introduce the quantity

$$W_p(\mu,\nu) := \left\{ \min_{\gamma\in\Pi(\mu,\nu)} \int_{X\times X} |x-y|^p \,\mathrm{d}\gamma(x,y) \right\}^{\frac{1}{p}},$$

which turned out to be a metric on $\mathscr{P}_p(X)$ and it metrizes the weak-$\star$ topology on $\mathscr{P}_p(X)$. We call $W_p(\mu,\nu)$ the *Wasserstein distance* between μ and ν. The space $\mathbb{W}_p(X) := (\mathscr{P}_p(X), W_p)$ is called *Wasserstein space* on X. Note that if X is not compact, one has to work with $\mathscr{P}_p(X)$ (the space of probability measures with finite pth order moments) instead of $\mathscr{P}(X)$ to have existence of an optimal plan between μ and ν and to avoid that $W_p(\mu,\nu) = +\infty$.

Let us summarize some results (adapting the setting for later use) in the following theorem.

Theorem 0.0.4. *Let $\Omega \subset \mathbb{R}^d$ be a given convex set and let $\varrho, g \in L^1(\Omega)$ be two non-negative probability densities on Ω. Then the following hold:*

(i) The problem

$$\frac{1}{2}W_2^2(\varrho,g) := \min\left\{ \int_{\Omega\times\Omega} \frac{1}{2}|x-y|^2 \,\mathrm{d}\gamma \;:\; \gamma\in\Pi(\varrho,g) \right\}, \tag{0.0.29}$$

has a unique solution, which is of the form $\gamma_{\hat{T}} := (id,\hat{T})_{\#}\varrho$, and $\hat{T}:\Omega\to\Omega$ is a solution of the problem

$$\min_{T_{\#}\varrho = g} \int_\Omega \frac{1}{2}|x-T(x)|^2\,\varrho(x)\,\mathrm{d}x\,. \tag{0.0.30}$$

(ii) The map $\hat{T}:\{\varrho>0\}\to\{g>0\}$ is a.e. invertible and its inverse $\hat{S} := \hat{T}^{-1}$ is a solution of the problem

$$\min_{S_{\#}g=\varrho} \int_\Omega \frac{1}{2}|x-S(x)|^2\,g(x)\,\mathrm{d}x. \tag{0.0.31}$$

(iii) $W_p(\cdot,\cdot)$ is a distance on the space $\mathscr{P}_p(\Omega)$ of probabilities over Ω with finite p*th order moment. In addition the followings are equivalent:*

a) $\mu_n \to \mu$ w.r.t. W_p;

b) $\mu_n \rightharpoonup \mu$ and $\int_\Omega |x|^p\,\mathrm{d}\mu_n(x) \to \int_\Omega |x|^p\,\mathrm{d}\mu(x)$;

c) $\int_\Omega \phi\,\mathrm{d}\mu_n(x) \to \int_\Omega \phi\,\mathrm{d}\mu(x)$ for all $\phi \in C^0(\Omega)$ with a growth at most of order p.

(iv) We have

$$\frac{1}{2}W_2^2(\varrho, g) = \max\left\{\int_\Omega \varphi(x)\varrho(x)\,\mathrm{d}x + \int_\Omega \psi(y)g(y)\,\mathrm{d}y : \ \varphi(x)+\psi(y) \le \frac{1}{2}|x-y|^2, \ \forall x,y\in\Omega\right\}. \tag{0.0.32}$$

(v) The optimal functions $\hat\varphi, \hat\psi$ in (0.0.32) are continuous, differentiable almost everywhere, Lipschitz if Ω is bounded, and such that:

- *$\hat T(x) = x - \nabla\hat\varphi(x)$ and $\hat S(x) = x - \nabla\hat\psi(x)$ for a.e. $x \in \Omega$; in particular, the gradients of the optimal functions are uniquely determined (even in the case of non-uniqueness of $\hat\varphi$ and $\hat\psi$) a.e. on $\{\varrho > 0\}$ and $\{g > 0\}$, respectively;*
- *the functions*

$$x \mapsto \frac{|x|^2}{2} - \hat\varphi(x) \quad \text{and} \quad x \mapsto \frac{|x|^2}{2} - \hat\psi(x),$$

are convex in Ω and hence $\hat\varphi$ and $\hat\psi$ are semi-concave;

- *$\hat\varphi(x) = \min_{y\in\Omega}\left\{\frac{1}{2}|x-y|^2 - \hat\psi(y)\right\}$ and $\hat\psi(y) = \min_{x\in\Omega}\left\{\frac{1}{2}|x-y|^2 - \hat\varphi(x)\right\}$;*
- *if we denote by χ^c the $c-$transform of a function $\chi : \Omega \to \mathbb{R}$ defined through $\chi^c(y) = \inf_{x\in\Omega}\frac{1}{2}|x-y|^2 - \chi(x)$, then the maximal value in (0.0.32) is also equal to*

$$\max\left\{\int_\Omega \varphi(x)\varrho(x)\,\mathrm{d}x + \int_\Omega \varphi^c(y)g(y)\,\mathrm{d}y, \ \varphi \in C^0(\Omega)\right\} \tag{0.0.33}$$

and the optimal φ is the same $\hat\varphi$ as above, and is such that $\hat\varphi = (\hat\varphi^c)^c$ a.e. on $\{\varrho > 0\}$.

(vi) If $g \in \mathscr{P}_2(\Omega)$ is given, the functional $W : \mathscr{P}_2(\Omega) \to \mathbb{R}$ defined through

$$W(\varrho) = \frac{1}{2}W_2^2(\varrho, g) = \max\left\{\int_\Omega \varphi(x)\varrho(x)\,\mathrm{d}x + \int_\Omega \varphi^c(y)g(y)\,\mathrm{d}y, \ \varphi \in C^0(\Omega)\right\}$$

is convex. Moreover, if $\{g > 0\}$ is a connected open set and $\chi = \tilde\varrho - \varrho$ is the difference between two probability measures, then we have

$$\lim_{\varepsilon\to 0}\frac{W(\varrho+\varepsilon\chi) - W(\varrho)}{\varepsilon} = \int_\Omega \hat\varphi\,\mathrm{d}\chi$$

where $\hat\varphi$ is the $c-$transform of the unique (up to additive constants) optimal function $\hat\psi$ in (0.0.32). As a consequence, $\hat\varphi$ is the first variation of W.

The only non-standard point is the last one (the computation of the first variation of W): it is sketched in [BS05], and a more detailed presentation will be part of [San15] (Section 7.2). Uniqueness of $\hat\psi$ is obtained from the uniqueness of its gradient and the connectedness of $\{g > 0\}$.

We will need some regularity results on optimal transport maps. The following results are due to L.A. Caffarelli, see [Caf92b, Caf92a].

Theorem 0.0.5. *Let $\Omega \subset \mathbb{R}^d$ be a bounded uniformly convex set with smooth boundary and let $\varrho, g \in L^1_+(\Omega)$ be two probability densities on Ω away from zero and infinity* [1]. *Then, using the notations from Theorem 0.0.4, we have:*

(i) $\hat{T} \in C^{0,\alpha}(\overline{\Omega})$ *and* $\hat{S} \in C^{0,\alpha}(\overline{\Omega})$.

(ii) If $\varrho \in C^{k,\beta}(\overline{\Omega})$ *and* $g \in C^{k,\beta}(\overline{\Omega})$, *then* $\hat{T} \in C^{k+1,\beta}(\overline{\Omega})$ *and* $\hat{S} \in C^{k+1,\beta}(\overline{\Omega})$.

Let us mention that recently the Sobolev regularity of the transport maps (and more generally of solutions of Monge-Ampère equations) has also been studied extensively by G. De Philippis, A. Figalli (see [DPF13, DPF13]) and independently by T. Schmidt ([Sch13]).

In many of our proofs we shall use some approximation techniques. To do this, we need a stability result

Theorem 0.0.6. *Let $\Omega \subset \mathbb{R}^d$ be a bounded convex set and let $\varrho_n \in L^1_+(\Omega)$ and $g_n \in L^1_+(\Omega)$ be two sequences of probability densities in Ω. Then, using the notations from Theorem 0.0.4, if $\varrho_n \rightharpoonup \varrho$ and $g_n \rightharpoonup g$ weakly as measures, then we have:*

(i) $W_2(\varrho, g) = \lim_{n\to\infty} W_2(\varrho_n, g_n)$.

(ii) *there exist two semi-concave functions φ, ψ such that $\nabla\hat{\varphi}_n \to \nabla\varphi$ and $\nabla\hat{\psi}_n \to \nabla\psi$ a.e. and $\nabla\varphi = \nabla\hat{\varphi}$ a.e. on $\{\varrho > 0\}$ and $\nabla\psi = \nabla\hat{\psi}$ a.e. on $\{g > 0\}$.*

If Ω is unbounded (for instance $\Omega = \mathbb{R}^d$), then the convergences $\varrho_n \rightharpoonup \varrho$ and $g_n \rightharpoonup g$ (weakly as measures) are not enough to guarantee (i) *but only imply $W_2(\varrho, g) \leq \liminf_{n\to\infty} W_2(\varrho_n, g_n)$. Yet,* (i) *is satisfied if $W_2(\varrho_n, \varrho), W_2(g_n, g) \to 0$, which is a stronger condition.*

Proof. The proof of *(i)* can be found in [Vil09]. We prove *(ii)*. (Actually this is a consequence of the Theorem 3.3.3. from [CS04], but for the sake of completeness we sketch its simple proof).

We first note that due to Theorem 0.0.4 *(v)* the sequences $\hat{\varphi}_n$ and $\hat{\psi}_n$ are equi-continuous. Moreover, since the Kantorovich potentials are uniquely determined up to a constant we may suppose that there is $x_0 \in \Omega$ such that $\hat{\varphi}_n(x_0) = \hat{\psi}_n(x_0) = 0$ for every $n \in \mathbb{N}$. Thus, $\hat{\varphi}_n$ and $\hat{\psi}_n$ are locally uniformly bounded in Ω and, by the Ascoli-Arzelà Theorem, they converge uniformly up to a subsequence

$$\hat{\varphi}_n \xrightarrow[n\to\infty]{} \varphi_\infty \qquad \text{and} \qquad \hat{\psi}_n \xrightarrow[n\to\infty]{} \psi_\infty,$$

to some continuous functions $\varphi_\infty, \psi_\infty \in C(\Omega)$, satisfying

$$\varphi_\infty(x) + \psi_\infty(y) \leq \frac{1}{2}|x-y|^2, \quad \text{for every} \quad x, y \in \Omega.$$

In order to show that φ_∞ and ψ_∞ are precisely Kantorovich potentials, we use the characterization of the potentials as solutions to the problem (0.0.32).

1. We say that ϱ and g are away from zero and infinity if there is some $\varepsilon > 0$ such that $\varepsilon \leq \varrho \leq 1/\varepsilon$ and $\varepsilon \leq g \leq 1/\varepsilon$ a.e. in Ω.

Indeed, let φ and ψ be such that $\varphi(x)+\psi(y) \le \frac{1}{2}|x-y|^2$ for every $x, y \in \Omega$. Then, for every $n \in \mathbb{N}$ we have

$$\int_\Omega \hat{\varphi}_n(x)\varrho_n(x)\,\mathrm{d}x + \int_\Omega \hat{\psi}_n(y)g_n(y)\,\mathrm{d}y \ge \int_\Omega \varphi(x)\varrho_n(x)\,\mathrm{d}x + \int_\Omega \psi(y)g_n(y)\,\mathrm{d}y,$$

and passing to the limit we obtain

$$\int_\Omega \varphi_\infty(x)\varrho(x)\,\mathrm{d}x + \int_\Omega \psi_\infty(y)g(y)\,\mathrm{d}y \ge \int_\Omega \varphi(x)\varrho(x)\,\mathrm{d}x + \int_\Omega \psi(y)g(y)\,\mathrm{d}y,$$

which proves that φ_∞ and ψ_∞ are optimal. In particular, the gradient of these functions coincide with those of $\hat{\varphi}$ and $\hat{\psi}$ on the sets where the densities are strictly positive.

We now prove that $\nabla\hat{\varphi}_n \to \nabla\varphi_\infty$ a.e. in Ω. We denote with $\mathcal{N} \subset \Omega$ the set of points $x \in \Omega$, such that there is a function among $\hat{\varphi}$ and $\hat{\varphi}_n$, for $n \in \mathbb{N}$, which is not differentiable at x. We note that by Theorem 0.0.4 *(v)* the set $\mathcal{N}$ has Lebesgue measure zero. Let now $x_0 \in \Omega \setminus \mathcal{N}$ and suppose, without loss of generality, $x_0 = 0$. Setting

$$\alpha_n(x) := \frac{|x|^2}{2} - \hat{\varphi}_n(x) + \hat{\varphi}_n(0) + x \cdot \nabla\varphi_\infty(0)$$

and

$$\alpha(x) := \frac{|x|^2}{2} - \varphi_\infty(x) + \varphi_\infty(0) + x \cdot \nabla\varphi_\infty(0),$$

we have that α_n are all convex and such that $\alpha_n(0) = 0$, and hence $\alpha_n(x) \ge \nabla\alpha_n(0)\cdot x$. Moreover, $\alpha_n \to \alpha$ locally uniformly and $\nabla\alpha(0) = 0$. Suppose by contradiction that $\lim_{n\to\infty} \nabla\alpha_n(0) \ne 0$. Then, there is a unit vector $p \in \mathbb{R}^d$ and a constant $\delta > 0$ such that, up to a subsequence, $p \cdot \nabla\alpha_n \ge \delta$ for every $n > 0$. Then, for every $t > 0$ we have

$$\frac{\alpha(pt)}{t} = \lim_{n\to\infty} \frac{\alpha_n(pt)}{t} \ge \liminf_{n\to\infty} \{p \cdot \nabla\alpha_n(0)\} \ge \delta,$$

which is a contradiction with the fact that $\nabla\alpha(0) = 0$. □

Let us state a simple lemma concerning properties of the functional

$$\mathscr{M}(\Omega) \ni \varrho \mapsto H(\varrho) = \begin{cases} \displaystyle\int_\Omega h(\varrho(x))\,\mathrm{d}x, & \text{if } \varrho \ll \mathscr{L}^d, \\ +\infty, & \text{otherwise.} \end{cases}$$

Lemma 0.0.7. *Let Ω be an open set and $h : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ be convex, l.s.c. and superlinear at $+\infty$. Then the functional $H : \mathscr{M}(\Omega) \to \mathbb{R} \cup \{+\infty\}$ is convex and lower semicontinuous with respect to the weak convergence of measures. Moreover if $h \in C^1$ then we have*

$$\lim_{\varepsilon\to 0} \frac{H(\varrho + \varepsilon\chi) - H(\varrho)}{\varepsilon} = \int_\Omega h'(\varrho)\,\mathrm{d}\chi$$

whenever $\rho, \chi \ll \mathscr{L}^d$, $H(\varrho) < +\infty$ and $H(\varrho + \varepsilon\chi) < +\infty$ at least for small ε. As a consequence, $h'(\varrho)$ is the first variation of H.

For this classical fact, and in particular for the semicontinuity, we refer to [But89] and [BB90].

We also use this lemma, together with point (vi) in Theorem 0.0.4 to deduce the following optimality conditions.

Corollary 0.0.8. *Let Ω be a bounded open set, $g \in L^1_+(\Omega)$ an absolutely continuous and strictly positive probability density on Ω, the potential $\hat{\varphi}$ and the functional W defined as in point (vi) in Theorem 0.0.4. Let $h : \mathbb{R} \to \mathbb{R}$ be a C^1 convex and superlinear function, and let $H : \mathscr{M}(\Omega) \to \mathbb{R} \cup \{+\infty\}$ be defined as above. Suppose that $\bar{\varrho}$ solves the minimization problem*

$$\min\{W(\varrho) + H(\varrho) \ : \ \varrho \in \mathscr{P}(\Omega)\}.$$

Then there exists a constant C such that

$$h'(\bar{\varrho}) = \max\{(C - \hat{\varphi}), h'(0)\}.$$

The proof of this fact is contained in [BS05] and in Section 7.2.3 of [San15]. We give a sketch here.

Proof. Take an arbitrary competitor $\tilde{\varrho}$, define $\varrho_\varepsilon := (1-\varepsilon)\bar{\varrho} + \varepsilon\tilde{\varrho}$ and $\chi = \tilde{\varrho} - \bar{\varrho}$ and write the optimality condition

$$0 \leq \lim_{\varepsilon \to 0} \frac{(H+W)(\bar{\varrho} + \varepsilon\chi) - (H+W)(\bar{\varrho})}{\varepsilon}.$$

This implies

$$\int_\Omega (\hat{\varphi} + h'(\bar{\varrho})) \, \mathrm{d}\tilde{\varrho} \geq \int_\Omega (\hat{\varphi} + h'(\bar{\varrho})) \, \mathrm{d}\bar{\varrho}$$

for any arbitrary competitor $\tilde{\varrho}$. This means that there is a constant C such that $\hat{\varphi} + h'(\bar{\varrho}) \geq C$ with $\hat{\varphi} + h'(\bar{\varrho}) = C$ on $\{\bar{\varrho} > 0\}$. The claim is just a re-writing of this fact, distinguishing the set where $\bar{\varrho} > 0$ (and hence $h'(\bar{\varrho}) \geq h'(0)$) and the set where $\bar{\varrho} = 0$. □

Following [AGS08], one can introduce the *tangent space* $\mathrm{Tan}_\mu \mathscr{P}_2(\mathbb{R}^d)$ to a measure $\mu \in \mathscr{P}_2(\mathbb{R}^d)$ as the closure in $L^2_\mu(\mathbb{R}^d;\mathbb{R}^d)$ of the set gradients of smooth, compactly supported functions, i.e.

$$\mathrm{Tan}_\mu \mathscr{P}_2(\mathbb{R}^d) := \overline{\{\nabla\phi \ : \ \phi \in C^\infty_c(\mathbb{R}^d)\}}^{L^2_\mu}.$$

The *subdifferential* of a functional $U : \mathscr{P}_2(\mathbb{R}^d) \to \mathbb{R}$ at $\mu \in \mathscr{P}_2(\mathbb{R}^d)$ can be defined as the set $D^-U(\mu)$ of elements $\xi \in \mathrm{Tan}_\mu \mathscr{P}_2(\mathbb{R}^d)$ such that

$$U(\nu) - U(\mu) \geq \inf_{\gamma \in \Pi_o(\mu,\nu)} \int_{\mathbb{R}^d \times \mathbb{R}^d} \xi(x) \cdot (y - x) \, \mathrm{d}\gamma(x,y) + o(W_2(\mu,\nu)),$$

$\forall \nu \in \mathscr{P}_2(\mathbb{R}^d)$. The *superdifferential* of U at $\mu \in \mathscr{P}_2(\mathbb{R}^d)$ is defined as the set $D^+U(\mu) := -D^-(-U)(\mu)$. It can be shown that if both $D^-U(\mu)$ and $D^+U(\mu)$ are non-empty sets, then they coincide and are reduced to a singleton $\{\xi\}$. In this case we say that U is *differentiable* at μ and $D_\mu U(\mu) := \xi$ is called the *Wasserstein gradient* of U at μ. In general, this is equal to the gradient of the first variation of U.

CURVES, GEODESICS IN $\mathbb{W}_p$ AND THE BENAMOU-BRENIER FORMULA

Another central pillar in the modern theory of OT is the work of F. Otto (see [Otto1]), studying the differential geometric features of the space $\mathbb{W}_p(\Omega)$ in the context of the porous medium equation. This is closely related to the work of J.-D. Benamou and Y. Brenier ([BB00]) which we detail in a moment.

From geometrical point of view, firstly it is important to understand the curves and geodesics in $\mathbb{W}_p(\Omega)$. A very good reference for this subject is the monograph of L. Ambrosio, N. Gigli and G. Savaré (see [AGS08]). In this context it is possible to make a strong link between absolutely continuous curves in $\mathbb{W}_p$ and solutions of the continuity equation

$$\partial_t \mu_t + \nabla \cdot (\mathbf{u}_t \mu_t) = 0.$$

Let us recall that a curve $(\mu_t)_{t\in[0,1]}$ in $\mathbb{W}_p(\Omega)$ is *absolutely continuous* if there exists $g \in L^1([0,1])$ such that

$$W_p(\mu_{t_1}, \mu_{t_2}) \leq \int_{t_1}^{t_2} g(s)\,\mathrm{d}s, \quad \text{for all } 0 \leq t_1 < t_2 \leq 1.$$

If $(\mu_t)_{t\in[0,1]}$ is a Lipschitz continuous curve then its metric derivative $|\dot{\mu}|_{W_p}(t)$ exists for almost every time $t \in [0,1]$ (as a consequence of Rademacher's theorem), and the above inequality remains true if one replaces g by $|\dot{\mu}|_{W_p}(t)$. Notice that any absolutely continuous curve can be reparametrized in time and becomes Lipschitz continuous. Thus the above reasoning holds for absolutely continuous curves as well. Let us recall that the *metric derivative* is defined (as is any metric space) by

$$|\dot{\mu}|_{W_p}(t) = \lim_{h\to 0} \frac{W_p(\mu_{t+h}, \mu_t)}{|h|},$$

provided the limit exists. Let us state the following very important characterization theorem (see for example [AGS08] or [San15], Theorem 5.14.)

Theorem 0.0.9. *Let $(\mu_t)_{t\in[0,1]}$ be an absolutely continuous curve in $\mathbb{W}_p(\Omega)$ (for $p > 1$ and $\Omega \subset \mathbb{R}^d$ compact). Then for a.e. $t \in [0,1]$ there exists a vector field $\mathbf{u}_t \in L^p_{\mu_t}(\Omega; \mathbb{R}^d)$ such that*

1. *the continuity equation $\partial_t \mu_t + \nabla \cdot (\mathbf{u}_t \mu_t) = 0$ is satisfied in the sense of distributions;*

2. *for a.e.* $t \in [0,1]$ *we have* $\|u_t\|_{L^p_{\mu_t}} \leq |\dot{\mu}|_{W_p}(t)$.

Conversely, if $(\mu_t)_{t\in[0,1]}$ *is a family of measures in* $\mathbb{W}_p(\Omega)$ *and for each* $t \in [0,1]$ *we have a vector field* $\mathbf{u}_t \in L^p_{\mu_t}(\Omega;\mathbb{R}^d)$ *with* $\int_0^1 \|\mathbf{u}_t\|_{L^p_{\mu_t}} \,\mathrm{d}t < +\infty$ *solving* $\partial_t\mu_t + \nabla\cdot(\mathbf{u}_t\mu_t) = 0$ *in the sense of distributions, then* $(\mu_t)_{t\in[0,1]}$ *is absolutely continuous in* $\mathbb{W}_p(\Omega)$ *and for a.e.* $t \in [0,1]$ *we have* $|\dot{\mu}|_{W_p}(t) \leq \|\mathbf{u}_t\|_{L^p_{\mu_t}}$.

It is well known (see [AGS08]) that $\mathbb{W}_p(\Omega)$ is a *geodesic space*, i.e. for all $\mu,\nu \in \mathscr{P}_p(\Omega)$ there exists a constant speed geodesic $(\mu_t)_{t\in[0,1]}$ in $\mathbb{W}_p(\Omega)$ for which $\mu_0 = \mu$ and $\mu_1 = \nu$. The notion that $(\mu_t)_{t\in[0,1]}$ is a *constant speed geodesic* in $\mathbb{W}_p(\Omega)$ (as in any geodesic space) means that

$$W_p(\mu_{t_1},\mu_{t_2}) = (t_2-t_1)W_p(\mu_0,\mu_1), \quad \text{for all } 0 \leq t_1 \leq t_2 \leq 1.$$

By the interpolation introduced by R.J. McCann (see [McC97]) one can characterize the constant speed geodesics in $\mathbb{W}_p(\Omega)$. Let us suppose that $\Omega \subset \mathbb{R}^d$ is convex and for $p \geq 1$ let $\mu,\nu \in \mathscr{P}_p(\Omega)$. Let moreover $\gamma_{\mathrm{opt}} \in \Pi_o(\mu,\nu)$ be an optimal transport plan in Problem (KP) for the cost $c(x,y) = |x-y|^p$ transporting μ onto ν. Then

$$\mu_t := ((1-t)x + ty)_{\#}\gamma_{\mathrm{opt}}, \quad \forall\, t \in [0,1] \tag{0.0.34}$$

gives a constant speed geodesic connecting μ to ν. In particular, if μ is a 'good' measure (meaning that it does not give mass to Borel sets with Hausdorff dimension less than or equal to $d-1$) γ_{opt} is induced by a map T, the solution of Problem (MP) with cost $c(x,y) = |x-y|^p$, and in this case the geodesic μ_t has the form of

$$\mu_t := ((1-t)\mathrm{id} + tT)_{\#}\mu, \quad \forall\, t \in [0,1]. \tag{0.0.35}$$

Inspired by problems from fluid mechanics and by the interpolations (0.0.34)-(0.0.35), in the seminal paper [BB00] J.-D. Benamou and Y. Brenier introduced a dynamic characterization of the distance W_p. This provided a *dynamical formulation* of the Monge-Kantorovich optimal transport problems (MP)-(KP). We refer to this formulation nowadays as the *Benamou-Brenier formula*. The basic idea behind this formula is that looking for an optimal transport map or plan from $\mu \in \mathscr{P}_p(\Omega)$ onto $\nu \in \mathscr{P}_p(\Omega)$ for the cost $c(x,y) = |x-y|^p$ is equivalent to looking for constant speed geodesics in $\mathbb{W}_p(\Omega)$ connecting μ to ν. Indeed, from optimal plans one can construct geodesics, and vice-versa from constant speed geodesics via their velocity fields one can reconstruct optimal transports.

One can find constant speed geodesics by minimizing the quantity

$$\int_0^1 |\dot{\mu}|_{W_p}(t)^p \,\mathrm{d}t$$

among all Lipschitz curves $(\mu_t)_{t\in[0,1]}$. On the other hand $|\dot\mu|_{W_p}(t) = \|\mathbf{u}_t\|_{L^p_{\mu_t}}$, where $\mathbf{u}_t$ is the optimal velocity field (with minimal $L^p_{\mu_t}$ norm) in the continuity equation $\partial_t\mu_t + \nabla\cdot(\mathbf{u}_t\mu_t) = 0$ with $\mu_0 = \mu$ and $\mu_1 = \nu$.

Thus the Benamou-Brenier formula reads as

$$W_p(\mu,\nu)^p = \min\left\{\int_0^1\int_\Omega |\mathbf{u}_t|^p\,\mathrm{d}\mu_t(x)\,\mathrm{d}t : \partial_t\mu_t + \nabla\cdot(\mathbf{u}_t\mu_t) = 0,\ \mu_0 = \mu, \mu_1 = \nu\right\}. \tag{BB}$$

To gain convexity in Problem (BB) one has to redefine the objective functional. For later uses, we call it *Benamou-Brenier functional* $\mathcal{B}_p : \mathscr{P}_p(\Omega)\times\mathscr{M}(\Omega;\mathbb{R}^d) \to [0,+\infty]$ the quantity defined as

$$\mathcal{B}_p(\mu,\mathbf{E}) := \begin{cases} \displaystyle\int_\Omega \frac{1}{p}\left|\frac{\mathrm{d}\mathbf{E}}{\mathrm{d}\mu}\right|^p \mathrm{d}\mu(x), & \text{if } \mathbf{E} \ll \mu, \\ +\infty, & \text{otherwise,}\end{cases}$$

where, recalling the notation, $\dfrac{\mathrm{d}\mathbf{E}}{\mathrm{d}\mu}$ denotes the Radon-Nikodým derivative of the measure $\mathbf{E}$ w.r.t. μ and $|\nu|$ denotes the total variation of the vector valued measure ν. We extend $\mathcal{B}_p$ to time dependent families of measures (using the same notation, which will be clear from the context) as $\mathcal{B}_p : L^\infty([0,1];\mathbb{W}_p(\Omega)) \times \mathscr{M}([0,1]\times\Omega;\mathbb{R}^d) \to [0,+\infty]$ [2] is given by

$$\mathcal{B}_p(\mu,\mathbf{E}) := \begin{cases} \displaystyle\int_0^1\int_\Omega \frac{1}{p}\left|\frac{\mathrm{d}\mathbf{E}_t}{\mathrm{d}\mu_t}\right|^p \mathrm{d}\mu_t(x)\,\mathrm{d}t, & \text{if } \mathbf{E}_t \ll \mu_t, \text{ a.e. } t\in[0,1], \\ +\infty, & \text{otherwise.}\end{cases}$$

It is well known that $\mathcal{B}_p$ is jointly convex and l.s.c. w.r.t. the weak-$\star$ convergence of measures (see Section 5.3.1 in [San15]) and that, if $\partial_t\mu_t + \nabla\cdot\mathbf{E}_t = 0$, then $\mathcal{B}_p(\mu,\mathbf{E}) < +\infty$ implies that $t\mapsto\mu_t$ is a curve in $W^{1,p}([0,1];\mathbb{W}_p(\Omega))$. Coming back to curves in Wasserstein spaces, we have seen that for any distributional solution μ_t (being a narrowly continuous curve in $\mathbb{W}_p(\Omega)$) of the continuity equation $\partial_t\mu_t + \nabla\cdot(\mathbf{u}_t\mu_t) = 0$ we have the relations

$$|\dot\mu|_{W_p}(t) \le \|\mathbf{u}_t\|_{L^p_{\mu_t}} \quad\text{and}\quad W_p(\mu_{t_1},\mu_{t_2}) \le \int_{t_1}^{t_2} |\dot\mu|_{W_p}(s)\,\mathrm{d}s.$$

For curves $(\mu_t)_{t\in[0,1]}$ which are also geodesics in $\mathbb{W}_p(\Omega)$ we have the equality

$$W_p(\mu,\nu) = \int_0^1 |\dot\mu|_{W_p}(t)\,\mathrm{d}t = \int_0^1 \|\mathbf{u}_t\|_{L^p_{\mu_t}}\,\mathrm{d}t.$$

The last equality is in fact exactly the Benamou-Brenier formula (BB) with the optimal velocity field $\mathbf{u}_t$ being the density of the optimal $\mathbf{E}_t$ w.r.t. the optimal

2. Observe that when Ω is compact, $\mu \in L^\infty([0,1];\mathbb{W}_p(\Omega))$ only means that $\mu = (\mu_t)_t$ is a time-dependent family of probability measures.

μ_t. This optimal velocity field $\mathbf{u}_t$ can be computed as $\mathbf{u}_t := (T - \mathrm{id}) \circ (T_t)^{-1}$, where $T_t := (1-t)\mathrm{id} + tT$ is the transport in McCann's interpolation (0.0.35) (we assumed here that the initial measure $\mu_0 = \mu$ is a 'good' measure, so that we can use transport maps instead of plans). This vector field is easily obtained if we consider that in this interpolation particles move with constant speed $T(x) - x$, but x represents here a Lagrangian coordinate, and not an Eulerian one: if we want to know the velocity at time t at a given point, we have to find out first the original position of the particle passing through that point at that time.

Since its first appearance, the Formula (BB) has been largely used in many applications and numerical methods related to optimal transport problems. Essentially this was the same idea which led F. Otto to settle down the first building blocks on the differential geometric features of the Wasserstein spaces $\mathbb{W}_p$. This direction has been built further by L. Ambrosio, N. Gigli and G. Savaré (see [AGS08]) and by other authors.

Recently the Formula (BB) has been successfully applied in the study of weak solutions of some Mean Field Game problems as well. We return to this question in Part ii.

DISPLACEMENT CONVEXITY AND GRADIENT FLOWS IN $\mathbb{W}_p$

Dealing with variational problems involving functionals defined on Wasserstein spaces − mainly for uniqueness reasons, but not only − it is very important to own a good notion of convexity. One can easily realize that the usual notion of convexity sometimes has to be replaced to be able to deal with a larger class of functionals. $\mathbb{W}_p(\Omega)$ being a geodesic space, it is natural to define a notion of convexity along geodesics. That is exactly what R.J. McCann used first in the study of variational problems for interacting gases (see [McC97]).

Let $\mathcal{F} : \mathbb{W}_p(\Omega) \to \overline{\mathbb{R}}$ be a functional. After McCann, we say that $\mathcal{F}$ is *displacement convex* if $[0,1] \ni t \mapsto \mathcal{F}(\mu_t)$ is convex, where $(\mu_t)_{t\in[0,1]}$ is any geodesic (connecting μ_0 to μ_1) given by the interpolation (0.0.34) or (0.0.35). In this setting – similarly as in convex analysis – one can introduce the notion of $\lambda-$*displacement convexity* of $\mathcal{F}$ for $\lambda \in \mathbb{R}$, if the inequality

$$\mathcal{F}(\mu_t) \le (1-t)\mathcal{F}(\mu_0) + t\mathcal{F}(\mu_1) - \frac{\lambda}{2}t(1-t)W_p^2(\mu_0,\mu_1), \quad \forall\, t \in [0,1]$$

holds for any geodesic $(\mu_t)_{t\in[0,1]}$.

One can easily check that for instance the interaction energy

$$\rho \mapsto \int_{\Omega\times\Omega} |x-y|^2 \,\mathrm{d}\rho(x)\,\mathrm{d}\rho(y) = 2\int_\Omega |x|^2 \,\mathrm{d}\rho(x) - 2\left(\int_\Omega x\,\mathrm{d}\rho(x)\right)^2$$

in $\mathscr{P}(\Omega)$ is not convex in the usual sense (it is more likely concave), but it is displacement convex (see Section 7.3.1. from [San15]).

In the forthcoming chapters we shall consider some local functionals of the type

$$\mathcal{F}(\rho) := \begin{cases} \int_\Omega f(\rho(x))\,\mathrm{d}x, & \text{if } \rho \ll \mathscr{L}^d, \\ +\infty, & \text{otherwise.} \end{cases}$$

R.J. McCann gave a sufficient condition for the displacement convexity of $\mathcal{F}$. This reads as follows: suppose that f is superlinear, convex and $f(0) = 0$, moreover $]0, +\infty[\ni s \mapsto s^{-d} f(s^d)$ is convex and non-increasing. Then $\mathcal{F}$ is displacement convex.

With this characterization one can check that for example the relative entropy functional, defined for any probability measure $\rho \in \mathscr{P}_2(\Omega)$ as

$$\mathcal{E}(\rho) := \begin{cases} \int_\Omega \rho(x) \log \rho(x)\,\mathrm{d}x, & \text{if } \rho \ll \mathscr{L}^d, \\ +\infty, & \text{otherwise.} \end{cases} \tag{Ent}$$

is l.s.c. and displacement convex in the $\mathbb{W}_2(\Omega)$ topology. Let us remark that this functional plays a crucial role also in the study of curvature notions in metric measure spaces. Indeed, roughly speaking, the displacement convexity (more precisely the $\lambda-$displacement convexity) of the relative entropy functional characterizes the Ricci curvature of a metric measure space.

Nevertheless, there is an unfortunate observation on the notion of displacement convexity: for a given probability measure $\nu \in \mathscr{P}_2(\Omega)$ the functional $\mathscr{P}_2(\Omega) \ni \mu \mapsto W_2^2(\mu, \nu)$ is not $\lambda-$displacement convex for any $\lambda \in \mathbb{R}$! However, in several variational problems involving $W_2^2(\cdot, \nu)$ (like in gradient flows, projection problems as in Chapter 1, etc.) this is a crucial issue. That is why one needed to push a bit further the notion of displacement convexity and introduce a more general notion: *convexity along generalized geodesics*. This is defined as follows: let $\nu \in \mathscr{P}_2(\Omega)$ be a 'good' measure (for which an optimal transport map onto any other probability measure always exists), which is called the *base measure*. Now for any $\mu_0, \mu_1 \in \mathscr{P}_2(\Omega)$ let us consider the optimal transport maps T_0 and T_1 in the transport of ν onto μ_0 and μ_1 respectively. We call a *generalized geodesic of base* ν the curve $(\mu_t)_{t\in[0,1]}$ defined as $\mu_t := ((1-t)T_0 + tT_1)_\# \nu$. We say that the functional $\mathcal{F}$ is convex along generalized geodesics if

$$\mathcal{F}(\mu_t) \le (1-t)\mathcal{F}(\mu_0) + t\mathcal{F}(\mu_1), \ \ \forall\, t \in [0,1].$$

One checks immediately that the functional $W_2^2(\cdot, \nu)$ satisfies this condition with base measure ν.

Notice that we discussed here only the case when ν is a 'good' measure. For more general cases and for the notion of $\lambda-$convexity along generalized geodesics, one has to work with so-called 3-plan interpolations of optimal plans from $\Pi(\nu, \mu_0)$ and $\Pi(\nu, \mu_1)$. We refer to [AGS08] for this construction. Moreover, let us remark that this notion is consistent in the following sense: if a functional is displacement convex, it is also convex along generalized geodesics.

The notion of *gradient flows* in the Wasserstein space $\mathbb{W}_p(\Omega)$ has an important role in the analysis of evolutive PDEs. This notion in metric spaces has been initiated by E. De Giorgi and later by L. Ambrosio (see [Amb95]). Nevertheless, it gained a lot of attention after the seminal paper or R. Jordan, D. Kinderlehrer and F. Otto ([JKO98]) and the paper of F. Otto on the porous medium equation (see [Otto1]). A nowadays classical reference for gradient flows in metric- and Wasserstein spaces is the monograph of L. Ambrosio, N. Gigli and G. Savaré (see [AGS08]).

Let us make an analogy with the euclidean case. Let $F : \mathbb{R}^d \to \mathbb{R}$ be a function and let us consider the following ODE

$$\begin{cases} \dot{x}(t) = -\nabla F(x(t)), & t > 0 \\ x(0) = x_0. \end{cases}$$

By the Cauchy-Lipschitz theory, if $F \in C^{1,1}$ there exists a unique global solution $x : [0, +\infty[\to \mathbb{R}^d$ of the above problem (actually uniqueness holds with a weaker condition, namely if F is $\lambda-$convex). Geometrically this curve follows the steepest descent of the function F. This geometric feature is a special property. To see why, let us consider an implicit Euler discretization of the above problem: let $\tau > 0$ be a time step and for $N > 0$ (on the time interval $[0, T]$, where $T = N\tau$) let us construct the points $x_0^\tau, \dots, x_N^\tau$ by $x_0^\tau := x_0$ and for $k \in \{0, \dots, N-1\}$

$$x_{k+1}^\tau := \mathrm{argmin}_{y \in \mathbb{R}^d} \left\{ F(y) + \frac{1}{2\tau}|x_k^\tau - y|^2 \right\}.$$

Now if we construct any reasonable interpolation between the points $x_0^\tau, \dots, x_N^\tau$ (piecewise constant, piecewise linear, etc.) this will converge to the solution of the ODE as the $\tau \downarrow 0$.

The special feature of this problem consists in the fact that this scheme can be written in any (reasonable) metric space, replacing the euclidean distance square by the square of the metric. In particular, if we consider the Wasserstein space $\mathbb{W}_2(\Omega)$, many well-known evolutive PDEs can be recovered as the gradient flows of well-chosen functionals. As an example, the heat equation corresponds to the gradient flow of the relative entropy functional. Let us see heuristically why is this true. For this let us consider an initial heat distribution $\rho_0 \in \mathscr{P}_2(\mathbb{R}^d)$ and for $\tau > 0$ the scheme $\rho_0^\tau := \rho_0$,

$$\rho_{k+1}^\tau := \mathrm{argmin}_{\rho \in \mathscr{P}_2(\mathbb{R}^d)} \left\{ \mathcal{E}(\rho) + \frac{1}{2\tau} W_2^2(\rho_k^\tau, \rho) \right\}$$

for $k > 0$. Now ρ_{k+1}^τ being the optimizer (by convexity arguments it is unique), it satisfies the first order optimality condition (given by Corollary 0.0.8): there exists a constant $C \in \mathbb{R}$ and a Kantorovich potential φ in the transport of ρ^τ_{k+1} onto ρ^τ_k such that

$$\log(\rho^\tau_{k+1}) + \frac{\varphi}{\tau} = C.$$

Taking the gradient of both sides and using Brenier's theorem, one obtains

$$\mathbf{v}^\tau_{k+1} := \frac{T - \mathrm{id}}{\tau} = -\frac{\nabla \rho^\tau_{k+1}}{\rho^\tau_{k+1}},$$

where $T = \mathrm{id} - \nabla\varphi$ is the optimal transport map in the transport of ρ^τ_{k+1} onto ρ^τ_k. Note that the above quantity can be seen as a discrete velocity (being the ratio between displacement and time). Now if we construct (suitable) interpolations between the ρ^τ_k's and between the $\mathbf{v}^\tau_k$'s, these will converge as $\tau \downarrow 0$ to the solution of the continuity equation $\partial_t \rho_t + \nabla \cdot (\mathbf{v}_t \rho_t) = 0$, where $\mathbf{v}_t$ can be identified with $-\nabla\rho_t/\rho_t$. Thus $\rho : [0, +\infty[\times \mathbb{R}^d \to \mathbb{R}$ heuristically solves the heat equation

$$\begin{cases} \partial_t \rho - \Delta\rho = 0, & \text{in }]0, +\infty[\times \mathbb{R}^d, \\ \rho(0, \cdot) = \rho_0. \end{cases}$$

For more details we refer to [San15, Chapter 8] or to [AGS08].

1

BV estimates in Optimal Transport with applications

In this chapter we study the BV regularity for solutions of certain variational problems in Optimal Transportation. We prove that the Wasserstein projection of a measure with BV density on the set of measures with density bounded by a given BV function f is of bounded variation as well and we also provide a precise estimate of its BV norm. Of particular interest is the case $f = 1$, corresponding to a projection onto a set of densities with an L^∞ bound, where we prove that the total variation decreases by projection. This estimate and, in particular, its iterations have a natural application to some evolutionary PDEs as, for example, the ones describing a crowd motion. In fact, as an application of our results, we obtain BV estimates for solutions of some non-linear parabolic PDE by means of optimal transportation techniques. We also establish some properties of the Wasserstein projection which are interesting in their own, and allow for instance to prove uniqueness of such a projection in a very general framework.

This chapter is based on a joint work with G. De Philippis, F. Santambrogio and B. Velichkov (see [DPMSV15]).

1.1 Introduction

Among variational problems involving optimal transportation and Wasserstein distances, a very recurrent one is the following

$$\min_{\varrho\in\mathscr{P}_2(\Omega)} \frac{1}{2}W_2^2(\varrho, g) + \tau F(\varrho)\,, \tag{1.1.1}$$

where F is a given functional on probability measures, $\tau > 0$ a parameter which can possibly be small, and g is a given probability in $\mathscr{P}_2(\Omega)$ (the space of probability measures on $\Omega \subseteq \mathbb{R}^d$ with finite second moment $\int_\Omega |x|^2 \, d\varrho(x) < +\infty$). This very instance of the problem is exactly the one we face in the time-discretization of the gradient flow of F in $\mathscr{P}_2(\Omega)$, where $g = \varrho_k^\tau$ is the measure at step k, and the optimal ϱ will be the next measure ϱ_{k+1}^τ. Under suitable assumptions, at the limit when $\tau \to 0$, this sequence converges to a curve of measures which is the gradient flow of F (see [AGS08, Amb95] for a general description of this theory).

The same problem also appears in other frameworks as well, for fixed τ. For instance in image processing, if F is a smoothing functional, this is a model to find a better (smoother) image ϱ which is not so far from the original g (the choice of the distance W_2 in this case can be justified by robustness arguments), see [LLSV15]. In some urban planning models (see [BS05, San07]) g represents the distribution of some resources and ϱ that of population, which from one side is attracted by the resources g and on the other avoids creating zones of high density thus guaranteeing enough space for each individual. In this case the functional F favors diffused measures, for instance $F(\varrho) = \int h(\varrho(x)) \, dx$, where h is a convex and superlinear function, which gives a higher cost to high densities of ϱ. Alternatively, g could represent the distribution of population, and ϱ that of services, to be chosen so that they are close enough to g but more concentrated. This effect can be obtained by choosing F that favors concentrated measures.

When F takes only the values 0 and $+\infty$, (1.1.1) becomes a projection problem. Recently, the projection onto the set [1] $\mathcal{K}_1$ of densities bounded above by the constant 1 has received lot of attention. This is mainly due to its applications in the time-discretization of evolution problems with density constraints typically associated to crowd motion. For a precise description of the associated model we refer to [RC11, MRCS10] and to Chapter 2, where a crowd is described as a population of particles which cannot overlap, and cannot go beyond a certain threshold density.

In this paper we concentrate on the case where $F(\varrho) = \int h(\varrho)$ for a convex integrand $h : \mathbb{R}_+ \to \mathbb{R} \cup \{+\infty\}$. The case of the projection on $\mathcal{K}_1$ is obtained by taking the following function:

$$h(\varrho) = \begin{cases} 0, & \text{if } 0 \le \varrho \le 1 \\ +\infty, & \text{if } \varrho > 1 \, , \end{cases}$$

1. Here and in the sequel we denote by $\mathcal{K}_f$ the set of absolutely continuous measure with density bounded by f:

$$\mathcal{K}_f := \{\varrho \in \mathscr{P}(\Omega) \, : \, \varrho \le f \, dx\}$$

We are interested in the estimates on the minimizer $\bar{\varrho}$ of (1.1.1). In general then can be divided into two categories: the ones which are independent of g (but depend on τ) and the ones uniform in τ (dependent on g). A typical example of the first type of estimate can be obtained by writing down the optimality conditions for (1.1.1). In the case $F(\varrho) = \int h(\varrho)$, we get $\varphi + \tau h'(\bar{\varrho}) = const$, where φ is the Kantorovich potential in the transport from $\bar{\varrho}$ to g (in fact this equality holds only $\bar{\varrho}-$a.e., but we skip the details and just recall the heuristic argument). On a bounded domain, φ is Lipschitz continuous with a universal Lipschitz constant depending only on the domain, and so is $\tau h'(\bar{\varrho})$. If h is strictly convex and C^1, then we can deduce the Lipschitz continuity for $\bar{\varrho}$. The bounds on the Lipschitz constant of $\bar{\varrho}$ do not really depend on g, but on the other hand they clearly degenerate as $\tau \to 0$. Another bound that one can prove is $\|\bar{\varrho}\|_{L^\infty} \leq \|g\|_{L^\infty}$ (see [CS05, San07]), which, on the contrary, is independent of τ.

In this Chapter we are mainly concerned with BV estimates. As we expect uniform bounds, in what follows we get rid of the parameter τ.

We recall that for every function $\varrho \in L^1$ and every open set A the total variation of ϱ in A is defined as

$$TV(\varrho, A) = \int_A |\nabla \varrho| \, dx = \sup \left\{ \int_A \varrho \, \mathrm{div}\xi \, dx \quad : \quad \xi \in C_c^1(A), \quad |\xi| \leq 1 \right\}.$$

Our main theorem reads as follows:

Theorem 1.1.1. *Let $\Omega \subset \mathbb{R}^d$ be a (possibly unbounded) convex set, $h : \mathbb{R}_+ \to \mathbb{R} \cup \{+\infty\}$ be a convex and l.s.c. function and $g \in \mathscr{P}_2(\Omega) \cap BV(\Omega)$. If $\bar{\varrho}$ is a minimizer of the following variational problem*

$$\min_{\varrho \in \mathscr{P}_2(\Omega)} \frac{1}{2} W_2^2(\varrho, g) + \int_\Omega h(\varrho(x)) \, dx \,,$$

then

$$\int_\Omega |\nabla \bar{\varrho}| \, dx \leq \int_\Omega |\nabla g| \, dx \,. \tag{1.1.2}$$

As we said, this covers the case of the Wasserstein projection of g on the subset $\mathcal{K}_1$ of $\mathscr{P}_2(\Omega)$ given by the measures with density less than or equal to 1. Starting from Theorem 1.1.1 and constructing an appropriate approximating sequence of functionals we are actually able to establish BV bounds for more general Wasserstein projections related to a prescribed BV function f. More precisely we have the following result.

Theorem 1.1.2. *Let $\Omega \subset \mathbb{R}^d$ be a (possibly unbounded) convex set, $g \in \mathscr{P}_2(\Omega) \cap BV(\Omega)$ and let $f \in BV_{\mathrm{loc}}(\Omega)$ be a function with*

$$\int_\Omega f \, dx \geq 1.$$

If

$$\bar\varrho = \operatorname{argmin}\Big\{ W_2^2(\varrho, g) \ : \ \varrho \in \mathscr{P}_2(\Omega),\ \varrho \in \mathcal{K}_f \Big\}, \tag{1.1.3}$$

then

$$\int_\Omega |\nabla \bar\varrho| \, dx \le \int_\Omega |\nabla g| \, dx + 2 \int_\Omega |\nabla f| \, dx. \tag{1.1.4}$$

We would like to spend some words on the BV estimate for the projection on the set $\mathcal{K}_1$, which is the original motivation for this paper. We note that this corresponds to the case

$$h(\varrho) = \begin{cases} 0, & \text{if } \varrho \in [0,1], \\ +\infty, & \text{if } \varrho > 1, \end{cases}$$

in Theorem 1.1.1 and to the case $f = 1$ in Theorem 1.1.2. In both cases we obtain that (1.1.2) holds.

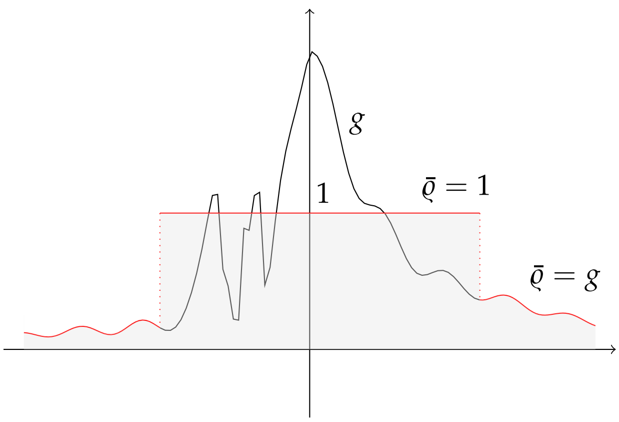

Figure 1: Projection in 1D

In dimension one the estimate (1.1.2) can be obtained by some direct considerations. In fact, by [Fig10] we have that the constraint $\bar\varrho \le 1$ is saturated, i.e. the projection is of the form

$$\bar\varrho(x) = \begin{cases} 1, & \text{if } x \in A, \\ g(x), & \text{if } x \notin A, \end{cases}$$

for an open set $A \subset \mathbb{R}$. Since we are in dimension one, A is a union of intervals and so it is sufficient to show that (1.1.2) holds in the case that A is just one interval, as in the picture on the left. In this case it is immediate to check that the total variation of g has not increased after the projection since $\bar\varrho = 1$ on A, while there is necessarily a point $x_0 \in A$ such that $g(x_0) \ge 1$.

In dimension $d \geq 2$ the estimate (1.1.2) is more involved essentially due to the fact that the projection tends to spread in all directions. This geometric phenomenon can be illustrated with the following simple example. Consider the function $g = (1+\varepsilon)\mathbb{1}_{B(0,R)}$, where $\varepsilon > 0$ and $R > 0$ are such that $(1+\varepsilon)|B(0,R)| = 1$. By the saturation of the constraint and symmetry considerations the projection $\bar{\varrho}$ of g is the characteristic function $\bar{\varrho} = \mathbb{1}_{B(0,\overline{R})}$, where $\overline{R} = (1+\varepsilon)^{1/d}R$. The total variation involves two opposite effects: the perimeter of the ball increases, but the height of the jump passes from $1+\varepsilon$ to 1. In fact we have

$$\int_{\mathbb{R}^d} |\nabla\bar{\varrho}|\,\mathrm{d}x = d\omega_d\overline{R}^{d-1} = d\omega_d R^{d-1}(1+\varepsilon)^{(d-1)/d} \leq d\omega_d R^{d-1}(1+\varepsilon)$$
$$= \int_{\mathbb{R}^d} |\nabla g|\,\mathrm{d}x.$$

Further explicit examples are difficult to construct. Even in the case $g = (1+\varepsilon)\mathbb{1}_\Omega$, where Ω is a union of balls, it is not trivial to compute the BV norm of the projection, which is the characteristic function of a union of (overlapping) balls.

The BV estimates are useful when the projection is treated as one time-step of a discretized evolution process. For instance, a BV bound allows to transform weak convergence in the sense of measures into strong L^1 convergence. Also, if we consider a PDE mixing a smooth evolution, such as the Fokker-Planck evolution, and some projection steps (in order to impose a density constraint, as in crowd motion issues, treated in details in Chapter 2), one could wonder which bounds on the regularity of the solution are preserved in time. From the fact that the discontinuities in the projected measure destroy any kind of $W^{1,p}$ norm, it is natural to look for BV bounds. Notice by the way that, for these kind of applications, proving $\int_\Omega |\nabla\bar{\varrho}| \leq \int_\Omega |\nabla g|$ (with no multiplicative coefficient nor additional term) is crucial in order to iterate this estimate at every step.

The structure of this chapter is as follows: in Section 1.2 we establish our *main inequality*, in Section 1.3 we prove Theorem 1.1.1 while in Section 1.4 we collect some properties of solution of (1.1.3) which can be interesting in their own and we we prove Theorem 1.1.2. Eventually, in Section 1.5 we present some applications of the above results, connections with other variational and evolution problems and some open questions.

1.2 THE MAIN INEQUALITY

In this section we establish the key inequality needed in the proof of Theorems 1.1.1 and 1.1.2.

Lemma 1.2.1. *Suppose that $\varrho, g \in L^1_+(\Omega)$ are smooth probability densities, which are bounded away from 0 and infinity, $\Omega \subset \mathbb{R}^d$ a bounded and uniformly convex domain and let $H \in C^2(\mathbb{R}^d)$ be a convex function. Then we have the following inequality*

$$\int_\Omega \Big(\varrho\,\nabla\cdot[\nabla H(\nabla\varphi)] - g\,\nabla\cdot[\nabla H(-\nabla\psi)]\Big)\,\mathrm{d}x \le 0, \tag{1.2.1}$$

where (φ, ψ) is a choice of Kantorovich potentials.

Proof. We first note that since ϱ and g are smooth and away from zero and infinity in Ω, Theorem 0.0.5 implies that φ, ψ are smooth as well.

Now using the identity $S(T(x)) \equiv x$ and that $S_\# g = \varrho$ we get

$$\begin{aligned}\int_\Omega \varrho(x)\,\nabla\cdot[\nabla H(\nabla\varphi(x))]\,\mathrm{d}x &= \int_\Omega g(x)\,\Big[\nabla\cdot[\nabla H(\nabla\varphi)]\Big](S(x))\,\mathrm{d}x\\ &= \int_\Omega g(x)\,\nabla\cdot\Big[\nabla H(\nabla\varphi\circ S)\Big](x)\,\mathrm{d}x\\ &+ \int_\Omega g(x)\,\Big(\Big[\nabla\cdot[\nabla H(\nabla\varphi)]\Big](S(x)) - \nabla\cdot[\nabla H(\nabla\varphi\circ S)](x)\Big)\,\mathrm{d}x,\end{aligned}$$

and, by the equality

$$-\nabla\psi(x) = S(x) - x = S(x) - T(S(x)) = \nabla\varphi(S(x)),$$

we obtain

$$\begin{aligned}\int_\Omega \Big(\varrho\,\nabla\cdot[\nabla H(\nabla\varphi)] - g\,\nabla\cdot[\nabla H(-\nabla\psi)]\Big)\,\mathrm{d}x =&\\ = \int_\Omega g(x)\,\Big(\Big[\nabla\cdot[\nabla H(\nabla\varphi)]\Big](S(x)) - \nabla\cdot[\nabla H(\nabla\varphi\circ S)](x)\Big)\,\mathrm{d}x&\\ = \int_\Omega \varrho(x)\,\Big(\nabla\cdot[\nabla H(\nabla\varphi)] - \Big[\nabla\cdot[\nabla H(\nabla\varphi)\circ S]\Big]\circ T\Big)\,\mathrm{d}x.&\end{aligned} \tag{1.2.2}$$

For simplicity we set

$$\begin{aligned}E &= \nabla\cdot(\nabla H(\nabla\varphi)) - \big[\nabla\cdot(\nabla H(\nabla\varphi)\circ S)\big]\circ T\\ &= \nabla\cdot\xi - [\nabla\cdot(\xi\circ S)]\circ T,\end{aligned} \tag{1.2.3}$$

where by ξ we denote the continuously differentiable function

$$\xi(x) = (\xi^1, \ldots, \xi^d) := \nabla H(\nabla\varphi(x)),$$

whose derivative is given by

$$D\xi = D\big(\nabla H(\nabla\varphi)\big) = D^2 H(\nabla\varphi)\cdot D^2\varphi.$$

We now calculate

$$\begin{aligned}\left[\nabla\cdot(\xi\circ S)\right]\circ T &= \sum_{i=1}^{d}\frac{\partial(\xi^i\circ S)}{\partial x^i}\circ T = \sum_{i=1}^{d}\sum_{j=1}^{d}\frac{\partial\xi^i}{\partial x^j}(S(T))\frac{\partial S^j}{\partial x^i}\circ T\\ &= \operatorname{tr}\left(D\xi\cdot(DT)^{-1}\right) = \operatorname{tr}\left(D^2H(\nabla\varphi)\cdot D^2\varphi\cdot(I_d-D^2\varphi)^{-1}\right),\end{aligned} \tag{1.2.4}$$

where the last two equality follow by $DS\circ T=(DT)^{-1}$ and we also used that $(DT)^{-1}=(I_d-D^2\varphi)^{-1}$, where I_d is the d-dimensional identity matrix.

By (1.2.3) and (1.2.4) we have that

$$\begin{aligned}E &= \operatorname{tr}\left[D^2H(\nabla\varphi)\cdot D^2\varphi\cdot\left(I_d-(I_d-D^2\varphi)^{-1}\right)\right]\\ &= -\operatorname{tr}\left[D^2H(\nabla\varphi)\cdot\left[D^2\varphi\right]^2\cdot(I_d-D^2\varphi)^{-1}\right].\end{aligned}$$

Since we have that

$$I_d-D^2\varphi\geq 0,$$

and that the trace of the product of two positive matrices is positive, we obtain $E\leq 0$, which together with (1.2.2) concludes the proof. □

Lemma 1.2.2. *Let $\Omega\subset\mathbb{R}^d$ be bounded and convex, $\varrho, g\in W^{1,1}(\Omega)$ be two probability densities and $H\in C^2(\mathbb{R}^d)$ be a radially symmetric convex function. Then the following inequality holds*

$$\int_\Omega\left(\nabla\varrho\cdot\nabla H(\nabla\varphi)+\nabla g\cdot\nabla H(\nabla\psi)\right)\mathrm{d}x\geq 0, \tag{1.2.5}$$

where (φ,ψ) is a choice of Kantorovich potentials.

Proof. Let us start observing that, due to the radial symmetry of H,

$$\nabla H(\nabla\psi) = -\nabla H(-\nabla\psi). \tag{1.2.6}$$

Step 1. Proof in the smooth case. Suppose that the probability densities ϱ and g are smooth and bounded away from zero and infinity and that Ω is uniformly convex. As in Lemma 1.2.1, we note that under these assumption on ϱ and g the Kantorovich potentials are smooth, hence after integration by part the left hand side of (1.2.5) becomes

$$\begin{aligned}\int_\Omega\left(\nabla\varrho\cdot\nabla H(\nabla\varphi)+\nabla g\cdot\nabla H(\nabla\psi)\right)\mathrm{d}x &= \int_{\partial\Omega}\left(\varrho\,\nabla H(\nabla\varphi)\cdot\mathbf{n}+g\,\nabla H(\nabla\psi)\cdot\mathbf{n}\right)\mathrm{d}\mathscr{H}^{d-1}\\ &\quad-\int_\Omega\left(\varrho\,\nabla\cdot\left[\nabla H(\nabla\varphi)\right]+g\,\nabla\cdot\left[\nabla H(\nabla\psi)\right]\right)\mathrm{d}x\\ &\geq\int_{\partial\Omega}\left(\varrho\,\nabla H(\nabla\varphi)+g\,\nabla H(\nabla\psi)\right)\cdot\mathbf{n}\,\mathrm{d}\mathscr{H}^{d-1},\end{aligned}$$

where we used Lemma 1.2.1 and (1.2.6). Moreover, by the radial symmetry of H one has that $\nabla H(z) = c(z)z$, for some $c(z) > 0$. Since the gradients of the Kantorovich potentials $\nabla\varphi$ and $\nabla\psi$ calculated in boundary points are pointing outward Ω (since $T(x) = x - \nabla\varphi(x) \in \Omega$, and $S(x) = x - \nabla\psi(x) \in \Omega$) and Ω is convex, we have that

$$\nabla H(\nabla\varphi(x)) \cdot \mathbf{n}(x) \geq 0 \qquad \text{and} \qquad \nabla H(\nabla\psi(x)) \cdot \mathbf{n}(x) \geq 0, \qquad \forall x \in \partial\Omega,$$

which concludes the proof of (1.2.5) if ϱ and g are smooth.

Step 2. Withdrawing smoothness and uniform convexity assumptions. We first note that for every $\varepsilon > 0$ there exist a uniformly convex domain Ω_ε such that $\Omega \subset \Omega_\varepsilon \subset \Omega'$ (where Ω' is a larger fixed convex domain) and $|\Omega_\varepsilon \setminus \Omega| \to 0$, and smooth nonnegative functions $\varrho_\varepsilon \in C^1(\overline{\Omega}')$ and $g_\varepsilon \in C^1(\overline{\Omega}')$ such that

$$\varrho_\varepsilon \xrightarrow[\varepsilon\to 0]{W^{1,1}(\Omega')} \varrho \qquad \text{and} \qquad g_\varepsilon \xrightarrow[\varepsilon\to 0]{W^{1,1}(\Omega')} g.$$

We will suppose that both ϱ_ε and g_ε are probability densities on Ω_ε. Moreover, by adding a positive constant and then multiplying by another one, we may assume that ϱ_ε and g_ε are probability densities away from zero:

$$\varrho_\varepsilon \geq \varepsilon, \qquad g_\varepsilon \geq \varepsilon \qquad \text{and} \qquad \int_{\Omega_\varepsilon} \varrho_\varepsilon \, dx = \int_{\Omega_\varepsilon} g_\varepsilon \, dx = 1.$$

Let $\varphi_\varepsilon \in C^{2,\beta}(\overline{\Omega}_\varepsilon)$ and $\psi_\varepsilon \in C^{2,\beta}(\overline{\Omega}_\varepsilon)$ be the Kantorovich potentials corresponding to the optimal transport maps between ϱ_ε and g_ε. By *Step 1* we have

$$\int_{\Omega_\varepsilon} \Big(\nabla\varrho_\varepsilon \cdot \nabla H(\nabla\varphi_\varepsilon) + \nabla g_\varepsilon \cdot \nabla H(\nabla\psi_\varepsilon)\Big) \, dx \geq 0. \tag{1.2.7}$$

Note that from the boundedness of Ω' we infer $|\nabla\varphi_\varepsilon|, |\nabla\psi_\varepsilon| \leq C$. Moreover, ∇H is locally bounded, which also implies $|\nabla H(\nabla\varphi_\varepsilon)|, |\nabla H(\nabla\psi_\varepsilon)| \leq C$. On the other hand, from $|\Omega_\varepsilon \setminus \Omega| \to 0$, supposing that the convergence $\nabla\varrho_\varepsilon \to \nabla\varrho$ and $\nabla g_\varepsilon \to \nabla g$ holds a.e. and is dominated, when we pass to the limit as $\varepsilon \to 0$ the integral restricted to $\Omega_\varepsilon \setminus \Omega$ is negligible. On Ω we use Theorem 0.0.6, the bounds on $|\nabla H(\nabla\varphi_\varepsilon)|, |\nabla H(\nabla\psi_\varepsilon)|$ and

$$\nabla\varphi_\varepsilon \xrightarrow[\varepsilon\to 0]{a.e.} \nabla\varphi \qquad \text{and} \qquad \nabla\psi_\varepsilon \xrightarrow[\varepsilon\to 0]{a.e.} \nabla\psi.$$

Passing to the limit as $\varepsilon \to 0$ in (1.2.7) we obtain (1.2.5), which concludes the proof. □

Remark 1.2.1. In Lemma 1.2.2 we can drop the convexity assumption on Ω if ϱ, g have compact support: indeed, it is enough to choose a ball $\Omega' \supset \Omega$ containing the supports of ϱ and g.

Remark 1.2.2. Lemma 1.2.2 also remains true in the case of compactly supported densities g and ϱ, even if we drop the assumption on H, that $H(z) = H(|z|)$. In this case the inequality becomes

$$\int_{\mathbb{R}^d} \Big(\nabla\varrho \cdot \nabla H(\nabla\varphi) - \nabla g \cdot \nabla H(-\nabla\psi)\Big)\, dx \geq 0.$$

Proof. The proof follows the same scheme of that of Lemma 1.2.2, first in the smooth case and then for approximation. We select a convex domain Ω large enough to contain the supports of ϱ and g in its interior: all the integrations and integration by parts are performed on Ω. The only difficulty is that we cannot guarantee the boundary term to be positive. Yet, we first take ϱ, g to be smooth and we approximate them by taking $\varrho_\varepsilon := \varepsilon\frac{1}{|\Omega|} + (1-\varepsilon)\varrho$ and $g_\varepsilon := \varepsilon\frac{1}{|\Omega|} + (1-\varepsilon)g$. For these densities and their corresponding potentials $\varphi_\varepsilon, \psi_\varepsilon$, we obtain the inequality

$$\int_\Omega \Big(\nabla\varrho_\varepsilon \cdot \nabla H(\nabla\varphi_\varepsilon) - \nabla g_\varepsilon \cdot \nabla H(-\nabla\psi_\varepsilon)\Big)\, dx \geq \int_{\partial\Omega} \Big(\varrho_\varepsilon \nabla H(\nabla\varphi_\varepsilon) - g_\varepsilon \nabla H(-\nabla\psi_\varepsilon)\Big) \cdot \mathbf{n}\, d\mathscr{H}^{d-1}.$$

We can pass to the limit (by dominated convergence as before) in this inequality, and notice that the r.h.s. tends to 0, since $|\nabla H(\nabla\varphi_\varepsilon)|, |\nabla H(\nabla\psi_\varepsilon)| \leq C$ and $\varrho_\varepsilon = g_\varepsilon = \varepsilon/|\Omega|$ on $\partial\Omega$. Once the inequality is proven for smooth ϱ, g, a new approximation gives the desired result. $\square$

We observe that a particular case of Theorem 1.2.2, which we present here as a corollary, could have been obtained in a very different way.

Corollary 1.2.3. *Let $\Omega \subset \mathbb{R}^d$ be a given bounded convex set and $\varrho, g \in W^{1,1}(\Omega)$ be two probability densities. Then the following inequality holds*

$$\int_\Omega (\nabla\varrho \cdot \nabla\varphi + \nabla g \cdot \nabla\psi)\, dx \geq 0, \tag{1.2.8}$$

where φ and ψ are the corresponding Kantorovich potentials.

Proof. The inequality (1.2.8) follows by setting $H(z) := \dfrac{1}{2}|z|^2$ in Lemman 1.2.2. Nevertheless, in this particular case, there is an alternate proof, using the geodesic convexity of the entropy functional, which we sketch below for $\Omega = \mathbb{R}^d$.

Let us recall the definition of the entropy functional $\mathcal{E} : \mathscr{P}_2(\mathbb{R}^d) \to \mathbb{R}$ defined by

$$\mathcal{E}(\varrho) = \begin{cases} \displaystyle\int_{\mathbb{R}^d} \varrho \log \varrho\, dx, & \text{if} \quad \varrho \ll \mathscr{L}^d, \\ +\infty, & \text{otherwise,} \end{cases}$$

and let us consider the geodesic

$$[0,1] \ni t \mapsto \varrho_t \in \mathscr{P}_2(\mathbb{R}^d), \qquad \varrho_0 = \varrho, \qquad \varrho_1 = g,$$

in the Wasserstein space $\mathbb{W}_2(\mathbb{R}^d)$. It is well known (see, for example, [AGS08]) that the map $t \mapsto \mathcal{E}(\varrho_t)$ is convex and that ϱ_t solves the continuity equation

$$\partial_t \varrho_t + \nabla \cdot (\varrho_t \mathbf{v}_t) = 0, \qquad \varrho_0 = \varrho, \qquad \varrho_1 = g,$$

associated to the vector field $\mathbf{v}_t = (T - \mathrm{id}) \circ ((1-t)\mathrm{id} + tT)^{-1}$ induced by the optimal transport map $T = \mathrm{id} - \nabla\varphi$ between ϱ and g. Now since the time derivative of $\mathcal{E}(\varrho_t)$ is increasing, we get

$$-\int_{\mathbb{R}^d} \nabla\varrho \cdot \nabla\varphi \, \mathrm{d}x = \int_{\mathbb{R}^d} \varrho \mathbf{v}_0 \cdot \frac{\nabla\varrho}{\varrho} \, \mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{\{t=0\}} \mathcal{E}(\varrho_t)$$

$$\leq \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{\{t=1\}} \mathcal{E}(\varrho_t) = \int_{\mathbb{R}^d} g \mathbf{v}_1 \cdot \frac{\nabla g}{g} \, \mathrm{d}x = \int_{\mathbb{R}^d} \nabla g \cdot \nabla\psi \, \mathrm{d}x,$$

which proves the claim. □

By approximating $H(z) = |z|$ with $H(z) = \sqrt{\varepsilon^2 + |z|^2}$, Lemma 1.2.2 has the following useful corollary, where we use the convention $\dfrac{z}{|z|} = 0$ for $z = 0$.

Corollary 1.2.4. *Let $\Omega \subset \mathbb{R}^d$ be a given bounded convex set and $\varrho, g \in W^{1,1}(\Omega)$ be two probability densities. Then the following inequality holds*

$$\int_\Omega \left(\nabla\varrho \cdot \frac{\nabla\varphi}{|\nabla\varphi|} + \nabla g \cdot \frac{\nabla\psi}{|\nabla\psi|} \right) \mathrm{d}x \geq 0, \tag{1.2.9}$$

where φ and ψ are the corresponding Kantorovich potentials.

1.3 BV ESTIMATES FOR MINIMIZERS

In this section we prove Theorem 1.1.1. Since we will need to perform several approximation arguments, and we want to use $\Gamma-$convergence, we need to provide uniqueness of the minimizers. The following easy lemma is well-known among specialists.

Lemma 1.3.1. *Let $g \in \mathscr{P}(\Omega) \cap L^1_+(\Omega)$, then the functional $\mu \mapsto W_2^2(\mu, g)$ is strictly convex on $\mathscr{P}_2(\Omega)$.*

Proof. Suppose by contradiction that there exist $\mu_0 \neq \mu_1$ and $t \in]0,1[$ are such that

$$W_2^2(\mu_t, g) = (1-t)W_2^2(\mu_0, g) + tW_2^2(\mu_1, g),$$

where $\mu_t = (1-t)\mu_0 + t\mu_1$. Let γ_0 be the optimal transport plan in the transport from μ_0 to g (pay attention to the direction: it is a transport map if we see it backward: *from g to μ_0*). As the starting measure is absolutely continuous, by Brenier's Theorem, γ_0 is of the form $(T_0, id)_{\#}g$. Analogously, take

$\gamma_1 = (T_1, id)_{\#}g$ optimal from μ_1 to g. Set $\gamma_t := (1-t)\gamma_0 + t\gamma_1 \in \Pi(\mu_t, g)$. We have

$$\begin{aligned}(1-t)W_2^2(\mu_0,g) + tW_2^2(\mu_1,g) = W_2^2(\mu_t,g) &\leq \int |x-y|^2 \, d\gamma_t \\ &= (1-t)\int |x-y|^2 \, d\gamma_0 + t\int |x-y|^2 \, d\gamma_1 \\ &= (1-t)W_2^2(\mu_0,g) + tW_2^2(\mu_1,g),\end{aligned}$$

which implies that γ_t is actually optimal in the transport from g to μ_t. Yet γ_t is not induced from a transport map, unless $T_0 = T_1$ a.e. on $\{g > 0\}$. This is a contradiction with $\mu_0 \neq \mu_1$ and proves strict convexity. □

Let us denote by $\mathcal{C}$ the class of convex l.s.c. function $h : \mathbb{R}_+ \to \mathbb{R} \cup \{+\infty\}$, finite in a neighborhood of 0 and with finite right derivative $h'(0)$ at 0, and superlinear at $+\infty$.

Lemma 1.3.2. *If $h \in \mathcal{C}$ there exists a sequence of C^2 convex functions h_n, superlinear at ∞, with $h_n'' > 0$, $h_n \leq h_{n+1}$ and $h(x) = \lim_n h_n(x)$ for every $x \in \mathbb{R}_+$.*

Moreover, if $h : \mathbb{R}_+ \to \mathbb{R} \cup \{+\infty\}$ is a convex l.s.c. superlinear function, there exists a sequence of functions $h_n \in \mathcal{C}$ with $h_n \leq h_{n+1}$ and $h(x) = \lim_n h_n(x)$ for every $x \in \mathbb{R}_+$.

Proof. Let us start from the case $h \in \mathcal{C}$. Set $\ell^+ := \sup\{x : h(x) < +\infty\} \in \mathbb{R}_+ \cup \{+\infty\}$. Let us define an increasing function $\xi_n : \mathbb{R} \to \mathbb{R}$ in the following way:

$$\xi_n(x) := \begin{cases} h'(0) & \text{for } x \in]-\infty, 0] \\ h'(x) & \text{for } x \in [0, \ell^+ - \frac{1}{n}] \\ h'(\ell^+ - \frac{1}{n}) & \text{for } \ell^+ - \frac{1}{n} \leq x < \ell^+, \\ h'(\ell^+ - \frac{1}{n}) + n(x - \ell^+) & \text{for } x \geq \ell^+, \end{cases}$$

where, if the derivative of h does not exist somewhere, we just replace it with the right derivative. (Notice that when $\ell^+ = +\infty$, the last two cases do not apply).

Let $q \geq 0$ be a C^1 function with $\mathrm{spt}(q) \subset [-1, 0]$, $\int q(t)\, dt = 1$ and let us set $q_n(t) = nq(nt)$. We define h_n as the primitive of the C^1 function

$$h_n'(x) := \int_{\mathbb{R}} \left(\xi_n(t) - \frac{1}{n}e^{-t}\right) q_n(t-x)\, dt,$$

with $h_n(0) = h(0)$. It is easy to check that all the required properties are satisfied: we have $h_n''(x) \geq \frac{1}{n}e^{-x}$, h_n is superlinear because $\lim_{x\to\infty} \xi_n(x) = +\infty$, and we have increasing convergence $h_n \to h$.

For the case of a generic function h, it is possible to approximate it with functions in $\mathcal{C}$ if we define $\ell^- := \inf\{x : h(x) < +\infty\} \in \mathbb{R}_+$ and take

$$h_n(x) = \begin{cases} h(\ell^- + \frac{1}{n}) + h'(\ell^- + \frac{1}{n})(x - \ell^- - \frac{1}{n}) + n|x - \ell^-| & \text{for } x \le \ell^- \\ h(\ell^- + \frac{1}{n}) + h'(\ell^- + \frac{1}{n})(x - \ell^- - \frac{1}{n}) & \text{for } x \in]\ell^-, \ell^- + \frac{1}{n}] \\ h(x) & \text{for } x \ge \ell^- + \frac{1}{n}. \end{cases}$$

In this case as well, it is easy to check that all the required properties are satisfied. □

Proof of Theorem 1.1.1.

Proof. Let us start from the case where g is $W^{1,1}$ and bounded from below, and h is C^2, superlinear, with $h'' > 0$, and Ω is a bounded convex set. A minimizer $\bar{\varrho}$ exists (by the compactness of $\mathscr{P}_2(\Omega)$ and by the lower semicontinuity of the functional with respect to the weak convergence of measures). Thanks to Corollary 0.0.8, there exists a Kantorovich potential φ for the transport from $\bar{\varrho}$ to g such that $h'(\bar{\varrho}) = \max\{C - \varphi, h'(0)\}$. This shows that $h'(\bar{\varrho})$ is Lipschitz continuous. Hence, $\bar{\varrho}$ is bounded. On bounded sets h' is a diffeomorphism with Lipschitz inverse, thanks to $h'' > 0$, which proves that $\bar{\varrho}$ itself is Lipschitz. Then we can apply Corollary 1.2.4 and get

$$\int_\Omega \left(\nabla \bar{\varrho} \cdot \frac{\nabla \varphi}{|\nabla \varphi|} + \nabla g \cdot \frac{\nabla \psi}{|\nabla \psi|} \right) dx \ge 0.$$

Yet, a.e. on $\{\nabla \bar{\varrho} \neq 0\}$ we have $h'(\bar{\varrho}) = C - \varphi$. Using also $h'' > 0$, we get that $\nabla \varphi$ and $\nabla \bar{\varrho}$ are vectors with opposite directions. Hence we have

$$\int_\Omega |\nabla \bar{\varrho}| \, dx \le \int_\Omega \nabla g \cdot \frac{\nabla \psi}{|\nabla \psi|} \, dx \le \int_\Omega |\nabla g| \, dx,$$

which is the desired estimate.

We can generalize to $h \in \mathcal{C}$ by using the previous lemma and approximating it with a sequence h_n. Thanks to monotone convergence we have $\Gamma-$convergence for the minimization problem that we consider. We also have compactness since $\mathscr{P}_2(\Omega)$ is compact, and uniqueness of the minimizer. Hence, the minimizers $\bar{\varrho}_n$ corresponding to h_n satisfy $\int_\Omega |\nabla \bar{\varrho}_n| \, dx \le \int_\Omega |\nabla g| \, dx$ and converge to the minimizer $\bar{\varrho}$ corresponding to h. By the semicontinuity of the total variation we conclude the proof in this case.

Similarly, we can generalize to other convex functions h, approximating them with functions in $\mathcal{C}$ (notice that this is only interesting if the function h allows the existence of at least a probability density with finite cost, i.e. if $h(1/|\Omega|) < +\infty$). Also, we can take $g \in BV$ and approximate it with $W^{1,1}$ functions bounded from below. If the approximation is done for instance by convolution, then we have a sequence with $W_2(g_n, g) \to 0$, which guarantees uniform convergence of the functionals, and hence $\Gamma-$convergence.

We can also handle the case of Ω unbounded and convex, by first taking g to be such that its support is a convex bounded set, and $h \in \mathcal{C}$. In this case the optimal $\bar{\varrho}$ must be compactly supported as well. Indeed, the optimality condition $h'(\bar{\varrho}) = \max\{C - \varphi, h'(0)\}$ imposes $\bar{\varrho} = 0$ on the set where $\varphi > C - h'(0)$. But on $\{\bar{\varrho} > 0\}$ we have $\varphi = \psi^c$, where ψ is the Kantorovich potential defined on $\mathrm{spt}(g)$, which is bounded. Hence φ grows at infinity quadratically, from $\varphi(x) = \inf_{y \in \mathrm{spt}(g)} \frac{1}{2}|x-y|^2 - \psi(y)$, which implies that there is no point x with $\bar{\varrho}(x) > 0$ too far from $\mathrm{spt}(g)$. Once we know that the densities are compactly supported, the same arguments as above apply (note that being Ω convex we ca assume that the densities are supported on a bounded convex set). Then one passes to the limit obtaining the result for any generic convex function h, and then we can also approximate g (as above, we select a sequence g_n of compactly supported densities converging to g in W_2). Notice that in this case the convergence is no more uniform on $\mathscr{P}_2(\Omega)$, but it is uniform on a bounded set $W_2(\varrho, g) \leq C$ which is the only one interesting in the minimization. □

1.4 PROJECTED MEASURES UNDER DENSITY CONSTRAINTS

1.4.1 *Existence, uniqueness, characterization, stability of the projected measure*

In this section we will take $\Omega \subset \mathbb{R}^d$ be a given closed set with negligible boundary, $f : \Omega \to [0, +\infty[$ a measurable function in $L^1_{\mathrm{loc}}(\Omega)$ with $\int_\Omega f \,\mathrm{d}x > 1$ and $\mu \in \mathcal{P}_2(\Omega)$ a given probability measure on Ω. We will consider the following projection problem

$$\min_{\varrho \in \mathcal{K}_f} W_2^2(\varrho, \mu), \tag{1.4.1}$$

where we set $\mathcal{K}_f = \left\{ \varrho \in L^1_+(\Omega) \ : \ \int_\Omega \varrho \,\mathrm{d}x = 1,\ \varrho \leq f \text{ a.e.} \right\}$.

This section is devoted to the study of the above projection problem. We first want to summarize the main known results. Most of these results are only available in the case $f = 1$.

Existence. The existence of a solution to Problem (1.4.1) is a consequence of the direct method of calculus of variations. Indeed, take a minimizing sequence ϱ^n; it is tight thanks to the bound $W_2(\varrho^n, \mu) \leq C$; it admits a weakly converging subsequence and the limit minimizes the functional $W_2(\cdot, \mu)$ because of its semicontinuity and of the fact that the inequality $\varrho \leq f$ is preserved. We note that from the existence point of view, the case $f \equiv 1$ and the general case do not show any significant difference.

Characterization. The optimality conditions, derived in [RC11] exploiting the strategy developed in [MRCS10] (in the case $f = 1$, but they are easy to adapt to the general case) state the following: if ϱ is a solution to the above problem and φ is a Kantorovich potential in the transport from ϱ to μ, then there exists a threshold $\ell \in \mathbb{R}$ such that

$$\varrho(x) = \begin{cases} f(x), & \text{if } \varphi(x) < \ell, \\ 0, & \text{if } \varphi(x) > \ell, \\ \in [0, f(x)], & \text{if } \varphi(x) = \ell. \end{cases}$$

In particular, this shows that $\nabla\varphi = 0$ $\varrho-$a.e. on $\{\varrho < f\}$ and, since $T(x) = x - \nabla\varphi(x)$, that the optimal transport T from ϱ to μ is the identity on such set. If $\mu = g\,dx$ is absolutely continuous, then one can write the Monge-Ampère equation

$$\det(DT(x)) = \varrho(x)/g(T(x))$$

and deduce $\varrho(x) = g(T(x)) = g(x)$ a.e. on $\{\varrho < f\}$. This suggests a sort of saturation result for the optimal ϱ, i.e. $\varrho(x)$ is either equal to $g(x)$ or to $f(x)$ (but one has to pay attention to the case $\varrho = 0$ and also to assume that g is absolutely continuous).

Uniqueness. For absolutely continuous measures $\mu = g\,dx$ and generic f the uniqueness of the projection follows by Lemma 1.3.1. In the specific case $f = 1$ and Ω convex the uniqueness was proved in [MRCS10, RC11] by a completely different method. In this case, as observed by A. Figalli, one can use displacement convexity along generalized geodesics. This means that if ϱ^0 and ϱ^1 are two solutions, one can take for every $t \in [0,1]$ the convex combination $T_t = (1-t)T_0 + tT_1$ of the optimal transport maps T_i from g to ϱ^i and the curve $t \mapsto \varrho_t := ((1-t)T_0 + tT_1)_{\#}\mu$ in $\mathcal{P}_2(\Omega)$, interpolating from ϱ^0 to ϱ^1. It can be proven that ϱ_t still satisfies $\varrho_t \leq 1$ (but this cannot be adapted to f, unless f is concave) and that $t \mapsto W_2^2(\varrho_t, g) < (1-t)W_2^2(\varrho^0, g) + tW_2^2(\varrho^1, g)$, which is a contradiction to the minimality. The assumption on μ can be relaxed but we need to ensure the existence of optimal transport maps: what we need to assume, is that μ gives no mass to "small" sets (i.e. $(d-1)-$dimensional); see [Gig11] for the sharp assumptions and notions about this issue. Thanks to this uniqueness result, we can define a projection operator $P_{\mathcal{K}_1} : \mathscr{P}_2(\Omega) \cap L^1(\Omega) \to \mathscr{P}_2(\Omega) \cap L^1(\Omega)$ through

$$P_{\mathcal{K}_1}[g] := \operatorname{argmin}\{W_2^2(\varrho, g) \ : \ \varrho \in \mathcal{K}_1\}.$$

Stability. From the same displacement interpolation idea, A. Roudneff-Chupin also proved ([RC11]) that the projection is Hölder continuous with exponent $1/2$ for the W_2 distance whenever Ω is a compact convex set. We do not develop the proof here, we just refer to Proposition 2.3.4 of [RC11]. Notice that the constant in the Hölder continuity depends a priori on the diameter of Ω. However, to be more precise, the following estimate is obtained (for g^0 and g^1 absolutely continuous)

$$W_2^2(P_{\mathcal{K}_1}[g^0], P_{\mathcal{K}_1}[g^1]) \leq W_2^2(g^0, g^1) + W_2(g^0, g^1)(\operatorname{dist}(g^0, \mathcal{K}_1) + \operatorname{dist}(g^1, \mathcal{K}_1)),$$

(1.4.2)

which shows that, even on unbounded domains, we have a local Hölder behavior.

In the rest of the section, we want to recover similar results in the largest possible generality, i.e. for general f, and without the assumptions on μ and Ω.

We will first get a saturation characterization for the projections, which will allow for a general uniqueness result. Continuity will be an easy corollary.

In order to proceed, we first need the following lemma.

Lemma 1.4.1. *Let ϱ be a solution of the Problem 1.4.1. Let moreover $\gamma \in \Pi(\varrho, \mu)$ be the optimal plan from ϱ to μ. If $(x_0, y_0) \in \mathrm{spt}(\gamma)$ then $\varrho = f$ a.e. in $B(y_0, R)$, where $R = |y_0 - x_0|$.*

Proof. Let us suppose that this is not true and there exists a compact set $K \subset B(y_0, R)$ with positive Lebesgue measure such that $\varrho < f$ a.e. in K. Let $\varepsilon := \mathrm{dist}(\partial B(y_0, R), K) > 0$.

By the definition of the support, for all $r > 0$ we have that

$$0 < \gamma(B(x_0, r) \times B(y_0, r)) \leq \int_{B(x_0,r)} \varrho \, \mathrm{d}x \leq \int_{B(x_0,r)} f \, \mathrm{d}x.$$

By the absolute continuity of the integral, for $r > 0$ small enough there exists $0 < \alpha \leq 1$ such that

$$\gamma(B(x_0, r) \times B(y_0, r)) = \alpha \int_K (f - \varrho) \, \mathrm{d}x =: \alpha m.$$

Now we construct the following measures $\tilde{\gamma}, \eta \in \mathscr{P}(\Omega \times \Omega)$ as

$$\tilde{\gamma} := \gamma - \gamma \llcorner (B(x_0, r) \times B(y_0, r)) + \eta$$

and

$$\eta := \alpha(f - \varrho) \, \mathrm{d}x \llcorner K \otimes (\pi^y)_\# \gamma \llcorner (B(x_0, r) \times B(y_0, r)).$$

It is immediate to check that $(\pi^y)_\# \tilde{\gamma} = \mu$. On the other hand

$$\tilde{\varrho} := (\pi^x)_\# \tilde{\gamma} = \varrho - \varrho \llcorner B(x_0, r) + \alpha(f - \varrho) \llcorner K \leq f$$

is an admissible competitor in Problem (1.4.1) and we have the following

$$\begin{aligned} W_2^2(\tilde{\varrho}, \mu) &\leq \int_{\Omega \times \Omega} |x - y|^2 \, \mathrm{d}\tilde{\gamma}(x, y) \\ &\leq W_2^2(\varrho, g) - \int_{B(x_0,r) \times B(y_0,r)} |x - y|^2 \, \mathrm{d}\gamma(x, y) + \int_{K \times B(y_0,r)} |x - y|^2 \, \mathrm{d}\eta(x, y) \\ &\leq W_2^2(\varrho, g) - (R - 2r)^2 \alpha m + (R - \varepsilon + r)^2 \alpha m. \end{aligned}$$

Now if we chose $r > 0$ small enough to have $R - 2r > R - \varepsilon + r$, i.e. $r < \varepsilon/3$ we get that

$$W_2^2(\tilde{\varrho}, g) < W_2^2(\varrho, g),$$

which is clearly a contradiction, hence the result follows. □

The following proposition establishes uniqueness of the projection on $\mathcal{K}_f$ as well as a very precise description of it. For a given measure μ we are going to denote by μ^{ac} the density of its absolutely continuous part with respect to the Lebesgue measure, i.e.

$$\mu = \mu^{\mathrm{ac}}\,\mathrm{d}x + \mu^{\mathrm{s}},$$

with $\mu^{\mathrm{s}} \perp \mathrm{d}x$. The following result recalls corresponding results in the *partial transport problem* ([Fig10]).

Proposition 1.4.2. *Let $\Omega \subset \mathbb{R}^d$ be a convex set and let $f \in L^1_{\mathrm{loc}}(\Omega)$, $f \geq 0$ be such that $\int_\Omega f \geq 1$. Then, for every probability measure $\mu \in \mathscr{P}(\Omega)$, there is a unique solution ϱ of the problem (1.4.1). Moreover, ϱ is of the form*

$$\varrho = \mu^{\mathrm{ac}}\mathbb{1}_B + f\mathbb{1}_{B^c}, \tag{1.4.3}$$

for a measurable set $B \subset \Omega$.

Proof. We first note that by setting $f = 0$ on Ω^c we can assume that $\Omega = \mathbb{R}^d$. Existence of a solution in Problem 1.4.1 follows by the direct methods in the calculus of variations by noticing that the set $\mathcal{K}_f$ is closed with respect to the weak convergence of measures.

Let us prove now the saturation result (1.4.3). Let us first premise the following fact: if $\mu, \nu \in \mathscr{P}(\Omega)$, $\gamma \in \Pi(\mu, \nu)$ and we define the set

$$A(\gamma) := \{x \in \Omega : \text{the only point } (x, y) \in \mathrm{spt}(\gamma) \text{ is } (x, x)\},$$

then

$$\mu \llcorner A(\gamma) \leq \nu \llcorner A(\gamma). \tag{1.4.4}$$

In particular $\mu^{\mathrm{ac}} \leq \nu^{\mathrm{ac}}$ for a.e. $x \in A(\gamma)$. To prove (1.4.4), let $\phi \geq 0$ continuous and write

$$\begin{aligned}\int_{A(\gamma)} \phi\,\mathrm{d}\mu = \int_\Omega \phi(x)\mathbb{1}_{A(\gamma)}(x)\,\mathrm{d}\gamma(x,y) &= \int_\Omega \phi(x)\mathbb{1}^2_{A(\gamma)}(x)\,\mathrm{d}\gamma(x,y)\\ &= \int_\Omega \phi(y)\mathbb{1}_{A(\gamma)}(y)\mathbb{1}_{A(\gamma)}(x)\,\mathrm{d}\gamma(x,y)\\ &\leq \int_\Omega \phi(y)\mathbb{1}_{A(\gamma)}(y)\,\mathrm{d}\gamma(x,y) = \int_{A(\gamma)} \phi\,\mathrm{d}\nu,\end{aligned}$$

where we used the fact that $\gamma-$a.e. $\mathbb{1}_{A(\gamma)}(x) > 0$ implies $x = y$.

Now, for an optimal transport plan $\gamma \in \Pi(\varrho, \mu)$, let us define

$$B := \mathrm{Leb}(f) \cap \mathrm{Leb}(\mu^{\mathrm{ac}}) \cap \mathrm{Leb}(\varrho) \cap \{\varrho < f\}^{(1)} \cap A(\gamma)^{(1)} \cap A(\tilde{\gamma})^{(1)}.$$

Here $\tilde{\gamma} \in \Pi(g, \varrho)$ is the transport plan obtained by seeing γ "the other way around", i.e. $\tilde{\gamma}$ is the image of γ through the map $(x, y) \mapsto (y, x)$ while $\mathrm{Leb}(h)$

is the set of Lebesgue points of h and for a set A we denote by $A^{(1)} := \mathrm{Leb}(\mathbb{1}_A)$ the set of its density one points.

Let now $x_0 \in B$ and let us consider the following two cases:

Case 1. $\varrho(x_0) < \mu^{ac}(x_0)$. Since, in particular, $\mu^{ac}(x_0) > 0$ and $x_0 \in \mathrm{Leb}(\mu^{ac})$ we have that $x_0 \in \mathrm{spt}(\mu)$. From Lemma 1.4.1 wee see that $(y_0, x_0) \in \mathrm{spt}(\gamma)$ implies $y_0 = x_0$. Indeed, if this were not the case there would exist a ball where $\varrho = f$ a.e. and x_0 would be in the middle of this ball; from $x_0 \in \mathrm{Leb}(f) \cap \mathrm{Leb}(\varrho)$ we would get $\varrho(x_0) = f(x_0)$ a contradiction with $x_0 \in B$. Hence, if we use the set $A(\tilde{\gamma})$ defined above with $\nu = \varrho$, we have $x_0 \in A(\tilde{\gamma})$. From $x_0 \in \mathrm{Leb}(\mu^{ac}) \cap \mathrm{Leb}(\varrho)$ we get $\mu^{ac}(x_0) \le \varrho(x_0)$, which is a contradiction.

Case 2. $\mu^{ac}(x_0) < \varrho(x_0)$. Exactly as in the previous case we have that $x_0 \in \mathrm{spt}(\varrho)$ and, by the Lemma 1.4.1, we have again that $(x_0, y_0) \in \mathrm{spt}(\gamma)$ implies $y_0 = x_0$. Indeed, otherwise x_0 would be on the boundary of a ball where $\varrho = f$ a contradiction with $x_0 \in \{\varrho < f\}^{(1)}$. Hence, we get $x_0 \in A(\gamma)$ and $\varrho(x_0) \le \mu^{ac}(x_0)$, again a contradiction.

Hence we get that $\mu^{ac} = \varrho$ for $x \in B$. By the definition of B,

$$B^c \subset_{\text{a.e.}} \{\varrho = f\} \cup A(\gamma)^c \cup A(\tilde{\gamma})^c ,$$

where a.e. refers to the Lebesgue measure. By applying Lemma 1.4.1, this implies that $\varrho = f$ a.e. on B^c, and concludes the proof of (1.4.3).

Uniqueness of the projection it is now an immediate consequence of the saturation property (1.4.3). Indeed, suppose that ϱ_0 and ϱ_1 were two different projections of a same measure g. Define $\varrho_{1/2} = \frac{1}{2}\varrho_0 + \frac{1}{2}\varrho_1$. Then, by convexity of $W_2^2(\cdot, \mu)$, we get that $\varrho_{1/2}$ is also optimal. But its density is not saturated on the set where the densities of ϱ_0 and ϱ_1 differ, in contradiction with (1.4.3). □

Corollary 1.4.3. *For fixed f, the map $P_{\mathcal{K}_f} : \mathscr{P}_2(\Omega) \to \mathscr{P}_2(\Omega)$ defined through*

$$P_{\mathcal{K}_f}[\mu] := \operatorname{argmin}\{W_2^2(\varrho, \mu) \ : \ \varrho \in \mathcal{K}_f\}$$

is continuous in the following sense: if $\mu_n \to \mu$ for the W_2 distance, then $P_{\mathcal{K}_f}[\mu_n] \rightharpoonup P_{\mathcal{K}_f}[\mu]$ in the weak convergence.

Moreover, in the case where $f = 1$ and Ω is a convex set, the projection is also locally $\frac{1}{2}$−Hölder continuous for W_2 on the whole $\mathscr{P}_2(\Omega)$ and satisfies (1.4.2).

Proof. This is just a matter of compactness and uniqueness. Indeed, take a sequence $\mu_n \to \mu$ w.r.t. W_2 and look at $P_{\mathcal{K}_f}[\mu_n]$. It is a tight sequence of measures since

$$W_2(P_{\mathcal{K}_f}[\mu_n], \mu) \le W_2(P_{\mathcal{K}_f}[\mu_n], \mu_n) + W_2(\mu_n, \mu) \le W_2(\varrho, \mu) + 2W_2(\mu_n, \mu) , \tag{1.4.5}$$

where $\varrho \in \mathcal{K}_f$ is any admissible measure. Hence we can extract a weakly converging subsequence to some measure $\tilde{\varrho} \in \mathcal{K}_f$ (recall that $\mathcal{K}_f$ is weakly closed). Moreover, by the lower semicontinuity of W_2 with respect to the weak convergence and since $W_2(\mu_n, \mu) \to 0$, passing to the limit in (1.4.5) we get

$$W_2(\tilde{\varrho}, \mu) \leq W_2(\varrho, \mu) \qquad \forall\, \varrho \in \mathcal{K}_f.$$

Uniqueness of the projection implies $\tilde{\varrho} = P_{\mathcal{K}_f}(\mu)$ and thus that the limit is independent on the extracted subsequence, this proves the desired continuity.

Concerning the second part of the statement, we take arbitrary μ^1 and μ^2 (not necessarily absolutely continuous) and we approximate them in the W_2 distance with absolutely continuous measures g_n^i ($i = 1, 2$; for instance by convolution), then we have, from (1.4.2)

$$W_2^2(P_{\mathcal{K}_1}[g_n^0], P_{\mathcal{K}_1}[g_n^1]) \leq W_2^2(g_n^0, g_n^1) + W_2(g_n^0, g_n^1)(\text{dist}(g_n^0, \mathcal{K}_1) + \text{dist}(g_n^1, \mathcal{K}_1)),$$

and we can pass to the limit as $n \to \infty$. □

The following technical lemma will be used in the next section and establishes the continuity of the projection with respect to f. To state it, for given $f \in L^1_{\text{loc}}$ and $\mu \in \mathcal{P}_2(\Omega)$ let us consider following functional

$$\mathcal{F}_f(\varrho) := \begin{cases} \frac{1}{2} W_2^2(\mu, \varrho), & \text{if } \varrho \in \mathcal{K}_f \\ +\infty, & \text{otherwise.} \end{cases}$$

Proposition 1.4.2 can be restated by saying that the functional $\mathcal{F}_f$ has a unique minimizer in $\mathcal{P}_2(\Omega)$.

Lemma 1.4.4. *Let $f_n, f \in L^1_{\text{loc}}(\Omega)$ with $\int_\Omega f_n \, \mathrm{d}x \geq 1$, $\int_\Omega f \, \mathrm{d}x \geq 1$ and let us assume that $f_n \to f$ in $L^1_{\text{loc}}(\Omega)$ and almost everywhere. Also assume $f_n \in \mathcal{P}_2(\Omega)$ if $\int_\Omega f_n \, \mathrm{d}x = 1$ and $f \in \mathcal{P}_2(\Omega)$ if $\int_\Omega f \, \mathrm{d}x = 1$. Then, for every $\mu \in \mathcal{P}_2(\Omega)$,*

(i) The sequence $(P_{\mathcal{K}_{f_n}}(\mu))_n$ is tight.

(ii) We have $P_{\mathcal{K}_{f_n}}(\mu) \rightharpoonup P_{\mathcal{K}_f}(\mu)$.

(iii) If $\int_\Omega f \, \mathrm{d}x > 1$, then $\mathcal{F}_{f_n}$ $\Gamma-$converges to $\mathcal{F}_f$ with respect to the weak convergence of measures.

Proof. Let us denote by $\bar{\varrho}_n$ the projection $P_{\mathcal{K}_{f_n}}(\mu)$ and let us start from proving its tightness, i.e. *(i)*. We fix $\varepsilon > 0$: there exists a radius R_0 such that $\mu(B(0, R_0)) > 1 - \dfrac{\varepsilon}{2}$ and $\int_{B(0,R_0)} f \, \mathrm{d}x > 1 - \dfrac{\varepsilon}{2}$. By L^1_{loc} convergence, there exists n_0 such that $\int_{B(0,R_0)} f_n > 1 - \varepsilon$ pour $n > n_0$. Now, take $R > 3R_0$ and suppose $\bar{\varrho}_n(B(0, R)^c) > \varepsilon$ for $n \geq n_0$. Then, the optimal transport T from $\bar{\varrho}_n$ to μ should move some mass from $B(0, R)^c$ to $B(0, R_0)$. Let us take a point $x_0 \in B(0, R)^c$ such that $T(x_0) \in B(0, R_0)$. From Lemma 1.4.1, this means that $\bar{\varrho}_n = f_n$ on the ball $B(T(x_0), |x_0 - T(x_0)|) \supset B(T(x_0), 2R_0) \supset B(0, R_0)$. But this means

$\int_{B(0,R_0)} \bar{\varrho}_n \,dx = \int_{B(0,R_0)} f_n \,dx > 1-\varepsilon$, and hence $\bar{\varrho}_n(B(0,R)^c) \leq \varepsilon$, which is a contradiction. This shows that $\bar{\varrho}_n$ is tight.

Now, if $\int_\Omega f \,dx = 1$, then the weak limit of $\bar{\varrho}_n$ (up to subsequences) can only be f itself, since it must be a probability density bounded from above by f and hence $f = P_{\mathcal{K}_f}(\mu)$. This proves *(ii)* in the case $\int_\Omega f \,dx = 1$. In the case $\int_\Omega f \,dx > 1$, this will be a consequence of *(iii)*. Notice that in this case we necessarily have $\int_\Omega f_n \,dx > 1$ for n large enough.

Let us prove *(iii)*. Since $\bar{\varrho}_n \leq f_n$ a.e., $\bar{\varrho}_n \rightharpoonup \bar{\varrho}$ and $f_n \to f$ in L^1_{loc} immediately implies that $\bar{\varrho} \leq f$ a.e., the $\Gamma-$liminf inequality simply follows by the lower semicontinuity of W_2.

Concerning the $\Gamma-$limsup, we need to prove that every density $\varrho \in \mathscr{P}_2(\Omega)$ with $\varrho \leq f$ a.e. can be approximated by a sequence $\varrho_n \leq f_n$ a.e. with $W_2(\varrho_n, \mu) \to W_2(\varrho, \mu)$. In order to do this let us define $\tilde{\varrho}_n := \min\{\varrho, f_n\}$. Note that $\tilde{\varrho}_n$ is not admissible since it is not a probability, because in general $\int_\Omega \tilde{\varrho}_n \,dx < 1$. Yet, we have $\int_\Omega \tilde{\varrho}_n \,dx \to 1$ since $\tilde{\varrho}_n \to \min\{\varrho, f\} = \varrho$ and this convergence is dominated by ϱ. We want to "complete" $\tilde{\varrho}_n$ so as to get a probability, stay admissible, and converge to ϱ in W_2, since this will imply that $W_2(\varrho_n, \mu) \to W_2(\varrho, \mu)$.

Let us select a ball B such that $\int_{B\cap\Omega} f \,dx > 1$ and note that we can find $\varepsilon > 0$ such that the set $\{f > \varrho + \varepsilon\} \cap B$ is of positive measure, i.e. $m := |\{f > \varrho + \varepsilon\} \cap B| > 0$. Since $f_n \to f$ a.e., the set $B_n := \{f_n > \varrho + \frac{\varepsilon}{2}\} \cap B$ has measure larger than $m/2$ for large n. Now take $B'_n \subset B_n$ with $|B'_n| = \frac{2}{\varepsilon}\left(1 - \int_\Omega \tilde{\varrho}_n \,dx\right) \to 0$, and define

$$\varrho_n := \tilde{\varrho}_n + \frac{\varepsilon}{2} \mathbb{1}_{B'_n}.$$

By construction, $\int_\Omega \varrho_n \,dx = 1$ and $\varrho_n \leq f_n$ a.e. since on B'_n we have $\tilde{\varrho}_n = \varrho$ and $\varrho + \frac{\varepsilon}{2} < f_n$ while on the complement of B'_n, $\tilde{\varrho}_n \leq f_n$ a.e. by definition. To conclude the proof we only need to check $W_2(\varrho_n, \varrho) \to 0$. This is equivalent (see, for instance, [AGS08] or [Vil09]) to

$$\int_\Omega \phi \varrho_n \,dx \to \int_\Omega \phi \varrho \,dx$$

for all continuous functions ϕ with such that $\phi \leq C(1 + |x|^2)$. Since $\varrho \in \mathcal{P}_2(\Omega)$ and $\tilde{\varrho}_n \leq \varrho$, thank to the dominated convergence theorem it is enough to show that $\int_\Omega \phi(\varrho_n - \tilde{\varrho}_n) \,dx \to 0$. But $\varrho_n - \tilde{\varrho}_n$ converges to 0 in L^1 and it is supported in $B'_n \subset B$. Since ϕ is bounded on B we obtain the desired conclusion. □

Remark 1.4.1. Let us conclude this section with the following *open question*: for $f = 1$ the projection is continuous and we can even provide Hölder bounds on

$P_{\mathcal{K}_1}$. The question whether $P_{\mathcal{K}_1}$ is 1-Lipschitz, as far as we know, is open. Let us underline that some sort of 1-Lipschitz results have been proven in [CC13] for solutions of similar variational problems, but seem impossible to adapt in this framework.

For the case $f \neq 1$ even the continuity of the projection with respect to the Wasserstein distance seems delicate.

1.4.2 *BV estimates for $P_{\mathcal{K}_f}$*

In this section, we prove Theorem 1.1.2. Notice that the case $f = 1$ has already been proven as a particular case of Theorem 1.1.1. To handle the general case, we develop a slightly different strategy, based on the standard idea to approximate L^∞ bounds with L^p penalizations.

Let $m \in \mathbb{N}$ and let us assume that $\inf f > 0$, for $\mu \in \mathcal{P}_2(\Omega)$, we define the approximating functionals $\mathcal{F}_m : L^1_+(\Omega) \to \mathbb{R} \cup \{+\infty\}$ by

$$\mathcal{F}_m(\varrho) := \frac{1}{2} W_2^2(\mu, \varrho) + \frac{1}{m+1} \int_\Omega \left(\frac{\varrho}{f}\right)^{m+1} \mathrm{d}x + \frac{\varepsilon_m}{2} \int_\Omega \left(\frac{\varrho}{f}\right)^2 \mathrm{d}x$$

and the limit functional $\mathcal{F}$ as

$$\mathcal{F}(\varrho) := \begin{cases} \frac{1}{2} W_2^2(\mu, \varrho), & \text{if } \varrho \in \mathcal{K}_f \\ +\infty, & \text{otherwise} \end{cases}$$

Here $\varepsilon_m \downarrow 0$ is a small parameter to be chosen later.

Lemma 1.4.5. *Let $\Omega \subset \mathbb{R}^d$ and $f : \Omega \to (0, +\infty)$ be a measurable function, bounded from below and from above by positive constants and let $\mu \in \mathcal{P}_2(\Omega)$. Then:*

(i) There are unique minimizers ϱ, ϱ_m in $L^1(\Omega)$ for each of the functionals $\mathcal{F}$ and $\mathcal{F}_m$, respectively.

(ii) The family of functionals $\mathcal{F}_m$ Γ-converges for the weak convergence of probability measures to $\mathcal{F}$, and the minimizers ϱ_m weakly converge to ϱ, as $m \to \infty$.

(iii) The minimizers ϱ_m of $\mathcal{F}_m$ satisfy

$$\varphi_m + \left(\frac{\varrho_m}{f}\right)^m \frac{1}{f} + \varepsilon_m \left(\frac{\varrho_m}{f}\right) \frac{1}{f} = 0, \tag{1.4.6}$$

for a suitable Kantorovich potential φ_m in the transport from ϱ_m to μ.

Proof. Existence and uniqueness of minimizers of $\mathcal{F}$ has been established in Proposition 1.4.2. Existence of minimizers of $\mathcal{F}_m$ is again a simple application of the direct methods in the calculus of variations and uniqueness follows from strict convexity.

Let us prove the $\Gamma-$convergence in *(ii)*. In order to prove the $\Gamma-$liminf inequality, let $\varrho_m \rightharpoonup \varrho$. If $\mathcal{F}_m(\varrho_m) \le C$, then for every $m_0 \le m$ and every finite measure set $A \subset \Omega$, we have

$$\|\varrho_m/f\|_{L^{m_0}(A)} \le |A|^{\frac{1}{m_0}-\frac{1}{m+1}}(C(m+1))^{\frac{1}{m+1}}.$$

If we pass to the limit $m \to \infty$, from $\frac{\varrho_m}{f} \rightharpoonup \frac{\varrho}{f}$, we get $\|\varrho/f\|_{L^{m_0}(A)} \le |A|^{\frac{1}{m_0}}$. Letting m_0 go to infinity we obtain $\|\varrho/f\|_{L^\infty} \le 1$, i.e. $\varrho \in \mathcal{K}_f$. Since

$$\mathcal{F}_m(\varrho_m) \ge \frac{1}{2}W_2^2(\mu,\varrho_m),$$

the lower semicontinuity of W_2^2 with respect to weak converges proves the $\Gamma-$liminf inequality.

In order to prove $\Gamma-$limsup, we use the constant sequence $\varrho_m = \varrho$ as a recovery sequence. Since we can assume $\varrho \le f$ (otherwise there is nothing to prove, since $\mathcal{F}(\varrho) = +\infty$), it is clear that the second and third parts of the functional tend to 0, thus proving the desired inequality.

The last part of the statement finally follows from Theorem 0.0.4 (vi) and Lemma 0.0.7, exactly as in Corollary 0.0.8. □

Proof of Theorem 1.1.2

Proof. Clearly we can assume that $TV(g,\Omega)$ and $TV(f,\Omega)$ are finite and that $\int_\Omega f\,\mathrm{d}x > 1$ since otherwise the conclusion is trivial.

Step 1. Assume that the support of g is compact, that $f \in C^\infty(\Omega)$ is bounded from above and below by positive constants, and let ϱ_m be the minimizer of $\mathcal{F}_m$. As in the proof of Theorem 1.1.1, we can use the optimality condition (1.4.6) to prove that ϱ is compactly supported. Also, the same condition imply that ϱ is Lipschitz continuous. Indeed, we can write (1.4.6) as

$$\varphi f + H_m'\left(\frac{\varrho}{f}\right) = 0,$$

where $H_m(t) = \frac{1}{m+1}t^{m+1} + \frac{\varepsilon_m}{2}t^2$. Since H_m is smooth and convex and H_m'' is bounded from below by a positive constant H_m' is invertible and

$$\varrho = f \cdot (H_m')^{-1}(-\varphi f),$$

where $(H_m')^{-1}$ is Lipschitz continuous. Since φ and f are locally Lipschitz, this gives Lipschitz continuity for ϱ on a neighborhood of its support.

Taking the derivative of the optimality condition (1.4.6) we obtain

$$\nabla\varphi_m + \left(m\left(\frac{\varrho_m}{f}\right)^{m-1} + \varepsilon_m\right)\frac{f\nabla\varrho_m - \varrho_m\nabla f}{f^3} - \left(\left(\frac{\varrho_m}{f}\right)^m + \varepsilon_m\frac{\varrho_m}{f}\right)\frac{\nabla f}{f^2} = 0.$$

Rearranging the terms we have

$$\nabla\varphi_m + A\nabla\varrho_m - B\nabla f = 0,$$

where by A and B we denote the (positive!) functions

$$A := \left(m\left(\frac{\varrho_m}{f}\right)^{m-1} + \varepsilon_m\right)\frac{1}{f^2}$$

and

$$B := \left(m\left(\frac{\varrho_m}{f}\right)^{m-1} + \varepsilon_m\right)\frac{\varrho_m}{f^3} + \left(\left(\frac{\varrho_m}{f}\right)^{m} + \varepsilon_m\frac{\varrho_m}{f}\right)\frac{1}{f^2}.$$

Now we will use the inequality from Corollary 1.2.4 for ϱ_m and g in the form

$$\int_\Omega |\nabla\varrho_m|\,dx \leq \int_\Omega |\nabla g|\,dx + \int_\Omega \nabla\varrho_m\cdot\left(\frac{\nabla\varrho_m}{|\nabla\varrho_m|} + \frac{\nabla\varphi_m}{|\nabla\varphi_m|}\right)dx.$$

In order to estimate the second integral on the right-hand side we use the inequality

$$\left|\frac{a}{|a|} - \frac{b}{|b|}\right| \leq \left|\frac{a}{|a|} - \frac{b}{|a|}\right| + \left|\frac{b}{|a|} - \frac{b}{|b|}\right| = \frac{|a-b|}{|a|} + \frac{|b|-|a|}{|a|} \leq \frac{2}{|a|}|a-b|, \quad (1.4.7)$$

for all non-zero $a, b \in \mathbb{R}^d$ (that we apply to $a = A\nabla\varrho_m$ and $b = -\nabla\varphi_m$), and we obtain

$$\begin{aligned}\int_\Omega |\nabla\varrho_m|\,dx &\leq \int_\Omega |\nabla g|\,dx + \int_\Omega |\nabla\varrho_m|\cdot\left|\frac{A\nabla\varrho_m}{A|\nabla\varrho_m|} + \frac{\nabla\varphi_m}{|\nabla\varphi_m|}\right|dx\\ &\leq \int_\Omega |\nabla g|\,dx + 2\int_\Omega \frac{1}{A}|A\nabla\varrho_m + \nabla\varphi_m|\,dx\\ &\leq \int_\Omega |\nabla g|\,dx + 2\int_\Omega \frac{B}{A}|\nabla f|\,dx.\end{aligned}$$

We must now estimate the ratio B/A. If we denote by λ the ratio ϱ_m/f we may write

$$\frac{B}{A} = \lambda + \lambda\frac{\varepsilon_m + \lambda^{m-1}}{\varepsilon_m + m\lambda^{m-1}} \leq \lambda\left(1 + \frac{1}{m}\right) + \frac{\varepsilon_m\lambda}{\varepsilon_m + m\lambda^{m-1}}.$$

Now, consider that

$$\max_{\lambda\in\mathbb{R}_+}\frac{\varepsilon_m\lambda}{\varepsilon_m + m\lambda^{m-1}} = \frac{m-2}{m-1}\left(\frac{\varepsilon_m}{m(m-2)}\right)^{1/(m-1)} =: \delta_m$$

is a quantity depending on m and tending to 0 if ε_m is chosen small enough (for instance $\varepsilon_m = 2^{-m^2}$). This allows to write

$$\int_\Omega |\nabla\varrho_m|\,dx \leq \int_\Omega |\nabla g|\,dx + 2\left(1 + \frac{1}{m}\right)\int_\Omega \frac{\varrho_m}{f}|\nabla f|\,dx + 2\delta_m\int_\Omega |\nabla f|\,dx.$$

In the limit, as $m \to +\infty$, we obtain

$$\int_\Omega |\nabla \varrho| \, dx \le \int_\Omega |\nabla g| \, dx + 2 \int_\Omega \frac{\varrho}{f} |\nabla f| \, dx.$$

Using the fact that $\varrho \le f$, we get

$$\int_\Omega |\nabla \varrho| \, dx \le \int_\Omega |\nabla g| \, dx + 2 \int_\Omega |\nabla f| \, dx.$$

Step 2. To treat the case $g, f \in BV_{\text{loc}}(\Omega)$ we proceed by approximation as in the proof of Theorem 1.1.1. To do this we just note that Corollary 1.4.3 and Lemma 1.4.4 give the desired continuity property of the projection with respect both to g and f, lower semicontinuity of the total variation with respect to the weak convergence then implies the conclusion. □

Remark 1.4.2. We conclude this section by underlining that the constant 2 in the inequality (1.1.4) cannot be replaced by any smaller constant. Indeed if $\Omega = \mathbb{R}$, $f = \mathbb{1}_{\mathbb{R}_+}$, $g = \dfrac{1}{n}\mathbb{1}_{[-n,0]}$ then $\varrho = P_{K_f}(g) = \mathbb{1}_{[0,1]}$ and $\displaystyle\int |\nabla \varrho| \, dx = 2$, $\displaystyle\int |\nabla f| \, dx = 1$, $\displaystyle\int |\nabla g| \, dx = \frac{2}{n}$.

1.5 APPLICATIONS

In this section we discuss some applications of Theorems 1.1.1 and 1.1.2 and we present some open problems.

1.5.1 *Partial transport*

The projection problem on $\mathcal{K}_f$ is a particular case of the so called *partial transport problem*, see [CM10, Fig09, Fig10, Ind13]. Indeed, the problem is to transport μ to a part of the measure f, which is a measure with mass larger than 1. As typical in the partial transport problem, the solution has an active region, which is given by f restricted to a certain set. This set satisfies a sort of interior ball condition, with a radius depending on the distance between each point and its image. In the partial transport case some regularity ($C^{1,\alpha}$) is known for the optimal map away from the intersection of the supports of the two measures.

A natural question is how to apply the technique that we developed here in the framework of more general partial transport problems (in general, both measures could have mass larger than 1 and could be transported only partially), and/or whether results or ideas from partial transport could be translated into the regularity of the free boundary in the projection.

1.5.2 *Shape optimization*

If we take a set $A \subset \mathbb{R}^d$ with $|A| < 1$ and finite second moment $\int_A |x|^2 \, dx < +\infty$, a natural question is which is the set B with volume 1 such that the uniform probability density on B is closest to that on A. This means solving a shape optimization problem of the form

$$\min\{W_2^2(\mathbb{1}_B, \frac{1}{|A|}\mathbb{1}_A) \; : \; |B| = 1\}.$$

The considerations in Section 1.4 show that solving such a problem is equivalent to solving

$$\min\{W_2^2(\varrho, \frac{1}{|A|}\mathbb{1}_A) \; : \; \varrho \in \mathscr{P}_2(\mathbb{R}^d)\}$$

and that the optimal ϱ is of the form $\varrho = \mathbb{1}_B$, $B \supset A$. Also, from our Theorem 1.1.2 (with $f = 1$), we deduce that if A is of finite perimeter, then the same is true for B, and $\mathrm{Per}(B) \leq \frac{1}{|A|}\mathrm{Per}(A)$ (i.e. the perimeter is bounded by the Cheeger ratio of A).

It is interesting to compare this problem with this perimeter bound with the problem studied in [Mil06], which has the same words but in different order: more precisely: here we minimize the Wasserstein distance and we try to get an information on the perimeter, in [Mil06] the functional to be minimized is a combination of W_2 and the perimeter. Hence, the techniques to prove any kind of results are different, because here W_2 cannot be considered as a lower order perturbation of the perimeter.

As a consequence, many natural questions arise: if A is a nice closed set, can we say that B contains A in its interior? if A is convex is B convex? what about the regularity of ∂B?

1.5.3 *Set evolution problems*

Consider the following problem. For a given set $A \subset \mathbb{R}^d$ we define $\varrho_0 = \mathbb{1}_A$. For a time interval $[0, T]$ and a time step $\tau > 0$ (and $N + 1 := [\frac{T}{\tau}]$) we consider the following scheme $\varrho_0^\tau := \varrho_0$ and

$$\varrho_{k+1}^\tau := P_{\mathcal{K}_1}\left[(1+\tau)\varrho_k^\tau\right], \; k \in \{0, \dots, N-1\}, \tag{1.5.1}$$

(here we extend the notion of Wasserstein distance and projection to measures with the same mass, even if different from 1: in particular, the mass of ϱ_k^τ will be $|A|(1+\tau)^k$ and at every step we project ϱ_k^τ on the set of finite positive measure, with the same mass of ϱ_k^τ, and with density bounded by 1, and we still denote this set by $\mathcal{K}_1$ and the projection operator in the sense of the quadratic Wasserstein distance onto this set by $P_{\mathcal{K}_1}$). We want to study the convergence of this algorithm as $\tau \to 0$. This is a very simplified model for the growth of a biological population, which increases exponentially in size (supposing that there is enough food: see [MRCS10, MRCS14] for a more sophisticated model) but is subject to a density constraint because each individual needs a certain amount of space. Notice that this scheme formally follows the same evolution as in the Hele-Shaw flow (this can be justified by the fact that, close to uniform density the W_2 distance and the H^{-1} distance are asymptotically the same).

Independently of the compactness arguments that we need to prove the convergence of the scheme, we notice that, for fixed $\tau > 0$, all the densities ϱ_k^τ are indeed indicator functions (this comes from the consideration in Section 1.4). Thus we have an evolution of sets. A natural question is whether this stays true when we pass to the limit as $\tau \to 0$. Indeed, we generally prove convergence of the scheme in the weak sense of measures, and it is well-known that, in general, a weak limit of indicator functions is not necessarily an indicator itself. However Theorem 1.1.2 provides an a priori bound the perimeter of these sets. This BV bound allows to transform weak convergence as measures into strong L^1 convergence, and to preserve the fact that these densities are indicator functions.

Notice on the other hand that the same result could not be applied in the case where the projection was performed onto $\mathcal{K}_f$, for a non-constant f. The reason lies in the term $2\int |\nabla f|$ in the estimate we provided. This means that, a priori, instead of being decreasing, the total variation could increase at each step of a fixed amount $2\int |\nabla f|$. When $\tau \to 0$, the number of iterations diverges and this does not allow to prove any BV estimate on the solution. Yet, a natural question would be to prove that the set evolution is well-defined as well, using maybe the fact that these sets are increasing in time.

1.5.4 *Crowd movement with diffusion*

In [MRCS10, RC11] crowd movement models where a density ϱ evolves according to a given vector field $\mathbf{v}$, but subject to a density constraint $\varrho \leq 1$ are studied. This means that, without the density constraint, the equation would be $\partial_t \varrho + \nabla \cdot (\varrho \mathbf{v}) = 0$, and a natural way to discretize the constrained equation would be to set $\tilde{\varrho}_{k+1}^\tau = (id + \tau \mathbf{v})_\# \varrho_k^\tau$ and then $\varrho_{k+1}^\tau = P_{\mathcal{K}_1}[\tilde{\varrho}_{k+1}^\tau]$.

What happens if we want to add some diffusion, i.e. if the continuity equation is replaced by a Fokker-Planck equation $\partial_t \varrho - \Delta \varrho + \nabla \cdot (\varrho \mathbf{v}) = 0$? among other possible methods, one discretization idea is the following: define $\tilde{\varrho}_{k+1}^\tau$ by following the unconstrained Fokker-Planck equation for time τ starting from ϱ_k^τ, and then project.

In order to get some compactness of the discrete curves we need to estimate the distance between ϱ_k^τ and $\tilde{\varrho}_{k+1}^\tau$. It is not difficult to see that the speed of the solution of the Heat Equation (and also of the Fokker-Planck equation) for the distance W_p is related to $\|\nabla \varrho\|_{L^p}$. It is well known that these parabolic equations regularize and so the L^p norm of the gradient will not blow up in time, but we have to keep into account the projections that we perform every time step τ. From the discontinuities that appear in the projected measures, one cannot expected that $W^{1,p}$ bounds on ϱ are preserved. The only reasonable bound is for $p = 1$, i.e. a BV bound, which is exactly what is provided in this paper.

The application to crowd motion with diffusion has been studied recently in [MS15a] and this is the subject of Chapter 2 of the present thesis.

1.5.5 *BV estimates for some degenerate diffusion equation*

In this subsection we apply our main Theorem 1.1.1 to establish BV estimates for for some degenerate diffusion equation. BV estimates for these equations are usually known and they can be derived by looking at the evolution in time of the BV norm of the solution. Theorem 1.1.1 allows to give an optimal transport proof of these estimates. Let $h : \mathbb{R}^+ \to \mathbb{R}$ be a given super-linear convex function and let us consider the problem

$$\begin{cases} \partial_t \varrho_t = \nabla \cdot (h''(\varrho_t)\rho_t \nabla \rho_t), & \text{in } (0,T] \times \mathbb{R}^d, \\ \varrho(0,\cdot) = \varrho_0, & \text{in } \mathbb{R}^d, \end{cases} \tag{1.5.2}$$

where ϱ_0 is a non-negative BV probability density. We remark that by the evolution for any $t \in (0,T]$ ϱ_t will remain a non-negative probability density. In the case $h(\rho) = \rho^m/(m-1)$ in equation (1.5.2) we get precisely the *porous medium equation* $\partial_t \rho = \Delta(\rho^m)$ (see [Váz07]).

Since the seminal work of F. Otto ([Otto1]) we know that the problem (1.5.2) can be seen as a gradient flow of the functional

$$\mathcal{F}(\varrho) := \int_{\mathbb{R}^d} h(\varrho)\,\mathrm{d}x$$

in the space $(\mathscr{P}_2(\mathbb{R}^d), W_2)$. As a gradient flow, this equation can be discretized in time through an implicit Euler scheme. More precisely let us take a time step $\tau > 0$ and let us consider the following scheme: $\varrho_0^\tau := \varrho_0$ and

$$\varrho_{k+1}^\tau := \mathrm{argmin}_\varrho \left\{ \frac{1}{2\tau} W_2^2(\varrho, \varrho_k^\tau) + \mathcal{F}(\varrho) \right\}, \; k \in \{0, \dots, N-1\}. \tag{1.5.3}$$

where $N := \left[\frac{T}{\tau}\right]$. Constructing piecewise constant and geodesic interpolations between the ϱ_k^τ's with the corresponding velocities and momentums, it is possible to show that as $\tau \to 0$ we will get a curve ϱ_t, $t \in [0,T]$ in $(\mathscr{P}_2(\mathbb{R}^d), W_2)$ which solves

$$\begin{cases} \partial_t \rho_t + \nabla \cdot (\varrho_t \mathbf{v}_t) = 0 \\ \mathbf{v}_t = -h''(\varrho_t)\nabla \varrho_t, \end{cases}$$

hence

$$\partial_t \varrho_t - \nabla \cdot (h''(\varrho_t)\varrho_t \nabla \varrho_t) = 0,$$

that is ϱ_t is a solution to (1.5.2), see [AGS08] for a rigorous presentation of these facts.

We now note that Theorem 1.1.1 implies that

$$\int_{\mathbb{R}^d} |\nabla \varrho_{k+1}^\tau| \, \mathrm{d}x \leq \int_{\mathbb{R}^d} |\nabla \varrho_k^\tau| \, \mathrm{d}x,$$

hence the total variation decreases for the sequence $\varrho_0^\tau, \ldots, \varrho_N^\tau$. As the estimations do not depend on $\tau > 0$ this will remain true also in the limit $\tau \to 0$. Hence (assuming uniqueness for the limiting equation) we get that for any $t, s \in [0, T], \ t > s$

$$TV(\varrho_t, \mathbb{R}^d) \leq TV(\varrho_s, \mathbb{R}^d),$$

and in particular for any $t \in [0, T]$

$$TV(\varrho_t, \mathbb{R}^d) \leq TV(\varrho_0, \mathbb{R}^d).$$

2

A diffusive model for macroscopic crowd motion with density constraints

In the spirit of the macroscopic crowd motion models with hard congestion (i.e. a strong density constraint $\rho \leq 1$) introduced by Maury *et al.* some years ago, we analyze a variant of the same models where diffusion of the agents is also taken into account. From the modeling point of view, this means that individuals try to follow a given spontaneous velocity, but are subject to a Brownian diffusion, and have to adapt to a density constraint which introduces a pressure term affecting the movement. From the PDE point of view, this corresponds to a modified Fokker-Planck equation, with an additional gradient of a pressure (only living in the saturated zone $\{\rho = 1\}$) in the drift. We prove existence and some estimates, based on optimal transport techniques.

This chapter is based on a joint work with F. Santambrogio (see [MS15a]).

2.1 Introduction

In the past few years modeling crowd behavior has become a very active field of applied mathematics. Beyond their importance in real life applications, these modeling problems serve as basic ideas to understand many other phenomena coming for example from biology (cell migration, tumor growth, pattern formations in animal populations, etc.), particle physics and economics. A first non-exhaustive list of references for these problems is [Cha07, CR05, CPT14b, Dog08, Hel92, HM95, Hug02, Hug03, MV07]. A very natural question in all these models is the problem of congestion phenomena: in many practical situations, very high quantities of individuals could try to occupy the same spot, which could be impossible, or lead to strong negative effects on the motion, because of natural limitations on the crowd density.

These phenomena have been studied by using different models, which could be either "microscopic" (based on ODEs on the motion of a high number of agents) or "macroscopic" (describing the agents via their density and velocity, typically with Eulerian formalism). Let us concentrate on the macroscopic models, where the density ρ plays a crucial role. These very same models can be characterized either by "soft congestion" effects (i.e. the higher the density the slower the motion), or by "hard congestion" (i.e. an abrupt threshold effect: if the density touches a certain maximal value, the motion is strongly affected, while nothing happens for smaller values of the density). See [MRCSV11] for comparison between the different classes of models. This last class of models, due to the discontinuity in the congestion effects, presents new mathematical difficulties, which cannot be analyzed with the usual techniques from conservation laws (or, more generally, evolution PDEs) used for soft congestion.

A very powerful tool to attack macroscopic hard-congestion problems is the theory of optimal transportation (see [Vil09, San15]), as we can see in [MRCS10, MRCSV11, RC11, San12a]. In this framework, the density of the agents solves a continuity equation (with velocity field taking into account the congestion effects), and can be seen as a curve in the Wasserstein space.

Our aim in this paper is to endow the macroscopic hard congestion models of [MRCS10, MRCSV11, RC11, San12a] with diffusion effects. In other words, we want to model a crowd, where every agent has a spontaneous velocity, a non-degenerate diffusion (i.e. a stochastic component in its motion) driven by a Brownian motion, and is subject to a density constraint. Implementing this new element into these models could give a better approximation of reality, when dealing with large populations. We also underline that one of the goals of this analysis [1] is to better "prepare" these hard congestion crowd motion models for a possible analysis in the framework of Mean Field Games (see [LL06a, LL06b, LL07], and also [San12b]). These MFG models usually involve a stochastic term, also implying regularizing effects, which are useful in the mathematical analysis of the corresponding PDEs.

2.1.1 *The existing first order models in the light of* Maury et al.

Some macroscopic models for crowd motion with density constraints and "hard congestion" effects were studied in [MRCSV11] and [MRCS10]. We briefly present them as follows:

1. The insertion of a diffusive behavoir in the model also leads to easier and more general uniqueness results. This is the subject of Chapter 3 of the present thesis.

- The density of the population in a bounded (convex) domain $\Omega \subset \mathbb{R}^d$ is described by a probability measure $\rho \in \mathcal{P}(\Omega)$. The initial density $\rho_0 \in \mathscr{P}(\Omega)$ evolves in time, and ρ_t denotes its value at each time $t \in [0, T]$.

- The spontaneous velocity field of the population is a given time dependent field, denoted by $\mathbf{u}_t$. It represents the velocity that each individual would like to follow in the absence of the others. Ignoring the density constraint, this would give rise to the continuity equation $\partial_t \rho_t + \nabla \cdot (\rho_t \mathbf{u}_t) = 0$. We observe that in the original work [MRCS10] the vector field $\mathbf{u}_t(x)$ was taken of the form $-\nabla D(x)$ (independent of time and of gradient form) but we try here to be more general (see [RC11] for the general case, which requires some extra regularity assumption).

- The set of admissible densities will be denoted by $\mathcal{K} := \{\rho \in \mathcal{P}(\Omega) : \rho \leq 1\}$. In order to guarantee that $\mathcal{K}$ is neither empty nor trivial, we suppose $|\Omega| > 1$. (In comparison with the models in Chapter 1, we drop the subscript 1 in the definition of $\mathcal{K}$, because in this setting we use only the constant density constraint 1).

- The set of admissible velocity fields with respect to the density ρ is characterized by the sign of the divergence of the velocity field on the saturated zone; formally we set

$$\mathrm{adm}(\rho) := \left\{ \mathbf{v} : \Omega \to \mathbb{R}^d : \nabla \cdot \mathbf{v} \geq 0 \text{ on } \{\rho = 1\} \text{ and } \mathbf{v} \cdot \mathbf{n} \leq 0 \text{ on } \partial\Omega \right\}.$$

- We consider the projection operator P in the $L^2(\mathscr{L}^d)$ or in the $L^2(\rho)$ sense (which will be the same, because the only relevant zone is $\{\rho = 1\}$).

- Finally we solve the following modified continuity equation for ρ

$$\partial_t \rho_t + \nabla \cdot \left(\rho_t P_{\mathrm{adm}(\rho_t)}[\mathbf{u}_t] \right) = 0, \tag{2.1.1}$$

  where the main point is that ρ is advected by a vector field, compatible with the constraints, which is the closest to the spontaneous one.

The problem in solving Equation (2.1.1) is that the projected field has very low regularity: it is a priori only L^2 in x, and it does not depend smoothly on ρ neither (since a density 1 and a density $1 - \varepsilon$ give very different projection operators). By the way, its divergence is not well-defined neither. To handle this issue we need to redefine the set of admissible velocities by duality:

$$\mathrm{adm}(\rho) = \left\{ \mathbf{v} \in L^2(\rho) : \int_\Omega \mathbf{v} \cdot \nabla p \leq 0, \ \forall p \in H^1(\Omega), p \geq 0, p(1 - \rho) = 0 \right\}.$$

With the help of this formulation we always have the orthogonal decomposition

$$\mathbf{u} = P_{\mathrm{adm}(\rho)}[\mathbf{u}] + \nabla p,$$

where
$$p \in \text{press}(\rho) := \left\{ p \in H^1(\Omega) : p \geq 0,\ p(1-\rho) = 0 \text{ a.e.} \right\}.$$

Indeed, the cones $\text{adm}(\rho)$ and $\nabla\text{press}(\rho)$ are dual to each other. Via this approach the continuity equation (2.1.1) can be rewritten as a system for (ρ, p) which is

$$\begin{cases} \partial_t \rho_t + \nabla \cdot (\rho_t(\mathbf{u}_t - \nabla p_t)) = 0 \\ p \geq 0,\ \rho \leq 1,\ p(1-\rho) = 0. \end{cases} \tag{2.1.2}$$

We can naturally endow this equation/system with initial condition $\rho(0,x) = \rho_0(x)$ ($\rho_0 \in \mathcal{K}$) and with Neumann boundary conditions.

2.1.2 *A diffusive counterpart*

The goal of our work is to study a second order model of crowd movements with hard congestion effect where beside the transport factor a non-degenerate diffusion is present as well. The diffusion is the consequence of a randomness (a Brownian motion) in the movement of the crowd.

With the ingredients that we introduced so far, we would like to modify the Fokker-Planck equation $\partial_t \rho_t - \Delta \rho_t + \nabla \cdot (\rho_t \mathbf{u}_t) = 0$ (always equipped with the natural Neumann boundary conditions on $\partial\Omega$) in order to take into account the density constraint $\rho_t \leq 1$. We observe that the Fokker Planck equation is derived from a motion $\mathrm{d}X_t = \mathbf{u}_t(X_t)\,\mathrm{d}t + \sqrt{2}\,\mathrm{d}B_t$, but is macroscopically represented by the advection of the density ρ_t by the vector field $-\dfrac{\nabla \rho_t}{\rho_t} + \mathbf{u}_t$. Projecting onto the set of admissible velocities raises a natural question: should we project only $\mathbf{u}_t$, and *then* apply the diffusion, or project the whole vector field, including $-\dfrac{\nabla \rho_t}{\rho_t}$? Indeed, this is not a real issue, as $\dfrac{\nabla \rho_t}{\rho_t} = 0$ on the saturated set $\{\rho_t = 1\}$. This corresponds to the fact that the Heat Kernel preserves the constraint $\rho \leq 1$. As a consequence, we consider the modified Fokker-Planck type equation

$$\begin{cases} \partial_t \rho_t - \Delta \rho_t + \nabla \cdot \left(\rho_t P_{\text{adm}(\rho_t)}[\mathbf{u}_t] \right) = 0, \\ \rho(0,x) = \rho_0(x), \text{ in } \Omega, \end{cases} \tag{2.1.3}$$

which can also be written equivalently as

$$\begin{cases} \partial_t \rho_t - \Delta \rho_t + \nabla \cdot (\rho_t(\mathbf{u}_t - \nabla p_t)) = 0 \\ p \geq 0,\ \rho \leq 1,\ p(1-\rho) = 0,\ \rho(0,x) = \rho_0(x), \text{ in } \Omega. \end{cases} \tag{2.1.4}$$

This equation describes the law of a motion where each particle solves the stochastic differential equation

$$\mathrm{d}X_t = (\mathbf{u}_t(X_t) - \nabla p_t(X_t))\,\mathrm{d}t + \sqrt{2}\,\mathrm{d}B_t,$$

where B_t is the standard d-dimensional Brownian motion. More precisely, if we know the solution X_t of the SODE above, and we define $\rho_t := \mathcal{L}(X_t)$, the law of the random variable X_t, this will solve the Fokker-Planck equation (2.1.4).

2.1.3 *Structure of the chapter and main results*

The main goal of this chapter is to provide an existence result, with some extra estimates, for the Fokker-Planck equation (3.1.3) via time discretization, using the so-called splitting method (the two main ingredients of the equation, i.e. the advection with diffusion on one hand, and the density constraint on the other hand, are treated one after the other). In Section 2.2 we recall some results on density-constrained crowd motion, in particular on the projection operator onto the set $\mathcal{K}$. In Section 2.3 we will provide the existence result we aim at, by a splitting scheme and some entropy bounds; the solution will be a curve of measures in $H^1([0,T];\mathbb{W}_2(\Omega))$. In Section 2.4 we will make use of BV estimates to justify that the solution we just built is also $\mathrm{Lip}([0,T];\mathbb{W}_1(\Omega))$ and satisfies a global BV bound $\|\rho_t\|_{BV} \leq C$: this requires to combine BV estimates on the Fokker-Planck equation (which are available depending on the regularity of the vector field $\mathbf{u}$) with BV estimates on the projection operator on $\mathcal{K}$ (which have been recently proven in [DPMSV15] and have been provided in Chapter 1). Section 2.5 presents a short review of alternative approaches, all discretized in time, but based either on gradient-flow techniques (the JKO scheme, see [JKO98]) or on different splitting methods. Finally, in Section 2.6, as a sort of an appendix, we detail the BV estimates on the Fokker-Planck equation (without any density constraint) that we could find; this seems to be a delicate matter, interesting in itself, and we are not aware of the sharp assumptions on the vector field $\mathbf{u}$ to guarantee the BV estimate that we need.

2.2 PROJECTION PROBLEMS IN WASSERSTEIN SPACES

Our analysis relies a lot on the projection operator $P_{\mathcal{K}}$ in the sense of W_2. Here $\mathcal{K} := \{\rho \in \mathcal{P}(\Omega) : \rho \leq 1\}$ and

$$P_{\mathcal{K}}[\mu] := \mathrm{argmin}_{\rho\in\mathcal{K}} \frac{1}{2} W_2^2(\mu,\rho).$$

We recall and summarize below (see [MRCS10, San12a, DPMSV15] and Chapter 1) the main properties of the projection $P_{\mathcal{K}}$ operator.

- As far as Ω is compact, for any probability measure μ, the minimizer in $\min_{\rho\in\mathcal{K}} \frac{1}{2} W_2^2(\mu,\rho)$ exists and is unique, and the operator $P_{\mathcal{K}}$ is continuous (it is even $C^{0,1/2}$ for the W_2 distance).
- The projection $P_{\mathcal{K}}[\mu]$ saturates the constraint $\rho \leq 1$ in the sense that for any $\mu \in \mathscr{P}(\Omega)$ there exists a measurable set $B \subseteq \Omega$ such that $P_{\mathcal{K}}[\mu] = \mathbb{1}_B + \mu^{ac}\mathbb{1}_{B^c}$, where μ^{ac} is the absolutely continuous part of μ.

- The projection is characterized in terms of a pressure field, in the sense that $\rho = P_{\mathcal{K}}[\mu]$ if and only if there exists a Lipschitz function $p \geq 0$, with $p(1-\rho) = 0$, and such that the optimal transport map T from ρ to μ is given by $T := \mathrm{id} - \nabla\varphi = \mathrm{id} + \nabla p$.

- There is (as proven in [DPMSV15], see also Chapter 1) a quantified BV estimate: if $\mu \in BV$ (in the sense that it is absolutely continuous and that its density belongs to $BV(\Omega)$), then $P_{\mathcal{K}}[\mu]$ is also BV and

$$TV(P_{\mathcal{K}}[\mu], \Omega) \leq TV(\mu, \Omega).$$

This last BV estimate will be crucial in Section 2.4, and it is important to have it in this very form (other estimates of the form $TV(P_{\mathcal{K}}[\mu], \Omega) \leq aTV(\mu, \Omega) + b$ would not be as useful as this one, as they cannot be easily iterated).

2.3 EXISTENCE VIA A SPLITTING-UP TYPE ALGORITHM *(main scheme)*

Similarly to the approach in [MRCSV11] (see the algorithm (13) and Theorem 3.5, but we drop the regularity assumption on the vector field, namely C^1 and we assume that it is merely L^∞) for a general, non-gradient vector field, we will build a theoretical algorithm, after time-discretization, to produce a solution of (3.1.3). In this section the spontaneous velocity field is a general vector field $\mathbf{u}_t : \Omega \to \mathbb{R}^d$ (not necessarily a gradient), which depends also on time. We will work on a time interval $[0, T]$ and in a bounded convex domain $\Omega \subset \mathbb{R}^d$ (the case of the flat torus is even simpler and we will not discuss it in details). We consider $\rho_0 \in \mathscr{P}^{\mathrm{ac}}(\Omega)$ to be given, which represents the initial density of the population, and we suppose $\rho_0 \in \mathcal{K}$.

2.3.1 *Splitting using the Fokker-Planck equation*

We assume here that $\mathbf{u} \in L^\infty([0, T] \times \Omega)^d$. Let us consider the following scheme.

Main scheme: Let $\tau > 0$ be a small time step with $N := [T/\tau]$. Let us set $\rho_0^\tau := \rho_0$ and for every $k \in \{1, \dots, N\}$ we define ρ_{k+1}^τ from ρ_k^τ in the following way. First we solve

$$\begin{cases} \partial_t \varrho_t - \Delta \varrho_t + \nabla \cdot (\varrho_t \mathbf{u}_{t+k\tau}) = 0, \\ \qquad\qquad\qquad\qquad t \in]0, \tau], \\ \varrho_0 = \rho_k^\tau, \end{cases} \tag{2.3.1}$$

equipped with the natural Neumann boundary condition $((\nabla(\ln \varrho_t) - \mathbf{u}_t) \cdot \mathbf{n} = 0$ a.e. on $\partial\Omega)$ and set $\rho_{k+1}^\tau = P_{\mathcal{K}}[\tilde{\rho}_{k+1}^\tau]$, where $\tilde{\rho}_{k+1}^\tau = \varrho_\tau$. See Figure 2 on the right.

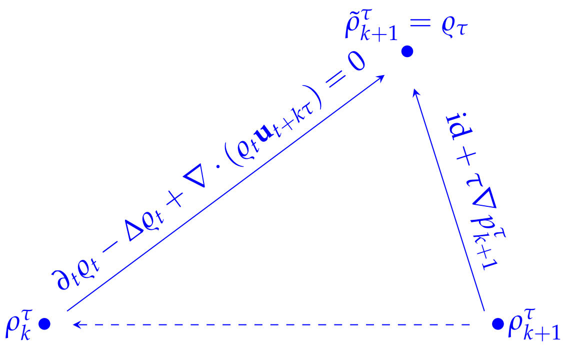

Figure 2: One time step in *Main scheme*: Fokker-Planck, then projection

This means: first follow the Fokker-Planck equation, ignoring the density constraint, for a time τ, then project. In order to state and prove the convergence of the scheme, we need to define some suitable interpolations of the discrete sequence of densities that we have just introduced.

First interpolation. We define the following curves of densities, velocities and momentums constructed with the help of the ρ_k^τ's. First set

$$\rho_t^\tau := \begin{cases} \varrho_{2(t-k\tau)}, & \text{if } t \in [k\tau, (k+1/2)\tau[\,, \\ (\mathrm{id} + 2((k+1)\tau - t)\nabla p_{k+1}^\tau)_\# \rho_{k+1}^\tau, & \text{if } t \in [(k+1/2)\tau, (k+1)\tau[\,, \end{cases}$$

where ϱ_t is the solution of the Fokker-Planck equation (2.3.1) with initial datum ρ_k^τ and ∇p_{k+1}^τ arises from the projection of $\tilde{\rho}_{k+1}^\tau$, more precisely $(\mathrm{id} + \tau \nabla p_{k+1}^\tau)$ is the optimal transport from ρ_{k+1}^τ to $\tilde{\rho}_{k+1}^\tau$. What are we doing? We are fitting into a time interval of length τ the two steps of our algorithm. First we follow the FP equation (2.3.1) at double speed, then we interpolate between the measure we reached and its projection following the geodesic between them. This geodesic is easily described as an image measure of ρ_{k+1}^τ through McCann's interpolation. By the construction it is clear that ρ_t^τ is a continuous curve in $\mathscr{P}(\Omega)$ for $t \in [0, T]$. We now define a family of time-dependent vector fields though

$$\mathbf{v}_t^\tau := \begin{cases} -2\frac{\nabla \varrho_{2(t-k\tau)}}{\varrho_{2(t-k\tau)}} + 2\mathbf{u}_t, & \text{if } t \in [k\tau, (k+1/2)\tau[\,, \\ -2\nabla p_{k+1}^\tau \circ (\mathrm{id} + 2((k+1)\tau - t)\nabla p_{k+1}^\tau)^{-1}, & \text{if } t \in [(k+1/2)\tau, (k+1)\tau[\,, \end{cases}$$

and finally let use define the curve of momentums simply as $\mathbf{E}_t^\tau := \rho_t^\tau \mathbf{v}_t^\tau$.

Second interpolation. We define another interpolation as follows. Set

$$\tilde{\rho}^\tau_t := \varrho_{t-k\tau}, \quad \text{if } t \in [k\tau, (k+1)\tau[,$$

where ϱ_t is (again) the solution of the Fokker-Planck equation (2.3.1) on the time interval $[0,\tau]$ with initial datum ρ^τ_k. Here we do not double its speed. We define the curve of velocities

$$\tilde{\mathbf{v}}^\tau_t := -\frac{\nabla \varrho_{t-k\tau}}{\varrho_{t-k\tau}} + \mathbf{u}_t, \quad \text{if } t \in [k\tau, (k+1)\tau[\,,$$

and we build the curve of momentums by $\tilde{\mathbf{E}}^\tau_t := \tilde{\rho}^\tau_t \tilde{\mathbf{v}}^\tau_t$.

Mind the two differences in the construction of ρ^τ_t and $\tilde{\rho}^\tau_t$ (hence in the construction of $\mathbf{v}^\tau_t$ and $\tilde{\mathbf{v}}^\tau_t$ and $\mathbf{E}^\tau_t$ and $\tilde{\mathbf{E}}^\tau_t$): 1) the first one is continuous in time, while the second one is not; 2) in the first construction we have taken into account the projection operator explicitly, while in the second one we see it just in an indirect manner (via the 'jumps' occurring at every time of the form $t = k\tau$).

Third interpolation. For each τ, we also define piecewise constant curves,

$$\begin{aligned} \hat{\rho}^\tau_t &:= \rho^\tau_{k+1}, \qquad && \text{if } t \in [k\tau, (k+1)\tau[, \\ \hat{\mathbf{v}}^\tau_t &:= \nabla p^\tau_{k+1}, \qquad && \text{if } t \in [k\tau, (k+1)\tau[, \end{aligned}$$

and $\hat{\mathbf{E}}^\tau_t := \hat{\rho}^\tau_t \hat{\mathbf{v}}^\tau_t$. We remark that $p^\tau_{k+1}(1-\rho^\tau_{k+1}) = 0$, hence the curve of momentums is just

$$\hat{\mathbf{E}}^\tau_t := \nabla p^\tau_{k+1}, \quad \text{if } t \in [k\tau, (k+1)\tau[.$$

In order to prove the convergence of the scheme above, we will obtain uniform $H^1([0,T];\mathbb{W}_2(\Omega))$ bounds for the curves ρ^τ. A key observation here is that the metric derivative (w.r.t. W_2) of the solution of the Fokker-Planck equation is comparable with the time differential of the entropy functional along the same solution (see Lemma 2.3.2). Now we state the main theorem of this section.

Theorem 2.3.1. *There exists a continuous curve $[0,T] \ni t \mapsto \rho_t \in \mathbb{W}_2(\Omega)$ and some measures $\mathbf{E}, \tilde{\mathbf{E}}, \hat{\mathbf{E}} \in \mathscr{M}([0,T]\times\Omega)$ such that the curves $\rho^\tau, \tilde{\rho}^\tau, \hat{\rho}^\tau$ converge uniformly in $\mathbb{W}_2(\Omega)$ to ρ and*

$$\mathbf{E}^\tau \overset{*}{\rightharpoonup} \mathbf{E}, \quad \tilde{\mathbf{E}}^\tau \overset{*}{\rightharpoonup} \tilde{\mathbf{E}}, \quad \hat{\mathbf{E}}^\tau \overset{*}{\rightharpoonup} \hat{\mathbf{E}}, \ \text{ in } \mathscr{M}([0,T]\times\Omega)^d, \ \text{ as } \tau \to 0.$$

Moreover $\mathbf{E} = \tilde{\mathbf{E}} - \hat{\mathbf{E}}$ and for a.e. t there exist $\mathbf{v}_t, \tilde{\mathbf{v}}_t, \hat{\mathbf{v}}_t \in L^2_{\rho_t}(\Omega)^d$ such that $\mathbf{E} = \rho\mathbf{v}$, $\tilde{\mathbf{E}} = \rho\tilde{\mathbf{v}}$, $\hat{\mathbf{E}} = \rho\hat{\mathbf{v}}$, $\int_0^T \left(\|\mathbf{v}_t\|^2_{L^2_{\rho_t}} + \|\tilde{\mathbf{v}}_t\|^2_{L^2_{\rho_t}} + \|\hat{\mathbf{v}}_t\|^2_{L^2_{\rho_t}} \right) \mathrm{d}t < +\infty$, $\mathbf{v} = \tilde{\mathbf{v}} - \hat{\mathbf{v}}$, $\tilde{\mathbf{E}}_t =$

$\rho_t \mathbf{u}_t - \nabla \rho_t$ and $\hat{\mathbf{v}}_t = \nabla p_t$ with $p_t \geq 0$ and $p_t(1-\rho_t) = 0$ a.e. As a consequence, ρ_t is a weak solution of the "modified" Fokker-Planck equation

$$\begin{cases} \partial_t \rho_t - \Delta \rho_t + \nabla \cdot (\rho_t(\mathbf{u}_t - \nabla p_t)) = 0, & \text{in }]0,T] \times \Omega, \\ p_t \geq 0, \ \rho_t \leq 1, \ p_t(1-\rho_t) = 0, & \text{in } [0,T] \times \Omega, \\ (\nabla(\ln \rho_t) - \mathbf{u}_t + \nabla p_t) \cdot \mathbf{n} = 0, & \text{on } [0,T] \times \partial\Omega, \\ \rho(t=0,\cdot) = \rho_0, \end{cases} \tag{2.3.2}$$

with the natural Neumann boundary conditions on $\partial\Omega$.

To prove this theorem we will use the following tools.

Lemma 2.3.2. *Let us consider a solution ϱ_t of the Fokker-Planck equation with the velocity field $\mathbf{u}_t$. Then for any time interval $]a,b[$ we have the following estimate*

$$\frac{1}{2}\int_a^b \int_\Omega \left| -\frac{\nabla \varrho_t}{\varrho_t} + \mathbf{u}_t \right|^2 \varrho_t \, \mathrm{d}x \, \mathrm{d}t \leq \mathcal{E}(\varrho_a) - \mathcal{E}(\varrho_b) + \frac{1}{2}\int_a^b \int_\Omega |\mathbf{u}_t|^2 \varrho_t \, \mathrm{d}x \, \mathrm{d}t \tag{2.3.3}$$

In particular this implies

$$\frac{1}{2}\int_a^b |\varrho_t'|^2_{W_2} \, \mathrm{d}t \leq \mathcal{E}(\varrho_a) - \mathcal{E}(\varrho_b) + \frac{1}{2}\int_a^b \int_\Omega |\mathbf{u}_t|^2 \varrho_t \, \mathrm{d}x \, \mathrm{d}t, \tag{2.3.4}$$

where $|\varrho_t'|_{W_2}$ denotes the metric derivative of the curve $t \mapsto \varrho_t \in \mathbb{W}_2(\Omega)$.

Proof. To prove this inequality, let us compute first

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(\varrho_t) &= \int_\Omega (\log \varrho_t + 1)\partial_t \varrho_t \, \mathrm{d}x = \int_\Omega \log \varrho_t (\Delta \varrho_t - \nabla \cdot (\varrho_t \mathbf{u}_t)) \, \mathrm{d}x \\ &= \int_\Omega \left(-\frac{|\nabla \varrho_t|^2}{\varrho_t} + \mathbf{u}_t \cdot \nabla \varrho_t \right) \mathrm{d}x, \end{aligned}$$

where we used the conservation of mass (hence $\int_\Omega \partial_t \varrho_t \, \mathrm{d}x = 0$) and the Neumann boundary conditions in the integration by parts. We now compare this with

$$\begin{aligned} \frac{1}{2}\int_\Omega \left| -\frac{\nabla \varrho_t}{\varrho_t} + \mathbf{u}_t \right|^2 \varrho_t \, \mathrm{d}x - \frac{1}{2}\int_\Omega |\mathbf{u}_t|^2 \varrho_t \, \mathrm{d}x &= \int_\Omega \left(\frac{1}{2}\frac{|\nabla \varrho_t|^2}{\varrho_t} - \nabla \varrho_t \cdot \mathbf{u}_t \right) \mathrm{d}x \\ &\leq \int_\Omega \left(\frac{|\nabla \varrho_t|^2}{\varrho_t} - \nabla \varrho_t \cdot \mathbf{u}_t \right) \mathrm{d}x \\ &= -\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(\varrho_t). \end{aligned}$$

This provides the first part of the statement, i.e. (2.3.3). If we combine this with the fact that the metric derivative of the curve $t \mapsto \varrho_t$ is always less or equal than the $L^2_{\varrho_t}$ norm of the velocity field in the continuity equation, we also get

$$\frac{1}{2}|\varrho_t'|^2_{W_2} - \frac{1}{2}\int_\Omega |\mathbf{u}_t|^2 \varrho_t \leq -\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(\varrho_t),$$

and hence (2.3.4). □

Corollary 2.3.3. *From the inequality (2.3.4) we deduce that*

$$\mathcal{E}(\varrho_b) - \mathcal{E}(\varrho_a) \leq \frac{1}{2}\int_a^b \int_\Omega |\mathbf{u}_t|^2 \varrho_t \,\mathrm{d}x\,\mathrm{d}t,$$

hence in particular for $\mathbf{u} \in L^\infty([0,T]\times\Omega)^d$ *we have that*

$$\mathcal{E}(\varrho_b) - \mathcal{E}(\varrho_a) \leq \frac{1}{2}\|\mathbf{u}\|^2_{L^\infty}(b-a).$$

In particular, if $\varrho_a \leq 1$*, then we have that*

$$\mathcal{E}(\varrho_b) \leq \frac{1}{2}\|\mathbf{u}\|^2_{L^\infty}(b-a).$$

The same estimate can be applied to the curve $\tilde{\rho}^\tau$*, with* $a = k\tau$ *and* $b \in]k\tau,(k+1)\tau[$*, thus obtaining* $\mathcal{E}(\tilde{\rho}^\tau_t) \leq C\tau$ *for every* t*.*

Lemma 2.3.4. *For any* $\rho \in \mathscr{P}(\Omega)$ *we have* $\mathcal{E}\left(P_{\mathcal{K}}[\rho]\right) \leq \mathcal{E}(\rho)$*.*

Proof. We can assume $\rho \ll \mathscr{L}^d$ otherwise the claim is straightforward. As we pointed out in Section 2.2, we know that there exists a measurable set $B \subseteq \Omega$ such that

$$P_{\mathcal{K}}[\rho] = \mathbb{1}_B + \rho\mathbb{1}_{B^c}.$$

Hence it is enough to prove that

$$\int_B \rho\log\rho\,\mathrm{d}x \geq 0 = \int_B P_{\mathcal{K}}[\rho]\log P_{\mathcal{K}}[\rho]\,\mathrm{d}x,$$

as the entropies on B^c coincide. As the mass of ρ and $P_{\mathcal{K}}[\rho]$ is the same on the whole Ω, and they coincide on B^c, we have $\displaystyle\int_B \rho(x)\,\mathrm{d}x = \int_B P_{\mathcal{K}}[\rho]\,\mathrm{d}x = |B|$.

Then, by Jensen's inequality we have

$$\frac{1}{|B|}\int_B \rho\log\rho\,\mathrm{d}x \geq \left(\frac{1}{|B|}\int_B \rho\,\mathrm{d}x\right)\log\left(\frac{1}{|B|}\int_B \rho\,\mathrm{d}x\right) = 0.$$

The entropy decay follows. □

To analyse the pressure field we will need the following result.

Lemma 2.3.5. *Let* $\{p^\tau\}_{\tau>0}$ *be a bounded sequence in* $L^2([0,T];H^1(\Omega))$ *and* $\{\rho^\tau\}_{\tau>0}$ *a sequence of piecewise constant curves valued in* $\mathbb{W}_2(\Omega)$*, which satisfy*

$$W_2(\rho^\tau(a),\rho^\tau(b)) \leq C\sqrt{b-a+\tau}$$

for all $a < b \in [0,T]$ *and* $\rho^\tau \leq C$ *for a fixed constant* C*. Suppose that*

$$p^\tau \geq 0,\ p^\tau(1-\rho^\tau) = 0,\ \rho^\tau \leq 1,$$

and that

$$p^\tau \rightharpoonup p \text{ weakly in } L^2([0,T];H^1(\Omega)) \text{ and } \rho^\tau \to \rho \text{ uniformly in } \mathbb{W}_2(\Omega).$$

Then $p(1-\rho) = 0$ *a.e.*

Proof. The proof of this result is the same as in Step 3 of Section 3.2 of [MRCS10] (see also [RC11]), hence we omit it. □

Lemma 2.3.6. *(i) For every $\tau > 0$ and k we have*

$$W_2^2(\rho_k^\tau, \tilde\rho_{k+1}^\tau), W_2^2(\rho_k^\tau, \rho_{k+1}^\tau) \le \tau C\left(\mathcal{E}(\rho_k^\tau) - \mathcal{E}(\rho_{k+1}^\tau)\right) + C\tau^2,$$

where $C > 0$ only depends on $\|\mathbf{u}\|_{L^\infty}$.

(ii) There exists a constant C, only depending on ρ_0 and $\|\mathbf{u}\|_{L^\infty}$, such that $\mathcal{B}_2(\rho^\tau, \mathbf{E}^\tau) \le C$, $\mathcal{B}_2(\tilde\rho^\tau, \tilde{\mathbf{E}}^\tau) \le C$ and $\mathcal{B}_2(\hat\rho^\tau, \hat{\mathbf{E}}^\tau) \le C$, where $\mathcal{B}_2$ stands for the Benamou-Brenier functional, see (BB).

(iii) For the curve $[0,T] \ni t \mapsto \rho_t^\tau$ we have that

$$\int_0^T |(\rho^\tau)'|_{W_2}^2(t)\,\mathrm{d}t \le C,$$

for a $C > 0$ independent of τ. In particular, we have a uniform Hölder bound on ρ_t:

$$W_2(\rho^\tau(a), \rho^\tau(b)) \le C\sqrt{b-a} \tag{2.3.5}$$

for every $b > a$.

(iv) $\mathbf{E}^\tau, \tilde{\mathbf{E}}^\tau, \hat{\mathbf{E}}^\tau$ are uniformly bounded sequences in $\mathscr{M}([0,T]\times\Omega)^d$.

Proof. (i) First by the triangle inequality and by the fact that $\rho_{k+1}^\tau = P_{\mathcal{K}}[\tilde\rho_{k+1}^\tau]$ we have that

$$W_2(\rho_k^\tau, \rho_{k+1}^\tau) \le W_2(\rho_k^\tau, \tilde\rho_{k+1}^\tau) + W_2(\tilde\rho_{k+1}^\tau, \rho_{k+1}^\tau) \le 2W_2(\rho_k^\tau, \tilde\rho_{k+1}^\tau). \tag{2.3.6}$$

We use (as before) the notation ϱ_t, $t \in [0,\tau]$ for the solution of the Fokker-Planck equation (2.3.1) with initial datum ρ_k^τ, in particular we have $\varrho_\tau = \tilde\rho_{k+1}^\tau$. Using Lemma 2.3.2 and since $\varrho_0 = \rho_k^\tau$ and $\varrho_\tau = \tilde\rho_{k+1}^\tau$, by (2.3.4) and using

$$W_2(\rho_k^\tau, \tilde\rho_{k+1}^\tau) \le \int_0^\tau |\varrho_t'|_{W_2}\,\mathrm{d}t$$

we obtain

$$\begin{aligned}
W_2^2(\rho_k^\tau, \tilde\rho_{k+1}^\tau) &\le \left(\tau^{\frac12}\left(\int_0^\tau |\varrho_t'|_{W_2}^2\,\mathrm{d}t\right)^{\frac12}\right)^2 \\
&\le 2\tau\left(\mathcal{E}(\varrho_0) - \mathcal{E}(\varrho_\tau)\right) + \tau\int_0^\tau\int_\Omega |\mathbf{u}_{k\tau+t}|^2\varrho_t\,\mathrm{d}x\,\mathrm{d}t \\
&\le 2\tau\left(\mathcal{E}(\rho_k^\tau) - \mathcal{E}(\tilde\rho_{k+1}^\tau)\right) + C\tau^2 \\
&\le 2\tau\left(\mathcal{E}(\rho_k^\tau) - \mathcal{E}(\rho_{k+1}^\tau)\right) + C\tau^2,
\end{aligned}$$

where $C > 0$ is a constant depending just on $\|\mathbf{u}\|_{L^\infty}$. We also used the fact that $\mathcal{E}(\rho_{k+1}^\tau) \le \mathcal{E}(\tilde\rho_{k+1}^\tau)$, a consequence of Lemma 2.3.4.

Now by the means of (2.3.6) we obtain

$$W_2^2(\rho_k^\tau,\rho_{k+1}^\tau) \le \tau C\left(\mathcal{E}(\rho_k^\tau)-\mathcal{E}(\rho_{k+1}^\tau)\right)+C\tau^2. \tag{2.3.7}$$

(ii) We use Lemma 2.3.2 on the intervals of type $[k\tau,(k+1/2)\tau[$ and the fact that on each interval of type $[(k+1/2)\tau,(k+1)\tau[$ the curve ρ_t^τ is a constant speed geodesic. In particular, on these intervals we have

$$|(\rho_t^\tau)'|_{W_2} = \|\mathbf{v}_t^\tau\|_{L^2_{\rho_t^\tau}} = 2\tau\|\nabla p_{k+1}^\tau\|_{L^2_{\rho_{k+1}^\tau}} = 2W_2(\rho_{k+1}^\tau,\tilde\rho_{k+1}^\tau).$$

On the other hand we also have

$$\tau^2\|\nabla p_{k+1}^\tau\|^2_{L^2_{\rho_{k+1}^\tau}} = W_2^2(\rho_{k+1}^\tau,\tilde\rho_{k+1}^\tau) \le W_2^2(\rho_k^\tau,\tilde\rho_{k+1}^\tau) \le \tau C\left(\mathcal{E}(\rho_k^\tau)-\mathcal{E}(\rho_{k+1}^\tau)\right)+C\tau^2.$$

Hence we obtain

$$\begin{aligned}
&\int_{k\tau}^{(k+1)\tau}\|\mathbf{v}_t^\tau\|^2_{L^2_{\rho_t^\tau}}\,\mathrm{d}t\\
&=\int_{k\tau}^{(k+1/2)\tau}\int_\Omega 4\left|-\frac{\nabla\varrho_{2(t-k\tau)}}{\varrho_{2(t-k\tau)}}+\mathbf{u}_{2t-k\tau}\right|^2\varrho_{2(t-k\tau)}(x)\,\mathrm{d}x\,\mathrm{d}t\\
&+4\int_{(k+1/2)\tau}^{(k+1)\tau}\int_\Omega|\nabla p_{k+1}^\tau|^2\rho_{k+1}^\tau\,\mathrm{d}x\,\mathrm{d}t\\
&\le C\left(\mathcal{E}(\rho_k^\tau)-\mathcal{E}(\rho_{k+1}^\tau)\right)+C\tau+2\tau\|\nabla p_{k+1}^\tau\|^2_{L^2_{\rho_{k+1}^\tau}}\\
&\le C\left(\mathcal{E}(\rho_k^\tau)-\mathcal{E}(\rho_{k+1}^\tau)\right)+C\tau.
\end{aligned}$$

Hence by adding up we obtain

$$\mathcal{B}_2(\rho^\tau,\mathbf{E}^\tau)\le\sum_k\left\{C\left(\mathcal{E}(\rho_k^\tau)-\mathcal{E}(\rho_{k+1}^\tau)\right)+C\tau\right\}=C\left(\mathcal{E}(\rho_0^\tau)-\mathcal{E}(\rho_{N+1}^\tau)\right)+CT\le C.$$

The estimate on $\mathcal{B}_2(\tilde\rho^\tau,\tilde{\mathbf{E}}^\tau)$ and $\mathcal{B}_2(\hat\rho^\tau,\hat{\mathbf{E}}^\tau)$ are completely analogous and descend from the previous computations.

(iii) The estimate on $\mathcal{B}_2(\rho^\tau,\mathbf{E}^\tau)$ implies a bound on $\int_0^T|(\rho_t^\tau)'|^2_{W_2}\,\mathrm{d}t$ because $\mathbf{v}^\tau$ is a velocity field for ρ^τ (i.e., the pair $(\rho^\tau,\mathbf{E}^\tau)$ solves the continuity equation). In addition estimate (2.3.5) follows from estimate (2.3.7).

(iv) In order to estimate the total mass of $\mathbf{E}^\tau$ we write

$$\begin{aligned}
|\mathbf{E}^\tau|([0,T]\times\Omega)&=\int_0^T\int_\Omega|\mathbf{v}_t^\tau|\rho_t^\tau\,\mathrm{d}x\,\mathrm{d}t\le\int_0^T\left(\int_\Omega|\mathbf{v}_t^\tau|^2\rho_t^\tau\,\mathrm{d}x\right)^{\frac12}\left(\int_\Omega\rho_t^\tau\,\mathrm{d}x\right)^{\frac12}\mathrm{d}t\\
&\le\sqrt{T}\left(\int_0^T\int_\Omega|\mathbf{v}_t^\tau|^2\rho_t^\tau\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac12}\le C.
\end{aligned}$$

The bounds on $\tilde{\mathbf{E}}^\tau$ and $\hat{\mathbf{E}}^\tau$ rely on the same argument. □

Proof of Theorem 2.3.1. We use the tools from Lemma 2.3.6.

Step 1. By the bounds on the metric derivative of the curves ρ_t^τ we get compactness, i.e. there exists a curve $[0,T] \ni t \mapsto \rho_t \in \mathscr{P}(\Omega)$ such that ρ_t^τ converges uniformly in $[0,T]$ w.r.t. W_2, in particular weakly-$\star$ in $\mathscr{P}(\Omega)$ for all $t \in [0,T]$. It is easy to see that $\tilde\rho^\tau$ and $\hat\rho^\tau$ are converging to the same curve. Indeed we have $\tilde\rho_t^\tau = \rho_{\tilde s(t)}^\tau$ and $\hat\rho_t^\tau = \rho_{\hat s(t)}^\tau$ for $|\tilde s(t) - t| \le \tau$ and $|\hat s(t) - t| \le \tau$, which implies $W_2(\rho_t^\tau, \tilde\rho_t^\tau), W_2(\rho_t^\tau, \hat\rho_t^\tau) \le C\tau^{\frac12}$. This provides the convergence to the same limit.

Step 2. By the boundedness of $\mathbf{E}^\tau, \tilde{\mathbf{E}}^\tau$ and $\hat{\mathbf{E}}^\tau$ in $\mathscr{M}([0,T]\times\Omega)^d$ we have the existence of $\mathbf{E}, \tilde{\mathbf{E}}, \hat{\mathbf{E}} \in \mathscr{M}([0,T]\times\Omega)^d$ such that $\mathbf{E}^\tau \overset{*}{\rightharpoonup} \mathbf{E}, \tilde{\mathbf{E}}^\tau \overset{*}{\rightharpoonup} \tilde{\mathbf{E}}, \hat{\mathbf{E}}^\tau \overset{*}{\rightharpoonup} \hat{\mathbf{E}}$ as $\tau \to 0$. Now we show that $\mathbf{E} = \tilde{\mathbf{E}} - \hat{\mathbf{E}}$. Indeed, let us show that for any $f \in \mathrm{Lip}([0,T]\times\Omega)^d$ test function we have that

$$\left| \int_0^T \int_\Omega f_t \cdot \left(\mathbf{E}_t^\tau - (\tilde{\mathbf{E}}_t^\tau + \hat{\mathbf{E}}_t^\tau) \right) (\mathrm{d}x, \mathrm{d}t) \right| \to 0,$$

as $\tau \to 0$. First for each $k \in \{0, \dots, N\}$ we have that

$$\begin{aligned}
\int_{k\tau}^{(k+1/2)\tau} \int_\Omega f_t \cdot \mathbf{E}_t^\tau(\mathrm{d}x, \mathrm{d}t) &= \int_{k\tau}^{(k+1)\tau} \int_\Omega f_{(t+k\tau)/2} \cdot (-\nabla\varrho_{t-k\tau} + \mathbf{u}_t\varrho_{t-k\tau})(\mathrm{d}x, \mathrm{d}t) \\
&= \int_{k\tau}^{(k+1)\tau} \int_\Omega f_t \cdot \tilde{\mathbf{E}}_t^\tau(\mathrm{d}x, \mathrm{d}t) \\
&+ \int_{k\tau}^{(k+1)\tau} \int_\Omega \left(f_{(t+k\tau)/2} - f_t \right) \cdot \tilde{\mathbf{E}}_t^\tau(\mathrm{d}x, \mathrm{d}t)
\end{aligned}$$

and

$$\begin{aligned}
&\int_{(k+1/2)\tau}^{(k+1)\tau} \int_\Omega f_t \cdot \mathbf{E}_t^\tau(\mathrm{d}x, \mathrm{d}t) = \\
&= \int_{k\tau}^{(k+1)\tau} \int_\Omega -f_{(t+(k+1)\tau)/2} \circ (\mathrm{id} + ((k+1)\tau - t)\nabla p_{k+1}^\tau) \cdot \nabla p_{k+1}^\tau \rho_{k+1}^\tau(\mathrm{d}x, \mathrm{d}t) \\
&= -\int_{k\tau}^{(k+1)\tau} \int_\Omega f_t \cdot \hat{\mathbf{E}}_t^\tau(\mathrm{d}x, \mathrm{d}t) \\
&+ \int_{k\tau}^{(k+1)\tau} \int_\Omega \left(f_t - f_{(t+(k+1)\tau)/2} \circ (\mathrm{id} + ((k+1)\tau - t)\nabla p_{k+1}^\tau) \right) \cdot \hat{\mathbf{v}}_t^\tau \hat\rho_t^\tau(\mathrm{d}x, \mathrm{d}t)
\end{aligned}$$

This implies that

$$\begin{aligned}
\left| \int_0^T \int_\Omega f_t \cdot (\mathbf{E}_t^\tau - \tilde{\mathbf{E}}_t^\tau + \hat{\mathbf{E}}_t^\tau)(\mathrm{d}x, \mathrm{d}t) \right| &\le \sum_k \int_{k\tau}^{(k+1)\tau} \mathrm{Lip}(f)\tau \int_\Omega |\tilde{\mathbf{E}}_t^\tau|(\mathrm{d}x, \mathrm{d}t) \\
&+ \sum_k \int_{k\tau}^{(k+1)\tau} \mathrm{Lip}(f)\tau \int_\Omega (1 + |\hat{\mathbf{v}}_t^\tau|)|\hat{\mathbf{E}}_t^\tau|(\mathrm{d}x, \mathrm{d}t) \\
&\le \tau C\,\mathrm{Lip}(f) \left(|\tilde{\mathbf{E}}^\tau|([0,T]\times\Omega) + |\hat{\mathbf{E}}^\tau|([0,T]\times\Omega) + \mathcal{B}_2(\hat\rho^\tau, \hat{\mathbf{E}}^\tau) \right) \\
&\le \tau C\,\mathrm{Lip}(f),
\end{aligned}$$

for a uniform constant $C > 0$. By an approximation argument we get similar estimate for all f continuous, not necessarily Lipschitz. Letting $\tau \to 0$ we prove the claim.

Step 3. The bounds on $\mathcal{B}_2(\rho^\tau, \mathbf{E}^\tau), \mathcal{B}_2(\tilde{\rho}^\tau, \tilde{\mathbf{E}}^\tau)$ and $\mathcal{B}_2(\hat{\rho}^\tau, \hat{\mathbf{E}}^\tau)$ pass to the limit by semicontinuity and allow to conclude that $\mathbf{E}, \tilde{\mathbf{E}}$ and $\hat{\mathbf{E}}$ are vector valued Radon measures absolutely continuous w.r.t. ρ. Hence there exist $\mathbf{v}_t, \tilde{\mathbf{v}}_t, \hat{\mathbf{v}}_t \in L^2_{\rho_t}(\Omega)$ such that $\mathbf{E} = \rho\mathbf{v}$, $\tilde{\mathbf{E}} = \rho\tilde{\mathbf{v}}$ and $\hat{\mathbf{E}} = \rho\hat{\mathbf{v}}$.

Step 4. We now look at the equations satisfied by $\mathbf{E}, \tilde{\mathbf{E}}$ and $\hat{\mathbf{E}}$. First we use $\partial_t\rho^\tau + \nabla\cdot\mathbf{E}^\tau = 0$, we pass to the limit as $\tau \to 0$, and we get

$$\partial_t\rho + \nabla\cdot\mathbf{E} = 0$$

satisfied in the sense of distributions.

Then, we use $\tilde{\mathbf{E}}^\tau = -\nabla\tilde{\rho}^\tau + \mathbf{u}_t\tilde{\rho}^\tau$, we pass to the limit again as $\tau \to 0$, and we get

$$\tilde{\mathbf{E}} = -\nabla\rho + \mathbf{u}_t\rho.$$

To justify the above limit, the only delicate point is passing to the limit the term $\mathbf{u}_t\tilde{\rho}^\tau$, since $\mathbf{u}$ is only L^∞, and $\tilde{\rho}^\tau$ converges weakly as measures, and we are a priori only allowed to multiply it by continuous functions. Yet, we remark that by Corollary 2.3.3 we have that $\mathcal{E}(\tilde{\rho}^\tau_t) \le C\tau$ for all $t \in [0,T]$. In particular, this provides, for each t, uniform integrability for $\tilde{\rho}^\tau_t$ and turns the weak convergence as measures into weak convergence in L^1. This allows to multiply by $\mathbf{u}_t$ in the weak limit.

Finally, we look at $\hat{\mathbf{E}}^\tau$. There exists a piecewise constant (in time) function p^τ (defined as p^τ_{k+1} on every interval $]k\tau,(k+1)\tau]$) such that $p^\tau \ge 0$, $p^\tau(1-\hat{\rho}^\tau) = 0$,

$$\int_0^T\int_\Omega |\nabla p^\tau|^2(\,\mathrm{d}x,\,\mathrm{d}t) = \int_0^T\int_\Omega |\nabla p^\tau|^2\hat{\rho}^\tau(\,\mathrm{d}x,\,\mathrm{d}t) = \int_0^T\int_\Omega |\hat{\mathbf{v}}^\tau|^2\hat{\rho}^\tau(\,\mathrm{d}x,\,\mathrm{d}t) \le C \tag{2.3.8}$$

and $\hat{\mathbf{E}}^\tau = \nabla p^\tau\hat{\rho}^\tau = \nabla p^\tau$. The bound (2.3.8) implies that ∇p^τ is uniformly bounded in $L^2([0,T];L^2(\Omega))$. Since for every t we have $|\{p^\tau_t = 0\}| \ge |\{\hat{\rho}^\tau_t < 1\}| \ge |\Omega| - 1$, we can use a version of Poincaré's inequality, and get a uniform bound for p^τ in $L^2([0,T];L^2(\Omega)) = L^2([0,T]\times\Omega)$. Hence there exists $p \in L^2([0,T]\times\Omega)$ such that $p^\tau \rightharpoonup p$ weakly in L^2 as $\tau \to 0$. In particular we have $\hat{\mathbf{E}} = \nabla p$. Moreover it is clear that $p \ge 0$ and by Lemma 2.3.5 we obtain $p(1-\rho) = 0$ a.e. as well. Indeed, the assumptions of the Lemma are easily checked: we only need to estimate $W_2(\hat{\rho}^\tau(a), \hat{\rho}^\tau(b))$ for $b > a$, but we have

$$W_2(\hat{\rho}^\tau(a), \hat{\rho}^\tau(b)) = W_2(\rho^\tau(k_a\tau), \rho^\tau(k_b\tau)) \le C\sqrt{k_b - k_a}$$

for $k_b\tau \le b + \tau$ and $k_a \ge a$. Once we have $\hat{\mathbf{E}} = \nabla p$ with $p(1-\rho) = 0$ a.e., $p \in L^2([0,T];H^1(\Omega))$ and $\rho \in L^\infty$, we can also write

$$\hat{\mathbf{E}} = \nabla p = \rho\nabla p.$$

If we sum up our results, using $\mathbf{E} = \tilde{\mathbf{E}} - \hat{\mathbf{E}}$, we have

$$\partial_t \rho - \Delta\rho + \nabla \cdot (\rho(\mathbf{u} - \nabla p)) = 0$$

together with

$$p \geq 0,\ \rho \leq 1,\ p(1-\rho) = 0$$

a.e. in $[0,T] \times \Omega$. As usual, this equation is satisfied in a weak sense, with Neumann boundary conditions and with the initial condition $\rho(t = 0, \cdot) = \rho_0$. □

2.4 UNIFORM $\mathrm{Lip}([0,T];\mathbb{W}_1))$ AND BV ESTIMATES

In this section we provide uniform estimates for the curves $\rho^\tau, \tilde{\rho}^\tau$ and $\hat{\rho}^\tau$ of the following form: we prove uniform BV (in space) bounds on $\tilde{\rho}^\tau$ (which implies the same bound for $\hat{\rho}^\tau$) and uniform Lipschitz bounds in time for the W_1 distance for ρ^τ. This is a small improvement compared to the previous section, both in time regularity (Lipschitz instead of H^1, but for W_1 instead of W_2) and in space (any higher order regularity of ρ was absent from the previous results). Nevertheless there is a price to pay for this improvement: we have to assume higher regularity for the velocity field. These uniform Lipschitz in time bounds are based both on BV estimates for the Fokker-Planck equation (see Lemma 2.6.1 from Section 2.6) and for the projection operator $P_{\mathcal{K}}$ (see [DPMSV15] and the previous Chapter 1). The assumption on $\mathbf{u}$ is essentially the following: we need to control the growth of the total variation of the solutions of the Fokker-Planck equation (2.3.1), and we need to iterate this bound along time steps.

We will discuss in Section 2.6 the different BV estimates on the Fokker-Planck equation that we were able to find. The desired estimate is true for $\mathbf{u}_t \in C^{1,1}(\Omega)$ and $\mathbf{u}_t(x) \cdot \mathbf{n}(x) = 0$ for $x \in \partial\Omega$, and seems to be an open problem if $\mathbf{u}$ is only Lipschitz continuous. We will also assume $\rho_0 \in BV(\Omega)$. Despite these extra regularity assumptions, we think these estimates have their own interest, exploiting some finer properties of the solutions of the Fokker-Planck equation and of the Wasserstein projection operator.

Before entering into the details of the estimates, we want to discuss why we concentrate on BV estimates (instead of Sobolev ones) and on W_1 (instead of W_p, $p > 1$). The main reason is the role of the projection operator: indeed, even if $\rho \in W^{1,p}(\Omega)$, we do not have in general $P_{\mathcal{K}}[\rho] \in W^{1,p}$ because the projection creates some jumps at the boundary of $\{P_{\mathcal{K}}[\rho] = 1\}$. This prevents from obtaining any $W^{1,p}$ estimate for $p > 1$. On the other hand, [DPMSV15] exactly proves a BV estimate on $P_{\mathcal{K}}[\rho]$ and paves the way to BV bounds for our equation. Concerning the regularity in time, we observe that the velocity field in the Fokker-Planck equation contains a term in $\nabla\rho/\rho$. Since the metric derivative in $\mathbb{W}_p$ is given by the L^p norm (w.r.t. ρ_t) of the velocity field, it is clear that estimates in $\mathbb{W}_p$ for $p > 1$ would require spatial $W^{1,p}$ estimates on the solution itself, which are impossible for $p > 1$. The precise result that we prove is

Theorem 2.4.1. *Supposing* $\|\mathbf{u}_t\|_{C^{1,1}} \leq C$ *and* $\rho_0 \in BV(\Omega)$*, we have* $\|\tilde{\rho}^\tau_t\|_{BV} \leq C$ *and* $W_1(\rho^\tau_k, \rho^\tau_{k+1}) \leq C\tau$*. As a consequence we also have*

$$\rho \in \mathrm{Lip}([0,T]; \mathbb{W}_1)) \cap L^\infty([0,T]; BV(\Omega)).$$

To prove this theorem we need the following lemmas.

Lemma 2.4.2. *Suppose* $\|\mathbf{u}_t\|_{\mathrm{Lip}} \leq C$ *and* $\mathbf{u}_t \cdot \mathbf{n} = 0$ *on* $\partial\Omega$*. Then for the solution* ϱ_t *of* (2.6.1) *with the vector field* $\mathbf{u}$ *we have the estimate*

$$\|\varrho_t\|_{L^\infty} \leq \|\varrho_0\|_{L^\infty} e^{Ct},$$

where $C = \|\nabla \cdot \mathbf{u}_t\|_{L^\infty}$*.*

Proof. Let us set $f : [0, +\infty[\times\Omega \to \mathbb{R}$, $f_t := \varrho_t e^{-Ct} \geq 0$ with a fixed constant $C > 0$. We have

$$\partial_t f_t = \partial_t \varrho_t e^{-Ct} - C f_t = e^{-Ct}(\Delta\varrho_t - \nabla \cdot (\varrho_t \mathbf{u}_t)) - C f_t,$$

which means that f_t is a solution of

$$\partial_t f_t = \Delta f_t - \nabla \cdot (f_t \mathbf{u}_t) - C f_t.$$

Now for a $p > 1$ let us denote $\mathcal{I}_p(t) := \|f_t\|^p_{L^p}$ and compute

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{I}_p(t) &= p\int_\Omega |f_t|^{p-1}\partial_t f_t \,\mathrm{d}x = p\int_\Omega |f_t|^{p-1}(\Delta f_t - \nabla \cdot (f_t \mathbf{u}_t) - C f_t)\,\mathrm{d}x \\
&= -pC\mathcal{I}_p(t) - p(p-1)\int_\Omega |f_t|^{p-2}|\nabla f_t|^2 \,\mathrm{d}x \\
&+ p(p-1)\int_\Omega |f_t|^{p-1}\nabla f_t \cdot \mathbf{u}_t \,\mathrm{d}x \\
&\leq -pC\mathcal{I}_p(t) - (p-1)\int_\Omega |f_t|^p \nabla \cdot \mathbf{u}_t \,\mathrm{d}x \\
&\leq ((p-1)\|\nabla \cdot \mathbf{u}_t\|_{L^\infty} - pC)\mathcal{I}_p(t)
\end{aligned}$$

By Grönwall's lemma we obtain that

$$\mathcal{I}_p(t) \leq e^{t((p-1)\|\nabla\cdot\mathbf{u}\|_{L^\infty} - pC)}\mathcal{I}_p(0),$$

hence

$$\mathcal{I}_p^{1/p}(t) \leq e^{t((p-1)/p\|\nabla\cdot\mathbf{u}\|_{L^\infty} - C)}\mathcal{I}_p^{1/p}(0),$$

and sending $p \to +\infty$ we get that

$$\|f_t\|_{L^\infty} \leq e^{t(\|\nabla\cdot\mathbf{u}\|_{L^\infty} - C)}\|f_0\|_{L^\infty}.$$

Using the definition of f_t we get the estimation

$$\|\varrho_t\|_{L^\infty} \leq e^{t\|\nabla\cdot\mathbf{u}\|_{L^\infty}}\|\varrho_0\|_{L^\infty},$$

which proves the claim. □

We remark that the above lemma implies in particular that after every step in the *Main scheme* we have $\tilde\rho^\tau_{k+1} \leq e^{\tau c} \leq 1 + C\tau$, where $c := \|\nabla\cdot\mathbf{u}\|_{L^\infty}$ and $C := c + o(\tau)$. Let us now present the following lemma as well.

Corollary 2.4.3. *Along the iterations of our* Main scheme, *for every k we have*

$$W_1(\tilde\rho^\tau_{k+1}, \rho^\tau_{k+1}) \leq \tau C$$

for a constant $C > 0$ independent of τ.

Proof. With the saturation property of the projection (see Section 2.2, Chapter 1 or [DPMSV15]), we know that there exists a measurable set $B \subseteq \Omega$ such that $\rho^\tau_{k+1} = \tilde\rho^\tau_{k+1}\mathbb{1}_B + \mathbb{1}_{\Omega\setminus B}$. On the other hand we know that

$$\begin{aligned} W_1(\tilde\rho^\tau_{k+1}, \rho^\tau_{k+1}) &= \sup_{f\in\mathrm{Lip}_1(\Omega),\, 0\leq f\leq \mathrm{diam}(\Omega)} \int_\Omega f(\tilde\rho^\tau_{k+1} - \rho^\tau_{k+1})\,\mathrm{d}x \\ &= \sup_{f\in\mathrm{Lip}_1(\Omega),\, 0\leq f\leq \mathrm{diam}(\Omega)} \int_{\Omega\setminus B} f(\tilde\rho^\tau_{k+1} - 1)\,\mathrm{d}x \leq \tau C\,|\Omega|\mathrm{diam}(\Omega). \end{aligned}$$

We used the fact that the competitors f in the dual formula can be taken positive and bounded by the diameter of Ω, just by adding a suitable constant. This implies as well that C is depending on c, $|\Omega|$ and $\mathrm{diam}(\Omega)$. □

Proof of Theorem 2.4.1. First we take care of the BV estimate. Lemma 2.6.1 in Section 2.6 guarantees, for $t\in]k\tau,(k+1)\tau[$, that we have $TV(\tilde\rho^\tau_t) \leq C\tau + e^{C\tau}TV(\rho^\tau_k)$. Together with the BV bound on the projection that we presented in Section 2.2 (taken from [DPMSV15]), this can be iterated, providing a uniform bound (depending on $TV(\rho_0)$, T and $\sup_t \|\mathbf{u}_t\|_{C^{1,1}}$) on $\|\tilde\rho^\tau_t\|_{BV}$. Passing this estimate to the limit as $\tau\to 0$ we get $\rho\in L^\infty([0,T];BV(\Omega))$.

Then we estimate the behavior in terms of W_1. We estimate

$$\begin{aligned} W_1(\rho^\tau_k, \tilde\rho^\tau_{k+1}) \leq \int_{k\tau}^{(k+1)\tau} |(\tilde\rho^\tau_t)'|_{W_1}\,\mathrm{d}t &\leq \int_{k\tau}^{(k+1)\tau}\int_\Omega \left(\frac{|\nabla\tilde\rho^\tau_t|}{\tilde\rho^\tau_t} + |\mathbf{u}_t|\right)\tilde\rho^\tau_t\,\mathrm{d}x\,\mathrm{d}t \\ &\leq \int_{k\tau}^{(k+1)\tau} \|\tilde\rho^\tau_t\|_{BV}\,\mathrm{d}t + C\tau \leq C\tau. \end{aligned}$$

Hence, we obtain

$$W_1(\rho^\tau_k, \rho^\tau_{k+1}) \leq W_1(\rho^\tau_k, \tilde\rho^\tau_{k+1}) + W_1(\tilde\rho^\tau_{k+1}, \rho^\tau_{k+1}) \leq \tau C.$$

This in particular means, for $b > a$,

$$W_1(\hat\rho^\tau(a), \hat\rho^\tau(b)) \leq C(b - a + \tau).$$

We can pass this relation to the limit, using that, for every t, we have $\hat\rho^\tau_t \to \rho_t$ in $\mathbb{W}_2(\Omega)$ (and hence also in $\mathbb{W}_1(\Omega)$, since $W_1 \leq W_2$), we get

$$W_1(\rho(a), \rho(b)) \leq C(b - a),$$

which means that ρ is Lipschitz continuous in $\mathbb{W}_1(\Omega)$. □

2.5 VARIATIONS ON A THEME: SOME REFORMULATIONS OF THE *main scheme*

In this section we propose some alternative approaches to study the problem (3.1.3). The general idea is to discretize in time, and give a way to produce a measure ρ^τ_{k+1} starting from ρ^τ_k. Observe that the interpolations that we proposed in the previous sections $\rho^\tau, \tilde\rho^\tau$ and $\hat\rho^\tau$ are only technical tools to state and prove a convergence result, and the most important point is exactly the definition of ρ^τ_{k+1}.

The alternative approaches proposed here explore different ideas, more difficult to implement than one that we presented in Section 2.3, and/or restricted to some particular cases (for instance when $\mathbf{u}$ is a gradient). They have their own modeling interest and this is the main reason justifying their sketchy presentation.

2.5.1 *Variant 1: transport, diffusion then projection.*

We recall that the original splitting approach for the equation without diffusion ([MRCSV11, RC11]) exhibited an important difference compared to what we did in Section 2.3. Indeed, in the first phase of each time step (i.e. before the projection) the particles follow the vector field $\mathbf{u}$ and $\tilde\rho^\tau_{k+1}$ was not defined as the solution of a continuity equation with advection velocity given by $\mathbf{u}_t$, but as the image of ρ^τ_k via a straight-line transport: $\tilde\rho^\tau_{k+1} := (\mathrm{id} + \tau\mathbf{u}_{k\tau})_\#\rho^\tau_k$. One can wonder whether it is possible to follow a similar approach here.

A possible way to proceed is the following: take a random variable X distributed according to ρ^τ_k, and define $\tilde\rho^\tau_{k+1}$ as the law of $X + \tau\mathbf{u}_{k\tau}(X) + B_\tau$, where B is a Brownian motion, independent of X. This exactly means that every particle moves starting from its initial position X, following a displacement ruled by $\mathbf{u}$, but adding a stochastic effect in the form of the value at time τ of a Brownian motion. We can check that this means

$$\tilde\rho^\tau_{k+1} := \eta_\tau \star \left((\mathrm{id} + \tau\mathbf{u}_{k\tau})_\#\rho^\tau_k\right),$$

where η_τ is a Gaussian kernel with zero-mean and variance τ, i.e. $\eta_\tau(x) := \dfrac{1}{\sqrt{4\tau\pi}}e^{-\frac{|x|^2}{4\tau}}$.

Then we define

$$\rho^\tau_{k+1} := P_\mathcal{K}\left[\tilde\rho_{k+1}\right].$$

Despite the fact that this scheme is very natural and essentially not that different from the *Main scheme*, we have to be careful with the analysis. First we have to quantify somehow the distance $W_p(\rho^\tau_k, \tilde\rho^\tau_{k+1})$ for some $p \geq 1$ and show that this is of order τ in some sense. Second, we need to be careful when performing the convolution with the heat kernel (or adding the Brownian motion, which is the same): this requires either to work in the whole space (which was not our framework) or in a periodic setting ($\Omega = \mathbb{T}^d$, the flat torus, which is qutie restrictive). Otherwise, the "explicit" convolution step should be replaced with some other construction, such as following the Heat equation (with Neumann boundary conditions) for a time τ. But this brings back to a situation very similar to the *Main scheme*, with the additional difficulty that we do not really have estimates on $(\mathrm{id} + \tau \mathbf{u}_{k\tau})_\# \rho_k^\tau$.

2.5.2 *Variant 2: gradient flow techniques for gradient velocity fields*

In this section we assume that the velocity field of the population is given by the opposite of the gradient of a function, $\mathbf{u}_t = -\nabla V_t$ a typical example is given when we take for V the distance function to the exit (see the discussions in [MRCS10] about this type of question). We start from the case where V does not depend on time, and we suppose $V \in W^{1,1}(\Omega)$. In this particular case – beside the splitting approach – the problem has a variational structure, hence it is possible to show the existence by the means of gradient flows in Wasserstein spaces.

Since the celebrated paper of Jordan, Kinderlehrer and Otto ([JKO98]) we know that the solutions of the Fokker-Planck equation (with a gradient vector field) can be obtained with the help of the gradient flow of a perturbed entropy functional with respect to the Wasserstein distance W_2. This formulation of JKO scheme was also used in [MRCS10] for the first order model with density constraints. It is easy to combine the JKO scheme with density constraints to study the second order/diffusive model. As a slight modification of the model from [MRCS10], we can consider the following discrete implicite Euler (or JKO) scheme. As usual, we fix a time step $\tau > 0$, $\rho_0^\tau = \rho_0$ and for all $k \in \mathbb{N}$ we just need to define ρ_{k+1}^τ. We take

$$\rho_{k+1}^\tau = \mathrm{argmin}_{\rho \in \mathcal{P}(\Omega)} \left\{ \int_\Omega V(x)\rho(x)\,\mathrm{d}x + \mathcal{E}(\rho) + I_\mathcal{K}(\rho) + \frac{1}{2\tau} W_2^2(\rho, \rho_k^\tau) \right\}, \tag{2.5.1}$$

where $I_\mathcal{K}$ is the indicator function of $\mathcal{K}$, which is

$$I_\mathcal{K}(x) := \begin{cases} 0, & \text{if } x \in \mathcal{K}, \\ +\infty, & \text{otherwise.} \end{cases}$$

The usual techniques from [JKO98, MRCS10] can be used to identify that the problem (3.1.3) is the gradient flow of the functional

$$\rho \mapsto J(\rho) := \int_\Omega V(x)\rho(x)\,\mathrm{d}x + \mathcal{E}(\rho) + I_\mathcal{K}(\rho)$$

and that the above discrete scheme converges (up to a subsequence) to a solution of (3.1.3), thus proving existence. The key estimate for compactness is

$$\frac{1}{2\tau} W_2^2(\rho_{k+1}^\tau, \rho_k^\tau) \le J(\rho_k^\tau) - J(\rho_{k+1}^\tau),$$

which can be summed up (as on the r.h.s. we have a telescopic series), thus obtaining the same bounds on $\mathcal{B}_2$ that we used in Section 2.3.

It is also possible to study a variant where V depends on time. We assume for simplicity that $V \in \mathrm{Lip}([0,T] \times \Omega)$ (this is a simplification, less regularity in space, such as $W^{1,1}$, could be sufficient). In this case we define

$$J_t(\rho) := \int_\Omega V_t(x)\rho(x)\,\mathrm{d}x + \mathcal{E}(\rho) + I_{\mathcal{K}}(\rho)$$

and

$$\rho^\tau_{k+1} = \mathrm{argmin}_{\rho \in \mathcal{P}(\Omega)} \left\{ J_{k\tau}(\rho) + \frac{1}{2\tau} W_2^2(\rho, \rho^\tau_k) \right\}, \tag{2.5.2}$$

The analysis proceeds similarly, with the only exception that the we get

$$\frac{1}{2\tau} W_2^2(\rho^\tau_{k+1}, \rho^\tau_k) \leq J_{k\tau}(\rho^\tau_k) - J_{k\tau}(\rho^\tau_{k+1}),$$

which is no more a a telescopic series. Yet, we have $J_{k\tau}(\rho^\tau_{k+1}) \geq J_{(k+1)\tau}(\rho^\tau_{k+1}) + \mathrm{Lip}(V)\tau$, and we can go on with a telescopic sum plus a remainder of the order of τ.

2.5.3 *Variant 3: transport then gradient flow-like step with the penalized entropy functional.*

We present now a different scheme, which combines some of the previous approaches. It could formally provide a solution of the same equation, but presents some extra difficulties.

We define now $\tilde{\rho}^\tau_{k+1} := (\mathrm{id} + \tau \mathbf{u}_{k\tau})_\# \rho^\tau_k$ and with the help of this we define

$$\rho^\tau_{k+1} := \mathrm{argmin}_{\rho \in \mathcal{K}}\, \mathcal{E}(\rho) + \frac{1}{2\tau} W_2^2(\rho, \tilde{\rho}^\tau_{k+1}).$$

In the last optimization problem we minimize a strictly convex and l.s.c. functionals, and hence we have existence and uniqueness of the solution. The formal reason for this scheme being adapted to the equation is that we perform a step of a JKO scheme in the spirit of [JKO98] (without the density constraint) or of [MRCS10] (without the entropy term). This should let a term $-\Delta\rho - \nabla \cdot (\rho \nabla p)$ appear in the evolution equation. The term $\nabla \cdot (\rho \mathbf{u})$ is due to the first step (the definition of $\tilde{\rho}^\tau_{k+1}$). To explain a little bit more for the unexperienced reader, we consider the optimality conditions for the above minimization problem. Following [MRCS10], we can say that $\rho \in \mathcal{K}$ is optimal if and only if there exists a constant $\ell \in \mathbb{R}$ and a Kantorovich potential φ for the transport from ρ to ρ^τ_k such that

$$\rho = \begin{cases} 1 & \text{on } \left(\ln \rho + \frac{\varphi}{\tau}\right) < \ell, \\ 0 & \text{on } \left(\ln \rho + \frac{\varphi}{\tau}\right) > \ell, \\ \in [0,1] & \text{on } \left(\ln \rho + \frac{\varphi}{\tau}\right) = \ell. \end{cases}$$

We then define $p = (\ell - \ln \rho - \frac{\varphi}{\tau})_+$ and we get $p \in \mathrm{press}(\rho)$. Moreover, $\rho -$ a.e. $\nabla p = -\frac{\nabla \rho}{\rho} - \frac{\nabla \varphi}{\tau}$. We then use the fact that the optimal transport is of the form $T = \mathrm{id} - \nabla \varphi$ and obtain a situation as is sketched in Figure 3.

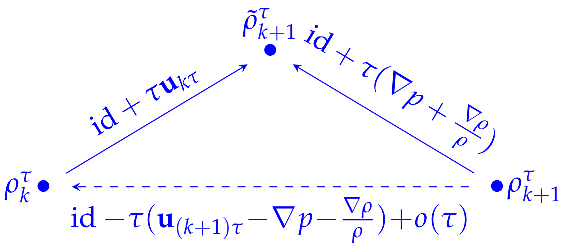

Figure 3: One time step in a hybrid scheme

Notice that $(\mathrm{id} + \tau \mathbf{u}_{k\tau})^{-1}(\mathrm{id} + \tau p) = \mathrm{id} - \tau(\mathbf{u}_{(k+1)\tau} - \nabla p) + o(\tau)$ provided $\mathbf{u}$ is regular enough. Formally we can pass to the limit $\tau \to 0$ and have

$$\partial_t \rho - \Delta \rho + \nabla \cdot (\rho(\mathbf{u} - \nabla p)) = 0.$$

Yet, this turns out to be quite naïve, because we cannot get proper estimates on $W_2(\rho^\tau_k, \rho^\tau_{k+1})$. Indeed, this is mainly due to the hybrid nature of the scheme, i.e. a gradient flow for the diffusion and the projection part on one hand and a free transport on the other hand. The typical estimate in the JKO scheme comes from the fact that one can bound $W_2(\rho^\tau_k, \rho^\tau_{k+1})^2/\tau$ with the opposite of the increment of the energy, and that this gives rise to a telescopic sum. Yet, this is not the case whenever the base point for a new time step is not equal to the previous minimizer. These kinds of difficulties are matter of current study, in particular for mixed systems and/or multiple populations.

2.6 BV-TYPE ESTIMATES FOR THE FOKKER-PLANCK EQUATION

Here we present some Total Variation (TV) decay results (in time) for the solutions of the Fokker-Planck equation. Some are very easy, some trickier. The goal is to look at those estimates which can be easily iterated in time and combined with the decay of the TV via the projection operator, as we did in Section 2.4.

Let us take a vector field $\mathbf{v} : [0, +\infty[\times \Omega \to \mathbb{R}^d$ (we will choose later which regularity we need) and consider in Ω the problem

$$\begin{cases} \partial_t \rho_t - \Delta \rho_t + \nabla \cdot (\rho_t \mathbf{v}_t) = 0, & \text{in }]0, +\infty[\times \Omega, \\ (\nabla(\ln \rho_t) - \mathbf{v}_t) \cdot \mathbf{n} = 0, & \text{on }]0, +\infty[\times \partial\Omega, \\ \rho(0, \cdot) = \rho_0, & \text{in } \Omega, \end{cases} \tag{2.6.1}$$

for $\rho_0 \in BV(\Omega) \cap \mathscr{P}(\Omega)$.

Lemma 2.6.1. *Suppose* $\|\mathbf{v}_t\|_{C^{1,1}} \leq C$ *for a.e.* $t \in [0, +\infty[$. *Suppose that either* $\Omega = \mathbb{T}^d$, *or that* Ω *is convex and* $\mathbf{v} \cdot \mathbf{n} = 0$ *on* $\partial\Omega$. *Then, we have the following total variation decay estimate*

$$\int_\Omega |\nabla \rho_t| \,\mathrm{d}x \leq C(t-s) + e^{C(t-s)} \int_\Omega |\nabla \rho_s| \,\mathrm{d}x, \quad \forall\, 0 \leq s \leq t, \tag{2.6.2}$$

where $C > 0$ *is a constant depending just on the* $C^{1,1}$ *norm of* $\mathbf{v}$.

Proof. First we remark that by the regularity of $\mathbf{v}$ the quantity $\|\mathbf{v}\|_{L^\infty} + \|D\mathbf{v}\|_{L^\infty} + \|\nabla(\nabla \cdot \mathbf{v})\|_{L^\infty}$ is uniformly bounded. Let us drop now the dependence on t in our notation and calculate in coordinates

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega |\nabla \rho| \,\mathrm{d}x &= \int_\Omega \frac{\nabla \rho}{|\nabla \rho|} \cdot \nabla(\partial_t \rho) \,\mathrm{d}x = \int_\Omega \frac{\nabla \rho}{|\nabla \rho|} \cdot \nabla(\Delta \rho - \nabla \cdot (\mathbf{v}\rho)) \,\mathrm{d}x \\
&= \int_\Omega \sum_j \frac{\rho_j}{|\nabla \rho|} \left(\sum_i \rho_{iij} - (\nabla \cdot (\mathbf{v}\rho))_j \right) \mathrm{d}x \\
&= - \int_\Omega \sum_{i,j,k} \left(\frac{\rho_{ij}^2}{|\nabla \rho|} - \frac{\rho_j \rho_k \rho_{ki} \rho_{ij}}{|\nabla \rho|^3} \right) \mathrm{d}x + B_1 \\
&\quad - \int_\Omega \sum_{j,i} \frac{\rho_j}{|\nabla \rho|} \left(\mathbf{v}^i_{ij} \rho + \mathbf{v}^i_i \rho_j + \mathbf{v}^i_j \rho_i + \mathbf{v}^i \rho_{ij} \right) \mathrm{d}x \\
&\leq B_1 + C + C \int_\Omega |\nabla \rho| \,\mathrm{d}x + \int_\Omega |\nabla \rho| |\nabla \cdot \mathbf{v}| \,\mathrm{d}x + B_2 \\
&\leq B_1 + B_2 + C + C \int_\Omega |\nabla \rho| \,\mathrm{d}x.
\end{aligned}$$

Here the B_i are the boundary terms, i.e.

$$B_1 := \int_{\partial\Omega} \sum_{i,j} \frac{\rho_j \mathbf{n}^i \rho_{ij}}{|\nabla \rho|} \,\mathrm{d}\mathscr{H}^{d-1} \quad \text{and} \quad B_2 := - \int_{\partial\Omega} \mathbf{v} \cdot \mathbf{n} |\nabla \rho| \,\mathrm{d}\mathscr{H}^{d-1}.$$

The constant $C > 0$ only depends on $\|\mathbf{v}\|_{L^\infty} + \|\nabla \cdot \mathbf{v}\|_{L^\infty} + \|\nabla(\nabla \cdot \mathbf{v})\|_{L^\infty}$. We used as well the fact that $- \int_\Omega \sum_{i,j,k} \left(\frac{\rho_{ij}^2}{|\nabla \rho|} - \frac{\rho_j \rho_k \rho_{ki} \rho_{ij}}{|\nabla \rho|^3} \right) \mathrm{d}x \leq 0$.

Now, it is clear that in the case of the torus the boundary terms B_1 and B_2 do not exist, hence we conclude by Grönwall's lemma. In the case of the convex domain we have $B_2 = 0$ (because of the assumption $\mathbf{v} \cdot \mathbf{n} = 0$) and $B_1 \leq 0$ because of the next Lemma 2.6.2. □

Lemma 2.6.2. *Suppose that* $\mathbf{u} : \Omega \to \mathbb{R}^d$ *is a smooth vector field with* $\mathbf{u} \cdot \mathbf{n} = 0$ *on* $\partial\Omega$, ρ *is a smooth function with* $\nabla \rho \cdot \mathbf{n} = 0$ *on* $\partial\Omega$, *and that* $\Omega \subset \mathbb{R}^d$ *is a smooth convex set parametrized as* $\Omega = \{h < 0\}$ *for a smooth convex function* h *with* $|\nabla h| = 1$ *on* $\partial\Omega$ *(so that* $\mathbf{n} = \nabla h$ *on* $\partial\Omega$*). Then we have, on the whole boundary* $\partial\Omega$,

$$\sum_{i,j} \mathbf{u}^i_j \rho_j \mathbf{n}^i = - \sum_{i,j} \mathbf{u}^i h_{ij} \rho_j.$$

In particular, we have $\sum_{i,j} \rho_{ij}\rho_j \mathbf{n}^i \leq 0$.

Proof. The Neumann boundary assumption on $\mathbf{u}$ means $\mathbf{u}(\gamma(t)) \cdot \nabla h(\gamma(t)) = 0$ for every curve γ valued in $\partial\Omega$ and for all t. Differentiating in t, we get

$$\sum_{i,j} \mathbf{u}^i_j(\gamma(t))(\gamma'(t))^j h_i(\gamma(t)) + \sum_{i,j} u^i(\gamma(t))h_{ij}(\gamma(t))(\gamma'(t))^j = 0.$$

Take a point $x_0 \in \partial\Omega$ and choose a curve γ with $\gamma(t_0) = x_0$ and $\gamma'(t_0) = \nabla\rho(x_0)$ (which is possible, since this vector is tangent to $\partial\Omega$ by assumption). This gives the first part of the statement. The second part, i.e. $\sum_{i,j} \rho_{ij}\rho_j \mathbf{n}^i \leq 0$, is obtained by taking $\mathbf{u} = \nabla\rho$ and using that $D^2h(x_0)$ is a positive definite matrix. □

Remark 2.6.1. *If we look attentively at the proof of Lemma 2.6.1, we can see that we did not really exploit the regularizing effects of the diffusion term in the equation. This means that the regularity estimate that we provide are the same that we would have without diffusion: in this case, the density ρ_t is obtained from the initial density as the image through the flow of* $\mathbf{v}$*. Thus, the density depends on the determinant of the Jacobian of the flow, hence on the derivatives of* $\mathbf{v}$*. It is normal that, if we want BV bounds on ρ_t, we need assumptions on two derivatives of* $\mathbf{v}$*.*

We would like to prove some form of BV estimates under weaker regularity assumptions on $\mathbf{v}$, trying to exploit the diffusion effects. In particular, we would like to treat the case where $\mathbf{v}$ is only $C^{0,1}$. As we will see in the following lemma, this degenerates in some sense.

Lemma 2.6.3. *Suppose that Ω is either the torus or a smooth convex set $\Omega = \{h < 0\}$ parameterized as a level set of a smooth convex function h. Let $\mathbf{v}_t : \Omega \to \mathbb{R}^d$ be a vector field for $t \in [0,T]$, Lipschitz and bounded in space, uniformly in time. In the case of a convex domain, suppose $\mathbf{v} \cdot \mathbf{n} = 0$ on $\partial\Omega$. Let $H : \mathbb{R}^d \to \mathbb{R}$ be given by $H(z) := \sqrt{\varepsilon^2 + |z|^2}$. Now let ρ_t (sufficiently smooth) be the solution of the Fokker-Planck equation with homogeneous Neumann boundary condition.*

Then there exists a constant $C > 0$ (depending on $\mathbf{v}$ and Ω) such that

$$\int_\Omega H(\nabla\rho_t)\,\mathrm{d}x \leq \int_\Omega H(\nabla\rho_0)\,\mathrm{d}x + C\varepsilon t + \frac{C}{\varepsilon}\int_0^t \|\rho_s\|^2_{L^\infty}\,\mathrm{d}s. \tag{2.6.3}$$

Proof. First let us discuss about some properties of H. It is smooth, its gradient is $\nabla H(z) = \dfrac{z}{H(z)}$ and it satisfies $\nabla H(z) \cdot z \leq H(z)$, $\forall z \in \mathbb{R}^d$. Moreover its Hessian matrix is given by

$$[H_{ij}(z)]_{i,j\in\{1,\dots,d\}} = \left[\frac{\delta^{ij}H^2(z) - z^i z^j}{H^3(z)}\right]_{i,j\in\{1,\dots,d\}} = \frac{1}{H(z)}I_d - \frac{1}{H^3(z)} z\otimes z,\ \forall\, z \in \mathbb{R}^d,$$

where $\delta^{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } j \neq j, \end{cases}$ is the Kronecker symbol. Note that, from this computation, the matrix $D^2 H \geq 0$ is bounded from above by $\frac{1}{H}$, and hence by ε^{-1}. Moreover we introduce a uniform constant $C > 0$ such that $\|\mathbf{v}\|^2_{L^\infty}|\Omega| + \|\nabla \cdot \mathbf{v}\|_{L^\infty} + \|D\mathbf{v}\|_{L^\infty} \leq C$.

Now to show the estimate of this lemma we calculate the quantity

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega H(\nabla\rho_t)\,\mathrm{d}x &= \int_\Omega \nabla H(\nabla\rho_t)\cdot\partial_t\nabla\rho_t\,\mathrm{d}x \\
&= \int_\Omega \nabla H(\nabla\rho_t)\cdot\nabla(\Delta\rho_t - \nabla\cdot(\mathbf{v}_t\rho_t))\,\mathrm{d}x \\
&= \int_\Omega \nabla H(\nabla\rho_t)\cdot\nabla\Delta\rho_t\,\mathrm{d}x - \int_\Omega \nabla H(\nabla\rho_t)\cdot\nabla(\nabla\cdot(\mathbf{v}_t\rho_t))\,\mathrm{d}x \\
&=: (I) + (II)
\end{aligned}$$

Now we study each term separately and for the simplicity we drop the t subscripts in the followings. We start from the case of the torus, where there is no boundary term in the integration by parts.

$$\begin{aligned}
(I) &= \int_\Omega \nabla H(\nabla\rho)\cdot\nabla\Delta\rho\,\mathrm{d}x = \int_\Omega \sum_{j,i} H_j(\nabla\rho)\rho_{jii}\,\mathrm{d}x \\
&= -\int_\Omega \sum_{j,i,k} H_{kj}(\nabla\rho)\rho_{ik}\rho_{ji}\,\mathrm{d}x
\end{aligned}$$

$$\begin{aligned}
(II) &= -\int_\Omega \nabla H(\nabla\rho)\cdot\nabla(\nabla\cdot(\mathbf{v}\rho))\,\mathrm{d}x = -\int_\Omega \sum_{i,j} H_j(\nabla\rho)(\mathbf{v}^i\rho)_{ij}\,\mathrm{d}x \\
&= \int_\Omega \sum_{i,j,k} H_{jk}(\nabla\rho)\rho_{ki}\mathbf{v}^i_j\rho\,\mathrm{d}x + \int_\Omega \sum_{i,j,k} H_{jk}(\nabla\rho)\rho_{ki}\mathbf{v}^i\rho_j\,\mathrm{d}x \\
&=: (II_a) + (II_b).
\end{aligned}$$

First look at the term (II_a). Since the matrix H_{jk} is positive definite, we can apply a Young inequality for each index i and obtain

$$\begin{aligned}
(II_a) = \int_\Omega \sum_{i,j,k} H_{jk}(\nabla\rho)\rho_{ki}\mathbf{v}^i_j\rho\,\mathrm{d}x &\leq \frac{1}{2}\int_\Omega \sum_{i,j,k} H_{jk}(\nabla\rho)\rho_{ki}\rho_{ij}\,\mathrm{d}x \\
&+ \frac{1}{2}\int_\Omega \sum_{i,j,k} H_{jk}(\nabla\rho)\mathbf{v}^i_j\mathbf{v}^i_k\rho^2\,\mathrm{d}x \\
&\leq \frac{1}{2}|(I)| + C\|\rho\|^2_{L^2}\|D^2H\|_{L^\infty}.
\end{aligned}$$

The L^2 norm in the second term will be estimated by the L^∞ norm for the sake of simplicity (see Remark 2.6.2 below).

For the term (II_b) we first make a point-wise computation

$$\begin{aligned}\sum_{i,j,k} H_{jk}(\nabla\rho)\rho_{ki}\mathbf{v}^i\rho_j &= \frac{1}{H^3(\nabla\rho)}\sum_i [D_i^2\rho\cdot(\varepsilon^2 I_d + |\nabla\rho|^2 I_d - \nabla\rho\otimes\nabla\rho)\cdot\nabla\rho]\mathbf{v}^i \\ &= \frac{\varepsilon^2}{H^3(\nabla\rho)}\sum_i \mathbf{v}^i D_i^2\rho\cdot\nabla\rho = -\varepsilon^2\sum_i \mathbf{v}^i\partial_i\left(\frac{1}{H(\nabla\rho)}\right).\end{aligned}$$

where $D_i^2\rho$ denotes the i^{th} row in the Hessian matrix of ρ and we used

$$\left(|\nabla\rho|^2 I_d - \nabla\rho\otimes\nabla\rho\right)\cdot\nabla\rho = 0.$$

Integrating by parts we obtain

$$(II_b) = \varepsilon^2\int_\Omega(\nabla\cdot v)\frac{1}{H(\nabla\rho)}\,\mathrm{d}x \le C\varepsilon^2\|1/H\|_{L^\infty} \le C\varepsilon,$$

where we used $H(z)\ge\varepsilon$.

Summing up all the terms we get and using $\|D^2H\|_{L^\infty}\le\varepsilon^{-1}$ we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega H(\nabla\rho_t)\,\mathrm{d}x \le -\frac{1}{2}|(I)| + C\|\rho_t\|^2_{L^\infty}\|D^2H\|_{L^\infty} + C\varepsilon \le C\varepsilon + C\|\rho_t\|^2_{L^\infty}\varepsilon^{-1},$$

which proves the claim.

If we switch to the case of a smooth bounded convex domain Ω, we have to handle boundary terms. These terms are

$$\int_{\partial\Omega}\sum_{i,j} H_j(\nabla\rho)\rho_{ij}\mathbf{n}^i - \int_{\partial\Omega}\sum_{i,j} H_j(\nabla\rho)\rho\mathbf{v}^i_j\mathbf{n}^i,$$

where we ignored those terms involving $\mathbf{n}^i\mathbf{v}^i$ (i.e., the integration by parts in (II_b), and the term $H_j(\nabla\rho)\rho_j\mathbf{n}^i\mathbf{v}^i$ in the integration by parts of (II_a)), since we already supposed $\mathbf{v}\cdot\mathbf{n}=0$. We use here Lemma 2.6.2, which provides

$$\begin{aligned}\sum_{i,j} H_j(\nabla\rho)\rho_{ij}\mathbf{n}^i - \rho H_j(\nabla\rho)\mathbf{v}^i_j n^i &= \frac{1}{H(\nabla\rho)}\sum_{i,j}\left(\rho_j\rho_{ij}\mathbf{n}^i - \rho\rho_j\mathbf{v}^i_j\mathbf{n}^i\right) \\ &= -\frac{1}{H(\nabla\rho)}\sum_{i,j}\left(\rho_j h_{ij}\rho_i - \rho\rho_j h_{ij}\mathbf{v}^i\right).\end{aligned}$$

If we use the fact that the matrix D^2h is positive definite and a Young inequality, we get $\sum_{i,j}\rho_j h_{ij}\rho_i\ge 0$ and

$$\rho\sum_{i,j}|\rho_j h_{ij}\mathbf{v}^i| \le \frac{1}{2}\sum_{i,j}\rho_j h_{ij}\rho_i + \frac{1}{2}\sum_{i,j}\rho^2\mathbf{v}^j h_{ij}\mathbf{v}^i,$$

which implies

$$\frac{1}{H(\nabla\rho)}\sum_{i,j}\left(\rho_j\rho_{ij}\mathbf{n}^i - \rho\rho_j\mathbf{v}^i_j\mathbf{n}^i\right) \le \frac{\rho^2}{H(\nabla\rho)}\|D^2h\|_{L^\infty}|\mathbf{v}|^2 \le \frac{C\|\rho\|^2_{L^\infty}}{\varepsilon}.$$

This provides the desired estimate on the boundary term. □

Remark 2.6.2. *In the above proof, we needed to use the L^∞ norm of ρ only in the boundary term. When there is no boundary term, the L^2 norm is enough, in order to handle the term (II_a). In both cases, the norm of ρ can be bounded in terms of the initial norm multiplied by e^{Ct}, where C bounds the divergence of $\mathbf{v}$. On the other hand, in the torus case, one only needs to suppose $\rho_0 \in L^2$ and in the convex case $\rho_0 \in L^\infty$. Both assumptions are satisfied in the applications to crowd motion with density constraints.*

We have seen that the constants in the above inequality depend on ε and explode as $\varepsilon \to 0$. This prevents us to obtain a clean estimate on the BV norm in this context, but at least proves that $\rho_0 \in BV \Rightarrow \rho_t \in BV$ for all $t > 0$ (to achieve this result, we just need to take $\varepsilon = 1$). Unfortunately, the quantity which is estimated is not the BV norm, but the integral $\int H(\nabla\rho)\,dx$. This is not enough for the purpose of the applications to Section 2.4, as it is unfortunately not true that the projection operator decreases the value of this other functional [2].

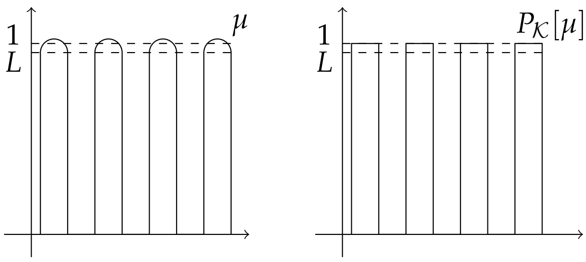

Figure 4: The counter-example to the decay of $\int H(\nabla\rho)\,dx$, which corresponds to the total legth of the graph

If we stay interested to the value of the BV norm, we can provide the following estimate.

Lemma 2.6.4. *Under the assumptions of Lemma 2.6.3, if we suppose $\rho_0 \in BV(\Omega) \cap L^\infty(\Omega)$, then, for $t \le T$, we have*

$$\int_\Omega |\nabla\rho_t|\,dx \le \int_\Omega |\nabla\rho_0|\,dx + C\sqrt{t}, \qquad (2.6.4)$$

where the constant C depends on $\mathbf{v}$, on T and on $\|\rho_0\|_{L^\infty}$.

2. Here is a simple counter-example: consider $\mu = g(x)\,dx$ a BV density on $[0,2] \subset \mathbb{R}$, with g defined as follows. Divide the interval $[0,2]$ into $2K$ intervals J_i of length $2r$ (with $2rK = 1$); call t_i the center of each interval J_i (i.e. $t_i = i2r + r$, for $i = 0, \ldots, 2K-1$) and set $g(x) = L + \sqrt{r^2 - (x-t_i)^2}$ on each J_i with i odd, and $g(x) = 0$ on J_i for i even, taking $L = 1 - \pi r/4$. It is not difficult to check that the projection of μ is equal to the indicator function of the union of all the intervals J_i with i odd, and that the value of $\int H(\nabla\rho)$ has increased by $K(2-\pi/2)r = 1 - \pi/4$, i.e. by a positive constant (see Figure 4).

Proof. Using the L^∞ estimate of Lemma 2.4.2, we will assume that $\|\rho_t\|_{L^\infty}$ is bounded by a constant (which depends on $\mathbf{v}$, on T and on $\|\rho_0\|_{L^\infty}$). Then, we can write

$$\begin{aligned}\int_\Omega |\nabla \rho_t| \, \mathrm{d}x \leq \int_\Omega H(\nabla \rho_t) \, \mathrm{d}x \leq \int_\Omega H(\nabla \rho_0) \, \mathrm{d}x + C\varepsilon t + \frac{Ct}{\varepsilon} \\ \leq \int_\Omega (|\nabla \rho_0| + \varepsilon) \, \mathrm{d}x + C\varepsilon t + \frac{Ct}{\varepsilon}.\end{aligned}$$

It is sufficient to choose, for fixed t, $\varepsilon = \sqrt{t}$, in order to prove the claim. □

Unfortunately, this $\sqrt{t}$ behavior is not suitable to be iterated, and the above estimate is useless for the sake of Section 2.4. The existence of an estimate (for $\mathbf{v}$ Lipschitz) of the form $TV(\rho_t) \leq TV(\rho_0) + Ct$, or $TV(\rho_t) \leq TV(\rho_0)e^{Ct}$, or even $f(TV(\rho_t)) \leq f(TV(\rho_0))e^{Ct}$, for any increasing function $f : R_+ \to \mathbb{R}_+$, seems to be an open question.

3

Uniqueness issues for evolutive equations with density constraints

This chapter contains some basic uniqueness results for evolutive equations under density constraints. First, we develop a rigorous proof of a well-known result in the case where the desired velocity field satisfies a monotonicity assumption. We prove the uniqueness of a solution for first order systems modeling crowd motion with hard congestion effects, introduced recently by *Maury et al.* The monotonicity of the velocity field implies that the 2−Wasserstein distance along two solutions is contractive, which in particular implies the uniqueness. In the case of diffusive models – proposed in Chapter 2 – we prove the uniqueness of a solution passing trough the dual equation, where we use some well-known parabolic estimates. In this case, by the regularization effect of the non-degenerate diffusion, the uniqueness follows even if the given velocity field is only L^∞ (as imposed in Chapter 2 for the existence result).

This chapter is based on a joint work with S. Di Marino (see [DMM15]).

3.1 Introduction

As we have seen in Chapter 2 (and can be observed also in the recent works [MRCS10, MRCS14, MRCSV11, MS15a, AKY14]), a very powerful tool to attack macroscopic hard-congestion problems – where we impose a density constraint on the density of the population – is the theory of optimal transport. In this framework, the density of the agents satisfies a modified continuity- or Fokker-

Planck equation (with a given velocity field taking into account the congestion effects) and can be seen as a curve in the Wasserstein space.

Our aim in this chapter is to prove some basic results for uniqueness in this setting. As far as we are aware of, this question is a missing puzzle in its full generality in the models studied in [MRCS10, MRCS14, MRCSV11, MS15a]. Let us remark that the uniqueness question is indispensable if one wants to include this type of models into a larger system and one aims to show existence results by fixed point methods, as it is done for instance for Mean Field Games in general (for example as in [Por15]).

We will treat two different cases, in which the approaches will be very different: in the first one we consider simply a crowd driven by a velocity field and with the density constraint. In this case it will be crucial the assumption that the velocity field is monotone in order to prove a contraction result along the solutions, that will imply uniqueness. In the second case we add a diffusive term, which is modeling some randomness in the crowd movement (see [MS15a] and Chapter 2 for recent developments and existence results in this setting); in this case we prove uniqueness passing to the dual problem and proving there existence for sufficiently generic data. In this case a major role is played by the regularizing effect of the Laplacian, that allows us to prove uniqueness even if the velocity is merely bounded.

3.1.1 *Admissible velocities and pressures*

In order to model crowd movement in the macroscopic setting with hard congestion, we work in a convex bounded domain $\Omega \subset \mathbb{R}^d$ with Lipschitz boundary such that $|\Omega| > 1$. The evolution of the crowd will be analyzed by the evolution of its density, which is assumed to be a probability measure on Ω. The condition we impose is simply a bound on the density of the crowd to be always less than 1. In particular the set of admissible measures will be denoted by $\mathcal{K}_1 := \{\rho \in \mathcal{P}(\Omega) : \rho \leq 1 \text{ a.e.}\}$.

As for the velocities we should have that the density is not increasing when it is saturated: informally we would say that $\mathbf{v}$ is an admissible velocity for the measure $\rho \in \mathcal{K}_1$ if $\nabla \cdot \mathbf{v} \geq 0$ in the set $\{\rho = 1\}$ and $\mathbf{v} \cdot \mathbf{n} \leq 0$ on $\partial\Omega$ where $\mathbf{n}$ is the outward normal. In order to make a rigorous definition we have to introduce the set of pressures:

$$\text{press}(\rho) := \{p \in H^1(\Omega) \ : \ p \geq 0, \ \ p(1-\rho) = 0 \text{ a.e.}\}.$$

Then, using the integration by parts formula, informally we should have that

$$0 \leq \int_\Omega p \nabla \cdot \mathbf{v} \, \mathrm{d}x - \int_{\partial\Omega} p \mathbf{v} \cdot \mathbf{n} \, \mathrm{d}\mathscr{H}^{d-1} = - \int_\Omega \mathbf{v} \cdot \nabla p \, \mathrm{d}x$$

and so we can define

$$\text{adm}(\rho) := \left\{ \mathbf{v} \in L^2(\Omega; \mathbb{R}^d) \ : \ \int_\Omega \mathbf{v} \cdot \nabla p \, \mathrm{d}x \leq 0 \ \text{ for every } p \in \text{press}(\rho) \right\}.$$

Now, in order to preserve the constraint $\rho \leq 1$, we impose that the velocity is always belonging to $\mathrm{adm}(\rho)$ and so a generic evolution equation with density constraint will be

$$\begin{cases} \partial_t \rho_t + \nabla \cdot (\rho_t \mathbf{v}_t) = 0 \\ \rho_t \leq 1, \, \mathbf{v}_t \in \mathrm{adm}(\rho_t). \end{cases}$$

One of the simplest such model is when we have a prescribed velocity field $\mathbf{u}_t$ and we want to impose in our model that the velocity is the nearest to $\mathbf{v}_t$, time by time. This describes a situation in which the crowd wants to have the velocity $\mathbf{u}_t$ but it cannot because of the density constraint and so it adapts its velocity in such a way to minimize some kind of global difference between $\mathbf{v}_t$ (the real velocity) and $\mathbf{u}_t$ (the desired velocity): this will result in an highly nonlocal and discontinuous effect. So the first order problem reads as

$$\begin{cases} \partial_t \rho_t + \nabla \cdot (\rho_t \mathbf{v}_t) = 0 \\ \rho_t \leq 1, \;\rho_{t=0} = \rho_0 \\ \mathbf{v}_t = P_{\mathrm{adm}(\rho_t)}[\mathbf{u}_t], \end{cases} \tag{3.1.1}$$

where the first equation is meant in the weak sense and the minimal hypothesis in order to have a well defined projection is $\mathbf{u} \in L^2([0,T] \times \mathbb{R}^d)$. Nevertheless in [MRCSV11, MRCS10] and [RC11] the following regularity hypotheses have been assumed to show the existence result: $\mathbf{u} \in C^1$ or $\mathbf{u} = -\nabla D$ for a $\lambda-$convex potential D and in both cases no dependence on time. In the following lemma we characterize the projection:

Lemma 3.1.1. *Let $\rho \in \mathcal{P}(\Omega)$ such that $\rho \leq 1$ a.e. and let $\mathbf{u} \in L^2(\Omega; \mathbb{R}^d)$. Then there exists $p \in \mathrm{press}(\rho)$ such that $P_{\mathrm{adm}(\rho)}[\mathbf{u}] = \mathbf{u} - \nabla p$. Furthermore p is characterized by*

(i) $\int_\Omega \nabla p \cdot (\mathbf{u} - \nabla p) \, \mathrm{d}x = 0;$

(ii) $\int_\Omega \nabla q \cdot (\mathbf{u} - \nabla p) \, \mathrm{d}x \leq 0,$ *for all* $q \in \mathrm{press}(\rho)$*;*

Proof. Let us set $K = \{\nabla p \,:\, p \in \mathrm{press}(\rho)\}$. It is easy to see that K is a closed cone in $L^2(\Omega; \mathbb{R}^d)$. Let us recall that the polar cone to K is defined as

$$K^o := \left\{ \mathbf{v} \in L^2(\Omega; \mathbb{R}^d) \;:\; \int_\Omega \mathbf{v} \cdot \nabla q \, \mathrm{d}x \leq 0, \; \forall \; \nabla q \in K \right\}.$$

By the definition of the admissible velocities we have $\mathrm{adm}(\rho) = K^o$. Moreau decomposition applied to K and K^o says that

$$\mathbf{u} = P_K[\mathbf{u}] + P_{K^o}[\mathbf{u}] \qquad \forall \mathbf{u} \in L^2(\Omega; \mathbb{R}^d).$$

This proves the thesis of (i) and (ii), since these are precisely the conditions of being the projection on a cone. □

Corollary 3.1.2. *Lemma 3.1.1* (i) *implies in particular that*

$$\int_\Omega |\nabla p|^2 \, \mathrm{d}x \leq \int_\Omega |\mathbf{u}|^2 \, \mathrm{d}x.$$

3.1.2 *The diffusive case*

In Chapter 2 (see also [MS15a]) we proposed a second order model for crowd motion. It consists of adding a non-degenerate diffusion to the movement and imposing the density constraint. This leads to a modified Fokker-Planck equation and with the notations introduced previously it reads as

$$\begin{cases} \partial_t \rho_t - \Delta \rho_t + \nabla \cdot (\rho_t \mathbf{v}_t) = 0 \\ \rho_t \leq 1, \ \rho_{t=0} = \rho_0 \\ \mathbf{v}_t = P_{\mathrm{adm}(\rho_t)}[\mathbf{u}_t], \end{cases} \tag{3.1.2}$$

where $\mathbf{u}_t$ is – as before – the desired given velocity field of the crowd. Introducing the pressure gradient in the characterization of the projection, one can write system (3.1.2) as

$$\begin{cases} \partial_t \rho_t - \Delta \rho_t + \nabla \cdot (\rho_t (\mathbf{u}_t - \nabla p_t)) = 0 \\ \rho_t \leq 1, \ \rho_{t=0} = \rho_0 \\ p_t \geq 0, p_t(1-\rho_t) = 0 \text{ a.e.}, \end{cases} \tag{3.1.3}$$

and in both cases the systems are equipped with natural Neumann boundary conditions on $\partial\Omega$.

Under the assumption that $\mathbf{u} \in L^\infty([0,T] \times \Omega)$ is has been shown (see Theorem 2.3.1) that the system (3.1.3) admits a solution $(\rho, p) \in L^\infty([0,T] \times \Omega) \times L^2([0,T]; H^1(\Omega))$. In addition $[0,T] \ni t \mapsto \rho_t$ is a continuous curve in the Wasserstein space $\mathbb{W}_2(\Omega)$.

3.2 MONOTONE VECTOR FIELDS AND THE FIRST ORDER CASE

Let $\Omega \subset \mathbb{R}^d$ be a bounded convex domain with Lipschitz boundary. In this section we suppose that the desired velocity field $\mathbf{u} : [0,T] \times \Omega \to \mathbb{R}^d$ of the crowd is a Borel monotone vector field, i.e. the following assumption is fulfilled:

There exists $\lambda \in \mathbb{R}$ such that for a.e. $x, y \in \Omega$

$$(\mathbf{u}_t(x) - \mathbf{u}_t(y)) \cdot (x - y) \leq \lambda |x-y|^2, \ \ \forall\, t \in [0,T]. \tag{H1}$$

The following results are well-known in community of researchers working with the above mentioned models. A first written version is essentially contained in [Més12] (Section 4.3.1.), nevertheless we simplified and clarified some of the proofs, hence we present them here. A key observation is the following lemma (see also Lemma 4.3.13. in [Més12]):

Lemma 3.2.1. *Let Ω be a convex bounded domain of $\mathbb{R}^d$ and let $\rho^1, \rho^2 \in \mathcal{P}(\Omega)$ two absolutely continuous measures such that $\rho^1 \le 1$ and $\rho^2 \le 1$ a.e. Take a Kantorovich potential φ from ρ^1 to ρ^2 and $p \in H^1(\Omega)$ such that $p \ge 0$ and $p(1-\rho^1) = 0$ a.e.*

Then

$$\int_\Omega \nabla\varphi \cdot \nabla p \,\mathrm{d}x = \int_\Omega \nabla\varphi \cdot \nabla p \,\mathrm{d}\rho^1 \ge 0.$$

To prove this result we consider the following lemma:

Lemma 3.2.2. *Let Ω be a convex bounded domain of $\mathbb{R}^d$ and let $\rho^1, \rho^2 \in \mathcal{P}(\Omega)$ two absolutely continuous measures such that $\rho^1 \le 1$ and $\rho^2 \le 1$ a.e. Take a Kantorovich potential φ from ρ^1 to ρ^2 and $p \in H^1(\Omega)$. Let $[0,1] \ni t \mapsto \rho_t$ be the geodesic connecting ρ^1 to ρ^2, with respect to the 2-Wasserstein distance W_2. Then we have that*

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{\{t=0\}} \int_\Omega p \,\mathrm{d}\rho_t = -\int_\Omega \nabla\varphi \cdot \nabla p \,\mathrm{d}\rho^1.$$

Proof. We know (using the interpolation introduced by R. McCann, see [McC97]) that $\rho_t = (x - t\nabla\varphi(x))_\# \rho_1$ for all $t \in [0,1]$ and so we have

$$\begin{aligned}\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{\{t=0\}} \int_\Omega p \,\mathrm{d}\rho_t &= \lim_{t\to 0} \int_\Omega \frac{p(x - t\nabla\varphi(x)) - p(x)}{t} \,\mathrm{d}\rho^1(x) \\ &= -\lim_{t\to 0} \int_\Omega \frac{1}{t}\int_0^t \nabla p(x - s\nabla\varphi(x)) \cdot \nabla\varphi(x) \,\mathrm{d}s \,\mathrm{d}\rho^1(x) \\ &= -\lim_{t\to 0} \int_\Omega A_t(\nabla p) \cdot \nabla\varphi \,\mathrm{d}\rho^1(x),\end{aligned}$$

where the second equality is easy to prove, for fixed t, by approximation via smooth functions and for $t \in [0,1]$ we denoted by $A_t : L^2(\Omega;\mathbb{R}^d) \to L^2_{\rho^1}(\Omega;\mathbb{R}^d)$ the linear operator

$$A_t(h)(x) = \frac{1}{t}\int_0^t h(x - s\nabla\varphi(x)) \,\mathrm{d}s.$$

Now as a general fact we will prove that $A_t(h) \to h$ strongly in $L^2_{\rho^1}(\Omega;\mathbb{R}^d)$ as $t \to 0$, for every $h \in L^2(\Omega;\mathbb{R}^d)$. First of all it is easy to see that $\|A_t\| \le 1$, in fact

$$\begin{aligned}\int_\Omega |A_t(h)|^2 \,\mathrm{d}\rho_1 &\le \frac{1}{t}\int_\Omega \int_0^t |h(x - s\nabla\varphi(x))|^2 \,\mathrm{d}s \,\mathrm{d}\rho^1(x) \\ &= \frac{1}{t}\int_0^t \int_\Omega |h|^2 \,\mathrm{d}\rho_s(x) \,\mathrm{d}s \le \int_\Omega |h|^2 \,\mathrm{d}x.\end{aligned}$$

Here we used the fact that since $\rho^1, \rho^2 \le 1$ a.e we have also $\rho_t \le 1$ a.e. for all $t \in [0,1]$. Now it is sufficient to note that for every $\varepsilon > 0$ there exists a Lipschitz function h_ε such that $\|h_\varepsilon - h\|_{L^2} \le \varepsilon$, and so we have

$$\begin{aligned}\|A_t(h) - h\|_{L^2_{\rho^1}} &\le \|A_t(h - h_\varepsilon)\|_{L^2_{\rho^1}} + \|h - h_\varepsilon\|_{L^2_{\rho^1}} + \|A_t(h_\varepsilon) - h_\varepsilon\|_{L^2_{\rho^1}} \\ &\le 2\varepsilon + tL\|\nabla\varphi\|_{L^2_{\rho^1}},\end{aligned}$$

where L is the Lipschitz constant of h_ε. Taking now the limit as t goes to 0 we obtain that $\limsup_{t\to 0} \|A_t(h) - h\|_{L^2_{\rho^1}} \leq 2\varepsilon$; by the arbitrariness of $\varepsilon > 0$ we conclude.

Now it is easy to finish the proof, in fact $\nabla p \in L^2(\Omega)$ and so

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{\{t=0\}} \int_\Omega p \,\mathrm{d}\rho_t = -\lim_{t\to 0} \int_\Omega A_t(\nabla p)\cdot \nabla\varphi \,\mathrm{d}\rho^1 = -\int_\Omega \nabla p \cdot \nabla \varphi \,\mathrm{d}\rho^1.$$

□

Proof of Lemma 3.2.1. Let $[0,1] \ni t \mapsto \rho_t$ be the Wasserstein geodesic between ρ^1 and ρ^2. We know that $\rho_t \leq 1$ a.e. for all $t \in [0,1]$ and in particular it is true that

$$\int_\Omega p \,\mathrm{d}\rho_t \leq \int_\Omega p \,\mathrm{d}x = \int_\Omega p \,\mathrm{d}\rho^1,$$

which means that the function $[0,1] \ni t \mapsto \int_\Omega p \,\mathrm{d}\rho_t$ has a local maximum in $t = 0$, hence its derivative in 0 is non-positive.

Given this, the claim follows using Lemma 3.2.2. □

Now we are in position to prove the main theorem of this section, namely:

Theorem 3.2.3. *Suppose $\Omega \subset \mathbb{R}^d$ is a bounded convex domain, $\mathbf{u}$ is a vector field satisfying Assumption (H1) and let $\rho_0 \leq 1$ a.e. be an admissible initial density. Let us suppose that there exist $(\rho^1, p^1), (\rho^2, p^2)$ two solutions to the system*

$$\begin{cases} \partial_t \rho_t + \nabla \cdot (\rho_t(\mathbf{u}_t - \nabla p_t)) = 0 & \text{in }]0,T] \times \Omega \\ \rho_t \leq 1,\ p_t \geq 0,\ (1-\rho_t)p_t = 0 & \text{a.e. in } \Omega,\ \forall t \in [0,T] \\ \rho_{t=0} = \rho_0, \end{cases} \tag{3.2.1}$$

$p^i \in L^2([0,T]; H^1(\Omega))$ for $i \in \{1,2\}$. Then, $\rho^1 = \rho^2$ and $p^1 = p^2$ a.e. In particular, under the same assumption, we can say that there exists a unique couple $(\rho, \mathbf{v})$ that solves (3.1.1).

Proof. We identify the two curves ρ^1_t and ρ^2_t as solutions of the continuity equation in (3.1.1) with the corresponding vector fields $\mathbf{v}^1_t$ and $\mathbf{v}^2_t$ (in particular $\mathbf{v}^i_t \in \mathrm{Tan}_{\rho^i_t}\mathscr{P}(\Omega)$ for all $t \in [0,T]$ and $\mathbf{v}^i_t := \mathbf{u}_t - \nabla p^i_t$). Let us compute and estimate $\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}W_2^2(\rho^1_t, \rho^2_t)$.

We recall (see [AGS08, Theorem 8.4.7.]) that for an absolutely continuous curve μ_t in $\mathcal{P}_2(\Omega)$ with its tangent vector field $\mathbf{w}_t$ and for $\nu \in \mathcal{P}_2(\Omega)$ a fixed measure we have the formula

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}W_2^2(\mu_t, \nu) = \int_{\Omega\times\Omega} (x-y)\cdot \mathbf{w}_t(x)\,\mathrm{d}\gamma, \quad \forall \gamma \in \Pi_o(\mu_t, \nu),$$

for $\mathscr{L}^1-$a.e. $t \in [0,T]$.

Using this formula in our case for $\mu_t = \rho^1_t$ and $\nu = \rho^2_s$, for a fixed s, then changing the roles of the two measures, we obtain (using also the Lemma 4.3.4 from [AGS08]) the following inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}W_2^2(\rho^1_t,\rho^2_t) \le \int_{\Omega\times\Omega}(x-y)\cdot[\mathbf{v}^1_t(x)-\mathbf{v}^2_t(y)]\,\mathrm{d}\gamma, \qquad \forall\gamma\in\Pi_o(\rho^1_t,\rho^2_t),$$

for $\mathscr{L}^1-$a.e. $t\in[0,T]$.

We also know that for absolutely continuous measures (assuming also that the boundary of Ω is negligible) there is an optimal transport map T^t such that $T^t_\#\rho^1_t = \rho^2_t$ for all $t\in[0,T]$, thus $\Pi_o(\rho^1_t,\rho^2_t)$ consists of only the optimal plan associated to T^t, that is, $\Pi_o(\rho^1_t,\rho^2_t) = \{(\mathrm{id},T^t)_\#\rho^1_t\}$. Using this, we can write the above formula in terms of transport maps, instead of plans, so we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}W_2^2(\rho^1_t,\rho^2_t) \le \int_\Omega (x-T^t(x))\cdot\left[\mathbf{v}^1_t(x)-\mathbf{v}^2_t(T^t(x))\right]\,\mathrm{d}\rho^1_t(x).$$

Let us use now the decomposition of the vector fields $\mathbf{v}^1_t$ and $\mathbf{v}^2_t$ with the introduced pressures, i.e. $\mathbf{v}_t = \mathbf{u}_t - \nabla p^1_t$ and $\mathbf{v}^2_t = \mathbf{u}_t - \nabla p^2_t$. Having in mind the monotonicity of $\mathbf{u}_t$, we have

$$\begin{aligned}
\int_\Omega (x-T^t(x))\cdot\left[\mathbf{v}^1_t(x)-\mathbf{v}^2_t(T^t(x))\right]\,\mathrm{d}\rho^1_t(x) = \\
&= \int_\Omega (x-T^t(x))\cdot\left[\mathbf{u}_t(x)-\mathbf{u}_t(T^t(x))\right]\,\mathrm{d}\rho^1_t(x)\\
&- \int_\Omega (x-T^t(x))\cdot\left[\nabla p^1_t(x)-\nabla p^2_t(T^t(x))\right]\,\mathrm{d}\rho^1_t(x)\\
&\le \lambda\int_\Omega |x-T^t(x)|^2\,\mathrm{d}\rho^1_t(x)\\
&- \int_\Omega (x-T^t(x))\cdot\left[\nabla p^1_t(x)-\nabla p^2_t(T^t(x))\right]\,\mathrm{d}\rho^1_t(x)\\
&= \lambda W_2^2(\rho^1_t,\rho^2_t) - \int_\Omega (x-T^t(x))\cdot\left[\nabla p^1_t(x)-\nabla p^2_t(T^t(x))\right]\,\mathrm{d}\rho^1_t(x)
\end{aligned}$$

Let us show that $\displaystyle\int_\Omega (x-T^t(x))\cdot\left[\nabla p^1_t(x)-\nabla p^2_t(T^t(x))\right]\,\mathrm{d}\rho^1_t(x) \ge 0$. We know that $x - T^t(x) = \nabla\varphi(x)$, for any Kantorovich potential φ in the transport of ρ^1_t onto ρ^2_t. Thus, using Lemma 3.2.1 we obtain $\displaystyle\int_\Omega \nabla\varphi(x)\cdot\nabla p^1_t(x)\,\mathrm{d}\rho^1_t(x) \ge 0$. Similarly $\displaystyle-\int_\Omega \nabla\varphi(x)\cdot\nabla p^2_t(T^t(x))\,\mathrm{d}\rho^1_t(x) \ge 0$. Indeed, $S^t := (T^t)^{-1}$ is be the optimal transport map between ρ^2_t and ρ^1_t. Then we have $S(y) = y - \nabla\psi(y)$ for any Kantorovich potential ψ in the transport from ρ^2_t onto ρ^1_t. This readily implies that $\nabla\varphi(S(y)) = -\nabla\psi(y)$ and so using the definition of $S^t_\#\rho^2_t = \rho^1_t$ we obtain

$$\begin{aligned}
-\int_\Omega \nabla\varphi(x)\cdot\nabla p^2_t(T^t(x))\,\mathrm{d}\rho^1_t(x) &= -\int_\Omega \nabla\varphi(S^t(y))\cdot\nabla p^2_t(y)\,\mathrm{d}\rho^2_t(y)\\
&= \int_\Omega \nabla\psi(y)\cdot\nabla p^2_t(y)\,\mathrm{d}\rho^2_t(y).
\end{aligned}$$

We use again Lemma 3.2.1 to show the non-negativity of this term as well. Thus, one obtains

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}W_2^2(\rho_t^1,\rho_t^2) \le \lambda W_2^2(\rho_t^1,\rho_t^2).$$

Grönwall's lemma implies that

$$W_2^2(\rho_t^1,\rho_t^2) \le e^{2\lambda t}W_2^2(\rho_0^1,\rho_0^2).$$

Since $\rho_0^1 = \rho_0^2 = \rho_0$ a.e., the above property implies that $\rho^1 = \rho^2$ a.e. in $[0,T]\times\Omega$. From this fact we can easily deduce that $\Delta(p_t^1 - p_t^2) = 0$, for a.e. $t \in [0,T]$ in the sense of distributions. Moreover, both p_t^1 and p_t^2 vanish a.e. in the set $\{\rho_t^1 = 1\}$ which has a positive Lebesgue measure greater than $|\Omega| - 1 > 0$. Thus, $p^1 = p^2$ a.e. in $[0,T]\times\Omega$. The thesis of the theorem follows. □

Remark 3.2.1. *The existence result for system (3.2.1) was obtained in different settings in the literature. On the one hand, if* $\mathbf{u} = -\nabla D$ *(for a reasonably regular potential D), the existence of a pair* (ρ, p) *can be obtained by gradient flow techniques in* $\mathbb{W}_2(\Omega)$ *(see [MRCS10, RC11]). On the other hand, if* $\mathbf{u}$ *is a general field with* C^1 *regularity, the existence result is proven with the help of a well-chosen splitting algorithm (see [MRCSV11, RC11]).*

Nevertheless, combining the techniques developed in Chapter 2 and [MS15a] on the one hand, and the well-known DiPerna-Lions-Ambrosio theory on the other hand, we expect to obtain existence result for (3.2.1) for more general vector fields with merely Sobolev regularity with one sided bounded divergence. This will be the subject of future research.

Remark 3.2.2. *The monotonicity assumption (H1) it is not surprising in this setting. We remark that the same assumption was required in [NPS11] to prove the contraction properties for a general class of transport costs along the solution of the Fokker-Planck equation in* $\mathbb{R}^d$

$$\partial_t\rho - \Delta\rho + \nabla\cdot(B\rho) = 0, \quad \rho(t=0,\cdot) = \rho_0,$$

where the velocity field $B : \mathbb{R}^d \to \mathbb{R}^d$ *was supposed to satisfy the monotonicity property (H1).*

3.3 THE GENERAL DIFFUSIVE CASE

We use Hilbert space techniques (similarly to the one developed in [Cro79, PQV14]; see also Section 3.1. from [Por15]) to study the uniqueness of a solution of the diffusive crowd motion model with density constraints described in Subsection 3.1.2 (for the details see Chapter 2). Since our objective is only to obtain uniqueness results (not necessarily via a contraction property), we can expect that this holds under more general assumptions in the presence of a non-degenerate diffusion in the model.

Let $\mathbf{u} : [0, T] \times \Omega \to \mathbb{R}^d$ be a given vector field, which represents again the desired velocity field of the crowd, $\Omega \subset \mathbb{R}^d$ a convex, bounded open set with Lipschitz boundary, $\rho_0 \in \mathscr{P}(\Omega)$ the initial density of the population such that $0 \le \rho_0 \le 1$ a.e. in Ω and let us consider the following problem

$$\begin{cases} \partial_t \rho_t - \Delta \rho_t + \nabla \cdot (P_{\mathrm{adm}(\rho_t)}[\mathbf{u}_t]\rho_t) = 0, & \text{in }]0, T] \times \Omega, \\ \rho_{t=0} = \rho_0, \quad 0 \le \rho_t \le 1, & \text{a.e. in } \Omega, \end{cases} \tag{3.3.1}$$

equipped with the natural homogeneous Neumann boundary condition. Equivalently the above system can be written as

$$\begin{cases} \partial_t \rho_t - \Delta \rho_t - \Delta p_t + \nabla \cdot (\mathbf{u}_t \rho_t) = 0, & \text{in }]0, T] \times \Omega, \\ \rho_{t=0} = \rho_0, & \text{in } \Omega \\ (\nabla \rho_t + \nabla p_t - \mathbf{u}_t \rho_t) \cdot \mathbf{n} = 0, & \text{on } \partial\Omega, \text{ for a.e. } t \in [0, T], \end{cases} \tag{3.3.2}$$

for an introduced pressure field $p_t \in \mathrm{press}(\rho_t)$. It has been shown in [MS15a] that under the assumption that $\mathbf{u} \in L^\infty([0, T] \times \Omega; \mathbb{R}^d)$ the systems (3.3.1) and (3.3.2) have a solution. More precisely there exist a continuous curve $[0, T] \ni t \mapsto \rho_t \in \mathbb{W}_2$ and $p_t \in \mathrm{press}(\rho_t)$ for all $t \in [0, T]$ (in particular $\rho \in L^\infty([0, T] \times \Omega)$ and $p \in L^2([0, T]; H^1(\Omega))$) such that (p, ρ) solves (3.3.2) in weak sense (see (3.3.3)).

Our aim in this section is to show that the solution (ρ, p) of (3.3.2) is unique. For a suitable test function $\phi : [0, T] \times \Omega \to \mathbb{R}$ with $\nabla \phi \cdot \mathbf{n} = 0$ a.e. on $[0, T] \times \partial\Omega$ and $\phi(T, \cdot) = 0$ a.e. in Ω, let us write the weak formulation of (3.3.2):

$$\int_0^T \int_\Omega [\rho \partial_t \phi + (\rho + p)\Delta\phi + \rho \mathbf{u} \cdot \nabla \phi] \, \mathrm{d}x \, \mathrm{d}t + \int_\Omega \rho_0(x)\phi(0, x) \, \mathrm{d}x = 0. \tag{3.3.3}$$

Taking $\phi \in C_c^\infty([0, T[\times\overline{\Omega})$, by density arguments the above formulation is meaningful for $\phi \in W^{1,1}([0, T]; L^1(\Omega)) \cap L^2([0, T]; H^2(\Omega))$.

Now let us suppose that Problem 3.3.2 has two solutions (ρ^1, p^1) and (ρ^2, p^2) with $\rho_0^1 = \rho_0^2 = \rho_0$. Writing the weak formulation (3.3.3) for the two pairs of solutions and taking the difference we obtain

$$\int_0^T \int_\Omega \left[(\rho^1 - \rho^2)\partial_t \phi + (\rho^1 - \rho^2 + p^1 - p^2)\Delta\phi + (\rho^1 - \rho^2)\mathbf{u} \cdot \nabla \phi \right] \mathrm{d}x \, \mathrm{d}t = 0. \tag{3.3.4}$$

We introduce the following notations

$$A := \frac{\rho^1 - \rho^2}{(\rho^1 - \rho^2) + (p^1 - p^2)} \quad \text{and} \quad B := \frac{p^1 - p^2}{(\rho^1 - \rho^2) + (p^1 - p^2)}.$$

Note that $0 \le A \le 1$ and $0 \le B \le 1$ a.e. in $[0, T] \times \Omega$ and $A + B = 1$. To be consistent with these bounds, we set $A = 0$ when $\rho^1 = \rho^2$, even if $p^1 = p^2$

and $B = 0$ when $p^1 = p^2$, even if $\rho^1 = \rho^2$. With these notations the weak formulation for the difference gives us

$$\int_0^T \int_\Omega ((\rho^1 - \rho^2) + (p^1 - p^2)) \left[A\partial_t\phi + (A+B)\Delta\phi + A\mathbf{u}\cdot\nabla\phi\right] \mathrm{d}x\,\mathrm{d}t = 0. \quad (3.3.5)$$

For a smooth function $G : [0,T]\times\Omega \to \mathbb{R}$ let us consider the dual problem

$$\begin{cases} A\partial_t\phi + (A+B)\Delta\phi + A\mathbf{u}\cdot\nabla\phi = AG, & \text{in } [0,T[\times\Omega, \\ \nabla\phi\cdot\mathbf{n} = 0 \text{ on } [0,T]\times\partial\Omega, & \phi(T,\cdot) = 0 \text{ a.e. in } \Omega. \end{cases} \quad (3.3.6)$$

Let us remark that if we are able to find a (reasonably regular) solution ϕ for this problem for any G smooth, we would get uniqueness of $\rho + p$, hence of ρ and p separately. Since the coefficients in (3.3.6) are not regular, we study a regularized problem. For $\varepsilon > 0$ let us consider A_ε and B_ε to be continuous approximations of A and B such that

$$\|A - A_\varepsilon\|_{L^r([0,T]\times\Omega)} < C(\Omega,r)\varepsilon, \;\; \varepsilon < A_\varepsilon \le 1$$

and

$$\|B - B_\varepsilon\|_{L^r([0,T]\times\Omega)} < C(\Omega,r)\varepsilon, \;\; \varepsilon < B_\varepsilon \le 1,$$

for $1 \le r < +\infty$, the value of which to be chosen later. Here $C(\Omega,r) > 0$ is a constant depending only on Ω and r. The regularized problem reads as follows

$$\begin{cases} \partial_t\phi_\varepsilon + (1 + B_\varepsilon/A_\varepsilon)\Delta\phi_\varepsilon + \mathbf{u}\cdot\nabla\phi_\varepsilon = G, & \text{in } [0,T[\times\Omega, \\ \nabla\phi_\varepsilon\cdot\mathbf{n} = 0 \text{ a.e. on } [0,T]\times\partial\Omega, & \phi_\varepsilon(T,\cdot) = 0 \text{ a.e. in } \Omega. \end{cases} \quad (3.3.7)$$

For all $\varepsilon > 0$ the above problem is uniformly parabolic and $B_\varepsilon/A_\varepsilon$ is continuous. Moreover G is smooth and $\mathbf{u} \in L^\infty([0,T]\times\Omega)$, thus by classical results (see for instance [LSU68, Kry08]) the problem has a (unique) solution $\phi_\varepsilon \in H^1([0,T];L^2(\Omega)) \cap L^2([0,T];H^2(\Omega))$. In particular ϕ_ε can be used as test function in (3.3.3). Now let us establish some standard uniform (in ε) estimates on ϕ_ε.

Lemma 3.3.1. *Let ϕ_ε be a solution of* (3.3.7). *Then there exists a constant*

$$C = C(T, \|\mathbf{u}\|_{L^\infty}, \|\nabla G\|_{L^2([0,T]\times\Omega)}) > 0$$

such that we have the following estimates, uniformly in $\varepsilon > 0$:

(i) $\sup_{t\in[0,T]} \|\nabla\phi_\varepsilon(t)\|_{L^2(\Omega)} \le C$;

(ii) $\|(B_\varepsilon/A_\varepsilon)^{\frac{1}{2}}\Delta\phi_\varepsilon\|_{L^2([0,T]\times\Omega)} \le C$;

(iii) $\|\Delta\phi_\varepsilon\|_{L^2([0,T]\times\Omega)} \le C$.

Proof. Let us multiply the first equation from (3.3.7) by $\Delta\phi_\varepsilon$ and integrate over $[t,T]\times\Omega$ for $0\le t<T$. We obtain

$$\frac{1}{2}\|\nabla\phi_\varepsilon(t)\|^2_{L^2(\Omega)}+\int_t^T\int_\Omega(1+B_\varepsilon/A_\varepsilon)|\Delta\phi_\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}t$$
$$=-\int_t^T\int_\Omega\nabla\phi_\varepsilon\cdot\nabla G\,\mathrm{d}x\,\mathrm{d}t-\int_t^T\int_\Omega\mathbf{u}\cdot\nabla\phi_\varepsilon\Delta\phi_\varepsilon\,\mathrm{d}x\,\mathrm{d}t \tag{3.3.8}$$

Hence by Young's inequality we have

$$\frac{1}{2}\|\nabla\phi_\varepsilon(t)\|^2_{L^2(\Omega)}\le\left(\frac{\|\mathbf{u}\|_{L^\infty}}{2\delta}+\frac{1}{2}\right)\int_t^T\|\nabla\phi_\varepsilon(s)\|^2_{L^2(\Omega)}\,\mathrm{d}s+\frac{1}{2}\|\nabla G\|^2_{L^2([0,T]\times\Omega)}$$
$$\le C+\frac{C}{2}\int_t^T\|\nabla\phi_\varepsilon(s)\|^2_{L^2(\Omega)}\,\mathrm{d}s$$

where $0<\delta\le 2/\|\mathbf{u}\|_{L^\infty}$ is a fixed constant and the constant $C>0$ is depending just on $\|\nabla G\|_{L^2([0,T]\times\Omega)}$ and $\|\mathbf{u}\|_{L^\infty}$. Hence by Grönwall's inequality we obtain

$$\frac{1}{2}\|\nabla\phi_\varepsilon(t)\|^2_{L^2(\Omega)}\le Ce^{C(T-t)},$$

which implies in particular that $\sup_{t\in[0,T]}\|\nabla\phi_\varepsilon(t)\|_{L^2(\Omega)}\le C$. Thus (i) follows.

On the other hand choosing $\delta:=2/\|u\|_{L^\infty}$ in Young's inequality used in (3.3.8) and using (i), we obtain

$$\int_t^T\int_\Omega(B_\varepsilon/A_\varepsilon)|\Delta\phi_\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}t\le C$$

hence $\|(B_\varepsilon/A_\varepsilon)^{\frac{1}{2}}\Delta\phi_\varepsilon\|_{L^2([0,T]\times\Omega)}\le C$, and thus (ii) follows.

By (3.3.8), (i) and (ii) easily imply (iii).

□

In particular, using ϕ_ε as test function in (3.3.5) one has

$$\int_0^T\int_\Omega(\rho^1-\rho^2)G\,\mathrm{d}x\,\mathrm{d}t=\int_0^T\int_\Omega(\rho^1-\rho^2+p^1-p^2)AG\,\mathrm{d}x\,\mathrm{d}t$$
$$=\int_0^T\int_\Omega(\rho^1-\rho^2+p^1-p^2)A\left[\partial_t\phi_\varepsilon+(1+B_\varepsilon/A_\varepsilon)\Delta\phi_\varepsilon+\mathbf{u}\cdot\nabla\phi_\varepsilon\right]\,\mathrm{d}x\,\mathrm{d}t$$
$$=\int_0^T\int_\Omega(\rho^1-\rho^2+p^1-p^2)A\left[\partial_t\phi_\varepsilon+(1+B_\varepsilon/A_\varepsilon)\Delta\phi_\varepsilon+\mathbf{u}\cdot\nabla\phi_\varepsilon\right]\,\mathrm{d}x\,\mathrm{d}t$$
$$-\int_0^T\int_\Omega(\rho^1-\rho^2+p^1-p^2)\left[A\partial_t\phi_\varepsilon+(A+B)\Delta\phi_\varepsilon+A\mathbf{u}\cdot\nabla\phi_\varepsilon\right]\,\mathrm{d}x\,\mathrm{d}t$$
$$=\int_0^T\int_\Omega(\rho^1-\rho^2+p^1-p^2)(B_\varepsilon/A_\varepsilon)(A-A_\varepsilon)\Delta\phi_\varepsilon\,\mathrm{d}x\,\mathrm{d}t$$
$$+\int_0^T\int_\Omega(\rho^1-\rho^2+p^1-p^2)(B_\varepsilon-B)\Delta\phi_\varepsilon\,\mathrm{d}x\,\mathrm{d}t$$
$$:=I^1_\varepsilon+I^2_\varepsilon$$

Now we shall prove that $|I^1_\varepsilon| \to 0$ and $|I^1_\varepsilon| \to 0$ as $\varepsilon \to 0$, which will lead to the uniqueness of ρ. First, let us recall that $0 \le \rho^1, \rho^2 \le 1$ a.e. in $[0,T] \times \Omega$, hence $\rho^1, \rho^2 \in L^\infty([0,T] \times \Omega)$. On the other hand $p^1, p^2 \in L^2([0,T]; H^1(\Omega))$ and by Corollary 3.1.2 we have that

$$\int_\Omega |\nabla p^i_t|^2 \, \mathrm{d}x \le \int_\Omega |\mathbf{u}_t|^2 \, \mathrm{d}t,$$

for almost every $t \in [0,T]$. This implies that (since $\mathbf{u}$ is bounded)

$$\text{ess} - \sup_{t\in[0,T]} \|\nabla p^i_t\|_{L^2(\Omega)} \le C.$$

In addition, p^i's being pressures one has $|\{p^i_t = 0\}| \ge |\{\rho^i_t < 1\}| \ge |\Omega| - 1 > 0$ for a.e. $t \in [0,T]$, and so by a version of Poincaré's inequality one obtains that $p^i \in L^\infty([0,T]; H^1(\Omega))$. By the Sobolev embedding theorem this means that $p^i \in L^\infty([0,T]; L^{2^*}(\Omega))$, $i \in \{1,2\}$, where $2^* = 2d/(d-2)$. This reasoning implies the following estimates

$$\begin{aligned}
|I^1_\varepsilon| &\le \|\rho^1 - \rho^2\|_{L^\infty([0,T]\times\Omega)} \cdot \|(B_\varepsilon/A_\varepsilon)^{1/2}(A - A_\varepsilon)\|_{L^2([0,T]\times\Omega)} \\
&\quad \times \|(B_\varepsilon/A_\varepsilon)^{1/2} \Delta\phi_\varepsilon\|_{L^2([0,T]\times\Omega)} \\
&\quad + \int_0^T \Big\{ \|p^1_t - p^2_t\|_{L^{2^*}(\Omega)} \cdot \|(B_\varepsilon/A_\varepsilon)^{1/2}(A - A_\varepsilon)\|_{L^r(\Omega)} \\
&\quad \times \|(B_\varepsilon/A_\varepsilon)^{1/2} \Delta\phi_\varepsilon\|_{L^2(\Omega)} \Big\} \, \mathrm{d}t \\
&\le C(1/\varepsilon)^{1/2} \varepsilon \\
&\quad + \|p^1 - p^2\|_{L^\infty(L^{2^*})} \cdot \|(B_\varepsilon/A_\varepsilon)^{1/2}(A - A_\varepsilon)\|_{L^2(L^r)} \cdot \|(B_\varepsilon/A_\varepsilon)^{1/2} \Delta\phi_\varepsilon\|_{L^2(L^2)} \\
&\le C\varepsilon^{1/2} \to 0, \quad \text{as } \varepsilon \to 0
\end{aligned}$$

and similarly

$$\begin{aligned}
|I^2_\varepsilon| &\le \|\rho^1 - \rho^2\|_{L^\infty([0,T]\times\Omega)} \cdot \|(A_\varepsilon/B_\varepsilon)^{1/2}(B - B_\varepsilon)\|_{L^2([0,T]\times\Omega)} \\
&\quad \times \|(B_\varepsilon/A_\varepsilon)^{1/2} \Delta\phi_\varepsilon\|_{L^2([0,T]\times\Omega)} \\
&\quad + \|p^1 - p^2\|_{L^\infty(L^{2^*})} \cdot \|(A_\varepsilon/B_\varepsilon)^{1/2}(B - B_\varepsilon)\|_{L^2(L^r)} \cdot \|(B_\varepsilon/A_\varepsilon)^{1/2} \Delta\phi_\varepsilon\|_{L^2(L^2)} \\
&\le C(1/\varepsilon)^{1/2} \varepsilon = C\varepsilon^{1/2} \to 0, \quad \text{as } \varepsilon \to 0,
\end{aligned}$$

where $r > 1$ is an exponent such that $\dfrac{1}{2} + \dfrac{1}{r} + \dfrac{1}{2^*} = 1$, i.e. $r = d$. Hence we obtained that

$$\int_0^T \int_\Omega (\rho^1 - \rho^2) G \, \mathrm{d}x \, \mathrm{d}t = 0, \quad \forall \, G \text{ smooth},$$

which in particular implies that $\rho^1 = \rho^2$ a.e. in $[0,T] \times \Omega$. This implies moreover by (3.3.4) that

$$\int_0^T \int_\Omega (p^1 - p^2) \Delta\phi \, \mathrm{d}x \, \mathrm{d}t = 0, \quad \forall \, \phi \text{ test function},$$

thus $\Delta(p^1_t - p^2_t) = 0$ in the sense of distributions, for a.e. $t \in [0, T]$. We conclude similarly as in the end of the proof of Theorem 3.2.3, hence one obtains that $p^1 = p^2$ a.e. in $[0, T] \times \Omega$. The result follows.

Part II

MEAN FIELD GAMES WITH DENSITY CONSTRAINTS

On the history of Mean Field Games

Mean Field Games (shortly MFG in the sequel) have been introduced in the mid 2000's and have their roots in the seminal work of J.-M. Lasry and P.-L. Lions (see [LL06a, LL06b, LL07]) and in the lectures delivered by P.-L. Lions at Collège de France the years after (see [Lio08]). In the same period M. Huang, R. P. Malhamé and P. E. Caines introduced similar models (see [HMC06]).

The main motivation of J.-M. Lasry and P.-L. Lions was to study the limit behavior of Nash equilibria for symmetric differential games with a very large number of identical "small" *players* (or *agents*). They managed this task in a very powerful and elegant way. Borrowing some tools from statistical physics, the main idea behind this theory is to see these models as continuum limit when the number of the agents tends to infinity. This is exactly what is happening in the derivation of Vlasov or Boltzmann equations for instance, when one wants to obtain unified models for large systems of interacting particles. This procedure is called *mean field limit* and here is where the name of Mean Field Games is coming from. The notion of "small" player refers to the fact that the contribution of a single individual to the entire model is negligible.

From the mathematical point of view, the MFG models are described with the help of some optimization problems, where the *density* of the agents enters as a parameter. More precisely, a typical agent is considering the following optimization problem:

$$u(t,x_0) = \inf_{\alpha} \mathbb{E}\left\{ \int_t^T L(\gamma(s),\alpha(s)) + f(\gamma(s), m(s,\gamma(s)))\,\mathrm{d}s + \Phi(\gamma(T), m(T,\cdot)) \right\}, \tag{3.3.9}$$

subject to

$$\begin{cases} \mathrm{d}\gamma(s) = \alpha(s)\,\mathrm{d}t + \sqrt{2\nu}\,\mathrm{d}B_s, & s \in]t,T], \\ \gamma(t) = x_0 \end{cases} \tag{3.3.10}$$

where $L : \mathbb{R}^d \times \mathbb{R}^d \to \overline{\mathbb{R}}$, $f : \mathbb{R}^d \times \mathscr{P}(\mathbb{R}^d) \to \overline{\mathbb{R}}$ and $\Phi : \mathbb{R}^d \times \mathscr{P}(\mathbb{R}^d) \to \overline{\mathbb{R}}$ are the datas of the problem representing the Lagrangian, the running cost and the final cost respectively. We emphasize the fact that the above optimal control problem is depending on the density $m \in \mathscr{P}(\mathbb{R}^d)$ of the agents. B_s denotes the $d-$dimensional Brownian motion. We call u the *value function* of the typical agent. Classical results from stochastic control theory imply that (at least heuristically) u solves a Hamilton-Jacobi-Bellman equation. Moreover, the optimal control $\alpha^* := -\nabla_p H(\cdot, \nabla u)$ is given in feedback form, where H is the Legendre-Fenchel transform of the Lagrangian L w.r.t. the second variable. Finally, Itō's lemma (and the Feynman-Kac formula) implies that the evolution of the density of the agents, $m(t,\cdot) := \mathrm{Law}(\gamma(t))$, is given by a Fokker-Plack (or Kolgomorov) equation with the optimal control α^* as velocity field. Thus, knowing the initial density $m_0 \in \mathscr{P}(\mathbb{R}^d)$ of the agents, the MFG system corresponding to the above model formally is given by the following coupled PDE system

$$\begin{cases} -\partial_t u(t,x) - \nu\Delta u(t,x) + H(x,\nabla u(t,x)) = f(x, m(t,\cdot)) & \text{in } (0,T] \times \mathbb{R}^d \\ \partial_t m(t,x) - \nu\Delta m(t,x) - \nabla \cdot \left(\nabla_p H(x,\nabla u(t,x)) m(t,x)\right) = 0 & \text{in } (0,T] \times \mathbb{R}^d, \\ m(0,x) = m_0, \quad u(T,x) = \Phi(x, m(T,\cdot)) & \text{in } \mathbb{R}^d. \end{cases} \tag{MFG}$$

If one chooses $\nu = 0$, the SODE in (3.3.10) becomes a simple ODE, while (MFG) turns into a first order system, where the Fokker-Planck equation is replaced by a continuity equation.

Let us remark that the solution of (MFG) produces a *Nash equilibrium*, in the sense that no agent will change his/her mind. Indeed, solving the control problem (3.3.10), the typical player "predicts" the evolution of the whole density m, and he/she uses this to solve his/her control problem (3.3.9)-(3.3.10). Using the optimal control $\alpha^* := -\nabla_p H(\cdot, \nabla u)$ one can obtain the "true" evolution of the density m. If the prediction was correct, it should correspond to the "true" density and in this case m is a Nash equilibrium. As good introductory bibliography to the subject we refer to [Car13a, GS14b].

POSSIBLE APPROACHES TO STUDY THE SYSTEM (MFG)

The well-posedness of the system (MFG) is not trivial. First, it is not clear in which sense the control problem (3.3.9)-(3.3.10) has to be considered: a priori m is a probability measure, hence integrating it along trajectories does not have any sense. Secondly, one could also attack directly the system (MFG) and show the well-posedness by PDE techniques. It this case one has to define a good notion of (weak) solutions, because we cannot expect always to have classical ones. Let us discuss now some possibilities to handle the question of existence and uniqueness.

Case 1 - Regularizing operators. Already in the seminal works of J.-M. Lasry and P.-L. Lions ([LL06a, LL06b, LL07]) it has been considered the case when

f and Φ are some regularizing operators in the m variable (for instance convolutions with smooth kernels). In this case the optimization problem (3.3.9)-(3.3.10) perfectly makes sense and it is possible to show the existence of a classical solution by a standard Schauder-type fixed point scheme. This method is well-suited for both first and second order systems.

Case 2 - Further classical solutions by PDE techniques. For local couplings (typically $f(x, m) = m^a$) D. Gomes and his teams studied the existence of classical solutions in various settings in a series of papers (see for instance [GPSM15, GPSM12, GM14, GPV14] and the references therein). The used techniques combine variational arguments and sharp PDE estimates. In these methods there is always an interplay (which has to be well-chosen) between the growth $a > 0$ of the coupling f and the growth of the Hamiltonian H in the gradient variable. We remark that these methods are relying on the parabolic structure of the PDE system (MFG), hence they can be used only in the second order case. Some connections between stationary MFG systems and the co-called Evans-Aronsson problem have been also recently studied in [GSM14].

Case 3 - Week solutions. In a recent paper (see [Por15]) A. Porretta showed the existence of weak solutions of second order systems like (MFG), with local couplings f and Φ having general order of growth in the m variable and for Hamiltonians with order of growth $1 < q \leq 2$ in the gradient variable. A key tool in his approach is the (new) characterization of the Fokker-Planck equation through weak renormalized solutions. Recently P.J. Graber used similar techniques to study (MFG)-type systems with soft congestion effects (see [Gra15]).

Case 4 - Further weak solutions through variational techniques. In the introductory papers [LL06a, LL06b, LL07] J.-M. Lasry and P.-L. Lions mentioned the fact that certain systems like (MFG) can be seen as optimality conditions for some optimal control problems with PDE constraints. This fact recalls also the dynamical formulation of the Monge-Kantorovich optimal transport problems introduced by J.-D. Benamou and Y. Brenier (see [BB00] and the (BB) formula discussed previously). This method is quite robust in the sense that it can be applied both for first and second order systems with local couplings f and Φ and Hamiltonians having a general class of order of growth. The rigorous analysis of different systems in this setting has been carried out recently mainly by P. Cardaliaguet and his collaborators in a series of papers (see [Car13b, CG15, Gra14, CGPT14]). The convex duality used as a main tool in this approach permits to understand the deeper phenomena behind the structure of system (MFG). In this sense we can see that the two equations in (MFG) are structurally well connected as well, since one is the dual of the other.

Our analysis in Chapter 4 and Chapter 5 will also rely on this approach, where we shall give more details on the formulation of the optimization problems à la Benamou-Brenier in our framework.

Monotonicity implies uniqueness. Assuming that the couplings f and Φ are monotone in the m variable (and the Hamiltonian is strictly convex in the gradient variable), J.-M. Lasry and P.-L. Lions showed the uniqueness of the solutions of the system (MFG). These monotonicity assumptions are

$$\int_{\mathbb{R}^d} [f(x, m_1(x)) - f(x, m_2(x))] \, \mathrm{d}(m_1(x) - m_2(x)) > 0 \;\; \forall \, m_1, m_2 \in \mathscr{P}_1(\mathbb{R}^d)$$

and

$$\int_{\mathbb{R}^d} [\Phi(x, m_1(x)) - \Phi(x, m_2(x))] \, \mathrm{d}(m_1(x) - m_2(x)) \geq 0 \;\; \forall \, m_1, m_2 \in \mathscr{P}_1(\mathbb{R}^d).$$

This technique can be used for first order models as well. Essentially this type of monotonicity property implies uniqueness in most of the previous cases.

THE MASTER EQUATION

Later in his course at Collège de France (see [Lio08]), P.-L. Lions noticed that the flow of measures solving the Fokker-Planck equation (from the system (MFG)) could be seen as a characteristic trajectory of a Hamilton-Jacobi equation lifted up to a state space that contains both the position of a typical agent and the distribution of the population, i.e. the space of probability measures. The solution of this non-linear PDE defined on an infinite dimensional space contains all the necessary information to describe entirely the Nash equilibria in the game, hence he called it the *master equation*. In a simple form this problem formally reads as

$$\begin{cases} -\partial_t U(t,x,\mu) + \frac{1}{2}|D_x U(t,x,\mu)|^2 + \langle D_\mu U, D_x U \rangle_{L^2_\mu} = f, & \text{in } [0,T[\times \mathbb{R}^d \times \mathscr{P}_2(\mathbb{R}^d), \\ U(T,x,\mu) = \Phi(x,\mu), \end{cases}$$

where $D_\mu U$ stands for the Wasserstein gradient of U. Defining a characteristic trajectory $[0,T] \ni t \mapsto m_t \in \mathscr{P}_2(\mathbb{R}^d)$ (in a suitable way), the pair $(u(t,x) := U(t,x,m_t), m_t)$ solves a classical first order quadratic system, similar to (MFG): the HJ equation in is satisfied in viscosity sense and the FP equation holds in the sense of distributions.

After the formal description of the above problem (see also the lecture notes of P. Cardaliaguet, [Car13a]), recently there have been carried out some deep works on its rigorous analysis (see for example the work of W. Gangbo and A. Święch [GŚ14a] on short time existence for first order models and the work of J.-F. Chassagneux, D. Crisan, F. Delarue, [CCD14] on the analysis second order models). For further reading on the master equation and connections with *mean field type control* problems we refer to the recent works of A. Bensoussan, J. Frehse and Ph. Yam (see [BFY15b, BFY15a, BFY13]).

LONG TIME AVERAGE OF MFGS

The so-called *stationary* or *ergodic* MFG models play an important role in the analysis developed in Chapter 4. These models were the first ones introduced by J.-M. Lasry and P.-L. Lions in [LL06a], where it has been also shown the convergence (as N tends to infinity) of Nash equilibria for N player stochastic ergodic differential games. A typical stationary MFG system can be written in the following form:

$$\begin{cases} -\nu\Delta u(x) + H(x, \nabla u(x)) - \lambda = f(x, m(x)) & \text{in } \mathbb{R}^d \\ -\nu\Delta m(x) - \nabla \cdot \big(\nabla_p H(x, \nabla u(x)) m(x)\big) = 0 & \text{in } \mathbb{R}^d, \\ \int_{\mathbb{R}^d} m(x)\,\mathrm{d}x = 1, \quad \int_{\mathbb{R}^d} u(x)\,\mathrm{d}x = 0, \quad m \geq 0. \end{cases} \tag{MFG_{stat}}$$

Later it has been understood (see [CLLP13, CLLP13]) that system (MFG_{stat}) corresponds to the "ergodic limit" of system (MFG). Indeed, with a well-chosen averaging procedure, one can show that the solutions of (MFG) converge (in a suitable sense as $T \to \infty$) to some solutions of (MFG_{stat}).

We remark also the fact that system (MFG_{stat}) (as its time-dependent version) possesses a variational formulation à la Benamou-Brenier which will be detailed in the next chapter.

4

Second order MFG with density constraints: the stationary case

In this chapter we study second order stationary Mean Field Game systems under density constraints on a bounded domain $\Omega \subset \mathbb{R}^d$. We show the existence of weak solutions for power-like Hamiltonians with arbitrary order of growth. Our strategy is a variational one, i.e. we obtain the Mean Field Game system as the optimality condition of a convex optimization problem, which has a solution. When the Hamiltonian has a growth of order $q' \in]1, d/(d-1)[$, the solution of the optimization problem is continuous which implies that the problem constraints are qualified. Using this fact and the computation of the subdifferential of the convex functional (BB) introduced by Benamou-Brenier (see [BB00]), we prove the existence of a solution of the MFG system. In the case where the Hamiltonian has a growth of order $q' \geq d/(d-1)$, the previous arguments do not apply and we prove the existence by means of an approximation argument.

This chapter is based on a joint work with F. Silva which has recently been accepted for publication to *J. Math. Pures Appl.* (see [MS15b]).

4.1 Introduction

Let us recall the (variational) formulation of some basic MFG models after which we shall motivate our results. As we discussed previously – in its simplest form – for symmetric differential games as the number of players tends to infinity, limits of Nash equilibria can be characterized in terms of the solution of the following coupled PDE system:

$$\begin{cases} -\partial_t u(t,x) - \nu\Delta u(t,x) + H(x, \nabla u(t,x)) = f[m(t)](x) & \text{in } [0,T[\times \mathbb{R}^d \\ \partial_t m(t,x) - \nu\Delta m(t,x) - \nabla\cdot(\nabla_p H(x,\nabla u(t,x))m(t,x)) = 0 & \text{in }]0,T]\times\mathbb{R}^d, \\ m(0,x) = m_0, \quad u(T,x) = g(x) & \text{in } \mathbb{R}^d. \end{cases} \tag{MFG}$$

where $H(x,\cdot)$ is convex. The Hamilton-Jacobi-Bellman (HJB) equation in (MFG) characterizes the value function $u[m]$ associated to a stochastic optimal control problem solved by a typical player whose cost function depends at each time t on the distribution $m(t,\cdot)$ of the other agents. We remark that this interaction can be global, e.g. if $f[m(t,\cdot)](x)$ is a convolution of $m(t,\cdot)$ with another function, or local, i.e. when $f[m(t)](x)$ can be identified to a function $f(x, m(t,x))$. The Fokker-Planck equation (FP) in (MFG) describes the evolution $m[u]$ of the initial distribution m_0 when all the agents follow the optimal feedback strategy computed by the typical agent.

For local couplings $f(\cdot, m)$, system (MFG) can be obtained (at least formally) as the optimality condition of problem

$$\begin{aligned} &\min \int_0^T \int_{\mathbb{R}^d} \left\{ m(t,x) L\left(x, \frac{\mathbf{w}(t,x)}{m(t,x)}\right) + F(x, m(t,x)) \right\} \mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}^d} g(x) m(T,x)\,\mathrm{d}x, \\ &\text{ s.t. } \partial_t m - \nu\Delta m + \nabla\cdot(\mathbf{w}) = 0, \quad m(0,x) = m_0, \end{aligned} \tag{4.1.1}$$

with $F(x,m) := \displaystyle\int_0^m f(x,m')\,\mathrm{d}m'$, $L(x,v) := H^*(x,-v)$ (where the Fenchel conjugate $H^*(x,v)$ is calculated on the second variable of H) and $m_0 \in L^\infty(\mathbb{R}^d)$ satisfying that $m_0 \geq 0$ and $\displaystyle\int_{\mathbb{R}^d} m_0\,\mathrm{d}x = 1$. This type of approach, including also the degenerate first order case ($\nu = 0$), has been studied extensively in the last years in a series of papers [Car13b, Gra14, CG15, CGPT14]. The optimization problem above recalls the so-called Benamou-Brenier formulation of the 2-Wasserstein distance between two probability measures, which gives a fluid mechanical or dynamical interpretation of the Monge-Kantorovich optimal transportation problem (see [BB00, CCN13] and also the formula (BB) presented before). We refer the reader to [ACCD12], [LST10] and the recent work [BC15] for some optimization methods to solve numerically (MFG) based on the formulation (4.1.1).

With a well-chosen time-averaging procedure, one can introduce stationary MFG systems as an ergodic limit of time dependent ones (see [CLLP13, CLLP12]),

$$\begin{cases} -\nu\Delta u(x) + H(x, \nabla u(x)) - \lambda &= f(x, m(x)) \quad \text{in } \mathbb{R}^d \\ -\nu\Delta m(x) - \nabla\cdot\left(\nabla_p H(x, \nabla u(x)) m(x)\right) &= 0 \qquad\qquad \text{in } \mathbb{R}^d, \\ \displaystyle\int_{\mathbb{R}^d} m(x)\,\mathrm{d}x = 1, \quad \int_{\mathbb{R}^d} u(x)\,\mathrm{d}x = 0, \quad m \geq 0. \end{cases} \tag{MFG_∞}$$

At least formally (MFG_∞) can be obtained as the first order optimality condition of the problem

$$\begin{aligned} &\min \int_{\mathbb{R}^d} \left\{ m(x) L\left(x, -\frac{\mathbf{w}(x)}{m(x)}\right) + F(x, m(x)) \right\} \mathrm{d}x, \\ &\text{s.t.} \quad -\Delta m + \nabla\cdot\mathbf{w} = 0, \quad \int_{\mathbb{R}^d} m(x)\,\mathrm{d}x = 1, \quad m \geq 0. \end{aligned} \tag{4.1.2}$$

The objective of this chapter is to rigorously study the optimization problem (4.1.2) with the additional constraint $m \leq 1$ a.e. Formally this should be linked to a system like (MFG_∞) with $m \leq 1$ *a.e.* and an additional Lagrange multiplier corresponding to the new constraint. Moreover, in view of the interpretation of (MFG) as a continuous Nash equilibria, we expect that our derivation of an MFG system with a density constraint is linked to symmetric games with a large number of players on which "hard congestion" constraints are imposed. Similar models in the framework of crowd motion, tumor growth, etc. have been already studied in the literature (see for instance [MRCS10, MRCS14] and Chapter 2). In the case of MFG systems, we refer the reader to the papers [BDFMW14] (for evolutive systems) and [GM14] (for stationary systems), in which "soft-congestion" effects, meaning that people slow down when they arrive to congested zones, are studied. Let us remark that in [Lio08] it is also explained how to study systems like (MFG) by means of a (degenerate) elliptic equation in space-time. However, this approach with the additional constraint $m \leq 1$ a.e. seems to be ineffective.

The question of hard congestion effects/density constraints for MFG systems was first raised in [San12b]. More precisely, in the cited reference the author asks if a MFG system can be obtained with the additional constraint that the density of the population does not exceed a given threshold, for instance 1. To the best of our knowledge, the analysis from this chapter is the first attempt to investigate this question. The stationary setting plays an important role in our study and we expect to extend our results to the dynamic case in some future research.

Let $\Omega \subset \mathbb{R}^d$ ($d \geq 2$) be a non-empty bounded open set with smooth boundary, satisfying a uniform interior ball condition, and such that the Lebesgue measure of Ω is strictly greater than 1. Moreover, let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a continuous function which is non-decreasing in the second variable and define $\ell_q : \mathbb{R} \times \mathbb{R}^d \to \overline{\mathbb{R}}$ (with $\overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$) and $\mathcal{B}_q : W^{1,q}(\Omega) \times L^q(\Omega)^d \to \overline{\mathbb{R}}$ as

$$\ell_q(a,b) := \begin{cases} \frac{1}{q}\frac{|b|^q}{a^{q-1}}, & \text{if } a > 0, \\ 0, & \text{if } (a,b) = (0,0), \\ +\infty, & \text{otherwise.} \end{cases} \qquad \mathcal{B}_q(m, \mathbf{w}) := \int_\Omega \ell_q(m(x), \mathbf{w}(x))\, \mathrm{d}x. \tag{4.1.3}$$

Let us remark that the choice of the notation $\mathcal{B}_q$ for the above functional is not random, since this is precisely the *Benamou-Brenier functional* – as in (BB) – restricted to the space $W^{1,q}(\Omega) \times L^q(\Omega)^d$.

We consider the problem

$$\begin{aligned} \min \quad & \mathcal{B}_q(m, \mathbf{w}) + \int_\Omega F(x, m(x))\, \mathrm{d}x, \\ \text{s.t.} \quad & -\Delta m + \nabla \cdot (\mathbf{w}) = 0 \ \text{ in } \Omega, \quad (\nabla m - \mathbf{w}) \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega, \\ & \int_\Omega m(x)\, \mathrm{d}x = 1, \quad 0 \le m \le 1, \end{aligned} \tag{P_q}$$

where, as before, $F(x,m)$ is an antiderivative of $f(x,m)$ with respect to the second variable. We divide our main results in two classes, depending on the value of q.

Case 1: $q > d$. In this case, using the classical direct method of the calculus of variations, we prove the existence of a solution $(m, \mathbf{w})$ of (P_q). Using that $m \in W^{1,q} \hookrightarrow C(\overline{\Omega})$, we are able to compute the subdifferential of $\mathcal{B}_r(m, \mathbf{w})$ for any $1 < r \le q$. It seems that this type of result is new in the literature. Moreover, the continuity of m allows us to prove that the constraints in (P_q) are qualified (see e.g. [BS00, Chapter 2]). Using the computation of the subdifferential with $r = q$ and classical arguments in convex analysis, we derive the existence of $u \in W^{1,s}(\Omega)$ ($s \in [1, d/(d-1)[$), $\lambda \in \mathbb{R}$ and two nonnegative regular measures μ and p such that

$$\begin{cases} -\Delta u + \frac{1}{q'}|\nabla u|^{q'} + \mu - p - \lambda = f(x,m), & \text{in } \Omega, \\ -\Delta m - \nabla \cdot \left(m |\nabla u|^{\frac{2-q}{q-1}} \nabla u \right) = 0, & \text{in } \Omega, \\ \nabla m \cdot \mathbf{n} = \nabla u \cdot \mathbf{n} = 0, & \text{on } \partial\Omega, \\ \int_\Omega m\, \mathrm{d}x = 1, \ \ 0 \le m \le 1, & \text{in } \Omega, \\ \operatorname{spt}(\mu) \subseteq \{m = 0\}, \quad \operatorname{spt}(p) \subseteq \{m = 1\}, \end{cases} \tag{MFG_q}$$

where the system of PDEs is satisfied in the weak sense, 'spt' denotes the support of a measure and $q' := q/(q-1)$. In the above system, p appears as a Lagrange multiplier associated to the constraint $m \le 1$ and can be interpreted as a sort of a "pressure" term. We also compute the dual problem associated to (P_q) recovering (MFG_q) by duality. Finally, in the open set $\{0 < m < 1\}$ we prove some local regularity results for the pair (m, u).

Case 2: $1 < q \leq d$. In this case, even if the existence of a solution still holds true, m is in general discontinuous, which implies that the arguments employed in the computation of the subdifferential of $\mathcal{B}_q(m, \mathbf{w})$ are no longer valid. Moreover, the discontinuity of m implies that the constraint $0 \leq m \leq 1$ is in general not qualified. In order to overcome these issues, we use an approximation argument. By adding the term $\varepsilon \mathcal{B}_r(m, \mathbf{w})$ with $r > d$ to the cost function and using the arguments in *Case 1* we obtain a system similar to (MFG_q) depending on $\varepsilon > 0$. Then, by means of some uniform bounds with respect to ε and recent results on estimates on the gradients for solutions of elliptic equations with measure data (see [Min07]), as $\varepsilon \downarrow 0$ we can prove the existence of limit points satisfying (MFG_q) where the concentration properties for p and μ have to be understood in a weak sense.

The structure of the chapter is as follows: in Section 4.2 we prove some preliminary results including the computation of the subdifferential of $\mathcal{B}_q(m, \mathbf{w})$. In Section 4.3 we define rigorously problem (P_q) for the case $q > d$ and we prove the existence of a solution as well as the qualification property of the constraints. In Section 4.4 we characterize the solutions of (P_q) in terms of (MFG_q) still in the case $q > d$. Moreover, we prove some local regularity results and we derive the dual problem. The uniqueness of the solutions is also discussed. In Section 4.5 we complete the proof of the previous statements for any $1 < q \leq d$ by means of an approximation argument. Finally, in Section A.3 – as a sort of an appendix – we recall some important results about elliptic equations with irregular right hand sides, used in the previous sections.

4.2 PRELIMINARY RESULTS

Let $q > 1$ be given and set $q' := q/(q-1)$. Consider the sets

$$A_{q'} := \{(a, b) \in \mathbb{R} \times \mathbb{R}^d : a + \tfrac{1}{q'}|b|^{q'} \leq 0\},$$

$$\mathcal{A}_{q'} := \left\{(a, b) \in L^\infty(\Omega) \times L^\infty(\Omega)^d, \;\; (a(x), b(x)) \in A_{q'}, \;\; \text{for a.e. } x \in \Omega\right\},$$

and recall the functions ℓ_q and $\mathcal{B}_q$ defined in (4.1.3). We have the following result.

Lemma 4.2.1. *Suppose that $q > d$ and let $1 < r \leq q$. Then, the following assertions hold true:*

(i) *The closure of $\mathcal{A}_{r'}$ in $\left(W^{1,q}(\Omega)\right)^* \times L^{q'}(\Omega)^d$ is given by*

$$\overline{\mathcal{A}_{r'}} = \left\{(\alpha, \beta) \in \mathscr{M}(\overline{\Omega}) \times L^{r'}(\Omega)^d \; : \; \alpha + \frac{1}{r'}|\beta|^{r'} \leq 0\right\}, \tag{4.2.1}$$

where the inequality in (4.2.1) *means that for every non-negative* $\phi \in C(\overline{\Omega})$ *we have that*

$$\int_\Omega \phi(x)\,\mathrm{d}\alpha(x) + \frac{1}{r'}\int_\Omega \phi(x)|\beta(x)|^{r'}\,\mathrm{d}x \le 0. \tag{4.2.2}$$

(ii) *Restricted to* $W^{1,q}(\Omega) \times L^q(\Omega)^d$*, the functional* $\mathcal{B}_r$ *is convex and l.s.c. Moreover, for every* $(m, \mathbf{w}) \in W^{1,q}(\Omega) \times L^q(\Omega)^d$*, it holds that*

$$\begin{aligned} \mathcal{B}_r(m,\mathbf{w}) &= \sup_{(\alpha,\beta)\in\mathcal{A}_{r'}} \int_\Omega [\alpha(x)m(x) + \beta(x)\cdot\mathbf{w}(x)]\,\mathrm{d}x \\ &= \sup_{(\alpha,\beta)\in\overline{\mathcal{A}_{r'}}} \left[\int_\Omega m(x)\,\mathrm{d}\alpha(x) + \int_\Omega \beta(x)\cdot\mathbf{w}(x)\,\mathrm{d}x\right], \end{aligned} \tag{4.2.3}$$

and $\mathcal{B}_r^*(\alpha,\beta) = \chi_{\overline{\mathcal{A}_{r'}}}(\alpha,\beta)$ *for all* $(\alpha,\beta) \in \left(W^{1,q}(\Omega)\right)^* \times L^{q'}(\Omega)^d$.

Proof. (i) Let $(\alpha_n, \beta_n) \in \mathcal{A}_{r'}$ converging to some (α,β) in $\left(W^{1,q}(\Omega)\right)^* \times L^{q'}(\Omega)^d$. Then, for any non-negative $\phi \in W^{1,q}(\Omega)$ we have that

$$\int_\Omega \phi(x)\alpha_n(x)\,\mathrm{d}x \le -\frac{1}{r'}\int_\Omega \phi(x)|\beta_n(x)|^{r'}\,\mathrm{d}x.$$

Since $\beta_n \to \beta$ in $L^{q'}(\Omega)^d$, except for some subsequence, $|\beta_n(x)|^{r'} \to |\beta(x)|^{r'}$ for a.e. $x \in \Omega$. Having positive integrands in the second integral, by Fatou's lemma we obtain

$$\langle \alpha, \phi\rangle_{(W^{1,q})^*, W^{1,q}} \le -\frac{1}{r'}\int_\Omega \phi(x)|\beta(x)|^{r'}\,\mathrm{d}x \quad \text{for all } \phi \in W^{1,q}(\Omega),\ \phi \ge 0.$$

In particular, letting $\phi \equiv 1$, we have that $\beta \in L^{r'}(\Omega)^d$ and by [Sch66, Chapitre I, Théorème V] we can extend α to a linear functional over $C(\overline{\Omega})$, i.e. to an element in $\mathscr{M}(\overline{\Omega})$, satisfying (4.2.2). This proves one inclusion in (4.2.1).

In order to prove the converse inclusion, let (α,β) be an element of the r.h.s. of (4.2.1). Equivalently,

$$\alpha^{\mathrm{ac}} + \frac{1}{r'}|\beta|^{r'} \le 0 \ \text{ a.e. in } \Omega \quad \text{and } \alpha^{\mathrm{s}} \le 0,$$

where α^{ac} and α^{s} denote the absolutely continuous and singular parts of α with respect to the Lebesgue measure, respectively. We shall construct different approximations for α^{ac} and β on the one hand and for α^{s} on the other hand. For $\gamma > 0$ and $x \in \mathbb{R}^d$ we set $B_\gamma(x) = \{y \in \mathbb{R}^d \ : \ |y-x| < \gamma\}$. Consider a mollifier $\eta : \mathbb{R}^d \to \mathbb{R}$ satisfying that $\eta \in C_c^\infty(\mathbb{R}^d)$, $\eta \ge 0$, $\int_{\mathbb{R}^d} \eta(x)\,\mathrm{d}x = 1$, $\mathrm{spt}(\eta) \subseteq B_1(0)$ and $\eta(x) = \eta(-x)$ for all $x \in \mathbb{R}^d$. Now, for $\varepsilon > 0$ set

$$\Omega_\varepsilon := \{x \in \Omega : \ \mathrm{dist}(x, \partial\Omega) > \varepsilon\}, \quad \eta_\varepsilon(x) := \frac{1}{\varepsilon^d}\eta(x/\varepsilon),$$

and for all $x \in \Omega$ and $i = 1, ..., d$, let us define

$$\tilde{\alpha}_\varepsilon(x) := \int_\Omega \eta_\varepsilon(x-y)\alpha^{ac}(y)\,\mathrm{d}y\mathbb{1}_{\Omega_\varepsilon}(x), \quad \tilde{\beta}^i_\varepsilon(x) := \int_\Omega \eta_\varepsilon(x-y)\beta^i(y)\,\mathrm{d}y\mathbb{1}_{\Omega_\varepsilon}(x).$$

By convexity and Jensen's inequality, for all $x \in \Omega_\varepsilon$ we have that

$$\tilde{\alpha}_\varepsilon(x) + \frac{1}{r'}|\tilde{\beta}_\varepsilon(x)|^{r'} \leq \left(\alpha + \frac{1}{r'}|\beta|^{r'}\right) * \eta_\varepsilon(x) \leq 0, \tag{4.2.4}$$

and so $(\tilde{\alpha}_\varepsilon, \tilde{\beta}_\varepsilon) \in \mathcal{A}_{r'}$ and one easily checks that $\tilde{\alpha}_\varepsilon \to \alpha^{\mathrm{ac}}$ in $\mathscr{M}(\overline{\Omega})$ and $\tilde{\beta}_\varepsilon \to \beta$ in $L^{q'}(\Omega)^d$.

In order to approximate α^{s} let us define the following kernel: for $x \in \overline{\Omega}$ and $\varepsilon > 0$ let us set $\rho^x_\varepsilon := \left(\mathbb{1}_{B_\varepsilon(x)\cap\overline{\Omega}}\right)/|B_\varepsilon(x) \cap \overline{\Omega}|$. Note that for all $x \in \overline{\Omega}$ we have that $\rho^x_\varepsilon \to \delta_x$ in $\mathscr{M}(\overline{\Omega})$ as $\varepsilon \downarrow 0$. Given $y \in \overline{\Omega}$ and $\varepsilon > 0$ let us define

$$\hat{\alpha}_\varepsilon(y) := \int_{\overline{\Omega}} \rho^x_\varepsilon(y) \, \mathrm{d}\alpha^{\mathrm{s}}(x).$$

Observe that for all $\varepsilon > 0$ the function $\hat{\alpha}_\varepsilon$ is non-positive and, due to our regularity assumption on $\partial\Omega$, we have that $\hat{\alpha}_\varepsilon \in L^\infty(\Omega)$. Let us show that $\hat{\alpha}_\varepsilon \to \alpha^{\mathrm{s}}$ in $\mathscr{M}(\overline{\Omega})$. For any $\phi \in C(\overline{\Omega})$, Fubini's theorem yields

$$\int_\Omega \phi(y)\hat{\alpha}_\varepsilon(y) \, \mathrm{d}y = \int_\Omega \phi(y) \int_{\overline{\Omega}} \rho^x_\varepsilon(y) \, \mathrm{d}\alpha^{\mathrm{s}}(x) \, \mathrm{d}y = \int_{\overline{\Omega}} \int_\Omega \phi(y)\rho^x_\varepsilon(y) \, \mathrm{d}y \, \mathrm{d}\alpha^{\mathrm{s}}(x)$$
$$\to \int_{\overline{\Omega}} \phi(x) \, \mathrm{d}\alpha^{\mathrm{s}}(x) \quad \text{as } \varepsilon \downarrow 0,$$

where we have used that ϕ is uniformly continuous in $\overline{\Omega}$ (since this set is compact) and so

$$\int_\Omega \phi(y)\rho^x_\varepsilon(y) \, \mathrm{d}y \to \phi(x) \quad \text{uniformly in } \overline{\Omega} \text{ as } \varepsilon \downarrow 0.$$

This proves the convergence of $\hat{\alpha}_\varepsilon$. Defining, $\alpha_\varepsilon := \hat{\alpha}_\varepsilon + \tilde{\alpha}_\varepsilon$ we have that $(\alpha_\varepsilon, \tilde{\beta}_\varepsilon) \in \mathcal{A}_{r'}$ and $(\alpha_\varepsilon, \tilde{\beta}_\varepsilon) \to (\alpha, \beta)$ in $\mathscr{M}(\overline{\Omega}) \times L^{q'}(\Omega)^d$. The embedding $\mathscr{M}(\overline{\Omega}) \hookrightarrow \left(W^{1,q}(\Omega)\right)^*$ implies that the convergence also holds in $(W^{1,q}(\Omega))^* \times L^{q'}(\Omega)^d$, from which assertion (i) follows.

In order to prove (ii), it suffices to show (4.2.3) (here we remark that by the Sobolev embedding we identify m with an element in $C(\overline{\Omega})$, hence the second integral is meaningful). Indeed, (4.2.3) shows that $\mathcal{B}_r$ is the supremum of linear and continuous functionals, hence it is convex and l.s.c. For $k \in \mathbb{N}$ set

$$A_{r',k} := \left\{(a,b) \in A_{r'} \; ; \; a \geq -k, \; \max_{i=1,\dots,d} |b_i| \leq k\right\}.$$

Since a.e. in Ω we have that

$$\lim_{k\to\infty} \sup_{(a,b)\in A_{r',k}} \{am(x) + b \cdot \mathbf{w}(x)\} = \sup_{(a,b)\in A_{r'}} \{am(x) + b \cdot \mathbf{w}(x)\},$$

and $(a,b) = (0,0) \in A_{r',k}$, by monotone convergence we have that

$$\mathcal{B}_r(m, \mathbf{w}) = \lim_{k\to\infty} \int_\Omega \sup_{(a,b)\in A_{r',k}} [am(x) + b \cdot \mathbf{w}(x)] \, \mathrm{d}x. \tag{4.2.5}$$

Note that if $|\{m<0\}|>0$, then by (4.2.1), we readily check that both sides in (4.2.3) are equal to $+\infty$. On the other hand, note that for every $(m,w)\in\mathbb{R}\times\mathbb{R}^d$ with $m\geq 0$ there exists a unique pair $(a(m,w),b(m,w))\in\mathbb{R}\times\mathbb{R}^d$ such that

$$\sup_{(a,b)\in A_{r',k}}\{am+b\cdot w\}=a(m,w)m+b(m,w)\cdot\mathbf{w}.$$

Indeed,

$$b(m,w)=\operatorname{argmax}_{|b|_\infty\leq k}\left\{-\frac{|b|^{r'}}{r'}m+b\cdot w\right\}\ \text{and}\ \ a(m,w)=-\frac{|b(m,w)|^{r'}}{r'},\tag{4.2.6}$$

which are well-defined by the strict concavity of the objective function. Moreover this implies that $\mathbb{R}_+\times\mathbb{R}^d\ni(m,w)\mapsto(a(m,w),b(m,w))\in\mathbb{R}\times\mathbb{R}^d$ is continuous and measurable and thus

$$W^{1,q}(\Omega)\times(L^q(\Omega))^d\ni(m,\mathbf{w})\mapsto(a(m,\mathbf{w}),b(m,\mathbf{w}))\in L^\infty(\Omega)\times L^\infty(\Omega)^d$$

is well defined. Therefore, defining

$$\mathcal{A}_{r',k}:=\left\{(a,b)\in L^\infty(\Omega)\times L^\infty(\Omega)^d,\ \ (a(x),b(x))\in A_{r',k}\ \text{ for a.e. } x\in\Omega\right\}$$

we get that

$$\int_\Omega\sup_{(a,b)\in A_{r',k}}[am(x)+b\cdot\mathbf{w}(x)]\,\mathrm{d}x=\sup_{(a,b)\in\mathcal{A}_{r',k}}\int_\Omega[a(x)m(x)+b(x)\cdot\mathbf{w}(x)]\,\mathrm{d}x,$$

which, together with (4.2.5), implies that

$$\begin{aligned}\mathcal{B}_r(m,\mathbf{w})&=\lim_{k\to\infty}\sup_{(a,b)\in\mathcal{A}_{r',k}}\int_\Omega[a(x)m(x)+b(x)\cdot\mathbf{w}(x)]\,\mathrm{d}x\\&=\sup_{(a,b)\in\mathcal{A}_{r'}}\int_\Omega[a(x)m(x)+b(x)\cdot\mathbf{w}(x)]\,\mathrm{d}x,\end{aligned}$$

proving the first equality in (4.2.3). The second equality follows from (i) and the continuity of the considered linear application. Finally, the identity $\mathcal{B}_r^*=\chi_{\overline{\mathcal{A}}_{r'}}$ is a consequence of (i) and (4.2.3). □

Remark 4.2.1. *We refer the reader to [San15, Chapter 5] for the proof of the semicontinuity of $\mathcal{B}_r$ in a more general setting.*

For $m\in W^{1,q}(\Omega)$ denote

$$E_0^m:=\left\{x\in\overline{\Omega}:m(x)=0\right\}\quad\text{and}\quad E_1^m=\left\{x\in\overline{\Omega}:m(x)>0\right\}.$$

Note that if $q>d$, then E_0^m is closed and E_1^m is open.

Theorem 4.2.2. *Let* $(m,\mathbf{w}) \in W^{1,q}(\Omega) \times L^q(\Omega)^d$ *(*$q > d$*) and* $1 < r \leq q$*. Suppose that* $\mathcal{B}_r(m,\mathbf{w}) < \infty$*. Then, if* $\mathbf{v} := (\mathbf{w}/m)\mathbb{1}_{E_1^m} \notin L^r(\Omega)^d$ *we have that* $\partial\mathcal{B}_r(m,\mathbf{w}) = \emptyset$*. Otherwise,* $\mathcal{B}_r$ *is subdifferentiable at* $(m,\mathbf{w})$ *and*

$$\partial\mathcal{B}_r(m,\mathbf{w}) = \left\{(\alpha,\beta) \in \overline{\mathcal{A}_{r'}} \ : \ \alpha \llcorner E_1^m = -\frac{1}{r'}|\mathbf{v}|^r \quad \text{and} \quad \beta \llcorner E_1^m = |\mathbf{v}|^{r-2}\mathbf{v}\right\}. \tag{4.2.7}$$

In particular, the singular part of α *is concentrated in* E_0^m*.*

Proof. First note that since $\mathcal{B}_r(m,\mathbf{w}) < \infty$, we have that $|\{m < 0\}| = 0$ and $\mathbf{w} = 0$ a.e. in E_0^m. By Lemma 4.2.1 for all $(m,\mathbf{w}) \in W^{1,q}(\Omega) \times L^q(\Omega)^d$, $m \geq 0$ we have that

$$\partial\mathcal{B}_r(m,\mathbf{w}) = \operatorname{argmax}_{(\alpha,\beta)\in\overline{\mathcal{A}_{r'}}}\left\{\int_\Omega m\,\mathrm{d}\alpha + \int_\Omega \beta\cdot\mathbf{w}\,\mathrm{d}x\right\}.$$

We claim that

$$\sup_{(\alpha,\beta)\in\overline{\mathcal{A}_{r'}}}\left\{\int_\Omega m\,\mathrm{d}\alpha + \int_\Omega \beta\cdot\mathbf{w}\,\mathrm{d}x\right\} = \sup_{\beta\in L^{r'}(\Omega)^d} -\frac{1}{r'}\int_\Omega m|\beta|^{r'}\,\mathrm{d}x + \int_\Omega \beta\cdot\mathbf{w}\,\mathrm{d}x. \tag{4.2.8}$$

Indeed, the inequality "$\geq$" is immediate. To show the converse inequality for every $\varepsilon > 0$ let $(\alpha_\varepsilon,\beta_\varepsilon) \in \overline{\mathcal{A}_{r'}}$ such that

$$\begin{aligned}\int_\Omega m(x)\,\mathrm{d}\alpha_\varepsilon(x) + \int_\Omega \beta_\varepsilon(x)\cdot\mathbf{w}(x)\,\mathrm{d}x \\ \geq \sup_{(\alpha,\beta)\in\overline{\mathcal{A}_{r'}}}\left\{\int_\Omega m(x)\,\mathrm{d}\alpha(x) + \int_\Omega \beta(x)\cdot\mathbf{w}(x)\,\mathrm{d}x\right\} - \varepsilon.\end{aligned}$$

Then, denoting by $\hat{s}$ the r.h.s. of (4.2.8), by (4.2.1) and the previous inequality we have that

$$\begin{aligned}\hat{s} &\geq -\frac{1}{r'}\int_\Omega m|\beta_\varepsilon|^{r'}\,\mathrm{d}x + \int_\Omega \beta_\varepsilon\cdot\mathbf{w}\,\mathrm{d}x \\ &\geq \sup_{(\alpha,\beta)\in\overline{\mathcal{A}_{r'}}}\left\{\int_\Omega m(x)\,\mathrm{d}\alpha(x) + \int_\Omega \beta(x)\cdot\mathbf{w}(x)\,\mathrm{d}x\right\} - \varepsilon\end{aligned}$$

and so (4.2.8) follows by letting $\varepsilon \to 0$. Let us prove now that if $\mathbf{v} \notin L^r(E_1^m)^d$, then $\partial\mathcal{B}_r(m,\mathbf{w}) = \emptyset$. We argue by contradiction supposing that there exists $(\hat{\alpha},\hat{\beta}) \in \partial\mathcal{B}_r(m,\mathbf{w})$. By (4.2.8) and the assumption $\mathbf{w} = 0$ a.e. in E_0^m, $\hat{\beta}$ must be a solution of the problem

$$\inf_{\beta\in L^{r'}(\Omega)^d} J(\beta), \quad \text{where } J(\beta) := \int_\Omega \left[\frac{1}{r'}|\beta|^{r'} - \mathbf{v}\cdot\beta\right] m\,\mathrm{d}x. \tag{4.2.9}$$

Since $m\mathbf{v} = \mathbf{w} \in L^q(\Omega)^d$ and $q \geq r$, we have that J is Fréchet differentiable and

$$0 = DJ(\hat{\beta})\beta = \int_{E_1^m} \left[|\hat{\beta}|^{r'-2}\hat{\beta} - \mathbf{v} \right] \cdot \beta m \, \mathrm{d}x \quad \text{for all } \beta \in L^{r'}(\Omega)^d, \tag{4.2.10}$$

which implies that, since β is arbitrary and $m > 0$ on E_1^m,

$$\mathbf{v}(x) = |\hat{\beta}(x)|^{r'-2}\hat{\beta}(x) \quad \text{for a.e. } x \in E_1^m, \tag{4.2.11}$$

which is a contradiction because $|\hat{\beta}|^{r'-2}\hat{\beta} \in L^r(E_1^m)^d$. Now, assume that $\mathbf{v} \in L^r(E_1^m)^d$ and let us prove that $\partial \mathcal{B}_r(m, \mathbf{w}) \neq \emptyset$. Define the functional

$$\hat{J} : L^{r'}_m(E_1)^d \to \mathbb{R}$$

(where $L^{r'}_m(E_1^m)$ denotes the space of measurable functions defined in E_1^m which are integrable on the power r' w.r.t. the measure m) as

$$\hat{J}(\beta) = \frac{1}{r'} \int_{E_1^m} |\beta|^{r'} m \, \mathrm{d}x - \int_{E_1^m} \mathbf{v} \cdot \beta m \, \mathrm{d}x.$$

Since $r' > 1$, we have that $\hat{J}$ is coercive, continuous and strictly convex. Since $L^{r'}_m(E_1^m)^d$ is a reflexive Banach space, classical results in convex analysis imply the existence of a unique $\bar{\beta} \in L^{r'}_m(E_1^m)^d$ such that $\hat{J}(\bar{\beta}) = \inf\{\hat{J}(\beta) \ : \ \beta \in L^{r'}_m(E_1^m)^d\}$. The first order optimality condition implies that $\bar{\beta}$ satisfies (4.2.10)-(4.2.11) and so

$$\bar{\beta}(x) = |\mathbf{v}(x)|^{r-2}\mathbf{v}(x) \quad \text{for a.e. } x \in E_1^m. \tag{4.2.12}$$

Since $\mathbf{v} \in L^r(E_1^m)^d$ we have that $\bar{\beta} \in L^{r'}(E_1^m)^d$. Moreover, using that $L^{r'}(E_1^m)^d \subseteq L^{r'}_m(E_1^m)^d$, relation (4.2.8) implies that $(-|\bar{\beta}|^{r'}/r', \bar{\beta}) \in \partial \mathcal{B}_r(m, \mathbf{w})$ and so

$$\partial \mathcal{B}_r(m, \mathbf{w}) \neq \emptyset.$$

Now, let $(\alpha, \beta) \in \partial \mathcal{B}_r(m, \mathbf{w})$. The expression for $\overline{\mathcal{A}_{r'}}$ in (4.2.1) implies that $(-(1/r')|\beta|^{r'}, \beta)$ attains the supremum on the r.h.s. of (4.2.8). Therefore, we must have

$$\int_\Omega m \, \mathrm{d}\alpha + \frac{1}{r'} \int_\Omega |\beta|^{r'} m \, \mathrm{d}x = \int_{E_1^m} m \, \mathrm{d}\alpha + \frac{1}{r'} \int_{E_1^m} |\beta|^{r'} m \, \mathrm{d}x = 0. \tag{4.2.13}$$

Let us prove that $\alpha \llcorner E_1^m$ is absolutely continuous w.r.t. the Lebesgue measure restricted to E_1^m. Let $B \in \mathscr{B}(E_1^m)$ a Borel set such that $|B| = 0$. Then, (4.2.13) implies that

$$\int_{E_1^m \setminus B} m \, \mathrm{d}\alpha + \frac{1}{r'} \int_{E_1^m \setminus B} |\beta|^{r'} m \, \mathrm{d}x + \int_B m \, \mathrm{d}\alpha = 0.$$

By a standard argument using Lusin's theorem (to approximate the $\mathbb{1}_{E_1^m \setminus B}$ by continuous functions) and (4.2.1) we must have that $\int_B m \, \mathrm{d}\alpha = 0$ and since

$m > 0$ on E_1^m we conclude that $\alpha(B) = 0$. Thus $\alpha \llcorner E_1^m \ll \mathscr{L}^d \llcorner E_1^m$. In particular, $\operatorname{spt}(\alpha^s) \subseteq E_0^m$ and, denoting still by α the density of α restricted to E_1^m, $\alpha(x) + (1/r')|\beta(x)|^{r'} \le 0$ for a.e. $x \in E_1^m$. Therefore, by (4.2.13) we have that

$$\int_{E_1^m} m\left[\alpha + \frac{1}{r'}|\beta|^{r'}\right] \mathrm{d}x = 0,$$

and since $m > 0$ on E_1^m, we conclude that $\alpha = -\frac{1}{r'}|\beta|^{r'}$ a.e. in E_1^m. Using (4.2.8) we get that β solves problem (4.2.9) and so $\beta = |\mathbf{v}|^{r-2}\mathbf{v}$ a.e. in E_1^m from which the result follows. □

Remark 4.2.2. *Note that redefining the domain of $\mathcal{B}_r$ as $C(\overline{\Omega}) \times L^q(\Omega)^d$, the above proof shows that the conclusions of the Theorem 4.2.2 are still valid in this setting.*

4.3 THE OPTIMIZATION PROBLEM

In this entire section we suppose that $q > d$. Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a continuous function in both variables and increasing in the second variable. Let us define the function

$$\Omega \times \mathbb{R} \ni (x, m) \mapsto F(x, m) := \int_0^m f(x, s)\, \mathrm{d}s \in \mathbb{R}.$$

Note that for every fixed $x \in \Omega$ the function $m \mapsto F(x, m)$ is convex. Let us define

$$\mathcal{F} : W^{1,q}(\Omega) \to \overline{\mathbb{R}} \quad \text{as} \quad \mathcal{F}(m) := \int_\Omega F(x, m(x))\, \mathrm{d}x. \tag{4.3.1}$$

Given $\mathbf{w} \in L^q(\Omega)^d$ we consider the following elliptic PDE

$$\begin{cases} -\Delta m + \nabla \cdot \mathbf{w} &= 0 \quad \text{in } \Omega, \\ (\nabla m - \mathbf{w}) \cdot \mathbf{n} &= 0 \quad \text{on } \partial\Omega. \end{cases} \tag{4.3.2}$$

We say that $m \in W^{1,q}(\Omega)$ is a weak solution of (4.3.2) if

$$\int_\Omega \nabla m(x) \cdot \nabla \varphi(x)\, \mathrm{d}x = \int_\Omega \mathbf{w}(x) \cdot \nabla \varphi(x)\, \mathrm{d}x \quad \forall\, \phi \in C^1(\overline{\Omega}) \tag{4.3.3}$$

By Lemma A.3.1 and Lemma A.3.2 in Section A.3 for a given $\mathbf{w} \in L^q(\Omega)^d$ equation (4.3.2) has a unique solution $m \in W^{1,q}(\Omega)$ satisfying that $\int_\Omega m(x)\, \mathrm{d}x = 1$. We consider the following optimization problem:

$$\inf_{(m,\mathbf{w}) \in \mathcal{K}_P} \mathcal{J}_q(m, \mathbf{w}) := \mathcal{B}_q(m, \mathbf{w}) + \mathcal{F}(m), \tag{P_q}$$

where the set of constraints $\mathcal{K}_P$ is defined as

$$\mathcal{K}_P := \Bigg\{ (m, \mathbf{w}) \in W^{1,q}(\Omega) \times L^q(\Omega)^d \text{ ; such that } (m, \mathbf{w}) \text{ satisfies (4.3.2)}, \\ \int_\Omega m(x)\, \mathrm{d}x = 1,\ m \le 1 \Bigg\}.$$

Remark 4.3.1. *Since $\mathcal{B}_q = \mathcal{B}_q + \chi_{\{m \geq 0\}}$, the constraint $m \geq 0$ is implicitly imposed in (P_q).*

Given $s \in [1, \infty[$ recall the notation $W^{k,s}_{\diamond}(\Omega) := \left\{ u \in W^{k,s}(\Omega) : \int_{\Omega} u = 0 \right\}$. Now, let us define $A : W^{1,q}(\Omega) \to \left(W^{1,q'}_{\diamond}(\Omega)\right)^*$ and $B : L^q(\Omega)^d \to \left(W^{1,q'}_{\diamond}(\Omega)\right)^*$ as

$$\langle\langle Am, \phi \rangle\rangle := \int_{\Omega} \nabla m(x) \cdot \nabla \phi(x) \, \mathrm{d}x, \quad \langle\langle B\mathbf{w}, \phi \rangle\rangle := -\int_{\Omega} \mathbf{w}(x) \cdot \nabla \phi(x) \, \mathrm{d}x,$$

for all $m \in W^{1,q}(\Omega)$, $\mathbf{w} \in L^q(\Omega)^d$ and $\phi \in W^{1,q'}_{\diamond}(\Omega)$, where we used $\langle\langle \cdot, \cdot \rangle\rangle$ to denote the duality product between $\left(W^{1,q'}_{\diamond}(\Omega)\right)^*$ and $W^{1,q'}_{\diamond}(\Omega)$. Since A and B are linear bounded operators, the adjoint operators $A^* : W^{1,q'}_{\diamond}(\Omega) \to \left(W^{1,q}(\Omega)\right)^*$ and $B^* : W^{1,q'}_{\diamond}(\Omega) \to L^{q'}(\Omega)^d$ are well-defined and given by

$$\langle A^* \phi, m \rangle = \int_{\Omega} \nabla \phi(x) \cdot \nabla m(x) \, \mathrm{d}x, \qquad \langle B^* \phi, w \rangle_{q',q} = -\int_{\Omega} \nabla \phi(x) \cdot \mathbf{w}(x) \, \mathrm{d}x,$$

where we have used $\langle \cdot, \cdot \rangle$ to denote the duality product between $(W^{1,q}(\Omega))^*$ and $W^{1,q}(\Omega)$ and $\langle \cdot, \cdot \rangle_{q',q}$ to denote the duality product between $L^{q'}(\Omega)^d$ and $L^q(\Omega)^d$. Now, let $I : W^{1,q}(\Omega) \to C(\overline{\Omega})$ be the Sobolev injection, which is well-defined since $q > d$ (see [Ada75]), and let $\mathcal{C} := \{z \in C(\overline{\Omega}) \; ; \; z \leq 1\}$. Let us set $X := W^{1,q}(\Omega) \times L^q(\Omega)^d$, $Y := \left(W^{1,q'}_{\diamond}(\Omega)\right)^* \times \mathbb{R} \times C(\overline{\Omega})$ and define the application $G : X \to Y$ as

$$G(m, \mathbf{w}) := \left(Am + B\mathbf{w}, \int_{\Omega} m(x) \, \mathrm{d}x - 1, Im \right).$$

By setting $\mathcal{K} := \{0\} \times \{0\} \times \mathcal{C} \subseteq Y$ we have that $\mathcal{K}_P$ can be rewritten as

$$\mathcal{K}_P = \{(m, \mathbf{w}) \in X \; : G(m, \mathbf{w}) \in \mathcal{K}\}.$$

Since A, B and I are linear bounded operators, we have that $\mathcal{K}_P$ is a closed and convex subset of $W^{1,q}(\Omega) \times L^q(\Omega)^d$.

Theorem 4.3.1. *Problem (P_q) has (at least) one solution $(m, \mathbf{w})$.*

Proof. Since $|\Omega| > 1$ we have that $(m, \mathbf{w}) := (1/|\Omega|, 0)$ belongs to $\mathcal{K}_P$ and the cost function is finite. Now, let $(m_k, \mathbf{w}_k) \in \mathcal{K}_P$ be a minimizing sequence. Since $m \leq 1$ a.e. in Ω and $\mathcal{B}_q(m_k, \mathbf{w}_k)$ is bounded uniformly in k, we get that $\|\mathbf{w}_k\|_{L^q}$ is bounded. Therefore, there exists $\mathbf{w} \in L^q(\Omega)^d$ such that, except for some subsequence, $\mathbf{w}_k \rightharpoonup \mathbf{w}$ weakly in $L^q(\Omega)^d$. In addition, by Lemma A.3.2 and the boundedness of $\mathbf{w}_k$ in $L^q(\Omega)^d$ we have that ∇m_k is uniformly bounded in $L^q(\Omega)^d$. Since $\int_{\Omega} m_k(x) \, \mathrm{d}x = 1$, Poincaré's inequality,

$$\left\| m_k - \frac{1}{|\Omega|} \right\|_{L^q} \leq C \|\nabla m_k\|_{L^q}$$

implies that m_k is bounded in $W^{1,q}(\Omega)$. Thus, there exists $m \in W^{1,q}(\Omega)$ such that, except for some subsequence, $m_k \rightharpoonup m$ weakly in $W^{1,q}(\Omega)$. Using these convergences, we get that $Am + B\mathbf{w} = 0$ and $\int_\Omega m(x)\,\mathrm{d}x = 1$. The continuous embedding I preserves the weak convergence and $\mathcal{C}$ is weakly closed in $C(\overline{\Omega})$. Thus, $Im \in \mathcal{C}$, which implies that $(m, \mathbf{w}) \in \mathcal{K}_P$.

Since $\mathcal{J}_q$ is convex and l.s.c. w.r.t the weak topology in $W^{1,q}(\Omega) \times L^q(\Omega)^d$ (by Lemma 4.2.1) we get that $\mathcal{J}_q(m, \mathbf{w}) = \inf\{\mathcal{J}_q(m_1, \mathbf{w}_1) \;:\; (m_1, \mathbf{w}_1) \in \mathcal{K}_P\}$. □

Now, we prove a *constraint qualification* result for problem (P_q) (see e.g. [BS00, Chapter 2]), which is crucial for deriving optimality conditions. We set

$$\mathrm{dom}(\mathcal{J}_q) := \{(m, \mathbf{w}) \in W^{1,q}(\Omega) \times L^q(\Omega)^d \;:\; \mathcal{J}_q(m, \mathbf{w}) < \infty\}.$$

Lemma 4.3.2. *We have that*

$$0 \in \mathrm{int}\left\{G(\mathrm{dom}(\mathcal{J}_q)) - \mathcal{K}\right\}. \tag{4.3.4}$$

Proof. We need to prove that for any given $(\delta_1, \delta_2, \delta_3) \in Y$ small enough there exists $(m, \mathbf{w}, c) \in \mathrm{dom}(\mathcal{J}_q) \times \mathcal{C}$ such that

$$Am + B\mathbf{w} = \delta_1, \quad \int_\Omega m(x)\,\mathrm{d}x = 1 + \delta_2, \quad I(m) - c = \delta_3. \tag{S}$$

We observe that $(m, 0) \in \mathrm{dom}(\mathcal{J}_q)$, for all $m \in W^{1,q}(\Omega) \cap \mathrm{dom}(\mathcal{F})$ non-negative, which implies that we can search the solution of (S) in the form $(m, 0, c) \in \mathrm{dom}(\mathcal{J}_q) \times \mathcal{C}$. First of all, note that for $m_0 := 1/|\Omega|$ we have that

$$Am_0 = 0, \qquad \int_\Omega m_0(x)\,\mathrm{d}x = 1, \qquad Im_0 = m_0 \in \mathrm{int}(\mathcal{C}).$$

By Lemma A.3.2 (see Section A.3), there exists $m_1 \in W^{1,q}(\Omega)$ such that

$$Am_1 = \delta_1 \qquad \text{and} \qquad \int_\Omega m_1(x)\,\mathrm{d}x = 1 + \delta_2. \tag{4.3.5}$$

Setting $\delta m := m_0 - m_1$, we obtain that $A\delta m = -\delta_1$ and $\int_\Omega \delta m(x)\,\mathrm{d}x = -\delta_2$. By Lemma A.3.2 the linear bounded operator $W^{1,q}(\Omega) \ni m \mapsto \left(Am, \int_\Omega m(x)\,\mathrm{d}x\right) \in \left(W_\diamond^{1,q'}(\Omega)\right)^* \times \mathbb{R}$ is surjective and so, by the Open Mapping Theorem, there exists $h \in W^{1,q}(\Omega)$ such that

$$Ah = 0, \qquad \int_\Omega h(x)\,\mathrm{d}x = 0 \quad \text{and} \quad \|h - \delta m\|_{W^{1,q}} = O\left(\|(\delta_1, \delta_2)\|_{\left(W_\diamond^{1,q'}(\Omega)\right)^* \times \mathbb{R}}\right).$$

In particular, as $q > d$ the Sobolev inequality implies that

$$\|I(h) - I(\delta m)\|_{L^\infty} = O\left(\|(\delta_1, \delta_2)\|_{\left(W_\diamond^{1,q'}(\Omega)\right)^* \times \mathbb{R}}\right). \tag{4.3.6}$$

Now, let $r := h - \delta m$ and for $\gamma > 0$ let us define $m_\gamma := m_1 + \gamma h$, which by construction solves (4.3.5). Since $m_0 \in \text{int}(\mathcal{C})$, if γ is near to one (and δ_1, δ_2 are small enough) then, by (4.3.6), $I(m_\gamma) = I(m_1) + \gamma I(\delta m) + \gamma I(r) \in \text{int}(\mathcal{C})$. Thus, if $\|\delta_3\|_{L^\infty}$ is small enough we have that $c := I(m_\gamma) - \delta_3 \in \text{int}(\mathcal{C})$. Thus, $(m_\gamma, 0, c)$ solves (S) and $(m_\gamma, 0) \in \text{dom}(\mathcal{J}_q)$. The result follows. □

4.4 OPTIMALITY CONDITIONS AND CHARACTERIZATION OF THE SOLUTIONS

The purpose of this section is to derive optimality conditions for problem (P_q) and, as a consequence, to obtain the existence of solutions of (MFG_q). As in the previous section we will assume in all the statements that $q > d$. Our strategy relies on a "direct method", which uses the constraint qualification condition established in Lemma 4.3.2 and the characterization of the subdifferential of $\mathcal{B}_q$ (see Theorem 4.2.2). The uniqueness and local regularity of the solutions are also discussed. Moreover, in Subsection 4.4.1 we formulate the associated dual problem, and we provide an alternative (but related) argument to derive optimality conditions.

Let us define the *Lagrangian* $\mathfrak{L} : W^{1,q}(\Omega) \times L^q(\Omega)^d \times W^{1,q'}_\diamond(\Omega) \times \mathscr{M}(\overline{\Omega}) \times \mathbb{R} \to \overline{\mathbb{R}}$ as

$$\begin{aligned}\mathfrak{L}(m, \mathbf{w}, u, p, \lambda) &:= \mathcal{J}_q(m, \mathbf{w}) - \langle\!\langle Am + B\mathbf{w}, u \rangle\!\rangle + \int_\Omega Im \, \mathrm{d}p + \lambda \left(\int_\Omega m \, \mathrm{d}x - 1 \right) \\ &= \mathcal{J}_q(m, \mathbf{w}) - \langle A^* u - I^* p - \lambda, m \rangle - \int_\Omega B^* u \cdot \mathbf{w} \, \mathrm{d}x - \lambda.\end{aligned} \tag{4.4.1}$$

Remark 4.4.1. *Since the inclusion $W^{1,q}(\Omega) \hookrightarrow C(\overline{\Omega})$ is dense, for every measure $p \in \mathscr{M}(\overline{\Omega})$ the adjoint of the injection operator $I^* p$ at p can be identified uniquely with the restriction of p to $W^{1,q}(\Omega)$. Thus, for notational convenience we will still write p for $I^* p$.*

Recall that for a Banach space X and a convex closed subset $K \subseteq X$, the *normal cone* to K at $x \in K$ is defined as

$$N_K(x) := \{x^* \in X^* : \langle x^*, z - x \rangle_{X^*, X} \leq 0, \forall z \in K\}, \tag{4.4.2}$$

where $\langle \cdot, \cdot \rangle_{X^*, X}$ denotes the duality pairing of X^* and X. Using [BS00, Example 2.63] we have

$$N_{\mathcal{K}}(G(m, \mathbf{w})) = \left\{ (u, \lambda, p) \in W^{1,q'}_\diamond(\Omega) \times \mathbb{R} \times \mathscr{M}_+(\overline{\Omega}) \; ; \; \text{spt}(p) \subseteq \{m = 1\} \right\}. \tag{4.4.3}$$

Now, we provide the first order optimality conditions associated to a solution $(m, \mathbf{w})$ of (P_q).

Theorem 4.4.1. *Let* $(m, \mathbf{w}) \in \mathcal{K}_P$ *be a solution of problem* (P_q)*. Then,*

$$\mathbf{v} := (\mathbf{w}/m)\mathbb{1}_{\{m>0\}} \in L^q(\Omega)^d$$

and there exists $(u, p, \lambda) \in W^{1,s}(\Omega) \times \mathscr{M}_+(\overline{\Omega}) \times \mathbb{R}$ *for all* $s \in]1, d/(d-1)[$ *and* $(\alpha, \beta) \in \partial\mathcal{B}_q(m, \mathbf{w})$ *such that* $A^*u \in \mathscr{M}(\overline{\Omega})$ *and the following optimality conditions hold true*

$$\begin{cases} \alpha + f(\cdot, m) - A^*u + p + \lambda & = & 0, \\ \beta & = & B^*u, \\ Am + B\mathbf{w} & = & 0, \\ \operatorname{spt}(p) \subseteq \{m = 1\}, \quad 0 \leq m \leq 1, & \displaystyle\int_\Omega m \, \mathrm{d}x = 1, & \displaystyle\int_\Omega u \, \mathrm{d}x = 0, \end{cases} \tag{4.4.4}$$

where the first equality holds in $\mathscr{M}(\overline{\Omega})$*. Conversely, if there exists* $(\alpha, \beta, u, p, \lambda) \in \partial\mathcal{B}_q(m, \mathbf{w}) \times W^{1,q'}(\Omega) \times \mathscr{M}_+(\overline{\Omega}) \times \mathbb{R}$ *such that* (4.4.4) *holds true, then* $(m, \mathbf{w})$ *solves* (P_q)*.*

Proof. By Lemma 4.3.2 the problem is qualified (see e.g. [BS00, Chapter 2]). Thus, by classical results in convex analysis (see e.g. [BS00, Theorem 2.158 and Theorem 2.165]; see also in Appendix A), we have the existence of $(u, p, \lambda) \in N_{\mathcal{K}}(G(m, \mathbf{w}))$ such that

$$(0, 0) \in \partial_{(m,\mathbf{w})}\mathfrak{L}(m, \mathbf{w}, u, p, \lambda). \tag{4.4.5}$$

Since $\mathcal{B}_q$ is finite at $(1/|\Omega|, 0)$ and the other terms appearing in $\mathfrak{L}$ are differentiable, by [ET76, Chapter 1, Proposition 5.6] and (4.4.5), we must have that $\mathcal{B}_q$ is subdifferentiable at $(m, \mathbf{w})$. Thus, by Theorem 4.2.2 and (4.4.2) we get that $\mathbf{v} \in L^q(\Omega)^d$ and there exists $(\alpha, \beta) \in \overline{\mathcal{A}_{q'}}$ such that (4.4.4) holds true, with the first equation being an equality in $\left(W^{1,q}(\Omega)\right)^*$. Since, except by A^*u, all the other terms can be identified with elements in $\mathscr{M}(\overline{\Omega})$, we have that A^*u can be identified to an element of $\mathscr{M}(\overline{\Omega})$. Using classical elliptic regularity theory (see [Sta65a, Théorème 9.1]) we get that $u \in W^{1,s}_\diamond(\Omega)$ for any $s \in]1, d/(d-1)[$. The fact that (4.4.4) is a sufficient condition follows also by the convexity of the problem (see [BS00, Theorem 2.158]). □

As a corollary we immediately obtain the following existence result for a MFG type system with density constraints

Corollary 4.4.2. *There exists* $(m, u, \mu, p, \lambda) \in W^{1,q}(\Omega) \times W^{1,s}_{\diamond}(\Omega) \times \mathscr{M}_+(\overline{\Omega}) \times \mathscr{M}_+(\overline{\Omega}) \times \mathbb{R}$ ($s \in]1, d/(d-1)[$) *such that*

$$\begin{cases} -\Delta u + \frac{1}{q'}|\nabla u|^{q'} + \mu - p - \lambda &= f(x,m) \quad \text{in } \Omega, \\ \nabla u \cdot \mathbf{n} &= 0 \quad \text{on } \partial\Omega, \\ -\Delta m - \nabla \cdot \left(m |\nabla u|^{\frac{2-q}{q-1}} \nabla u \right) &= 0 \quad \text{in } \Omega, \\ \nabla m \cdot \mathbf{n} &= 0 \quad \text{on } \partial\Omega, \\ \int_\Omega m \, \mathrm{d}x = 1, & \quad 0 \le m \le 1, \ \text{in } \Omega, \\ \operatorname{spt}(\mu) \subseteq \{m = 0\}, & \quad \operatorname{spt}(p) \subseteq \{m = 1\}, \end{cases} \tag{MFG_q}$$

where the coupled system for (u, m) *is satisfied in the following weak sense: for all* $\varphi \in C^1(\overline{\Omega})$

$$\begin{gathered} \int_\Omega \nabla u \cdot \nabla \varphi \, \mathrm{d}x + \int_\Omega \frac{1}{q'} |\nabla u|^{q'} \varphi \, \mathrm{d}x - \lambda \int_\Omega \varphi \, \mathrm{d}x + \int_\Omega \varphi \, \mathrm{d}(\mu - p) = \int_\Omega f(x, m(x)) \varphi(x) \, \mathrm{d}x, \\ \int_\Omega \left(\nabla m + m |\nabla u|^{\frac{2-q}{q-1}} \nabla u \right) \cdot \nabla \varphi \, \mathrm{d}x = 0. \end{gathered} \tag{4.4.6}$$

Let us define $E_2^m := \{x \in \overline{\Omega} : 0 < m(x) < 1\}$. Note that by the continuity of m, E_2^m is an open set.

Remark 4.4.2 (The uniqueness of the solutions)**.** *Assuming that the coupling* f *is strictly increasing in its second variable, the objective functional in* (P_q) *becomes strictly convex in the* m *variable (and the set* $\mathcal{K}_P$ *is convex). Thus, the function* $m \in W^{1,q}(\Omega)$ *in* (MFG_q) *is unique, which implies also the uniqueness of* $\mathbf{w} \in L^q(\Omega)^d$*. In particular,* $\nabla u \in L^{q'}(\Omega)^d$ *is also unique on* E_1^m*. The first identity in* (4.4.6) *with* $\varphi \in C_c^1(E_2^m)$ *implies the uniqueness of* $\lambda \in \mathbb{R}$*. If* $\varphi \in C_c^1(E_1^m)$ *we obtain*

$$\int_\Omega \nabla u \cdot \nabla \varphi \, \mathrm{d}x + \int_\Omega \frac{1}{q'} |\nabla u|^{q'} \varphi \, \mathrm{d}x - \lambda \int_\Omega \varphi \, \mathrm{d}x - \int_\Omega \varphi \, \mathrm{d}p = \int_\Omega f(x, m(x)) \varphi(x) \, \mathrm{d}x,$$

which together with the condition $\operatorname{spt}(p) \subseteq \{m = 1\}$ *yields the uniqueness of* p*. Using [BP06, Theorem 3.4] we obtain that on* E_1^m $u \in W^{1,q'}(\Omega)$ *is unique on up to additive constants which may differ on each connected component of the set* E_1^m*. In general, we cannot expect uniqueness for* μ*.*

Now, let us discuss some interior regularity properties for the solutions on the open set E_2^m. Our approach is based on a bootstrapping argument.

Proposition 4.4.3. *There exists* $\gamma_0 \in]0, 1[$ *such that*

$$u \in C^{1,\gamma_0}_{\mathrm{loc}}(E_2^m) \quad \text{and} \quad m \in C^{1,\gamma_0}_{\mathrm{loc}}(E_2^m). \tag{4.4.7}$$

If in addition, $f \in C^{j,\gamma_1}_{\mathrm{loc}}(\overline{\Omega} \times \mathbb{R})$ *for* $j \in \{0, 1\}$ *and some* $\gamma_1 \in]0, 1[$*, we have that for some* $\gamma_2 \in]0, 1[$

$$u \in C^{2+j,\gamma_2}_{\mathrm{loc}}(E_2^m). \tag{4.4.8}$$

Proof. Step 1. We show that there exists $k > d$ such that $u \in W^{2,k}_{\mathrm{loc}}(E^m_2)$. By the classical Sobolev embeddings, this implies that $u \in C^{1,\gamma}_{\mathrm{loc}}(E^m_2)$ (for some $\gamma \in]0,1[$). Let $r_1 \in]q', d/(d-1)[$. Since $u \in W^{1,r_1}_\diamond(\Omega)$ we have that $|\nabla u|^{q'} \in L^{r_1/q'}(\Omega)$. The continuity of f and the density constraint on m imply that $f(\cdot, m(\cdot)) \in L^\infty(\Omega)$. Thus, denoting by $\delta_1 := r_1/q'$, classical regularity theory (see [GT83]) yields $u \in W^{2,\delta_1}_{\mathrm{loc}}(E^m_2)$. In particular, the Sobolev inequality (see e.g. [Ada75]) yields $u \in W^{1,\frac{d\delta_1}{d-\delta_1}}_{\mathrm{loc}}(E^m_2)$ and so $|\nabla u|^{q'} \in L^{\frac{d\delta_1}{q'(d-\delta_1)}}_{\mathrm{loc}}(E^m_2)$. We easily check that $\delta_2 := d\delta_1/q'(d-\delta_1) > \delta_1$ and so $u \in W^{2,\delta_2}_{\mathrm{loc}}(E^m_2)$. Let us define the sequence $\delta_{i+1} := \frac{d\delta_i}{(d-\delta_i)q'}$. Since $\delta_{i+1} - \delta_i \geq (q' + d - dq')/(d-\delta_i)q'$ and $q' + d - dq' > 0$, after a finite number of steps we get the existence of $i^* \geq 2$ such that $k := \delta_{i^*} > d$ and $u \in W^{2,k}_{\mathrm{loc}}(E^m_2)$.

Step 2. Let us prove that $m \in C^{1,\gamma_0}_{\mathrm{loc}}(E^m_2)$ for some $\gamma_0 \in]0,1[$. Since $m \in W^{1,q}(\Omega)$ and $q > d$, we already have that m is Hölder continuous. Having $u \in C^{1,\gamma}_{\mathrm{loc}}(E^m_2)$, this implies that $\nabla u \in C^{0,\gamma}_{\mathrm{loc}}(E^m_2)^d$, hence $m|\nabla u|^{\frac{2-q}{q-1}}\nabla u \in C^{0,\hat\gamma}_{\mathrm{loc}}(E^m_2)^d$, for some $\hat\gamma \in]0,1[$. Using a Schauder-type estimate (see [GM12, Theorem 5.19]) we get that $m \in C^{1,\gamma'}_{\mathrm{loc}}(E^m_2)$ for some $\gamma' \in]0,1[$.

Step 3. Using the above regularity for m, if $f \in C^{0,\gamma_1}_{\mathrm{loc}}(\overline{\Omega} \times \mathbb{R})$, the local Hölder regularity for $\frac{1}{q'}|\nabla u|^{q'}$ and [GT83, Corollary 6.9] imply that $u \in C^{2,\gamma''}_{\mathrm{loc}}(E^m_2)$ for some $\gamma'' \in]0,1[$. Finally, if $f \in C^{1,\gamma_1}_{\mathrm{loc}}(\overline{\Omega} \times \mathbb{R})$, the local Hölder regularity of ∇m and of D^2u imply that $u \in C^{3,\gamma'''}_{\mathrm{loc}}(E^m_2)$ for some $\gamma''' \in]0,1[$. □

Remark 4.4.3. *Let us discuss on the 'sharpness' of this regularity. Since the Hamiltonian is sub-quadratic, its Hessian has a singularity in 0. In general ∇u could vanish, which means that $D^2H(\nabla u) \cdot D^2u$ in general is not in $C^{0,\gamma}$ even if $u \in C^{2,\gamma}$. This regularity is crucial to go one step further in the regularity of m. By the coupling $f(\cdot, m)$ (and the singularity of D^2H) this prevents us to expect more regularity for u as well.*

4.4.1 *The dual problem*

In order to write explicitly the dual problem we will need the following Lemma concerning the Legendre-Fenchel transform of $\mathcal{F}$.

Lemma 4.4.4. *Let $\mathcal{F}$ be defined by (4.3.1). Then its Legendre-Fenchel transform $\mathcal{F}^* : \big(W^{1,q}(\Omega)\big)^* \to \overline{\mathbb{R}}$ is given by*

$$\mathcal{F}^*(m^*) = \begin{cases} \displaystyle\int_\Omega F^*(x, m^*(x))\,\mathrm{d}x, & \text{if } m^* \in \mathscr{M}^{ac}(\overline{\Omega}), \\ +\infty, & \text{otherwise,} \end{cases} \tag{4.4.9}$$

where F^* denotes the Legendre-Fenchel transform of F w.r.t. the second variable.

Proof. The result is a consequence of [Bré72, Section 2]. □

We recall that given a Banach space X and a convex closed set $K \subseteq X$, the *support function* $\sigma_K : X^* \to \overline{\mathbb{R}}$ is defined as

$$\sigma_K(x^*) := \sup_{x \in K} \langle x^*, x \rangle_{X^*,X} \qquad \forall x^* \in X^*.$$

Proposition 4.4.5. *The dual problem of* (P_q) *(in the sense of convex analysis) has at least one solution and can be written as*

$$-\min_{(u,p,\lambda,a)\in\mathcal{K}_D} \left\{ \int_\Omega F^*(x,a)\,\mathrm{d}x + \lambda + p(\overline{\Omega}) \right\} \tag{PD_q}$$

where

$$\mathcal{K}_D := \left\{ (u,p,\lambda,a) \in W^{1,q'}_\diamond(\Omega) \times \mathscr{M}_+(\overline{\Omega}) \times \mathbb{R} \times \mathscr{M}^{ac}(\overline{\Omega}) : \right.$$
$$\left. A^*u + \frac{1}{q'}|B^*u|^{q'} - p - \lambda \leq a \right\}$$

where the inequality has to be understood in the sense of measures.

Proof. The dual problem of (P_q) can be written as

$$\max_{\substack{(u,p,\lambda)\in \\ W^{1,q'}_\diamond(\Omega)\times\mathscr{M}(\overline{\Omega})\times\mathbb{R}}} \left\{ \inf_{\substack{(m,\mathbf{w})\in \\ W^{1,q}(\Omega)\times L^q(\Omega)^d}} \mathfrak{L}(m,\mathbf{w},u,p,\lambda) - \sigma_{\mathcal{K}}(u,\lambda,p) \right\}, \tag{4.4.10}$$

where $\mathfrak{L}$ is defined in (4.4.1) and we recall that $\mathcal{K} := \{0\} \times \{0\} \times \mathcal{C}$. The fact that we have a max instead of a sup in (4.4.10) is justified by Lemma 4.3.2 and [BS00, Theorem 2.165]. Now, note that

$$\sigma_{\mathcal{K}}(u,\lambda,p) = \sigma_{\mathcal{C}}(p) = \begin{cases} p(\overline{\Omega}) & \text{if } p \in \mathscr{M}_+(\overline{\Omega}), \\ +\infty & \text{otherwise.} \end{cases} \tag{4.4.11}$$

On the other hand, we have that

$$\begin{aligned} \inf_{(m,\mathbf{w})} \mathfrak{L}(m,\mathbf{w},u,p,\lambda) &= -\sup_{(m,\mathbf{w})} -\mathfrak{L}(m,\mathbf{w},u,p,\lambda) \\ &= -\sup_{(m,\mathbf{w})} \left\{ \langle A^*u - p - \lambda, m\rangle + \int_\Omega B^*u\cdot\mathbf{w}\,\mathrm{d}x - \mathcal{J}_q(m,\mathbf{w}) \right\} \\ &\quad - \lambda \\ &= -\mathcal{J}_q^*(A^*u - p - \lambda, B^*u) - \lambda. \end{aligned}$$

Since there exists $(m, \mathbf{w}) \in \mathrm{dom}(\mathcal{B}_q)$ at which $\mathcal{F}$ is continuous (take for example $(m, \mathbf{w}) = (1/|\Omega|, 0)$), for any $(\alpha, \beta) \in \left(W^{1,q}(\Omega)\right)^* \times L^{q'}(\Omega)^d$ we have that (see e.g. [ABM06, Theorem 9.4.1])

$$\begin{aligned} \mathcal{J}_q^*(\alpha, \beta) &= \left(\mathcal{B}_q + \mathcal{F}\right)^*(\alpha, \beta), \\ &= \inf_{a \in \left(W^{1,q}(\Omega)\right)^*} \left\{ \mathcal{B}_q^*(\alpha - a, \beta) + \mathcal{F}^*(a) \right\}, \\ &= \inf_{a \in \left(W^{1,q}(\Omega)\right)^*} \left\{ \chi_{\overline{\mathcal{A}_{q'}}}(\alpha - a, \beta) + \mathcal{F}^*(a) \right\}, \\ &= \inf_{a \in \mathscr{M}^{ac}(\overline{\Omega})} \left\{ \int_\Omega F^*(x, a(x))\, \mathrm{d}x : \; \alpha + \frac{1}{q'}|\beta|^{q'} \leq a \right\}, \end{aligned} \tag{4.4.12}$$

where we have used Lemma 4.2.1 and Lemma 4.4.4. Let us prove that the above minimization problem has a solution. First, by (4.4.9) the integral functional is l.s.c. with respect to the weak$-\star$ topology of measures. Let us take a minimizing sequence $a_n \in L^1(\Omega)$. There exists a constant $C > 0$ such that

$$C \geq \int_\Omega F^*(x, a_n(x))\, \mathrm{d}x \geq \int_\Omega \left[a_n(x) y(x) - F(x, y(x))\right] \mathrm{d}x, \quad \forall y \in L^\infty(\Omega).$$

By choosing $y(x) = \mathrm{sgn}(a_n(x))$ (which is equal to 1 if $a_n(x) \geq 0$ and -1 if not), we obtain that a_n is bounded in $L^1(\Omega)$. Therefore, when the sequence a_n is identified to a sequence of measures, we get a weakly$-\star$ convergent subsequence to some $a \in \mathscr{M}(\overline{\Omega})$. The constraint is convex and closed with respect to this convergence, so by the lower semicontinuity of the objective functional we have that a is a solution and, by Lemma 4.4.4, $a \in \mathscr{M}^{ac}(\overline{\Omega})$ as well. Using this result and (4.4.10), (4.4.11) and (4.4.12), the conclusion follows. □

Using the dual problem, let us provide an alternative, but related way, to obtain first order necessary and sufficient optimality conditions. By Theorem 4.3.1 and Proposition 4.4.5 we know that that there exist $(m, \mathbf{w}) \in \mathcal{K}_P$ and $(u, p, \lambda, a) \in \mathcal{K}_D$ optimizers for (P_q) and (PD_q) respectively. Moreover, since Lemma 4.3.2 implies that Problem (P_q) is qualified, by [BS00, Theorem 2.165] Problem (PD_q) has the same value as Problem (P_q). Therefore,

$$\begin{gathered} \mathcal{B}_q(m, \mathbf{w}) + \mathcal{F}(m) = -\int_\Omega F^*(\cdot, a)\, \mathrm{d}x - \lambda - \sigma_{\mathcal{C}}(p), \\ Am + B\mathbf{w} = 0, \quad m \leq 1, \quad \int_\Omega m\, \mathrm{d}x = 1, \quad A^*u - p - \lambda - a + \frac{1}{q'}|B^*u|^{q'} \leq 0. \end{gathered} \tag{4.4.13}$$

Using the above relations, we obtain

$$
\begin{aligned}
0 &= \mathcal{F}(m) + \int_\Omega F^*(\cdot, a)\,\mathrm{d}x + \mathcal{B}_q(m, \mathbf{w}) + \chi_{\overline{\mathcal{A}_{q'}}}(A^*u - p - \lambda - a, B^*u) \\
&+ \lambda + \sigma_{\mathcal{C}}(p), \\
&= \mathcal{F}(m) + \int_\Omega F^*(\cdot, a)\,\mathrm{d}x + \mathcal{B}_q(m, \mathbf{w}) + \mathcal{B}_q^*(A^*u - p - \lambda - a, B^*u) \\
&+ \lambda + \sigma_{\mathcal{C}}(p), \\
&\geq \langle a, m \rangle_{\mathscr{M}(\overline{\Omega}), C(\overline{\Omega})} + \mathcal{B}_q(m, \mathbf{w}) + \mathcal{B}_q^*(A^*u - p - \lambda - a, B^*u) \\
&+ \lambda \int_\Omega m\,\mathrm{d}x + \sigma_{\mathcal{C}}(p), \\
&\geq \langle a, m \rangle_{\mathscr{M}(\overline{\Omega}), C(\overline{\Omega})} + \langle A^*u - p - \lambda - a, m \rangle_{\mathscr{M}(\overline{\Omega}), C(\overline{\Omega})} + \int_\Omega B^*u \cdot \mathbf{w}\,\mathrm{d}x \\
&+ \lambda \int_\Omega m\,\mathrm{d}x + \sigma_{\mathcal{C}}(p), \\
&\geq \langle A^*u - p, m \rangle_{\mathscr{M}(\overline{\Omega}), C(\overline{\Omega})} + \int_\Omega B^*u \cdot \mathbf{w}\,\mathrm{d}x + \langle p, m \rangle_{\mathscr{M}(\overline{\Omega}), C(\overline{\Omega})}, \\
&= \langle A^*u, m \rangle + \langle B^*u, \mathbf{w} \rangle_{q', q}, \\
&= \langle\!\langle Am, u \rangle\!\rangle + \langle\!\langle B\mathbf{w}, u \rangle\!\rangle = 0.
\end{aligned}
$$

This means that all the inequalities in the previous list are actually equalities. Thus,

(i) $\mathcal{F}(m) + \mathcal{F}^*(a) = \langle a, m \rangle$ and so, using the fact that $\mathcal{F}$ is differentiable on $W^{1,q}(\Omega)$, we have $a = f(\cdot, m)$.

(ii) $\mathcal{B}_q(m, \mathbf{w}) + \mathcal{B}_q^*(A^*u - p - \lambda - a, B^*u) = \langle A^*u - p - \lambda - a, m \rangle + \langle B^*u, \mathbf{w} \rangle_{q', q}$, namely

$$(A^*u - p - \lambda - a, B^*u) \in \partial \mathcal{B}_q(m, \mathbf{w})$$

(iii) $\sigma_{\mathcal{C}}(p) = \langle p, m \rangle_{\mathscr{M}(\overline{\Omega}), C(\overline{\Omega})}$, which implies that $p \in N_{\mathcal{C}}(m)$.

Using (4.4.3), (4.4.13) and (i)-(iii) we recover system (4.4.4).

4.5 TREATING LESS REGULAR CASES VIA AN APPROXIMATION ARGUMENT

In this section we provide the proof of the existence of a solution of a suitable form of (MFG_q) when $1 < q \leq d$. Note that given $\mathbf{w} \in L^q(\Omega)^d$ the solution m of (4.3.2) is in general discontinuous. Because of the constraint $0 \leq m \leq 1$, this implies that problem (P_q) is in general not qualified (see [BS00, Chapter 2]) and thus the arguments in the previous section are no longer valid. To handle this issue, we propose an approach which is based on a regularization argument.

Let us fix $1 < q \leq d$ and $r > d$. For $\varepsilon > 0$ define $\mathcal{J}_{q,\varepsilon} : W^{1,r}(\Omega) \times L^r(\Omega)^d \to \overline{\mathbb{R}}$ as

$$\mathcal{J}_{q,\varepsilon}(m, \mathbf{w}) := \mathcal{J}_q(m, \mathbf{w}) + \varepsilon \mathcal{B}_r(m, \mathbf{w}).$$

Following the arguments in the proof of Theorem 4.3.1, problem

$$\inf_{(m,\mathbf{w})\in\mathcal{K}_P} \mathcal{J}_{q,\varepsilon}(m,\mathbf{w}) \tag{$P_{q,\varepsilon}$}$$

admits at least one solution $(m_\varepsilon, \mathbf{w}_\varepsilon)$. Since $m_\varepsilon \in C(\overline{\Omega})$, Problem ($P_{q,\varepsilon}$) is qualified. Moreover, since both $\mathcal{B}_q$ and $\mathcal{B}_r$ are continuous at $(\hat{m}, \hat{\mathbf{w}}) := (1/|\Omega|, 0)$, by [ET76, Chapter 1, Proposition 5.6] we have that

$$\partial(\mathcal{B}_q(m,\mathbf{w}) + \varepsilon\mathcal{B}_r(m,\mathbf{w})) = \partial\mathcal{B}_q(m,\mathbf{w}) + \varepsilon\partial\mathcal{B}_r(m,\mathbf{w})$$

for all $(m,\mathbf{w}) \in W^{1,r}(\Omega) \times L^r(\Omega)^d$. Therefore, exactly as in the proof of Theorem 4.4.1, if we define $\mathbf{v}_\varepsilon := (\mathbf{w}_\varepsilon/m_\varepsilon)\mathbb{1}_{E_1^{m_\varepsilon}}$ we have that $\mathbf{v}_\varepsilon \in L^r(\Omega)^d$ and there exist $(u_\varepsilon, p_\varepsilon, \lambda_\varepsilon) \in W^{1,s}_\diamond(\Omega) \times \mathscr{M}_+(\overline{\Omega}) \times \mathbb{R}$ ($s \in]1, d/(d-1)[$), $(\alpha_{\varepsilon,q}, \beta_{\varepsilon,q}) \in \partial\mathcal{B}_q(m_\varepsilon, \mathbf{w}_\varepsilon)$ and $(\alpha_{\varepsilon,r}, \beta_{\varepsilon,r}) \in \partial\mathcal{B}_r(m_\varepsilon, \mathbf{w}_\varepsilon)$ such that

$$\begin{cases} \alpha_{\varepsilon,q} + \varepsilon\alpha_{\varepsilon,r} - A^* u_\varepsilon + f(x,m_\varepsilon) + p_\varepsilon + \lambda_\varepsilon & = & 0, \\ \beta_{\varepsilon,q} + \varepsilon\beta_{\varepsilon,r} & = & B^* u_\varepsilon, \\ Am_\varepsilon + B\mathbf{w}_\varepsilon & = & 0, \\ \int_\Omega m_\varepsilon \, dx = 1, \quad 0 \le m_\varepsilon \le 1, & & \operatorname{spt}(p_\varepsilon) \subseteq \{m_\varepsilon = 1\}. \end{cases} \tag{4.5.1}$$

Now, for $\varepsilon \ge 0$, let us define $F_{q,\varepsilon}, G_{q,\varepsilon}, H_{q,\varepsilon} : \mathbb{R}^d \to \mathbb{R}$ as

$$F_{q,\varepsilon}(z) := \frac{1}{q}|z|^q + \frac{\varepsilon}{r}|z|^r, \; G_{q,\varepsilon}(z) := \frac{1}{q'}|z|^q + \frac{\varepsilon}{r'}|z|^r, \text{ and } H_{q,\varepsilon}(z) := G_{q,\varepsilon}(\nabla F^*_{q,\varepsilon}(z)).$$

For notational convenience, we set and $H_q := H_{q,0}$. Elementary arguments in convex analysis show that $H_{q,\varepsilon} \to H_q$ uniformly over compact sets. System (4.5.1) can be written in the following alternative form:

Proposition 4.5.1. *There exists $\tilde{\alpha}_\varepsilon \in \mathscr{M}_-(\overline{\Omega})$ such that*

$$\begin{cases} -\Delta u_\varepsilon + H_{q,\varepsilon}(-\nabla u_\varepsilon) - p_\varepsilon - \tilde{\alpha}_\varepsilon - \lambda_\varepsilon & = & f(x,m_\varepsilon), \text{ in } \Omega, \\ -\Delta m_\varepsilon + \nabla\cdot\left(m_\varepsilon \nabla F^*_{q,\varepsilon}(-\nabla u_\varepsilon)\right) & = & 0, \text{ in } \Omega, \\ \nabla m_\varepsilon \cdot \mathbf{n} = 0 & & \nabla u_\varepsilon \cdot \mathbf{n} = 0, \text{ on } \partial\Omega. \\ 0 \le m_\varepsilon \le 1, & & \int_\Omega m_\varepsilon \, dx = 1, \\ \operatorname{spt}(p_\varepsilon) \subseteq \{m_\varepsilon = 1\}, & & \operatorname{spt}(\tilde{\alpha}_\varepsilon) \subseteq \{m_\varepsilon = 0\}. \end{cases} \tag{$MFG_{q,\varepsilon}$}$$

Proof. By Theorem 4.2.2 we have that

$$\alpha_{\varepsilon,q} \llcorner E_1^{m_\varepsilon} = -\frac{1}{q'}|\mathbf{v}_\varepsilon|^q, \quad \alpha_{\varepsilon,r} \llcorner E_1^{m_\varepsilon} = -\frac{1}{r'}|\mathbf{v}_\varepsilon|^r,$$

$$\beta_{\varepsilon,q} \llcorner E_1^{m_\varepsilon} = |\mathbf{v}_\varepsilon|^{q-2}\mathbf{v}_\varepsilon, \quad \beta_{\varepsilon,r} \llcorner E_1^{m_\varepsilon} = |\mathbf{v}_\varepsilon|^{r-2}\mathbf{v}_\varepsilon.$$

On the other hand, since $\nabla u_\varepsilon \in L^{r'}(\Omega)^d$ we have that $\overline{\mathbf{v}}_\varepsilon := \nabla F^*_{q,\varepsilon}(-\nabla u_\varepsilon) \in L^r(\Omega)^d$. Using that $\nabla F_{q,\varepsilon}(\mathbf{v}_\varepsilon) = \beta_{\varepsilon,q} + \varepsilon\beta_{\varepsilon,r} = -\nabla u_\varepsilon$ in $E_1^{m_\varepsilon}$ and that $\nabla F^{-1}_{q,\varepsilon} = \nabla F^*_{q,\varepsilon}$, we get that $\overline{\mathbf{v}}_\varepsilon = \mathbf{v}_\varepsilon$ in $E_1^{m_\varepsilon}$. Therefore, there exists $\xi_\varepsilon \in L^{q'}(\Omega)^d$ such that $\operatorname{spt}(\xi_\varepsilon) \subseteq E_0^{m_\varepsilon}$ and a.e. in Ω

$$\beta_{\varepsilon,q} = |\overline{\mathbf{v}}_\varepsilon|^{q-2}\overline{\mathbf{v}}_\varepsilon + \xi_\varepsilon \quad \text{and} \quad \beta_{\varepsilon,r} = \frac{1}{\varepsilon}(\nabla F_{q,\varepsilon}(\overline{\mathbf{v}}_\varepsilon) - \beta_{\varepsilon,q}) = |\overline{\mathbf{v}}_\varepsilon|^{r-2}\overline{\mathbf{v}}_\varepsilon - (1/\varepsilon)\xi_\varepsilon.$$

Using the convexity of $\frac{1}{q'}|\cdot|^{q'}$ and $\frac{1}{r'}|\cdot|^{r'}$, we easily check that

$$\frac{1}{q'}|\beta_{\varepsilon,q}|^{q'} \geq \frac{1}{q'}|\overline{\mathbf{v}}_\varepsilon|^q + \overline{\mathbf{v}}_\varepsilon \cdot \xi_\varepsilon \;\text{ and }\; \frac{\varepsilon}{r'}|\beta_{\varepsilon,r}|^{r'} \geq \frac{\varepsilon}{r'}|\overline{\mathbf{v}}_\varepsilon|^r - \overline{\mathbf{v}}_\varepsilon \cdot \xi_\varepsilon.$$

Hence

$$-\frac{1}{q'}|\beta_{\varepsilon,q}|^{q'} - \frac{\varepsilon}{r'}|\beta_{\varepsilon,r}|^{r'} \leq -\frac{1}{q'}|\overline{\mathbf{v}}_\varepsilon|^q - \frac{\varepsilon}{r'}|\overline{\mathbf{v}}_\varepsilon|^r = -H_{q,\varepsilon}(-\nabla u_\varepsilon),$$

with an equality a.e. in $E_1^{m_\varepsilon}$. In particular, we have the existence of a positive measure γ_ε such that $\operatorname{spt}(\gamma_\varepsilon) \subseteq \operatorname{spt}(\xi_\varepsilon) \subseteq E_0^{m_\varepsilon}$ and

$$-\frac{1}{q'}|\beta_{\varepsilon,q}|^{q'} - \frac{\varepsilon}{r'}|\beta_{\varepsilon,r}|^{r'} = -H_{q,\varepsilon}(-\nabla u_\varepsilon) - \gamma_\varepsilon.$$

Since the definition of $(\alpha_{\varepsilon,q}, \beta_{\varepsilon,q})$ and $(\alpha_{\varepsilon,r}, \beta_{\varepsilon,r})$ implies the existence of two positive measures $\tilde{\alpha}_{\varepsilon,q}$ and $\tilde{\alpha}_{\varepsilon,r}$ such that $\operatorname{spt}(\tilde{\alpha}_{\varepsilon,q}) \subseteq E_0^{m_\varepsilon}$, $\operatorname{spt}(\tilde{\alpha}_{\varepsilon,r}) \subseteq E_0^{m_\varepsilon}$ and

$$\alpha_{\varepsilon,q} = -\frac{1}{q'}|\beta_{\varepsilon,q}|^{q'} - \tilde{\alpha}_{\varepsilon,q} \quad \text{and} \quad \alpha_{\varepsilon,r} = -\frac{1}{r'}|\beta_{\varepsilon,r}|^{r'} - \tilde{\alpha}_{\varepsilon,r},$$

the result follows by setting $\tilde{\alpha}_\varepsilon := -\tilde{\alpha}_{\varepsilon,q} - \varepsilon\tilde{\alpha}_{\varepsilon,r} - \gamma_\varepsilon$. □

Now we present the main theorem of this section.

Theorem 4.5.2. *There exists* $(m, u, p, \mu, \lambda) \in W^{1,q}(\Omega) \times W^{1,q'}_\diamond(\Omega) \times \mathscr{M}_+(\overline{\Omega}) \times \mathscr{M}_+(\overline{\Omega}) \times \mathbb{R}$ *such that*

$$\begin{cases} -\Delta u + \frac{1}{q'}|\nabla u|^{q'} + \mu - p - \lambda &= f(x,m) \quad \text{in } \Omega, \\ \nabla u \cdot \mathbf{n} &= 0 \quad \text{on } \partial\Omega, \\ -\Delta m - \nabla \cdot \left(m|\nabla u|^{\frac{2-q}{q-1}}\nabla u \right) &= 0 \quad \text{in } \Omega, \\ \nabla m \cdot \mathbf{n} &= 0 \quad \text{on } \partial\Omega, \\ \int_\Omega m \,\mathrm{d}x = 1, & \quad 0 \leq m \leq 1, \;\text{ in } \Omega, \end{cases} \tag{MFG_q}$$

where the coupled system for (u, m) *is satisfied in the weak sense (see (4.4.6)). Moreover, defining*

$$\langle \mu - p, m \rangle := \lambda + \int_\Omega \left[f(x,m) - \frac{1}{q'}|\nabla u|^{q'} \right] m \,\mathrm{d}x - \int_\Omega \nabla m \cdot \nabla u \,\mathrm{d}x \tag{4.5.2}$$

we have the inequality

$$\int_\Omega \mathrm{d}p + \langle \mu - p, m\rangle \le 0. \tag{4.5.3}$$

Proof. Step 1: Bounds for λ_ε, p_ε and $\tilde\alpha_\varepsilon$. Note first that the second equation in (4.5.1) and Theorem 4.2.2 imply that $\mathbf{w}_\varepsilon = m_\varepsilon \mathbf{v}_\varepsilon$ a.e. in Ω. Also, in the set $E_1^{m_\varepsilon}$ we have that $\nabla F_{q,\varepsilon}(\mathbf{v}_\varepsilon) = -\nabla u_\varepsilon$ and so in $E_1^{m_\varepsilon}$ the identities $\mathbf{v}_\varepsilon = \nabla F^*_{q,\varepsilon}(-\nabla u_\varepsilon)$ and $H_{q,\varepsilon}(-\nabla u_\varepsilon) = G_{q,\varepsilon}(\mathbf{v}_\varepsilon)$ hold true. Now, by the second and third equations in (4.5.1) we get that

$$\begin{aligned}\int_\Omega \nabla u_\varepsilon \cdot \nabla m_\varepsilon \,\mathrm{d}x &= \int_\Omega \nabla u_\varepsilon \cdot \mathbf{w}_\varepsilon \,\mathrm{d}x \\ &= -\int_\Omega m_\varepsilon \nabla F_{q,\varepsilon}(\mathbf{v}_\varepsilon)\cdot \mathbf{v}_\varepsilon \,\mathrm{d}x = -\int_\Omega m_\varepsilon(|\mathbf{v}_\varepsilon|^q + \varepsilon|\mathbf{v}_\varepsilon|^r)\,\mathrm{d}x.\end{aligned} \tag{4.5.4}$$

and so taking m_ε as test function in the first equation of (4.5.1), we obtain

$$\begin{aligned}\lambda_\varepsilon + \int_\Omega m_\varepsilon \,\mathrm{d}p_\varepsilon &= \int_\Omega \left(G_{q,\varepsilon}(\mathbf{v}_\varepsilon)m_\varepsilon + \nabla u_\varepsilon \cdot \nabla m_\varepsilon - f(x,m_\varepsilon)m_\varepsilon\right)\,\mathrm{d}x \\ &= \int_\Omega \left(-\frac{1}{q}|\mathbf{v}_\varepsilon|^q m_\varepsilon - \varepsilon\frac{1}{r}|\mathbf{v}_\varepsilon|^r m_\varepsilon - f(x,m_\varepsilon)m_\varepsilon\right)\,\mathrm{d}x,\end{aligned}$$

which implies that

$$\lambda_\varepsilon + \int_\Omega m_\varepsilon \,\mathrm{d}p_\varepsilon = -\mathcal{B}_q(m_\varepsilon,\mathbf{w}_\varepsilon) - \varepsilon\mathcal{B}_r(m_\varepsilon,\mathbf{w}_\varepsilon) - \int_\Omega f(x,m_\varepsilon)m_\varepsilon\,\mathrm{d}x. \tag{4.5.5}$$

The optimality of $(m_\varepsilon, \mathbf{w}_\varepsilon)$ yields

$$0 \le \mathcal{B}_q(m_\varepsilon,\mathbf{w}_\varepsilon) + \varepsilon\mathcal{B}_r(m_\varepsilon,\mathbf{w}_\varepsilon) \le \mathcal{J}_{q,\varepsilon}(1/|\Omega|,0) - \mathcal{F}(m_\varepsilon). \tag{4.5.6}$$

Thus, since f is continuous, $0 \le m_\varepsilon \le 1$, (4.5.5) and the fact that $\mathrm{spt}(p_\varepsilon) \subseteq \{m_\varepsilon = 1\}$ yield the existence of a constant $c_1 > 0$ (independent of ε) such that

$$-c_1 \le \lambda_\varepsilon + \int_\Omega m_\varepsilon\,\mathrm{d}p_\varepsilon = \lambda_\varepsilon + |p_\varepsilon| \le c_1. \tag{4.5.7}$$

On the other hand, by taking $1 - m_\varepsilon$ as test function in the first equation of (4.5.1), a similar computation using (4.5.4) yields

$$\begin{aligned}(|\Omega|-1)\lambda_\varepsilon - |\tilde\alpha_\varepsilon| &= \int_\Omega H_{q,\varepsilon}(-\nabla u_\varepsilon)\,\mathrm{d}x + \mathcal{B}_q(m_\varepsilon,\mathbf{w}_\varepsilon) + \varepsilon\mathcal{B}_r(m_\varepsilon,\mathbf{w}_\varepsilon) \\ &\quad - \int_\Omega f(x,m_\varepsilon)(1-m_\varepsilon)\,\mathrm{d}x,\end{aligned} \tag{4.5.8}$$

from which

$$(|\Omega|-1)\lambda_\varepsilon - |\tilde\alpha_\varepsilon| \ge -\int_\Omega f(x,m_\varepsilon)(1-m_\varepsilon)\,\mathrm{d}x \ge c_2, \tag{4.5.9}$$

where $c_2 > 0$ is independent of ε. Since $|\Omega| > 1$, inequalities (4.5.7)-(4.5.9) imply that λ_ε is uniformly bounded w.r.t. ε and so p_ε and $\tilde{\alpha}_\varepsilon$ are uniformly bounded w.r.t. ε in $\mathscr{M}(\overline{\Omega})$.

Step 2: Convergence of ∇u_ε and m_ε. By (4.5.8), as a function of ε we have that $H_{q,\varepsilon}(-\nabla u_\varepsilon)$ is uniformly bounded in $L^1(\Omega)$ which implies that u_ε is bounded in $W^{1,q'}(\Omega)$ and that $-\Delta u_\varepsilon$ is bounded in $\mathscr{M}(\overline{\Omega})$. On the one hand, the boundedness of u_ε in $W^{1,q'}(\Omega)$ implies the existence of $u \in W^{1,q'}(\Omega)$ such that up to some subsequence u_ε converges weakly to u in $W^{1,q'}(\Omega)$. In particular, $\int_\Omega u \, dx = 0$. On the other hand, the boundedness of $-\Delta u_\varepsilon$ in $\mathscr{M}(\overline{\Omega})$ and [Min07, Theorem 1.3 with $p = 2$] (for the precise result see Appendix A) imply the existence of $s \in]0, 1[$ and $\delta_0 > 0$ such that ∇u_ε is uniformly bounded in $W^{s,1+\delta_0}_{\text{loc}}(\Omega)^d$. By [DNPV12, Corollary 7.2] we can extract a subsequence such that $\nabla u_\varepsilon \to \nabla u$ a.e. in Ω and so $H_{q,\varepsilon}(-\nabla u_\varepsilon) \to \frac{1}{q'}|\nabla u|^{q'}$ a.e. in Ω.

Now, in order to establish the convergence for m_ε, note that inequality (4.5.6) and the fact that $0 \le m_\varepsilon \le 1$ imply that $\mathbf{w}_\varepsilon$ is uniformly bounded in $L^q(\Omega)^d$ for all $\varepsilon > 0$. This means that, up to some subsequence, $\mathbf{w}_\varepsilon$ is converging weakly in $L^q(\Omega)^d$. Since Lemma A.3.2 implies that $\|\nabla m_\varepsilon\|_{L^q} \le c\|\mathbf{w}_\varepsilon\|_{L^q}$ (for a constant $c > 0$ independent of ε), by Poincaré's inequality we get that m_ε is uniformly bounded in $W^{1,q}(\Omega)$. Extracting a subsequence again, there exists m such that m_ε converges weakly to m in $W^{1,q}(\Omega)$. By the compact Sobolev embedding, we get strong convergence in $L^q(\Omega)$, which implies that $0 \le m \le 1$ a.e. in Ω and $\int_\Omega m \; dx = 1$.

Step 3: The limit equations. The weak formulation of the second equation in ($MFG_{q,\varepsilon}$) yields

$$\int_\Omega \nabla m_\varepsilon \cdot \nabla \varphi \, dx = -\int_\Omega m_\varepsilon \nabla F^*_{q,\varepsilon}(-\nabla u_\varepsilon) \cdot \nabla \varphi \, dx, \quad \text{for all } \varphi \in C^\infty(\overline{\Omega}).$$

Since, extracting a subsequence, $\mathbf{w}_\varepsilon = m_\varepsilon \nabla F^*_\varepsilon(-\nabla u_\varepsilon)$ converges weakly in $L^q(\Omega)^d$ to some $\mathbf{w}$, the weak convergence of m_ε in $W^{1,q}(\Omega)$ implies that

$$\int_\Omega \nabla m \cdot \nabla \varphi \, dx = \int_\Omega \mathbf{w} \cdot \nabla \varphi \, dx, \quad \text{for all } \varphi \in C^\infty(\overline{\Omega}).$$

Moreover, extracting a subsequence again, we get that

$$m_\varepsilon(x) \nabla F^*_{q,\varepsilon}(-\nabla u_\varepsilon(x)) \to -m(x)|\nabla u(x)|^{\frac{2-q}{q-1}} \nabla u(x) \quad \text{for almost every } x \in \Omega.$$

The latter equality and Egorov's theorem imply that $\mathbf{w} = -m|\nabla u|^{\frac{2-q}{q-1}} \nabla u$ from which the second equation in (MFG_q) follows.

On the other hand, the weak formulation of the first equation in ($MFG_{q,\varepsilon}$) reads

$$\begin{aligned}\int_\Omega \nabla u_\varepsilon \cdot \nabla \varphi \, dx \quad &+ \int_\Omega H_{q,\varepsilon}(-\nabla u_\varepsilon) \varphi \, dx - \lambda_\varepsilon \int_\Omega \varphi \, dx \\ &- \int_\Omega \varphi \, d(p_\varepsilon + \tilde{\alpha}_\varepsilon) = \int_\Omega f(x, m_\varepsilon(x)) \varphi(x) \, dx,\end{aligned} \tag{4.5.10}$$

for any test function $\varphi \in C^1(\overline{\Omega})$. The continuity of f and the dominated convergence theorem imply that

$$\lim_{\varepsilon\to 0}\int_\Omega f(x,m_\varepsilon(x))\varphi(x)\,dx = \int_\Omega f(x,m(x))\varphi(x)\,dx \quad \text{for all } \varphi \in C(\overline{\Omega}).$$

The previous steps imply that we only need to study the limit behavior of the second term in (4.5.10). Since $H_{q,\varepsilon}(-\nabla u_\varepsilon)$ is bounded in $L^1(\Omega)$, there exists $\gamma \in \mathscr{M}(\overline{\Omega})$ such that, extracting a subsequence, for all $\varphi \in C(\overline{\Omega})$, $\int_\Omega H_{q,\varepsilon}(-\nabla u_\varepsilon)\varphi\,dx \to \int_\Omega \varphi\,d\gamma$. Fatou's lemma implies that

$$\int_\Omega \frac{1}{q'}|\nabla u|^{q'}\varphi\,dx \le \liminf_{\varepsilon\to 0}\int_\Omega H_{q,\varepsilon}(-\nabla u_\varepsilon)\varphi\,dx = \int_\Omega \varphi\,d\gamma \quad \forall\ \varphi \in C(\overline{\Omega}),\ \ \varphi \ge 0.$$

Defining, $\rho \in \mathscr{M}_+(\overline{\Omega})$ as $d\rho := d\gamma - \frac{1}{q'}|\nabla u|^{q'}\,dx$, we obtain that

$$\int_\Omega \varphi\,d\rho + \int_\Omega \frac{1}{q'}|\nabla u|^{q'}\varphi\,dx = \lim_{\varepsilon\to 0}\int_\Omega H_{q,\varepsilon}(-\nabla u_\varepsilon)\varphi\,dx \quad \text{for all } \varphi \in C(\overline{\Omega}). \tag{4.5.11}$$

Thus passing to the limit in (4.5.10) as $\varepsilon \to 0$ we get

$$\int_\Omega \nabla u\cdot\nabla\varphi\,dx + \int_\Omega \frac{1}{q'}|\nabla u|^{q'}\varphi\,dx - \lambda\int_\Omega \varphi\,dx - \int_\Omega \varphi\,d(p+\tilde{\alpha}-\rho) = \int_\Omega f(x,m(x))\varphi(x)\,dx.$$

Setting, $\mu := \rho - \tilde{\alpha} \in \mathscr{M}_+(\overline{\Omega})$ we obtain the weak form of the first equation in (MFG_q).

Step 4: Proof of (4.5.3). By ($MFG_{q,\varepsilon}$) and (4.5.4) we have

$$\begin{aligned}
0 &= \int_\Omega (1-m_\varepsilon)\,dp_\varepsilon - \int_\Omega m_\varepsilon\,d\tilde{\alpha}_\varepsilon = \int_\Omega dp_\varepsilon + \int_\Omega m_\varepsilon\,d(-\tilde{\alpha}_\varepsilon - p_\varepsilon),\\
&= \int_\Omega dp_\varepsilon + \int_\Omega \left[\lambda_\varepsilon + f(x,m_\varepsilon) - H_{q,\varepsilon}(-\nabla u_\varepsilon)\right] m_\varepsilon\,dx - \int_\Omega \nabla u_\varepsilon\cdot\nabla m_\varepsilon\,dx,\\
&= \int_\Omega dp_\varepsilon + \int_\Omega \left[\frac{1}{q}|\mathbf{v}_\varepsilon|^q + \frac{\varepsilon}{r}|\mathbf{v}_\varepsilon|^r + \lambda_\varepsilon + f(x,m_\varepsilon)\right] m_\varepsilon\,dx,\\
&\ge \int_\Omega dp_\varepsilon + \int_\Omega \left[\frac{1}{q}|\mathbf{v}_\varepsilon|^q + \lambda_\varepsilon + f(x,m_\varepsilon)\right] m_\varepsilon\,dx.
\end{aligned}$$

By Fatou's lemma we have

$$\int_\Omega \frac{1}{q}|\nabla u|^{q'} m\,dx = \int_\Omega \frac{1}{q}|\nabla u|^{q\left(\frac{2-q}{q-1}+1\right)} m\ dx \le \liminf_{\varepsilon\to 0}\int_\Omega \frac{1}{q}|\mathbf{v}_\varepsilon|^q m_\varepsilon\,dx.$$

Thus, letting $\varepsilon \to 0$ and using (4.5.2), we get

$$\begin{aligned}
0 &\ge \int_\Omega dp + \int_\Omega \left[\frac{1}{q}|\nabla u|^{q'} + \lambda + f(x,m)\right] m\,dx\\
&= \int_\Omega dp + \int_\Omega \nabla u\cdot\nabla m\,dx + \int_\Omega |\nabla u|^{q'} m\,dx + \langle \mu - p, m\rangle.
\end{aligned}$$

By taking $u \in W^{1,q'}_{\diamond}(\Omega)$ as test function in the second equation of (MFG_q) we obtain that

$$\int_{\Omega}\left[\nabla m\cdot\nabla u + m|\nabla u|^{\frac{2-q}{q-1}}|\nabla u|^2\right]\,\mathrm{d}x = \int_{\Omega}\left[\nabla m\cdot\nabla u + m|\nabla u|^{q'}\right]\,\mathrm{d}x = 0,$$

from which (4.5.3) follows. □

Remark 4.5.1. *Inequality (4.5.3) is a sort of "weak concentration property". In fact, by an approximation argument it is easy to see that we can take* $C(\overline{\Omega})\cap W^{1,q}(\Omega)$ *for the space of test functions in the first equation of* (MFG_q)*. Thus, if m is continuous, we would have that* $\langle\mu - p, m\rangle = \int_{\Omega} m\,\mathrm{d}(\mu - p)$ *and so (4.5.3) would imply that*

$$\int_{\Omega} m\,\mathrm{d}\mu = 0 \quad\text{and}\quad \int_{\Omega}(1-m)\,\mathrm{d}p = 0,$$

i.e

$$\mathrm{spt}(\mu)\subseteq\{m=0\}\quad\text{and}\quad\mathrm{spt}(p)\subseteq\{m=1\},$$

as in Corollary 4.4.2.

4.6 SOME POSSIBLE EXTENSION

Let us notice that in Section 4.4 we derived optimality conditions for Problem (P_q) in a direct way, using the characterization of the subdifferential of the functional $\mathcal{B}_q$. This means that one can easily remove the convexity assumption on the function F in the second variable, thus also on the functional $\mathcal{F}$ (see (4.3.1)). In this section we analyze some generalizations in this direction. This is also the subject of an ongoing collaboration with F.J. Silva. Let us define the problem setting (which is unchanged, except for $\mathcal{F}$) for this section. Similarly as before, we consider the optimal control problem

$$\inf\ \mathcal{B}_q(m,\mathbf{w}) + \mathcal{F}(m), \tag{P_q}$$

subject to the constraints

$$\begin{cases} -\Delta m + \nabla\cdot\mathbf{w} &= 0 \ \text{ in } \Omega,\\ (\nabla m - \mathbf{w})\cdot\mathbf{n} &= 0 \ \text{ on } \partial\Omega,\end{cases}\qquad \int_{\Omega} m(x)\,\mathrm{d}x = 1, \qquad 0\le m\le 1.$$

where $\mathcal{F} : W^{1,q}(\Omega)\to\overline{\mathbb{R}}$ and $\mathcal{B}_q : W^{1,q}(\Omega)\times L^q(\Omega)^d\to\overline{\mathbb{R}}$ is defined as before. Under rather general assumptions on $\mathcal{F}$, if $q > d$ we can prove the existence of a solution $(m,\mathbf{w})$ of (P_q). Using the expression of the subdifferential of $\mathcal{B}_q$, provided in Theorem 4.2.2, and supposing that $\mathcal{F}$ is differentiable at m with functional derivative denoted as $D\mathcal{F}(m)$ we obtain the existence of $(u, \lambda, \mu, p) \in W^{1,q'}_{\diamond}(\Omega) \times \mathbb{R} \times \mathscr{M}_+(\overline{\Omega}) \times \mathscr{M}_+(\overline{\Omega})$ such that they satisfy the following MFG system in a weak sense

$$\begin{cases} -\Delta u + \frac{1}{q'}|\nabla u|^{q'} + \mu - p - \lambda = D\mathcal{F}(m), & \text{in } \Omega, \\ -\Delta m - \nabla \cdot \left(m|\nabla u|^{\frac{2-q}{q-1}} \nabla u \right) = 0, & \text{in } \Omega, \\ \nabla m \cdot \mathbf{n} = \nabla u \cdot \mathbf{n} = 0, & \text{on } \partial\Omega, \\ \int_\Omega m \, \mathrm{d}x = 1, \quad 0 \leq m \leq 1, & \text{in } \Omega, \\ \mathrm{spt}(\mu) \subseteq \{m = 0\}, \quad \mathrm{spt}(p) \subseteq \{m = 1\}. \end{cases} \tag{MFG_q}$$

In contrast to the analysis performed in the previous sections, we are able to provide the existence of a solution of (MFG_q) without the assumption that $\mathcal{F}$ is *convex* and a *local* function of m. The result can be extended to the case $1 < q \leq d$ with the approximation argument in Section 4.5.

The second extension is that – using the ideas developed in the previous sections – we can prove the existence of a solution of (P_q) when no upper bound for the density are imposed. The main argument is that if $q > d$ if we replace the constraint $m \leq 1$ by $m \leq \gamma$, with γ large enough, then the solution of the corresponding problem is bounded by a constant *independent* of γ. Thus, if $q > d$ then the following system (similarly as in Section 4.4) admits at least one solution

$$\begin{cases} -\Delta u + \frac{1}{q'}|\nabla u|^{q'} + \mu - \lambda = D\mathcal{F}(m), & \text{in } \Omega, \\ -\Delta m - \nabla \cdot \left(m|\nabla u|^{\frac{2-q}{q-1}} \nabla u \right) = 0, & \text{in } \Omega, \\ \nabla m \cdot \mathbf{n} = \nabla u \cdot \mathbf{n} = 0, & \text{on } \partial\Omega, \\ \int_\Omega m \, \mathrm{d}x = 1, \quad m \geq 0, & \text{in } \Omega, \\ \mathrm{spt}(\mu) \subseteq \{m = 0\}. \end{cases} \tag{$MFG_{\infty,q}$}$$

Note that this system is a 'classical' one (as the ones introduced initially in [LL06a]), in the sense that there is no density constraint $m \leq 1$ imposed. Thus we give a new proof of existence of weak solutions for stationary second order MFG systems.

4.6.1 *The case with density constraint but without convexity*

We will assume the following hypotheses for $\mathcal{F} : W^{1,q}(\Omega) \to \overline{\mathbb{R}}$:

(F1) $\mathcal{F}$ is Gâteaux-differentiable.

(F2) $\mathcal{F}$ is weakly lower semicontinuous.

(F3) there exists $C_\mathcal{F} > 0$ such that $-C_\mathcal{F} \leq \mathcal{F}(m)$ for all $m \in W^{1,q}(\Omega)$ and $\mathcal{F}(1/|\Omega|) \leq C_\mathcal{F}$.

Let us underline that no convexity assumptions are imposed on $\mathcal{F}$.

Remark 4.6.1 (Examples of coupling functions $\mathcal{F}$)**.** *Typical examples of functions $\mathcal{F}$ satisfying the previous assumptions are:*

$$\mathcal{F}(m) = \int_\Omega F_1(x, m(x), \nabla m(x))\,\mathrm{d}x + \int_\Omega F_2(x, (K_0 \star m)(x), (K_1 \star \nabla m)(x))\,\mathrm{d}x,$$

where $K_0 : \mathbb{R}^d \to \mathbb{R}$ and $K_1 : \mathbb{R}^d \to \mathbb{R}^d$ are Lipschitz convolution kernels. F_1 and $F_2 : \Omega \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ are measurable in the first variable, differentiable in the second variable and bounded from below. Moreover they are differentiable and weakly l.s.c. w.r.t. the third variable.

The above general assumptions on $\mathcal{F}$ are sufficient to show the existence of a solution (by the direct method of calculus of variations) of the new Problem (P_q). Moreover, using the differentiability assumption on $\mathcal{F}$ together with the characterization of the subdifferential of $\mathcal{B}_q$ (see Theorem 4.2.2) we can derive system (MFG_q) as necessary optimality conditions for the solutions of problem (P_q). The analysis relies on the one in Section 4.4, however a clever reformulation is needed because $\mathcal{F}$ is not convex anymore.

If the exponent in the cost $\mathcal{B}_q$ is small, i.e. $1 < q \le d$ the constraint for the density in the problem (P_q) is not qualified, hence a priori we are not able to derive first order optimality conditions. To overcome these issues, we propose the same approximation procedure as the one in Section 4.5. To be able to pass to the limit after the regularization process, we need more assumptions on $\mathcal{F}$. Namely let us assume

(F4) $D\mathcal{F}$ is a bounded operator from $W^{1,q}(\Omega) \to \mathscr{M}(\overline{\Omega})$ in the sense that for any $\|m\|_{W^{1,q}} \le C$, there exists $\overline{C} > 0$ depending only on C such that $\|D\mathcal{F}(m)\|_{\mathscr{M}} \le \overline{C}$. Note that we assume that the range of $D\mathcal{F}$ to be $\mathscr{M}(\overline{\Omega})$ instead of $\left(W^{1,q}(\Omega)\right)^*$.

(F5) $D\mathcal{F}$ is a continuous linear operator in a weaker sense, meaning

$$\langle D\mathcal{F}(m_\varepsilon), \phi\rangle \to \langle D\mathcal{F}(m), \phi\rangle,$$

as $\varepsilon \to 0$ and $m_\varepsilon \to m$ strongly in L^{q^*} and for all $\phi \in C^0(\overline{\Omega})$ test function.

Under these additional assumptions, the approximation procedure goes along the same lines as in Section 4.5. The main difference is that we work with general class of interactions $\mathcal{F}$.

4.6.2 *The case without the density constraint*

Now, we consider the case where the upper bound constraint $m \le 1$ is no longer considered, i.e. we consider the problem

$$\min \mathcal{J}_q(m, w), \qquad (\hat{\mathscr{P}}_q)$$

subject to the constraints

$$\begin{cases} -\Delta m + \nabla \cdot \mathbf{w} &= 0 \quad \text{in } \Omega, \\ (\nabla m - \mathbf{w}) \cdot \mathbf{n} &= 0 \quad \text{on } \partial\Omega, \end{cases} \qquad \int_\Omega m(x)\,\mathrm{d}x = 1.$$

The crucial point is the existence result when $q > d$.

Theorem 4.6.1. *If $q > d$ then problem ($\hat{\mathscr{P}}_q$) has at least one solution.*

Proof. Given $\gamma > 1/|\Omega|$ let us consider the auxiliary problem

$$\min \ \mathcal{J}_q(m, w) \tag{$\hat{\mathscr{P}}_{q,\gamma}$}$$

subject to the constraints

$$\begin{cases} -\Delta m + \nabla \cdot \mathbf{w} &= 0 \quad \text{in } \Omega, \\ (\nabla m - \mathbf{w}) \cdot \mathbf{n} &= 0 \quad \text{on } \partial\Omega, \end{cases} \qquad \int_\Omega m(x)\,\mathrm{d}x = 1, \quad m \le \gamma$$

By Proposition 4.3.1 we now that ($\hat{\mathscr{P}}_{q,\gamma}$) has at least one solution. We will show that any such solution $(m_\gamma, \mathbf{w}_\gamma)$ satisfies that $\|m_\gamma\|_\infty \le \ell$ for some constant $\ell > 0$ which is independent of γ. This will prove the result since any solution $(m_\ell, \mathbf{w}_\ell)$ of ($\hat{\mathscr{P}}_{q,\ell}$) solves ($\hat{\mathscr{P}}_q$). Indeed, if there is a feasible $(m, \mathbf{w})$ for problem ($\hat{\mathscr{P}}_q$) such that $\mathcal{J}_q(m, \mathbf{w}) < \mathcal{J}_q(m_\ell, \mathbf{w}_\ell)$ then since there exists $\ell' > 0$ such that $\|m\|_{L^\infty} \le \ell'$ (because $m \in W^{1,q}(\Omega)$) we have that $\mathcal{J}_q(m_{\ell'}, \mathbf{w}_{\ell'}) \le \mathcal{J}_q(m, \mathbf{w})$, where $(m_{\ell'}, \mathbf{w}_{\ell'})$ is a solution of ($\hat{\mathscr{P}}_{q,\ell'}$), and $m_{\ell'} \le \ell$ which contradicts the optimality of $(m_\ell, \mathbf{w}_\ell)$.

Let us denote by $c_0 > 0$ a constant such that $\|m\|_{L^\infty} \le c_0 \|m\|_{W^{1,q}}$ for all $m \in W^{1,q}(\Omega)$ (we know that it exists by the classical Sobolev embeddings results) and by $c_1 > 0$ the constant in Lemma A.3.2. Thus, any solution of $(m, \mathbf{w})$ of the PDE the constraint satisfies that $\|m\|_{L^\infty} \le c_0 c_1 \|\mathbf{w}\|_{L^q}$. Let $(m_\gamma, \mathbf{w}_\gamma)$ be a solution of ($\hat{\mathscr{P}}_{q,\gamma}$). Then, $\mathbf{w}_\gamma$ vanishes a.e. in $\{m_\gamma = 0\}$. The density constraint implies that $1/(q\gamma^{q-1}) \le 1/(qm(x)^{q-1})\mathbb{1}_{\{m>0\}}$ and so

$$\|\mathbf{w}_\gamma\|_{L^q}^q = \int_{\{m>0\}} |\mathbf{w}_\gamma|^q \,\mathrm{d}x \le q\gamma^{q-1}\mathcal{B}_q(m_\gamma, \mathbf{w}_\gamma) = q\gamma^{q-1}\left[\mathcal{J}_q(m_\gamma, \mathbf{w}_\gamma) - \mathcal{F}(m_\gamma)\right].$$

By the optimality of $(m_\gamma, \mathbf{w}_\gamma)$ and **(F3)** we deduce, setting $(\hat{m}, \hat{\mathbf{w}}) := (1/|\Omega|, 0)$,

$$\|\mathbf{w}_\gamma\|_{L^q}^q \le q\gamma^{q-1}\left[\mathcal{J}_q(\hat{m}, \hat{\mathbf{w}}) - \mathcal{F}(m_\gamma)\right] \le qC_{\mathcal{F}}\gamma^{q-1} \quad \text{and so} \quad \|\mathbf{w}_\gamma\|_{L^q} \le q^{\frac{1}{q}} C_{\mathcal{F}}^{\frac{1}{q}} \gamma^{\frac{1}{q'}}.$$

Thus, $\|m_\gamma\|_{L^\infty} \le \hat{c}\gamma^{\frac{1}{q'}}$, where $\hat{c} := c_0 c_1 q^{\frac{1}{q}} C_{\mathcal{F}}^{\frac{1}{q}}$. Iterating the previous argument yields

$$\|m_\gamma\|_{L^\infty} \le \hat{c}^{\sum_{i=0}^k (\frac{1}{q'})^i} \gamma^{(\frac{1}{q'})^{k+1}} \qquad \forall\, k \in \mathbb{N},$$

and so, letting $k \uparrow \infty$ and using that $q' > 1$, we get $\|m_\gamma\|_{L^\infty} \le \hat{c}^q$. The result follows. □

Finally, the derivation of the first order necessary optimality conditions, i.e. system ($MFG_{\infty,q}$) can be performed in the same way as in the case of (MFG_q).

5

First order MFG with density constraints: Pressure equals Price

In this chapter we study first order Mean Field Game systems under density constraints as optimality conditions of two optimization problems in duality. A weak solution of the system contains an extra term, an additional price imposed on the saturated zones. We show that this price corresponds to the pressure field from the models of incompressible Euler's equations *à la Brenier*. By this observation we could obtain a minimal regularity, which allows to write optimality conditions at the level of single agent trajectories and to define a weak notion of Nash equilibrium for our model.

This chapter is based on a joint work with P. Cardaliaguet and F. Santambrogio (see [CMS15]).

5.1 Introduction

5.1.1 *The MFG system*

Introduced by Lasry-Lions [LL06a, LL06b, LL07] (see also Huang-Malhamé-Caines [HMC06]) the mean field game system (in short, MFG system) describes a differential game with infinitely many identical players who interact through their repartition density. We write here the system of MFG in the specific case where there is no diffusion, i.e. in the first-order case

$$\begin{cases} \text{(i)} & -\partial_t u + H(x, Du) = f(x, m) & \text{in } (0, T) \times \mathbb{T}^d \\ \text{(ii)} & \partial_t m - \nabla \cdot (m D_p H(x, Du)) = 0 & \text{in } (0, T) \times \mathbb{T}^d \\ \text{(iii)} & u(T, x) = g(x), \quad m(0, x) = m_0(x) & \text{in } \mathbb{T}^d. \end{cases} \tag{5.1.1}$$

Here, to avoid the discussion of the boundary data, we work for simplicity with periodic boundary conditions, i.e., in the torus $\mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d$. The Hamiltonian $H : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}$ is typically convex with respect to the last variable and the coupling map $f : \mathbb{T}^d \times [0, +\infty[$ is nondecreasing with respect to the last variable.

Let us briefly describe the interpretation of (5.1.1). In the above backward-forward system, $u = u(t, x)$ is the value function associated to any tiny player while $m = m(t, x)$ is the density of players at time t and at position x. The value function $u(t, x)$ is formally given by

$$u(t, x) = \inf_{\gamma} \int_t^T L(\gamma(s), \dot{\gamma}(s)) + f(\gamma(s), m(s, \gamma(s)))\, ds + g(\gamma(T))$$

where the player minimizes over the paths $\gamma : [t, T] \to \mathbb{T}^d$ with $\gamma(t) = x$, L is the Fenchel conjugate of H with respect to the last variable, $f = f(x, m(t, x))$ is the running cost and $g : \mathbb{T}^d \to \mathbb{R}$ is the terminal cost at the terminal time $t = T$. The running cost f couples the two equations.

At the initial time $t = 0$, the initial distribution is m_0 (a probability measure on $\mathbb{T}^d$). Then the density evolvs according to the motion of the players. Since – by standard argument in optimal control – it is optimal for the players to play $\dot{\gamma}(s) = -D_p H(\gamma(s), Du(s, \gamma(s)))$, the evolution of the density is given by the continuity equation (5.1.1)-(ii).

Note that each tiny players acts as if he knew the evolution of the players' density $m = m(t, x)$ (he somehow "forecasts" it, this is what is usually called "rational expectations"). The mean field game system corresponds to an equilibrium situation in where the "forecast" of the players is correct: the solution of the continuity equation is indeed $m = m(t, x)$. In terms of game theory, this corresponds to a Nash equilibrium.

Existence and uniqueness of solutions for the above problem are discussed by P.-L. Lions in [Lio08] (through a reduction to an elliptic equation in time-space when the coefficients are smooth and the coupling blows down at 0) and in Cardaliaguet [Car13b], Graber [Gra14], Cardaliaguet-Graber [CG15], Cardaliaguet–Graber-Porretta-Tonon [CGPT14] (following an approach by variational methods suggested in [LL07] and also inspired by Benamou-Brenier [BB00]). Recently, in [BC15] Benamou and Carlier used similar variational techniques to study an augmented Lagrangian scheme for MFG problems.

5.1.2 *The problem with a density constraint*

Our objective in this chapter is to study the behavior of the MFG system when there is a *density constraint*, i.e., when the density m cannot exceed some given value $\overline{m} > 1/|\mathbb{T}^d| = 1$. Namely: $0 \leq m(t,x) \leq \overline{m}$ at any point (t,x). In other words, the players pay an infinite price when the density goes above $\overline{m}$: $f(x,m) = +\infty$ if $m > \overline{m}$. The question of how to model this situation was first introduced by Santambrogio [San12b] and then investigated Mészáros and Silva in [MS15b] (see also Chapter 4) in the framework of stationary second order models. We emphasize the fact that imposing a density constraint will result a so-called "hard congestion" effect in the model. Models of MFGs where so-called "soft congestion" (meaning that agents slow down when they reach zones with high density) effects have been studied recently by Gomes and Mitake in [GM14], by Gomes and Voskanyan in [GV15] and by Burger, Di Francesco, Markowich and Wolfram in [BDFMW14].

Coming back to our model, there are several issues in the interpretation of system (5.1.1) when there is a density constraint. Indeed, the above interpretation does not make sense anymore for the following reason: if, on the one hand, the constraint $m \leq \overline{m}$ is fulfilled, then the minimization problem of the agents (due to the fact that they are considered negligible against the others) does not see this constraint and the pair (u,m) is the solution of a standard MFG system; but this solution has no reason to satisfy the constraint, and there is a contradiction. On the other hand, if there are places where $m(t,x) > \overline{m}$, then the players do not go through these places because their cost is infinite there: but then the density at such places is zero, and there is again a contradiction. So, in order to understand the MFG system with a density constraint, one has to change our point of view. We shall see that there are several ways to understand more deeply the phenomena behind this question.

Perhaps the simplest approach is to go through an approximation argument: let us consider the solution $(u^\varepsilon, m^\varepsilon)$ corresponding to a running cost f^ε which is finite everywhere, but tends to infinity as ε tends to 0 when $m > \overline{m}$. In other words, $f^\varepsilon(x,m) \to f(x,m)$ if $m \leq \overline{m}$ and $f^\varepsilon(x,m) \to +\infty$ if $m > \overline{m}$, as $\varepsilon \to 0$. In this case the MFG system with a density constraint should simply be the limit configuration (a limit which should be proven to be well-defined).

We indeed show that the pair $(u^\varepsilon, m^\varepsilon)$ has (up to subsequences) a limit (u,m) which satisfies (in a weak sense) the following system:

$$\left\{\begin{array}{llcll} \text{(i)} & -\partial_t u + H(x,Du) & = & f(x,m) + \beta & \text{in } (0,T)\times\mathbb{T}^d \\ \text{(ii)} & \partial_t m - \nabla\cdot(mD_pH(x,Du)) & = & 0 & \text{in } (0,T)\times\mathbb{T}^d \\ \text{(iii)} & u(T,x) = g(x) + \beta_T, & & m(0,x) = m_0(x) & \text{in } \mathbb{T}^d \\ \text{(iv)} & 0 \leq m \leq \overline{m} & & & \text{in } [0,T]\times\mathbb{T}^d \end{array}\right. \tag{5.1.2}$$

Beside the expected density constraint (iv), two extra terms appear: β in (i) and β_T in (iii). These two quantities turn out to be nonnegative and concentrated on the set $\{m = \overline{m}\}$. They formally correspond to an extra price payed by the players to go through zones where the concentration is saturated, i.e., where $m = \overline{m}$. In other words, the new optimal control problem for the players is now (formally)

$$u(t,x) = \inf_{\substack{\gamma \\ \gamma(t)=x}} \int_t^T L(\gamma(s),\dot\gamma(s)) + f(\gamma(s), m(s,\gamma(s)) + \beta(s,\gamma(s))\,\mathrm{d}s \\ + g(\gamma(T)) + \beta_T(\gamma(T)), \tag{5.1.3}$$

and thus (still formally) satisfies the dynamic programming principle: for any $0 \le t_1 \le t_2 < T$,

$$u(t_1,x) = \inf_{\substack{\gamma \\ \gamma(t_1)=x}} \int_{t_1}^{t_2} L(\gamma(s),\dot\gamma(s)) + f(\gamma(s), m(s,\gamma(s)) + \beta(s,\gamma(s))\,\mathrm{d}s \\ + u(t_2,\gamma(t_2)). \tag{5.1.4}$$

The "extra prices" β and β_T discourage too many players to be attracted by the area where the constraint is saturated, thus ensuring the density constraints (iv) to be fulfilled.

5.1.3 *The variational method*

Another way to see the problem is the following: it is known (see [LL07]) that the solution (u,m) to (5.1.1) can be obtained by variational methods at least when f is finite everywhere. More precisely, the value function u is (formally) given as a minimizer of the functional

$$\mathcal{A}(u) := \int_0^T \int_{\mathbb{T}^d} F^*(x, -\partial_t u + H(x,Du))\,\mathrm{d}x\,\mathrm{d}t - \int_{\mathbb{T}^d} u(0,x)\,\mathrm{d}m_0(x),$$

subject to the constraint that $u(T,x) = g(x)$, where $F = F(x,m)$ is an antiderivative of $f = f(x,m)$ with respect to m and F^* is its Legendre-Fenchel conjugate w.r.t. the second variable. In the same way m is (formally) given as a minimizer of the problem

$$\mathcal{B}(m,\mathbf{w}) := \int_{\mathbb{T}^d} g(x)m(T,x)\,\mathrm{d}x + \int_0^T\int_{\mathbb{T}^d} m(t,x)H^*\left(x, -\frac{\mathbf{w}}{m}\right) + F(x,m(t,x))\,\mathrm{d}x\,\mathrm{d}t$$

subject to the constraint

$$\partial_t m + \nabla\cdot(\mathbf{w}) = 0 \quad \text{in } (0,T)\times\mathbb{T}^d, \qquad m(0) = m_0,$$

where H^* is the convex conjugate of H with respect to the last variable. It turns out that both problems make perfectly sense, even when $f(x,m) = +\infty$ if

$m > \overline{m}$. In fact, if f^ε is a finite approximation of f as before, one can expect the minimizers of $\mathcal{A}^\varepsilon$ and $\mathcal{B}^\varepsilon$ (corresponding to f^ε) to converge to the minimizers of $\mathcal{A}$ and $\mathcal{B}$ as $\varepsilon \to 0$ (as a simple consequence of $\Gamma-$convergence). This is precisely what happens. Note that, as $f(x,m) = +\infty$ for $m > \overline{m}$, $F(x,m)$ has the same property, so that $F^*(x,m)$ is linear on $[\overline{m}, +\infty)$. This linear behavior explains the appearance of the terms β and β_T described above.

5.1.4 *Connections between MFGs with density constraints and the incompressible Euler's equations à la Brenier*

One of the main contributions of this chapter is the determination of some strong connections between our model of MFG with density constraint and the incompressible Euler's equations studied by Brenier (see [Bre99]) and later by Ambrosio and Figalli (see [AF09]). This fact helps to understand the deeper phenomena hiding in our models. Actually this connection is not that surprising. Firstly, the incompressibility constraint in the model of Brenier to study perfect fluids will introduce the pressure field. Morally the same effect happens imposing density constraint for MFG. Secondly, both Brenier's model and ours have a variational structure, similar also to the one introduced by Benamou and Brenier in [BB00]. Therefore, the terms β and β_T, that we call "additional prices/costs" for the agents (appearing only if they pass through saturated zones) in (5.1.2) correspond to a sort of pressure field from fluid mechanics. This observation motivates the title of this chapter as well.

5.1.5 *Variational models for the incompressible Euler's equations*

To better see the connections with our models of MFG with density constraints, let us describe in a nutshell some variational models which describe the evolution of the velocity field of perfect incompressible fluids driven by the Euler's equation. The first description goes back to the mid 18th century, when Euler wrote down the equations linking the evolution of the velocity fields of a perfect fluid and its pressure gradient. Modeling a moving fluid in a smooth domain $\Omega \subset \mathbb{R}^d$ is given by

$$\begin{cases} \partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\nabla p, & \text{in } [0,T]\times\Omega, \\ \nabla\cdot\mathbf{u} = 0, & \text{in } [0,T]\times\Omega, \\ \mathbf{u}\cdot\mathbf{n} = 0, & \text{on } [0,T]\times\partial\Omega. \end{cases} \tag{5.1.5}$$

The incompressibility condition for the fluid is actually encoded in the equation $\nabla\cdot\mathbf{u} = 0$, i.e. the velocity field is divergence-free.

Assuming that the velocity field $\mathbf{u}$ is smooth, we can associate a flow g to it, given by the characteristic equation

$$\begin{cases} \dot{g}(t,x) = \mathbf{u}(t,g(t,x)), \\ g(0,x) = x. \end{cases}$$

The incompressibility constraint translated to this level says that at each time $t \in [0,T]$, $g(t,\cdot) : \Omega \to \Omega$ is a measure-preserving diffeomorphism, which is $g(t,\cdot)_{\#}\mu_\Omega = \mu_\Omega$, where $\mu_\Omega := \mathscr{L}^d \llcorner \Omega / \mathscr{L}^d(\Omega)$. Writing system (5.1.5) in terms of g, one obtains

$$\begin{cases} \ddot{g}(t,x) = -\nabla p(t,g(t,x)), \\ g(0,x) = x, \\ g(t,\cdot) \in \mathrm{SDiff}(\Omega), \end{cases} \tag{5.1.6}$$

where $\mathrm{SDiff}(\Omega)$ denotes the space of measure-preserving diffeomorphisms of Ω. The groundbreaking idea of V. Arnold ([Arn66]) was to see $\mathrm{SDiff}(\Omega)$ as an infinite dimensional manifold with the metric inherited from its embedding into $L^2(\Omega;\Omega)$ and to look for solutions to (5.1.6) by solving a geodesic problem on $\mathrm{SDiff}(\Omega)$, i.e.

$$\inf T \int_0^T \int_\Omega \frac{1}{2}|\dot{g}(t,x)|^2 \,\mathrm{d}\mu_\Omega(x)\,\mathrm{d}t$$

among all path $[0,T] \ni t \mapsto g(t,\cdot) \in \mathrm{SDiff}(\Omega)$ connecting two diffeomorphisms, $g(0,\cdot) = f$ to $g(T,\cdot) = h$ (typically f is taken as id). In this minimization problem, the pressure gradient in (5.1.6) actually arises as a Lagrange multiplier for the incompressibility constraint.

Without entering too much into the details, one can remark that after the introduction of this idea by V. Arnold, many deep mathematical phenomena have been investigated in this direction. Some "negative" result found by Shnirelman (the infimum in the above problem is not always attained; there are $h \in \mathrm{SDiff}(\Omega)$ which cannot be connected to id with a finite energy) lead Y. Brenier (see [Bre89]) to relax Arnold's minimization problem (morally as it happened for Kantorovich while relaxing Monge's problem). He was looking for optimal *generalized incompressible flows* $\boldsymbol{\eta} \in \mathscr{P}_2(\Gamma)$ (Borel probability measures on the set Γ of continuous paths in Ω) solutions to

$$\inf T \int_0^T \int_\Gamma \frac{1}{2}|\dot{\gamma}(t)|^2 \,\mathrm{d}\boldsymbol{\eta}(\gamma)\,\mathrm{d}t$$

under the constraints $(e_0, e_T)_{\#}\boldsymbol{\eta} = (\mathrm{id}, h)_{\#}\mu_\Omega$ and $(e_t)_{\#}\boldsymbol{\eta} = \mu_\Omega$ for all $t \in [0,T]$ (here $e_t : \Gamma \to \Omega$ denotes the evaluation map $e_t(\gamma) = \gamma(t)$). In this context one can relax the regularity of h as well, taking it to be just a measure-preserving map instead of a diffeomorphism. Along the existence theory provided in [Bre89], Y. Brenier showed also a consistency of his model: smooth solutions to (5.1.5) are optimal even in his larger class of generalized incompressible flows, provided the pressure gradient satisfies some well-chosen bounds.

The model of Y. Brenier has been further generalized in different directions. Noticing that $(e_0, e_t)_\#\boldsymbol{\eta}$ is a measure-preserving plan, i.e. a probability measure on $\Omega \times \Omega$, L. Ambrosio and A. Figalli developed similar models (see [AF09]), where, between others, they were considering a minimizing geodesic problem connecting measure-preserving plans ν_1, ν_2. Later a similar model of both Eulerian and Lagrangian nature has been studied by Y. Brenier ([Bre99]), which allowed him in particular to show that the pressure gradient ∇p is a locally finite (vector valued) measure on $(0, T) \times \Omega$. Improving the techniques used by him, L. Ambrosio and A. Figalli showed (see [AF08]) that p belongs actually to the more regular space $L^2_{\text{loc}}((0, T); BV_{\text{loc}}(\Omega))$. This result lead them to define some finer (but weaker) optimality conditions for the minimizing geodesic problem along single particle trajectories.

Using similar techniques as in [Bre99] and [AF09, AF08] we show that β is an $L^2_{\text{loc}}((0, T); BV(\mathbb{T}^d)) \hookrightarrow L^{d/(d-1)}_{\text{loc}}((0, T) \times \mathbb{T}^d)$ function (while a priori it was only supposed to be a measure) and β_T is $L^1(\mathbb{T}^d)$. With the help of an example we show that this local integrability cannot be extended up to the final time $t = T$, showing also some sort of sharpness of the result. This regularity property will allow us to give a clearer (weak) meaning to the control problem (5.1.4), obtaining optimality conditions along single agent trajectories. Our techniques to proceed with the analysis rely on the properties of measures defined on paths, that we shall call *density-constrained flows* in our context, and we are exploiting some properties of a Hardy-Littlewood type maximal functional as well.

After this analysis we deduce the existence of a *local weak Nash equilibrium* for our model.

The chapter is organized in the following way. We first introduce our main notation and assumptions (Section 5.2). Then we discuss the two optimization problems for the functionals $\mathcal{A}$ and $\mathcal{B}$ described above (Section 5.3). We introduce the definition of the MFG system with a density constraint, present our main existence result as well as the approximation by standard MFG systems in Section 5.4. In Section 5.5, by means of an example, we study some finer properties of a solution (m, u, β, β_T) of the MFG system with density constraints. Section 5.6 is devoted to the proof of the $L^{d/(d-1)}_{\text{loc}}$ integrability of the additional price β under some additional assumptions on the Hamiltonian and the coupling. Finally, having in hand this integrability property, we introduce in Section 5.7 the optimal density-constrained flows and derive optimality conditions along single agent paths, which allow in particular to study the existence of the local weak Nash equilibrium.

5.2 NOTATIONS, ASSUMPTIONS AND PRELIMINARIES

We consider the MFG system with a density constraint (5.1.2) under the assumption that all the maps are periodic in space. Typical conditions are

– (Condition on the density constraint)

$$\text{The density constraint } \overline{m} \text{ is larger than } 1 = 1/\mathscr{L}^d(\mathbb{T}^d). \tag{H1}$$

– (Conditions on the initial and final conditions)
m_0 is a probability measure on $\mathbb{T}^d$ which is absolutely continuous with respect to Lebesgue measure and there exists $\overline{c} > 0$ such that

$$0 \le m_0 < \overline{m} - \overline{c}, \text{ a.e. on } \mathbb{T}^d.$$

$$\text{Moreover } g : \mathbb{T}^d \to \mathbb{R} \text{ is a } C^1 \text{ function on } \mathbb{T}^d. \tag{H2}$$

– (Conditions on the Hamiltonian)
$H : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}$ is continuous in both variables, convex and differentiable in the second variable, with D_pH continuous in both variables. Moreover, H has superlinear growth in the gradient variable: there exist $r > 1$ and $C > 0$ such that

$$\frac{1}{rC}|p|^r - C \le H(x,p) \le \frac{C}{r}|p|^r + C. \tag{H3-1}$$

We denote by $H^*(x,\cdot)$ the Fenchel conjugate of $H(x,\cdot)$, which, due to the above assumptions, satisfies

$$\frac{1}{r'C}|q|^{r'} - C \le H^*(x,q) \le \frac{C}{r'}|q|^{r'} + C, \tag{H3-2}$$

where r' is the conjugate of r. We will also denote by L the Lagrangian given by $L(x,q) = H^*(x,-q)$, which thus satisfies the same bounds as H^*.

– (Conditions on the coupling)
Let f be continuous on $\mathbb{T}^d \times [0,\overline{m}]$, non-decreasing in the second variable with

$$f(x,0) = 0. \tag{H4}$$

We define F so that $F(x,\cdot)$ is an antiderivative of $f(x,\cdot)$ on $[0,\overline{m}]$, that is,

$$F(x,m) = \int_0^m f(x,s)\,\mathrm{d}s, \quad \forall\, m \in [0,\overline{m}], \tag{5.2.1}$$

and extend F to $+\infty$ on $(-\infty,0) \cup (\overline{m},+\infty)$. It follows that F is continuous on $\mathbb{T}^d \times (0,\overline{m})$, is convex and differentiable in the second variable. We also define $F^*(x,\cdot)$ to be the Fenchel conjugate of $F(x,\cdot)$ for each x. Note that

$$F^*(x,\alpha) \ge \alpha\overline{m} - F(x,\overline{m}) \tag{5.2.2}$$

and in particular $F^*(\cdot,\alpha) = 0$ for all $\alpha \le 0$. Following the approach of Cardaliaguet-Carlier-Nazaret [CCN13] (see also Cardaliaguet [Car13b], Graber [Gra14] or Cardaliaguet-Graber [CG15]) it seems that the solution to (5.1.2) can be obtained as the system of optimality conditions for optimal control problems.

5.2.1 *Measures on curves and a superposition principle*

Let us denote by Γ the set of absolutely continuous curves $\gamma : [0,T] \to \mathbb{T}^d$. We denote by $\mathscr{P}(\Gamma)$ the set of Borel probability measures defined on Γ. Let us set $\mathscr{P}_r(\Gamma)$ $(r \geq 1)$ to be the subset of $\mathscr{P}(\Gamma)$ such that

$$\int_\Gamma \int_0^T |\dot\gamma(s)|^r \,\mathrm{d}s \,\mathrm{d}\boldsymbol{\eta}(\gamma) < +\infty$$

and $m_t := (e_t)_\# \boldsymbol{\eta}$ is $L^\infty(\mathbb{T}^d)$ for all $\boldsymbol{\eta} \in \mathscr{P}_r(\Gamma)$, where $e_t : \Gamma \to \mathbb{T}^d$ denotes the evaluation map $e_t(\gamma) := \gamma(t)$ for all $t \in [0,T]$.

Now we state a well-known result, a connection between the solutions of the continuity equation and the measures on paths, called *superposition principle*, which can be considered as a weaker version of the DiPerna-Lions-Ambrosio theory (see for instance Theorem 8.2.1. from [AGS08]).

Theorem 5.2.1. *Let $\mu : [0,T] \to \mathscr{P}(\mathbb{T}^d)$ be a narrowly continuous solution of the continuity equation $\partial_t \mu + \nabla \cdot (\mathbf{v}\mu) = 0$, $\mu_0 \in \mathscr{P}(\mathbb{T}^d)$ for a velocity field $\mathbf{v} : [0,T] \times \mathbb{T}^d \to \mathbb{R}^d$ satisfying $\int_0^T \int_{\mathbb{T}^d} |\mathbf{v}_t|^2 \,\mathrm{d}\mu_t \,\mathrm{d}t < +\infty$. Then there exists $\boldsymbol{\eta} \in \mathscr{P}_2(\Gamma)$ such that*

(i) $\mu_t = (e_t)_\# \boldsymbol{\eta}$ *for all* $t \in [0,T]$;

(ii) *we have the energy inequality*

$$\int_\Gamma \int_0^T |\dot\gamma(t)|^2 \,\mathrm{d}t \,\mathrm{d}\boldsymbol{\eta}(\gamma) \leq \int_0^T \int_{\mathbb{T}^d} |\mathbf{v}|^2 \,\mathrm{d}\mu_t \,\mathrm{d}t;$$

(iii) $\dot\gamma(t) = \mathbf{v}(t, \gamma(t))$, *for* $\boldsymbol{\eta}$*−a.e.* γ *and a.e.* $t \in [0,T]$.

5.3 OPTIMAL CONTROL PROBLEMS

The first problem is an optimal control of Hamilton-Jacobi equations: denote by $\mathcal{K}_P$ the set of functions $u \in C^1([0,T] \times \mathbb{T}^d)$ such that $u(T,x) = g(x)$. Let us define on $\mathcal{K}_P$ the functional

$$\mathcal{A}(u) = \int_0^T \int_{\mathbb{T}^d} F^*(x, -\partial_t u + H(x, Du)) \,\mathrm{d}x \,\mathrm{d}t - \int_{\mathbb{T}^d} u(0,x) \,\mathrm{d}m_0(x). \quad (5.3.1)$$

Then we have our first optimal control problem.

Problem 5.3.1 (Optimal control of HJ)**.** *Find* $\inf\limits_{u \in \mathcal{K}_P} \mathcal{A}(u)$.

It is easy to check that one can restrict the optimization to the class of minimizers such that $-\partial_t u + H(x, Du) \geq 0$, because $F^*(x, \alpha) = 0$ for $\alpha \leq 0$ (see Lemma 3.2 in [Car13b]).

The second problem is an optimal control problem for the continuity equation: define $\mathcal{K}_D$ to be the set of all pairs $(m, \mathbf{w}) \in L^1([0,T] \times \mathbb{T}^d) \times L^1([0,T] \times$

$\mathbb{T}^d;\mathbb{R}^d)$ such that $m \geq 0$ almost everywhere, $\int_{\mathbb{T}^d} m(t,x)\,dx = 1$ for a.e. $t \in [0,T]$, and

$$\begin{cases} \partial_t m + \nabla \cdot (\mathbf{w}) &= 0 \quad \text{in } (0,T) \times \mathbb{T}^d \\ m(0,\cdot) &= m_0 \quad \text{in } \mathbb{T}^d. \end{cases}$$

in the sense of distributions. Because of the integrability assumption on w, it follows that $t \mapsto m(t)$ has a unique representative such that $\int_{\mathbb{T}^d} m(t,x)u(x)\,dx$ is continuous on $[0,T]$ for all $u \in C^0(\mathbb{T}^d)$ (cf. [AGS08]). It is to this representative that we refer when we write $m(t)$, and thus $m(t)$ is well-defined as a probability density for all $t \in [0,T]$.

Define the functional

$$\mathcal{B}(m,\mathbf{w}) = \int_{\mathbb{T}^d} g(x)m(T,x)\,dx + \int_0^T \int_{\mathbb{T}^d} m(t,x)L\left(x, \frac{\mathbf{w}(t,x)}{m(t,x)}\right) + F(x,m(t,x))\,dx\,dt \tag{5.3.2}$$

on $\mathcal{K}_D$. Recall that L is defined just after (H3-2). We follow the convention that

$$mL\left(x, \frac{\mathbf{w}}{m}\right) = \begin{cases} +\infty, & \text{if } \; m = 0 \text{ and } \mathbf{w} \neq 0, \\ 0, & \text{if } \; m = 0 \text{ and } \mathbf{w} = 0. \end{cases} \tag{5.3.3}$$

Since $m \geq 0$, the second integral in (5.3.2) is well-defined in $(-\infty, \infty]$ by the assumptions on F and L. The first integral is well-defined and necessarily finite by the continuity of g and the fact that $m(T,x)\,dx$ is a probability measure.

We next state the "dual problem" as

Problem 5.3.2 (Dual Problem)**.** *Find* $\inf_{(m,\mathbf{w}) \in \mathcal{K}_D} \mathcal{B}(m,\mathbf{w})$.

Proposition 5.3.1. *Problems 5.3.1 and 5.3.2 are in duality, i.e.*

$$\inf_{u \in \mathcal{K}_P} \mathcal{A}(u) = - \min_{(m,\mathbf{w}) \in \mathcal{K}_D} \mathcal{B}(m,\mathbf{w}) \tag{5.3.4}$$

Moreover, the minimum on the right-hand side is achieved by a pair $(m,\mathbf{w}) \in \mathcal{K}_D$ *having* $m \in L^\infty([0,T] \times \mathbb{T}^d)$ *and* $\mathbf{w} \in L^{r'}([0,T] \times \mathbb{T}^d;\mathbb{R}^d)$.

Proof. The proof relies on the Fenchel-Rockafellar duality theorem (see for example [ET76]) and basically follows the lines of the proof of Lemma 2.1 from [Car13b], hence we omit it. The integrability of $(m,\mathbf{w})$ is just coming from the density constraint and from the growth condition of H^*. □

Remark 5.3.1. *If* f *is strictly increasing with respect to the second variable in* $\mathbb{T}^d \times (0, \overline{m})$*, then the minimizer* $(m,\mathbf{w})$ *is unique.*

In general one cannot expect Problem 5.3.1 to have a solution. This motivates us to relax it and search for solutions in a larger class. For this let us first state the following observation.

Lemma 5.3.2. *Let* (u_n) *be a minimizing sequence for Problem 5.3.1 and set* $\alpha_n = -\partial_t u_n + H(x, Du_n)$*. Then* (u_n) *is bounded in* $BV([0,T]\times\mathbb{T}^d)\cap L^r([0,T]\times\mathbb{T}^d)$*, the sequence* (α_n) *is bounded in* $L^1([0,T]\times\mathbb{T}^d)$*, with* $\alpha_n \geq 0$ *a.e., while* (Du_n) *is bounded in* $L^r([0,T]\times\mathbb{T}^d)$*. Finally, there exists a Lipschitz continuous function* $\psi : [0,T]\times\mathbb{T}^d \to \mathbb{R}$ *such that* $\psi(T,\cdot) = g$ *and* $u_n \geq \psi$ *for any* n*.*

Proof. As $F^*(\cdot,\alpha) = 0$ for $\alpha \leq 0$, we can assume without loss of the generality that $\alpha_n \geq 0$. By comparison, $u_n \geq \psi$ where ψ is the unique Lipschitz continuous viscosity solution to

$$\begin{cases} -\partial_t\psi + H(x, D\psi) &= 0 \quad \text{in } (0,\mathrm{T})\times\mathbb{T}^{\mathrm{d}} \\ \psi(T,x) &= g(x) \quad \text{in } \mathbb{T}^{\mathrm{d}}. \end{cases}$$

So (u_n) is uniformly bounded from below. Integrating the equation for (u_n) on $[0,T]\times\mathbb{T}^d$ and using the fact that $H \geq -C$ and the fact that g is bounded, we get (up to redefining the constant $C > 0$)

$$\int_{\mathbb{T}^d} u_n(0,x)\,\mathrm{d}x \leq \int_0^T\int_{\mathbb{T}^d} \alpha_n\,\mathrm{d}x\,\mathrm{d}t + C.$$

So, by (5.2.2) and for n large enough,

$$\begin{aligned} \inf_{u\in\mathcal{K}_P}\mathcal{A}(u) + 1 &\geq \int_0^T\int_{\mathbb{T}^d} F^*(x,\alpha_n)\,\mathrm{d}x\,\mathrm{d}t - \int_{\mathbb{T}^d} u_n(0,x)m_0\,\mathrm{d}x \\ &\geq \int_0^T\int_{\mathbb{T}^d} \overline{m}\alpha_n\,\mathrm{d}x\,\mathrm{d}t - \int_{\mathbb{T}^d} u_n(0,x)m_0\,\mathrm{d}x - C \\ &\geq \int_{\mathbb{T}^d} u_n(0,x)(\overline{m} - m_0)\,\mathrm{d}x - C. \end{aligned}$$

By (H2) $\overline{m} - m_0 > \bar{c}$ and $u_n(0,\cdot)$ is bounded from below: this shows that $(u_n(0,\cdot))$ is bounded in $L^1(\mathbb{T}^d)$. Thus, as $\alpha_n \geq 0$, we also have that (α_n) is bounded in $L^1([0,T]\times\mathbb{T}^d)$. Then integrating the equation $\alpha_n = -\partial_t u_n + H(x, Du_n)$ over $[t,T]\times\mathbb{T}^d$ and using the lower bound on H, we get on the one hand

$$\int_{\mathbb{T}^d} u_n(t,x)\,\mathrm{d}x \leq \int_t^T\int_{\mathbb{T}^d} \alpha_n\,\mathrm{d}x\,\mathrm{d}t + C,$$

which, in view of the lower bound on u_n, gives an $L^\infty([0,T])$ bound on $\langle u_n(t,\cdot)\rangle = \int_{\mathbb{T}^d} u_n(t,x)\,\mathrm{d}x$. We integrate again the equation $\alpha_n = -\partial_t u_n + H(x, Du_n)$ over $[0,T]\times\mathbb{T}^d$ and use the coercivity of H and Poincaré's inequality to get

$$\begin{aligned} C \geq & \int_0^T \int_{\mathbb{T}^d} \alpha_n \,\mathrm{d}x\,\mathrm{d}t + C \geq \int_0^T \int_{\mathbb{T}^d} H(x, Du_n)\,\mathrm{d}x\,\mathrm{d}t \\ \geq & (1/C) \int_0^T \int_{\mathbb{T}^d} |Du_n|^r \,\mathrm{d}x\,\mathrm{d}t - C \\ \geq & (1/C) \int_0^T \int_{\mathbb{T}^d} |u_n - \langle u_n(t,\cdot)\rangle|^r \,\mathrm{d}x\,\mathrm{d}t - C \\ \geq & (1/C) \int_0^T \int_{\mathbb{T}^d} |u_n|^r \,\mathrm{d}x\,\mathrm{d}t - C. \end{aligned}$$

In particular (Du_n) and (u_n) are bounded in $L^r([0,T]\times\mathbb{T}^d)$. Thus $\partial_t u_n = -\alpha_n + H(x, Du_n)$ is bounded in $L^1([0,T]\times\mathbb{T}^d)$. The result follows. □

By the results of Lemma 5.3.2 we introduce a relaxation of the Problem 5.3.1. Let us denote by $\mathcal{K}_R$ the set of pairs (u,α) such that $u \in BV([0,T]\times\mathbb{T}^d)$ with $Du \in L^r([0,T]\times\mathbb{T}^d;\mathbb{R}^d)$ and $u(T^-,\cdot) \geq g$ a.e., α is a nonnegative measure on $[0,T]\times\mathbb{T}^d$, and, if we extend (u,α) by setting $u = g$ and $\alpha := H(\cdot, Dg)\,\mathrm{d}x\,\mathrm{d}t$ on $]T, T+1[\times\mathbb{T}^d$, then the pair (u,α) satisfies

$$-\partial_t u + H(x, Du) \leq \alpha$$

in the sense of distribution in $(0, T+1)\times\mathbb{T}^d$. Note that the extension of (u,α) to $[0, T+1]\times\mathbb{T}^d$ just expresses the fact that $u(T^+) = g$ and that α compensates the possible jump from $u(T^-)$ to g. We set

$$\mathcal{A}(u,\alpha) = \int_0^T \int_{\mathbb{T}^d} F^*(x, \alpha^{ac}(t,x))\,\mathrm{d}x\,\mathrm{d}t + \overline{m}\alpha^s([0,T]\times\mathbb{T}^d) - \int_{\mathbb{T}^d} u(0^+, x)\,\mathrm{d}m_0(x).$$

where α^{ac} and α^s are respectively the absolutely continuous part and the singular part of the measure α.

Problem 5.3.3 (Relaxed Problem)**.** *Find* $\inf\limits_{(u,\alpha)\in\mathcal{K}_R} \mathcal{A}(u,\alpha)$.

Let us consider the following result as a counterpart of Lemma 2.7 from [CG15] in our case.

Lemma 5.3.3. *Let $(m, \mathbf{w}) \in \mathcal{K}_D$ such that $m \in L^\infty([0,T]\times\mathbb{T}^d)$ and $(u,\alpha) \in \mathcal{K}_R$ an arbitrary competitor for Problem 5.3.3. Then, for all $t \in [0,T]$, we have*

$$\begin{aligned} \int_0^t \int_{\mathbb{T}^d} -mH^*\left(x, -\frac{\mathbf{w}}{m}\right)\mathrm{d}x\,\mathrm{d}t \leq & \int_{\mathbb{T}^d} m(t,x)u(t^-,x)\,\mathrm{d}x - \int_{\mathbb{T}^d} m_0(x)u(0^+,x)\,\mathrm{d}x \\ & + \int_0^t \int_{\mathbb{T}^d} \alpha^{ac} m\,\mathrm{d}x\,\mathrm{d}t + \overline{m}\alpha^s([0,t]\times\mathbb{T}^d) \end{aligned}$$

and

$$\int_t^T \int_{\mathbb{T}^d} -mH^*\left(x, -\frac{\mathbf{w}}{m}\right) \mathrm{d}x\,\mathrm{d}t \leq \int_{\mathbb{T}^d} m(T,x)g(x)\,\mathrm{d}x - \int_{\mathbb{T}^d} m(t,x)u(t^+,x)\,\mathrm{d}x + \int_t^T \int_{\mathbb{T}^d} \alpha^{ac} m\,\mathrm{d}x\,\mathrm{d}t + \overline{m}\alpha^s([t,T]\times\mathbb{T}^d).$$

Moreover we can take $t = 0$ in the above inequalities. If, finally, equality holds in the second inequality when $t = 0$, then $\mathbf{w} = -mD_pH(\cdot, Du)$ a.e. and

$$\limsup_{\varepsilon\to 0} m_\varepsilon(t,x) = \overline{m} \ \ \textit{for}\ \alpha^s - \textit{a.e.}\ (t,x) \in [0,T]\times\mathbb{T}^d,$$

where m_ε is any standard mollification of m.

Proof. We prove the result only for $t = 0$, the general case follows from a similar (and simpler) argument. We first extend the pairs (u,α) and $(m,\mathbf{w})$ to $(0,T+1)\times\mathbb{T}^d$ by setting $u = g$ and $\alpha := H(\cdot, Dg)\,\mathrm{d}x\,\mathrm{d}t$, $m(s,x) = m(T,x)$, $\mathbf{w}(s,x) = 0$ on $(T,T+1)\times\mathbb{T}^d$. Note that

$$\partial_t m + \nabla\cdot\mathbf{w} = 0 \quad \text{and} \quad -\partial_t u + H(x,Du) \leq \alpha \quad \text{on} \quad (0,T+1)\times\mathbb{T}^d.$$

We smoothen the pair $(m,\mathbf{w})$ in a standard way into $(m_\varepsilon, \mathbf{w}_\varepsilon)$: $m_\varepsilon := m\star\rho_\varepsilon$ and $\mathbf{w}_\varepsilon := \mathbf{w}\star\rho_\varepsilon$, where the mollifier ρ has a support in the unit ball of $\mathbb{R}^{d+1}$ and $\rho_\varepsilon := \varepsilon^{-d-1}\rho(\cdot/\varepsilon)$. Then, for any $\eta > \varepsilon$, we have, since $m_\varepsilon \leq \overline{m}$,

$$\begin{aligned}\int_\eta^{T+\eta}\int_{\mathbb{T}^d} u\partial_t m_\varepsilon + m_\varepsilon H(x,Du)\,\mathrm{d}x\,\mathrm{d}t - \left[\int_{\mathbb{T}^d} m_\varepsilon u\,\mathrm{d}x\right]_{\eta^+}^{T+\eta} \\ \leq \int_\eta^{T+\eta}\int_{\mathbb{T}^d} m_\varepsilon\,\mathrm{d}\alpha \leq \int_\eta^{T+\eta}\int_{\mathbb{T}^d}\alpha^{ac} m_\varepsilon\,\mathrm{d}x\,\mathrm{d}t + \overline{m}\alpha^s([\eta,T+\eta]\times\mathbb{T}^d)\end{aligned} \tag{5.3.5}$$

where, as $\partial_t m_\varepsilon + \nabla\cdot\mathbf{w}_\varepsilon = 0$,

$$\int_\eta^{T+\eta}\int_{\mathbb{T}^d} u\partial_t m_\varepsilon\,\mathrm{d}x\,\mathrm{d}t = \int_\eta^{T+\eta}\int_{\mathbb{T}^d} Du\cdot\mathbf{w}_\varepsilon\,\mathrm{d}x\,\mathrm{d}t.$$

In the followings, we shall consider only such η's for which no jump of u occurs, in particular $\int_{\mathbb{T}^d} u(\eta^-,x)\,\mathrm{d}x = \int_{\mathbb{T}^d} u(\eta^+,x)\,\mathrm{d}x = \int_{\mathbb{T}^d} u(\eta,x)\,\mathrm{d}x$. So, by convexity of H, (5.3.5) and the above equality,

$$\begin{aligned}\int_\eta^{T+\eta}\int_{\mathbb{T}^d} -m_\varepsilon H^*\left(x, -\frac{\mathbf{w}_\varepsilon}{m_\varepsilon}\right)\mathrm{d}x\,\mathrm{d}t \ &\leq \int_\eta^{T+\eta}\int_{\mathbb{T}^d}\mathbf{w}_\varepsilon\cdot Du + m_\varepsilon H(x,Du)\,\mathrm{d}x\,\mathrm{d}t \\ &\leq \left[\int_{\mathbb{T}^d} m_\varepsilon u\,\mathrm{d}x\right]_\eta^{T+\eta} + \int_\eta^{T+\eta}\int_{\mathbb{T}^d}\alpha^{ac} m_\varepsilon\,\mathrm{d}x\,\mathrm{d}t + \overline{m}\alpha^s([\eta,T+\eta]\times\mathbb{T}^d).\end{aligned}$$

We multiply the inequality $-\partial_t u + H(x,Du) \leq \alpha$ by $m_\varepsilon(\eta,\cdot)$ and integrate on $(0,\eta)\times\mathbb{T}^d$ to get, as m_ε is bounded by $\overline{m}$ and H is bounded from below,

$$\int_{\mathbb{T}^d} u(0^+,x)m_\varepsilon(\eta,x)\,\mathrm{d}x \leq \int_{\mathbb{T}^d} u(\eta,x)m_\varepsilon(\eta,x)\,\mathrm{d}x + C\eta + \overline{m}\alpha((0,\eta)\times\mathbb{T}^d).$$

Note that $\overline{m}\alpha([0,\eta)\times\mathbb{T}^d)=\overline{m}\alpha^s([0,\eta)\times\mathbb{T}^d)+o(1)$, where $o(1)\to 0$ as $\eta\to 0$. So

$$\int_\eta^{T+\eta}\int_{\mathbb{T}^d}-m_\varepsilon H^*\left(x,-\frac{\mathbf{w}_\varepsilon}{m_\varepsilon}\right)\mathrm{d}x\,\mathrm{d}t\le$$
$$\le\int_{\mathbb{T}^d}[m_\varepsilon(T+\eta,x)u(T+\eta,x)-u(0^+,x)m_\varepsilon(\eta,x)]\,\mathrm{d}x$$
$$+\int_\eta^{T+\eta}\int_{\mathbb{T}^d}\alpha^{ac}m_\varepsilon\,\mathrm{d}x\,\mathrm{d}t+\overline{m}\alpha^s([0,T+\eta]\times\mathbb{T}^d)+o(1).$$

We now let $\varepsilon\to 0$. By convergence of $(m_\varepsilon,\mathbf{w}_\varepsilon)$ to $(m,\mathbf{w})$ in $L^q\times L^{r'}$, $\forall q\ge 1$, and by the fact that

$$\lim_{\varepsilon\to 0}\int_\eta^{T+\eta}\int_{\mathbb{T}^d}m_\varepsilon H^*(x,-\mathbf{w}_\varepsilon/m_\varepsilon)\,\mathrm{d}x\,\mathrm{d}t=\int_\eta^{T+\eta}\int_{\mathbb{T}^d}mH^*(x,-\mathbf{w}/m)\,\mathrm{d}x\,\mathrm{d}t,$$

(see the proof of Lemma 2.7. from [CG15]) we obtain for all $\eta>0$, chosen above,

$$\int_\eta^{T+\eta}\int_{\mathbb{T}^d}-mH^*\left(x,-\frac{\mathbf{w}}{m}\right)\mathrm{d}x\,\mathrm{d}t\le$$
$$\le\int_{\mathbb{T}^d}[m(T+\eta,x)u(T+\eta,x)-u(0^+,x)m(\eta,x)]\,\mathrm{d}x$$
$$+\int_\eta^{T+\eta}\int_{\mathbb{T}^d}\alpha^{ac}m\,\mathrm{d}x\,\mathrm{d}t+\overline{m}\alpha^s([0,T+\eta]\times\mathbb{T}^d)+o(1).$$

By definition of the extension of the maps u and m,

$$\int_{\mathbb{T}^d}[m(T+\eta)u(T+\eta)-u(0^+,x)m(\eta,x)]\,\mathrm{d}x$$
$$+\int_\eta^{T+\eta}\int_{\mathbb{T}^d}\alpha^{ac}m\,\mathrm{d}x\,\mathrm{d}t+\overline{m}\alpha^s([0,T+\eta]\times\mathbb{T}^d)$$
$$=\int_{\mathbb{T}^d}[m(T,x)g(x)-u(0^+,x)m(\eta,x)]\,\mathrm{d}x$$
$$+\int_\eta^{T+\eta}\int_{\mathbb{T}^d}\alpha^{ac}m\,\mathrm{d}x\,\mathrm{d}t+\overline{m}\alpha^s([0,T]\times\mathbb{T}^d).$$

We finally let $\eta\to 0$ and get

$$\int_0^T\int_{\mathbb{T}^d}-mH^*\left(x,-\frac{\mathbf{w}}{m}\right)\mathrm{d}x\,\mathrm{d}t$$
$$\le\int_{\mathbb{T}^d}[m(T,x)g(x)-u(0^+,x)m_0(x)]\,\mathrm{d}x+\int_0^T\int_{\mathbb{T}^d}\alpha^{ac}m\,\mathrm{d}x\,\mathrm{d}t$$
$$+\overline{m}\alpha^s([0,T]\times\mathbb{T}^d)$$

thanks to the $L^\infty-$weak-$\star$ continuity of $t\mapsto m(t)$ and the L^1 integrability of $u(0^+,\cdot)$.

The proof of the equality $\mathbf{w}=-mD_pH(\cdot,Du)$ when equality holds in the above inequality follows exactly the proof of the corresponding statement in

[CG15], so we omit it. Note that, if equality holds, then all the above inequalities must become equalities as ε and then η tend to 0. In particular, from inequality (5.3.5), we must have

$$\lim_{\eta\to 0}\limsup_{\varepsilon\to 0}\int_\eta^{T+\eta}\int_{\mathbb{T}^d} m_\varepsilon(t,x)\,\mathrm{d}\alpha^s(t,x) = \overline{m}\alpha^s([0,T]\times\mathbb{T}^d).$$

By Fatou's lemma, this implies that

$$\overline{m}\alpha^s([0,T]\times\mathbb{T}^d) \le \int_0^T\int_{\mathbb{T}^d}\limsup_{\varepsilon\to 0} m_\varepsilon(t,x)\,\mathrm{d}\alpha^s(t,x),$$

where the right-hand side is also bounded above by the left-hand side since $m_\varepsilon \le \overline{m}$. So $\limsup_{\varepsilon\to 0} m_\varepsilon = \overline{m}$ α^s-a.e. □

Proposition 5.3.4. *We have*

$$\inf_{u\in\mathcal{K}_P}\mathcal{A}(u) = \min_{(u,\alpha)\in\mathcal{K}_R}\mathcal{A}(u,\alpha). \tag{5.3.6}$$

Moreover, if (u,α) is a minimum of $\mathcal{A}$, then $\alpha = \alpha \llcorner (0,T)\times\mathbb{T}^d + (u(T^-,\cdot)-g)\,\mathrm{d}\left(\delta_T\otimes\mathscr{H}^d\llcorner\mathbb{T}^d\right)$.

Proof. We follow here [Gra14]. Inequality $\ge$ is obvious and we now prove the reverse one. Let us fix $(u,\alpha)\in\mathcal{K}_R$ and let $(m,\mathbf{w})$ be an optimal solution for the dual problem. Then

$$\begin{aligned}\mathcal{A}(u,\alpha) &= \int_0^T\int_{\mathbb{T}^d} F^*(x,\alpha^{ac})\,\mathrm{d}x\,\mathrm{d}t + \overline{m}\alpha^s([0,T]\times\mathbb{T}^d) - \int_{\mathbb{T}^d} m_0(x)u(0^+,x)\,\mathrm{d}x\\ &\ge \int_0^T\int_{\mathbb{T}^d}(m\alpha^{ac} - F(x,m))\,\mathrm{d}x\,\mathrm{d}t + \overline{m}\alpha^s([0,T]\times\mathbb{T}^d) - \int_{\mathbb{T}^d} m_0(x)u(0^+,x)\,\mathrm{d}x\\ &\ge \int_0^T\int_{\mathbb{T}^d}(-mH^*(x,-\mathbf{w}/m) - F(x,m))\,\mathrm{d}x\,\mathrm{d}t - \int_{\mathbb{T}^d} m(T,x)g(x)\,\mathrm{d}x\\ &= -\mathcal{B}(m,\mathbf{w})\end{aligned}$$

where the last inequality comes from Lemma 5.3.3. By optimality of $(m,\mathbf{w})$ and (5.3.4) we obtain therefore

$$\mathcal{A}(u,\alpha) \ge \inf_{u\in\mathcal{K}_P}\mathcal{A}(u),$$

which shows equality (5.3.6).

To prove that the problem in the right-hand side of (5.3.6) has a minimum we consider a minimizing sequence (u_n) for Problem 5.3.1. We extend $u_n = g$ on $(T,T+1)\times\mathbb{T}^d$ and set $\alpha_n := -\partial_t u_n + H(x,Du_n)$ on $[0,T+1]\times\mathbb{T}^d$ and note that, in view of Lemma 5.3.2, there is a subsequence, again denoted by (u_n,α_n), such that (u_n) converges in L^1 to a BV map u, (Du_n) converges weakly in L^r, and (α_n) converges in sense of measures to α on $(0,T+1)\times\mathbb{T}^d$. As $u_n \ge \psi$ on $[0,T]\times\mathbb{T}^d$, we also have $u\ge\psi$ $(0,T)\times\mathbb{T}^d$, so that $u(T^-,\cdot)\ge\psi(T,\cdot) = g$.

By convexity of H with respect to p, the pair (u,α) belongs to $\mathcal{K}_R$. One easily shows by standard relaxation that

$$\mathcal{A}(u,\alpha) \le \liminf_{n\to\infty} \mathcal{A}(u_n).$$

Hence (u,α) is a minimum.

Let us finally check that $\alpha = \alpha \llcorner (0,T)\times\mathbb{T}^d + (u(T^-,\cdot)-g)\,\mathrm{d}\left(\delta_T\otimes\mathscr{H}^d\llcorner\mathbb{T}^d\right)$. Indeed, by definition of $\mathcal{K}_R$, we can extend (u,α) by setting

$$(u,\alpha) := (g, H(\cdot, Dg)) \ \text{ on } \ (T,T+1)\times\mathbb{T}^d$$

and the following inequality holds in the sense of measure in $(0,T+1)\times\mathbb{T}^d$:

$$-\partial_t u + H(x,Du) \le \alpha.$$

Let $\phi\in C^\infty(\mathbb{T}^d)$, with $\phi\ge 0$, be a test function. We multiply the above inequality by ϕ and integrate on $(T-\eta,T+\eta)\times\mathbb{T}^d$ to get

$$\begin{aligned}\int_{\mathbb{T}^d}\phi(x)u((T-\eta)^+,x)\,\mathrm{d}x + \int_{T-\eta}^{T}\int_{\mathbb{T}^d}\phi H(x,Du)\,\mathrm{d}x\,\mathrm{d}t \\ \le \int_{\mathbb{T}^d} g\phi\,\mathrm{d}x + \int_{T-\eta}^{T}\int_{\mathbb{T}^d}\phi\,\mathrm{d}\alpha.\end{aligned}$$

Letting $\eta\to 0$ along a suitable sequence such that $u((T-\eta)^+,\cdot)\to u(T^-,\cdot)$ in L^1, we obtain:

$$\int_{\mathbb{T}^d}\phi u(T^-,x)\,\mathrm{d}x \le \int_{\mathbb{T}^d} g\phi\,\mathrm{d}x + \int_{\mathbb{T}^d}\phi\,\mathrm{d}(\alpha\llcorner\{T\}\times\mathbb{T}^d).$$

This means that $u(T^-,\cdot)\le g+\alpha\llcorner\{T\}\times\mathbb{T}^d$. Let us now replace α by

$$\tilde\alpha := \alpha\llcorner(0,T)\times\mathbb{T}^d + (u(T^-,\cdot)-g)\,\mathrm{d}\left(\delta_T\otimes\mathscr{H}^d\llcorner\mathbb{T}^d\right).$$

We claim that the pair $(u,\tilde\alpha)$ still belongs to $\mathcal{K}_R$. For this we just have to check that, if we extend $(u,\tilde\alpha)$ to $(0,T+1)\times\mathbb{T}^d$ as before, then

$$-\partial_t u + H(x,Du)\le\tilde\alpha \ \text{ on } \ (0,T+1)\times\mathbb{T}^d$$

holds in the sense of distributions. Let $\phi\in C_c^\infty((0,T+1)\times\mathbb{T}^d)$ with $\phi\ge 0$. Then

$$\begin{aligned}\int_0^{T+1}\int_{\mathbb{T}^d} u\partial_t\phi + \phi H(x,Du)\,\mathrm{d}x\,\mathrm{d}t &= \int_0^{T}\int_{\mathbb{T}^d} u\partial_t\phi + \phi H(x,Du)\,\mathrm{d}x\,\mathrm{d}t \\ &\quad + \int_T^{T+1}\int_{\mathbb{T}^d} g\partial_t\phi + \phi H(x,Dg)\,\mathrm{d}x\,\mathrm{d}t \\ &\le \int_0^{T}\int_{\mathbb{T}^d}\phi\,\mathrm{d}(\alpha\llcorner(0,T)\times\mathbb{T}^d) \\ &\quad + \int_{\mathbb{T}^d}(u(T^-,x)-g)\phi(T,x)\,\mathrm{d}x \\ &\quad + \int_T^{T+1}\int_{\mathbb{T}^d}\phi H(x,Dg)\,\mathrm{d}x\,\mathrm{d}t \\ &\le \int_0^{T+1}\int_{\mathbb{T}^d}\phi\,\mathrm{d}\tilde\alpha.\end{aligned}$$

This proves that the pair $(u, \tilde{\alpha})$ belongs to $\mathcal{K}_R$. In particular, $\mathcal{A}(u, \alpha) \leq \mathcal{A}(u, \tilde{\alpha})$, so that

$$\overline{m}\alpha^s(\{T\} \times \mathbb{T}^d) \leq \overline{m} \int_{\mathbb{T}^d} u(T^-, x) - g(x) \, dx.$$

Since we have proved that $u(T^-, \cdot) \leq g + \alpha \llcorner \{T\} \times \mathbb{T}^d$, we have therefore an equality in the above inequality, which means that $\alpha \llcorner \{T\} \times \mathbb{T}^d = (u(T^-, \cdot) - g) \, dx$. □

5.4 THE MFG SYSTEM WITH DENSITY CONSTRAINTS

In this section, we study the existence of solutions for the MFG system with density constraints:

$$\begin{cases} \text{(i)} & -\partial_t u + H(x, Du) = f(x, m) + \beta & \text{in } (0, T) \times \mathbb{T}^d \\ \text{(ii)} & \partial_t m - \nabla \cdot \big(m D_p H(x, Du)\big) = 0 & \text{in } (0, T) \times \mathbb{T}^d \\ \text{(iii)} & u(T, x) = g(x) + \beta_T, \quad m(0, x) = m_0(x) & \text{in } \mathbb{T}^d \\ \text{(iv)} & 0 \leq m \leq \overline{m} & \text{in } (0, T) \times \mathbb{T}^d \end{cases} \tag{5.4.1}$$

under the assumptions on H, f, g and m_0 stated in Section 5.2. We also study the approximation of the solution of this system by the solution of the classical MFG system.

5.4.1 *Solutions of the MFG system with density constraints*

Definition 5.4.1. *We say that (u, m, β, β_T) is a solution to the MFG system (5.4.1) if*

1. *Integrability conditions: β is a nonnegative Radon measure on $(0, T) \times \mathbb{T}^d$, $\beta_T \in L^1(\mathbb{T}^d)$ is nonnegative, $u \in BV([0, T] \times \mathbb{T}^d) \cap L^r([0, T] \times \mathbb{T}^d)$, $Du \in L^r([0, T] \times \mathbb{T}^d; \mathbb{R}^d)$, $m \in L^1([0, T] \times \mathbb{T}^d)$ and $0 \leq m \leq \overline{m}$ a.e.,*
2. *The following inequality*

$$-\partial_t u + H(x, Du(t, x)) \leq f(x, m) + \beta \tag{5.4.2}$$

holds in $(0, T) \times \mathbb{T}^d$ in the sense of measures, with the boundary condition on $\mathbb{T}^d$

$$g \leq u(T^-, \cdot) = g + \beta_T \text{ a.e.}$$

Moreover, $\beta^{ac} = 0$ a.e. in $\{m < \overline{m}\}$ and

$$\limsup_{\varepsilon \to 0} m_\varepsilon(t, x) = \overline{m} \qquad \beta^s - a.e. \text{ if } t < T \text{ and a.e. in } \{\beta_T > 0\} \text{ if } t = T, \tag{5.4.3}$$

where m_ε is any standard mollification of m.

3. *Equality*

$$\partial_t m - \nabla \cdot (m D_p H(x, Du(t,x))) = 0, \qquad m(0) = m_0$$

holds in the sense of distribution,

4. *Equality*

$$\int_0^T \int_{\mathbb{T}^d} m(-H(x,Du) + Du \cdot D_p H(x,Du) + f(x,m) + \beta^{ac})\,\mathrm{d}x\,\mathrm{d}t$$
$$+\overline{m}\beta^s((0,T)\times\mathbb{T}^d) + \overline{m}\int_{\mathbb{T}^d} \beta_T\,\mathrm{d}x = \int_{\mathbb{T}^d} m_0(x)u(0^+,x) - m(T,x)g(x)\,\mathrm{d}x$$

holds.

Let us recall that, in the above definition, β^{ac} and β^s denote the absolutely continuous part and the singular part of the measure β.

Some comments on the definition are now in order. Equality (5.4.3) is a weak way of stating that $m = \overline{m}$ in the support of β and β_T respectively, while the last requirement formally says that equality $-\partial_t u + H(x, Du(t,x)) = f(x,m) + \beta$ holds.

We can state the main result of this section.

Theorem 5.4.1. *Let $(u,\alpha) \in \mathcal{K}_R$ be a solution of the relaxed Problem 5.3.3 and $(m,\mathbf{w}) \in \mathcal{K}_D$ be a solution of the dual Problem 5.3.2. Then $\alpha \geq f(\cdot,m)$ as measures, and, if we set*

$$\beta := \alpha \llcorner (0,T)\times\mathbb{T}^d - f(\cdot,m)\,\mathrm{d}x\,\mathrm{d}t$$

and $\beta_T := \alpha \llcorner \{T\}\times\mathbb{T}^d$, the quadruplet (u,m,β,β_T) is a solution of the MFG system (5.4.1).

Conversely, let (u,m,β,β_T) be a solution of the MFG system (5.4.1). Let us set

$$\alpha := f(\cdot,m)\,\mathrm{d}x\,\mathrm{d}t + \beta + \beta_T\,\mathrm{d}(\delta_T \otimes \mathscr{H}^d \llcorner \mathbb{T}^d) \tag{5.4.4}$$

and $\mathbf{w} = -mD_pH(x,Du)$. Then the pair (u,α) is a solution of the relaxed problem while the pair $(m,\mathbf{w})$ is a solution of the dual problem.

The proof of this results goes along the same lines as in [CG15] (Theorem 3.5). However for the completeness (and because of some differences) we sketch it here.

Proof. Let $(u,\alpha) \in \mathcal{K}_R$ be a solution of the Problem 5.3.3 and $(m,\mathbf{w}) \in \mathcal{K}_D$ be the solution of Problem 5.3.2. First, by the definition of Legendre-Fenchel transform we have for a.e. $(t,x) \in [0,T]\times\mathbb{T}^d$

$$F^*(x,\alpha^{ac}(t,x)) + F(x,m(t,x)) - \alpha^{ac}(t,x)m(t,x) \geq 0. \tag{5.4.5}$$

On the other hand by optimality we have that

$$
\begin{aligned}
0 &= \mathcal{A}(u,\alpha) + \mathcal{B}(m,\mathbf{w}) \\
&= \int_0^T \int_{\mathbb{T}^d} F^*(x,\alpha^{ac}(t,x))\,\mathrm{d}x\,\mathrm{d}t + \overline{m}\alpha^s([0,T]\times\mathbb{T}^d) \\
&- \int_{\mathbb{T}^d} u(0^+,x)\,m_0(x)\,\mathrm{d}x + \int_{\mathbb{T}^d} g(x)m(T,x)\,\mathrm{d}x \\
&+ \int_0^T \int_{\mathbb{T}^d} m(t,x)H^*\left(x, -\frac{\mathbf{w}(t,x)}{m(t,x)}\right) + F(x,m(t,x))\,\mathrm{d}x\,\mathrm{d}t \\
&\geq \int_0^T \int_{\mathbb{T}^d} \alpha^{ac} m\,\mathrm{d}x\,\mathrm{d}t + \overline{m}\alpha^s([0,T]\times\mathbb{T}^d) \\
&+ \int_{\mathbb{T}^d} g(x)m(T,x) - u(0^+,x)\,m_0(x)\,\mathrm{d}x \\
&+ \int_0^T \int_{\mathbb{T}^d} m(t,x)H^*\left(x, -\frac{\mathbf{w}(t,x)}{m(t,x)}\right)\mathrm{d}x\,\mathrm{d}t \;\geq\; 0,
\end{aligned}
$$

where we used the Lemma 5.3.3 for the last inequality. This means that all the inequalities in the previous lines are equalities. In particular we have an equality in (5.4.5), which implies

$$
\alpha^{ac}(t,x) \in \partial_m F(x,m(t,x)) \qquad \text{a.e.}
$$

As $\partial_m F(x,m(t,x)) = \{f(x,m(t,x))\}$ for $0 < m(t,x) < \overline{m}$ a.e., we have $\alpha^{ac}(t,x) = f(x,m(t,x))$ a.e. in $\{0 < m < \overline{m}\}$. Moreover, as $\partial_m F(x,0) = (-\infty,0]$ and $\alpha^{ac} \geq 0$, we also have $\alpha^{ac} = 0 = f(\cdot,0)$ a.e. in $\{m = 0\}$. Finally, since $\partial_m F(x,\overline{m}) = [f(x,\overline{m}),+\infty)$, $\alpha^{ac} \geq f(x,m(t,x))$ a.e. on $\{m = \overline{m}\}$. Therefore $\alpha^{ac} \geq f(\cdot,m)$ a.e. Let us set $\beta := \alpha \llcorner (0,T)\times\mathbb{T}^d - f(\cdot,m)\,\mathrm{d}x\,\mathrm{d}t$ and $\beta_T := \alpha \llcorner \{T\}\times\mathbb{T}^d$. From Proposition 5.3.4 we know that $\beta_T = u(T^-,\cdot) - g$.

Since equality holds in the above inequalities, there is an equality in the inequality of Lemma 5.3.3: thus point 4 holds in Definition 5.4.1. Moreover, by Lemma 5.3.3, we have that $\mathbf{w} = -mD_pH(\cdot,Du)$ a.e. and (5.4.3) holds. In conclusion, since $(u,\alpha) \in \mathcal{K}_R$ and $(m,\mathbf{w}) \in \mathcal{K}_D$, the quadruplet (u,m,β,β_T) is a solution to the MFG system (5.4.1).

Now let us prove the converse statement. For this let us take a solution (m,u,β,β_T) of the MFG system (5.4.1) in the sense of the Definition 5.4.1. Let us define α as in (5.4.4) and $\mathbf{w} := -mD_pH(\cdot,Du)$. We shall prove that (u,α) is a solution for the Problem 5.3.3 and $(m,\mathbf{w})$ is a solution of Problem 5.3.2. For the first one, one easily checks, following the argument of Proposition 5.3.4, that $(u,\alpha) \in \mathcal{K}_R$. Let us now consider a competitor $(\tilde{u},\tilde{\alpha}) \in \mathcal{K}_R$. Using the equality in Lemma 5.3.3 for $(u, \alpha, m, -mD_pH(\cdot, Du))$ and the inequality for $(\tilde{u}, \tilde{\alpha}, m, -mD_pH(\cdot, Du))$ we have

$$\begin{aligned}\mathcal{A}(\tilde{u}, \tilde{\alpha}) &= \int_0^T \int_{\mathbb{T}^d} F^*(x, \tilde{\alpha}^{ac}(x))\,\mathrm{d}x\,\mathrm{d}t + \overline{m}\tilde{\alpha}^s([0,T]\times\mathbb{T}^d) \\ &\quad - \int_{\mathbb{T}^d} \tilde{u}(0^+, x)m_0(x)\,\mathrm{d}x \\ &\geq \int_0^T \int_{\mathbb{T}^d} F^*(x, \alpha^{ac}(x)) + m(\tilde{\alpha}^{ac} - \alpha^{ac})\,\mathrm{d}x\,\mathrm{d}t + \overline{m}\tilde{\alpha}^s([0,T]\times\mathbb{T}^d) \\ &\quad - \int_{\mathbb{T}^d} \tilde{u}(0^+, x)m_0(x)\,\mathrm{d}x \\ &\geq \int_0^T \int_{\mathbb{T}^d} F^*(x, \alpha^{ac}(x))\,\mathrm{d}x\,\mathrm{d}t + \overline{m}\alpha^s([0,T]\times\mathbb{T}^d) \\ &\quad - \int_{\mathbb{T}^d} u(0^+, x)m_0(x)\,\mathrm{d}x,\end{aligned}$$

thus (u, α) is a minimizer for the Problem 5.3.3.

In a similar manner we can show that $(m, \mathbf{w})$ is a solution for Problem 5.3.2. Hence the statement of the theorem follows. □

We now briefly discuss the uniqueness of the approximation of solutions to the MFG system with density constraints. If $F = F(x, m)$ is strictly convex on $[0, \overline{m}]$ with respect to the m variable, then, as $H^* = H^*(x, q)$ is strictly convex with respect to q (because $H = H(x, p)$ is C^1 in p), we can conclude that the dual Problem 5.3.2 has a unique minimizer. In particular, in this case, the m component of the MFG system (5.4.1) is unique. We do not expect uniqueness of the u component: this is not the case in the “classical setting", i.e., without density constraint (see, however, the discussion in [Car13b]). For this reason, the fact that one can approximate *any* solution of the MFG system (5.4.1) by regular maps with suitable property is not straightforward. This is the aim of the next Lemma, needed in the sequel, where we explain that the β component of any solution can be approached by a minimizing sequence of Lipschitz maps with some optimality property.

Lemma 5.4.2. *Let (u, m, β, β_T) be a solution to the MFG system (5.4.1). Then there exists Lipschitz continuous maps (u_n, α_n) such that*

(i) u_n satisfies a.e. and in the viscosity sense,

$$-\partial_t u_n + H(x, Du_n) = \alpha_n \text{ in } (0,T)\times\mathbb{T}^d,$$

(ii) the pair (u_n, α_n) is a minimizing sequence for Problem 5.3.1 and Problem 5.3.3,

(iii) (u_n) is bounded from below, is bounded in $BV([0,T]\times\mathbb{T}^d)\cap L^r([0,T]\times\mathbb{T}^d)$ and (Du_n) is bounded in $L^r([0,T]\times\mathbb{T}^d)$,

(iv) (u_n) converges to some $\tilde{u} \geq u$ in $L^1((0,T)\times\mathbb{T}^d)$ with $\tilde{u} = u$ m−a.e.,

(v) (α_n) is bounded in $L^1((0,T)\times\mathbb{T}^d)$ and converges in measure on $[0,T]\times\mathbb{T}^d$ to α defined from (β, β_T) by (5.4.4),

(vi) $(\tilde{u}, m, \beta, \beta_T)$ is a solution to the MFG system (5.4.1),

Proof. Let us define α as in (5.4.4) and recall that, by Theorem 5.4.1, (u, α) is a minimum in the relaxed Problem 5.3.3. In particular, (u, α) belongs to $\mathcal{K}_R$, which means that, if we extend (u, α) by $(g, H(\cdot, Dg))$ in $[T, T+1] \times \mathbb{T}^d$, then

$$-\partial_t u + H(x, Du) \leq \alpha \text{ in } (0, T+1) \times \mathbb{T}^d.$$

For $\eta \in (0,1)$ we set $u^\eta(t,x) := u(t+\eta, x)$ and $\alpha^\eta := (\tau_\eta)_\# \alpha$, where $\tau_\eta : (0, T+1) \times \mathbb{T}^d \to (-\eta, T+1-\eta) \times \mathbb{T}^d$ is the time shift $\tau_\eta(t,x) = (t-\eta, x)$. We then smoothen u^η into $u^\eta \star \rho_\varepsilon$ where $\varepsilon \in (0, \eta/2)$, ρ is a standard even mollifier with compact support in $\mathbb{R}^{d+1}$ and $\rho_\varepsilon(\cdot) := \varepsilon^{-d-1}\rho(\cdot/\varepsilon)$. We note that $u^\eta \star \rho_\varepsilon(t,x)$ for $t \in [T-\varepsilon, T]$ is a mollified version of g. We finally slightly modify $u^\eta \star \rho_\varepsilon$ so that it satisfied the boundary condition: let $\zeta : \mathbb{R} \to [0,1]$ be smooth, increasing, with $\zeta(s) = 0$ for $s \leq -1$ and $\zeta(s) = 1$ for $s \geq 0$. Set $\zeta_\varepsilon(s) = \zeta(\varepsilon^{-1}s)$, $u^{\eta,\varepsilon}(t,x) = (1 - \zeta_\varepsilon(t-T))u^\eta \star \rho_\varepsilon(t,x) + \zeta_\varepsilon(t-T)g(x)$. Then $u^{\eta,\varepsilon}(T,x) = g(x)$ and

$$-\partial_t u^{\eta,\varepsilon} + H(x, Du^{\eta,\varepsilon}) \leq \alpha^{\eta,\varepsilon} \text{ in } (0, \mathrm{T}+1) \times \mathbb{T}^{\mathrm{d}},$$

where

$$\alpha^{\eta,\varepsilon} := \Big[(\alpha^\eta - H(\cdot, Du^\eta)) \star \rho_\varepsilon + H(x, Du^{\eta,\varepsilon}) - \varepsilon^{-1}\zeta'_\varepsilon(t-T)(g(x) - g \star \rho_\varepsilon(t,x))\Big]_+ .$$

As $\varepsilon \to 0$, $u^{\eta,\varepsilon}$ is bounded in BV and converges to u^η in L^1 while $\alpha^{\eta,\varepsilon}$ is non-negative, bounded in L^1 and converges to α^η as a measure. We have

$$\mathcal{A}(u^{\eta,\varepsilon}, \alpha^{\eta,\varepsilon}) = \int_0^T \int_{\mathbb{T}^d} F^*\left(x, \alpha^{\eta,\varepsilon}(t,x))\right) \mathrm{d}x\,\mathrm{d}t - \int_{\mathbb{T}^d} u \star \rho_\varepsilon(\eta, x) m_0(x)\,\mathrm{d}x.$$

As $\varepsilon \to 0$, the first integral in the right-hand side converges to

$$\int_0^T \int_{\mathbb{T}^d} F^*(x, (\alpha^\eta)^{ac}(t,x))\,\mathrm{d}x\,\mathrm{d}t + \overline{m}(\alpha^\eta)^s((0,T] \times \mathbb{T}^d)$$
$$= \int_0^T \int_{\mathbb{T}^d} F^*(x, \alpha^{ac}(t+\eta, x))\,\mathrm{d}x\,\mathrm{d}t + \overline{m}\alpha^s([\eta, T+\eta] \times \mathbb{T}^d).$$

Pick now a sequence (η_n) tending to 0, such that $u \star \rho_\varepsilon(\eta_n, \cdot)$ converges in L^1 to $u(\eta_n, \cdot)$ as $\varepsilon \to 0$ (this is the case for a.e. η) and $(u(\eta_n, \cdot))$ tends in L^1 to $u(0^+, \cdot)$ as $n \to +\infty$: then

$$\limsup_n \limsup_{\varepsilon \to 0} \mathcal{A}(u^{\eta_n,\varepsilon}) \leq \limsup_n \mathcal{A}(u^{\eta_n}, \alpha^{\eta_n}) = \mathcal{A}(u, \alpha).$$

As (u, α) is a minimum in the relaxed Problem 5.3.3, we can find $\varepsilon_n \to 0$ such that $(u^{\eta_n,\varepsilon_n}, \alpha^{\eta_n,\varepsilon_n})$ is a minimizing sequence for Problem 5.3.3 thanks to Proposition 5.3.4.

Let now $\tilde{u}_n$ be the viscosity solution to

$$\begin{cases} -\partial_t u + H(x, Du) = \alpha^{\varepsilon_n,\eta_n} \text{ in } (0, \mathrm{T}) \times \mathbb{T}^{\mathrm{d}} \\ u(T,x) = g(x) \text{ in } \mathbb{T}^{\mathrm{d}}. \end{cases}$$

Standard results on viscosity solutions imply that $\tilde{u}_n$ is Lipschitz continuous (because so are $\alpha^{\varepsilon_n,\eta_n}$ and g), satisfies the equation a.e. and, by comparison, is such that $\tilde{u}_n \geq u^{\eta_n,\varepsilon_n}$. Therefore

$$\mathcal{A}(\tilde{u}_n, \alpha^{\varepsilon_n,\eta_n}) \leq \mathcal{A}(u^{\eta_n,\varepsilon_n}, \alpha^{\eta_n,\varepsilon_n}),$$

so that $(\tilde{u}_n, \alpha^{\varepsilon_n,\eta_n})$ is also a minimizing sequence for Problem 5.3.3. By Lemma 5.3.2, $(\tilde{u}_n)$ is bounded from below, is bounded in $BV([0,T] \times \mathbb{T}^d) \cap L^r([0,T] \times \mathbb{T}^d)$ and (Du_n) is bounded in $L^r([0,T] \times \mathbb{T}^d)$. Up to a subsequence, $(\tilde{u}_n)$ converges to a BV map $\tilde{u}$ in L^1 such that $\tilde{u} \geq u$. Note that, as in the proof of Proposition 5.3.4, $(\tilde{u}, \alpha)$ is also a minimizer of Problem 5.3.3, so that, by Theorem 5.4.1 , $(\tilde{u}, m, \beta, \beta_T)$ is also a solution to the MFG system (5.4.1). In particular, by (4) in the definition of solution, the inequalities of Lemma 5.3.3 must be equalities for $(\tilde{u}, \alpha)$ and (u, α) so that,

$$\int_{\mathbb{T}^d} m(t,x)u(t,x)\,\mathrm{d}x = \int_{\mathbb{T}^d} m(t,x)\tilde{u}(t,x)\,\mathrm{d}x \qquad \text{for a.e. } t \in [0,T].$$

As $\tilde{u} \geq u$, this implies that $\tilde{u} = u$ $m-$a.e. In conclusion the pair $(\tilde{u}_n, \alpha^{\eta_n,\varepsilon_n})$ satisfies our requirements. □

5.4.2 *Approximation by classical MFG systems*

We now study to what extent the solution of the MFG system with density constraint introduced above can be obtained as the limit of the solutions of classical MFG systems. For this, we assume that $f^\varepsilon : \mathbb{T}^d \times [0, +\infty) \to \mathbb{R}$ is a continuous function for each $\varepsilon > 0$, increasing with respect to m, with $f^\varepsilon(\cdot, 0) = 0$, and which fulfills the growth condition: there exists $\theta > 1 + d/r$ and C, $C_\varepsilon > 0$ such that

$$C^{-1}m^{\theta-1} - C \leq f^\varepsilon(x,m) \leq C_\varepsilon m^{\theta-1} + C_\varepsilon.$$

We consider $(u^\varepsilon, m^\varepsilon)$ the solution to the classical MFG system

$$\begin{cases} \text{(i)} & -\partial_t u^\varepsilon + H(x, Du^\varepsilon) = f^\varepsilon(x, m^\varepsilon) & \text{in } (0,T) \times \mathbb{T}^d \\ \text{(ii)} & \partial_t m^\varepsilon - \nabla \cdot \left(m^\varepsilon D_p H(x, Du^\varepsilon)\right) = 0 & \text{in } (0,T) \times \mathbb{T}^d \\ \text{(iii)} & u^\varepsilon(T,x) = g(x), \quad m^\varepsilon(0,x) = m_0(x) & \text{in } \mathbb{T}^d \end{cases} \tag{5.4.6}$$

Following Cardaliaguet [Car13b], Cardaliaguet-Graber [CG15], we know that the MFG system (5.4.6) has a unique (weak) solution $(u^\varepsilon, m^\varepsilon)$: namely, $(u^\varepsilon, m^\varepsilon) \in C^0([0,T] \times \mathbb{T}^d) \times L^\theta([0,T] \times \mathbb{T}^d)$ and the following hold:

(i) the following integrability conditions hold:

$$Du^\varepsilon \in L^r, \; m^\varepsilon H^*(\cdot, D_pH(\cdot, Du^\varepsilon)) \in L^1 \quad \text{and} \quad m^\varepsilon D_pH(\cdot, Du^\varepsilon)) \in L^1.$$

(ii) Equation (5.4.6)-(i) holds in the following sense: the inequality

$$-\partial_t u^\varepsilon + H(x, Du^\varepsilon) \le f(x, m^\varepsilon) \quad \text{in } (0,T)\times\mathbb{T}^d, \tag{5.4.7}$$

holds in the sense of distributions, with $u^\varepsilon(T,\cdot) = g$,

(iii) Equation (5.4.6)-(ii) holds:

$$\partial_t m^\varepsilon - \mathrm{div}(m^\varepsilon D_pH(x, Du^\varepsilon)) = 0 \text{ in } (0,T)\times\mathbb{T}^d, \quad m^\varepsilon(0) = m_0 \tag{5.4.8}$$

in the sense of distributions,

(iv) The following equality holds:

$$\int_0^T \int_{\mathbb{T}^d} m^\varepsilon(t,x)\left(f(x, m^\varepsilon(t,x)) + H^*(x, D_pH(x, Du^\varepsilon)(t,x))\right) \mathrm{d}x\,\mathrm{d}t \\ + \int_{\mathbb{T}^d} m^\varepsilon(T,x)g(x) - m_0(x)u^\varepsilon(0,x)\,\mathrm{d}x = 0. \tag{5.4.9}$$

In addition, u^ε is Hölder continuous and in $W^{1,1}$, and equality

$$-\partial_t u^\varepsilon + H(x, Du^\varepsilon) = f(x, m^\varepsilon) \text{ holds a.e. in } (0,T)\times\mathbb{T}^d$$

(see Cardaliaguet-Porretta-Tonon [CPT14a]).

Let us now suppose that $f^\varepsilon(x,m) \to f(x,m)$ uniformly with respect to x for any $m \le \overline{m}$ and $f^\varepsilon(x,m) \to +\infty$ uniformly with respect to x for any $m > \overline{m}$ as $\varepsilon \to 0^+$.

Proposition 5.4.3. *Under the above assumptions,*

1. *the family (u^ε) is bounded in $BV([0,T]\times\mathbb{T}^d) \cap L^r([0,T]\times\mathbb{T}^d)$ while (Du^ε) is bounded in $L^r([0,T]\times\mathbb{T}^d)$, the family $(\alpha^\varepsilon := -\partial_t u^\varepsilon + H(\cdot, Du^\varepsilon))$ is bounded in $L^1([0,T]\times\mathbb{T}^d)$, with $\alpha^\varepsilon \ge 0$ a.e., (m^ε) is bounded in $L^\theta([0,T]\times\mathbb{T}^d)$ while $(\mathbf{w}^\varepsilon)$ is bounded in $L^{r'}([0,T]\times\mathbb{T}^d)$.*
2. *If (u, m, α) is any cluster point for the weak convergence of $(u^\varepsilon, m^\varepsilon, \alpha^\varepsilon)$, then $\alpha \ge f(\cdot, m)$ and, if we set $\beta := \alpha \llcorner (0,T)\times\mathbb{T}^d$ and $\beta_T := u(T^-,\cdot) - g$, then the quadruplet (u, m, β, β_T) is a solution of the MFG system with density constraint (5.4.1).*

Proof. The proof is a straightforward adaptation of our previous constructions. According to [CGPT14], we know that $(u^\varepsilon, \alpha^\varepsilon)$ is minimizer over $\mathcal{K}_R$ of the functional

$$\mathcal{A}^\varepsilon(u,\alpha) = \int_0^T \int_{\mathbb{T}^d} (F^\varepsilon)^*(x,\alpha)\,\mathrm{d}x\,\mathrm{d}t - \int_{\mathbb{T}^d} u(0^+, x)\,\mathrm{d}m_0(x),$$

where $F^\varepsilon(x,m) := \displaystyle\int_0^m f^\varepsilon(x,s)\,\mathrm{d}s$ and $(F^\varepsilon)^*$ is the Fenchel conjugate of F^ε with respect to the last variable. Then, by convexity,

$$(F^\varepsilon)^*(x,\alpha) \ge \alpha\overline{m} - F^\varepsilon(x, \overline{m}),$$

where, by our assumptions, $F^\varepsilon(x,\overline{m})$ converges uniformly with respect to x to $F(x,\overline{m})$. Let ψ be the Lipschitz continuous viscosity solution to

$$\begin{cases} -\partial_t\psi + H(x,D\psi) &= 0 \\ \psi(T,x) &= g(x). \end{cases}$$

It is also an a.e. solution, so that $(\psi,0)$ belongs to $\mathcal{K}_R$. Then

$$\mathcal{A}^\varepsilon(u^\varepsilon,\alpha^\varepsilon) \le \mathcal{A}^\varepsilon(\psi,0) \le -\int_{\mathbb{T}^d}\psi(0,x)\,\mathrm{d}m_0(x) \le C.$$

So $(\mathcal{A}^\varepsilon(u^\varepsilon,\alpha^\varepsilon))$ is bounded from above and one can then argue exactly as in the proof of Lemma 5.3.2 to obtain the bounds on (u^ε) and (α^ε) as well as a bound for $(\mathcal{A}^\varepsilon(u^\varepsilon,\alpha^\varepsilon))$.

Following [CGPT14], we also know that the pair

$$(m^\varepsilon,\mathbf{w}^\varepsilon) := (m^\varepsilon, -m^\varepsilon D_pH(\cdot,Du^\varepsilon))$$

is a minimizer over $\mathcal{K}_D$ of

$$\mathcal{B}^\varepsilon(m,\mathbf{w}) = \int_{\mathbb{T}^d} g(x)m(T,x)\,\mathrm{d}x + \int_0^T\int_{\mathbb{T}^d} m(t,x)L\left(x,\frac{\mathbf{w}(t,x)}{m(t,x)}\right) + F^\varepsilon(x,m(t,x))\,\mathrm{d}x\,\mathrm{d}t$$

Since, by [CGPT14], $\mathcal{A}^\varepsilon(u^\varepsilon,\alpha^\varepsilon) = -\mathcal{B}^\varepsilon(m^\varepsilon,\mathbf{w}^\varepsilon)$, $(\mathcal{B}^\varepsilon(m^\varepsilon,\mathbf{w}^\varepsilon))$ is bounded. From our assumption on f^ε we have therefore that (m^ε) is bounded in $L^\theta([0,T]\times\mathbb{T}^d)$ while (w^ε) is bounded in $L^{r'}([0,T]\times\mathbb{T}^d)$.

Let finally (u,α) be a cluster point of $(u^\varepsilon,\alpha^\varepsilon)$ and $(m,\mathbf{w})$ be a cluster point of $(m^\varepsilon,w^\varepsilon)$ for the weak convergence. Then standard arguments from the theory of $\Gamma-$convergence show that (u,α) minimizes $\mathcal{A}$ while $(m,\mathbf{w})$ minimizes $\mathcal{B}$, so that, if we set $\beta := \alpha \llcorner (0,T)\times\mathbb{T}^d$ and $\beta_T := u(T^-,\cdot) - g$, the quadruplet (u,m,β,β_T) is a solution of the MFG system (5.4.1) according to Theorem 5.4.1. □

5.5 NO COUPLING, POWER-LIKE HAMILTONIANS AND $m_0 < \overline{m}$

In this section, we study through an example some finer properties of the solutions of (5.4.1). Let us consider $f(x,m)\equiv 0$, $H(x,p) = \frac{1}{s}|p|^s$ $(s>1)$ and $T=1$. The terminal cost g is a given smooth function. We assume that the initial density of the population satisfies $m_0 < \overline{m} - c$ a.e. in $\mathbb{T}^d$ for a given constant $0<c<\overline{m}$. For simplicity, let us set $s=2$. In this case the functional $\mathcal{B}$ for the Problem 5.3.2 has the form

$$\mathcal{B}(m,\mathbf{w}) = \int_0^1\int_{\mathbb{T}^d}\frac{1}{2}\frac{|\mathbf{w}|^2}{m} + F(x,m)\,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{T}^d} g(x)m(1,x)\,\mathrm{d}x,$$

where we use the convention (5.3.3). Let us also chose $F(x,m)\equiv 0$ for $m\in[0,\overline{m}]$ and $F\equiv+\infty$ otherwise. This functional recalls the one introduced by Bemamou and Brenier to give a dynamical formulation for the Monge-Kantorovich optimal transportation problem (see [BB00]). Only a constraint on the density m and a penalization on the final value have been added.

Indeed, forgetting for a while the density constraint, Problem 5.3.2 can be reformulated as

$$\min\left\{\frac{1}{2}W_2^2(m_0,m_1)+\int_{\mathbb{T}^d} g m_1\,\mathrm{d}x\ :\ m_1\in\mathscr{P}(\mathbb{T}^d),\ m_1\le\overline{m}\right\}. \tag{5.5.1}$$

We remark that the above formulation gives always a geodesic curve connecting m_0 and m_1 (thus m_t is defined for all $t\in[0,1]$). Since the admissible set in the above problem is geodesically convex (and $\mathbb{T}^d$ is a convex set), the density constraint is satisfied as soon as it is satisfied at the terminal time. Hence the problem in (5.5.1) is completely equivalent to Problem 5.3.2. Actually we can prove something more: if the initial density satisfies strictly the constraint, then saturation may happen only in the final time. This is a straightforward result, nevertheless for the completeness we prove it here.

Lemma 5.5.1. *Let $m_0 < \overline{m} - c$ (for a given constant $0 < c < \overline{m}$) a.e. in $\mathbb{T}^d$ and m_1 be the solution of the Problem 5.5.1. Let (m_t) be the geodesic connecting m_0 to m_1. For any $\tau\in(0,1)$ there exists $\theta>0$ such that $m_t<\overline{m}-\theta$ a.e. in $\mathbb{T}^d$ for all $t\in[0,1-\tau]$.*

Proof. As we mentioned before, since the admissible set in Problem 5.5.1 is geodesically convex, we have that $m_t\le\overline{m}$ a.e. in $\mathbb{T}^d$ for all $t\in[0,1]$. On the other hand, since $m_t\ll\mathscr{H}^d\llcorner\mathbb{T}^d$ for all $t\in[0,1]$ we know that there exist optimal transport maps $T^t,S^t:\mathbb{T}^d\to\mathbb{T}^d$ such that $(T^t)_\# m_0=m_t$ and $(S^t)_\# m_t=m_0$ with $T^t\circ S^t=\mathrm{id}$ for all $t\in[0,1]$. The maps (S_t) and (T_t) are given in terms of S^1 and T^1 by McCann's interpolation, which is $T_t:=(1-t)\mathrm{id}+tT^1$, and $S_t=t\mathrm{id}+(1-t)S^1$. Moreover T_t and S_t are countably Lipschitz, hence we can write down the Jacobian equation

$$\det(DT_t)=\frac{m_0}{m_t\circ T_t},$$

from where the density m_t is given by

$$m_t=\frac{m_0}{\det(DT_t)}\circ S_t. \tag{5.5.2}$$

Using the concavity of $\det^{1/d}$ (for positive definite matrices) we obtain that

$$\begin{aligned}\det(DT_t)&=\det\left((1-t)I_d+tDT^1\right)\ge\left((1-t)+t\det(DT^1)^{1/d}\right)^d\\&=\left((1-t)+t\left(\frac{m_0}{m_1\circ T^1}\right)^{1/d}\right)^d\\&\ge\left((1-t)+t\left(\frac{m_0}{\overline{m}}\right)^{1/d}\right)^d.\end{aligned}$$

Hence by (5.5.2) we have that

$$m_t \le \frac{m_0 \circ S_t}{\left((1-t) + t\left(\frac{m_0 \circ S_t}{\overline{m}}\right)^{1/d}\right)^d}.$$

Let us set $\lambda := (m_0 \circ S_t)/\overline{m} < 1$. Then, for any $\tau \in (0,1)$ there exists $\theta > 0$ such that

$$(1+\theta)\lambda^{1/d} < (1-t) + t\lambda^{1/d}, \quad \forall t \in [0, 1-\tau]$$

provided $\lambda \le (\overline{m} - c)/\overline{m}$ (in other words $m_0 < \overline{m} - c$). Thus this implies that $(1+\theta)m_t < \overline{m}$ a.e. in $\mathbb{T}^d$ for all $t \in [0, 1-\tau]$. Notice that this property may fail for $t = 1$. □

5.5.1 *Some properties of β, β_1 and u*

Let us discuss now some further properties of β, β_1 and u.

Proposition 5.5.2. *Let (u, m, β, β_1) be a solution of the MFG system with density constraints and let us assume that we are in the setting of this section. Then $\beta \equiv 0$ and u and β_1 are bounded.*

Proof. From Theorem 5.4.1, we know that the pair $(m, -mD_pH(\cdot, Du))$ is a minimizer of $\mathcal{B}$. In view of Lemma 5.5.1, we have therefore $m(t,x) < \overline{m}$ for a.e. $(t,x) \in (0,T) \times \mathbb{T}^d$. By Definition 5.4.1, this implies that $\beta^{ac} = 0$. Recall on the other hand that

$$\limsup_{\varepsilon \to 0} m_\varepsilon(t,x) = \overline{m} \qquad \beta^s - \text{a.e. if } t < 1,$$

where m_ε is any standard mollification of m. But, still by Lemma 5.5.1, for any $\tau \in (0,1)$, there exists $\theta > 0$ such that $m \le \overline{m} - \theta$ on $(0, 1-\tau)$. Thus $m_\varepsilon(t,x) \le \overline{m} - \theta$, so that the restriction of β^s to $(0, 1-\tau) \times \mathbb{T}^d$ is zero, hence $\beta^s = 0$ on $(0,1) \times \mathbb{T}^d$.

Let us now check that u is bounded. We note that, as $\beta = 0$ and H satisfies the growth condition (H3-1), u satisfies a.e. the inequality $-\partial_t u + |Du|^2/2 \le 0$ in $(0,1) \times \mathbb{T}^d$. Thus, if we mollify u in the usual way, u_ε is a classical sub solution to $-\partial_t u_\varepsilon + |Du_\varepsilon|^2/2 \le 0$ on $(\varepsilon, 1-\varepsilon) \times \mathbb{T}^d$. By Hopf's formula we get therefore

$$u_\varepsilon(t,x) \le \inf_{y \in \mathbb{T}^d} \left\{ u_\varepsilon(1-\varepsilon, y) + C\frac{|x-y|^2}{(1-\varepsilon-t)} + C(1-t) \right\} \; \forall (t,x) \in (\varepsilon, 1-\varepsilon) \times \mathbb{T}^d.$$

Hence

$$u_\varepsilon(t,x) \le \inf_{y \in \mathbb{T}^d} \{u_\varepsilon(1-\varepsilon, y) + C\} \qquad \forall (t,x) \in (\varepsilon, 1/2) \times \mathbb{T}^d.$$

Recalling that $\int_{\mathbb{T}^d} u(t,x)\,\mathrm{d}x$ is bounded for a.e. t (see the proof of Lemma 5.3.2), we also have that $\int_{\mathbb{T}^d} u_\varepsilon(t,x)\,\mathrm{d}x$ is bounded as well for all t and therefore $\inf_{y\in\mathbb{T}^d} u_\varepsilon(1,y)$ is bounded from above. So we have proved that u_ε is bounded from above by a constant C_0 on $(\varepsilon,1/2)\times\mathbb{T}^d$, where C_0 is independent of ε. This shows that u is bounded from above by C_0 on $(0,1/2)\times\mathbb{T}^d$.

Let us set $z(t,x) := (C_0 + \|H(\cdot,0)\|_{L^\infty}) \vee \|g\|_{L^\infty} - \|H(\cdot,0)\|_{L^\infty}(1-t)$. Then z is a subsolution to $-\partial_t z + H(x,Dz) \le 0$ which satisfies $z(1,\cdot) \ge g$ and $z(0,\cdot) \ge C_0 \ge u(0,\cdot)$. Therefore the map $\tilde{u}(t,x) := u(t,x) \wedge z(t,x)$ is still a subsolution (because $H = H(x,p)$ is convex with respect to p), which satisfies $\tilde{u}(0,\cdot) = u(0,\cdot)$ a.e. and $g(x) \le \tilde{u}(1^-,x) \le u(1^-,x)$. Let us set $\tilde{\alpha} := (\tilde{u}(1^-,\cdot) - g)\,\mathrm{d}(\delta_1 \otimes \mathscr{H}^d \llcorner \mathbb{T}^d)$. Then the pair $(\tilde{u},\tilde{\alpha})$ belongs to $\mathcal{K}_R$ and by optimality of (u,α) we have

$$
\begin{aligned}
\mathcal{A}(u,\alpha) &= \int_{\mathbb{T}^d} \left(u(1^-,x) - g(x)\right)\,\mathrm{d}x - \int_{\mathbb{T}^d} m_0(x)u(0,x)\,\mathrm{d}x \\
&\le \mathcal{A}(\tilde{u},\tilde{\alpha}) = \int_{\mathbb{T}^d} \left(\tilde{u}(1^-,x) - g(x)\right)\,\mathrm{d}x - \int_{\mathbb{T}^d} m_0(x)\tilde{u}(0,x)\,\mathrm{d}x.
\end{aligned}
$$

As $\tilde{u}(0,\cdot) = u(0,\cdot)$ and $\tilde{u}(1^-,x) \le u(1^-,x)$, this proves that $\tilde{u}(1^-,x) = u(1^-,x)$ a.e., which means that $u(1^-,\cdot)$ is bounded from above. Since we already know that u is bounded from below (see the proof of Lemma 5.3.2), we have established that $u(1^-,\cdot)$ is bounded. By Hopf's formula, this entails the boundedness of u on $(0,1)\times\mathbb{T}^d$, from where the boundedness of β_1 follows as well.

□

Remark 5.5.1 (Nash-type equilibrium)**.** *For this example a notion of Nash equilibrium can be formulated by the means of (m,β_1), i.e. by the means of the "additional price" β_1 to be payed by the agents at the final time, and the value of which is precisely $\beta_1 = (u(1^-,\cdot) - g)$. This price clearly has to be payed only if they arrive to the saturated zone at the final time. Let us postpone the precise definition and the details on the question of the Nash equilibrium, which will be established for more general cases in Section 5.7 (see Definition 5.7.3).*

5.6 REGULARITY OF THE "ADDITIONAL PRICE" β

In this section we show that the measure β is absolutely continuous and belongs to $L^{d/(d-1)}_{\mathrm{loc}}((0,T)\times\mathbb{T}^d)$. In this respect, our model recalls those studied by Brenier (see [Bre99]) and later by Ambrosio-Figalli (see [AF09, AF08]), where they analyzed the motion of incompressible perfect fluids driven by the Euler's equations.

We will see in the next section that this regularity is essential in order to define Nash equilibria in our context. For this, we assume in addition to the previous hypotheses the following conditions: there exists $\lambda > 0$ such that

– (Assumption for H): H and H^* are of class C^2 with

$$\lambda I_d \leq D^2_{pp}H \leq \lambda^{-1}I_d \quad \text{and} \quad \lambda I_d \leq D^2_{qq}H^* \leq \lambda^{-1}I_d, \tag{HP1-1}$$

$$|D^2_{xx}H^*(x,p)| \leq C(1+|p|^2), \qquad |D^2_{xp}H^*(x,p)| \leq C(1+|p|). \tag{HP1-2}$$

– (Assumption on F): f is of class C^2 on $\mathbb{T}^d \times [0,\overline{m}]$ and, for any $m \in [0,\overline{m}]$ and $\alpha \geq 0$,

$$F(x,m) + F^*(x,\alpha) - \alpha m \geq \frac{\lambda}{2}|\alpha_1 - f(x,m)|^2 + p(\overline{m}-m), \tag{HP2}$$

where $p = (\alpha - f(x,\overline{m}))_+$ and $\alpha_1 = \alpha - p$.

– (Assumption on g):

$$g \text{ is of class } C^2. \tag{HP3}$$

A canonical example that we can have in mind is the case when $F(x,m) = f(x)m$ if $m \in [0,\overline{m}]$ and $+\infty$ otherwise and $H(x,p) = |p|^2/2$.

Theorem 5.6.1. *Let (u,m,β,β_T) be a solution of the MFG system (5.4.1). Under the above assumptions, $f(\cdot,m(\cdot,\cdot)) \in H^1_{\text{loc}}((0,T)\times\mathbb{T}^d)$ and β is absolutely continuous in $(0,T)\times\mathbb{T}^d$ with $\beta \in L^2_{\text{loc}}((0,T);BV(\mathbb{T}^d)) \hookrightarrow L^{d/(d-1)}_{\text{loc}}((0,T)\times\mathbb{T}^d)$.*

The proof is largely inspired by the works of Brenier (see [Bre99]) and Ambrosio-Figalli (see [AF08]) on the incompressible Euler's equations.

Proof of the Theorem 5.6.1. By abuse of notion, we use $\mathcal{B}(m',\mathbf{v}')$ meaning precisely $\mathcal{B}(m',m'\mathbf{v}')$ for any admissible pair $(m',m'\mathbf{v}')$ in the dual problem ($\mathbf{v}'$ denoting the velocity field).

Throughout the proof, (u,m,β,β_T) is a fixed solution of the MFG system (5.4.1) and we define α by (5.4.4) and set $\mathbf{w} = -mD_pH(x,Du)$. Recall that $(m,\mathbf{w})$ is a minimizer for $\mathcal{B}$. We also set $\mathbf{v} := \mathbf{w}/m$ and construct competitors $(m^{\delta,\eta}, m^{\delta,\eta}\mathbf{v}^{\delta,\eta})$ in the following way: let us fix $0 < t_1 < t_2 < T$ and let $\zeta \in C^\infty_c((0,T);[0,1])$ be a smooth cut-off such that $\zeta \equiv 1$ on $[t_1,t_2]$; for $\eta > 0$ small and $\delta \in \mathbb{R}^d$ small (such that $t+\zeta(t)\eta \in (0,T)$ for all $t \in [0,T]$), we denote

$$m^{\delta,\eta}(t,x) := m(t+\zeta(t)\eta, x+\zeta(t)\delta)$$

the time-space translation of the density and let

$$\mathbf{v}^{\delta,\eta}(t,x) := \mathbf{v}(t+\zeta(t)\eta, x+\zeta(t)\delta)(1+\eta\zeta'(t)) - \zeta'(t)\delta$$

the velocity field associated to $m^{\delta,\eta}$. Indeed, by construction $(m^{\delta,\eta}, m^{\delta,\eta}\mathbf{v}^{\delta,\eta})$ solves the continuity equation, and satisfies the other constraints.

Step 0. Let us collect some tools now.

First, we have

$$\mathcal{B}(m^{\delta,\eta}, \mathbf{v}^{\delta,\eta}) \leq \mathcal{B}(m, \mathbf{v}) + C(\eta^2 + |\delta|^2). \tag{5.6.1}$$

Indeed, let us denote by ξ_η the inverse map of $t \mapsto t + \zeta(t)\eta$. Then, after changing variables,

$$\begin{aligned}\mathcal{B}(m^{\delta,\eta}, &\mathbf{v}^{\delta,\eta}) \\ = \int_0^T \int_{\mathbb{T}^d} &\left[m(s,y)H^*\left(y - \zeta(\xi_\eta(s))\delta, -\mathbf{v}(s,y)(1 + \eta\zeta'(\xi_\eta(s))) + \zeta'(\xi_\eta(s))\delta\right)\right) \\ &+ F(y - \zeta(\xi_\eta(s))\delta, m(s,y))\big] \, \xi'_\eta(s) \, \mathrm{d}y \, \mathrm{d}s + \int_{\mathbb{T}^d} g(y) m(T,y) \, \mathrm{d}y.\end{aligned}$$

In view of our C^2 regularity assumptions on H^*, F and g, the map $(\eta, \delta) \mapsto \mathcal{B}(m^{\delta,\eta}, \mathbf{v}^{\delta,\eta})$ is C^2. We obtain (5.6.1) by optimality of $(m, \mathbf{v})$.

Second, by stationarity of the problem for $\mathcal{B}$ (it is enough to consider perturbations of form $(m^{0,\eta}, \mathbf{v}^{0,\eta})$ for ζ with compact support, not necessarily 1 on $[t_1, t_2]$), we have

$$\int_{\mathbb{T}^d} \{m(H^*(x, -\mathbf{v}(t,x)) + D_q H^*(x, -\mathbf{v}(t,x)) \cdot \mathbf{v}(t,x)) + F(x, m(t,x))\} \mathrm{d}x = const.$$

From our assumption on H^*, we have

$$H^*(x, -\mathbf{v}) + D_q H^*(x, -\mathbf{v}) \cdot \mathbf{v} \leq H^*(x, 0) - \frac{\lambda}{2}|\mathbf{v}|^2.$$

Thus

$$\mathrm{ess} - \sup_{t \in [0,T]} \int_{\mathbb{T}^d} m(t,x) |\mathbf{v}(t,x)|^2 \, \mathrm{d}x \leq C. \tag{5.6.2}$$

By (HP1-1), we have $D^2_{qq} H^* \leq (1/\lambda) I_d$ and therefore (5.6.2) implies

$$\mathrm{ess} - \sup_{t \in [0,T]} \int_{\mathbb{T}^d} m(t,x) |D_q H^*(x, -\mathbf{v}(t,x))|^2 \, \mathrm{d}x \leq C. \tag{5.6.3}$$

Third, for any smooth map (u', α'), with $\alpha' \geq 0$, and $(m', \mathbf{w}') \in \mathcal{K}_D$ (where $\mathbf{v}' = \mathbf{w}'/m'$) competitor for the primal and the dual problems respectively, we have

$$\begin{aligned}\mathcal{A}(u', \alpha') + \mathcal{B}(m', \mathbf{v}') \geq &\int_0^T \int_{\mathbb{T}^d} \{m'(H(x, Du') + H^*(x, -\mathbf{v}') + \mathbf{v}' \cdot Du')\} \, \mathrm{d}x \, \mathrm{d}t \\ &+ \int_0^T \int_{\mathbb{T}^d} \{F(x, m') + F^*(x, \alpha') - \alpha' m'\} \, \mathrm{d}x \, \mathrm{d}t.\end{aligned}$$

In view of our assumptions on (HP1-1)-(HP1-2) and (HP2), we have the key inequality

$$\begin{aligned}\mathcal{A}(u', \alpha') + \mathcal{B}(m', \mathbf{v}') \geq \; &\frac{\lambda}{4} \int_0^T \int_{\mathbb{T}^d} m'(t,x) |Du' - D_q H^*(x, -\mathbf{v}')|^2 \, \mathrm{d}x \, \mathrm{d}t \\ &+ \frac{\lambda}{4} \int_0^T \int_{\mathbb{T}^d} m'(t,x) |\mathbf{v}' + D_p H(x, Du')|^2 \, \mathrm{d}x \, \mathrm{d}t \\ &+ \int_0^T \int_{\mathbb{T}^d} \left\{ \frac{\lambda}{2} |\alpha'_1 - f(x, m')|^2 + p'(\overline{m} - m') \right\} \, \mathrm{d}x \, \mathrm{d}t\end{aligned}$$

(5.6.4)

where $p' = (\alpha' - f(x, \overline{m}))_+$ and $\alpha'_1 = \alpha' - p'$.

With the help of these tools let us show now the statements of the theorem.

Step 1. We first check that $f(\cdot, m) \in H^1_{\mathrm{loc}}((0,T) \times \mathbb{T}^d)$. Let us fix $(m', \mathbf{v}')$ to be a smooth competitor for $\mathcal{B}$ and let (u_n, α_n) be the minimizing sequence for Problem 5.3.3 defined in Lemma 5.4.2: we know that (α_n) is bounded in L^1 and converges to the (nonnegative) measure α defined from (β, β_T) by (5.4.4). Then, passing to the limit in the inequality

$$\mathcal{A}(u_n, \alpha_n) + \mathcal{B}(m', \mathbf{v}') \geq \int_0^T \int_{\mathbb{T}^d} \left\{ F(x, m') + F^*(x, \alpha_n) - \alpha_n m' \right\} \mathrm{d}x\, \mathrm{d}t,$$

we get

$$\inf_{\mathcal{K}_R} \mathcal{A} + \mathcal{B}(m', \mathbf{v}') \geq \int_0^T \int_{\mathbb{T}^d} \left\{ F(x, m') + F^*(x, \alpha^{ac}) - \alpha^{ac} m' \right\} \mathrm{d}x\, \mathrm{d}t$$
$$+ \int_0^T \int_{\mathbb{T}^d} (\overline{m} - m')\, \mathrm{d}\alpha^s.$$

In view of the proof of Theorem 5.4.1, we have $\alpha^{ac} \geq f(\cdot, m)$, with an equality in $\{m < \overline{m}\}$. So, if we set as above $p = (\alpha^{ac} - f(x, \overline{m}))_+$ and $\alpha_1^{ac} = \alpha^{ac} - p$, then $\alpha_1^{ac} = f(\cdot, m)$. By (HP2), this implies that

$$\inf_{\mathcal{K}_R} \mathcal{A} + \mathcal{B}(m', \mathbf{v}') \geq \int_0^T \int_{\mathbb{T}^d} \frac{\lambda}{2} |f(x, m) - f(x, m')|^2 \, \mathrm{d}x\, \mathrm{d}t,$$

an inequality which remains true for any $(m', \mathbf{v}') \in \mathcal{K}_D$ (not necessarily smooth ones). Adding $\inf_{\mathcal{K}_R} \mathcal{A}$ to inequality (5.6.1) and using the duality $\inf_{\mathcal{K}_R} \mathcal{A} + \min_{\mathcal{K}_D} \mathcal{B} = 0$ we have

$$\inf_{\mathcal{K}_R} \mathcal{A} + \mathcal{B}(m^{\delta,\eta}, \mathbf{v}^{\delta,\eta}) \leq C(\eta^2 + |\delta|^2), \tag{5.6.5}$$

which implies

$$\int_{t_1}^{t_2} \int_{\mathbb{T}^d} |f(x, m) - f(x, m^{\delta,\eta})|^2 \, \mathrm{d}x\, \mathrm{d}t \leq C(\eta^2 + |\delta|^2)$$

and the regularity of f in x allows to conclude $f(\cdot, m) \in H^1_{\mathrm{loc}}((0,T) \times \mathbb{T}^d)$.

Step 2. Let (u_n, α_n) be the minimizing sequence defined by Lemma 5.4.2. Without loss of generality, we can assume that

$$\mathcal{A}(u_n, \alpha_n) - \inf_{\mathcal{K}_R} \mathcal{A} \leq 1/n. \tag{5.6.6}$$

We set $p_n := (\alpha_n - f(\cdot, \overline{m}))_+$ and $\alpha_{1,n} := \alpha_n - p_n$. For $\varphi : [0,T] \times \mathbb{T}^d \to \mathbb{R}$ and for $\eta > 0$ small, let us define the average of φ on the $[t, t+\eta]$ by

$$\varphi^\eta(t, x) := \int_0^1 \varphi(t + \theta\eta, x)\, \mathrm{d}\theta,$$

which is well-defined on $[t_1, t_2] \times \mathbb{T}^d$. With this procedure, we similarly define the functions p_n^η, α_n^η, etc. Let us take moreover $\sigma \in C^\infty([t_1, t_2]; [0, +\infty))$.

The aim of this step consists in estimating the quantity

$$I := \int_{t_1}^{t_2} \int_{\mathbb{T}^d} \sigma(t)\overline{m}|p_n^\eta(t, x+\delta) - p_n^\eta(t, x)| \, dx \, dt.$$

Namely we prove that

$$\begin{aligned} I \quad & \leq C\|\sigma\|_{L^2} \left(|\delta| + \left(1 + \frac{|\delta|}{\eta}\right) (1/n + |\delta|^2 + \eta^2)^{1/2} \right) \\ & +C\|\sigma\|_{L^\infty}(1/n + \eta^2 + |\delta|^2)^{1/2} \\ & \times \left[(1/n + \eta^2 + |\delta|^2)^{1/2} + |\delta| \left(1 + (1/n + \eta^2 + |\delta|^2)^{1/2} \right) \right] \\ & +C \left\{ \|\sigma\|_{L^2}^2 + \|\sigma\|_{L^\infty} \left[(1/n + |\delta|^2 + \eta^2) + (1/n + |\delta|^2 + \eta^2)^{1/2} \right] \right\}^{1/2} \\ & \times (1/n + |\delta|^2 + \eta^2)^{1/2} \\ & =: X(\sigma, 1/n, \delta, \eta). \end{aligned} \tag{5.6.7}$$

We will show in the last two steps that this inequality easily entails the desired estimates on p and β.

The proof of (5.6.7) is quite long and relies on the combination of (5.6.4), (5.6.5) and (5.6.6) which imply that

$$\begin{aligned} 1/n + C(\eta^2 + |\delta|^2) \geq \; & \frac{\lambda}{4} \int_{t_1}^{t_2} \int_{\mathbb{T}^d} m^{\delta,\eta}(t,x) |Du_n - D_q H^*(x, -\mathbf{v}^{\delta,\eta})|^2 \, dx \, dt \\ & + \frac{\lambda}{4} \int_{t_1}^{t_2} \int_{\mathbb{T}^d} m^{\delta,\eta}(t,x) |\mathbf{v}^{\delta,\eta} + D_p H(x, Du_n)|^2 \, dx \, dt \\ & + \int_{t_1}^{t_2} \int_{\mathbb{T}^d} \left\{ \frac{\lambda}{2} |\alpha_{1,n} - f(x, m^{\delta,\eta})|^2 + p_n(\overline{m} - m^{\delta,\eta}) \right\} dx \, dt \end{aligned} \tag{5.6.8}$$

We have

$$\begin{aligned} I \leq & \int_{t_1}^{t_2} \int_{\mathbb{T}^d} \sigma(t)(\overline{m} - m(t,x)) |p_n^\eta(t, x+\delta) - p_n^\eta(t,x)| \, dx \, dt \\ & + \int_{t_1}^{t_2} \int_{\mathbb{T}^d} \sigma(t) m(t,x) |p_n^\eta(t, x+\delta) - p_n^\eta(t,x)| \, dx \, dt \\ & =: I_{01} + I_{02} \end{aligned}$$

where the the first term can be estimated as follows:

$$\begin{aligned}
I_{01} &\leq \|\sigma\|_{L^\infty} \int_{t_1}^{t_2} \int_{\mathbb{T}^d} (\overline{m} - m(t,x)) \left\{ |p_n^\eta(t, x+\delta)| + |p_n^\eta(t,x)| \right\} \, \mathrm{d}x \, \mathrm{d}t \\
&= \|\sigma\|_{L^\infty} \int_0^1 \mathrm{d}\theta \int_{t_1}^{t_2} \int_{\mathbb{T}^d} (\overline{m} - m(t,x)) \\
&\qquad\qquad \times \left\{ p_n(t+\theta\eta, x+\delta) + p_n(t+\theta\eta, x) \right\} \, \mathrm{d}x \, \mathrm{d}t \\
&\leq \|\sigma\|_{L^\infty} \int_0^1 \mathrm{d}\theta \int_0^T \int_{\mathbb{T}^d} (\overline{m} - m^{-\delta,-\theta\eta}) p_n \, \mathrm{d}x \, \mathrm{d}t \\
&+ \|\sigma\|_{L^\infty} \int_0^1 \mathrm{d}\theta \int_0^T \int_{\mathbb{T}^d} (\overline{m} - m^{0,-\theta\eta}) p_n \, \mathrm{d}x \, \mathrm{d}t.
\end{aligned}$$

Now by (5.6.8) we obtain that

$$I_{01} \leq C \|\sigma\|_{L^\infty} (1/n + |\delta|^2 + \eta^2).$$

For the second term we have

$$\begin{aligned}
I_{02} &\leq \int_{t_1}^{t_2} \int_{\mathbb{T}^d} \sigma(t) m(t,x) |\alpha_n^\eta(t, x+\delta) - \alpha_n^\eta(t,x)| \, \mathrm{d}x \, \mathrm{d}t \\
&+ \int_{t_1}^{t_2} \int_{\mathbb{T}^d} \sigma(t) \overline{m} |\alpha_{1,n}^\eta(t, x+\delta) - \alpha_{1,n}^\eta(t,x)| \, \mathrm{d}x \, \mathrm{d}t \\
&:= I_1 + I_2.
\end{aligned}$$

To estimate the term I_1, let us compute

$$\begin{aligned}
&\alpha_n^\eta(t, x+\delta) - \alpha_n^\eta(t,x) = \\
&= \int_0^1 -\partial_t u_n(t+\theta\eta, x+\delta) + H(x+\delta, Du_n(t+\theta\eta, x+\delta)) \, \mathrm{d}\theta \\
&- \int_0^1 -\partial_t u_n(t+\theta\eta, x) + H(x, Du_n(t+\theta\eta, x)) \, \mathrm{d}\theta \\
&= -\eta^{-1} \int_0^1 [Du_n(t+\eta, x+s\delta) - Du_n(t, x+s\delta)] \cdot \delta \, \mathrm{d}s \\
&+ \int_0^1 \int_0^1 D_x H(x+s\delta, Du_n(t+\theta\eta, x+s\delta)) \cdot \delta \, \mathrm{d}s \, \mathrm{d}\theta \\
&+ \int_0^1 \mathrm{d}\theta \int_0^1 D_p H(x+s\delta, \xi_s) \cdot [Du_n(t+\theta\eta, x+\delta) - Du_n(t+\theta\eta, x)] \, \mathrm{d}s \, \mathrm{d}\theta
\end{aligned}$$

where $\xi_s := (1-s) Du_n(t+\theta\eta, x) + s Du_n(t+\theta\eta, x+\delta)$. Thus,

$$\begin{aligned}
&|\alpha_n^\eta(t, x+\delta) - \alpha_n^\eta(t,x)| \leq \\
&\leq |\delta| \eta^{-1} \int_0^1 |Du_n(t+\eta, x+s\delta) - Du_n(t, x+s\delta)| \, \mathrm{d}s \\
&+ |\delta| \int_0^1 \int_0^1 |D_x H(x+s\delta, Du_n(t+\theta\eta, x+s\delta))| \, \mathrm{d}s \, \mathrm{d}\theta \\
&+ \int_0^1 \int_0^1 |D_p H(x+s\delta, \xi_s)| \, |Du_n(t+\theta\eta, x+\delta) - Du_n(t+\theta\eta, x)| \, \mathrm{d}s \, \mathrm{d}\theta.
\end{aligned}$$

In view of our assumption (HP1-2) on $D_x H$ and $D_p H$:

$$\begin{aligned}
I_1 &= \int_{t_1}^{t_2}\int_{\mathbb{T}^d}\sigma(t)m(t,x)|\alpha_n^\eta(t,x+\delta)-\alpha_n^\eta(t,x)|\,\mathrm{d}x\,\mathrm{d}t\\
&\leq |\delta|\eta^{-1}\int_{t_1}^{t_2}\int_{\mathbb{T}^d}\int_0^1\sigma(t)m(t,x)\\
&\qquad\qquad\times|Du_n(t+\eta,x+s\delta)-Du_n(t,x+s\delta)|\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}t\\
&+C|\delta|\int_{t_1}^{t_2}\int_{\mathbb{T}^d}\int_0^1\int_0^1\sigma(t)m(t,x)\left\{1+|Du_n(t+\theta\eta,x+s\delta)|^2\right\}\mathrm{d}s\,\mathrm{d}\theta\,\mathrm{d}x\,\mathrm{d}t\\
&+C\int_{t_1}^{t_2}\int_{\mathbb{T}^d}\int_0^1\sigma(t)m(t,x)\left\{1+|Du_n(t+\theta\eta,x)|+|Du_n(t+\theta\eta,x+\delta)|\right\}\\
&\qquad\qquad\times|Du_n(t+\theta\eta,x+\delta)-Du_n(t+\theta\eta,x)|\,\mathrm{d}\theta\,\mathrm{d}x\,\mathrm{d}t\\
&:= I_{11}+I_{12}+I_{13}.
\end{aligned}$$

For I_{11}, we have

$$\begin{aligned}
I_{11} &\leq |\delta|\eta^{-1}\int_{t_1}^{t_2}\int_{\mathbb{T}^d}\int_0^1\Big\{\sigma(t)m(t,x)\big(|Du_n(t+\eta,x+s\delta)-D_qH^*(x,-\mathbf{v}(t,x))|\\
&\qquad\qquad+|D_qH^*(x,-\mathbf{v}(t,x))-Du_n(t,x+s\delta)|\big)\Big\}\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}t\\
&\leq |\delta|\eta^{-1}\int_{t_1+\eta}^{t_2+\eta}\int_{\mathbb{T}^d}\int_0^1\sigma^{0,-\eta}(t)m^{-s\delta,-\eta}(t,x)\\
&\qquad\qquad\times|Du_n(t,x)-D_qH^*(x-s\delta,-\mathbf{v}^{-s\delta,-\eta}(t,x))|\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}t\\
&+|\delta|\eta^{-1}\int_{t_1}^{t_2}\int_{\mathbb{T}^d}\int_0^1\sigma(t)m^{-s\delta,0}(t,x)\\
&\qquad\qquad\times|Du_n(t,x)-D_qH^*(x-s\delta,-\mathbf{v}^{-s\delta,0}(t,x))|\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}t
\end{aligned}$$

By Cauchy-Schwarz and (5.6.8) we obtain:

$$I_{11}\leq C|\delta|\eta^{-1}\|\sigma\|_{L^2}(1/n+\eta^2+|\delta|^2)^{1/2}.$$

We now estimate I_{12}, that we bound from above as follows:

$$\begin{aligned}
I_{12} &\leq C\|\sigma\|_{L^2}|\delta|\\
&\quad+C|\delta|\int_{t_1}^{t_2}\int_{\mathbb{T}^d}\int_0^1\int_0^1\sigma(t)m(t,x)|D_qH^*(x,-\mathbf{v}(t,x))|^2\,\mathrm{d}s\,\mathrm{d}\theta\,\mathrm{d}x\,\mathrm{d}t\\
&\quad+C|\delta|\int_{t_1}^{t_2}\int_{\mathbb{T}^d}\int_0^1\int_0^1\sigma(t)m(t,x)\\
&\qquad\qquad\times\left\{|Du_n(t+\theta\eta,x+s\delta)|^2-|D_qH^*(x,-\mathbf{v}(t,x))|^2\right\}\mathrm{d}s\,\mathrm{d}\theta\,\mathrm{d}x\,\mathrm{d}t
\end{aligned}$$

The second term can be estimated by (5.6.3), while, for the third one, we use the inequality $|a|^2 - |b|^2 \le |a-b|^2 + |a-b||b|$ to get:

$$
\begin{aligned}
I_{12} \le\ & C\|\sigma\|_{L^2}|\delta| \\
& + C\|\sigma\|_{L^\infty}|\delta| \int_{t_1}^{t_2}\int_{\mathbb{T}^d}\int_0^1\int_0^1 m(t,x) \\
& \qquad \times |Du_n(t+\theta\eta, x+s\delta) - D_qH^*(x, -\mathbf{v}(t,x))|^2 \,\mathrm{d}s\,\mathrm{d}\theta\,\mathrm{d}x\,\mathrm{d}t \\
& + 2C\|\sigma\|_{L^\infty}|\delta| \int_{t_1}^{t_2}\int_{\mathbb{T}^d}\int_0^1\int_0^1 \Big\{ m(t,x) \\
& \qquad \times |Du_n(t+\theta\eta, x+s\delta) - D_qH^*(x, -\mathbf{v}(t,x))| \\
& \qquad \times |D_qH^*(x, -\mathbf{v}(t,x))| \Big\} \,\mathrm{d}s\,\mathrm{d}\theta\,\mathrm{d}x\,\mathrm{d}t \\
\le\ & C\|\sigma\|_{L^2}|\delta| \\
& + C\|\sigma\|_{L^\infty}|\delta| \int_{t_1+\theta\eta}^{t_2+\theta\eta}\int_{\mathbb{T}^d}\int_0^1\int_0^1 m^{-s\delta,-\theta\eta}(t,x) \\
& \qquad \times |Du_n(t,x) - D_qH^*(x-s\delta, -\mathbf{v}^{-s\delta,-\theta\eta}(t,x))|^2 \,\mathrm{d}s\,\mathrm{d}\theta\,\mathrm{d}x\,\mathrm{d}t \\
& + 2C\|\sigma\|_{L^\infty}|\delta| \int_{t_1+\theta\eta}^{t_2+\theta\eta}\int_{\mathbb{T}^d}\int_0^1\int_0^1 \Big\{ m^{-s\delta,-\theta\eta}(t,x) \\
& \qquad \times |Du_n(t,x) - D_qH^*(x-s\delta, -\mathbf{v}^{-s\delta,-\theta\eta}(t,x))| \\
& \qquad \times |D_qH^*(x-s\delta, -\mathbf{v}^{-s\delta,-\theta\eta}(t,x))| \Big\} \,\mathrm{d}s\,\mathrm{d}\theta\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
$$

As before, using the energy estimates (5.6.3) and (5.6.8) together with a Cauchy-Schwarz inequality in the last integral, we obtain

$$
I_{12} \le C\|\sigma\|_{L^2}|\delta| + C\|\sigma\|_{L^\infty}|\delta| \left\{ (1/n + \eta^2 + |\delta|^2) + C(1/n + \eta^2 + |\delta|^2)^{1/2} \right\}.
$$

It is easy to see that with the help of the estimations for I_{11} and I_{12} we can estimate I_{13} as well. Hence we obtain

$$
\begin{aligned}
I_{13} \le\ & C\|\sigma\|_{L^2}(1/n + |\delta|^2 + \eta^2)^{1/2} \\
& + C\left\{ C\|\sigma\|_{L^2}^2 + C\|\sigma\|_{L^\infty}\left[(1/n + |\delta|^2 + \eta^2) + (1/n + |\delta|^2 + \eta^2)^{1/2}\right]\right\}^{1/2} \\
& \times (1/n + |\delta|^2 + \eta^2)^{1/2}.
\end{aligned}
$$

Let us now take care of I_2. Setting $f^\eta(t,x) := \displaystyle\int_0^1 f(x, m(t+\theta\eta, x))\,\mathrm{d}\theta$, we have

$$\begin{aligned}
I_2 &= \int_{t_1}^{t_2}\int_{\mathbb{T}^d}\sigma(t)\overline{m}|\alpha_{1,n}^{\eta}(t,x+\delta)-\alpha_{1,n}^{\eta}(t,x)|\,\mathrm{d}x\,\mathrm{d}t\\
&\le \int_{t_1}^{t_2}\int_{\mathbb{T}^d}\sigma(t)\overline{m}|\alpha_{1,n}^{\eta}(t,x+\delta)-f^{\eta}(t,x+\delta)|\,\mathrm{d}x\,\mathrm{d}t\\
&+\int_{t_1}^{t_2}\int_{\mathbb{T}^d}\sigma(t)\overline{m}|f^{\eta}(t,x+\delta)-f^{\eta}(t,x)|\,\mathrm{d}x\,\mathrm{d}t\\
&+\int_{t_1}^{t_2}\int_{\mathbb{T}^d}\sigma(t)\overline{m}|f^{\eta}(t,x)-\alpha_{1,n}^{\eta}(t,x)|\,\mathrm{d}x\,\mathrm{d}t\\
&:= I_{21}+I_{22}+I_{23}.
\end{aligned}$$

Since

$$\begin{aligned}
I_{21} &\le C\int_{t_1}^{t_2}\int_{\mathbb{T}^d}\int_0^1\sigma(t)\\
&\qquad\times|\alpha_{1,n}(t+\theta\eta,x+\delta)-f(x+\delta,m(t+\theta\eta,x+\delta))|\,\mathrm{d}\theta\,\mathrm{d}x\,\mathrm{d}t\\
&\le C\int_0^1\int_{t_1-\theta\eta}^{t_2-\theta\eta}\int_{\mathbb{T}^d}\sigma^{-\theta\eta}(t)|\alpha_{1,n}(t,x)-f(x,m(t,x))|\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}\theta
\end{aligned}$$

we obtain by Cauchy-Schwarz and (5.6.8):

$$I_{21}\ \le C\|\sigma\|_{L^2}(1/n+|\delta|^2+\eta^2)^{1/2}.$$

The term I_{23} can be treated in the same way. For I_{22}, we have

$$\begin{aligned}
I_{22} &\le C\|\sigma\|_{L^2}\int_0^1\mathrm{d}\theta\\
&\times\left(\int_{t_1}^{t_2}\int_{\mathbb{T}^d}|f(x+\delta,m(t+\theta\eta,x+\delta))-f(x,m(t+\theta\eta,x))|^2\,\mathrm{d}x\,\mathrm{d}t\right)^{1/2}\\
&\le C\|\sigma\|_{L^2}|\delta|
\end{aligned}$$

because $f(\cdot,m(\cdot,\cdot))$ is in $H^1_{\mathrm{loc}}((0,T)\times\mathbb{T}^d)$.

Putting the above inequalities together gives (5.6.7).

Step 3. We now show that the sequence $p_n := (\alpha_n - f(\cdot,\overline{m}))_+$ belongs to the space $L^2([t_1,t_2];BV(\mathbb{T}^d))$. Let us take a test function $\psi\in C_c^\infty((0,T)\times\mathbb{T}^d)$, $e\in\mathbb{R}^n$ with $|e|=1$, $\eta>0$ small and let us set $\delta:=\eta e$. We estimate

$$\begin{aligned}
&\int_{t_1}^{t_2}\int_{\mathbb{T}^d}\sigma^{-\eta}(t)\frac{\psi^{-\eta}(t,x)-\psi^{-\eta}(t,x-\delta)}{\eta}p_n(t,x)\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad=\int_{t_1-\eta}^{t_2-\eta}\int_{\mathbb{T}^d}\sigma(t)\frac{\psi(t,x)-\psi(t,x-\delta)}{\eta}p_n^{\eta}(t,x)\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad=\int_{t_1-\eta}^{t_2-\eta}\int_{\mathbb{T}^d}\sigma(t)\psi(x)\frac{p_n^{\eta}(t,x)-p_n^{\eta}(t,x+\delta)}{\eta}\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad\le\|\psi\|_{L^\infty}\frac{1}{\eta}\int_{t_1-\eta}^{t_2-\eta}\int_{\mathbb{T}^d}\sigma(t)|p_n^{\eta}(t,x)-p_n^{\eta}(t,x+\delta)|\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad\le\|\psi\|_{L^\infty}\frac{1}{\eta}X(\sigma,1/n,\eta e,\eta).
\end{aligned}$$

First let us recall that $p_n \overset{*}{\rightharpoonup} p$ as $n \to +\infty$ in $\mathscr{M}([0,T]\times\mathbb{T}^d)$, which allows us to pass to the limit in the above inequality as $n \to +\infty$ and obtain

$$\int_{t_1}^{t_2}\int_{\mathbb{T}^d} \sigma^{-\eta}(t)\frac{\psi^{-\eta}(t,x)-\psi^{-\eta}(t,x-\delta)}{\eta}\,\mathrm{d}p(t,x) \le \|\psi\|_{L^\infty}\frac{1}{\eta}X(\sigma,0,\eta e,\eta).$$

Now sending $\eta \to 0$ and recalling (5.6.7) we have

$$\int_{t_1}^{t_2}\int_{\mathbb{T}^d} \sigma(t)\nabla\psi(t,x)\cdot e\,\mathrm{d}p(t,x) \le C\|\psi\|_{L^\infty}\|\sigma\|_{L^2}.$$

Therefore we obtain that $p \in L^2([t_1,t_2];BV(\mathbb{T}^d)) \hookrightarrow L^2([t_1,t_2];L^{d/(d-1)}(\mathbb{T}^d))$ and in particular, by the arbitrariness of t_1 and t_2 and by an injection we have $p \in L^{d/(d-1)}_{\mathrm{loc}}((0,T)\times\mathbb{T}^d)$.

Step 4. Conclusion: as $0 \le \alpha_n \le p_n + f(\cdot,\overline{m})$ and (α_n) converges to α defined by (5.4.4), we have $0 \le \beta \le p + f(\cdot,\overline{m})$ in $(0,T)\times\mathbb{T}^d$. This proves that β is absolutely continuous and belongs to $L^{d/(d-1)}_{\mathrm{loc}}((0,T)\times\mathbb{T}^d)$. □

Remark 5.6.1. *Note that by the example provided in the Section 5.5 we have the sharpness of the above integrability result in the following sense: we cannot expect a bound for p_n in $L^{d/(d-1)}_{\mathrm{loc}}((0,T]\times\mathbb{T}^d)$, i.e., up to the final time, because of the occurrence of a possible jump at $t = T$. Assumption* **(H2)**, *$m_0 < \overline{m}$, prohibits this local integrability up to $t = 0$ as well.*

5.7 NASH EQUILIBRIA FOR MFG WITH DENSITY CONSTRAINTS

Let us suppose in this section the additional assumptions (HP1-1)-(HP1-2), (HP2) and (HP3) as in Section 5.6. To define a proper notion of Nash equilibrium, we use the techniques for measures on paths, corresponding the trajectories of single agents. This will also allow us to clarify the meaning of the control problem (5.1.4). The used machinery is inspired by [AF09] (Section 6) and also by [Car13b] (Section 4.3) and [CCN13] (Section 4). We remark also some similarities of this approach with the works modeling traffic congestion and Wardrop equilibria (see [BCS10, CJS08]).

5.7.1 *Density-constrained flows and a first optimality condition*

Let us recall that Γ denotes the set of absolutely continuous curves $\gamma : [0,T] \to \mathbb{T}^d$ and $\mathscr{P}_2(\Gamma)$ the set of Borel probability measures $\tilde{\boldsymbol{\eta}}$ defined on Γ such that

$$\int_\Gamma\int_0^T |\dot\gamma(s)|^2\,\mathrm{d}s\,\mathrm{d}\tilde{\boldsymbol{\eta}}(\gamma) < +\infty.$$

We call $\tilde{\boldsymbol{\eta}}$ an *almost density-constrained flow* if there exists $C = C(\tilde{\boldsymbol{\eta}}) > 0$ such that $0 \le \tilde{m}_t \le C(\tilde{\boldsymbol{\eta}})$ a.e. in $\mathbb{T}^d$ for all $t \in [0,T]$, where $\tilde{m}_t := (e_t)_\#\tilde{\boldsymbol{\eta}}$. If $C(\tilde{\boldsymbol{\eta}}) \le \overline{m}$

(the density constraint, given by our model) then we call $\tilde{\boldsymbol{\eta}}$ a *density-constrained flow*. Let us recall moreover that we use the definition of the Lagrangian as $L(x, \mathbf{v}) = H^*(x, -\mathbf{v})$.

In the whole section we consider a solution (u, m, β, β_T) of the MFG system (5.4.1). By Theorem 5.4.1 and Theorem 5.6.1 this corresponds to (u, α) and $(m, \mathbf{w})$ solutions of Problem 5.3.3 and Problem 5.3.2 respectively, where

$$\alpha = f(\cdot, m)\,\mathrm{d}x\,\mathrm{d}t + \beta\,\mathrm{d}x\,\mathrm{d}t + \beta_T\,\mathrm{d}(\delta_T \otimes \mathscr{H}^d \llcorner \mathbb{T}^d) \quad \text{and} \quad \mathbf{w} = -mD_pH(x, Du).$$

Let us state the following results (in the spirit of Lemma 4.6.-4.8. from [Car13b]) which characterize the density-constrained flows.

Lemma 5.7.1. *Let $\tilde{\boldsymbol{\eta}} \in \mathscr{P}_2(\Gamma)$ be an almost density-constrained flow and set $\tilde{m}_t := (e_t)_\# \tilde{\boldsymbol{\eta}}$. Then*

(i) for all $0 \le t_1 < t_2 \le T$ we have

$$\int_{\mathbb{T}^d} u(t_1^+, x)\tilde{m}(t_1, x)\,\mathrm{d}x \le \int_{\mathbb{T}^d} u(t_2^-, x)\tilde{m}(t_2, x)\,\mathrm{d}x + \int_\Gamma \int_{t_1}^{t_2} L(\gamma(t), \dot{\gamma}(t))\,\mathrm{d}t\,\mathrm{d}\tilde{\boldsymbol{\eta}}(\gamma) + \int_{t_1}^{t_2} \int_{\mathbb{T}^d} (f(x, m(t,x)) + \beta(t,x))\,\tilde{m}(t,x)\,\mathrm{d}x\,\mathrm{d}t.$$

(ii) In particular, for all $0 \le t_1 < T$

$$\int_{\mathbb{T}^d} u(t_1^+, x)\tilde{m}(0, x)\,\mathrm{d}x \le \int_{\mathbb{T}^d} (g(x) + \beta_T(x))\tilde{m}(T, x)\,\mathrm{d}x + \int_\Gamma \int_0^T L(\gamma(t), \dot{\gamma}(t))\,\mathrm{d}t\,\mathrm{d}\tilde{\boldsymbol{\eta}}(\gamma) + \int_0^T \int_{\mathbb{T}^d} (f(x, m(t,x)) + \beta(t,x))\,\tilde{m}(t,x)\,\mathrm{d}x\,\mathrm{d}t.$$

Proof. Let us recall that u satisfies, in the sense of measures,

$$-\partial_t u + H(x, Du) \le \alpha \qquad \text{in } (0, \mathrm{T}) \times \mathbb{T}^{\mathrm{d}},$$

where α belongs to $L^{d/(d-1)}_{\mathrm{loc}}((0,T) \times \mathbb{T}^d)$ thanks to Theorem 5.6.1. If we regularize u into u_n and α into α_n by convolution (with a compact support in $B_{1/n}(0)$), we obtain

$$-\partial_t u_n + H(x, Du_n) \le \alpha_n + r_n \qquad \text{in } (1/n, T - 1/n) \times \mathbb{T}^d,$$

where

$$r_n(t,x) = H(x, Du_n(t,x)) - H(\cdot, Du) \star \rho_n(t,x).$$

Note that (r_n) tends to 0 in $L^1((0,T) \times \mathbb{T}^d)$ and when H does not depend on the x variable one also has $r_n \le 0$. Let us fix $0 < t_1 < t_2 < T$ and n large. Now for any $\gamma \in H^1([0,T])$ we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left(u_n(t, \gamma(t)) - \int_t^T L(\gamma(s), \dot{\gamma}(s))\,\mathrm{d}s\right) &\ge \partial_t u_n(t, \gamma(t)) - H(\gamma(t), Du_n(t, \gamma(t))) \\ &\ge -\alpha_n(t, \gamma(t)) - r_n(t, \gamma(t)). \end{aligned} \tag{5.7.1}$$

Integrating this inequality on $[t_1, t_2]$, then over Γ w.r.t. $\tilde{\boldsymbol{\eta}}$, we obtain

$$\int_{\mathbb{T}^d} u_n(t_1,x)\tilde{m}(t_1,x)\,\mathrm{d}x \leq \int_{\mathbb{T}^d} u_n(t_2,x)\tilde{m}(t_2,x)\,\mathrm{d}x + \int_\Gamma \int_{t_1}^{t_2} L(\gamma(t),\dot{\gamma}(t))\,\mathrm{d}t\,\mathrm{d}\tilde{\boldsymbol{\eta}}(\gamma)$$
$$+ \int_{\mathbb{T}^d}\int_{t_1}^{t_2} [\alpha_n(t,x) + r_n(t,x)]\tilde{m}(t,x)\,\mathrm{d}t\,\mathrm{d}x.$$

We recall the fact that $\tilde{m} \in L^\infty([0,T]\times\mathbb{T}^d)$. Since (u_n) strongly converges in L^1 to $u \in BV([0,T]\times\mathbb{T}^d)$, we have the existence of $J \subset (0,T)$ of full measure such that for every $t_1, t_2 \in J, t_1 < t_2$, the first two integrals pass to the limit as $n \to +\infty$. By the strong convergence in $L^{d/(d-1)}([t_1,t_2]\times\mathbb{T}^d)$ of (α_n) to α and in $L^1([t_1,t_2]\times\mathbb{T}^d)$ of (r_n) to zero, we can pass to the limit as $n \to +\infty$ is the last integral as well. So, for a.e. $0 < t_1 < t_2 < T$, we have

$$\int_{\mathbb{T}^d} u(t_1,x)\tilde{m}(t_1,x)\,\mathrm{d}x \leq \int_{\mathbb{T}^d} u(t_2,x)\tilde{m}(t_2,x)\,\mathrm{d}x + \int_\Gamma \int_{t_1}^{t_2} L(\gamma(t),\dot{\gamma}(t))\,\mathrm{d}t\,\mathrm{d}\tilde{\boldsymbol{\eta}}(\gamma)$$
$$+ \int_{\mathbb{T}^d}\int_{t_1}^{t_2} \alpha(t,x)\tilde{m}(t,x)\,\mathrm{d}t\,\mathrm{d}x.$$

In order to show that the inequality holds for any $t_1 < t_2$, let us now check that

$$\mathrm{ess}-\lim_{t'\to t^\pm} \int_{\mathbb{T}^d} u(t',x)\tilde{m}(t',x)\,\mathrm{d}x = \int_{\mathbb{T}^d} u(t^\pm,x)\tilde{m}(t,x)\,\mathrm{d}x,$$

where $u(t^\pm,\cdot)$ is understood in the sense of trace and $\tilde{m}(t,\cdot)$ is the (bounded) density of the continuous representative of the map $t \mapsto \tilde{m}(t,\cdot)\,\mathrm{d}x$ (for the L^∞ weak$-\star$ convergence).The above limit basically follows from the trace properties of BV functions, but for the sake of completeness let us sketch it below. Let u_n be a standard mollification in space of u. As u is in BV, $u(t',\cdot)$ converges in L^1 to $u(t^\pm,\cdot)$ as $t' \to t^\pm$, so that $u_n(t',\cdot)$ uniformly converges to $u_n(t^\pm,\cdot)$. Let us write $\int_{\mathbb{T}^d} u(t',x)\tilde{m}(t',x)\,\mathrm{d}x$ as

$$\int_{\mathbb{T}^d} u_n(t',x)\tilde{m}(t',x)\,\mathrm{d}x + \int_{\mathbb{T}^d}(u(t',x) - u_n(t',x))\tilde{m}(t',x))\,\mathrm{d}x. \qquad (5.7.2)$$

By uniform convergence of $u_n(t',\cdot)$, the first term in (5.7.2) converges to

$$\int_{\mathbb{T}^d} u_n(t^\pm,x)\tilde{m}(t,x)\,\mathrm{d}x,$$

which is arbitrary close to $\int_{\mathbb{T}^d} u(t^\pm,x)\tilde{m}(t,x)\,\mathrm{d}x$ for n large. As for the second term in (5.7.2), it is bounded by $\|u(t',\cdot) - u_n(t',\cdot)\|_{L^1}\|m\|_{L^\infty}$, which, by L^1 convergence of $u(t',\cdot)$ to $u(t^\pm,\cdot)$, tends to 0 uniformly in t'. This proves (i).

For (ii), we just apply (i) for $t_2 = T$, since $u(T^-,\cdot) = g + \beta_T$. □

Definition 5.7.1. *We say that an $\boldsymbol{\eta} \in \mathscr{P}_2(\Gamma)$ is an* optimal density-constrained flow *associated to the solution (u, m, β, β_T) if $m(t,\cdot) = (e_t)_\#\boldsymbol{\eta}$, for all $t \in [0,T]$ and the following energy equality holds*

$$\begin{aligned}\int_{\mathbb{T}^d} u(0^+,x)m_0(x)\,\mathrm{d}x &= \int_{\mathbb{T}^d} g(x)m(T,x)\,\mathrm{d}x + \overline{m}\int_{\mathbb{T}^d}\beta_T\,\mathrm{d}x \\ &+ \int_\Gamma\int_0^T L(\gamma(t),\dot\gamma(t))\,\mathrm{d}t\,\mathrm{d}\boldsymbol{\eta}(\gamma) \\ &+ \int_0^T\int_{\mathbb{T}^d}(f(x,m(t,x))+\beta(t,x))\,m(t,x)\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

Note that the above definition is the reformulation in terms of density-constrained flows of the energy equality from Definition 5.4.1 point (4).

Remark 5.7.1. *Let us observe that for an optimal density-constrained flow $\boldsymbol{\eta}$, the energy equality in Definition 5.7.1 holds for any $0 \le t_1 < t_2 \le T$ as well, i.e.*

$$\begin{aligned}\int_{\mathbb{T}^d} u(t_1^+,x)m(t_1,x)\,\mathrm{d}x &= \int_{\mathbb{T}^d} u(t_2^-,x)m(t_2,x)\,\mathrm{d}x + \int_\Gamma\int_{t_1}^{t_2} L(\gamma(t),\dot\gamma(t))\,\mathrm{d}t\,\mathrm{d}\boldsymbol{\eta}(\gamma) \\ &+ \int_{t_1}^{t_2}\int_{\mathbb{T}^d}(f(x,m(t,x))+\beta(t,x))\,m(t,x)\,\mathrm{d}x\,\mathrm{d}t. \qquad (5.7.3)\end{aligned}$$

This can be easily deduced using the inequalities from Lemma 5.7.1 three times on the intervals $[0,t_1],[t_1,t_2]$ and $[t_2,T]$ together with the global equality from Definition 5.7.1 and the fact that $(\partial_t u)^s$ is a non-negative measure (by the fact that α does not have a singular part in $(0,T)\times\mathbb{T}^d$ and one has in the sense of measures $-\partial_t u + H(\cdot,Du) \le \alpha$), i.e. one has always $u(t^-,\cdot) \le u(t^+,\cdot)$ a.e. for all $t \in (0,T)$.

Identity (5.7.3) implies also that $(\partial_t u)^s = 0$ on the support of m, more precisely

$$\int_{\mathbb{T}^d} u(t^+,x)m(t,x)\,\mathrm{d}x = \int_{\mathbb{T}^d} u(t^-,x)m(t,x)\,\mathrm{d}x,$$

for all $t \in (0,T)$.

The following proposition gives the existence result for an optimal density-constrained flow $\boldsymbol{\eta}$.

Proposition 5.7.2. *There exists at least one optimal density-constrained flow $\boldsymbol{\eta} \in \mathscr{P}_2(\Gamma)$ in the sense of the Definition 5.7.1.*

Proof. The proof uses the same construction and goes along the same lines as in [Car13b]. Nevertheless, we discuss the main steps here.

We construct a family $(\boldsymbol{\eta}_\varepsilon)_{\varepsilon>0}$ of density-constrained flows by

$$\int_\Gamma \Psi(\gamma)\,\mathrm{d}\boldsymbol{\eta}_\varepsilon(\gamma) := \int_{\mathbb{T}^d}\Psi(X_\varepsilon^x)m_0(x)\,\mathrm{d}x,$$

for any bounded and continuous map $\Psi : \Gamma \to \mathbb{R}$, where X_ε^x is the solution of the Cauchy problem

$$\begin{cases} \dot x(t) = \dfrac{\mathbf{w}_\varepsilon(t,x(t))}{m_\varepsilon(t,x(t))}, & \text{a.e in } [0,T], \\ x(0) = x, \end{cases}$$

$(m_\varepsilon, \mathbf{w}_\varepsilon)$ being a standard mollification of $(m, \mathbf{w})$ such that $0 < m_\varepsilon \le \overline{m}$. One easily checks that $m_\varepsilon(t, \cdot) = (e_t)_\# \boldsymbol{\eta}_\varepsilon$.

Using Lemma 4.7. from [Car13b] we obtain that the family $(\boldsymbol{\eta}_\varepsilon)_{\varepsilon>0}$ is tight. Denoting by $\boldsymbol{\eta}$ the limit of a suitable subsequence of it, this is an optimal density-constrained flow in the sense of Definition 5.7.1. The proof of this statement goes exactly as for Lemma 4.8. in [Car13b], using the equality (4) from Definition 5.4.1 and the inequality (ii) from Lemma 5.7.1. □

5.7.2 *Optimality conditions on the level of single agent trajectories*

In this subsection our aim is to show that the optimal density-constrained flows are actually concentrated on paths which are optimal (in some weak sense) for the control problem (5.1.4) (see Definition 5.7.2). We will show that they satisfy a weak dynamic programming principle.

Let us recall that $\beta \in L^2_{\rm loc}((0, T); BV(\mathbb{T}^d))$ and $\beta_T \in L^1(\mathbb{T}^d)$. In order to handle the evaluation of β along single agent paths we shall work with specific representative of it (which is defined everywhere in $\mathbb{T}^d$).

For an $L^1_{\rm loc}$ function $h : \mathbb{T}^d \to \mathbb{R}$ we define the specific representative of h by

$$\hat{h}(x) := \limsup_{\varepsilon \downarrow 0} h_\varepsilon(x), \tag{5.7.4}$$

where

$$h_\varepsilon(x) := \int_{\mathbb{R}^d} h(x + \varepsilon y)\rho(y)\,\mathrm{d}y$$

and ρ being the heat kernel

$$\rho(y) := (2\pi)^{-d/2} e^{-|y|^2/2}. \tag{5.7.5}$$

We use this specific regularization via the heat kernel because of the semigroup property $(h_\varepsilon)_{\varepsilon'} = h_{\varepsilon+\varepsilon'}$ we shall profit on later.

To treat passages to limit (in the regularization, as $\varepsilon \downarrow 0$, similarly as in Section 6 from [AF09]) we will need some uniform point-wise bounds on β_ε, hence we shall use the properties of the Hardy-Littlewood-type maximal function defined with the help of the heat kernel (5.7.5). Thus for any $h \in L^1(\mathbb{T}^d)$ we set

$$(Mh)(x) := \sup_{\varepsilon>0} \int_{\mathbb{R}^d} |h(x + \varepsilon y)|\rho(y)\,\mathrm{d}y.$$

Let us state some basic properties of the maximal functional M that we will use in our setting. First because of the semigroup property we have

$$Mh_\varepsilon = \sup_{\varepsilon'>0} |h_\varepsilon|_{\varepsilon'} \le \sup_{\tilde{\varepsilon}>0} |h|_{\tilde{\varepsilon}} = Mh.$$

Secondly it is well-known that M leaves invariant any L^p space with $1 < p \le +\infty$ and there exists $C_p > 0$ such that

$$\|Mh\|_{L^p(\mathbb{T}^d)} \le C_p \|h\|_{L^p(\mathbb{T}^d)}.$$

Let us recall that by Theorem 5.6.1 we have that $M\beta \in L^{d/(d-1)}_{\rm loc}((0,T)\times\mathbb{T}^d) \hookrightarrow L^1_{\rm loc}((0,T)\times\mathbb{T}^d)$. The integrability property we need is only $M\beta \in L^1_{\rm loc}((0,T)\times\mathbb{T}^d)$, but to guarantee this, $\beta\in L^1_{\rm loc}((0,T)\times\mathbb{T}^d)$ is not enough.

Let us set moreover $\alpha(t,x) := f(x,m(t,x)) + \beta(t,x)$ for a.e. $(t,x)\in[0,T]\times\mathbb{T}^d$ and let us use its representative $\hat\alpha$ (obtained as in (5.7.4)) in the sequel.

Definition 5.7.2. *Given $0<t_1<t_2<T$, we say that a path $\gamma\in H^1([0,T];\mathbb{T}^d)$ with $M\hat\alpha(\cdot,\gamma)\in L^1_{\rm loc}((0,T))$ is* minimizing *on the time interval $[t_1,t_2]$ in the problem (5.1.4) if we have*

$$\hat u(t_2^+,\gamma(t_2)) + \int_{t_1}^{t_2} L(\gamma(t),\dot\gamma(t)) + \hat\alpha(t,\gamma(t))\,\mathrm{d}t \le \hat u(t_2^-,\gamma(t_2)+\omega(t_2)) + \\ + \int_{t_1}^{t_2} L(\gamma(t)+\omega(t),\dot\gamma(t)+\dot\omega(t)) + \hat\alpha(t,\gamma(t)+\omega(t))\,\mathrm{d}t,$$

for all $\omega\in H^1([t_1,t_2];\mathbb{T}^d)$ such that $\omega(t_1)=0$ and $M\hat\alpha(\cdot,\gamma+\omega)\in L^1([t_1,t_2])$.

Remark 5.7.2. *Let us notice that for any density-constrained flow $\tilde{\boldsymbol\eta}$ the integrability property $M\hat\alpha(\cdot,\gamma)\in L^1_{\rm loc}((0,T))$ is natural, since it is satisfied $\tilde{\boldsymbol\eta}$-a.e., if $M\hat\alpha\in L^1_{\rm loc}((0,T)\times\mathbb{T}^d)$. Indeed, by this we have*

$$\int_\Gamma\int_{t_1}^{t_2} M\hat\alpha(t,\gamma(t))\,\mathrm{d}t\,\mathrm{d}\tilde{\boldsymbol\eta}(\gamma) = \int_{t_1}^{t_2}\int_{\mathbb{T}^d} M\hat\alpha(t,x)\tilde m(t,x)\,\mathrm{d}x\,\mathrm{d}t < +\infty,$$

for all $0<t_1<t_2<T$, where $\tilde m(t,\cdot)\,\mathrm{d}x = (e_t)_\#\tilde{\boldsymbol\eta}$.

Theorem 5.7.3. *For any $0<t_1<t_2<T$, any optimal density-constrained flow $\boldsymbol\eta$ is concentrated on minimizing paths on the time interval $[t_1,t_2]$ for the problem (5.1.4) in the sense of the Definition 5.7.2.*

Proof. We follow here Ambrosio-Figalli [AF09]. Let us take an optimal density-constrained flow $\boldsymbol\eta$ given by Proposition 5.7.2, fix $0<t_1<t_2<T$ and $y\in\mathbb{T}^d$, take $\omega\in H^1([t_1,t_2];\mathbb{T}^d)$ with $\omega(t_1)=0$ and $\chi\in C^1_c((0,T);[0,1])$ with $\chi>0$ on $(t_1,t_2]$ and $\chi(t_1)=0$ a smooth cut-off function. Let us take a Borel subset $E\subset\Gamma$ such that $\boldsymbol\eta(E)$ is positive. For $\varepsilon>0$ and $y\in\mathbb{T}^d$ we introduce the map $T_{\varepsilon,y}:\Gamma\to\Gamma$ by

$$T_{\varepsilon,y}(\gamma) := \begin{cases} \gamma, & \text{if } \gamma\notin E,\\ \gamma+\omega+\varepsilon\chi y, & \text{if } \gamma\in E.\end{cases}$$

Now let us define $\boldsymbol\eta_{\varepsilon,y} := (T_{\varepsilon,y})_\#\boldsymbol\eta$, which in particular is an admissible density-constrained flow satisfying the inequalities from Lemma 5.7.1. In addition let us remark that, $(e_{t_1})_\#\boldsymbol\eta_{\varepsilon,y} = (e_{t_1})_\#\boldsymbol\eta = m(t_1,\cdot)\,\mathrm{d}x$.

Using the inequality (i) from Lemma 5.7.1 for $\boldsymbol{\eta}_{\varepsilon,y}$ (on the time interval $[t_1,t_2]$) and the equality (5.7.3) for $\boldsymbol{\eta}$ (on the same interval $[t_1,t_2]$) we obtain

$$\begin{aligned}\int_E \Big[\hat{u}(t_2^+,\gamma(t_2)) + \int_{t_1}^{t_2} L(\gamma(t),\dot\gamma(t)) + \hat\alpha(t,\gamma(t))\,\mathrm{d}t\Big]\,\mathrm{d}\boldsymbol{\eta}(\gamma) \le \\ \le \int_E \Big[\hat{u}(t_2^-,\gamma(t_2)+\omega(t_2)+\varepsilon\chi(t_2)y) \\ + \int_{t_1}^{t_2} L(\gamma(t)+\omega(t)+\varepsilon\chi(t)y,\dot\gamma(t)+\dot\omega(t)+\varepsilon\dot\chi(t)y)\,\mathrm{d}t\Big]\,\mathrm{d}\boldsymbol{\eta}(\gamma) \\ + \int_E\int_{t_1}^{t_2} \hat\alpha(t,\gamma(t)+\omega(t)+\varepsilon\chi(t)y)\,\mathrm{d}t\,\mathrm{d}\boldsymbol{\eta}(\gamma).\end{aligned}$$

where we are allowed to use any representative of u and α, thus we use the specially constructed ones $\hat u$ and $\hat\alpha$. Let us average this last inequality w.r.t. the variable y using the kernel ρ introduced in (5.7.5). We obtain

$$\begin{aligned}\int_E \Big[\hat{u}(t_2^+,\gamma(t_2)) + \int_{t_1}^{t_2} L(\gamma(t),\dot\gamma(t)) + \hat\alpha(t,\gamma(t))\,\mathrm{d}t\Big]\,\mathrm{d}\boldsymbol{\eta}(\gamma) \le \\ \int_E\int_{\mathbb{R}^d} \Big[\hat{u}(t_2^-,\gamma(t_2)+\omega(t_2)+\varepsilon\chi(t_2)y) \\ + \int_{t_1}^{t_2} L(\gamma(t)+\omega(t)+\varepsilon\chi(t)y,\dot\gamma(t)+\dot\omega(t)+\varepsilon\dot\chi(t)y)\,\mathrm{d}t\Big]\rho(y)\,\mathrm{d}y\,\mathrm{d}\boldsymbol{\eta}(\gamma) \\ + \int_E\int_{t_1}^{t_2} \hat\alpha_{\varepsilon\chi(t)}(t,\gamma(t)+\omega(t))\,\mathrm{d}t\,\mathrm{d}\boldsymbol{\eta}(\gamma).\end{aligned}$$

Now choosing $\mathcal{D}\subset H^1([t_1,t_2];\mathbb{T}^d)$ a dense subset with $\omega(t_1)=0$ for all $\omega\in\mathcal{D}$, by the arbitrariness of E for $\boldsymbol{\eta}$-almost every curve $\gamma\in\Gamma$ we deduce that

$$\begin{aligned}&\hat{u}(t_2^+,\gamma(t_2)) + \int_{t_1}^{t_2} L(\gamma(t),\dot\gamma(t)) + \hat\alpha(t,\gamma(t))\,\mathrm{d}t \\ &\le \int_{\mathbb{R}^d}\int_{t_1}^{t_2} L(\gamma(t)+\omega(t)+\varepsilon\chi(t)y,\dot\gamma(t)+\dot\omega(t)+\varepsilon\dot\chi(t)y)\rho(y)\,\mathrm{d}t\,\mathrm{d}y \\ &\qquad + \hat{u}_{\varepsilon\chi(t_2)}(t_2^-,\gamma(t_2)+\omega(t_2)) + \int_{t_1}^{t_2}\hat\alpha_{\varepsilon\chi(t)}(t,\gamma(t)+\omega(t))\,\mathrm{d}t,\end{aligned}$$

for all $\omega\in\mathcal{D}$ and $\varepsilon = 1/n$. By a density argument the above inequality holds for any $\omega\in H^1([t_1,t_2];\mathbb{T}^d)$ with $\omega(t_1)=0$. We finally let $\varepsilon\downarrow 0$. As $M\hat\alpha(t,\gamma+\omega)\in L^1([t_1,t_2])$ and using the domination $|\alpha_\varepsilon|\le M\hat\alpha$, we can pass to the limit in the last term of the above inequality. By the dominate convergence theorem we can also pass to the limit in the first term thanks to the growth property and the continuity of L. Thus the result follows. □

Remark 5.7.3. *The global version of Theorem 5.7.3 (to arrive up to the initial time 0 and the final time T) remains an open question. This is mainly due to the local integrability property for the additional price $\beta\in L^2_{\mathrm{loc}}((0,T);BV(\mathbb{T}^d))$ we are aware of for the moment. Let us remark that an integrability property $\beta\in L^{1+\varepsilon}([0,T]\times\mathbb{T}^d)$ for some $\varepsilon>0$ would be enough to conclude in the global version.*

The *notion of Nash equilibria* has now a clearer formulation. Since we are able to give a weak meaning for the optimization problem along single agent trajectories, a solution (u, m, β, β_T) of the MFG system with density constraints gives the following notion of equilibrium.

Definition 5.7.3 (Local weak Nash equilibria). *Let (u, m, β, β_T) be a solution of the MFG system with density constraints in the sense of Definition 5.4.1 on $[0, T] \times \mathbb{T}^d$. We say that (m, β, β_T) is a* local weak Nash equilibrium*, if there exists an optimal density-constrained flow $\boldsymbol{\eta} \in \mathscr{P}_2(\Gamma)$ in the sense of Definition 5.7.1 (constructed with the help of (m, β, β_T)) which is concentrated on locally minimizing paths for Problem (5.1.4) in the sense of Definition 5.7.2. In particular one has that $m_t = (e_t)_\# \boldsymbol{\eta}$ and $0 \le m_t \le \overline{m}$ a.e. in $\mathbb{T}^d$ for all $t \in [0, T]$.*

Remark 5.7.4. *Let us remark that by Proposition 5.7.2 and Theorem 5.7.3 for any solution (u, m, β, β_T) for the MFG system with density constraints obtained with the additional assumptions (HP1-1)-(HP1-2), (HP2) and (HP3) the triple (m, β, β_T) is always a local weak Nash equilibrium in the sense of the above definition.*

5.7.3 *The case without density constraint*

Let us have a few words on the Nash equilibrium and on the optimality condition on the level of single agent trajectories in the case when we do not impose density constraints. More precisely, our aim is to clarify Remark 4.9. from [Car13b].

Let us recall that in Section 4.3. from [Car13b] it was considered a class of flows $\tilde{\boldsymbol{\eta}} \in \mathscr{P}_{r'}(\mathbb{T}^d)$ such that $\tilde{m} \in L^q([0, T] \times \mathbb{T}^d)$ where $\tilde{m}_t := (e_t)_\# \tilde{\boldsymbol{\eta}}$, where $r' > 1$ is the growth of the Lagrangian L in the velocity variable, while $q - 1$ (where $q > 1$) is the growth of the continuous coupling f in the second variable. Because of this growth condition and since $m \in L^q([0, T] \times \mathbb{T}^d)$ we have first that $\alpha(t, x) := f(x, m(t, x)) \in L^{q'}([0, T] \times \mathbb{T}^d)$. Moreover Lemma 5.7.1 and Proposition 5.7.2 hold with $\beta \equiv 0$ and $\beta_T \equiv 0$, since we did not impose any density constraint (see the corresponding Lemma 4.6-4.8 from [Car13b]).

The difference, compared to our analysis in the previous section, is that we can consider globally minimizing paths in Definition 5.7.2. More precisely, by the global integrability property of $\hat{\alpha}$, and hence $M\hat{\alpha} \in L^{q'}([0, T] \times \mathbb{T}^d)$ we allow curves $\gamma \in W^{1,r'}([0, T])$ (and their variations) such that $M\hat{\alpha}(\cdot, \gamma) \in L^{q'}([0, T])$. This is once again a natural class, since for any flow $\tilde{\boldsymbol{\eta}}$, with the above described properties, satisfies that

$$\int_\Gamma \int_{t_1}^{t_2} M\hat{\alpha}(t, \gamma(t))\, \mathrm{d}t\, \mathrm{d}\tilde{\boldsymbol{\eta}}(\gamma) = \int_{t_1}^{t_2} \int_{\mathbb{T}^d} M\hat{\alpha}(t, x) \tilde{m}(t, x)\, \mathrm{d}x\, \mathrm{d}t < +\infty,$$

for all $0 \le t_1 < t_2 \le T$, since $M\hat{\alpha} \in L^{q'}([0, T] \times \mathbb{T}^d)$ and $\tilde{m} \in L^q([0, T] \times \mathbb{T}^d)$ where $\tilde{m}_t = (e_t)_\# \tilde{\boldsymbol{\eta}}$.

By these observations in the statement of Theorem 5.7.3 one can change now the word “locally” to “globally” and the proof goes along the same lines.

Part III

APPENDIX

A

Convex analysis, Γ−convergence and basic Calderón-Zygmund theory

A.1 Classical results from convex analysis

We collect here some well-known results from convex analysis. In what follows, let X and Y be normed Banach spaces, $f : X \to \overline{\mathbb{R}}$ is a convex, l.s.c. and proper function. We denote $\mathrm{dom}(f) := \{x \in X : f(x) < +\infty\}$. We define its *Legendre-Fenchel transformation* as $f^* : X^* \to \overline{\mathbb{R}}$,

$$f^*(x^*) := \sup_{x \in X} \{\langle x^*, x\rangle_{X^*,X} - f(x)\}, \quad \forall x^* \in X^*.$$

The *subdifferential* of f in a point $x \in X$ is defined as

$$\partial f(x) := \{x^* \in X^* : f(y) \geq f(x) + \langle x^*, y - x\rangle_{X^*,X} \quad \forall y \in X\}.$$

We have the following identity

$$f(x) + f^*(x^*) \geq \langle x^*, x\rangle_{X^*,X}, \quad \forall x \in X, x^* \in X^*$$

with equality if and only if $x^* \in \partial f(x)$ or equivalently $x \in \partial f^*(x^*)$.

Let us characterize the Legendre-Fenchel transformation of a sum through the inf-convolution, provided a qualification condition holds.

Lemma A.1.1. *Let $f, g : X \to \overline{\mathbb{R}}$ be two convex and l.s.c. functions. If the following qualification condition*

$$\exists x_0 \in \mathrm{dom}(g) \text{ such that } f \text{ is continuous and finite in } x_0 \qquad (Q_1)$$

holds, then

$$(f+g)^*(x^*) = \inf_{z^* \in X^*} \{f^*(x^* - z^*) + g^*(z^*)\},$$

for all $x^* \in X^*$.

Another useful result is concerning the subdifferential of the sum of two convex functions is the following (see for instance [ABM06, Theorem 9.5.4.]).

Theorem A.1.2. *Let* $f, g : X \to \overline{\mathbb{R}}$ *be two closed convex proper functions. Then*

a) the following inclusion is always true:

$$\partial f + \partial g \subset \partial(f+g).$$

b) Moreover, if the qualification assumption (Q_1) *holds, then we have*

$$\partial f(x) + \partial g(x) = \partial(f+g)(x),$$

for all $x \in X$.

Let us introduce two more notions. Let $K \subseteq X$ be a non-empty closed convex set. The *support function* of the set K is $\sigma_K : X^* \to \overline{\mathbb{R}}$ and defined as

$$\sigma_K(x^*) := \sup_{x \in K} \langle x^*, x\rangle_{X^*,X}, \quad \forall x^* \in X^*.$$

The *normal cone* to the set K in $x \in K$ is defined as

$$N_K(x) := \{x^* \in X^* : \langle x^*, z - x\rangle_{X^*,X} \leq 0, \forall z \in K\}.$$

These last to notions are linked to χ_K as $\chi_K^*(x^*) = \sigma_K(x^*)$ and $\partial\chi_K(x) = N_K(x)$.

Now let us consider the following optimization problem:

$$\min_{x \in X} f(x), \tag{CP}$$

subject to $G(x) \in K$, where $f : X \to \overline{\mathbb{R}}$ is a convex, l.s.c. and proper function, $G : X \to Y$ is a linear operator and $K \subseteq Y$ is a closed convex set.

We define the Lagrangian as $\mathfrak{L} : X \times Y^* \to \overline{\mathbb{R}}$,

$$\mathfrak{L}(x, \lambda) := f(x) + \langle \lambda, G(x)\rangle_{Y^*,Y}$$

and dual problem as

$$\sup_{\lambda \in Y^*} \inf_{x \in X} \mathfrak{L}(x, \lambda) - \sigma_K(\lambda), \tag{CPD}$$

One has the following general result:

Theorem A.1.3 ([BS00]). *If the qualification condition*

$$0 \in \operatorname{int}\left(G(\operatorname{dom}(f)) - K\right) \tag{Q_2}$$

holds, then the dual problem (CPD) *has at least one solution* $\overline{\lambda}$*, moreover the values of the primal* (CP) *and the dual* (CPD) *are equal and the following optimality conditions hold for any solution* $\overline{x}$ *of* (CP)*:*

$$\mathfrak{L}(\overline{x},\overline{\lambda}) = \inf_{x\in X} \mathfrak{L}(x,\overline{\lambda})$$

and

$$\overline{\lambda} \in N_K(G(\overline{x})),$$

where $N_K(y)$ *denotes the normal cone to the set* K *in the point* y*.*

Let us state another characterization theorem (see for instance [ABM06, Theorem 9.5.5]).

Theorem A.1.4. *Let* X *be a normed space, let* $f : X \to \overline{\mathbb{R}}$ *be a convex, closed and proper function, and let* $C \subset X$ *be a closed convex non-empty subset. We assume that one of the following qualification condition holds:*

(i) f *is continuous at some point of* C*;*

(ii) $\operatorname{dom}(f) \cap \operatorname{int} C \neq \emptyset$.

Then the following statements are equivalent:

(1) u *is an optimal solution of the minimization problem* $\min\{f(v) : v \in C\}$*;*

(2) u *is a solution of the equation* $0 \in \partial f(u) + N_C(u)$*;*

(3) there exists $u^* \in X^*$ *such that* $u \in C; u^* \in \partial f(u); \langle u^*, v-u\rangle \geq 0,\ \forall v \in C$.

A.2 SOME WORDS ON $\Gamma-$CONVERGENCE

In order to handle our approximation procedures (in Chapter 1), we need to spend some words on the notion of $\Gamma-$ *convergence* (see [DM93]).

Definition A.2.1. *On a metric space* X *let* $F_n : X \to \mathbb{R} \cup \{+\infty\}$ *be a sequence of functions. We define the two lower-semicontinuous functions* F^- *and* F^+ *(called* $\Gamma - \liminf$ *and* $\Gamma - \limsup$ *of this sequence, respectively) by*

$$F^-(x) := \inf\{\liminf_{n\to\infty} F_n(x_n) \ : \ x_n \to x\},$$
$$F^+(x) := \inf\{\limsup_{n\to\infty} F_n(x_n) \ : \ x_n \to x\}.$$

Should F^- *and* F^+ *coincide, then we say that* F_n *actually* $\Gamma-$converges *to the common value* $F = F^- = F^+$*.*

This means that, when one wants to prove $\Gamma-$convergence of F_n towards a given functional F, one has actually to prove two distinct facts: first we need $F^- \geq F$ (this is called $\Gamma-$liminf inequality, i.e. we need to prove $F(x) \leq \liminf_n F_n(x_n)$ for any approximating sequence $x_n \to x$) and then $F^+ \leq F$ (this is called $\Gamma-$limsup inequality, i.e. we need to find a *recovery sequence* $x_n \to x$ such that $\limsup_n F_n(x_n) \leq F(x)$).

The definition of Γ−convergence for a continuous parameter $\varepsilon \to 0$ obviously passes through the convergence to the same limit for any subsequence $\varepsilon_n \to 0$.

Among the properties of Γ−convergence we have the following ones:

– if there exists a compact set $K \subset X$ such that $\inf_X F_n = \inf_K F_n$ for any n, then F attains its infimum and $\inf F_n \to \min F$,
– if $(x_n)_n$ is a sequence of minimizers for F_n admitting a subsequence converging to x, then x minimizes F (in particular, if F has a unique minimizer x and the sequence of minimizers x_n is compact, then $x_n \to x$),
– if F_n is a sequence Γ−converging to F, then $F_n + G$ will Γ−converge to $F + G$ for any continuous function $G : X \to \mathbb{R} \cup \{+\infty\}$.

In the sequel we will need the following two easy criteria to guarantee Γ−convergence.

Proposition A.2.1. *If each F_n is l.s.c. and $F_n \to F$ uniformly, then F_n Γ−converges to F.*

If each F_n is l.s.c., $F_n \leq F_{n+1}$ and $F(x) = \lim_n F_n(x)$ for all x, then F_n Γ−converges to F.

We will essentially apply the notion of Γ−convergence in the space $X = \mathscr{P}(\Omega)$ endowed with the weak convergence [1] (which is indeed metrizable on this bounded subset of the Banach space of measures), instead in the space $\mathscr{P}_2(\Omega)$ endowed with the W_2 convergence, which lacks compactness whenever Ω is not compact.

A.3 CALDERÓN-ZYGMUND TYPE THEORY

In this section we recall some classical results about the regularity of solutions of elliptic equations with irregular r.h.s. Recall that we set $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ for the duality product between $(W^{1,q'}_\diamond(\Omega))^*$ $(q > 1)$ and $W^{1,q'}_\diamond(\Omega)$. The following surjectivity result holds true.

Lemma A.3.1. *For any $f \in (W^{1,q'}_\diamond(\Omega))^*$ the weak formulation of*

$$\mathrm{div}(F) = f \ \text{ in } \Omega, \ \ F \cdot n = 0 \ \text{ in } \partial\Omega, \quad \text{i.e.} \quad -\int_\Omega F(x) \cdot \nabla \varphi(x)\, \mathrm{d}x = \langle\!\langle f, \varphi \rangle\!\rangle \tag{A.3.1}$$

for all $\varphi \in W^{1,q'}_\diamond(\Omega)$, has at least one solution $F \in L^q(\Omega)^d$.

Proof. Let us consider the problem

$$\min_{u \in W^{1,q'}_\diamond} \frac{1}{q'} \int_\Omega |\nabla u|^{q'}\, \mathrm{d}x - \langle\!\langle f, u \rangle\!\rangle.$$

1. We recall that a family of probability measure μ_n weakly converges to a probability measure μ in Ω if $\int \phi\, \mathrm{d}\mu_n \to \int \phi\, \mathrm{d}\mu$ for all $\phi \in C_b(\Omega)$, where $C_b(\Omega)$ is the space of continuous and bounded functions on Ω.

Since the cost function is strictly convex, coercive and weakly lower semicontinuous, we have the existence of a unique $u \in W^{1,q'}_\diamond(\Omega)$ such that

$$\int_\Omega |\nabla u(x)|^{q'-2}\nabla u(x)\cdot\nabla\phi(x)\,dx = \langle\langle f,\varphi\rangle\rangle \quad \forall\,\varphi\in W^{1,q'}_\diamond(\Omega).$$

The result follows by defining $F = -|\nabla u|^{q'-2}\nabla u \in L^q(\Omega)^d$. □

Now, given $f \in (W^{1,q'}_\diamond(\Omega))^*$, let us consider the equation

$$-\Delta m = f \quad \text{in } \Omega, \quad \nabla m\cdot n = 0 \quad \text{in } \partial\Omega. \tag{A.3.2}$$

We say that $m \in W^{1,q}(\Omega)$ is a weak solution of (A.3.2) if

$$\int_\Omega \nabla m(x)\nabla\varphi(x)\,dx \;=\; \langle\langle f,\varphi\rangle\rangle \quad \forall\,\varphi\in W^{1,q'}_\diamond(\Omega). \tag{A.3.3}$$

Lemma A.3.2. *Assume that $q > d$ and let $a \in \mathbb{R}$. Then, there exists a unique weak solution (A.3.2) satisfying that $\int_\Omega m\,dx = a$. Moreover, there exists a constant $c > 0$, independent of (a, f), such that for any F solving (A.3.1) we have that*

$$\|\nabla m\|_{L^q} \le c\|F\|_{L^q}. \tag{A.3.4}$$

Sketch of the proof: Noticing that (A.3.3) is invariant if a constant is added to m, it suffices to prove the result for $a = 0$. Since $q > d$ we have that $q' < 2$ and so, by the Lax-Milgram theorem, existence and uniqueness for (A.3.3) holds in $W^{1,2}_\diamond(\Omega)$. Using interpolation results due to Stampacchia (see [Sta64] and [Sta65b]), estimate (A.3.4) holds if Dirichlet-boundary conditions were considered (see e.g. [GM12, Theorem 7.1]). This argument yields the desired local regularity for m, which can be extended up to the boundary (which we recall that it is assumed to be regular) using classical reflexion arguments.

□

Finally let us recall the following result about elliptic equations with measure data. This is a well-known result, we refer to [Min07] which contains the following statement for nonlinear problems as well.

Theorem A.3.3 ([Min07], Theorem 1.2)**.** *Let $f \in \mathscr{M}(\overline{\Omega})$. Then the unique solution $u \in W^{1,1}_0(\Omega)$ of the problem*

$$-\Delta u = f \ \text{ in } \Omega, \quad u = 0 \ \text{ on } \partial\Omega \tag{A.3.5}$$

has the following regularity properties:

(i) $\nabla u \in W^{1-\varepsilon,1}_{\text{loc}}(\Omega)^d$, *for all* $\varepsilon \in (0,1)$.

(ii) *More generally,* $\nabla u \in W^{\frac{\sigma(r)-\varepsilon}{r},r}_{\text{loc}}(\Omega)^d$, *for all* $\varepsilon \in (0,\sigma(r))$ *where* $1 \le r < \frac{d}{d-1}$ *and* $\sigma(r) := d - r(d-1)$.

Remark A.3.1. *We remark that the result about the uniqueness of the (renormalized) solution of the problem* (A.3.5) *can be found in [DMMOP99]. Moreover since the regularity results in Theorem A.3.3 are local, these remain true if we use homogeneous Neumann boundary conditions instead of Dirichlet ones. In this context the solution is unique up to an additive constant.*

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