Flows of singular vector fields and applications to fluid and kinetic equations

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Contents

Introduction

Several physical phenomena arising in fluid dynamics and kinetic equations can be modeled by the transport PDE

$$\partial_t u + b \cdot \nabla u = 0 \tag{0.1}$$

where $b(t,x) : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ is a velocity field, and $u(t,x) : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ is the the physical quantity that evolves in time. Such quantities are the vorticity of a fluid, or the density of a collection of particles advected by a velocity field which is highly irregular, in the sense that it has a derivative given by a distribution and a nonlinear dependence on the solution u. The theory of characteristics provides a link between this PDE and the ODE

$$\begin{cases} \frac{dX}{dt}(t,x) = b(t, X(t,x)) \\ X(0,x) = x, \end{cases} \tag{0.2}$$

where $X(t,x) : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ is the *flow map* of the particle trajectories. In the classical setting, b is Lipschitz with respect to the spatial variable, and Cauchy-Lipschitz theory identifies a unique flow $X(t,x) : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ which solves (0.2) and inherits the Lipschitz regularity of b. Furthermore, if u_0 is a solution at initial time to (0.1), then it is transported by the flow solving (0.2). The question of well-posedness of (0.1) is more complex when the velocity field is no longer Lipschitz in its second variable, but has only a weak derivative which is merely integrable or a measure. The most well-known developments in recent years have been [25] and [5], wherein well-posedness of (0.2) was shown in the *almost everywhere* sense, under the assumption that $b \in W^{1,1}_{\text{loc}}$ or $b \in BV$, respectively, with bounded divergence. These approaches exploited the link between (0.2) and (0.1) and relied on what is called the *renormalization* property of the vector field; roughly speaking, that given a bounded distributional solution u to (0.1), u^2 is also a solution, and so are many other nonlinear compositions of u. This property is intrinsically linked to well-posedness: should renormalization hold, then solutions of (0.1) are unique and stable. However, a weak sense differentiability of the vector field is needed to give a positive answer: in order to prove that b has the renormalization property, a regularization procedure is introduced for the PDE, leading to a commutator estimate. In order for the 'error term' to converge to zero in a suitably strong sense, the Sobolev (or BV) regularity of b is essential.

The more recent development in [22] has been well-posedness of (0.2) via *quantitative estimates* for the flow which rely only on the Sobolev regularity and growth of b (without assumptions on the divergence). Out of the smooth context, one replaces the notion of a classical flow with that of an almost-everywhere map solving (0.2) in a suitable weak sense. This is called a *regular Lagrangian flow* and is measure-preserving in the sense that it does not concentrate trajectories. Equivalently there is a constant L such that

$$\mathcal{L}^d((X(t,\cdot)^{-1}(B)) \leq L\mathcal{L}^d(B), \qquad \text{for every Borel } B \subset \mathbb{R}^d,$$

a condition which holds for instance for vector fields with bounded divergence. The difference in this approach is that it identifies an equivalence class of solutions to (0.1) which, like their smooth counterpart, are transported by regular Lagrangian flows. Because the flows are measure preserving, the flows also preserve equi-integrability of approximations of the initial data. The approach in [22] gives stability, compactness (and therefore existence), and a mild Lusin-type Lipschitz regularity for the regular Lagrangian flow. In comparison to the the literature in [**25, 5**], one obtains explicit quantitative rates in the estimates for stability and compactness. These bounds depend only on the compressibility constants of the Lagrangian flow, not on the divergence of the vector field, which could in principle be unbounded. The more recent developments have been generalized for when b is less than Sobolev, or more precisely has a gradient given by a *singular integral*, a common regularity in fluid and kinetic equations, and will support the main results of this thesis summarized in the following sections 1, 2, 3.

1. The Euler equation

A further development in [**16**] has broadened the Lagrangian approach to give the same quantitative stability for Lagrangian flows associated to vector fields which are no longer Sobolev or BV, but have a gradient given by the singular integral of an integrable function. This has significance for the incompressible Euler equation in 2 dimensions, which is an old problem in fluid dynamics. The equation for an inviscid fluid are given by

$$\begin{cases} \partial_t v + \operatorname{div}(v \otimes v) + \nabla p = 0 \\ \operatorname{div} v = 0, \end{cases} \tag{0.3}$$

where $v(t,x)$ is the velocity, representing the speed of a particle at position x and time t, and $p(t,x)$ is the scalar pressure, that sustains the incompressibility constraint $\operatorname{div} v = 0$. It can be written as the vorticity formulation

$$\partial_t \omega + \operatorname{div}(v\,\omega) = 0 \tag{0.4}$$

where ω is the vorticity, v is the velocity given by the coupling $\operatorname{curl} v = \omega$. The velocity can be written via the Biot Savart law a convolution with the vorticity, making the problem nonlocal (and the PDE nonlinear). In case of L^1 vorticities, the gradient of the velocity is no longer locally integrable, as it is the singular integral of an L^1 function. The usual strategy for proving existence of solutions to (0.3) is by smoothing the initial data, and using estimates that enable passing to the limit in its weak formulation. For initial velocities belonging to $H^s, s > 2$, well-posedness of solutions was proved in [**72**]. Existence and uniqueness of solutions to (0.3) is known for vorticities in $L^1 \cap L^\infty$, and was first proved in [**74**]. For compactly supported initial vorticities in L^p, with $1 < p < \infty$, existence was first proved in [**26**]. In all cases the summability of the vorticity imply at best that the velocity field is Sobolev. Sobolev embeddings guarantee strong convergence in L^2_{loc} for the approximated velocities, when the vorticity has some integrability higher than L^1. In the case of measure vorticities with distinguished sign, the velocity is void of any Sobolev regularity, and has gradient given by the singular integral of a measure. This is generally insufficient for strong convergence of the velocities in L^2_{loc}[1]: the approximated velocities may concentrate. However, concentrations may occur for sequences whose limit *still* satisfies (0.3), in spite of the lack of strong L^2_{loc} convergence: this is referred to as concentration-cancellation and has been studied in [**28**], [**12**], and [**24**].

We will address the question whether initial vorticities in L^1 give rise to weak solutions which are transported by flows. Under the bounds in this setting, using the compactness estimates of [**16**], we show that Lagrangian flows associated to velocities whose curl are equi-integrable are strongly precompact, and thus stable under approximation. In contrast to [**12**], [**24**] [**34**], [**69**], we rely only on the Lagrangian formulation, so that existence of solutions which are naturally associated to flows is a consequence. In this setting we can also allow for velocities with locally infinite kinetic energy. Without using strong convergence of the velocities, we are able to prove the Lagrangian flows converge anyway, and can nevertheless deduce strong compactness of the solutions a posteriori.

[1] See for instance Example 11.2.1 in [**12**].

2. The Vlasov Poisson equation

The second problem we address are Lagrangian solutions to the Vlasov Poisson equation with L^1 data. The PDE is given by

$$\partial_t f + v \cdot \nabla_x f + E \cdot \nabla_v f = 0, \tag{0.5}$$

where $f(t,x,v) \geq 0$ is the distribution function, $t \geq 0$, $x, v \in \mathbb{R}^N \times \mathbb{R}^N$, and

$$E(t,x) = \nabla_x \Delta^{-1} \left(\int f(t,x,v) dv \right) \tag{0.6}$$

is the force field. If we denote by $\rho(t,x) = \int f(t,x,v)dv$ the density, and

$$b(t,x,v) = (b_1, b_2)(t,x,v) = (v, E(t,x)) = (v, \nabla_x \Delta^{-1} \rho) \tag{0.7}$$

the associated vector field on $(0,T) \times \mathbb{R}^N \times \mathbb{R}^N$, then the Vlasov equation can be written in the form of the transport equation $\partial_t f + b \cdot \nabla_{x,v} f = 0$. A solution involves the couple (f, E). Observe that the regularity we are dealing with is worse than what has previously been discussed: the first component b_1 is Lipschitz but has no decay in x, the second b_2 involves the nonlinearity $E(t,x)$ which is highly singular in x: it has a gradient given by the singular integral in $\mathbb{R}^N_x$ of the density. b has (x,v) differential given by

$$Db = \begin{pmatrix} D_1 b^1 & D_2 b^1 \\ D_1 b^2 & D_2 b^2 \end{pmatrix} = \begin{pmatrix} 0 & \mathrm{Id} \\ S_1 \rho & 0 \end{pmatrix}, \tag{0.8}$$

where the index 1 stands for x, 2 for v, and S_1 is a singular integral operator on $\mathbb{R}^N_x$. Apart from the nonlinearity, the difficulty of this system lies in the fact that an equation on phase space $\mathbb{R}^N_x \times \mathbb{R}^N_v$ is coupled with a 'split' vector field (0.7) whose non-trivial component b_2 has weak spatial regularity and does not decay in $\mathbb{R}^N_v$. Considering solutions with data in L^1, integration with respect to v means only an $L^1(\mathbb{R}^N_x)$ bound on ρ (and no decay in $\mathbb{R}^N_v$) survives, which does not give good bounds for E. Global weak solutions to the VP system were proved to exist in [**7, 30, 31**], with only $f^0 \in L^1(\mathbb{R}^6)$, $f^0 \log^+ f^0 \in L^1$, $|v|^2 f^0 \in L^1$, $E^0 \in L^2$. Related results with *weak* initial data have been obtained in [**54, 39, 76, 52**]. Even weaker solutions were considered in [**46, 47, 48**], where the distribution function is a measure. However, these solutions do not have well-defined characteristics. We seek to extend the existence result of [**30**] to initial data in L^1 with finite energy avoiding the $L \log^+ L$ assumption. Our weak solutions are Lagrangian (in the same spirit as solutions to the Euler equation) and involve a well-defined flow. We use the theory of Lagrangian flows for transport equations with vector fields having weak regularity, developed in [**25, 5, 3, 22, 4**], and recently in [**16, 2, 14**]. It enables to consider force fields that are *not* in $W^{1,1}_{\mathrm{loc}}$, nor in BV_{loc}. In this context we will prove stability results with strongly or weakly convergent initial distribution function. The flow is proved to converge strongly anyway, and the class of solutions considered is stable. The split nature of the vector field is the motivational setting for the next stability result: the main problem has been to generalize the previous results of [**22, 16**] to *anisotropic* vector fields, such as the one considered in (0.7). In this we go beyond the regularity setting where the vector field has gradient given by the singular integral of an L^1 function, and it theoretically allows us to consider measure densities. However, these do not give a good notion of solution, since the Lagrangian flows are defined only almost everywhere.

3. Anisotropic vector fields

We consider the following anisotropic vector fields: those for which the gradient is given by the singular integral of a measure in some directions, and the singular integral of an L^1 function in others. Apart from stability and compactness of Lagrangian solutions to the transport equation, this allows us to prove existence of Lagrangian solutions to the two and three-dimensional Vlasov Poisson non-linear equations with L^1 data. We study general vector fields of the form $b(t,x) = (b_1, b_2)(t, x_1, x_2)$, where the components b_1 and b_2 have a 'split' regularity. We we write $\mathbb{R}^N = \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ with coordinates x_1 and x_2, and split analogously the vector field according to $b=(b_1,b_2)$. We will consider the case in which D_1b_2 is a singular integral (in $\mathbb{R}^{n_1}$) of a measure, while D_1b_1, D_2b_1 and D_2b_2 are singular integrals (in $\mathbb{R}^{n_1}$) of integrable functions:

$$Db=\begin{pmatrix} S*L^1 & S*L^1 \\ S*\mathcal{M} & S*L^1 \end{pmatrix}$$

(in fact our assumptions are slightly more general: see assumption **(R2a)-(R2b)** in Chapter 5, Section 18). This technical regularity assumption is motivated by the structure of the differential of (0.7). In [23], an integral functional measuring a logarithmic distance between two flows, X and $\bar{X}$, is introduced. This allows some a priori estimates measuring 'non-uniqueness' of the flow to be derived. We use a functional in which the two directions are weighted by parameters according to their degree of regularity. We modify the functional so that it depends on two (small) parameters δ_1 and δ_2, with $\delta_1\le\delta_2$:

$$\Phi_{\delta_1,\delta_2}(s)=\int_{B_r}\log\left(1+\left|\left(\frac{|X_1(s,x)-\bar{X}_1(s,x)|}{\delta_1},\frac{|X_2(s,x)-\bar{X}_2(s,x)|}{\delta_2}\right)\right|\right)dx\,, \tag{0.9}$$

where δ_1,δ_2 are parameters to be chosen later, and the integral is localized over a fixed compact set. It is clear that for a given $\gamma>0$ we have the lower bound

$$\Phi_{\delta_1,\delta_2}(s)\ge\int_{\{|X-\bar{X}|\ge\gamma\}}\log\left(1+\frac{\gamma}{\delta_1}\right)dx=\mathcal{L}^N(\{|X-\bar{X}|\ge\gamma\})\log\left(1+\frac{\gamma}{\delta_1}\right).$$

This gives the estimate

$$\mathcal{L}^N(\{|X-\bar{X}|\ge\gamma\})\le\frac{\Phi_\delta(s)}{\log\left(1+\frac{\gamma}{\delta_1}\right)}\,. \tag{0.10}$$

A strategy for proving stability (and uniqueness) is thus to derive upper bounds on the functional $\Phi_\delta(s)$ which blow up in δ slower than $\log(1/\delta)$ as $\delta\to0$. Differentiating and integrating in time yields the interpolation

$$\Phi_{\delta_1,\delta_2}(s)\le\iint\min\left\{\frac{2\|b\|_\infty}{\delta_1}\,;\,\frac{\left(\frac{1}{\delta_1}|b_1(X)-b_1(\bar{X})|,\frac{1}{\delta_2}|b_2(X)-b_2(\bar{X})|\right)}{\left|\left(\frac{X_1-\bar{X}_1}{\delta_1},\frac{X_2-\bar{X}_2}{\delta_2}\right)\right|}\right\}dxdt\,. \tag{0.11}$$

We remark that this integral is performed over a suitable localization with respect to the sublevels where the flows are not too large. An estimate for the size of this set is crucial in the final estimate. However, the complication of the anisotropic difference quotient in (0.11) requires the use of a modified operator to estimate the directional increments of b. This is complicated by the fact that just as a classical maximal function estimates the difference quotients in the BV case, the grand maximal function is an approximation of the identity in all x,y variables which is not bounded when composed with a singular integral in x variables. This is resolved by the use of tensor products of maximal functions. One relevant technical point in the proof is the estimate for the anisotropic difference quotients showing up when differentiating (5.59). We need an estimate of the form:

$$|b(x)-b(y)|\lesssim\left|\left(\frac{x_1-y_1}{\delta_1},\frac{x_2-y_2}{\delta_2}\right)\right|\Big[U(x)+U(y)\Big]\,, \tag{0.12}$$

where U is a suitable function of the derivative of b. This is complicated by the fact that, as in the classical case, one expects to use a maximal function in x_1 and x_2 in order to estimate the difference quotients, but however this would not match (in terms of persistence of cancellations) with the presence of a singular integral in the variable x_1 only. This is resolved in Section 19 by the use of tensor products of maximal functions, and will result in the proof of (5.60) together with a bound of the form

$$\|U\|\le\delta_1\|D_1b\|+\delta_2\|D_2b\|\,.$$

We then use the equi-integrability of the L^1 components of D_1b_1, D_2b_2, D_2b_1 which gives a remainder in L^2 that can be controlled much in the same way as [16]. After an interpolation estimate on the minimum in (0.11), from the estimate in (0.12), we derive the weighted upper bound

$$\Phi_{\delta_1,\delta_2}(s) \lesssim \left[\frac{\delta_1}{\delta_2}\|D_1b_2\|_{\mathcal{M}}+\frac{\delta_2}{\delta_1}\|D_2b_1\|_{L^1}+\|D_1b_1\|_{L^1}+\|D_2b_2\|_{L^1}\right]\log\left(\frac{1}{\delta_2}\right). \tag{0.13}$$

The last step is to achieve 'smallness' of this bound relative to the parameter $\log(1/\delta)$ is exploiting the equi-integrability of g to gain L^1 smallness up to an L^2 remainder. The L^1 components $\|D_2b_1\|_{L^1}$, $\|D_1b_1\|_{L^1}$ and $\|D_2b_2\|_{L^1}$ (although the derivatives themselves are not L^1) can be assumed to be small, since the singular integral of the L^1 function preserves the equi-integrability estimates as in [16]. This is not the case for $\|D_1b_2\|_{\mathcal{M}}$. However, this is mitigated by the coefficient δ_1/δ_2, since we can choose δ_1 to go to zero faster than $\delta_2 : \delta_1 \ll \delta_2$.

This regularity setting does not include the BV case, but the anisotropic functional introduced is a first step toward this open problem, since it allows to compensate for the lack of equi-integrability of the measure-part derivative with a 'weighted' functional, allowing for a part of a derivative to be the singular integral of a measure. Observe that the last step is where the estimate fails for BV, or when more than one component is the singular integral of a measure. This is due to the lack of equi-integrability of measures and is required to send all parameters to zero. An full extension of this procedure to the BV case would answer positively the following conjecture:

Conjecture 3.1 (Bressan). *Let $b_n \in C^1([0,T)\times\mathbb{R}^N)$ be smooth vector fields and denote by X_n the solution of the ODEs*

$$\begin{cases} \frac{dX_n}{dt}(t,x) = b_n(t, X_n(t,x)) \\ X_n(0,x) = x, \end{cases} \tag{0.14}$$

Assume that X_n satisfy for some constant $C > 0$

$$\frac{1}{C} \leq \det(\nabla_x(X_n)(t,x)) \leq C,$$

and that $||b_n||_\infty$ and $||Db_n||_{L^1(\mathbb{R}^N)}$ are uniformly bounded. Then the sequence X_n is strongly precompact in L^1_{loc}.

Our result is the following stability estimate. For two Lagrangian flows X and $\bar{X}$ associated to b and $\bar{b}$ in the regularity setting described above, for every $\gamma > 0$ and $\eta > 0$ there exist $\lambda > 0$ such that

$$\mathcal{L}^N\left(\{|X-\bar{X}|>\gamma\}\right) \lesssim ||b-\bar{b}||_{L^1((0,T)\times B_\lambda)}+\eta. \tag{0.15}$$

The corollaries are the following. We have (apart from uniqueness) an explicit rate of stability for a sequence X_n of Lagrangian flows associated to vector fields b_n in the above regularity setting, that converge in $L^1_{\text{loc}}([0,T]\times\mathbb{R}^N)$ to b. We have as well strong compactness in L^1_{loc} for a sequence X_n of Lagrangian flows associated to vector fields b_n, and hence we arrive at existence of a Lagrangian flow associated to a vector field b in our setting, with suitable bounds on the divergence to guarantee that the compression constants of the flows are uniformly bounded. This will be applied in Chapter 7, where we apply the stability results to the vector field in (0.7).

3.1. Plan of the thesis. In Chapter 1 we will review the Calderón Zygmund theory of singular integrals. The classical theorem of singular integral operators on L^1 and L^p will be proved along with an interpolation theorem using the Calderón Zygmund decomposition on $\mathbb{R}^d$. Attention will be paid to singular kernels of *fundamental type*, which appear in the context of the Euler and Vlasov Poisson PDE. In Chapter 2 we will review the DiPerna Lions [25] theory of renormalization and well-posedness of bounded weak solutions to the transport equation under the Sobolev regularity of b. We will give the proof of strong L^1_{loc} convergence of the commutator estimate. We also remark on the extension of the renormalization scheme

to the BV setting thanks to Ambrosio [5].
In Chapter 3 we will review the existence of classical solutions to the 2 dimensional Euler equation and link the study of the ODE in (0.2) to the vorticity equation. The potential theory involves the estimates coming from the study of singular integral operators from Chapter 1. These will also be used to summarize the results for weak solutions in the settings of [74, 26, 24], in which we will exploit the Aubin-Lions type arguments to show convergence of the velocity field under the integrability assumptions on the vorticity. The uniqueness result for vorticities in $\omega \in L^1 \cap L^\infty$ will also be proved using estimates on the L^2 energy. We end with Delort's existence proof of vortex sheets (or measure vorticities) with distinguished sign.
Chapter 4 will be devoted to the classical existence of smooth solutions to the Vlasov Poisson equation in 3 dimensions, a Lagrangian proof of existence using the characteristics which was first done in [55]. Similarly to Chapter 3, it involves estimates from potential theory but requires also an estimate on the *moments* of f, first proved in [38]. We also recall growth and regularity bounds on E. The classical existence proof involves first a local existence result (using an iterative scheme) which is then shown to be global.
Chapter 5 will be devoted to the Lagrangian flow compactness estimates discussed in [22, 16, 14]. A crucial estimate involves a composition of the Hardy-Littlewood grand maximal function with the singular integral satisfies sufficient cancellations with singular kernels which make the composition operator MS well defined, and bounded from $L^1 \to M^1$. This means we have the bound

$$|||MSf|||_{M^1} \leq C_{N,S}||f||_{L^1} \tag{0.16}$$

where M^1 denotes the weak Lebesgue space. Additionally, the difference quotients of b are estimated in terms of the grand maximal function of the derivative. In particular, when $Db = Sg$, where Sg is the singular integral of an L^1 function g, one has

$$\frac{|b(X) - b(\bar{X})|}{|X - \bar{X}|} \lesssim MSg(X) + MSg(\bar{X})\,, \tag{0.17}$$

which is the vital step in the stability estimate for vector fields whose gradient is given by a singular integral, and will be applied to our stated problem on the Euler quation. The first result of this thesis is the stability of Lagrangian flows associated to anisotropic vector fields, with consequences for compactness and existence of the flows.
Chapter 6 involves the second result of this thesis, which is existence of several classes of weak solutions to the Euler equation when the vorticity is L^1 summable. As stated, this puts us out of the historical context which relies on absolute convergence of the velocity $v = K * \omega$ in order to prove existence of solutions. However, the derivative Dv is in the setting of [16], and we may apply compactness results to deduce stability of vorticity approximations. An interesting property is that here we require only distributional convergence of the velocities, which suffices anyway for strong compactness of the associated flows. Since Lagrangian solutions of the Euler equation are defined as weak solutions associated to Lagrangian flows, their existence follows. These are in particular solutions in the renormalized DiPerna Lions sense.
Chapter 7 is the final result of this thesis and is the application of Chapter 6, more specifically the stability estimates for anisotropic vector fields, to the Vlasov Poisson equation, in order to prove existence and compactness of Lagrangian flows to the characteristic ODEs. This implies existence of Lagrangian solutions in L^1. Although the estimate (0.15) allows to consider measure densities, the reason we do not consider measure solutions is that the Lagrangian flows are defined only almost everywhere. We will also need to prove strong compactness of the force field, using the bounds from singular integrals to control the time derivative of E, and an abstract lemma which allows to control the spatial increments of E. We remark that we require a finite energy condition in order to prove a bound on the size of the superlevels of the flow, which we need to conclude the estimate in (0.11).
Finally, we remark that in many theorems in the classical framework we do not state the result under the sharpest possible assumptions. For instance, an assumption that the initial data are compactly supported simplifies the proofs but is not necessary, and at several points we will assume C^1 regularity rather than

Lipschitz. Some of the existence lemmas in the smooth framework will not be proven here, but results from Cauchy Lipschitz theory are used often in our regularization arguments.

CHAPTER 1

Preliminaries

4. The Picard Lindelöf theorem for ODEs

We begin by recalling the following classical theorem from Cauchy-Lipschitz theory. The starting point in the theory of ODE is when the right hand side of the ODE (0.2) depends on the solution in a local manner, i.e. $t \in I \subset \mathbb{R}$, and $b(t, \cdot)$ is locally Lipschitz. The following theorem provides local existence and uniqueness: for any (t_0, x_0) in the region where b is continuous in both variables and locally Lipschitz in its second argument, there is a neighborhood of (t_0, x_0) such that the ODE has a unique local solution in this neighborhood. Furthermore, the solutions are C^1 in this neighborhood.

THEOREM 4.1 (Picard Lindelöf/Cauchy Lipschitz). *Let $(X, ||\cdot||_X)$ be a Banach space, and let $t_0 \in \mathbb{R}$, and $x_0 \in X$ be given. Consider the ODE*

$$\dot{\gamma}(t) = b(t, \gamma(t)), \qquad \gamma(t_0) = x_0. \tag{1.1}$$

Suppose that $b : \mathbb{R} \times X \to X$ is bounded and continuous on some region

$$Q_{\alpha,\beta} = \{(t, y) : |t - t_0| \leq \alpha, ||x - x_0||_X \leq \beta\}.$$

Suppose that b is Lipschitz with respect to x, uniformly in time on $Q_{\alpha,\beta}$. Then there exists $\delta > 0$ and a function γ belonging to $C^1([t_0 - \delta, t_0 + \delta]; X)$ which is the unique solution to (1.1).

We remark that the modern version of this proof is based on Banach's fixed point theorem and constructs a solution by iteration, a method which has persisted in many proofs of construction of local-in-time solutions to both PDE and ODE. (See for instance section 13.) If we consider the solution $\gamma(t)$ as a function of time and the initial point, we can define the *classical flow* of a smooth and bounded vector field $b : I \times \mathbb{R}^d \to \mathbb{R}^d$ as the map $X(t, x) : I \times \mathbb{R}^d \to \mathbb{R}^d$ satisfying, for all $(t, x) \in I \times \mathbb{R}^d$,

$$\begin{cases} \frac{dX}{dt}(t, x) = b(t, X(t, x)) \\ X(t_0, x) = x. \end{cases} \tag{1.2}$$

Existence and uniqueness of the flow follows from Theorem 4.1. Denoting by $X(s, t, x)$ the flow of b starting at time $s \in I$, the following semigroup property holds: for every $t_0, t_1, t_2 \in I$,

$$X(t_2, t_0, x) = X(t_2, t_1, X(t_1, t_0, x)).$$

For vector fields with weaker regularity, it is reasonable to expect existence (via an approximation scheme) but not uniqueness of solutions.

THEOREM 4.2 (Cauchy-Peano). *In the region $C \subset I \times \mathbb{R}^d$ where b is continuous in both variables, for any $(t_0, x_0) \in C$, there exists a local C^1 solution in a neighborhood of this point.*

REMARK 4.3. *Let $L \subset I \times \mathbb{R}^d$ denote the open region (possibly empty) where b is continuous in both variables and locally Lipschitz with respect to x. Then in the region $L \subset C$ the local solutions are unique and Lipschitz. The regularity is lost when they 'leave' L. When they reach the boundary, still within C, the* *flow may separate into several solutions. Thus for the solution to be able to leave C, it must happen that b is nonlinear in its spatial variable. (Otherwise $L = I \times \mathbb{R}^d$.) The typical example is*

$$\dot{\gamma}(t) = \sqrt{|\gamma(t)|}, \qquad \gamma(0) = 0.$$

Indeed there are solutions $\gamma(t) = 0$, $\gamma(t) = \frac{1}{4}t^2$, and infinitely many others.

5. Notation and background material

Throughout the paper we will denote by $B_R \equiv B_R(0)$. We will denote by $L^0(\mathbb{R}^d)$ the space of all measurable real valued functions on $\mathbb{R}^d$, defined a.e. with respect to the Lebesgue measure, endowed with the convergence in measure defined below. We denote by $L^0_{\text{loc}}(\mathbb{R}^d)$ the same space, endowed with local convergence in measure. The space $\log L(\mathbb{R}^d)$ contains all functions $u : \mathbb{R}^d \to \mathbb{R}$ such that $\int_{\mathbb{R}^d} \log(1 + |u(x)|)dx < \infty$, with $\log L_{\text{loc}}(\mathbb{R}^d)$ defined accordingly. We refer to $\mathcal{B}(E, F)$ the space of bounded functions between sets E and F. We denote by $\mathcal{S}(\mathbb{R}^d)$ the Schwartz space of rapidly decreasing functions, and the dual $\mathcal{S}'(\mathbb{R}^d)$ the space of tempered distributions.

DEFINITION 5.1. *We say that a sequence of measurable functions $u_n : \mathbb{R}^d \to \mathbb{R}$ converges globally in measure in $\mathbb{R}^d$ to a measurable function $u : \mathbb{R}^d \to \mathbb{R}$ if for every $\gamma > 0$ there holds*

$$\mathcal{L}^N(\{x \in \mathbb{R}^d : |u_n(x) - u(x)| > \gamma\}) \to 0, \qquad \text{as } n \to \infty.$$

Similarly, we say that the sequence u_n converges locally in measure to u if for every $\gamma > 0$ and every $r > 0$ there holds

$$\mathcal{L}^N(\{x \in B_r : |u_n(x) - u(x)| > \gamma\}) \to 0, \qquad \text{as } n \to \infty.$$

We study several bounded operators on L^p which do not remain bounded on L^1, and satisfy only weak bounds. To that end we begin with the definition of *weak L^p* spaces. We introduce the following pseudo-norm:

DEFINITION 5.2. *Let u be a measurable function on $\Omega \subset \mathbb{R}^d$. For $1 \le p < \infty$, we set*

$$|||u|||^p_{M^p(\Omega)} = \sup_{\lambda > 0} \{\lambda^p \mathcal{L}^d(\{x \in \Omega \ : \ |u(x)| > \lambda\})\}$$

and define the weak Lebesgue space $M^p(\Omega)$ as the space consisting of all such measurable functions $u : \Omega \to \mathbb{R}$ with $|||u|||_{M^p(\Omega)} < \infty$. For $p = \infty$, we set $M^\infty(\Omega) = L^\infty(\Omega)$.

REMARK 5.3. *We remark that the weak Lebesgue spaces $M^p(\Omega)$ are normalizable for $p > 1$, but not for $p = 1$. For clarity we denote the pseudonorm with $|||\cdot|||_{M^p}$.*

REMARK 5.4. *For any vector field $f \in L^1_x(M^1_y)$ we have the inequality*

$$|||f(x, y)|||_{M^1_{x,y}} \le ||f(x, y)||_{L^1_x(M^1_y)}.$$

However, a Fubini-type inequality of the form

$$|||f(x, y)|||_{M^1_{x,y}} \le ||f(x, y)||_{M^1_x(M^1_y)}$$

does not *hold on the product space. This can be seen by considering the characteristic function on the set $\{(x, y) \in \mathbb{R}^2 : 0 < x \le 1, 0 < y \le 1/x\}$.*

One such operator that is bounded only in the weak sense on L^1 is the classical (local and global) maximal function.

DEFINITION 5.5. *Let u be an integrable function defined on $\mathbb{R}^d$. The maximal function of u is defined as*

$$Mu(x) = \sup_{r > 0} \fint_{B_r(x)} |u(y)| dy, \qquad \text{for every } x \in \mathbb{R}^d.$$

For u a finite measure we define

$$Mu(x) = \sup_{r>0} \fint_{B_r(x)} d|u|(y), \qquad \text{for every } x \in \mathbb{R}^d.$$

The local maximal function of u is defined as

$$M_\lambda u(x) = \sup_{0<r<\lambda} \fint_{B_r(x)} |u(y)| dy,$$

which is finite a.e for a function $u \in L^1_{\text{loc}}$ or a locally finite measure.

The maximal function Mu is finite almost everywhere, but its norm is bounded only on L^p.

PROPOSITION 5.6. *For every $1 < p \leq \infty$ we have the strong estimate*

$$||Mu||_{L^p(\mathbb{R}^d)} \leq C_{d,p} ||u||_{L^p(\mathbb{R}^d)},$$

with only the weak estimate for $p = 1$,

$$|||Mu|||_{M^1(\mathbb{R}^d)} \leq C_d ||u||_{\mathcal{M}(\mathbb{R}^d)}.$$

When $u \in L^1(\mathbb{R}^d)$ is not identically zero, $Mu \notin L^1(\mathbb{R}^d)$. In fact, $Mu \in L^1_{\text{loc}}(\mathbb{R}^d)$ if and only if $|u|\log^+|u| \in L^1_{\text{loc}}(\mathbb{R}^d)$, as we have in the following lemma.

LEMMA 5.7. *Let $\lambda > 0$. The local maximal function of a function $u \in L^1_{\text{loc}}(\mathbb{R}^d)$ is finite for a.e $x \in \mathbb{R}^d$ and*

$$\int_{B_\rho(0)} M_\lambda u(y) dy \leq c_{n,\rho} + c_n \int_{B_{\rho+\lambda}(0)} |u(y)| \log(2 + |u(y)|) dy.$$

For $p > 1$ and $\rho > 0$ we have

$$\int_{B_\rho(0)} (M_\lambda u(y))^p dy \leq c_{n,p} \int_{B_{\rho+\lambda}(0)} |u(y)|^p dy.$$

This is false for $p = 1$, where we have only the weak estimate, for all $\alpha > 0$

$$|\{y \in B_\rho(0) : M_\lambda u(y) > \alpha\}| \leq \frac{c_n}{\alpha} \int_{B_{\rho+\lambda}(0)} |u(y)| dy.$$

We recall the following lemma which states that the maximal function is the 'largest' of all approximations of the identity.

LEMMA 5.8. *Let $\psi : (0, \infty) \to [0, \infty)$ be a nonincreasing function and assume*

$$I \equiv \int_{\mathbb{R}^d} \psi(|y|) dy < \infty.$$

Then for every $u \in L^1_{\text{loc}}(\mathbb{R}^d)$ and $\varepsilon > 0$ we have

$$\int_{\mathbb{R}^d} |u(x-y)| \frac{1}{\varepsilon^d} \psi\left(\frac{|y|}{\varepsilon}\right) dy \leq I \cdot Mu(x) \qquad \text{for every } x \in \mathbb{R}^d.$$

We recall a classical theorem on the difference quotients of a BV function.

LEMMA 5.9. *Let $u \in BV(\mathbb{R}^d)$, and denote by Du is the distributional derivative of u. There exists an $\mathcal{L}^d$-negligible set $\mathcal{N} \subset \mathbb{R}^d$ such that*

1 For every $x, y \in \mathbb{R}^d \setminus \mathcal{N}$,

$$|u(x) - u(y)| \leq C_d |x-y| \Big((MDu)(x) + (MDu)(y) \Big).$$

2 For every $x, y \in \mathbb{R}^d \setminus \mathcal{N}$ with $|x-y| \leq \lambda$, we have the local inequality

$$|u(x)-u(y)| \leq C_d|x-y|\Big((M_\lambda Du)(x)+(M_\lambda Du)(y)\Big).$$

We remark on the significance of this lemma since it offers a pointwise bound on the increments of a BV function. This is particularly useful for performing estimates on Sobolev functions, since the operator MDu is bounded on L^p for $p>1$ whenever Du is. The critical case is of course $p=1$, when only the weak estimate for MDu holds. We next recall an interpolation lemma for functions belonging to $M^1 \cap M^p$, which allows to interpolate between the two spaces. The useful estimate is that the L^1 norm depends only logarithmically on the M^p norm. This implies that functions in M^1 are 'not too far' from being in L^1.

LEMMA 5.10. *(Interpolation Lemma.) Let $u : \Omega \mapsto [0,+\infty)$ be a nonnegative measurable function, where $\Omega \subset \mathbb{R}^d$ has finite measure. Then for every $1<p<\infty$, we have the interpolation estimate*

$$||u||_{L^1(\Omega)} \leq \frac{p}{p-1}|||u|||_{M^1(\Omega)}\left[1+\log\left(\frac{|||u|||_{M^p(\Omega)}}{|||u|||_{M^1(\Omega)}}\mathcal{L}^N(\Omega)^{1-\frac{1}{p}}\right)\right].$$

We also state a crucial lemma on the characterization of a uniformly integrable family of functions. It states that, up to a remainder in L^2, uniformly equiintegrable sequences of functions have arbitrarily small norm in L^1.

LEMMA 5.11 (Equi-integrability). *Consider a family $\{\varphi_i\}_{i\in I} \subset L^1(\Omega)$ which is bounded in $L^1(\Omega)$. Then this family is equi-integrable if and only if for every $\varepsilon>0$, there exists a constant C_ε and a Borel set $A_\varepsilon \subset \Omega$ with finite measure such that for every $i \in I$ one can write*

$$\begin{aligned}
&\varphi_i = \varphi_i^1 + \varphi_i^2,\\
&||\varphi_i||_{L^1(\Omega)} \leq \varepsilon \quad \textit{and} \quad \operatorname{spt}(\varphi_i^2) \subset A_\varepsilon, \quad ||\varphi_i^2||_{L^2(\Omega)} \leq C_\varepsilon.
\end{aligned}$$

There are many variants of the following lemma, can be seen as a generalization of Rellich-Kondrachov compactness theorem when a sequence of functions with Sobolev spatial regularity has an additional time regularity.

LEMMA 5.12 (Aubin Lions). *Let $m<s$. Suppose u_n is a sequence in $L^\infty([0,T];H^s(\mathbb{R}^d))$ such that*

(1) u_n is uniformly bounded in $L^\infty([0,T];H^s(\mathbb{R}^d))$,
(2) $\partial_t u_n$ is uniformly bounded in $L^\infty([0,T];H^m_{\mathrm{loc}}(\mathbb{R}^d))$.

Then u_n is strongly precompact in $L^\infty([0,T];H^r_{\mathrm{loc}}(\mathbb{R}^d))$ for any $m<r<s$.

We also recall a classical weak form of the Aubin Lions lemma, in the spirit of Kruzkov [41, Lemma 5].

LEMMA 5.13. *Let Ω be an open subset of $\mathbb{R}^N$ and $T>0$, $1 \leq p < \infty$. Assume that S is a bounded subset of $L^p((0,T)\times\Omega)$, such that*

(1) S is locally L^p-precompact in space, i.e. for any compact subset $K \subset \Omega$,

$$\int_0^T\int_K |u(t,x+h)-u(t,x)|^p\,dxdt \to 0 \text{ as } h \to 0, \text{ uniformly for } u \in S. \tag{1.3}$$

(2) For $u \in S$, $\partial_t u$ is bounded in $L^\infty((0,T);\mathcal{D}'(\Omega))$. This means that for any $\varphi \in C_c^\infty(\Omega)$ and any $u \in S$, the map $t \mapsto \langle \partial_t u, \varphi\rangle$ belongs to $L^\infty((0,T))$ and

$$|\langle \partial_t u, \varphi\rangle| \leq C_\varphi, \text{ for a.e. } t \in (0,T) \text{ and all } u \in S, \tag{1.4}$$

where $C_\varphi \geq 0$ depends on the support of φ and a finite number of L^∞ norms of derivatives of φ (but not on u).

Then S is precompact in $L^p_{\mathrm{loc}}((0,T)\times\Omega)$.

PROOF. According to the Riesz-Fréchet-Kolmogorov criterion of precompactness in L^p and taking into account (1), we have to prove that for any compact sets $L \subset (0,T)$ and $K \subset \Omega$,

$$\int_L \int_K |u(t+\tau,x)-u(t,x)|^p \, dxdt \to 0 \text{ as } \tau \to 0, \text{ uniformly for } u \in S. \tag{1.5}$$

For $\varepsilon > 0$, define $u_\varepsilon = \rho_\varepsilon *_x u$, where ρ_ε is a mollifier sequence in space. Then because of (i), $u_\varepsilon - u$ can be made arbitrary small in $L^p((0,T)\times K)$ for ε small enough, uniformly for $u \in S$. But for fixed ε, because of (ii), $\partial_t u_\varepsilon$ is bounded in $L^\infty((0,T)\times K)$, uniformly for $u \in S$. It follows that $\|u_\varepsilon(\cdot+\tau,\cdot)-u_\varepsilon\|_{L^p(L\times K)} \leq C_\varepsilon|\tau|$. Decomposing

$$u(\cdot+\tau,\cdot)-u = (u(\cdot+\tau,\cdot)-u_\varepsilon(\cdot+\tau,\cdot)) + (u_\varepsilon(\cdot+\tau,\cdot)-u_\varepsilon) + (u_\varepsilon - u), \tag{1.6}$$

we conclude that $u(\cdot+\tau,\cdot)-u \to 0$ in $L^p(L\times K)$ as $\tau \to 0$, uniformly for $u \in S$, i.e. (1.5) holds, and this concludes the proof of the lemma. $\square$

We review three convolution operator inequalities used in potential theory, beginning with Young's inequalities.

THEOREM 5.14 (Young's inequality). *Let $1 \leq p,q,r \leq \infty$ with $1+\frac{1}{r} = \frac{1}{p}+\frac{1}{q}$. Then for all $f \in L^p(\mathbb{R}^d)$ and $g \in L^q(\mathbb{R}^d)$ the convolution $f * g$ belongs to $L^r(\mathbb{R}^d)$ with*

$$||f*g||_{L^r(\mathbb{R}^d)} \leq ||f||_{L^p(\mathbb{R}^d)}||g||_{L^q(\mathbb{R}^d)}.$$

THEOREM 5.15 (weak Young's inequality). *Let $1 < p,q,r < \infty$ with $1+\frac{1}{r} = \frac{1}{p}+\frac{1}{q}$. Then for all $f \in L^p(\mathbb{R}^d)$ and $g \in M^q(\mathbb{R}^d)$ the convolution $f*g$ belongs to $L^r(\mathbb{R}^d)$ with*

$$||f*g||_{L^r(\mathbb{R}^d)} \leq ||f||_{L^p(\mathbb{R}^d)}|||g|||_{M^q(\mathbb{R}^d)}.$$

We end with the following Theorem which gives a control on the potential of an integrable function.

THEOREM 5.16 (Hardy Littlewood Sobolev inequality). *Let $0 < \alpha < d$. Given a function $u \in L^1_{\text{loc}}(\mathbb{R}^d)$, define the Riesz potential of u as*

$$I_\alpha(u)(x) := \int_{\mathbb{R}^d} \frac{u(y)}{|x-y|^{d-\alpha}} dy, \qquad x \in \mathbb{R}^d.$$

The integral is well defined provided $u \in L^p(\mathbb{R}^d)$ with $1 \leq p < \frac{d}{\alpha}$. We have the following decay estimates on $I_\alpha(u)$:
Sub-critical case: Let $1 < p < q < \infty$ and $q = \frac{dp}{d-\alpha p}$. Then

$$\|I_\alpha(u)\|_{L^q(\mathbb{R}^d)} \leq C_{\alpha,d,p} \|u\|_{L^p(\mathbb{R}^d)}.$$

Critical case: For $p=1$ and $q = \frac{d}{d-\alpha}$ we have the weak estimate

$$|||I_\alpha(u)|||_{M^q(\mathbb{R}^d)} \leq C_{\alpha,d} \|u\|_{L^1(\mathbb{R}^d)}.$$

We recall an interpolation theorem from [65] for nonlinear operators. Since most of the operators we study will not be bounded on L^1, we define precisely what it means to be bounded from $L^1 \to M^1$.

DEFINITION 5.17. *Let $T : L^p(\mathbb{R}^d) \to L^q(\mathbb{R}^d)$ be an operator. We say that*

- *T is of type (p,q) if there exists $A > 0$ so that*

$$||Tf||_{L^q(\mathbb{R}^d)} \leq A||f||_{L^p(\mathbb{R}^d)}, \qquad \forall f \in L^p(\mathbb{R}^d)$$

- *T is of weak type (p,q) if there exists $A > 0$ such that for all $\alpha > 0$*

$$\mathcal{L}^d(\{x \in \mathbb{R}^d : |Tf| > \alpha\}) \leq \left(\frac{A||f||_{L^p(\mathbb{R}^d)}}{\alpha}\right)^q, \qquad \forall f \in L^p(\mathbb{R}^d).$$

We remark that if T is of type (p,q) then it is of weak type (p,q). In particular the latter is equivalent to saying T is bounded from L^p to M^q. We then have the following interpolation result.

THEOREM 5.18. *Let $1 < r < \infty$ and $T : L^1(\mathbb{R}^d) + L^r(\mathbb{R}^d) \to L^0(\mathbb{R}^d)$ be a sub-additive mapping, meaning that*

$$|T(f+g)(x)| \leq |Tf(x)| + |Tg(x)|.$$

Suppose T is of weak type $(1,1)$ with constant A_1 and weak type (r,r) with constant A_r. Then T is of type (p,p) for all $p \in (1,r)$ with constant depending only on A_1, A_r, p, r.

6. Singular integrals

In this section we review some classical literature on the Calderón-Zygmund theory of singular integral operators. These are operators on L^2 of the form

$$\widehat{Su}(\xi) = \hat{K}\hat{u}(\xi) \tag{1.7}$$

where $\hat{K}$ is a bounded multiplier, and $u \in L^2(\mathbb{R}^d)$. These comprise a class of convolution operators commuting with translations that are bounded on L^2. The operator S consists of a kernel K possessing a non-integrable singularity at a finite point (the origin) as well as at infinity. The kernels also satisfy certain growth and regularity conditions, but it is the local singularity at the origin and the cancellation condition that is its most crucial characterization. It is important to note that in the representation formula (1.7), K is generally not a function, and its Fourier transform is in the sense of distributions. An important result due to Stein [65] states that if S is a translation invariant operator bounded on L^2, then S is necessarily of the form

$$Su(x) = K * u(x),$$

for an appropriate tempered distribution $K \in \mathcal{S}'(\mathbb{R}^d)$, whose Fourier transform is bounded. Since there are distributions arising neither from functions nor measures, writing (1.7) as a convolution should be understood in the principal value sense, that is

$$Su(x) = \lim_{\varepsilon \to 0} \int_{\mathbb{R}^d} \mathbb{1}_{|x-y|>\varepsilon} K(x-y)u(y)dy. \tag{1.8}$$

If K satisfies a local cancellation conditon (for instance that K is odd, if $d=1$) then this limit exists in L^p. A fundamental property of singular kernels is that they extend via convolution to bounded operators on L^p, for $1 < p < \infty$. This is not true for $p = 1$. However, a substitute result, namely a weak bound from L^1 into M^1 exists. The techniques for proving this weak-type result were initiated by Besicovitch and Titchmarsh in the case of the one dimensional Hilbert transform, and were further developed by Calderón and Zygmund's treatment of the n-dimensional theory. The rest of the chapter will be devoted to the presentation of those methods.

DEFINITION 6.1. *We say that K is a singular kernel on $\mathbb{R}^d$ if*

(1) $K \in \mathcal{S}'(\mathbb{R}^d)$ and $\hat{K} \in L^\infty(\mathbb{R}^d)$,
(2) $K|_{\mathbb{R}^d \setminus \{0\}} \in L^1_{\text{loc}}(\mathbb{R}^d \setminus \{0\})$ and there exists a constant $A \geq 0$ such that

$$\int_{|x|>2|y|} |K(x-y) - K(x)| dx \leq A$$

for every $y \in \mathbb{R}^d$.

We next give a sufficient cancellation, growth and regularity condition for kernels $K \in L^1_{\text{loc}}(\mathbb{R}^d \setminus \{0\})$ so that the associated distribution is a singular kernel.

PROPOSITION 6.2. *Consider a function $K \in L^1_{\text{loc}}(\mathbb{R}^d \setminus \{0\})$ satisfying the following conditions:*

(1) There exists a constant $A \geq 0$ such that

$$\int_{|x|>2|y|} |K(x-y) - K(x)|dx \leq A \quad \textit{for every } y \in \mathbb{R}^d;$$

(2) There exists a constant $A_0 \geq 0$ such that

$$\int_{|x|\leq R} |x||K(x)|dx \leq A_0 R \quad \textit{for every } R > 0;$$

(3) There exists a constant $A_2 \geq 0$ such that

$$\left|\int_{R_1<x<R_2} K(x)dx\right| \leq A_2 \quad \textit{for every } 0 < R_1 < R_2 < \infty.$$

Then K can be extended to a tempered distribution on $\mathbb{R}^d$ which is a singular kernel, unique up to a constant times δ_0. Conversely, every singular kernel on $\mathbb{R}^d$ has a restriction on $\mathbb{R}^d \setminus \{0\}$ that satisfies the previous three conditions.

We define the following particular class of singular kernels, satisfying conditions (1)-(3) above.

Definition 6.3. *A kernel K is a* singular kernel of fundamental type *in $\mathbb{R}^d$ if the following properties hold:*

(1) $K|_{\mathbb{R}^d\setminus\{0\}} \in C^1(\mathbb{R}^d \setminus \{0\})$,
(2) There exists a constant $C_0 \geq 0$ such that

$$|K(x)| \leq \frac{C_0}{|x|^d} \qquad x \in \mathbb{R}^d \setminus \{0\}\,, \tag{1.9}$$

(3) There exists a constant $C_1 \geq 0$ such that

$$|\nabla K(x)| \leq \frac{C_1}{|x|^{d+1}} \qquad x \in \mathbb{R}^d \setminus \{0\}\,, \tag{1.10}$$

(4) There exists a constant $A_1 \geq 0$ such that

$$\left|\int_{R_1<|x|<R_2} K(x)dx\right| \leq A_1 \qquad \textit{for every } 0 < R_1 < R_2 < \infty. \tag{1.11}$$

In particular, these conditions are sufficient to extend the function defined on $\mathbb{R}^d \setminus \{0\}$ to a singular kernel K on $\mathbb{R}^d$, unique up to addition of a multiple of a Dirac delta at the origin, and which satisfies the estimates in Definition 6.1. Since $\hat{K} \in L^\infty(\mathbb{R}^d)$ we may consider the action of a singular kernel on L^2 in Fourier variables. By a density argument one can extend this operator to L^p, satisfying the same bounds.

Theorem 6.4. *(Calderón Zygmund.) Let K be a singular kernel and define*

$$Su = K * u \quad \textit{for } u \in L^2(\mathbb{R}^d)$$

in the sense of multiplication in the Fourier variable. Then for every $1 < p < \infty$ we have the strong estimate

$$||Su||_{L^p(\mathbb{R}^d)} \leq C_{d,p}(A + ||\hat{K}||_{L^\infty})||u||_{L^p(\mathbb{R}^d)}, \quad u \in L^p \cap L^2(\mathbb{R}^d), \tag{1.12}$$

and for $p = 1$ the weak estimate

$$|||Su|||_{M^1(\mathbb{R}^d)} \leq C_d(A + ||\hat{K}||_{L^\infty})||u||_{L^1(\mathbb{R}^d)}, \quad u \in L^1 \cap L^2(\mathbb{R}^d). \tag{1.13}$$

One has in addition the rough estimates

$$C_{d,p} \leq \frac{c_d}{p-1}, \qquad 1 < p < 2,$$
$$C_{d,p} \leq c_d\, p\,, \qquad 2 < p < \infty.$$

For a given singular kernel K, we will call the associated operator S defined in Theorem 6.5 a *singular integral operator* on $\mathbb{R}^d$. S can be extended to the whole $L^p(\mathbb{R}^d)$ for any $1 < p < \infty$ with values in $L^p(\mathbb{R}^d)$, still satisfying the same estimate. We define then the Fréchet space $\mathcal{R}(\mathbb{R}^d) = \cap_{m\in\mathbb{N},\, 1<p<\infty} W^{m,p}(\mathbb{R}^d)$ and its dual $\mathcal{R}'(\mathbb{R}^d) \subset \mathcal{S}'(\mathbb{R}^d)$. Since Theorem 6.4 implies all singular integral operators are bounded on $\mathcal{R}(\mathbb{R}^d)$, by duality we can define the operator S also $\mathcal{R}'(\mathbb{R}^d) \to \mathcal{R}'(\mathbb{R}^d)$. In particular it enables us to define Su for $u \in \mathcal{M}(\mathbb{R}^d)$. The result Su is in $\mathcal{R}'(\mathbb{R}^d) \subset \mathcal{S}'(\mathbb{R}^d)$.
For $p = 1$, S extends to the whole $L^1(\mathbb{R}^d)$ with values in $M^1(\mathbb{R}^d)$, with the same estimate as in the theorem. Since a function in M^1 is not generally integrable, and hence it cannot define a distribution, one cannot identify the values of $S^D u$ as a distribution and $S^{M^1} u$ as an M^1 function. For all $u \in L^1(\mathbb{R}^d)$, the operator $S^D : L^1(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ is an extension of S and defines a tempered distribution via the formula

$$\langle S^D u, \varphi\rangle = \langle u, \tilde{S}\varphi\rangle \tag{1.14}$$

for every $\varphi \in \mathcal{S}(\mathbb{R}^d)$.
This is well defined, since for $\varphi \in \mathcal{S}(\mathbb{R}^d)$, $\tilde{S}\varphi \in H^q(\mathbb{R}^d)$ and which belongs to $C_0(\mathbb{R}^d)$ when $q > d/2$. $S^D u \in \mathcal{S}'(\mathbb{R}^d)$ can likewise be defined for $u \in \mathcal{M}(\mathbb{R}^d)$. Since $\hat{K} \in L^\infty$, and since $\langle \hat{u}, S\varphi\rangle_{\mathcal{S}',\mathcal{S}} = \langle u, \widehat{S\varphi}\rangle_{\mathcal{S}',\mathcal{S}} = \langle u, \hat{K}\hat{\varphi}\rangle_{\mathcal{S}',\mathcal{S}}$, (1.14) is equivalent to the definition in Fourier variables

$$\widehat{S^D u} = \hat{K}\hat{u}$$

in $\mathcal{S}'(\mathbb{R}^d)$.

PROOF OF THEOREM 6.4. Because of its significance for the kernels considered in this paper, we prove Thereom 6.4 in the case when K is a singular kernel of fundamental type.
Step 1. S is of weak type $(2,2)$. Since $\hat{K} \in L^\infty$ it follows by Plancherel identity that for $u \in L^1 \cap L^2$,

$$||Su||_{L^2(\mathbb{R}^d)} \leq c||u||_{L^2(\mathbb{R}^d)}.$$

S admits a unique extension to all of L^2, where the above inequality still holds. By Chebyshev's inequality we get

$$\mathcal{L}^d\{x \in \mathbb{R}^d : |Su(x)| > \alpha\} \leq (c^2/\alpha^2) \int_{\mathbb{R}^d} |u|^2 dx, \qquad \forall u \in L^2(\mathbb{R}^d).$$

Step 2. S is of weak type $(1,1)$. We seek a constant C such that

$$\mathcal{L}^d\{x \in \mathbb{R}^d : |Su(x)| > \alpha\} \leq (c^2/\alpha) \int_{\mathbb{R}^d} |u| dx, \qquad \forall u \in L^1(\mathbb{R}^d).$$

We fix $\alpha > 0$ and apply a Calderón Zygmund decomposition on $|u|$. We decompose $\mathbb{R}^d$ into closed cubes $\{I_k\}_{k=1}^\infty$ with mutually disjoint interiors such that for each k,

$$\begin{cases} \alpha < \fint_{I_k} |u| \leq 2^d \alpha, \\ |u| \leq \alpha \qquad \text{a.e outside } \cup_k I_k. \end{cases}$$

We set

$$b_k = \left(u - \fint_{I_k} u\right) \mathbf{1}_{I_k}, \qquad \text{and}$$

$$g = \begin{cases} u, & x \notin \cup_k I_k, \\ \fint_{I_k} u, & x \in I_k. \end{cases} \tag{1.15}$$

By construction we get that $g, b_k \in L^1(\mathbb{R}^d)$ and additionally the estimates

$$\text{spt}\,(b_k) \subset I_k, \quad \int_{I_k} b_k = 0, \quad ||b_k||_{L^1(\mathbb{R}^d)} \leq 2\int_{I_k} |u|, \quad \sum_k ||b_k||_{L^1(\mathbb{R}^d)} \leq 2||u||_{L^1(\mathbb{R}^d)}. \tag{1.16}$$

Moreover, we have the decomposition in $L^1(\mathbb{R}^N)$:

$$u = g + b$$

where $b = \sum_k b_k$. These have following properties:

$$b_k(x) = 0 \qquad \forall x \notin \cup_k I_k, \tag{1.17}$$

$$||g||_{L^1(\mathbb{R}^d)} \leq ||u||_{L^1(\mathbb{R}^d)}, \tag{1.18}$$

$$||g||_{L^\infty(\mathbb{R}^d)} \leq 2^d \alpha, \tag{1.19}$$

$$\mathcal{L}^d(\cup_k I_k) \leq \sum_k \mathcal{L}^d(I_k) \leq \frac{1}{\alpha}||u||_{L^1(\mathbb{R}^d)}. \tag{1.20}$$

Since $Su = Su + \sum_k Sb_k$, it follows that

$$\mathcal{L}^d\{x \in \mathbb{R}^d : |Su(x)| > \alpha\} \leq \mathcal{L}^d\{x \in \mathbb{R}^d : |Sg(x)| > \alpha/2\} + \sum_k \mathcal{L}^d\{x \in \mathbb{R}^d : |Sb_k(x)| > \alpha/2\}.$$

Step 3. We estimate Sg.

$$||g||^2_{L^2(\mathbb{R}^d)} = \int\limits_{x \notin \cup_k I_k} |g(x)|^2 dx + \int\limits_{x \in \cup_k I_k} |g(x)|^2 dx \tag{1.21}$$

$$\leq \int\limits_{x \notin \cup_k I_k} \alpha |g(x)| dx + c^2 \alpha^2 \mathcal{L}^d(\cup_k I_k) \tag{1.22}$$

$$\leq (c^2 + 1)\alpha ||u||_{L^1(\mathbb{R}^d)}. \tag{1.23}$$

Applying step 1 to Sg we obtain

$$\mathcal{L}^d\{x \in \mathbb{R}^d : |Sg(x)| > \alpha\} \leq \frac{c}{\alpha}||u||_{L^1(\mathbb{R}^d)}.$$

Step 4. We estimate Sb_k. Let y_k denote the center of the cube I_k. Since b_k are supported on I_k and have zero average on I_k, for $x \notin \cup_k I_k$ we can write

$$Sb_k(x) = \int_{I_k} K(x-y) b_k(y) dy \tag{1.24}$$

$$= \int_{I_k} [K(x-y) - K(x-y_k)] b_k(y) dy. \tag{1.25}$$

For $y \in I_k$ we have the estimate

$$|K(x-y) - K(x-y_k)| \leq \int_0^1 |\nabla K(s(x-y) + (1-s)(x-y_k))||y - y_k| ds \tag{1.26}$$

$$\leq \int_0^1 \frac{c|y-y_k|}{|x-y+s(y-y_k)|^{d+1}} ds \leq c_d \sup_{\bar{y} \in [y, y_k]|} \frac{\text{diam}(I_k)}{|x-\bar{y}|^{d+1}} \leq c_d \frac{\text{diam}(I_k)}{|x-y_k|^{d+1}}. \tag{1.27}$$

In the last line one observes that the diameter of I_k is proportional to its distance from the complement of $\cup I_k$. If x is a fixed point outside of $\cup_k I_k$, the distances $\{|x-y|\}$ as y varies over I_k are all lower bounded by $\frac{1}{2}|x-y_k|$. Hence from (1.16) we have

$$|Sb_k(x)| \leq c_d \frac{\text{diam}(I_k)}{|x-y_k|^{d+1}} \int_{I_k} |b(y)| dy \leq c_d \frac{\text{diam}(I_k)}{|x-y_k|^{d+1}} ||u||_{L^1(\mathbb{R}^d)}. \tag{1.28}$$

Thus it suffices to prove that

$$\int_{x \notin \cup_k I_k} |Sb_k(x)| dx \leq c_d ||u||_{L^1(\mathbb{R}^d)}.$$

Observe that (1.28) is indeed integrable at infinity. Using polar coordinates centered at y_k one has

$$\int_{x\notin\cup_k I_k} |Sb_k(x)|dx \le c_d||u||_{L^1(\mathbb{R}^d)} \int_{|x-y_k|\ge 2\mathrm{diam}(I_k)} \frac{\mathrm{diam}(I_k)}{|x-y_k|^{d+1}}dx \le c_d||u||_{L^1(\mathbb{R}^d)}. \tag{1.29}$$

It follows from a layer cake decomposition that for every $\alpha > 0$

$$\mathcal{L}^d\{x\notin \cup_k I_k : |Sb_k(x)|> \alpha/2\} \le \frac{2c_d}{\alpha}||u||_{L^1(\mathbb{R}^d)}.$$

Combining this with step 3 and the fact that

$$\mathcal{L}^d(\cup_k I_k) \le \frac{1}{\alpha}||u||_{L^1(\mathbb{R}^d)},$$

we obtain that S is of weak type $(1,1)$.

Step 5.(The L^p inequalities.) For $1 < p < 2$, we verify the hypothesis of the interpolation Theorem 5.18 with $r = 2$. S is linear and well-defined on $L^1(\mathbb{R}^d) + L^2(\mathbb{R}^d)$ by the preceding arguments. It is of weak type $(1,1)$ and of weak type $(2,2)$ with bounds depending only on $||\hat{K}||_\infty$ and d. Thus for every $u \in L^p(\mathbb{R}^d)$, $1 < p < 2$,

$$||Su||_{L^p(\mathbb{R}^d)}\le A_p||u||_{L^p(\mathbb{R}^d)}, \tag{1.30}$$

where A_p depends only on $||\hat{K}||_\infty, p$ and d.
For $2 < p < \infty$, we use the duality between L^p and L^q for $\frac{1}{p}+\frac{1}{q} = 1$. Let $u \in L^p(\mathbb{R}^d)$. Then it follows from (1.30) that we can estimate

$$\begin{aligned}
||Su||_{L^p(\mathbb{R}^d)} &= \sup_{g\in L^{p'}} \left\{ \left|\int_{\mathbb{R}^d} (Su)g dx\right| : ||g||_{L^{p'}(\mathbb{R}^d)}\le 1\right\} \\
&= \sup_{g\in L^{p'}} \left\{ \left|\int_{\mathbb{R}^d} u(Sg) dx\right| : ||g||_{L^{p'}(\mathbb{R}^d)}\le 1\right\} \\
&\le ||u||_{L^p(\mathbb{R}^d)} \sup_{g\in L^{p'}} \{||Sg||_{L^{p'}(\mathbb{R}^d)}\colon ||g||_{L^{p'}(\mathbb{R}^d)}\le 1\} \\
&\le C_{p'}||u||_{L^p(\mathbb{R}^d)}.
\end{aligned}$$

□

We remark that condition (1) in Proposition 6.2 ensures (1.24) is immediately integrable outside the union of cubes I_k. One has

$$\int_{x\notin(\cup_k I_k)^c} |K(x-y)-K(x-y_k)|dx \le \int_{|x|\ge 2|y|} |K(x-y)-K(x)|dx \le A.$$

(1) is historically [65] the regularity condition assumed for a singular kernel. It is implied by a Hölder regularity on K as well as by (3) in Definition (6.3). However, using the decay of ∇K outside of the origin gives a better picture of the Calderón Zygmund geometry outside of the singular set $\cup_k I_k$. For a proof of the optimal constants $C_{d,p}$, see [65].
The range of p in Theorem 6.4 is sharp, since the operator S is not bounded on L^∞. However, if the function u has an additional Hölder regularity, one has the following interpolation estimate.

Lemma 6.5. *Let S be the operator defined in Theorem 6.4. Let $u \in C_c^\alpha(\mathbb{R}^d)$ for some $\alpha \in (0,1)$ be supported in some ball of radius $R > 0$. Then for every $\varepsilon > 0$ there exists a constant $c > 0$ independent of R, u, such that*

$$||Su||_{L^\infty(\mathbb{R}^d)}\le c\left[||u||_{C^\alpha(\mathbb{R}^d)}\varepsilon^\alpha + \max(1,\log(R/\varepsilon))||u||_{L^\infty(\mathbb{R}^d)}\right],$$

and

$$||Su||_{C^\alpha(\mathbb{R}^d)}\le c||u||_{C^\alpha(\mathbb{R}^d)}.$$

We recall also an interpolation lemma for singular integrals which states a necessary condition for an L^2-bounded operator to be bounded on $L^1 \to M^1$. From the proof of Theorem 6.4 we see that Sg is bounded on L^2, and outside of $\cup_k I_k$, $Sb_k(x)$ has a decay rate which is integrable at infinity. This combined with an a priori estimate on the size of $\cup_k I_k$ allows one to prove via the Calderón Zygmund decomposition that Su is bounded in M^1. This suggests that a sufficient condition for a general operator T to be bounded from $L^1 \to M^1$ is that for x out of the support of some suitable function u, Tu decays sufficiently fast at infinity, so that its integral outside of the support of u is proportional to the L^1 norm of u.

LEMMA 6.6. *Let $T_+ : L^2 \to L^2$ be a nonlinear operator satisfying*

(1) $T_+(u) \leq 0$ for every $u \in L^2(\mathbb{R}^d)$;
(2) $T_+(u+v) \leq T_+(u) + T_+(v)$ for every $u, v \in L^2(\mathbb{R}^d)$;
(3) $T_+(\lambda u) = |\lambda| T_+(u)$ for every $u \in L^2(\mathbb{R}^d)$ and $\lambda \in \mathbb{R}$;
(4) There exists a constant $P_2 \geq 0$ such that

$$||T_+(u)||_{L^2(\mathbb{R}^d)} \leq P_2 ||u||_{L^2(\mathbb{R}^d)}$$

for every $u \in L^2(\mathbb{R}^d)$;

(5) There exists a constant $P_1 \geq 0$ such that if $u \in L^2(\mathbb{R}^d)$ satisfies $\operatorname{spt} u \subset \bar{B}_R(x_0)$ for some $x_0 \in \mathbb{R}^d, R > 0$, and $\int_{\mathbb{R}^d} u = 0$, then

$$\int\limits_{|x-x_0|>2R} T_+(u)dx \leq P_1 ||u||_{L^1(\mathbb{R}^d)}.$$

Then there exists a constant C_d, depending only on dimension d, such that for every $u \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$,

$$|||T_+(u)|||_{M^1(\mathbb{R}^d)} \leq C_d(P_1 + P_2) ||u||_{L^1(\mathbb{R}^d)}.$$

PROOF. **Step 1.** We apply the Calderón Zygmund decomposition. Let $\alpha > 0$. For any $u \in L^2(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$, we decompose $\mathbb{R}^d$ into closed cubes $\{I_k\}_{k=1}^\infty$ with disjoint interiors such that for each k,

$$\begin{cases} \alpha < \fint_{I_k} |u| \leq 2^d \alpha, \\ |u| \leq \alpha \qquad \text{a.e outside } \cup_k I_k. \end{cases}$$

Let b_k and g be defined as in (1.15). Let $B_k = B_{r_k}(y_k)$ be a ball containing I_k centered at y_k, such that for some c_d we have

$$\mathcal{L}^N(B_k) \leq c_d \mathcal{L}^d(I_k).$$

Set

$$V_k = B_{2r_k}(y_k), \qquad V = \cup_k V_k.$$

It follows that

$$\mathcal{L}^d(V) \leq \sum_k \mathcal{L}^d(V_k) \leq \sum_k 2^N c_N \mathcal{L}^d(I_k) \leq 2^d c_d \frac{1}{\alpha} ||u||_{L^1(\mathbb{R}^d)}. \tag{1.31}$$

Since $\operatorname{spt}(b_k) \subset \bar{B}_k$ and $\int_{\mathbb{R}^d} b_k = 0$, we have from assumption (5) that

$$\int_{\mathbb{R}^d \setminus \bar{V}_k} T_+(b_k) \leq P_1 ||b_k||_{L^1(\mathbb{R}^d)}. \tag{1.32}$$

Step 2. Fix $m \in \mathbb{N}$. By subadditivity of T_+,

$$T_+\left(g + \sum_{k=1}^m b_k\right) \leq T_+(g) + \sum_{k=1}^m T_+(b_k). \tag{1.33}$$

Then by (1.16) and (1.32) we get

$$\left\| \sum_{k=1}^{m} T_+(b_k) \right\|_{L^1(\mathbb{R}^d \setminus V)} \leq \sum_{k=1}^{m} P_1 ||b_k||_{L^1(\mathbb{R}^d)} \leq 2P_1 ||u||_{L^1(\mathbb{R}^d)}. \tag{1.34}$$

Since from (1.17) we have

$$||g||_{L^2(\mathbb{R}^d)} \leq \left(2^d \alpha ||u||_{L^1(\mathbb{R}^d)}\right)^{1/2},$$

it follows that

$$||T_+(g)||_{L^2(\mathbb{R}^d)} \leq P_2 \left(2^d \alpha ||u||_{L^1(\mathbb{R}^d)}\right)^{1/2}. \tag{1.35}$$

Using (1.31), (1.34), (1.32), and (1.35) we can estimate for every $\lambda > 0$ the superlevels

$$\begin{aligned}
&\mathcal{L}^d\left(\left\{x \in \mathbb{R}^d : T_+\left(g + \sum_{k=1}^{m} b_k\right)(x) > \lambda\right\}\right) \\
&\leq \mathcal{L}^d\left(\left\{x \in \mathbb{R}^d : T_+(g)(x) > \frac{\lambda}{2}\right\}\right) + \mathcal{L}^d\left(\left\{x \in V : \sum_{k=1}^{m} T_+(b_k)(x) > \frac{\lambda}{2}\right\}\right) \\
&+ \mathcal{L}^d\left(\left\{x \notin V : \sum_{k=1}^{m} T_+(b_k)(x) > \frac{\lambda}{2}\right\}\right) \\
&\leq \frac{1}{(\lambda/2)^2} P_2^2 2^d \alpha ||u||_{L^1(\mathbb{R}^N)} + \mathcal{L}^d(V) + \frac{1}{\lambda/2} 2P_1 ||u||_{L^1(\mathbb{R}^d)} \\
&\leq \left[\frac{\alpha}{\lambda^2} 2^{d+2} P_2^2 + \frac{1}{\alpha} 2^d c_d + \frac{1}{\lambda} 4P_1\right] ||u||_{L^1(\mathbb{R}^d)}.
\end{aligned} \tag{1.36}$$

Since by definition of b_k we have

$$\left|\sum_{k=1}^{m} b_k\right| \leq |u| + |g| \in L^2(\mathbb{R}^d),$$

we deduce by Dominated Convergence that

$$g + \sum_{k=1}^{m} b_k \to g + \sum_{k=1}^{\infty} b_k = u, \qquad m \to \infty, \text{ in } L^2(\mathbb{R}^d).$$

But this clearly implies that

$$T_+\left(g + \sum_{k=1}^{m} b_k\right) \to T_+(u), \tag{1.37}$$

in $L^2(\mathbb{R}^d)$. Up to a subsequence, (1.37) holds pointwise a.e. in $\mathbb{R}^d$, so by Fatou's lemma

$$\mathcal{L}^d(\{x \in \mathbb{R}^d : T_+(u)(x) > \lambda\}) \leq \left[\frac{\alpha}{\lambda^2} 2^{d+2} P_2^2 + \frac{1}{\alpha} 2^d c_d + \frac{1}{\lambda} 4P_1\right] ||u||_{L^1(\mathbb{R}^d)}. \tag{1.38}$$

Optimizing in α gives

$$\mathcal{L}^N(\{x \in \mathbb{R}^d : T_+(u)(x) > \lambda\}) \leq \frac{1}{\lambda}\left[2^{d+2}\sqrt{c_d} P_2 + 4P_1\right] ||u||_{L^1(\mathbb{R}^d)}. \tag{1.39}$$

□

CHAPTER 2

The transport equation in the Sobolev setting

We describe the classical well-posedness problem of the Cauchy problem for the transport equation

$$\partial_t u + b \cdot \nabla u = 0, \qquad (t,x) \in I \times \mathbb{R}^d,$$

where $I \subset \mathbb{R}$ is an interval of times, and when $b(t,x)$ is not Lipschitz but rather has Sobolev or BV regularity. For this reason we will describe a weak formulation of the equation in the distributional sense. If $u \in L^\infty(I \times \mathbb{R}^d)$ and b has locally summable divergence, then we can give a distributional meaning to the terms $\partial_t u$ and $b \cdot \nabla u$, with

$$\langle b \cdot \nabla u, \varphi \rangle = -\langle bu, \nabla \varphi \rangle - \langle u \operatorname{div} b, \varphi \rangle, \tag{2.1}$$

for every $\varphi \in C_c^\infty([0,T] \times \mathbb{R}^d)$. We present a computation in order to formalize the idea of renormalized solutions, which gives a fundamental characterization of the chain rule outside of the smooth setting. The theory of DiPerna and Lions links this renormalization property inherently with well-posedness of the transport equation. We present the weak formulation of the equation under the assumption that b and its divergence are locally integrable.

A strategy for uniqueness. We present an exploratory computation motivating the concept of renormalization, to show uniqueness for the Cauchy problem

$$\begin{cases} \partial_t u(t,x) + b(t,x) \cdot \nabla u(t,x) = 0 \\ u(0,x) = u^0(x). \end{cases} \tag{2.2}$$

We neglect for the moment any regularity assumptions on b, and proceed with a formal argument. Let b be a divergence free vector field. Multiplying (2.2) by $2u$, we obtain

$$\text{Step 1.)} \qquad 2u\partial_t u + 2u\, b \cdot \nabla u = 0,$$

which we re write as

$$\text{Step 2.)} \qquad \partial_t u^2 + b \cdot \nabla u^2 = 0.$$

Integrating on $\mathbb{R}^d$ for fixed time $t \in [0,T]$ and using the fact that $\operatorname{div} b = 0$ we get

$$\text{Step 3.)} \qquad \frac{d}{dt} \int_{\mathbb{R}^d} u(t,x)^2 dx = -\int_{\mathbb{R}^d} \operatorname{div}\,(b(t,x)u(t,x)^2) dx = 0,$$

which implies that the $L^2(\mathbb{R}^d)$ norm of u is conserved:

$$\text{Step 4.)} \qquad \frac{d}{dt} ||u(t,\cdot)||_{L^2(\mathbb{R}^d)} = 0.$$

This implies that if the initial data is $u^0 = 0$, the only solution is $u \equiv 0$, which is clearly necessary and sufficient for uniqueness. However, this formal argument fails in a weaker setting. Since solutions of (2.2) are not smooth in general, application of the chain rule in step 2 is not justified. The second issue to deduce from step 4 that $||u(t,\cdot)||_{L^2(\mathbb{R}^d)} = 0$ when the initial datum in the formulation of (2.2) is meant in a weaker, distributional sense. We require that

$$||u(t,\cdot)||_{L^2(\mathbb{R}^d)} \to ||u^0||_{L^2(\mathbb{R}^d)}, \qquad t \to 0,$$

but this is equivalent to a strong continuity property of the solution that does not follow from the weak formulation.

7. Weak solutions

Let b be a vector field belonging to $L^1_{\mathrm{loc}}([0,T]\times\mathbb{R}^d)$ such that $\operatorname{div} b \in L^1_{\mathrm{loc}}([0,T]\times\mathbb{R}^d)$, and let $u^0 \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$.

DEFINITION 7.1. *Given $T>0$, we say that a function $u \in L^\infty_{\mathrm{loc}}([0,T];\mathbb{R}^d)$ is a weak solution in $[0,T]$ of (2.2) if the following identity holds for all $\varphi \in C^\infty_c([0,T)\times\mathbb{R}^d)$:*

$$\int_0^T\int_{\mathbb{R}^d} u(t,x)[\partial_t\varphi(t,x)+\varphi(t,x)\operatorname{div} b(t,x)+b(t,x)\cdot\nabla\varphi(t,x)]dxdt = -\int_{\mathbb{R}^d} u^0(x)\varphi(0,x)dx. \tag{2.3}$$

Observe that the local boundedness assumption on u implies that $\partial_t u$ has meaning as a distribution, and the weak formulation is consistent with smooth solutions to (2.2), as can be seen by multiplying (2.2) with test functions $\varphi \in C^\infty_c([0,T)\times\mathbb{R}^d)$ and integrating over $[0,T]\times\mathbb{R}^d$. In order to recover the initial datum u^0 from a weak solution u, we come to the notion of weak continuity of weak solutions. Let u be a weak solution to (2.2). If we consider the weak form (2.3) for u by testing against tensor products of functions $\varphi(t)\phi(x)$ with $\varphi \in C^1_c((0,T))$ and $\phi \in C^1_c(\mathbb{R}^d)$ then we get

$$\frac{d}{dt}\int_{\mathbb{R}^d} u(t,x)\phi(x)dx = \int_{\mathbb{R}^d} u(t,x)[\phi(x)\operatorname{div} b(t,x)+b(t,x)\cdot\nabla\phi(x)]dx, \qquad \text{in } \mathcal{D}'([0,T]). \tag{2.4}$$

If we consider functions $\phi \in C^1_c(\mathbb{R}^d)$ with $\operatorname{spt}\phi \in B_R$, then we obtain the following estimate:

$$\left|\frac{d}{dt}\int_{\mathbb{R}^d} u(t,x)\phi(x)dx\right| \le ||\phi||_{C^1(\mathbb{R}^d)} V_R(t), \tag{2.5}$$

where $V_R(t)$ is the function belonging to $L^1([0,T])$ given by

$$V_R(t) = ||u||_\infty \int_{B_R} (|\operatorname{div} b(t,x)|+|b(t,x)|)dx. \tag{2.6}$$

This in turn implies

$$\left|\int_{\mathbb{R}^d} (u(t,x)-u(s,x))\phi(x)dx\right| \le ||\phi||_{C^1(\mathbb{R}^d)}\int_s^t V_R(\tau)d\tau, \tag{2.7}$$

for every $\phi \in C^1_c(B_R)$ and almost every $s,t\in[0,T]$. Thus $u(t,\cdot)$ can be extended uniquely to a continuous function $\bar u_R(t,\cdot) \in [C^1_c(B_R)]'$. Repeating this argument for $R\in\mathbb{N}$ and since $u \in L^\infty([0,T]\times\mathbb{R}^d)$ we get that $\bar u(t,\cdot)$ is continuous in $[L^1(\mathbb{R}^d)]'$. We thus have the following.

LEMMA 7.2. *(Weak continuity in time.) The map $t\mapsto u(t,\cdot)$ is weakly$-*$ continuous from $[0,T]$ into $L^\infty(\mathbb{R}^d)$.*

We remark that the assumption $t\mapsto u(t,\cdot)$ is weakly$-*$ continuous (up to a modification on a negligible set of times) in the $L^\infty(\mathbb{R}^d)$ topology is a reasonable one, in order to define the weak solution $u(t,\cdot)$ at the endpoints of $[0,T]$ and give a sense to the initial datum u^0. In general, one cannot expect strong continuity of the solution with respect to time.

7.1. Existence of weak solutions. Since (2.2) is a linear equation, existence of weak solutions is trivial: a smooth approximation of the vector field allows passage to the limit.

THEOREM 7.3. *(Existence.) Let $b \in L^\infty([0,T]\times\mathbb{R}^d)$ with $\operatorname{div} b \in L^1_{\mathrm{loc}}([0,T]\times\mathbb{R}^d)$ and let $u^0 \in L^\infty(\mathbb{R}^d)$. Then there exists a weak solution $u \in L^\infty([0,T]\times\mathbb{R}^d)$ to (2.2).*

PROOF. Let $\rho_\varepsilon \in C_c^\infty(\mathbb{R}^d)$ be the standard mollifier and let $\eta_\varepsilon \in C_c^\infty(\mathbb{R}^{d+1})$ be a mollifier on $\mathbb{R}^{d+1}$. Denote by $u_0^\varepsilon = u^0 * \rho_\varepsilon$ and $b_\varepsilon = b * \eta_\varepsilon$. Since b_ε is smooth, there is a solution u_ε uniquely determined by the u_ε^0 to the equation

$$\begin{cases} \partial_t u + b_\varepsilon \cdot \nabla u = 0 \\ u(0,\cdot) = u_\varepsilon^0. \end{cases}$$

Clearly u_ε is uniformly bounded in $L^\infty([0,T]\times\mathbb{R}^d)$, so up to a subsequence we have that u_ε is weakly$-*$ convergent to a limit u in $L^\infty([0,T]\times\mathbb{R}^d)$. Passing to the limit as $\varepsilon \to 0$ in the weak formulation (7.1) shows that u is a weak solution.

□

8. Renormalization

With weak solutions of low regularity, the application of the chain rule in order to write

$$\partial_t u^2 = 2u\partial_t u, \qquad \nabla u^2 = 2u\nabla u,$$

is not justified. We begin the next discussion with a remark. Let b be a Lipschitz vector field. Then the smooth solution to (2.2) $u \in C^1([0,T]\times\mathbb{R}^d)$ satisfies for any $\beta \in C^1(\mathbb{R})$,

$$\partial_t \beta(u) + b \cdot \nabla\beta(u) = \beta'(u)[\partial_t u + b\cdot\nabla u]. \tag{2.8}$$

This implies that $\beta(u)$ is a smooth solution with initial datum $\beta(u^0)$. We now define a class of weak solutions which satisfy such a rule, in the sense of distributions. Only the integrability in time (and not the regularity) of b does plays a role, so we will denote by $I \subset \mathbb{R}$ a generic (and possibly infinite) interval of times.

DEFINITION 8.1. *(Renormalized solutions.) Let $b \in L^1(I; L^1_{\text{loc}}(\mathbb{R}^d))$ be such that $\operatorname{div} b \in L^1(I; L^1_{\text{loc}}(\mathbb{R}^d))$. Let $u \in L^\infty(I\times\mathbb{R}^d)$ be a weak solution of the transport equation with initial datum u^0. Then u is a renormalized solution if*

$$\begin{cases} \partial_t\beta(u) + b\cdot\nabla\beta(u) = 0 \\ \beta(u(0,\cdot)) = \beta(u^0) \end{cases} \tag{2.9}$$

holds in the sense of distributions for every bounded function $\beta \in C^1(\mathbb{R})$, where the distribution $b\cdot\nabla u$ is defined according (2.1).

We say that b has the *r*enormalization property if every bounded solution of the transport equation with vector field b is a renormalized solution. It turns out that this property is intrinsically tied to the well-posednesss problem: renormalization implies well-posedness. Under certain incompressibility assumptions (such as $\operatorname{div} b \in L^\infty$) renormalization also implies *stability* of solutions and thereby existence of solutions by approximation. DiPerna and Lions proved that all distributional solutions are renormalized when there is Sobolev regularity of the space variables.

THEOREM 8.2. *Let $b : [0,T]\times\mathbb{R}^d \to \mathbb{R}^d$ be a bounded vector field such that $\operatorname{div} b \in L^1([0,T]; L^\infty(\mathbb{R}^d))$. If b has the renormalization property, then bounded solutions of (2.2) are unique. Moreover, if b_n and u_n^0 are smooth and uniformly bounded approximations such that $b_n \to b$ and $u_n^0 \to u^0$ strongly in $L^1_{\text{loc}}(\mathbb{R}^d)$, then the solutions u_n associated to b_n with initial data u_n^0 converge strongly in $L^1_{\text{loc}}(\mathbb{R}^d)$ to the solution u associated to b.*

PROOF. By linearity it suffices to show that the only bounded solution to

$$\begin{cases} \partial_t u + b\cdot\nabla u = 0 \\ u(0,\cdot) = 0 \end{cases} \qquad \text{in } \mathcal{D}'([0,T]\times\mathbb{R}^d)$$

is $u = 0$. Since b has the renormalization property, we have the additional information

$$\begin{cases} \partial_t \beta(u) + b \cdot \nabla \beta(u) = 0 \\ \beta(u)(0,\cdot) = 0 \end{cases} \qquad (2.10) \qquad \text{in } \mathcal{D}'([0,T] \times \mathbb{R}^d),$$

for any bounded $\beta \in C^1(\mathbb{R})$. Fix $R > 0$ and $\eta > 0$ and let $\phi \in C_c^\infty([0,T) \times \mathbb{R}^d)$ such that $\phi \equiv 1$ on $[0, T-\eta] \times B_R(0)$ with the property

$$\partial_t \phi(t,x) \le -||b||_\infty |\nabla \phi(t,x)| \tag{2.11}$$

on $[0,T] \times \mathbb{R}^d$. Let $\beta \in C^1(\mathbb{R})$ be positive and define

$$f(t) := \int_{\mathbb{R}^d} \beta(u(t,x))\phi(t,x)dx.$$

By (2.10), $f(0) = 0$. Let $\varphi \in C_c([0,T))$ be a positive test function. Then we can estimate the distributional derivative $\frac{d}{dt} f(t)$ by testing (2.10) against $\varphi(t)\phi(t,x)$:

$$\begin{aligned} &-\int_0^T f(t)\varphi'(t)dt \\ &= \int_0^T \int_{\mathbb{R}^d} \beta(u(t,x))\varphi(t)[\partial_t \phi(t,x) + b(t,x) \cdot \nabla \phi(t,x)]dxdt + \int_0^T f(t) \operatorname{div} b(t,x)\varphi(t)dt \\ &\le \int_0^T f(t)||\operatorname{div} b(t,\cdot)||_\infty \varphi(t)dt. \end{aligned}$$

Observe that the first integral in the second line is negative from the choice of ϕ in (2.11). Thus

$$\frac{d}{dt} f(t) \le ||\operatorname{div} b(t,\cdot)||_\infty f(t) \qquad \text{in } \mathcal{D}'([0,T]).$$

Since $\operatorname{div} b \in L^1([0,T]; L^\infty(\mathbb{R}^d))$, Gronwall's inequality yields that $f(t) = 0$ for all $t \in [0,T]$. By varying $\phi(t,x)$ we deduce that $\beta(u(t,\cdot)) = 0$ for every $t \in [0,T]$ and every admissible β and hence $u(t,\cdot) = 0$ for every $t \in [0,T]$.

It is clear that up to a subsequence, u_n is weakly$-*$ compact in $L^\infty([0,T] \times \mathbb{R}^d)$ and the limit u is a weak solution. Since this solution must be unique, the whole sequence converges to u. Since b_n is smooth, it has the renormalization property, therefore u_n^2 is a solution of (2.2) with initial datum $(u_n^0)^2$. But then u_n^2 converges in $L^\infty([0,T] \times \mathbb{R}^d) - w*$ to a unique solution with initial data $(u^0)^2$. By the renormalization property, this solution must be u^2. Since both u_n and u_n^2 converge in $L^\infty([0,T] \times \mathbb{R}^d) - w*$ to u and u^2 respectively, we deduce by Radon-Riesz theorem that u_n converges to u strongly in $L^1_{\text{loc}}([0,T] \times \mathbb{R}^d)$. □

We now come to the seminal result of DiPerna and Lions, in which it is proven that every vector field with Sobolev regularity has the renormalization property. We present the regularization argument (which uses a radial convolution kernel) and exploits the Sobolev regularity of b in the term $b \cdot \nabla u$.

THEOREM 8.3. *Let $b \in L^1_{\text{loc}}(I; W^{1,1}_{\text{loc}}(\mathbb{R}^d))$ and let $u \in L^\infty_{\text{loc}}(I \times \mathbb{R}^d)$ be a weak solution of the transport equation. Then u is a renormalized solution.*

PROOF. We fix an even convolution kernel $\rho_\varepsilon \in C_c^\infty(\mathbb{R}^d)$. Denote by $u_\varepsilon = u * \rho_\varepsilon$. We convolve the transport equation, and note that we have a commutator term r_ε when convolving with the term $b \cdot \nabla u$:

$$\partial_t u_\varepsilon + b \cdot \nabla u_\varepsilon = b \cdot \nabla u_\varepsilon - (b \cdot \nabla u) * \rho_\varepsilon := r_\varepsilon. \tag{2.12}$$

By smoothness of u_ε w.r.t. x, we have from the PDE that $\partial_t u_\varepsilon \in L^1_{\text{loc}}$, therefore for every fixed $\varepsilon > 0$, u_ε belongs to $W^{1,1}_{\text{loc}}(I \times \mathbb{R}^d)$. Thus we can apply Stampacchia's Chain rule for Sobolev spaces [64], to get

$$\beta'(u_\varepsilon) r_\varepsilon = \frac{d}{dt} \beta(u_\varepsilon) + b \cdot \nabla \beta(u_\varepsilon). \tag{2.13}$$

When $\varepsilon \to 0$ convergence in the distributional sense of all terms in the right hand side above to (2.9) is trivial. On the other hand, $\beta'(u_\varepsilon)$ is locally equibounded, and from the identity (2.1) it follows that r_ε converges to zero distributionally. In order to ensure distributional convergence of the product $\beta'(u_\varepsilon)r_\varepsilon$ we would like that $r_\varepsilon \to 0$ in $L^1_{\mathrm{loc}}(\mathbb{R}^d)$. It was proven by DiPerna and Lions that this is indeed the case, and this is where the Sobolev regularity becomes essential.

PROPOSITION 8.4. *(Strong convergence of the commutator.) If $u \in L^\infty_{\mathrm{loc}}(I \times \mathbb{R}^d)$ and b is a bounded vector field belonging to $b \in L^1_{\mathrm{loc}}(I; W^{1,1}_{\mathrm{loc}}(\mathbb{R}^d))$, then $r_\varepsilon \to 0$ strongly in $L^1_{\mathrm{loc}}(I \times \mathbb{R}^d)$.*

PROOF. From the definition of $b \cdot \nabla u$ and the convolution of a distribution with a smooth function, we have

$$r_\varepsilon(t,x) = \int_{\mathbb{R}^d} u(t,z)(b(t,z) - b(t,x)) \cdot \nabla \rho_\varepsilon(x-z)dz - (u \operatorname{div} b) * \rho_\varepsilon(x). \tag{2.14}$$

Changing variables $z \mapsto x - \varepsilon y$ we can write

$$r_\varepsilon(t,x) = \int_{\mathbb{R}^d} u(t, x-\varepsilon y)\frac{(b(t,x-\varepsilon y) - b(t,x))}{\varepsilon} \cdot \nabla\rho(y)dy - (u \operatorname{div} b) * \rho_\varepsilon(x). \tag{2.15}$$

Next we use the continuity of translations in L^p and the strong convergence of difference quotients (a property which indeed characterizes Sobolev functions.) For any $f \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^d)$,

$$\lim_{\varepsilon \to 0} \frac{f(x+\varepsilon z) - f(x)}{\varepsilon} = Df(x)z \qquad \text{in } L^1_{\mathrm{loc}}(\mathbb{R}^d). \tag{2.16}$$

Thus we obtain that r_ε converges strongly in $L^1_{\mathrm{loc}}(I \times \mathbb{R}^d)$ to

$$-u(t,x)\int_{\mathbb{R}^d} (Db(t,x)y) \cdot \nabla\rho(y)dy - u(t,x)\operatorname{div} b(t,x).$$

The elementary identity

$$\int_{\mathbb{R}^d} y_i \frac{\partial \rho}{\partial y_j}(y)dy = -\delta_{ij}$$

shows this limit is 0. □

REMARK 8.5. *We remark here that the last line of the proof can be neglected: since r_ε tends distributionally to zero, the point of the proof of Proposition 8.4 is the strength of the convergence.*

The renormalization strategy of Theorem 8.3 has become an important technique to proving well-posedness of the transport equation. Since the Sobolev regularity of b only enters in the last step (2.16) of the commutator proposition, the general argument may be applied to a weaker regularity setting involving a distributional derivative. Indeed the renormalization property was proved for vector fields of bounded variation in [5].[1] The main difference is that here $Db = D^a b + D^s b$, where $D^a b$ and $D^s b$ denote the absolutely continuous and singular part of Db respectively. The difference quotient (2.16) does not converge strongly in L^1_{loc} due to the part of the derivative which is the singular part of the measure. One has instead

$$\frac{b(t,x-\varepsilon y) - b(t,x)}{\varepsilon} = b^1_{\varepsilon,y}(t,x) + b^2_{\varepsilon,y}(t,x),$$

where

$$\begin{aligned} &b^1_{\varepsilon,y}(t,x) \to D^a b(t,x)y, \qquad \text{strongly in } L^1_{\mathrm{loc}}(\mathbb{R}^d),\\ &\limsup_{\varepsilon\to 0} \int_K |b^2_{\varepsilon,y}(t,x)|dx \le |D^s b(t,\cdot)y|(K) \qquad \forall K \Subset \mathbb{R}^d. \end{aligned}$$

[1] The renormalization scheme has been used in various other regularity settings, for instance with vector fields such that the symmetric part of the derivative is absolutely continuous w.r.t $\mathcal{L}^d$ [20], and for special vector fields with bounded deformation [6].

Thus the commutator r_ε corresponds to two integrals, the first involving $b^1_{\varepsilon,y}(t,x)$, and the other with $b^2_{\varepsilon,y}(t,x)$. Under suitable bounds on the divergence, the first part converges strongly in L^1_{loc} as in the previous proof. It is however the error term in the commutator involving the singular part of the derivative that is more complex. This relies on an anisotropic regularization procedure on the derivative of b to control the error term. This leads to the following theorem, which we state without proof.

THEOREM 8.6 (Ambrosio). *Let b be a bounded vector field in $L^1_{\mathrm{loc}}(I; BV(\mathbb{R}^d))$, such that $\operatorname{div} b \in L^1((0,T); L^1(\mathbb{R}^d))$. Then b has the renormalization property.*

CHAPTER 3

The Euler equation

The incompressible Euler equations for the motion of an inviscid fluid are given by

$$\left\{ \begin{array}{l} \partial_t v + \operatorname{div}(v \otimes v) + \nabla p = 0 \\ v(0, \cdot) = v^0(x), \\ \operatorname{div} v = 0 \end{array} \right. \tag{3.1}$$

where $v(t, x)$ is the velocity vector representing the speed of a particle at position x and time t, and $p(t, x)$ is the pressure. We consider the two-dimensional setting. The incompressible Euler equations may be rewritten as the transport equation for the scalar vorticity ω, advected by the velocity v, where the coupling is given by

$$\omega = \operatorname{curl} v. \tag{3.2}$$

This gives the vorticity formulation

$$\left\{ \begin{array}{l} \partial_t \omega + \operatorname{div}(v\omega) = 0 \\ \omega(0, \cdot) = \omega^0(x). \end{array} \right. \tag{3.3}$$

The coupling (3.2) can written via the Biot-Savart law as the convolution

$$v(t, x) = \frac{1}{2\pi} \int\limits_{\mathbb{R}^2} \frac{(x - y)^{\perp}}{|x - y|^2} \omega(t, y) dy = K * \omega\,(t, x), \tag{3.4}$$

where we denote by $K(x) = \frac{1}{2\pi} \frac{x^{\perp}}{|x|^2} = \left(\frac{-x_2}{|x|^2}, \frac{x_1}{|x|^2} \right)$ the Biot-Savart kernel. We remark that $K \in L^p_{\mathrm{loc}}(\mathbb{R}^2)$ for $p < 2$ and $K \in L^q(\mathbb{R}^2 \setminus B_1(0))$ for $q > 2$, so in order for (3.4) to converge absolutely one would require that $\omega \in L^{p'}(\mathbb{R}^2) \cap L^{q'}(\mathbb{R}^2)$ for $p' > 2$ and $q' < 2$, for all times. Alternatively, for vorticities in L^p_c with $p > 2$, the velocity is bounded.

In this chapter we begin with the smooth setting with a formulation of (3.3) as an integrodifferential equation, so that the study of smooth solutions reduces to the study of an ODE. We discuss the formulations of weak solutions when all terms in (3.1)-(3.3) make sense in the integral formulation, and study the classical existence proofs of solutions associated to vorticities in $L^1 \cap L^\infty$, $L^1 \cap L^p$ for $p > 1$, and finally for vortex sheet initial data, which are measure vorticities belonging to $H^{-1}(\mathbb{R}^2)$. Because the velocity field is coupled with the vorticity via (3.2), it is recovered from the vorticity via the nonlocal operator K. This gives a regularity on v which is no more than $W^{1,p}$ for $p < \infty$, with an extension to $p = \infty$ only if the vorticity has an additional Hölder regularity and compact support. We begin by summarizing the regularity and integrability properties of v.

Regularity of the velocity field. We summarize some estimates for the vector field v given by (3.4). The Biot Savart kernel K belongs to $L^1_{\mathrm{loc}}(\mathbb{R}^2)$ and has a distributional derivative given by the following singular kernel.

(I) For $i, j = 1, 2$, we have

$$\partial_{x_j} K^i(x) = \partial_{x_j} \frac{1}{2\pi} \left(\frac{-x_2}{|x|^2}, \frac{x_1}{|x|^2} \right)_i . \tag{3.5}$$

The Fourier transform of (3.5) is bounded and is given by

$$\widehat{\partial_{x_j} K^i}(\xi) = \frac{1}{2\pi} \xi_j \left(\frac{-\xi_2}{|\xi|^2}, \frac{\xi_1}{|\xi|^2} \right)_i \in L^\infty(\mathbb{R}^2). \tag{3.6}$$

This kernel satisfies the conditions of definition (6.3), thus its associated singular integral operator has an extension on L^p for $1 < p < \infty$. For $p = 1$ the kernel $\partial_{x_j} K^i$ defines a tempered distribution $S^i_j \in \mathcal{S}'(\mathbb{R}^2)$ via the formula

$$\langle S^i_j u, \varphi \rangle = \langle u, \tilde{S}^i_j \varphi \rangle \qquad \forall \varphi \in \mathcal{S}(\mathbb{R}^2), \tag{3.7}$$

where $\tilde{S}$ is the singular integral operator associated to the kernel $\partial_{x_j} K^i(-x)$. Thus for $i, j = 1, 2$, we have

$$(Dv(t,x))_{ij} = \partial_{x_j} v^i(t,x) = S^i_j \omega(t,x) \qquad \text{in } \mathcal{S}'((0,T) \times \mathbb{R}^2), \tag{3.8}$$

where S^i_j is the singular integral operator associated to the kernels $\partial_{x_j} K^i$, applied to the function ω. The Calderón Zygmund estimate from 6.4 gives the bound

$$||Dv||_{L^\infty((0,T);L^p(\mathbb{R}^2))} \leq c_p ||\omega_n||_{L^\infty((0,T);L^p(\mathbb{R}^2))}, \qquad \forall 1 < p < \infty.$$

An elementary computation shows that

(II) div $v(t,x) = 0$ in $\mathcal{D}'((0,T) \times \mathbb{R}^2)$.

(III) Vorticities bounded in $L^\infty([0,T]; L^p(\mathbb{R}^2))$ for any $1 \leq p < \infty$ are associated to velocities bounded in $L^\infty([0,T]; L^1(\mathbb{R}^2)) + L^\infty([0,T]; L^\infty(\mathbb{R}^2))$. Indeed the weak Hardy Littlewood Sobolev inequality gives the following estimate for any $1 \leq p < 2$ and $q = \frac{2p}{2-p}$:

$$||v||_{L^\infty((0,T);L^q(\mathbb{R}^2))} \leq c ||\omega||_{L^\infty((0,T);L^p(\mathbb{R}^2))}, \tag{3.9}$$

with the weak estimate for $p = 1$:

$$|||v|||_{L^\infty((0,T);M^2(\mathbb{R}^2))} \leq c ||\omega||_{L^\infty((0,T);L^1(\mathbb{R}^2))}. \tag{3.10}$$

(3.9) and (3.10) imply in particular the embedding $v(t,x) \in L^p_{\text{loc}}((0,T) \times \mathbb{R}^2)$ for any $1 \leq p < 2$.

Remark 8.7. *One generally can assume L^2_{loc} integrability of smooth solutions to (3.1), as can be shown by the following energy estimate. If v belongs to $C^1([0,T) \times \mathbb{R}^2)$ and solves (3.1) with initial datum $v^0 \in L^2(\mathbb{R}^2)$, then we can multiply (3.1) by v, use the Chain Rule and integrate, to obtain*

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^2} |v(t,x)|^2 dx = - \int_{\mathbb{R}^2} [((v \cdot \nabla) v) \cdot v](t,x) dx. \tag{3.11}$$

Integrating the right side by parts, we get the conservation of energy:

$$\int_{\mathbb{R}^2} |v(t,x)|^2 dx = \int_{\mathbb{R}^2} |v(0,x)|^2 dx, \tag{3.12}$$

for all $t > 0$. Hence, sufficiently smooth solutions of (3.1) conserve the energy. One might wonder if the L^p norms of the vorticity are also conserved. It turns out that in two dimensions this is indeed the case, when the initial datum is smooth. The nature of smooth solutions to (3.3) will be discussed in the next section.

9. Smooth solutions

Global in time existence of smooth solutions is known for two dimensions. A celebrated result by Beale-Kato-Majda roughly states that if the vorticity remains bounded in space and integrable for all times, i.e. the quantity

$$\int_0^T ||\omega(t,\cdot)||_{L^\infty} dt$$

is finite for every T, then the flow associated to the velocity field (and hence the solution) exists globally in time. One immediate consequence of this is that the 2D Euler equation has global in time existence of smooth solutions: the vorticty is conserved along particle trajectories: $||\omega(t,\cdot)||_\infty \leq ||\omega^0||_\infty$, and hence does not become unbounded. In this chapter we will restrict ourselves to the study in two dimensions.

We summarize the theory in the classical setting, first by recalling the formulation of the Euler equations as an integrodifferential equation involving particle trajectories. For a given smooth velocity field $v(s,x)$, and for any $s \in [t,T]$ the fluid particle trajectories $X(s,t,x) \in C^1([0,T] \times [0,T] \times \mathbb{R}^2)$ satisfy

$$\frac{dX}{ds}(s,t,x) = v(s, X(s,t,x)), \qquad X(t,t,x) = x, \tag{3.13}$$

where v is given by the convolution in (3.4). Note that due to the incompressibility condition the maps $X(s,t,\cdot)$ are volume-preserving. They satisfy the group property

$$X(r,t,X(t,s,x)) = X(r,s,x)$$

and in particular $X(s,t,X(t,s,x)) = x$, so that $X^{-1}(s,t,\cdot) = X(t,s,\cdot)$. We will consider $s \in [0,T]$ instead of $s \in [t,T]$. The theory of characteristics implies that the vorticity is transported according to the following classical formula.

Lemma 9.1 (Vorticity transport in 2D). *Let $X(s,t,x)$ be the smooth particle trajectories corresponding to a smooth divergence-free velocity field $v(s,x)$. Then the vorticity $\omega = \operatorname{curl} v$ solving (3.3) is given by*

$$\omega(t,x) = \omega^0\Big(X(s=0,t,x)\Big), \qquad \textit{for all } t \in [0,T].$$

We may write this as

$$\omega(t,x) = \omega^0(X^{-1}(t,x)),$$

or equivalently

$$\omega(t,X(t,x)) = \omega^0(x).$$

In particular, the vorticity ω is conserved along particle trajectories.

Remark 9.2. *The divergence free condition ensures that the trajectory $X(t,x)$ is measure preserving, so that $||\omega(t,\cdot)||_{L^p(\mathbb{R}^2)} = ||\omega^0||_{L^p(\mathbb{R}^2)}$ for any $1 \leq p \leq \infty$.*

Remark 9.3. *Let $L = ||v||_\infty$ denote the Lipschitz constant of the map $t \mapsto X(t,x) \in C^1(\mathbb{R};\mathbb{R}^2)$. If $\omega^0 \in C^1_c(\mathbb{R}^2)$, then using the fact that*

$$|X(t,x) - Id| \leq Lt, \qquad \forall x \in \mathbb{R}^2,\ t \geq 0$$

we deduce that $\omega(t,\cdot) \in C^1_c(\mathbb{R}^2)$ for any $t \geq 0$, with

$$\operatorname{spt} \omega(t,\cdot) \subset \operatorname{spt} \omega^0 + B_{Lt}.$$

In other words, the vorticity is transported with finite speed. Equivalently, for any time $T > 0$ and radius R there exist $R_T > 0$ such that

$$\sup_{t\in[0,T]} \int_{B_R} |\omega(t,x)| dx = \int_{B_{R_T}} |\omega^0(x)| dx.$$

Lemma 9.1 allows us to re-write (3.13) with a change of variable $y \mapsto X(t,y)$ so that

$$v(s,x) = \int_{\mathbb{R}^2} K(x - X(s,t,y))\omega^0(y)dy \tag{3.14}$$

and

$$\begin{cases} \frac{dX}{ds}(s,t,x) = \int\limits_{\mathbb{R}^2} K(X(s,t,x) - X(s,t,y))\omega^0(y)dy, \\ X(t,t,0) = x. \end{cases} \tag{3.15}$$

Observe that any solution to (3.15) defines a velocity by (3.14) and a vorticity according to Lemma 9.1. For sufficiently smooth solutions, it turns out that the particle-trajectory formulation is equivalent to the Euler equation (3.3).

PROPOSITION 9.4. *Let $v^0(x)$ be a smooth velocity field satisfying $\operatorname{div} v^0 = 0$ and let $\omega^0 = \operatorname{curl} v^0$. Suppose that $X \in C^1([0,T]\times\mathbb{R}^2)$ is a solution to (3.15) on some time interval $[0,T]$. Let v be the associated velocity field given by (3.14). Then the integrodifferential equation (3.15) is equivalent to the 2D Euler equation (3.3) for sufficiently smooth solutions $(\omega, v) \in C^1([0,T]; C_c(\mathbb{R}^2)) \times C^1([0,T]\times\mathbb{R}^2)$.*

REMARK 9.5. *Proposition 9.4 gives a necessary and sufficient reduction from (3.3) to the ODE in (3.15), since it dictates that sufficiently regular solutions are associated to flows. This equivalence simplifies the proof of local in time existence of (3.3), since one can construct a solution to (3.3) assuming that nonlinear operator on the right hand side of (3.15) satisfies a Lipschitz property. One might ask if Propostion 9.4 holds for a weaker class of solutions with less regularity. In the following section we will introduce several formulations of weak solution show that this need not be the case.*

We now define smooth solutions to (3.3) with the coupling given by the Biot Savart law.

DEFINITION 9.6. *Let $(\omega^0, v^0) \in C_c(\mathbb{R}^2)\times C^1(\mathbb{R}^2)$ with $\omega^0 = \operatorname{curl} v^0$. We say the couple (ω, v) is a smooth solution to (3.3) in $[0,T)$ with initial data (ω^0, v^0), if*

(1) $(\omega, v) \in C^1([0,T); C_c(\mathbb{R}^2)) \times C^1([0,T)\times\mathbb{R}^2)$,
(2) $\operatorname{curl} v = \omega$,
(3) for all $t \in [0,T)$,

$$\omega(t,x) = \omega^0\Big(X(s=0,t,x)\Big), \qquad \textit{for all } t\in[0,T], \tag{3.16}$$

where $X(s,t,x) \in C^1([0,T)^2\times\mathbb{R}^2)$ is a solution to (3.15).

We now state the necessary and sufficient criterion for smooth solutions to (3.15) to exist for all time.

THEOREM 9.7 (Beale-Kato-Majda). *Let $N = 2$ or 3. Let $\omega^0 = \operatorname{curl} v^0$ be a compactly supported initial vorticity, with $\operatorname{div} v^0 = 0$ and $\omega^0 \in C^{0,\alpha}(\mathbb{R}^N)$ for some $\alpha \in (0,1)$. Let $\omega(t,\cdot)$ be a local in time smooth solution to the Euler equation. Suppose that for any $T>0$ there exists M such that*

$$\int_0^T ||\omega(t,\cdot)||_{L^\infty(\mathbb{R}^N)}dt \le M.$$

Then the corresponding particle trajectory $X(t,x)$ belongs to $C^1([0,\infty)\times\mathbb{R}^N)$, i.e. the solution exists globally in time.

We remark that this criterion implies global existence immediately two dimensions. This is because a vorticity associated to an L^∞ initial datum cannot become unbounded. Theorem (9.7) implies a continuation criterion for construction of a smooth solution to the ODE (3.15): a solution can be continued for as long as all quantities depending on $||\omega(t,\cdot)||_\infty$ remain bounded. Thus we have the following.

THEOREM 9.8 (Global existence of smooth solutions). *Suppose that* (ω^0, v^0) *is a compactly supported datum with* $\omega^0 = \operatorname{curl} v^0$, $\operatorname{div} v^0 = 0$, *and* $\omega^0 \in C^{0,\alpha}(\mathbb{R}^2)$ *for some* $\alpha \in (0,1)$. *Then there exists a unique volume-preserving particle trajectory* $X(t,x) \in C^1([0,\infty) \times \mathbb{R}^2)$ *solving* (3.15), *and hence a unique smooth solution* (ω, v) *to* (3.3).

REMARK 9.9. *To obtain a global smooth solution we require an additional Hölder regularity on the vorticity at initial time: it follows from* (3.16) *that the smooth solution inherits the* $C^{0,\alpha}$ *regularity of* ω^0 *at later times. For initial data* ω^0 *belonging to* $C^1_c(\mathbb{R}^2)$, *the smoothness persists.*

PROOF OF THEOREM 9.8. We outline the proof which can be found for instance in [12]. Let $\tilde{R} > 0$ be such that $\operatorname{spt} \omega^0 \subset B_{\tilde{R}}$. We verify that under the assumptions of the theorem, the right hand side of (3.15) is bounded and locally Lipschitz. Define for the purpose of this proof the modified Hölder norm

$$||f||_{C^{1,\alpha}(\mathbb{R}^2)} = |f(0)| + ||\nabla_x f||_{L^\infty(\mathbb{R}^2)} + ||\nabla_x f||_\alpha,$$

where we denote by $||\cdot||_\alpha$ the seminorm

$$||f||_\alpha = \sup_{x,y\in\mathbb{R}^2, x\neq y} \frac{|f(x)-f(y)|}{|x-y|^\alpha}.$$

Let

$$F(X(s,t,x)) = \int\limits_{\mathbb{R}^2} K(X(s,t,x) - X(s,t,y))\omega^0(y)dy.$$

Observe that F depends only on $X(s,t,x)$ and not s and $X(s,t,x)$ separately. Then it remains to prove that the autonomous operator $F : O_R \to B$ is bounded and Lipschitz, where

$$B = \{X : \mathbb{R}^2 \to \mathbb{R}^2 : \quad ||X(\cdot)||_{C^{1,\alpha}} < \infty\},$$

and $O_R \subset B$ denotes the open set

$$O_R = \left\{ X \in B : \inf_\alpha \det \nabla_x X(\cdot) > \frac{1}{2}, \;\; ||X(\cdot)||_{C^{1,\alpha}} \leq R \right\}.$$

Local existence and uniqueness of a particle trajectory solution to (3.15) follows from Picard Lindelöf (Theorem 4.1). We remark on the technical assumptions on O_R and B. B is a Banach algebra consisting of functions on $\mathbb{R}^2$ with Hölder continuous gradient. The condition $\det \nabla_x X(\cdot) > \frac{1}{2}$ ensures that the locally invertible mappings in O_R are invertible on $\mathbb{R}^2$ and allow the change of variable $x \mapsto X^{-1}(x)$. For any $X \in O_R$, we have the following inequalities:

$$\begin{aligned} &||(\nabla_x X)^{-1}||_\alpha \leq c||\nabla_x X||^3_\alpha, \\ &||X^{-1}||_{C^{1,\alpha}(\mathbb{R}^2)} \leq c||X||^3_{C^{1,\alpha}(\mathbb{R}^2)}, \quad \text{and} \\ &||\omega^0 \circ X^{-1}||_\alpha \leq c||\omega^0||_\alpha ||X||^{3\alpha}_{C^{1,\alpha}(\mathbb{R}^2)}. \end{aligned} \tag{3.17}$$

After a change of variable we can write

$$F \circ X(x) = \tilde{F} \circ X^{-1}(x),$$

where

$$\tilde{F}(x) = \int_{\mathbb{R}^2} K(x-y)[\omega^0(X^{-1}(y)) \det \nabla_x X^{-1}(y)]dy.$$

Applying Lemma 6.5 with $\varepsilon = 1$ and from the estimates in (3.17) it follows that F is bounded on O_R, since

$$\begin{aligned} ||\nabla_x[\tilde{F} \circ X^{-1}]||_\alpha &\leq c||[\omega^0 \circ X^{-1}] \det \nabla_x X^{-1}(\cdot)||_\alpha \\ &\leq c(R,\tilde{R})||\omega^0||_\alpha. \end{aligned} \tag{3.18}$$

Lipschitz continuity of F on $(O_R, ||\cdot||_{C^{1,\alpha}})$ follows from a similar estimate. Let $X, \bar{X} \in O_R$. Indeed the Hölder continuity of $\bar{X}$ and X implies that for any $z \in \mathbb{R}$,

$$\frac{d}{dz}F(X(x) + z\,\bar{X}(x))|_{z=0} = \int\limits_{\mathbb{R}^2} K(X(x) - X(y))\nabla_x \bar{X}(y)\omega^0(y)\,dy$$
$$+ \int_{\mathbb{R}^2} (\nabla_x K)(X(x) - X(y))[\bar{X}(x) - \bar{X}(y)]\nabla_x X(y)\omega^0(y)\,dy.$$

Observe that the last integral is well defined, since by changing variable one verifies the singularity of the kernel $\nabla_x K$ is cancelled by the Hölder regularity of $\bar{X}$. The two terms can then be estimated in the same way as (3.18). This eventually gives

$$||(\nabla_x F)(X) \cdot \bar{X}||_{C^{1,\alpha}(\mathbb{R}^2)} \leq c(R, \tilde{R})||\omega^0||_\alpha ||\bar{X}||_{C^{1,\alpha}(\mathbb{R}^2)},$$

so that $\nabla_x F$ is a bounded linear operator on O_R. Thus the ODE (3.15) has a unique solution $\tilde{X}(t,x) \in C^1([0,T_R); O_R)$ for every $R > 0$, and hence a smooth solution (ω, v) in $[0, T_R)$. Applying Lemma 9.1 gives that

$$\int_0^T ||\omega(t,\cdot)||_{L^\infty(\mathbb{R}^2)}dt = \int_0^T ||\omega^0||_{L^\infty(\mathbb{R}^2)}dt,$$

and by Theorem 9.7 the solution is global. □

10. Weak solutions

It is often the case in the physical setting that v, ω and p are not differentiable, so we need to consider the weak formulation.

DEFINITION 10.1. *Let $v^0 \in L^2_{\text{loc}}(\mathbb{R}^2)$. We say that $v \in L^\infty([0,T]; L^2_{\text{loc}}(\mathbb{R}^2))$ is a weak solution of the Euler velocity formulation (3.1) with initial datum v^0 if for all $\varphi \in C^1_c([0,T] \times \mathbb{R}^2)$ with $\operatorname{div}\varphi = 0$ there holds*

$$\int_0^T \int\limits_{\mathbb{R}^2} \varphi_t \cdot v + \nabla\varphi : v \otimes v\,dxdt + \int\limits_{\mathbb{R}^2} \varphi(0,x)v^0(x)dx = 0, \tag{3.19}$$

and v is divergence free in the sense of distributions.

We also consider a vorticity formulation that makes sense when vorticities are discontinuous but have sufficient integrability. If we consider test functions of the form $\varphi = -\nabla^\perp\phi$, one should in principle recover the weak formulation for (3.3) via the coupling (3.2). However, observe that if $v \in L^p$ with $p > 1$, Calderón Zygmund theory discussed in section 6.1 and Sobolev embedding give that $v \in L^{2p/(2-p)}$. In order for the product $v\omega$ to have well-defined local integral, one would require that $\omega \in L^p$ for $p \geq \frac{4}{3}$. This leads us to the second weak formulation for the vorticity.

DEFINITION 10.2. *Let $p \geq 4/3$. Given $\omega^0 \in L^1(\mathbb{R}^2) \cap L^p(\mathbb{R}^2)$, we say (v,ω) is a weak solution to the vorticity formulation on $[0,T]$ with initial datum $\omega^0(x)$ if*

(1) $\omega \in L^\infty([0,T]; L^1(\mathbb{R}^2) \cap L^p(\mathbb{R}^2))$;

(2) $v = K * w$,

(3) $\forall\varphi \in C^1([0,T]; C^1_c(\mathbb{R}^2))$ *there holds*

$$\int_0^T \int_{\mathbb{R}^2} \omega(t,x)(\partial_t\varphi(t,x) + v(t,x)\cdot\nabla\varphi(t,x))dxdt = \int_{\mathbb{R}^2} \varphi(T,x)\omega(T,x)dx - \int_{\mathbb{R}^2} \varphi(0,x)\omega^0(x)dx. \tag{3.20}$$

Weak solutions form a much larger class than classical solutions: indeed, every classical solution fulfills Definition 10.2, but weak solutions in general need not be C^1. Existence and uniqueness of weak solutions in a bounded domain with bounded initial datum was first proved by Yudovich, using a smooth approximation argument.

THEOREM 10.3. *Let $\omega^0 \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$. Then there exists a weak solution (ω, v) to the vorticity formulation in the sense of definition 10.2.*

PROOF. Let $\rho \in C_c^\infty(\mathbb{R}^2)$ be a positive mollifier with $\int_{\mathbb{R}^2} \rho = 1$, and denote by $\rho_n(x) = n^2\rho(\cdot/n)$, $\omega_0^n(x) = \rho_n(x) * \omega^0(x)$. Then by Theorem 9.8 we obtain a unique global smooth solution (v_n, ω_n) with $v_n = K * \omega_n$ to the Euler equation, which in particular satisfies the conditions of definition 10.2. The next step is to derive estimates from potential theory to get uniform bounds on the quantities $||\omega_n||_{L^\infty}+||\omega_n||_{L^1}$ and $||v_n||_{L^\infty}$. In fact, one has[1]

$$||v_n(t,\cdot)||_{L^\infty(\mathbb{R}^2)}\leq c(||\omega_n(t,\cdot)||_{L^\infty(\mathbb{R}^2)}+||\omega_n(t,\cdot)||_{L^1(\mathbb{R}^2)}) \leq c'(||\omega^0||_{L^\infty}+||\omega^0||_{L^1}), \tag{3.21}$$

and there exist $\omega(t,\cdot) \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ and $v(t,\cdot) = K(\cdot) * \omega(t,\cdot)$ such that for any $t \in [0,T]$ and $1 \leq p < \infty$, we can extract a subsequence

$$\begin{aligned} \omega_n(t,\cdot) &\rightharpoonup \omega(t,\cdot) \qquad \text{in } L^p(\mathbb{R}^2) - w; \\ v_n(t,\cdot) &\rightharpoonup^* v(t,\cdot) \qquad \text{in } L^\infty_{\mathrm{loc}}(\mathbb{R}^2) - w - *,\ n \to \infty. \end{aligned}$$

To prove strong compactness in space of the velocities, we use the Calderón Zygmund estimate

$$||Dv_n||_{L^\infty((0,T);L^p(\mathbb{R}^2))}\leq c_p||\omega_n||_{L^\infty((0,T);L^p(\mathbb{R}^2))}.$$

By (3.21), the right hand side is uniformly bounded for any $1 < p < \infty$. Thus v_n is uniformly bounded in $L^\infty([0,T]; W^{1,p}_{\mathrm{loc}}(\mathbb{R}^2))$ for any $1 < p < \infty$, and hence also in $L^q_{\mathrm{loc}}(\mathbb{R}^2)$, with $q = \frac{2p}{2-p} > 2$ by the Sobolev embedding. By Rellich's theorem we obtain that v_n is strongly precompact in $L^2_{\mathrm{loc}}(\mathbb{R}^2; L^\infty_{\mathrm{loc}}([0,T]) - w*)$. To prove strong compactness in time, we use the uniform control on the time derivative of v_n. Since v_n are smooth solutions, we have that for any $s > 2$,

$$\partial_t v_n \in L^\infty([0,T]; H^{-s}_{\mathrm{loc}}(\mathbb{R}^2)),$$

with uniform bounds. (See for instance the proof of Theorem 10.5.) For every $t \in [0,T]$, the following uniform in n estimate holds:

$$\sup_{\varphi\in H^s(\mathbb{R}^2),||\varphi||_{H^s}\leq 1} |\langle v_n(t+\tau,\cdot),\varphi\rangle - \langle v_n(t,\cdot),\varphi\rangle| \leq \tau \sup_{0<t\leq T}|\langle \partial_t v_n(t,\cdot),\varphi\rangle|\leq C\tau. \tag{3.22}$$

It follows from an Aubin Lions argument (see Theorem 29.2) that after extracting another subsequence, $v_n \to v$ strongly in $L^2_{\mathrm{loc}}([0,T] \times \mathbb{R}^2)$. It is straightforward to show that the limit (v, ω) is a weak solution. By the uniform bounds in (3.21), it follows from weak convergence of the product $\omega_n v_n$ that

$$\lim_{n\to\infty} \int_0^T \int_{\mathbb{R}^2} \omega_n(t,x)(\partial_t\varphi(t,x)+v_n(t,x)\cdot\nabla\varphi(t,x))dxdt = \int_0^T \int_{\mathbb{R}^2} \omega(t,x)(\partial_t\varphi(t,x)+v\cdot\nabla\varphi(t,x))dxdt. \tag{3.23}$$

□

We have proven Theorem 10.3 using the derivative bounds on the velocity arising from potential theory. This regularity setting allows us to conclude that the velocities converge strongly in L^2_{loc}, from which it follows immediately that the limit must satisfy the weak formulation (3.20). Because the approximations are smooth, they are classical solutions for which the vorticity is transported by a flow. However, it is far from obvious that the limiting velocity has a curl which is associated to a flow as in (3.16). In [74], the limiting vorticity is constructed by means of particle trajectories in (3.13). In particular, the limit vorticity is transported by a flow. The next question is whether different regularizations may produce some weak

[1] See [74] or Lemma 13.3 for a similar computation.

solutions which are not associated to flows. The next result states that in the setting of vorticities in $L^1 \cap L^\infty$, the velocity associated to the difference of any two solutions ω_1 and ω_2 decays sufficiently fast and has finite L^2 energy, which rules out this possibility.

THEOREM 10.4. *Let* $\omega^0 \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$. *Then the weak solution* $(\omega, v) \in L^\infty([0,T]; L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)) \times L^\infty([0,T]; L^\infty(\mathbb{R}^2))$ *is unique.*

PROOF. For simplicity we prove the theorem for solutions with compact support.
Step 1. Any weak solution $\omega \in L^1 \cap L^\infty$ with compactly supported initial datum $w_0(x)$ satisfies

$$\int_{\mathbb{R}^2} \omega(t,x)dx = \int_{\mathbb{R}^2} \omega^0(x)dx. \tag{3.24}$$

This follows since the corresponding velocity v is bounded in $L^\infty(\mathbb{R}^2)$. The support of the vorticity has finite propagation speed so that there is an increasing bounded function $R(t)$ such that $\operatorname{spt}\omega(t,\cdot) \subset \{x \in \mathbb{R}^2 : |x| \leq R(t)\}$. From the identity (3.20), we see that any test function $\varphi(t,x)$ identically 1 on $0 \leq t \leq T, |x| \leq R(t)$ gives zero on the left-hand side.
Step 2. Let $(\omega_j, v_j)_{j=1}^2$ denote two weak solutions with the same initial data $\operatorname{curl} v_j^0 = \omega^0 \in L_c^\infty(\mathbb{R}^2)$. The velocities solve the distributional formulation

$$\begin{cases} \partial_t v + v \cdot \nabla v = -\nabla p, \\ \operatorname{div} v = 0. \end{cases}$$

Since the support of w_j are contained in $B_{R_j(t)}$, we have the following expansion for v_j, whenever $|x| \geq 2R_j(t)$,

$$v_j(t,x) = \frac{c}{|x|} \int_{\mathbb{R}^2} \omega_j(t,y)dy + O(|x|^{-2}).$$

Then by (3.24) the integral terms in $v = v_1 - v_2$ cancel each other and $v(t,x) = O(|x|^{-2})$ for $|x| \geq 2\max_j R_j(T)$. This gives

$$E(t) = \int_{\mathbb{R}^2} |v(t,x)|^2 dx < \infty.$$

Combining this with the fact that v satisfies the distributional identity

$$\partial_t v + v_1 \cdot \nabla v + v \cdot \nabla v_2 = -\nabla(p_1 - p_2),$$

and taking the L^2 inner product of this equation with v, we may integrate by parts to obtain

$$\frac{1}{2}\frac{d}{dt}E(t) - \int_{\mathbb{R}^2} v^2 \nabla \cdot v_1 dx + \int_{\mathbb{R}^2} (v \cdot \nabla v_2) v dx = \int_{\mathbb{R}^2} (p_1 - p_2) \nabla \cdot v \, dx,$$

whence it is clear that only the first and third terms survive. Let $1 < p < \infty$. Using Hölder's inequality gives

$$\begin{aligned} \frac{d}{dt}E(t) &\leq 2\int_{\mathbb{R}^2} |v(t,x)|^2 |\nabla v_2(t,x)| dx \leq 2||\nabla v_2(t,\cdot)||_{L^p(\mathbb{R}^2)} \left(\int_{\mathbb{R}^2} |v(t,x)|^{2p/(p-1)} dx \right)^{(p-1)/p} \\ &\leq 2||\nabla v_2(t,\cdot)||_{L^p(\mathbb{R}^2)} \left(||v(t,\cdot)||_{L^\infty(\mathbb{R}^2)}^{2/(p-1)} \int_{\mathbb{R}^2} |v(t,x)|^2 dx \right)^{(p-1)/p}. \end{aligned}$$

Using the Hólder inequality and Calderón-Zygmund Theorem 6.4, one has for $2 < p < \infty$ the interpolated estimate

$$||\nabla v_2(t,\cdot)||_{L^p(\mathbb{R}^2)} \leq cp\, ||\omega^0||_{L^1(\mathbb{R}^2)}^{1/p} ||\omega^0||_{L^\infty(\mathbb{R}^2)}^{1-1/p}.$$

This implies

$$\frac{d}{dt}E(t) \leq MpE(t)^{1-1/p}, \tag{3.25}$$

for some constant M depending on $||\omega^0||_{L^\infty(\mathbb{R}^2)}$ and $||\omega^0||_{L^1(\mathbb{R}^2)}$.
Step 3. We conclude by the following argument. Since $E(0) = 0$, $E(t) = 0$ is a trivial solution to 3.25.

Though this inequality does not have unique solutions, it possesses the maximal solution $\bar{E}(t) = (Mt)^p$, and that any other solution $E(t)$ satisfies $E(t) \leq \bar{E}(t)$. We want to prove this maximal solution can be made small enough.
Define an interval $I_M = [0, T^*]$ where $T*$ is chosen small enough so that $T^* \leq \frac{1}{2M}$. Then for every $t \in I_M$,

$$E(t) \leq \left(\frac{1}{2}\right)^p,$$

which is arbitrarily small for sufficiently large p. This implies that $E(t) \equiv 0$ for every $t \in I_M$. Repeating this argument for a partition of $[0, T]$ we conclude that $E(t) = 0$ for all $t \in [0, T]$ and thus $v_1 = v_2$ almost everywhere.

□

10.1. Existence of weak solutions with vorticity in $L^1 \cap L^p$. The following classical result (see [12]) states that given a smooth approximating sequence of solutions (ω_n, v_n) to (3.1), a uniform bound on the kinetic energy guarantees strong convergence in L^1 of the velocity fields. The obvious question that arises is whether this convergence is strong enough to pass to the limit in the velocity formulation. It turns out to be sufficient if the vorticities possess an additional $L^p, p > 1$ control: in fact, the velocities converge strongly in L^2_{loc}.

THEOREM 10.5. *Let (v_n) be a family of smooth solutions to (3.1). Let $\omega_n = \operatorname{curl} v_n$, and assume that for any $R > 0$*

$$\max_{0 \leq t \leq T} \int_{\mathbb{R}^2} |\omega_n(t,x)| dx + \int_{B_R} |v_n(t,x)|^2 dx \leq C(R,T). \tag{3.26}$$

Then up to subsequences (v_n) is strongly compact in $L^1_{\text{loc}}([0,T] \times \mathbb{R}^2)$, and (ω_n) is weakly-$*$ compact in $L^\infty([0,T]; \mathcal{M}(\mathbb{R}^2))$, with limits (v, ω) satisfying

$$\max_{0 \leq t \leq T} \int_{B_R} |v(t,x)|^2 dx \leq C(R,T), \qquad \operatorname{div} v = 0,$$
$$\operatorname{curl} v = \omega.$$

PROOF. **Step 1.** From assumption (3.26), up to a subsequence there is $\omega \in L^\infty([0,T]; \mathcal{M}(\mathbb{R}^2))$ such that $\omega_n \rightharpoonup^* \omega$ weakly-$*$ in $\mathcal{M}(\mathbb{R}^2)$. By finiteness of the kinetic energy first show that for all $s \in \mathbb{N}$ with $s > 1$, for any cutoff $\phi \in C_c^\infty(\mathbb{R}^2)$, and $t_1, t_2 \in [0,T]$

$$||\phi v_n(t_1,\cdot) - \phi v_n(t_2,\cdot)||_{H^{-s-1}(\mathbb{R}^2)} \leq C|t_1 - t_2|. \tag{3.27}$$

Since v_n are smooth, divergence-free solutions we can estimate

$$\begin{aligned}
||\phi v_n(t_1,\cdot) - \phi v_n(t_2,\cdot)||_{H^{-s-1}(\mathbb{R}^2)} &= \left\|\phi \int_{t_1}^{t_2} \frac{\partial}{\partial s} v_n(s,\cdot) ds\right\|_{H^{-s-1}(\mathbb{R}^2)} \\
&\leq |t_2 - t_1| \sup_{0 \leq t \leq T} \left\|\phi \frac{\partial v_n}{\partial s}(s,\cdot)\right\|_{H^{-s-1}(\mathbb{R}^2)} \leq |t_2 - t_1| \sup_{0 \leq t \leq T} ||\phi(v_n \cdot \nabla v_n)(s,\cdot)||_{H^{-s-1}(\mathbb{R}^2)} \\
&\leq |t_2 - t_1| \big(||\operatorname{div}(\phi(v_n \otimes v_n))(s,\cdot)||_{H^{-s-1}(\mathbb{R}^2)} + (||(v_n \otimes v_n)(s,\cdot) : \nabla\phi||_{H^{-s-1}(\mathbb{R}^2)}\big) \\
&\leq |t_2 - t_1| ||\phi(v_n \otimes v_n)(s,\cdot)||_{H^{-s}(\mathbb{R}^2)} \leq |t_2 - t_1| ||\phi(v_n \otimes v_n)(s,\cdot)||_{L^1(\mathbb{R}^2)}.
\end{aligned} \tag{3.28}$$

It follows that for all $\phi \in C_c^\infty(\mathbb{R}^2)$, and $t_1, t_2 \in [0,T]$,

$$||\phi \omega_n(t_1,\cdot) - \phi \omega_n(t_2,\cdot)||_{H^{-s-2}(\mathbb{R}^2)} \leq C|t_1 - t_2|.$$

On the other hand, by the dual version of the Sobolev embedding we have, for all $s > 1$:

$$\sup_{0\le t\le T} ||\phi\omega_n(t,\cdot)||_{H^{-s}(\mathbb{R}^2)} \le C \sup_{0\le t\le T} ||\phi\omega_n(t,\cdot)||_{L^1(\mathbb{R}^2)} \le C(\phi). \tag{3.29}$$

Thus we can apply Aubin Lions Lemma to get existence of $\omega \in C([0,T]; H^{-s}_{\text{loc}}(\mathbb{R}^2))$ such that for any $s > 1$ and $\phi \in C_c^\infty(\mathbb{R}^2)$, one has up to a subsequence

$$\sup_{0\le t\le T} ||\phi\omega_n(t,\cdot) - \phi\omega(t,\cdot)||_{H^{-s}(\mathbb{R}^2)} \to 0. \tag{3.30}$$

Step 2. We show that v_n is a Cauchy sequence in L^1. Let $\rho \in C_c^\infty(\mathbb{R})$ be a radial cutoff function identical to one on B_1 and zero for $|x| > 2$. Set $\rho_\delta(x) = \rho(x/\delta)$. Then we estimate

$$\begin{aligned} ||v_n - v_m||_{L^1((0,T)\times\mathbb{R}^2)} &\le \int_{\mathbb{R}^2} ||\rho_\delta(x-y)K(x-y)(\omega_n-\omega_m)(t,y)||_{L^1((0,T)\times\mathbb{R}^2)} dy \\ &+ \int_{\mathbb{R}^2} ||(\rho_R-\rho_\delta)(x-y)K(x-y)(\omega_n-\omega_m)(t,y)||_{L^1((0,T)\times\mathbb{R}^2)} dy \\ &+ \int_{\mathbb{R}^2} ||(1-\rho_R)(x-y)K(x-y)(\omega_n-\omega_m)(t,y)||_{L^1((0,T)\times\mathbb{R}^2)} dy \\ &= I_1 + I_2 + I_3. \end{aligned}$$

By Young's inequality

$$\begin{aligned} I_1 &\le ||\rho_\delta K||_{L^1(\mathbb{R}^2)} ||\omega_n - \omega_m||_{L^1((0,T)\times\mathbb{R}^2)} \le c\delta, \text{ and} \\ I_3 &\le ||(1-\rho_R)K * (\omega_n-\omega_m)||_{L^\infty((0,T)\times\mathbb{R}^2)} \le c||(1-\rho_R)K||_{L^\infty(\mathbb{R}^2)} \le cR^{-1}. \end{aligned}$$

For I_2, we note that for fixed $x \in B_R$, $(\rho_R - \rho_\delta)K(x-\cdot) \in C_c^\infty(\mathbb{R}^2)$. Let $\phi = \rho_{2R}$ and $s > 1$. Then from (3.30) we get the following pointwise convergence:

$$\begin{aligned} &\int_{\mathbb{R}^2} |(\rho_R-\rho_\delta)(x-y)K(x-y)[\omega_n(t,y)-\omega_m(t,y)]| dy \\ &\le ||(\rho_R-\rho_\delta)K||_{H^s(\mathbb{R}^2)} ||\phi\omega_n - \phi\omega_m||_{L^\infty((0,T);H^{-s}(\mathbb{R}^2))} \to 0. \end{aligned}$$

Applying Dominated Convergence gives that $I_2 \to 0$ as $n, m \to \infty$. Choosing δ and R first such that $I_1 + I_3$ is small yields the result.

□

REMARK 10.6. *We remark that the assumption $v^0 \in L^2_{\text{loc}}(\mathbb{R}^2)$ is required in the estimate (3.28) in order to apply Aubin-Lions and deduce convergence of ω_n in $L^\infty((0,T); H^{-s}_{\text{loc}}(\mathbb{R}^2))$. The L^1 control on the vorticity gives only the M^2 bound on v_n in (3.10), therefore an approximation of solutions to the velocity formulation requires the L^2 energy estimate in (3.26). While this automatically implies that v_n is weakly precompact in L^2_{loc}, in general (3.26) does not imply that the sequence is strongly precompact, see for instance Example 11.2.1 in [12]. This is due to the fact that a pointwise converging, weakly convergent sequence in L^2 may still concentrate. However, concentrations may occur for sequences whose limit* still *satisfies (3.1), in spite of the lack of strong L^2_{loc} convergence: this is referred to as concentration-cancellation and has been studied in [28, 12]. This happens for instance if the vorticity is a measure with distinguished sign [24] and will be discussed in the next sections.*

We now use Theorem 10.5 to prove the following existence result. Given a uniform control in $L^1 \cap L^p$, $p > 1$ of the vorticity approximations, the corresponding velocities converge strongly in L^2_{loc}. In the weak velocity formulation, this allows for passage to the limit in the non-linear term $v \otimes v$. The crucial point is that the L^p control gives a control on the gradient of v (a singular integral of the vorticity) also in L^p, so that by the Sobolev embedding v belongs to a higher $L^{p'}$, $p' > 2$ space.

THEOREM 10.7. *(DiPerna Majda.) Let $(v^0, \omega^0) \in L^2_{\text{loc}}(\mathbb{R}^2) \times L^1(\mathbb{R}^2) \cap L^p(\mathbb{R}^2)$ for some $p > 1$, with $\omega^0 = \operatorname{curl} v^0$. Then there exists $(v, \omega) \in L^\infty([0,T); L^2_{\text{loc}}(\mathbb{R}^2)) \times L^\infty([0,T); L^1(\mathbb{R}^2) \cap L^p(\mathbb{R}^2))$, with $\omega = \operatorname{curl} v$, solving the weak velocity formulation* (3.19).

PROOF. We regularize v^0 with the standard mollifier $\rho \in C_c^\infty$, $\int \rho = 1$, $\rho \geq 1$, with $\rho_n = n^2 \rho(x/n)$. Let $v_n^0 = v^0 * \rho_n$. Then $\omega_n^0 = \operatorname{curl} v_n^0$. Let v_n be the unique smooth solution to (3.1) with initial datum v_n^0 and $\operatorname{curl} v_n = \omega_n$. For any $R > 0$, the solutions (ω_n, v_n) satisfy the uniform estimate

$$\max_{0 \leq t \leq T} \left(\int_{\mathbb{R}^2} |\omega_n| dx + \int_{\mathbb{R}^2} |\omega_n|^p dx + \int_{B_R} |v_n|^2 dx \right) \leq C_{R,T}. \tag{3.31}$$

By Theorem 10.5, there exists $v \in L^\infty((0,T); L^2_{\text{loc}}(\mathbb{R}^2))$ and a subsequence still denoted v_n such that $v_n \to v$ strongly in $L^1_{\text{loc}}((0,T) \times \mathbb{R}^2.)$ From 3.31, we have as well that $v_n \rightharpoonup^* v$ in $L^\infty((0,T); L^2_{\text{loc}}(\mathbb{R}^2)) - w*$. We show that this convergence is in fact strong. We have the following Calderón Zygmund estimate for the gradient of v. For every $t \in [0,T]$,

$$||\nabla v_n(t, \cdot)||_{L^p(\mathbb{R}^2)} \leq C_p ||\omega_n||_{L^p(\mathbb{R}^2)}. \tag{3.32}$$

We write

$$v_n = K_1 * \omega_n + K_2 * \omega_n. \tag{3.33}$$

where the truncations K_1 and K_2 are defined as $K_1(x) = K(x)\mathbf{1}_{B_1(0)}$ and $K_2(x) = K(x)\mathbf{1}_{B_1(0)^c}$. Since $K_1 \in L^1(\mathbb{R}^2), K_2 \in L^\infty(\mathbb{R}^2)$ and $\omega_n \in L^p(\mathbb{R}^2)$, we have by Young's inequality

$$||K_1 * \omega_n||_{L^p(\mathbb{R}^2)} \leq C_p. \tag{3.34}$$

and

$$||K_2 * \omega_n||_{L^\infty(\mathbb{R}^2)} \leq C. \tag{3.35}$$

From (3.34) and (3.35) it follows that v_n is uniformly bounded in $L^\infty((0,T); L^p_{\text{loc}}(\mathbb{R}^2))$. By the Sobolev embedding theorem with $p' = \frac{2p}{2-p}$ we have for any $R > 0$

$$\sup_{0 \leq t \leq T} ||v_n(t, \cdot)||_{L^{p'}(B_R)} \leq C. \tag{3.36}$$

Since $p' > 2$, this implies in particular the interpolation estimate for $0 < \lambda < 1$:

$$||v_n - v||_{L^2((0,T) \times B_R)} \leq C ||v_n - v||^{1-\lambda}_{L^1((0,T) \times B_R)} ||v_n - v||^\lambda_{L^{p'}((0,T) \times B_R)}. \tag{3.37}$$

By the convergence of v_n in $L^1_{\text{loc}}((0,T) \times \mathbb{R}^2)$, we deduce that $v_n \to v$ strongly in $L^2_{\text{loc}}((0,T) \times \mathbb{R}^2)$. Writing the difference of the nonlinear terms in (3.19) as

$$\nabla \varphi : v_n \otimes v_n - \nabla \varphi : v \otimes v = \nabla \varphi : (v_n - v) \otimes v_n + \nabla \varphi : v \otimes (v_n - v),$$

it is clear that

$$\lim_{n \to \infty} \int_0^T \int_{\mathbb{R}^2} \varphi_t \cdot v_n + \nabla \varphi : v_n \otimes v_n \, dxdt = \int_0^T \int_{\mathbb{R}^2} \varphi_t \cdot v + \nabla \varphi : v \otimes v \, dxdt.$$

$\square$

10.2. Existence of vortex sheets with distinguished sign. Given smooth approximating sequences (v_n, ω_n) without a uniform $L^1 \cap L^p$ control on the vorticity, it is not true in general that the velocities converge strongly in L^2_{loc}. The L^1 convergence does not forbid concentrations to occur in the limit of v_n. The natural question is whether measure vorticities still give rise to approximations whose limit is a solution to the velocity formulation. The answer is positive in case that the vorticity that does not change sign, since concentration-cancellations occur in the nonlinearity $v_n \otimes v_n$ so that the limit velocity satisfies the formulation (3.19). It is an outstanding open problem whether vorticities of mixed sign give rise to a velocity that solves (3.19). When a vorticity has two signs, nearby vortices may screen each other by canceling out higher order effects on more distant velocities, which in turn drive a stronger concentration of vorticity, leading to instability. In the case of vorticities with distinguished sign, all fluid elements spin in the same direction so that no screening effects at small scales can occur. In this section we summarize Delort's proof for existence of weak solutions associated to measure data. These correspond to initial vorticities in $\mathcal{M} \cap H^{-1}$ which do not change sign.

THEOREM 10.8. *(Delort.) Let (v^0, ω^0) be initial data such that $\omega^0 = \operatorname{curl} v^0$, with ω^0 belonging to $\mathcal{M}(\mathbb{R}^2) \cap H^{-1}(\mathbb{R}^2)$. Assume additionally that $\omega^0 \geq 0$ and v^0 has locally finite kinetic energy, i.e. that*

$$\forall R > 0 : \qquad \int_{B_R} |v^0|^2 dx \leq C(R).$$

Then there exists a weak solution $(v, \omega) \in L^\infty([0,T]; L^2_{\text{loc}}(\mathbb{R}^2)) \times L^\infty([0,T]; \mathcal{M}(\mathbb{R}^2))$, $\omega = \operatorname{curl} v$, associated to (v^0, ω^0), solving the weak velocity formulation (3.19).

PROOF. **Step 1.** We assume for simplicity that ω^0 has compact support. We regularize v^0 with the standard mollifier $\rho \in C_c^\infty$, $\int \rho = 1$, $\rho \geq 1$, with $\rho_n = n^2 \rho(x/n)$. Let $v_n^0 = v^0 * \rho_n$. Then $\omega_n^0 = \operatorname{curl} v_n^0$, $\omega_n^0 \geq 0$, and for any $R > 0$,

$$\int_{B_R} |v_n^0|^2 dx + \int |\omega_n^0| dx \leq C(R^2).$$

We have that $v_n^0 \to v^0$ strongly in $L^2_{\text{loc}}(\mathbb{R}^2)$ and $\omega_n^0 \to \omega^0$ in $H_c^{-1}(\mathbb{R}^2)$. Let v_n be the unique smooth solution to (3.1) with $\operatorname{curl} v_n = \omega_n$. The solutions (ω_n, v_n) satisfy the uniform estimate

$$\max_{0 \leq t \leq T} \left(\int_{\mathbb{R}^2} |\omega_n| dx + \int_{B_R} |v_n|^2 dx \right) \leq C_{R,T}. \tag{3.38}$$

By Theorem 10.5, there exists $v \in L^\infty([0,T], L^2_{\text{loc}}(\mathbb{R}^2))$ with curl $v = \omega \in L^\infty([0,T]; \mathcal{M}(\mathbb{R}^2))$ and subsequences (v_n, ω_n) such that $v_n \rightharpoonup v$ in $L^2_{\text{loc}}(\mathbb{R}^2) - w$ and $\omega_n \rightharpoonup^* \omega$ in $L^\infty([0,T]; \mathcal{M}(\mathbb{R}^2)) - w*$, with the additional convergence $v_n \to v$ in $L^1_{\text{loc}}([0,T] \times \mathbb{R}^2) - s$.

Step 2. We exploit the special non-linear structure of equation (3.19). Since we test with functions $\varphi \in C_c^\infty([0,T] \times \mathbb{R}^2)$ with $\operatorname{div} \varphi = 0$, it is equivalent to substitute, for $\eta \in C_c^\infty([0,T] \times \mathbb{R}^2)$,

$$\varphi = \nabla^\perp \eta = (-\eta_{x_2}, \eta_{x_1}).$$

If we substitute this in (3.19) the equation reduces to

$$\int_0^T \int_{\mathbb{R}^2} \eta_{tx_2} v_1 - \eta_{tx_1} v_2 \, dxdt - \int_0^T \int_{\mathbb{R}^2} \eta_{x_1 x_2}(v_2^2 - v_1^2) + (\eta_{x_2 x_2} - \eta_{x_1 x_1})(v_1 v_2) dxdt = 0. \tag{3.39}$$

Thus we need only to show that the quantities $v_{2,n}^2 - v_{1,n}^2$ and $v_{1,n} v_{2,n}$ converge weakly to $v_2^2 - v_1^2$ and $v_1 v_2$ respectively. (Observe that these quantities are antisymmetric and their convergences do not imply distributional convergence of $|v_n|^2$.) Since the vorticities ω_n do not satisfy a uniform L^p control, $p > 1$, it is a priori possible that concentration occurs in the limit of v_n. Using the fact that the vorticity is of fixed sign we show that concentration-cancellation occurs in the limit. Since $\omega_n^0(x) \geq 0$ it follows that $\omega(t,x) \geq 0$

for $t \geq 0$. By a rotation of $\pi/4$, $v_{2,n}^2 - v_{1,n}^2$ becomes $v_n^1 v_n^2$. By rotational invariance of (3.19) it suffices to prove that for every $\varphi \in C_c^\infty((0,T) \times \mathbb{R}^2)$,

$$\lim_{n\to\infty} \int_0^T \int_{\mathbb{R}^2} v_n^1 v_n^2(t,x)\varphi(t,x)dxdt = \int_0^T \int_{\mathbb{R}^2} v^1 v^2(t,x)\varphi(t,x)dxdt. \tag{3.40}$$

Step 3. We define the vorticity maximal function as

$$|\omega_n|(B_R) = \int_{B_R(x)} |\omega_n(y)|dy.$$

We show that if this decays sufficiently fast as $R \to 0$, the quadratic quantity $v_n^1 v_n^2$ can be controlled in the limit. We recall without proof an estimate from [24] which gives a decay rate on the circulation for L^1 vorticities. All vorticity sequences which are uniformly bounded in L^1, with associated velocities uniformly in L^2_{loc}, satisfy the following decay estimate on the vorticity maximal function.

Lemma 10.9. *Assume that (ω_n^0, v_n^0) are smooth, $\omega_n^0 \geq 0$ has compact support and (ω_n^0, v_n^0) satisfies the estimate in (3.38). Them the vorticity maximal function has the following decay rate for any $\delta < 1/2$:*

$$\max_{0\leq t\leq T, x_0\in\mathbb{R}^2} \int_{B_\delta(x_0)} \omega_n(t,x)dx \leq C_T|\log(2\delta)|^{-1/2}, \tag{3.41}$$

where C_T depends only on T, the quantity in (3.38) and the support of ω_n^0. In particular, the limiting vorticity ω satisfies the same estimate: for $\delta < 1/4$ there holds

$$\max_{0\leq t\leq T, x_0\in\mathbb{R}^2} \int_{B_\delta(x_0)} d\omega(t,x) \leq C_T|\log(4\delta)|^{-1/2}. \tag{3.42}$$

Next we use the Biot-Savart law to re-write the expression (3.40)

$$\int_{\mathbb{R}^2} v_n^1 v_n^2(t,x)\varphi(t,x)dx = \iint_{\mathbb{R}^2\times\mathbb{R}^2} F_\varphi(x,y)\omega_n(t,x)\omega_n(t,y)dxdy,$$

where $F_\varphi(x,y)$ is the function on $\mathbb{R}^4$ given by

$$F_\varphi(x,y) = \frac{-1}{4\pi^2}\frac{\partial^2}{\partial_{x_1}\partial_{x_2}} \int_{\mathbb{R}^2} \log|x-z|\log|y-z|\varphi(z)dz.$$

One can check that $F_\varphi(x,y)$ is a bounded function on $\mathbb{R}^2 \times \mathbb{R}^2$, continuous outside the diagonal, and tends to zero at infinity. (See [24].) We test (3.40) on test functions of the form $\phi(t)\psi(x)$, with $\phi \in C_c^\infty(\mathbb{R}^+), \psi \in C_c^\infty(\mathbb{R}^2)$. Let $\rho(x) \in C_c^\infty$ be a fixed radial positive cutoff function equal to 1 on $B_1(0)$ and identically zero for $|x|> 2$. Fix $0 < \delta < 1$. Then we write

$$\int_0^T \int_{\mathbb{R}^2} v_n^1 v_n^2(t,x)\phi(t)\psi(x)dxdt \tag{3.43}$$

$$= \int_0^T \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \phi(t)\left[1-\rho\left(\frac{|x-y|}{\delta}\right)\right]F_\psi(x,y)\omega_n(t,x)\omega_n(t,y)dxdydt \tag{3.44}$$

$$+ \int_0^T \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \phi(t)\rho\left(\frac{|x-y|}{\delta}\right)F_\psi(x,y)\omega_n(t,x)\omega_n(t,y)dxdydt \tag{3.45}$$

$$= I_1 + I_2. \tag{3.46}$$

For fixed δ and $t \in [0,T]$, we have that $\phi(t)\left[1-\rho\left(\frac{|x-y|}{\delta}\right)\right]F_\varphi(x,y) \in C_0(\mathbb{R}^4)$. We have by step 1 that $\omega_n(t,x)\otimes\omega_n(t,y) \rightharpoonup \omega(t,x)\otimes\omega(t,y)$ in $L^\infty((0,T);\mathcal{M}(\mathbb{R}^2)\times\mathcal{M}(\mathbb{R}^2))$. Thus we may pass to the limit in

$$\lim_{n\to\infty} I_1 = \int_0^T \phi(t)dt \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\left[1-\rho\left(\frac{|x-y|}{\delta}\right)\right]F_\varphi(x,y)d\omega(t,x)d\omega(t,y). \tag{3.47}$$

Step 4. We apply Lemma 10.9 to estimate I_2. By non-negativity of ω_n and the estimate in (3.38),

$$|I_2| \leq C(\varphi, ||\phi||_\infty)\int_0^T \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\rho\left(\frac{|x-y|}{\delta}\right)F_\varphi(x,y)\omega_n(t,x)\omega_n(t,y)\,dxdydt \tag{3.48}$$

$$\leq C(\varphi, ||\phi||_\infty, T)\left[\log\left(\frac{1}{2\delta}\right)\right]^{-1/2}\int_0^T\int_{\mathbb{R}^2}\omega_n(t,y)\,dydt \tag{3.49}$$

$$\leq C(\varphi, ||\phi||_\infty, T)\left[\log\left(\frac{1}{2\delta}\right)\right]^{-1/2}. \tag{3.50}$$

Lemma 10.9 and the convergence of ω_n imply that the same estimate holds when ω_n is replaced by ω. Since $\delta > 0$ can be chosen arbitrarily small, we conclude the argument. $\square$

REMARK 10.10. *We remark that equi-integrability of an approximating sequence of vorticities would imply also that concentration-cancellation occurs for the corresponding approximated sequence of velocities, since it implies a decay of the last integral in* (3.48)*. The point here is that in spite of the lack of equi-integrability of the measure vorticity, a signed measure has maximal function decaying to zero at a logarithmic rate. This corresponds to the circulation of a positive vorticity in a ball of radius δ going to zero as $\delta \to 0$. Indeed the decay of the vorticity maximal function in Lemma 10.9 is false for measures without distinguished sign.*

The solution constructed in [**24**] is global in time. However, the uniqueness of a weak vorticity solution with measure data is still an open question, in spite of numerical evidence that suggests the contrary [**56**]. The uniqueness result presented in Theorem 10.4 is only known for initial vorticities in L^∞ or very close to L^∞ [**71**]. By contrast, a pioneering work [**62**] showed existence of a nontrivial weak solution in L^2 to the velocity formulation, compactly supported in space and time.

CHAPTER 4

The Vlasov Poisson Equation

We introduce the Cauchy problem for the classical Vlasov-Poisson system

$$\partial_t f + v \cdot \nabla_x f + E \cdot \nabla_v f = 0, \tag{4.1}$$

$$f(0,x,v) = f^0(x,v)\,, \tag{4.2}$$

where $f(t,x,v) \geq 0$ is the distribution function, $t \geq 0$, $x, v \in \mathbb{R}^N$, and

$$E(t,x) = -\nabla_x U(t,x) \tag{4.3}$$

is the force field. The potential U satisfies the Poisson equation

$$-\Delta_x U = \omega(\rho(t,x) - \rho_b(x)), \tag{4.4}$$

with $\omega = +1$ for the electrostatic (repulsive) case, $\omega = -1$ for the gravitational (attractive) case, and where the density ρ of particles is defined through

$$\rho(t,x) = \int_{\mathbb{R}^N} f(t,x,v)dv, \tag{4.5}$$

and $\rho_b \geq 0$, $\rho_b \in L^1(\mathbb{R}^N)$ is an autonomous background density. Since we are in the whole space, the relation (4.3) together with the Poisson equation (4.4) yield the equivalent relation

$$E(t,x) = \frac{\omega}{|S^{N-1}|}\frac{x}{|x|^N} * (\rho(t,x) - \rho_b(x)), \tag{4.6}$$

where the convolution is in the space variable.

In this chapter we review classical solutions to the initial value problem 4.1. Local existence was first due to [7]. This provides not only a unique local existence result for compactly supported initial data but also establishes a criterion condition, which shows how a solution may cease to exist after finite time. As long as the 'maximal velocity' of a solution, defined as

$$P(t) = \sup\{|v| \colon (x,v) \in \operatorname{spt} f(t), 0 \leq t < T\}$$

is bounded, the solution continues to exist. Two simultaneous proofs of global existence of smooth solutions for both the repulsive and attractive cases were proved independently in [55] and [44]. The proof of [55] will be studied here. In this we show that this maximal velocity grows at most polynomially in time.

11. Conservation of mass and energy

We recall some basic identities related to the VP system. Integrating (4.1) with respect to v and noting that the last term is in v-divergence form we obtain the *local conservation of mass*

$$\partial_t \rho(t,x) + \operatorname{div}_x(J(t,x)) = 0, \tag{4.7}$$

where the current J is defined by

$$J(t,x) = \int_{\mathbb{R}^N} v f(t,x,v)\,dv. \tag{4.8}$$

Integrating again with respect to x, we obtain the global conservation of mass

$$\frac{d}{dt}\iint\limits_{\mathbb{R}^N\times\mathbb{R}^N} f(t,x,v)dxdv = \frac{d}{dt}\int\limits_{\mathbb{R}^N}\rho(t,x)dx = 0. \tag{4.9}$$

Multiplying (4.1) by $\frac{|v|^2}{2}$, integrating in x and v, we get after integration by parts in v

$$\frac{d}{dt}\iint\limits_{\mathbb{R}^N\times\mathbb{R}^N}\frac{|v|^2}{2}f(t,x,v)dxdv - \iint\limits_{\mathbb{R}^N\times\mathbb{R}^N} E\cdot vf\,dxdv = 0. \tag{4.10}$$

Using (4.6) and (4.7), one has

$$\partial_t E + \sum_{k=1}^{N}\frac{\partial}{\partial x_k}\left(\frac{\omega}{|S^{N-1}|}\frac{x}{|x|^N} * J_k\right) = 0, \tag{4.11}$$

or in other words

$$\partial_t E = \omega\nabla_x(-\Delta_x)^{-1}\mathrm{div}_x J, \tag{4.12}$$

which means that $\partial_t E$ is the gradient component of $-\omega J$, by the Helmholtz-Weyl projection. We deduce that

$$\int_{\mathbb{R}^N} E\cdot\partial_t E\,dx = -\omega\int_{\mathbb{R}^N} E\cdot J\,dx. \tag{4.13}$$

Using (4.8) in (4.10), we obtain the *conservation of energy*

$$\frac{d}{dt}\left[\iint\limits_{\mathbb{R}^N\times\mathbb{R}^N}\frac{|v|^2}{2}f(t,x,v)dxdv + \frac{\omega}{2}\int\limits_{\mathbb{R}^N}|E(t,x)|^2dx\right] = 0\,. \tag{4.14}$$

The total conserved energy is the sum of the kinetic energy and of the potential energy multiplied by the factor $\omega = \pm 1$. In particular, in the electrostatic case $\omega = +1$ we deduce from (4.14) a uniform bound in time on both the kinetic and the potential energy, assuming that they are finite initially. In the gravitational case $\omega = -1$ it is not possible to exclude that the individual terms of the kinetic and potential energy become unbounded in finite time, while the sum remains constant. Indeed it is known that it does not happen in three dimensions as soon as f^0 is sufficiently integrable, but we cannot exclude this a priori for only L^1 solutions: see Proposition 13.7.

Remark 11.1. *Note that the assumption $E^0 \in L^2$ is satisfied in 3 dimensions as soon as $\rho^0 - \rho_b \in L^{6/5}$ (See Lemma 13.3). However, in one or two dimensions, for E^0 to be in L^2 it is necessary that $\int(\rho^0-\rho_b)dx = 0$, as is can be seen in Fourier variable. It is also necessary that $\rho^0 - \rho_b$ has enough decay at infinity. Thus in one or two dimensions, in order to have finite energy, ρ_b cannot be zero identically.*

12. Regularity of the velocity field

Recall that the Vlasov Poisson equation is a transport equation with the vector field

$$b(t,x,v) = (b_1,b_2)(t,x,v) = (v, E(t,x)) = (v, -\omega\nabla_x\Delta^{-1}\rho). \tag{4.15}$$

We use the results from section 6.1, chapter 1 to give estimates for the regularity and growth of the electric field $E(t,x)$ as defined in (4.3) and establish bounds on the vector field b.

12.1. Local integrability. If $\rho-\rho_b \in L^p$ with $p>1$, we have the estimates from the Hardy-Littlewood-Sobolev inequality:

$$\begin{aligned}||\nabla(-\Delta)^{-1}(\rho(t,x)-\rho_b(x))||_{L^q(\mathbb{R}^N)} &\leq \left|\left|\frac{1}{|S^{N-1}|}|x|^{1-N} * |\rho(t,x)-\rho_b(x)|\right|\right|_{L^q(\mathbb{R}^N)} \\ &\leq c_N||\rho(t,x)-\rho_b(x)||_{L^p(\mathbb{R}^N)},\end{aligned} \tag{4.16}$$

where $q=\frac{Np}{N-p}$. If $\rho-\rho_b \in L^1(\mathbb{R}^N)$, we have the weak inequality:

$$|||\nabla(-\Delta)^{-1}(\rho(t,x)-\rho_b(x))|||_{M^{\frac{N}{N-1}}(\mathbb{R}^N)} \leq c_N||\rho(t,x)-\rho_b(x)||_{L^1(\mathbb{R}^N)}. \tag{4.17}$$

It follows that

$$|||E|||_{L^\infty((0,T);M^{\frac{N}{N-1}}(\mathbb{R}^N))} \leq c_N||\rho-\rho_b||_{L^\infty((0,T);L^1(\mathbb{R}^N))}. \tag{4.18}$$

12.2. Spatial regularity. Since $b_1=v$ is smooth, the only non-trivial gradient is the one of $b_2=E$, indeed the differential matrix of the vector field is given by

$$Db=\begin{pmatrix} D_x b_1 & D_v b_1 \\ D_x b_2 & D_v b_2\end{pmatrix}=\begin{pmatrix} 0 & \mathrm{Id} \\ D_x E & 0\end{pmatrix}. \tag{4.19}$$

We have by (4.15)

$$(D_xE)_{ij} \equiv \partial_{x_j}E_i=-\omega\partial^2_{x_ix_j}((-\Delta_x)^{-1}(\rho-\rho_b)) \quad \text{for } 1\leq i,j\leq N. \tag{4.20}$$

It is well-known that the operator $\partial^2_{x_ix_j}(-\Delta_x)^{-1}$ is a singular integral operator. Its kernel is

$$K_{ij}(x)=-\frac{1}{|S^{N-1}|}\frac{\partial}{\partial_{x_j}}\left(\frac{x_i}{|x|^N}\right), \tag{4.21}$$

it is given outside of the origin by

$$K_{ij}(x)=\frac{1}{|S^{N-1}|}\left(N\frac{x_ix_j}{|x|^{N+2}}-\frac{\delta_{ij}}{|x|^N}\right), \qquad \text{for } x\in\mathbb{R}^N\backslash\{0\}. \tag{4.22}$$

The kernel satisfies the conditions of definition 6.3 and $\hat{K}_{ij}(\xi)=-\xi_i\xi_j/|\xi|^2$. Thus (each component of) D_xE is a singular integral of the density. From (4.19) it is clear that b is divergence free.

12.3. Time regularity. According to (4.12), ∂_tE is a singular integral of the current J defined by (4.8). Using the bounds available for solutions with finite mass and energy

$$||f(t,\cdot)||_{L^1(\mathbb{R}^N_x\times\mathbb{R}^N_v)}, \iint |v|^2 f(t,x,v)dxdv \leq C, \tag{4.23}$$

and since $|v|\leq 1+|v|^2$, we get that $J\in L^\infty((0,T);L^1(\mathbb{R}^N_x))$. Hence ∂_tE is a singular integral of an $L^\infty((0,T);L^1(\mathbb{R}^N_x))$ function. In particular,

$$\partial_tE \in L^\infty((0,T);\mathcal{S}'(\mathbb{R}^N)). \tag{4.24}$$

13. Smooth solutions in 3D

We first consider classical solutions of the Vlasov Poisson system (4.1),(4.2), (4.3), where all derivatives exist in the classical sense. Because of remark 11.1, we can from now on assume that $\rho_b=0$. Two simultaneous proofs of global existence for both the repulsive and attractive case of classical existence were given by Pfaffelmoser [55], and by Lions and Perthame [44].

13.1. Local existence. In order to construct a smooth solution we consider the characteristic flows. The following lemma is from Cauchy Lipschitz theory. Let $\rho = \rho_f$ be defined as in (4.5) with $N = 3$. Define

$$U(t,x) := \omega \int_{\mathbb{R}^3} \frac{\rho(t,y)}{|x-y|} dy, \tag{4.25}$$

provided that the integral is finite. We let $Z(s,t,x,v) = (X,V)(s,t,x,v)$ be the characteristic flow on $\mathbb{R}^N_x \times \mathbb{R}^N_v$ associated to the vector field b given by (4.15). The following lemma is a consequence of Picard Lindelöf.

Lemma 13.1. *Let $I \subset \mathbb{R}$ and $E \equiv -\nabla_x U \in C^1(I \times \mathbb{R}^N_x)$ be continuously differentiable with respect to x and bounded on $\bar{I} \times \mathbb{R}^N_x$ for every $\bar{I} \Subset I$. Let $b = (v, -\omega E)$. Then for every $t \in I$ and $z = (x,v) \in \mathbb{R}^N_x \times \mathbb{R}^N_v$ there exists a unique solution $Z(s,t,x,v) = (X,V)(s,t,x,v)$ of the characteristic*

$$\dot{\mathcal{Z}}(s,t,x,v) = b(s, \mathcal{Z}(s,t,x,v)) \tag{4.26}$$

$$\mathcal{Z}(t,t,x,v) = (x,v). \tag{4.27}$$

The flow Z satisfies the following

1. *$Z : I \times I \times \mathbb{R}^N_x \times \mathbb{R}^N_v \to \mathbb{R}^N_x \times \mathbb{R}^N_v$ belongs to $C^1(I \times I \times \mathbb{R}^N_x \times \mathbb{R}^N_v)$,*
2. *For all $s,t \in I$ the mapping $Z(s,t,\cdot) : \mathbb{R}^N_x \times \mathbb{R}^N_v \to \mathbb{R}^N_x \times \mathbb{R}^N_v$ is a C^1-diffeomorphism with inverse $Z(t,s,\cdot)$ and is measure-preserving.*

Moreover, for every $f^0 \in C^1(\mathbb{R}^N_x \times \mathbb{R}^N_v)$ the function

$$f(t,x,v) = f^0(Z(0,t,x,v)), \qquad t \in I, \tag{4.28}$$

is the unique solution of (4.1) in $C^1(I \times \mathbb{R}^N_x \times \mathbb{R}^N_v)$ with initial datum f^0. Moreover for every $1 \le p \le \infty$ and $t \in I$,

$$||f(t,\cdot)||_{L^p(\mathbb{R}^N_x \times \mathbb{R}^N_v)} = ||f^0||_{L^p(\mathbb{R}^N_x \times \mathbb{R}^N_v)}.$$

Using Lemma 13.1 we can define precisely the notion of a classical solution to (4.1).

Definition 13.2. *We say $f : I \times \mathbb{R}^3 \times \mathbb{R}^3 \to [0,\infty)$ is a smooth solution of the Vlasov Poisson equation in some interval of times $I \subset \mathbb{R}$ with initial datum f^0 if:*

1. *f is given by the formula in (4.28),*
2. *The induced density $\rho(t,x)$ and potential $U(t,x)$ associated to f by (4.5) and (4.4) belong to $C^1(I \times \mathbb{R}^3)$, with $U(t,\cdot)$ belonging to $C^2(\mathbb{R}^3)$, and U is given by the convolution in (4.25),*
3. *for every $\bar{I} \Subset I$, $E(t,x)$ belongs to $L^\infty(\bar{I} \times \mathbb{R}^3)$,*
4. *The functions f, E, ρ satisfy (4.1)-(4.6) with $\rho_b = 0$.*

We recall some estimates from potential theory, which improve the integrability bounds we derived in section 12 when the densities are sufficiently smooth. We drop for the moment the dependence in time to derive spatial bounds on the potential and force field. The properties of the singular kernel in (4.21) combined with the regularity of ρ give better bounds on the electric field E and its spatial gradient. These quantities are important to control in order to construct a local solution.

Lemma 13.3. *Let $\rho = \rho_f$ be defined as in (4.5) with $N = 3$ and assume $\rho \in C^1_c(\mathbb{R}^3)$. Define*

$$U(x) := \omega \int_{\mathbb{R}^3} \frac{\rho(y)}{|x-y|} dy.$$

Then the following holds.

(1) *If $\rho \in C^1_c(\mathbb{R}^3)$ then U is the unique solution of*

$$\Delta U = 4\pi\omega\rho, \qquad \lim_{|x|\to\infty} U(x) = 0,$$

in $C^2(\mathbb{R}^3)$. Moreoever

$$\nabla_x U(x) = \omega \int_{\mathbb{R}^3} \frac{x-y}{|x-y|^3}\rho(y)dy,$$
$$U(x) = O(|x|^{-1}),\ \nabla_x U(x) = O(|x|^{-2}), \qquad \text{as } |x|\to\infty.$$

(2) *For any $1 \le p < 3$,*

$$||\nabla_x U||_{L^\infty(\mathbb{R}^3_x)} \le c_p ||\rho||^{p/3}_{L^p(\mathbb{R}^3_x)} ||\rho||^{1-p/3}_{L^\infty(\mathbb{R}^3_x)}.$$

(3) *If $\rho \in L^{6/5}(\mathbb{R}^3_x)$ then $U \in L^6(\mathbb{R}^3_x)$ and $\nabla_x U \in L^2(\mathbb{R}^3_x)$, with a control on the norm given by*

$$||U||_{L^6(\mathbb{R}^3_x)} \le c||\rho||_{L^{6/5}(\mathbb{R}^3_x)}, \qquad ||\nabla_x U||_{L^2(\mathbb{R}^3_x)} \le c||\rho||^2_{L^{6/5}(\mathbb{R}^3_x)}.$$

(4) *The second order weak derivative of U, which we denote $\nabla_x E(x)$, satisfies, for any $0 < r \le R$,*

$$||\nabla_x E||_{L^\infty(\mathbb{R}^3_x)} \le c[R^{-3}||\rho||_{L^1(\mathbb{R}^3_x)} + r||\nabla_x \rho||_{L^\infty(\mathbb{R}^3_x)} + (1+\log(R/r))||\rho||_{L^\infty(\mathbb{R}^3_x)}],$$

with $c > 0$ independent of $\rho, R,$ and r, and

$$||\nabla_x E||_{L^\infty(\mathbb{R}^3_x)} \le c[1 + ||\rho||_{L^\infty(\mathbb{R}^3_x)}(1+\log^+(||\nabla_x\rho||_{L^\infty(\mathbb{R}^3_x)}) + ||\rho||_{L^1(\mathbb{R}^3_x)}].$$

PROOF. We omit the proof of (1) which can be found for instance in [33]. (3) follows directly from the weak Young inequality (Theorem 5.15) and Hardy Littlewood Sobolev inequality with $d = 3$. (4) is due to the cancellation properties of the singular kernel in (4.21). If $\rho \in C^1_c$ then we have

$$\partial_{x_i}\partial_{x_j} U(x) = \frac{4\pi}{3}\omega\rho(t,x)\delta_{ij} - \omega \int\limits_{|x-y|\le r} K_{ij}(x-y)[\rho(y)-\rho(x)]dy \tag{4.29}$$

$$- \omega \int\limits_{|x-y|>r} \rho(y)K_{ij}(x-y)dy, \tag{4.30}$$

where K is the kernel in (4.22). Observe that the C^1 regularity of ρ kills one power of the singularity of (4.29) so the former integral exists. We let $0 < r \le R$ and compute

$$\begin{aligned} |\partial_{x_i}\partial_{x_j} U(x)| &\le \frac{4\pi}{3}||\rho||_\infty + ||\nabla\rho||_\infty \int\limits_{|x-y|\le r} \frac{4}{|x-y|^2}dy \\ &+ \int\limits_{r<|x-y|\le R} \frac{4}{|x-y|^3}|\rho(y)|dy + \int_{|x-y|>R} \frac{4}{|x-y|^3}|\rho(y)|dy \\ &\le c\left(||\rho||_\infty + r\,||\nabla\rho||_\infty + ||\rho||_\infty \log(R/r) + R^{-3}||\rho||_{L^1}\right). \end{aligned}$$

The last statement follows from choosing $R = 1$ and $r = 1/||\nabla\rho||_\infty$ if $||\nabla\rho||_\infty \ge 1$, otherwise $r = 1$. For (2) we use Hölder's inequality to estimate for any $R > 0$, and any p' such that $\frac{1}{p} + \frac{1}{p'} = 1$,

$$\begin{aligned} |\nabla_x U| &\le \int\limits_{|x-y|\le R} \frac{|\rho(y)|}{|x-y|^2}dy + \int\limits_{|x-y|>R} \frac{|\rho(y)|}{|x-y|^2}dy \\ &\le 4\pi R||\rho||_\infty + \left(\frac{4\pi}{2p'-3}R^{3-2p'}\right)^{\frac{1}{p'}} ||\rho||_{L^p}. \end{aligned}$$

Since $1 \le p < 3$, $p' > 3/2$. Optimizing in R gives the result. □

We next recall a local classical result first proved in [7] that gives unique local existence for compactly supported initial data. A continuation criterion is introduced: a classical solution can be extended as long as its maximal velocity support or its density remain bounded, and thus a possible breakdown by shock formation is ruled out. The quantity

$$P(t) = \sup\{|v|\colon (x,v) \in \operatorname{spt} f(t), 0 \leq t < T\},$$

is used to control $||\rho(t)||_\infty, ||\rho(t)||_{L^1}, ||\nabla_x U||_\infty$ locally in time, and thus implies a control on the derivatives $\nabla_x Z$ of the flow.

THEOREM 13.4. *Let $f^0 \in C^1_c(\mathbb{R}^N_x \times \mathbb{R}^N_v)$ be non-negative. Then there exists a smooth solution f of (4.1) on some time interval $[0,T)$ with initial datum f^0. For all $t \in [0,T)$ the function $f(t,\cdot)$ is compactly supported and non-negative. If $T > 0$ is chosen maximal and if*

$$P(t) = \sup\{|v|\colon (x,v) \in \operatorname{spt} f(t), 0 \leq t < T\} < \infty,$$

or

$$\sup\{\rho(t,x) : 0 \leq t < T, x \in \mathbb{R}^3\} < \infty,$$

then the solution is global, i.e. $T = \infty$.

PROOF. We sketch the proof which relies on an iteration scheme. We show that the quantities necessary to continue a classical solution can remain bounded as long as $P(t)$ does. Let $z = (x,v) \in \mathbb{R}^6$. We fix R^0 and P^0 so that $f^0(x,v) = 0$ if $|x|\geq R^0$ or $|v|\geq P^0$. Then the zero-th iterate is defined as $f^0(t,z) = f^0(z)$, for $t \geq 0$, $z \in \mathbb{R}^N_x \times \mathbb{R}^N_v$. One defines the $n+1$-th iterate by setting

$$f_{n+1}(t,z) := f^0(Z_n(0,t,z)),$$

where $Z_n(s,t,z) = (X_n,V_n)(s,t,x,v)$ is the solution of the characteristic system

$$\dot{x} = v, \qquad \dot{v} = -\nabla_x U_n(s,x),$$

with $Z_n(t,t,z) = z$, $\rho_n = \rho_{f_n}$, and $U_n = U_{\rho_n}$. Let $P^0(t) := P^0$ and for each n set

$$P_{n+1}(t) := \sup\{|V_n(s,0,z)|\colon z \in \operatorname{spt} f^0, s \in [0,T]\} \tag{4.31}$$

By Lemma 13.1 and 13.3, the sequence f_n satisfies the following bounds for all $t \geq 0$:

$$f_n \in C^1([0,\infty) \times \mathbb{R}^N_x \times \mathbb{R}^N_v),\ ||f(t,\cdot)||_\infty = ||f^0||_\infty,\ ||f(t,\cdot)||_{L^1} = ||f^0||_{L^1},$$
$$\text{and } f_n(t,x,v) = 0 \text{ if } |x|\geq R_0 + \int_0^t P_n(s)ds \text{ or } |v|\geq P_n(t).$$

One has also $\rho_n \in C^1([0,\infty) \times \mathbb{R}^3)$ with

$$||\rho_n(t,\cdot)||_{L^1} = ||f^0||_{L^1}, \qquad ||\rho_n(t,\cdot)||_\infty \leq \frac{4\pi}{3}||f^0||_\infty P^3_n(t), \qquad \forall t \geq 0. \tag{4.32}$$

This implies $\nabla_x U_n \in C^1([0,\infty) \times \mathbb{R}^3)$ with the bound given by Lemma 13.3

$$||\nabla_x U_n(t,\cdot)||_\infty \leq C(f^0) P^2_n(t), \tag{4.33}$$

with $C(f^0)$ depending on $||f^0||_1$ and $||f^0||_\infty$. (4.33) turns out to be the crucial quantity in bounding $P_n(t)$ and the length of the time interval on which the iterates converge will depend on $C(f^0)$. Let $P : [0,\delta] \to (0,\infty)$ denote the maximal solution of the integral equation

$$\mathcal{P}(t) = P^0 + C(f^0)\int_0^t \mathcal{P}^2(s)ds,$$

where $\delta := \left(P^0 C(f^0)\right)^{-1}$ and $0 \leq t < \delta$. From the characteristics one has for any $0 \leq s \leq t < \delta$ and $z \in \operatorname{spt} f^0$,

$$|V_n(s,0,z)| \leq |v| + \int_0^s ||\nabla_x U_n(\tau,\cdot)||_\infty d\tau \leq P^0 + C(f^0)\int_0^s P^2_n(\tau)d\tau \leq P(t).$$

This together with (4.31) implies inductively that $P_n(t) \leq P(t)$, which establishes a local in time control on $P_n(t)$ and hence also on the density and electric field. To continue the smooth solution we show that all derivatives are under control as long as P is. Fix $0 < \delta_0 < \delta$. We show the iterates converge uniformly on any compact $[0, \delta_0]$. To control $\nabla_x \rho_n$ we estimate with

$$|\nabla_x \rho_{n+1}(t,x)| \leq \int_{|v| \leq P(t)} |\nabla_x [f^0(Z_n(0,t,x,v))]| dv \leq C(f^0) ||\nabla_x Z_n(0,t,\cdot)||_\infty.$$

Differentiating the characteristic system Z_n with respect to x one can apply a Gronwall argument to $\nabla_x Z_n$, which is then bounded by an exponential in time integral of $\nabla_x E_n$. But this implies that

$$||\nabla_x \rho_{n+1}(t,\cdot)||_\infty \leq C \exp\left(\int_0^t ||\nabla_x E_n(\tau,\cdot)||_\infty d\tau\right). \tag{4.34}$$

The vital step here is the estimate in Lemma 13.8 (4), wherein $||\nabla_x E_n||_\infty$ depends only logarithmically on $||\nabla_x \rho_n||_\infty$, so that (4.34) and (4.32) imply that

$$||\nabla_x E_{n+1}(t,\cdot)||_\infty \leq c \left(\int_0^t ||\nabla_x E_n(\tau,\cdot)||_\infty d\tau\right). \tag{4.35}$$

Induction gives the bound

$$||\nabla_x E_{n+1}(t,\cdot)||_\infty \leq C e^{Ct}.$$

This closes the estimate

$$||\nabla_x \rho_{n+1}(t,\cdot)||_\infty + ||\nabla_x E_{n+1}(t,\cdot)||_\infty \leq C, \tag{4.36}$$

for a sufficiently large C, not depending on $t \in [0, \delta_0]$ or n. (4.36) implies the Gronwall estimate

$$|Z_n(s) - Z_{n-1}(s)| \leq c \int_s^t ||\nabla_x U_n(\tau,\cdot) - \nabla_x U_{n-1}(\tau,\cdot)||_\infty d\tau. \tag{4.37}$$

The bound from (4.37) yields that the sequence f_n is uniformly Cauchy and converges to some function $f \in C_c([0,T) \times \mathbb{R}^N_x \times \mathbb{R}^N_v)$, since the supports of ρ_n and f_n are uniformly bounded in n. This follows from the estimates

$$\begin{aligned}
&|f_{n+1}(t,z) - f_n(t,z)| \leq c|Z_n(0,t,z) - Z_{n-1}(0,t,z)| \\
&\leq c \int_0^t ||\nabla_x U_n(\tau,\cdot) - \nabla_x U_{n-1}(\tau,\cdot)||_\infty d\tau \\
&\leq c \int_0^t ||\rho_n(\tau,\cdot) - \rho_{n-1}(\tau,\cdot)||_\infty^{2/3} ||\rho_n(\tau,\cdot) - \rho_{n-1}(\tau,\cdot)||_{L^1}^{1/3} d\tau \\
&\leq c \int_0^t ||f_n(\tau,\cdot) - f_{n-1}(\tau,\cdot)||_\infty d\tau.
\end{aligned} \tag{4.38}$$

The uniform limit f has the properties

$$f(t,x,v) = 0 \text{ if } |x| \geq R^0 + \int_0^t P(s) ds \text{ or } |v| \geq P(t), \tag{4.39}$$

and $\rho_n \to \rho \equiv \rho_f$, $U_n \to U \equiv U_f$ uniformly. To show f is a classical solution, we need to show that

$$U,\ \nabla_x U,\ \nabla_x E \in C([0,\delta_0] \times \mathbb{R}^3).$$

This follows from Lemma 13.3, since for any $m, n \in \mathbb{N}$, and for any $0 < r \leq R$,

$$\begin{aligned}
||\nabla_x E_n(t,\cdot) - \nabla_x E_m(t,\cdot)||_\infty \leq\ & c\,[R^{-3} ||\rho_n(t,\cdot) - \rho_m(t,\cdot)||_{L^1} + r ||\nabla_x \rho_n(t,\cdot) - \nabla_x \rho_m(t,\cdot)||_\infty \\
&+ (1 + \log(R/r)) ||\rho_n(t,\cdot) - \rho_m(t,\cdot)||_\infty],
\end{aligned}$$

and

$$||\nabla_x U_n(t,\cdot) - \nabla_x U_m(t,\cdot)||_\infty \leq c_p ||\rho_n(t,\cdot) - \rho_m(t,\cdot)||_{L^1}^{1/3} ||\rho_n(t,\cdot) - \rho_m(t,\cdot)||_\infty^{2/3}.$$

Observe that for any $0 < r \leq R$ the sequences $\nabla_x U_n$ and $\nabla_x E_n$ are uniformly Cauchy. (Since f has compact support the L^1 difference of ρ_n can be estimated by the L^∞ difference, which converges to zero.) The L^∞ difference of the derivatives of ρ_n can be bounded by a constant, but are compensated for by the factor r in front which can be chosen arbitrarily small. This allows to conclude that the sequence Z_n is precompact in $C^1([0,\delta_0] \times [0,\delta_0] \times \mathbb{R}_x^N \times \mathbb{R}_v^N)$, converging uniformly to the characteristic flow Z associated to (v, E). Then the solution f belongs to $C^1([0,\delta_0] \times \mathbb{R}_x^N \times \mathbb{R}_v^N)$ and $f(t,x,v) = \lim_{n\to\infty} f^0(Z_n(0,t,z)) = f^0(Z(0,t,z))$. Lastly, we prove the solution is global by contradiction. Suppose instead that $f \in C^1([0,T) \times \mathbb{R}_x^N \times \mathbb{R}_v^N)$ is the maximal solution obtained by the previous construction. Assume that $P(t)$ is finite but $T < \infty$. By Lemma 13.1, $||f(t,\cdot)||_\infty = ||f^0||_\infty$ and $||f(t,\cdot)||_{L^1} = ||f^0||_{L^1}$ for all $0 \leq t < T$. Using the procedure above for the new initial value problem starting at $t_0 = t$ with datum $f(t^0)$, if t is sufficently close to T then one can extend the solution beyond T. This follows since $C(f(t^0)) = C(f^0)$ and the integral equation

$$\mathcal{P}(\tau) = P^0 + C(f(t^0)) \int_{t_0}^{\tau} \mathcal{P}^2(s)ds$$

has a maximal solution on some interval $I = [t^0, t^0 + \delta']$ of length δ' independent of t^0. Since f vanishes for $|v| > P(t)$, the iterates P_n will be bounded by P as before on I. Repeating the previous estimates on I shows that the solution must exist here. We remark that the a priori bound on ρ implies a bound on $\nabla_x U$ and hence on P as well. □

Remark 13.5. *Following the construction one can check that the solution f is unique. If f and g are two solutions with the same initial datum then the estimates for the iterates $f_n - f_{n-1}$ can be repeated for $f - g$ to give*

$$||f(t) - g(t)||_\infty \leq C \int_0^t ||f(s) - g(s)||_\infty ds.$$

Remark 13.6. *The condition that f^0 is compactly supported can be relaxed for data with sufficient decay at infinity. Because of this assumption, the solution constructed in the theorem satisfies stronger bounds than required by definition 13.2, since Z measure preserving implies that f stays compactly supported for all time.*

13.2. Energy estimates. Recall that the problem with the conservation of energy is that in the gravitational case $\omega = -1$ the potential energy does not have a definite sign so the individual terms may become unbounded in finite time although their sum is constant. In the repulsive case $\omega = +1$ both terms are clearly bounded. The conservation of energy plays a crucial role in stability analysis of approximating solutions, and a rather surprising result given by Horst [38] states that in *both* gravitational and repulsive cases, the potential and kinetic energies for smooth solutions remain bounded. The reason is that both the spatial density ρ_f and the field ∇U_f can be bounded by the kinetic energy of f, which is a second order moment in velocity of f, while ρ_f is a zero order moment. This will be seen in the following result.

Proposition 13.7. *Let f be a classical solution to (4.1) on $[0,T)$ with spatial density ρ. Then for all $t \in [0,T)$, $\rho(t,\cdot) \in L^{5/3}(\mathbb{R}_x^3)$, and both the kinetic and potential energies are (individually) bounded:*

$$\frac{1}{2} \iint |v|^2 f(t,x,v)dvdx, \ \frac{1}{2}\int |\nabla_x U(t,x)|^2 dx \leq C$$

$$||\rho(t,\cdot)||_{L^{5/3}(\mathbb{R}_x^3)} \leq C ||f(t,\cdot)||_{L^\infty(\mathbb{R}_x^3 \times \mathbb{R}_v^3)}^{2/5},$$

where C depends only on $||f^0||_{L^1}$ and $||f^0||_{L^\infty}$ and its kinetic energy.

In order to prove this energy estimate we first derive a general result on the $k-th$ order moments of an integrable function. This allows us to prove that the spatial density ρ, a zero order moment of f, is bounded in an appropriately scaled norm by the kinetic energy, which is a second order moment (in velocity) of f. For $k \geq 0$ and for a non-negative, measurable function $f : \mathbb{R}^3_x \times \mathbb{R}^3_v \to [0, \infty)$ we denote the $k-th$ order moment density by

$$m_k(f)(x) := \int_{\mathbb{R}^3} |v|^k f(x,v) dv$$

and the $k-th$ order moment in velocity of f by

$$M_k(f) := \int_{\mathbb{R}^3} m_k(f)(x) dx = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |v|^k f(x,v) dv dx.$$

LEMMA 13.8. *Let $1 \leq p, q \leq \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$, $0 \leq k' \leq k \leq \infty$ and set*

$$r := \frac{k + 3/q}{k' + 3/q + (k-k')/p}.$$

If $f \in L^p(\mathbb{R}^3_x \times \mathbb{R}^3_v)$ is non-negative and $M_k(f) < \infty$ then $m_{k'}(f) \in L^r(\mathbb{R}^3_x)$ and there is $c = c(k, k', p) > 0$ such that

$$||m_{k'}(f)||_{L^r(\mathbb{R}^3_x)} \leq c||f||^{(k-k')/(k+3/q)}_{L^p(\mathbb{R}^3_x \times \mathbb{R}^3_v)} M_k(f)^{(k'+3/q)/(k+3/q)}.$$

PROOF. We split the integral on $\mathbb{R}^3_v$ into sets corresponding to small and large velocities and optimize with respect to the chosen split parameter. For any $R > 0$, we can write

$$\begin{aligned} m_{k'}(f)(x) &\leq \int_{|v| \leq R} |v|^{k'} f(x,v) dv + \int_{|v|>R} |v|^{k'} f(x,v) dv \\ &\leq ||f(x,\cdot)||_{L^p(\mathbb{R}^3_v)} \left(\int_{|v| \leq R} |v|^{k'q} dv \right)^{1/q} + R^{k'-k} \int_{|v|>R} |v|^k f(x,v) dv \\ &\leq c||f(x,\cdot)||_{L^p(\mathbb{R}^3_v)} R^{k'+3/q} + R^{k'-k} m_k(f)(x). \end{aligned}$$

Choosing $R = [m_k(f)(x)/||f(x,\cdot)||_{L^p(\mathbb{R}^3_v)}]^{1/(k+3/q)}$, we minimize the right hand side with respect to R. This gives the estimate

$$m_{k'}(f)(x) \leq c(||f(x,\cdot)||_{L^p(\mathbb{R}^3_v)})^{(k-k')/(k+3/q)} (m_k(f)(x))^{(k-k')/(k+3/q)}.$$

Raising this to the power r and integrating over $\mathbb{R}^3_x$ we obtain the estimate of the lemma by applying Hölder's inequality. $\square$

PROOF OF PROPOSITION 13.7. The case $\omega = +1$ is obvious, since the kinetic and potential energy are both non-negative and bounded. By Lemma 13.8 with $k = 2, k' = 0, p = \infty, q = 1, r = 5/3$, we have

$$||\rho(t,\cdot)||_{L^{5/3}(\mathbb{R}^3_x)} \leq c||f(t,\cdot)||^{2/5}_{L^\infty(\mathbb{R}^3_x \times \mathbb{R}^3_v)} M_2(f(t,\cdot))^{3/5} := c||f(t,\cdot)||^{2/5}_{L^\infty(\mathbb{R}^3_x \times \mathbb{R}^3_v)} E_k(f(t))^{3/5}.$$

For the case $\omega = -1$ we apply Lemma 13.3 part (3) and Lemma 13.8 with $k = 2, k' = 0, p = 9/7, r = 6/5$, to get

$$|E_p(f(t))| := ||\nabla U(t,\cdot)||^2_{L^2(\mathbb{R}^3_x)} \leq c||\rho(t,\cdot)||^2_{L^{6/5}(\mathbb{R}^3_x)} \leq c||f(t,\cdot)||^{3/2}_{L^{9/7}(\mathbb{R}^3_x \times \mathbb{R}^3_v)} M_2(f(t))^{1/2}. \tag{4.40}$$

$$= cE_k(f(t))^{1/2}. \tag{4.41}$$

By the conservation of energy this implies the kinetic energy is also bounded:

$$E_k(f(t)) - cE_k(f(t))^{1/2} \leq E_k(f(t)) + E_p(f(t)) \leq E_k(f^0).$$

$\square$

13.3. Global existence. We use Theorem 13.4 to prove global existence, first proved for general smooth data by Lions and Perthame [44] and then Pfaffelmoser [55]. The former uses a less strenuous bound on the third order moments $M_3(f)$ in order to bound the maximal interval of existence $P(t)$. Indeed we have from Lemma 13.3 with $p = \frac{5}{3}$, Proposition 13.7 and (4.32) that

$$||\nabla_x U(t,\cdot)||_\infty \leq c||\rho(t,\cdot)||_\infty^{4/9} \leq cP(t)^{4/3},$$

which implies the Gronwall estimate

$$P(t) \leq P(0) + \int_0^t ||\nabla_x U(s,\cdot)||_\infty ds \leq P(0) + c\int_0^t P^{4/3}(s)ds.$$

This bound is improved with an additional estimate for $M_3(f(t))$ by splitting the integral over x and then over v to obtain a global bound for $P(t)$. The a priori energy estimates in Proposition 13.7 combined with compactness properties of the operator Δ^{-1} are used to prove global existence [44]. These solutions are not known to be unique nor are they known to satisfy the conservation laws. On the other hand, the latter proof [55] avoids an estimate from Lemma 13.8 and uses a more elegant Lagrangian estimate along a characteristic to measure the increase of velocity. In this method one fixes a characteristic (X, V) along which the increase in velocity during the time interval $[t-\Delta, t]$ is estimated. The integral is split onto three domains of (x, v) concurrently. Rather than estimating the maximum value of $|\nabla_x U|$, the integral of this quantity is considered on a small time interval. The aim is to understand the total effect which one particle (the source particle) has on another (the target particle) on a given short interval. The first set is where velocities are bounded, either absolutely or with respect to the target particle, the second, along which velocities are unbounded, *and* wherein the particles in the integral are close to the target particle so that the singularity of the kernel $x/|x|^3$ is strong, and the third, the 'ugliest' set which contains the complement. On this set the time integral is exploited in a delicate way in order to bound it with the kinetic energy.

THEOREM 13.9. *(Pfaffelmoser.) Let $f^0 \in C_c^1(\mathbb{R}_x^N \times \mathbb{R}_v^N)$ be non-negative. Then there exists a global classical solution f of (4.1) with initial datum f^0.*

PROOF. **Step 1.** Let f be the solution, and $[0, T)$ be the (positive) maximal interval of existence provided by Theorem 13.4. The following arguments will also apply backwards in time. Let

$$P(t) = \max\{|v| \colon (x, v) \in \operatorname{spt} f(s), 0 \leq s \leq t\}.$$

We prove that this is bounded. We fix a characteristic $(X, V)(t)$ with $(X, V)(0) \in \operatorname{spt} f^0$, and let $0 \leq \Delta \leq t < T$. We estimate the increase in velocity along the single fixed characteristic $X(s)$:

$$|V(t) - V(t-\Delta)| \leq \int_{t-\Delta}^{t} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{f(s,y,w)}{|y - X(s)|^2} dwdyds. \tag{4.42}$$

Changing variable $y \mapsto X(s,t,x,v), w \mapsto V(s,t,x,v)$, and noting that f is constant along flow lines, (4.42) becomes

$$|V(t) - V(t-\Delta)| \leq \int_{t-\Delta}^{t} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{f(t,x,v)}{|X(s,t,x,v) - X(s)|^2} dvdxds. \tag{4.43}$$

Fix parameters $0 < p < P(t)$ and $r > 0$ to be specified later. We split the integration domain into the following three sets, the "good", the "bad", and the "ugly": a set where the velocities are bounded, either relatively or absolutely, a set where velocities are large and the particle in the integral is close to the singularity, and a final set in which we can bound the integral with the kinetic energy.

$$
\begin{aligned}
G =&\{(s,x,v)\in[t-\Delta,t]\times\mathbb{R}^N_x\times\mathbb{R}^N_v : |v|\le p\}\\
&\cup\{(s,x,v)\in[t-\Delta,t]\times\mathbb{R}^N_x\times\mathbb{R}^N_v : |v-V(t)|\le p\},\\
B =&\{(s,x,v)\in[t-\Delta,t]\times\mathbb{R}^N_x\times\mathbb{R}^N_v : |v|> p\}\\
&\cap\{(s,x,v)\in[t-\Delta,t]\times\mathbb{R}^N_x\times\mathbb{R}^N_v : |v-V(t)|> p\}\\
&\cap\Big[\{(s,x,v)\in[t-\Delta,t]\times\mathbb{R}^N_x\times\mathbb{R}^N_v : |X(s,t,x,v)-X(s)|\le r|v|^{-3}\}\\
&\cup\{(s,x,v)\in[t-\Delta,t]\times\mathbb{R}^N_x\times\mathbb{R}^N_v : |X(s,t,x,v)-X(s)|\le r|v-V(t)|^{-3}\}\Big],\\
U =&\{(s,x,v)\in[t-\Delta,t]\times\mathbb{R}^N_x\times\mathbb{R}^N_v : |v|> p\}\\
&\cap\{(s,x,v)\in[t-\Delta,t]\times\mathbb{R}^N_x\times\mathbb{R}^N_v : |v-V(t)|> p\}\\
&\cap\{(s,x,v)\in[t-\Delta,t]\times\mathbb{R}^N_x\times\mathbb{R}^N_v : |X(s,t,x,v)-X(s)|> r|v|^{-3}\}\\
&\cap\{(s,x,v)\in[t-\Delta,t]\times\mathbb{R}^N_x\times\mathbb{R}^N_v : |X(s,t,x,v)-X(s)|> r|v-V(t)|^{-3}\}.
\end{aligned}
\tag{4.44}
$$

G is a good set because it is not too large. On B both the velocity of the source particle and its velocity relative to the target particle are large. At the same time the distance between the two particles is small in comparison to one of these velocities. On U, the time integral is important: the target particle cannot remain for long in region where the density is very large or where the distance to the source particle is too small, due to its high velocity.

Step 2. We choose a length of the interval $[t-\Delta,t]$ such that the velocities do not change much. By Lemma 13.3 (2) and Proposition 13.7 there is $c>0$ so that

$$||\nabla_x U(t,\cdot)||_\infty\le c||\rho(t,\cdot)||_{L^{5/3}(\mathbb{R}^N_x)}||\rho(t,\cdot)||_\infty^{4/9}\le cP^{4/3}(t). \tag{4.45}$$

We fix an increment in time $\Delta=\Delta(t)$.

$$\Delta := \min\left\{t,\frac{p}{4cP(t)^{4/3}}\right\},$$

then

$$|V(s,t,x,v)-v|\le \Delta cP(t)^{4/3}\le\frac{1}{4}p,\qquad \forall s\in[t-\Delta,t],x,v\in\mathbb{R}^6. \tag{4.46}$$

Step 3. For $(s,x,v)\in G$ we have after changing variables back $X(s,t,x,v)\mapsto y, V(s,t,x,v)\mapsto w$ that

$$\int_G\frac{f(t,x,v)}{|X(s,t,x,v)-X(s)|^2}dvdxds\le\int_{t-\Delta}^{t}\int_{\mathbb{R}^3}\int_{|w|<2p\,\cup\,|w-V(s)|<2p}\frac{f(s,y,w)}{|y-X(s)|^2}dwdyds \tag{4.47}$$

If we let

$$\rho'(s,y):=\int_{|w|<2p\,\cup\,|w-V(s)|<2p}f(s,y,w)dw\le Cp^3, \tag{4.48}$$

and $||\rho'(s,\cdot)||_{L^{5/3}}\le||\rho(s,\cdot)||_{L^{5/3}}\le c$. We can estimate using (4.45)

$$
\begin{aligned}
\int_{t-\Delta}^{t}\int_{\mathbb{R}^3}\int_{|w|<2p\,\cup\,|w-V(s)|<2p}\frac{f(s,y,w)}{|y-X(s)|^2}dwdyds&\le\int_{t-\Delta}^{t}\int_{\mathbb{R}^3}\frac{\rho'(s,y)}{|y-X(s)|^2}dyds\\
&\le Cp^{4/3}\Delta.
\end{aligned}
\tag{4.49}
$$

Step 4. For $(s,x,v)\in B$ we have from (4.46) the following

$$[p/2<|w|<2|v|]\cap[p/2<|w-V(s)|<2|v-V(t)|]$$
$$\cap\left[|y-X(s)|<8r|w|^{-3}\cup|y-X(s)|<8r|w-V(s)|^{-3}\right].$$

On the other hand on the domain of integration we have $|w|\leq P(t)|$ and $|w-V(s)|\leq 2P(t)$. Since $||f(s,\cdot)||_\infty=||f^0||_\infty$ we can change variable and bound the following:

$$\begin{aligned}&\int_G \frac{f(t,x,v)}{|X(s,t,x,v)-X(s)|^2}dvdxds\\
&\leq \int\limits_{t-\Delta}^{t}\ \int\limits_{p/2<|w|\leq P(t)}\ \int\limits_{|y-X(s)|<8r|w|^{-3}} \frac{f(s,y,w)}{|y-X(s)|^2}dydwds\\
&+\int\limits_{t-\Delta}^{t}\ \int\limits_{p/2<|w-V(s)|\leq 2P(t)}\ \int\limits_{|y-X(s)|<8r|w-V(s)|^{-3}} \frac{f(s,y,w)}{|y-X(s)|^2}dydwds\\
&\leq Cr\log\left(\frac{4P(t)}{p}\right)\Delta.\end{aligned}$$

Step 5. To estimate the integral on U we integrate with respect to time first and try to bound $|X(s,t,x,v)-X(s)|$ from below linearly in time. Let $(x,v)\in\mathbb{R}^6$ with $|v-V(t)|>p$. We define

$$d(s)=X(s,t,x,v)-X(s),\qquad s\in[t-\Delta,t].$$

We compute the Taylor expansion of d around a minimal point $s_0\in[t-\Delta,t]$:

$$|d(s_0)|=\min\{|d(s)|: t-\Delta\leq s\leq t\}.$$

Let

$$\bar{d}(s)=d(s_0)+(s-s_0)\dot{d}(s_0),\qquad s\in[t-\Delta,t].$$

Then

$$d(s_0)=\bar{d}(s_0),\ \dot{d}(s_0)=\dot{\bar{d}}(s_0) \tag{4.50}$$

and so

$$|\ddot{d}(s)-\ddot{\bar{d}}(s)|=|\dot{V}(s,t,x,v)-\dot{V}(s)|\leq 2||\nabla_x U(s)||_\infty\leq 2cP(t)^{4/3}.$$

After computing the second order Taylor expansion of $d(s)-\bar{d}(s)$ around s_0 and using (4.50) we get

$$|d(s)-\bar{d}(s)|\leq cP(t)^{4/3}(s-s_0)^2\leq cP(t)^{4/3}\Delta|s-s_0| \tag{4.51}$$

$$\leq\frac{1}{4}p|s-s_0|<\frac{1}{4}|v-V(t)||s-s_0|. \tag{4.52}$$

On the other hand by (4.46)

$$|\dot{d}(s_0)|=|V(s_0,t,x,v)-V(s_0)|\geq|v-V(t)|-p/2>1/2|v-V(t)|.$$

By definition of s_0, we compute the Taylor expansion of $|d(s)|^2$ around s_0;

$$0\leq\frac{1}{2}(|d(s)|^2-|d(s_0)|^2)=(s-s_0)d(s_0)\cdot\dot{d}(s_0)+O((s-s_0)^2),$$

so that

$$(s-s_0)d(s_0)\cdot\dot{d}(s_0)\geq 0.$$

Hence for all $s\in[t-\Delta,t]$, we have that

$$|\bar{d}(s)|^2\geq\frac{1}{4}|v-V(t)|^2|s-s_0|^2.$$

Together with (4.51) this gives

$$|d(s)| \geq \frac{1}{4}|v - V(t)||s - s_0|, \qquad \forall s \in [t-\Delta, t] \tag{4.53}$$

and $(x, v) \in B$. Next we define

$$\sigma_1(\xi) = \begin{cases} \xi^{-2}, & \xi > r|v|^{-3} \\ (r|v|^{-3})^{-2}, & \xi \leq r|v|^{-3}, \end{cases} \tag{4.54}$$

and

$$\sigma_2(\xi) = \begin{cases} \xi^{-2}, & \xi > r|v - V(t)|^{-3} \\ (r|v - V(t)|^{-3})^{-2}, & \xi \leq r|v - V(t)|^{-3}. \end{cases} \tag{4.55}$$

From the definition of U we have that σ_i are non-increasing and using (4.53) we get

$$|d(s)|^{-2}\mathbf{1}_U(s, x, v) \leq \sigma_i(|d(s)|) \leq \sigma_i\left(\frac{1}{4}|v - V(t)||s - s_0|\right), \qquad \forall s \in [t-\Delta, t],$$

for $i = 1, 2$. □

Step 6. We can now estimate the time integral on U. Calculating the integral $\int_0^\infty \sigma_i(\xi)d\xi$ for $i = 1, 2$ we get

$$\begin{aligned} \int_{t-\Delta}^{t} |d(s)|^{-2}\mathbf{1}_U(s, x, v)ds &\leq 8|v - V(t)|^{-1}\int_0^\infty \sigma_i(\xi)d\xi \\ &\leq 16|v - V(t)|^{-1}\min\{r^{-1}|v|^3, r^{-1}|v - V(t)|^3\} \\ &\leq 16r^{-1}|v|^2. \end{aligned} \tag{4.56}$$

Thus finally

$$\begin{aligned} \int_U \frac{f(t, x, v)}{|X(s, t, x, v) - X(s)|^2}dvdxds &\leq \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{t-\Delta}^{t} |d(s)|^{-2}\mathbf{1}_U(s, x, v)dsdvds \\ &\leq Cr^{-1}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}|v|^2 f(t, x, v)dvdx \leq Cr^{-1}. \end{aligned}$$

Adding up the estimates on G, B, U and by definition of Δ we get the following control:

$$\begin{aligned} |V(t) - V(t-\Delta)| &\leq C\left(p^{4/3} + r\log(4P(t)/p) + r^{-1}\Delta^{-1}\right)\Delta \\ &= C\left(p^{4/3} + r\log(4P(t)/p) + r^{-1}\max\{1/t, , 4cP(t)^{4/3}/p\}\right)\Delta. \end{aligned}$$

Step 7. We now choose p and r so that the sum above is the same order of $P(t)$. Setting

$$p = P(t)^{4/11}, \ r = P(t)^{16/33},$$

we can assume without loss of generality that $P(t) \geq 1$ so that $p \leq P(t)$. We make the following observation. Since

$$\lim_{t \to T} P(t) = \infty,$$

if $T < \infty$, and $P(t)$ is non decreasing, there exists a unique $T^* \in (0, T)$ so that

$$1/t \leq 4C^* P(t)^{4/3}/p = 4C^* P(t)^{32/33}, \qquad t \geq T^*.$$

Hence for all $t \geq T^*$,

$$|V(t) - V(t-\Delta)| \leq CP(t)^{16/33} \log(P(t))\Delta.$$

Thus for any $\varepsilon > 0$ there exists a modified constant $C > 0$ so that

$$|V(t) - V(t-\Delta)| \leq CP(t)^{16/33+\varepsilon}\Delta, \qquad \forall t \geq T^*. \tag{4.57}$$

We partition the time interval $[T^*, t]$. Let $t > T^*$ and define $t_0 = t$ and $t_{i+1} = t_i - \Delta(t_i)$ for $t_i \geq T^*$. Since $t_i - t_{i+1} = \Delta(t_i) \geq \Delta(t_0)$, we can find $k \in \mathbb{N}$ so that

$$t_k < T^* \leq t_{k-1} < ... < t_0 = t.$$

From the estimate (4.57) we deduce

$$\begin{aligned} |V(t) - V(t_k)| &\leq \sum_{i=1}^{k} |V(t_{i-1}) - V(t_i)| \leq CP(t)^{16/33+\varepsilon} \sum_{i=1}^{k} (t_{i-1} - t_i) \\ &\leq CP(t)^{16/33+\varepsilon} t. \end{aligned}$$

Now by definition of $P(t)$ we have

$$P(t) \leq P(t_k) + CP(t)^{16/33+\varepsilon} t,$$

Note that $P(t_k) \leq \sup_{s\in[0,T^*]} |V(s)| \leq C$ and $P(t_k) \leq P(t)$. This eventually implies that for any $\delta > 0$ there exists $C > 0$ such that

$$P(t) \leq (1+t)^{33/17+\delta}, \qquad t \in [0, T), \tag{4.58}$$

so by Theorem 13.4 we conclude.

Remark 13.10. *The bound in (4.58) has been improved in [38] to*

$$P(t) \leq C(1+t)\log(2+t),$$

which is valid in both cases $\omega = \pm 1$ and is the sharpest bound known so far. In the repulsive case, it was shown in [57] that

$$P(t) \leq C(1+t)^{2/3}.$$

Remark 13.11. *Theorem 13.9 provides limited information on the asymptotic behavior of the solution for large times. This is not surprising since the proof is valid for both the repulsive and attractive cases while the asymptotic behavior can be expected to be different. In particular in the plasma physics case ($\omega = +1$) particles repel each other, so the spatial density should decay as $t \to \infty$, whereas in the case $\omega = -1$ static solutions are known to exist [8, 9, 57], so there can be no decay on f. On the other hand in the plasma physics case, certain solutions [10, 32] have been known to decay polynomially in time as $t \to \infty$. Whether all smooth solutions decay in time in the repulsive case remains an open problem.*

CHAPTER 5

Estimates for Lagrangian flows

14. Vector fields with Sobolev regularity, $p > 1$

In this section we show that the results from DiPerna Lions theory can be recovered from a priori estimates from the Lagrangian formulation

$$
\begin{cases} \dfrac{dX}{ds}(s,x) = b(s, X(s,x)), \ s \in [0,T], \\ X(0,x) = x, \end{cases} \tag{5.1}
$$

under suitable growth conditions of the field b and an appropriate notion of flows when b is non-smooth. We summarize the quantitative estimates derived in [23] for $W^{1,p}$ vector fields, with $p > 1$. These will allow us to recover the existence, uniqueness, and stability of Lagrangian solutions to the transport equation. The novelty of the approach offered in [23] is that using a purely Lagrangian derivation, from just the definition of such flows, one is able to derive quantitative regularity and stability of regular Lagrangian flows, as well as propagation of mild regularity for weak solutions to the transport equation. As opposed to the renormalization scheme in section 2 which is used to prove uniqueness, we exploit the ODE (5.1) to get an explicit rate on the decay on the set where two flows associated to b differ, and this estimate relies *only* on the regularity and growth of b. For simplicity, we assume that $b \in W^{1,p} \cap L^\infty$. As in [23], we summarize the estimate for the superlevels of the function $X(t,x) - \bar{X}(t,x)$ of Lagrangian flows associated to b and $\bar{b}$ which depend only on the L^∞ and $W^{1,p}$ norms of b and $\bar{b}$. In fact, this can be relaxed to a more general growth condition on b, a technical modification we postpone for section 16 in order to illustrate the initial analysis more clearly. The estimate precedes the following corollaries:

1. Existence, uniqueness, stability, and compactness of Lagrangian flows,
2. Approximate differentiability of the Lagrangian flow.

We remark that the missing point in these estimates is the case when $p = 1$, due to the fact that the maximal function of an L^1 function is no longer in L^1. However, this is resolved in section 16, where we outline the analogous estimates for a vector field whose derivative is the singular integral of an L^1 function.
For locally summable vector fields, we begin with the following definition of a flow map associated to a weakly differentiable and bounded vector field.

Definition 14.1. *Let $b \in L^1_{\text{loc}}([0,T] \times \mathbb{R}^d)$. A map $X : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ is a regular Lagrangian flow for the bounded vector field b if*
1) for a.e. $x \in \mathbb{R}^d$ the map $t \mapsto X(t,x)$ is an absolutely continuous integral solution of $\dot{\gamma}(t) = b(t, \gamma(t))$ for $t \in [0,T]$ with $\gamma(0) = x$.
2. There exists a constant L independent of t such that

$$\mathcal{L}^d(X(t,.)^{-1}(A)) \le L\mathcal{L}^d(A)$$

for every Borel set $A \subset \mathbb{R}^d$. The constant L is called the compressibility constant of X.

REMARK 14.2. *The bounded compression constant corresponds to a lower bound on the Jacobian of the flow, and is equivalent to*

$$\int_{\mathbb{R}^N} \varphi(X(s,x))dx \leq L \int_{\mathbb{R}^N} \varphi(x)dx \tag{5.2}$$

for all measurable non-negative φ.

14.1. Definition of the quantity $\Phi_\delta(t)$ and an exploratory computation. We begin by introducing a 'uniqueness' functional which measures an integral distance between two flows. Let b be a vector field satisfying the assumptions of Theorem 14.5. Let X and $\bar{X}$ be two regular Lagrangian flows associated to the same vector field b. Let $\delta > 0$ be some small parameter. Consider the functional

$$\Phi_\delta(t) = \int\limits_{B_r} \log\left(\frac{|X(t,x) - \bar{X}(t,x)|}{\delta} + 1\right) dx. \tag{5.3}$$

At $t = 0$, $\Phi_\delta \equiv 0$. It is clear that if X and $\bar{X}$ are not in the same equivalence class of Lagrangian flows that there exists a $\gamma > 0$ giving the lower bound

$$\Phi_\delta(t) \geq \int\limits_{B_r \cap \{|X(t,x) - \bar{X}(t,x)| \geq \gamma\}} \log\left(1 + \frac{\gamma}{\delta}\right) dx = \mathcal{L}^N(\{x \in B_r : |X(t,x) - \bar{X}(t,x)| \geq \gamma\}) \log\left(1 + \frac{\gamma}{\delta}\right), \tag{5.4}$$

which yields

$$\mathcal{L}^N(\{x \in B_r : |X(t,x) - \bar{X}(t,x)| \geq \gamma\}) \leq \frac{\Phi_\delta(t)}{\log\left(1 + \frac{\gamma}{\delta}\right)}. \tag{5.5}$$

If the ratio on the right hand side goes to zero as $\delta \to 0$, then we must have that $X = \bar{X}$ almost everywhere. This is achieved if Φ_δ grows slower than $\log(1/\delta)$ as $\delta \to 0$. This is immediate if, for instance, $\Phi_\delta \leq C$. If we differentiate (5.3) with respect to time, we can compute

$$\Phi'_\delta(t) \leq \int\limits_{B_r} \left|\frac{dX(t,x)}{dt} - \frac{d\bar{X}(t,x)}{dt}\right| (|X(t,x) - \bar{X}(t,x)| + \delta)^{-1} dx = \int\limits_{B_r} \frac{|b(t, X(t,x) - b(t, \bar{X}(t,x)|}{|X(t,x) - \bar{X}(t,x)| + \delta} dx. \tag{5.6}$$

Let $\tilde{R} = 2r + T(||b||_\infty + ||\bar{b}||_\infty)$. Then Lemma 5.9 implies that

$$\Phi'_\delta(t) \leq c_d \int\limits_{B_r} M_{\tilde{R}} Db(t, X(t,x)) + M_{\tilde{R}} Db(t, \bar{X}(t,x)) dx.$$

Using the estimates from Lemma 5.7 this gives us

$$\begin{aligned} \Phi'_\delta(t) &\leq c_d \tilde{r}^{d-d/p}(\bar{L} + L)||M_{\tilde{R}} Db(t,.)||_{L^p(\mathbb{R}^d)} \\ &\leq c_d \tilde{r}^{d-d/p}(\bar{L} + L)||Db(t,.)||_{L^p(\mathbb{R}^d)}. \end{aligned}$$

Now for any $\tau \in [0,T]$, we integrate the expression for $\Phi'(t)$ over $[0,\tau]$ to recover the desired upper bound

$$\Phi_\delta(t) \leq c_d \tilde{r}^{d-d/p}(\bar{L} + L)||Db||_{L^1((0,\tau);L^p(\mathbb{R}^d))} \leq C. \tag{5.7}$$

Thus any Lagrangian flow associated to a Sobolev vector field must be unique. If we broaden the approach to account for Lagrangian flows associated to two distinct vector fields b and $\bar{b}$, we obtain an explicit rate on the decay of the left hand side of (5.5) as $\delta \to 0$, depending on the regularity and growth of b and $\bar{b}$. Thus the L^1 stability of Lagrangian flows are controlled by the upper bound on Φ_δ. We have the following estimate on the L^1 distance of the flows in terms of the logarithmic L^1 distance of the vector fields.

THEOREM 14.3. *(Stability.) Let b and $\bar{b}$ be bounded vector fields belonging to $L^1([0,T];W^{1,p}(\mathbb{R}^N))$ for some $p > 1$. Let X and $\bar{X}$ be regular Lagrangian flows associated to b and $\bar{b}$ respectively with compressibility constants L and $\bar{L}$. Then for every time $\tau \in [0,T]$, we have*

$$||X(\tau,.) - \bar{X}(\tau,.)||_{L^1(B_r)} \leq C|\log(||b-\bar{b}||_{L^1([0,\tau]\times B_R)})|^{-1} \tag{5.8}$$

where $R = r + T||b||_\infty$ and C depends only on $\tau, r, ||b||_\infty, ||\bar{b}||_\infty, L, \bar{L}$ and $||Db||_{L^1(L^p)}$.

REMARK 14.4. *The constant C depends on the regularity properties of b only and not $\bar{b}$. The Theorem is also valid if $\bar{b}$ is a vector field which is merely bounded, and b has the required Sobolev regularity.*

PROOF OF THEOREM 14.3. Set $\delta = ||b-\bar{b}||_{L^1([0,\tau]\times B_R)}$ and consider the integral functional in (5.3). Differentiating with respect to time we have the additional first term

$$\Phi_\delta'(t) \leq \frac{1}{\delta}||b(t,\bar{X}(t,\cdot)) - \bar{b}(t,\bar{X}(t,\cdot))||_{L^1(B_r)} + \int_{B_r} \frac{|b(t,X(t,x) - b(t,\bar{X}(t,x))|}{|X(t,x)-\bar{X}(t,x)|+\delta}dx.$$

Changing variable $\bar{X}(t,x) \mapsto x$ in the first term above and using the estimate in (5.7) we have the upper bound

$$\Phi_\delta(t) \leq \frac{\bar{L}}{\delta}||b-\bar{b}||_{L^1((0,\tau)\times B_{\tilde{r}})} + c_d\tilde{r}^{d-d/p}(\bar{L}+L)||Db||_{L^1((0,\tau);L^p(\mathbb{R}^d))} \leq C_1. \tag{5.9}$$

Now fix $\eta > 0$. Using the Chebyshev inequality, we find a measurable set $K \subset B_r$ such that $|B_r \setminus K| \leq \eta$ and for all $x \in K$,

$$\log\left(\frac{|X(t,x)-\bar{X}(t,x)|}{\delta}+1\right) \leq \frac{C_1}{\eta}.$$

We split the integral (5.8) in the following manner.

$$\begin{aligned}\int_{B_r} |X(\tau,x) - \bar{X}(\tau,x)|dx &= \int_{B_r\setminus K} |X(\tau,x) - \bar{X}(\tau,x)|dx + \int_K |X(\tau,x)-\bar{X}(\tau,x)|dx \\ &\leq \eta(||X(\tau,.)||_{L^\infty(B_r)} + ||\bar{X}(\tau,.)||_{L^\infty(B_r)}) + \int_K |X(\tau,x)-\bar{X}(\tau,x)|dx \\ &\leq \eta C_2 + c_n r^n \delta(\exp(C_1/\eta)) \leq C_3(\eta + \delta\exp(C_1/\eta)),\end{aligned}$$

where C_1, C_2, C_3 depend on $T, r, ||b||_\infty, ||\bar{b}||_\infty, L, \bar{L}$, and $||Db||_{L^1(L^p)}$. We can assume $\delta < 1$. If we set $\eta = 2C_1|\log\delta|^{-1} = 2C_1(-\log(\delta))^{-1}$, we get $\exp(C_1/\eta) = \delta^{-1/2}$, so that

$$\int_{B_r} |X(\tau,x) - \bar{X}(\tau,x)|dx \leq C_3(2C_1|\log\delta|^{-1} + \delta^{1/2}) \leq C|\log\delta|^{-1},$$

with C depending on $\tau, r, ||b||_\infty, ||\bar{b}||_\infty, L, \bar{L}$, and $||Db||_{L^1(L^p)}$. □

14.2. Approximate differentiability of the flow. Similar to the quantity defined in (5.3), we define an integral to measure the Lipschitz continuity of a regular Lagrangian flow. For $0 \leq t \leq T$, $0 < r < 2R$, and $x \in B_R$, define

$$Q(t,x,r) = \fint_{B_r(x)} \log\left(\frac{|X(t,x)-X(t,y)|}{r}+1\right)dy.$$

Differentiating in time gives

$$\frac{dQ}{dt}(t,x,r) \leq \fint_{B_r(x)} \frac{|b(t,X(t,x)) - b(t,X(t,y))|}{|X(t,x)-X(t,y)|+r}dy. \tag{5.10}$$

Setting $\tilde{R} = 4R + 2T||b||_{L^\infty}$ gives $|X(t,x) - X(t,y)| \leq \tilde{R}$. Applying Lemma 5.9 we have the estimate

$$\frac{dQ}{dt}(t,x,r) \leq c_d M_{\tilde{R}} Db(t, X(t,x)) + c_d \fint_{B_r(x)} M_{\tilde{R}} Db(t, X(t,y)) dy. \tag{5.11}$$

Integrating (5.11) in time, and taking the supremum over $0 < r < 2R$ and $0 \leq t \leq T$, we get

$$\sup_{0\leq t\leq T} \sup_{0<r<2R} Q(t,x,r) \leq c + c_d \int_0^T M_{\tilde{R}} Db(t, X(t,x)) dt + c_d \int_0^T \sup_{0<r<2R} \fint_{B_r(x)} M_{\tilde{R}} Db(t, X(t,y)) dy dt. \tag{5.12}$$

Observe that the latter term on the right hand side can be estimated with a composite maximal function of $M_{\tilde{R}} Db$. Taking the L^p norm over B_R we can estimate (5.12) in the same way as (5.7). This gives the following result.

THEOREM 14.5. *Let b be a bounded vector field belonging to $L^1([0,T]; W^{1,p}(\mathbb{R}^d))$ for $p > 1$, and let X be a regular Lagrangian flow associated to b. Let L be the compressibility constant of X. For every $p > 1$ define the integral quantity*

$$A_p(R,X) = \left[\int_{B_R(0} \left(\sup_{0\leq t\leq T} \sup_{0<r<2R} \fint_{B_r(x)} \log\left(\frac{|X(t,x) - X(t,y)|}{r} + 1 \right) dy \right)^p dx \right]^{1/p}$$

Then

$$A_p(R,X) \leq C(R, L, ||Db||_{L^1(L^p)}).$$

PROOF. Changing variable $X(t,x) \mapsto x$ in (5.12) and applying Lemma 5.7 twice yields

$$\begin{aligned} A_p(R,X) &\leq c_{p,R} + c_d \left\| \int_0^T M_{\tilde{R}} Db(t, X(t,x)) dt \right\|_{L^p(B_R)} + c_d \left\| \int_0^T \sup_{0<r<2R} \fint_{B_r(x)} M_{\tilde{R}} Db(t, X(t,y)) dy dt \right\|_{L^p(B_R)} \\ &\leq c_{d,p} L^{1/p} ||Db||_{L^1((0,T);L^p(B_{R+\tilde{R}+T||b||_\infty}))} + c_d \int_0^T \| M_{2R}[(M_{\tilde{R}} Db) \circ (t, X(t,.))](x) \|_{L^p(B_R)} \, dt \\ &\leq C(R, L, ||Db||_{L^1(L^p)}). \end{aligned}$$

□

The bound on $A_p(R,X)$ can be used to prove the approximate differentiability of the flow associated to a vector field satisfying the assumptions of Theorem 14.5, since it implies the Lipschitz constants of a sequence of approximating flows are bounded up to a set of small measure.

THEOREM 14.6. *Let b be a bounded vector field belonging to $L^1([0,T]; W^{1,p}(\mathbb{R}^d))$ for some $p > 1$, and let X be a regular Lagrangian flow associated to b. Then $X(t,\cdot)$ is approximately differentiable $\mathcal{L}^d$-a.e. in $\mathbb{R}^d$, for every $t \in [0,T]$.*

PROOF. Theorem 14.6 is a consequence of the following property: For every $\varepsilon > 0$ and $R > 0$, we can find a set $K \subset B_R$ such that $\mathcal{L}^d(B_R \setminus K) \leq \varepsilon$ and for any $0 \leq t \leq T$ we have

$$\mathrm{Lip}(X(t,\cdot)|_K) \leq \exp \frac{c_d A_p(R,X)}{\varepsilon^{1/p}}.$$

Fixing $\varepsilon > 0, R > 0$, and denoting by M the constant $M = \frac{c_d A_p(R,X)}{\varepsilon^{1/p}}$, we have from (5.12) existence of a set $K \subset B_R$ such that $\mathcal{L}^d(B_R \setminus K) \leq \varepsilon$ and

$$\sup_{0\leq t\leq T}\sup_{0<r<2R} \fint_{B_r(x)} \log\left(\frac{|X(t,x)-X(t,y)|}{r}+1\right) dy \leq M \qquad \forall x \in K.$$

If we fix $x, y \in K$ and set $r = |x-y|$ then

$$\begin{aligned}
&\log\left(\frac{|X(t,x)-X(t,y)|}{r}+1\right) = \fint_{B_r(x)\cap B_r(y)} \log\left(\frac{|X(t,x)-X(t,y)|}{r}+1\right) dz \\
&\leq c_d \fint_{B_r(x)} \log\left(\frac{|X(t,x)-X(t,z)|}{r}+1\right) dz + c_d \fint_{B_r(y)} \log\left(\frac{|X(t,y)-X(t,z)|}{r}+1\right) dz \\
&\leq c_d M.
\end{aligned}$$

We have then clearly that

$$\mathrm{Lip}(X(t,\cdot)|_K) \leq \exp(c_d M).$$

Applying Theorem 3.1.16 of [35] yields that $X(t,\cdot)$ is approximately differentiable $\mathcal{L}^d$-a.e. □

Using this Lipschitz property of the flow, one may prove the following result on compactness of the flows, which serves as a substitute for Ascoli Arzela when the uniform Lipschitz bounds hold up to a set of arbitrarily small measure.

THEOREM 14.7. *(Compactness.) Let b_n be a sequence of uniformly bounded vector fields in $L^1([0,T]; W^{1,p}(\mathbb{R}^d))$ for some $p > 1$. For each n, let X_n be a regular Lagrangian flow associated to b_n, with uniformly bounded compression constant L_n. Then the sequence X_n is strongly precompact in $L^1_{\mathrm{loc}}([0,T] \times \mathbb{R}^2)$.*

PROOF. Fix $\delta, R > 0$. Since b_n is uniformly bounded we deduce that X_n is uniformly bounded in $L^\infty([0,T] \times B_R)$. By Theorem 14.6, for every n we find a set $K_{n,\delta}$ such that $\mathcal{L}^d(B_R \setminus K_{n,\delta} \leq \delta)$ and for every t fixed,

$$\mathrm{Lip}(X_n(t,\cdot)|_{K_{n,\delta}}) \leq \exp\frac{c_d A_p(R,X_n)}{\delta^{1/p}} \leq C^\delta(R). \tag{5.13}$$

The uniform boundedness of X_n and $\mathrm{Lip}(X_n(t,\cdot)|_{K_{n,\delta}})$ in L^∞ allow us to apply (Lemma C1, [22]) to conclude X_n is precompact in measure in $[0,T] \times B_R$, and therefore also precompact in $L^1([0,T] \times B_R)$. □

REMARK 14.8. *(A more direct method to compactness.) The stability estimate in Theorem 14.3 provides an alternate way to show compactness, without using the stronger Lipschitz property of the flow. If X_n is a sequence of regular Lagrangian flows with uniformly bounded compression constants, associated to uniformly bounded vector fields b_n in $L^1([0,T]; W^{1,p}(\mathbb{R}^d))$, then X_n is strongly precompact in $L^1_{\mathrm{loc}}([0,T] \times \mathbb{R}^d)$. Indeed, applying Theorem 14.3 to the flows $X_n(t,x)$ and $X_n(t,x+h)-h$ relative to vector fields $b_n(t,x)$ and $b_n(t,x+h)$ for a fixed parameter $h \in \mathbb{R}^d$, we have for every $t \in [0,T]$:*

$$\begin{aligned}
&||X_n(t,\cdot) - X_n(t,\cdot+h) - h||_{L^1(B_R)} \leq C|\log(||b_n(\tau,\cdot) - b_n(\tau,\cdot+h)||_{L^1([0,t]\times B_R)})|^{-1} \\
&\leq \frac{C}{||Db_n||_{L^p}|\log(h)|},
\end{aligned}$$

and we conclude by the Riesz Fréchet Kolmogorov theorem.

REMARK 14.9. *We summarize three methods to prove compactness of the flow: the first one, using a uniform control on the quantity $A_p(R,X)$ to gain the approximate differentiability property, and the second, using the more general stability estimate of two Lagrangian flows as in Remark 14.8. The third alternative* *is a variant of the first argument that works with a slightly weaker assumption than $Db \in L^p_{\rm loc}(\mathbb{R}^d)$ and was considered in [22]. If we assume the regularity assumption*

$$\forall \lambda > 0, \qquad M_\lambda Db \in L^1([0,T]; L^1_{\rm loc}(\mathbb{R}^d)), \tag{5.14}$$

which is equivalent to $Db \in L\log L_{\rm loc}(\mathbb{R}^d)$, then we can define the quantity, for $R>0$, $0<r<2R$,

$$a(r,R,X) = \int_{B_R} \sup_{0\le t\le T} \fint_{B_r} \log\left(\frac{|X(t,x)-X(t,y)|}{r}+1\right) dydx,$$

which is 'smaller' than $A_p(R,X)$ since we do not take the supremum over r. Proceeding as in the proof of Theorem 14.5 with $\tilde{R} = 3R + 2T||b||_\infty$ one can compute

$$a(r,R,X) \le c_d L||M_{\tilde{R}} Db||_{L^1([0,T];L^1(B_{\tilde{R}}(0))} = C(R). \tag{5.15}$$

By concavity of the log *function , for $0 \le z \le \tilde{R}$ we have*

$$\log\left(\frac{z}{r}+1\right) \ge \frac{\log\left(\frac{\tilde{R}}{r}+1\right)}{\tilde{R}} z.$$

Now since $|X(t,x) - X(t,x+z)| \le \tilde{R}$ we get a control on the spatial increments of $X(t,x)$ by the estimate

$$\fint_{B_R} \sup_{0\le t\le T} \int_{B_r} |X(t,x)-X(t,x+z)| dxdz \le \frac{\tilde{R}}{\log\left(\frac{\tilde{R}}{r}+1\right)} a(r,R,X) \le g(r). \tag{5.16}$$

where $g(r)$ satisfies $g(r) \to 0$ for $r \to 0$. Combined with the elementary estimate

$$|X(t+\tau,x) - X(t,x)| \le \tau ||b||_\infty,$$

one can argue with Riesz Frechet Kolmogorov theorem.

Corollary 14.10. *(Existence.) Let b be a bounded vector field satisfying (5.14) such that $[\operatorname{div} b]^- \in L^1([0,T]; L^\infty(\mathbb{R}^d))$. Then there exists a regular Lagrangian flow associated to b.*

Proof. We regularize b by convolution, and note that the regularized sequences b_n and Db_n satisfy the bounds of Remark 14.8. For the smooth flows X_n associated to b_n, we have the uniform bound on the compression constants L_n

$$L_n = \exp\left(\int_0^T ||[\operatorname{div} b_n(t,\cdot)]^-||_\infty dt\right).$$

□

Remark 14.11. *Following the proof of Theorem 14.3, we see that a sufficient criterion for stability of the flows is (5.7), or in other words, that the quantity Φ_δ is upper bounded by a constant. This happens when the difference quotients of b are bounded in $L^1_{\rm loc}$, and corresponds to $b \in W^{1,p}(\mathbb{R}^d)$ for $p>1$. Indeed the argument fails for $p=1$ since MDb is no longer in $L^1_{\rm loc}$, and the constant upper bound in (5.7) is no longer possible. One might ask whether these estimates can be modified when the bound for Φ_δ depends in some way on δ, but blows up slower than $\log(1/\delta)$. Observe that in (5.6) we have neglected the presence of the δ parameter in the denominator of the difference quotient of b, when in fact we have the 'smaller' estimate*

$$\frac{|b(t,X(t,x) - b(t,\bar{X}(t,x)|}{|X(t,x)-\bar{X}(t,x)|+\delta} \le \min\left\{\frac{|b(t,X(t,x))|+|b(t,\bar{X}(t,x))|}{\delta}, \frac{|b(s,X(t,x)) - b(t,\bar{X}(t,x))|}{|X(t,x)-\bar{X}(t,x)|}\right\}. \tag{5.17}$$

This will be studied further in section 16, when we study the case in which Db is no longer in L^p for $p>1$.

REMARK 14.12. *The estimates derived in Chapter 5 may be relaxed for Sobolev vector fields satisfying more a more general growth condition, namely that* $\frac{b(t,x)}{1+|x|}$ *belongs to* $L^1_t(L^1_x + L^\infty_x)$.[1] *The integral quantity in* (5.3) *is then measured over a suitable set where the flow is bounded. We then have to consider the size of the set where the flow may be large. The stability results follow, up to a modification in the proofs which will be considered in more detail in the next section.*

15. Difference quotients for vector fields whose gradient is the singular integral of an L^1 function

In this section we will review the more recent work in [16], where the vector field has a gradient given by the singular integral of an L^1 function. This makes the quantity $\Phi_\delta(t)$ introduced in Theorem 14.3 more difficult to estimate, by the fact that the classical maximal function cannot be composed with a singular integral operator on L^1. The first step is to seek a 'milder' approximation to the identity than the classical function M, for instance a convolution operator, and prove that this sufficiently smooth average satisfies cancellations in the convolution with a singular kernel K.

15.1. Cancellations in maximal functions and singular integrals. In this section we recall the key estimate in [16] that permits the composition of a singular integral operator and maximal function. Given two singular kernels K_1 and K_2 with associated operators S_1 and S_2, we can consider the composition S_2S_1, where $\widehat{S_2S_1u} = \hat{K}_2\hat{K}_1\hat{u}$ is a well defined operator which satisfies the estimates of Theorem 6.4. This estimate does not arise from composition, since S_1u fails to be in L^1. However, the cancellations between the kernels K_1 and K_2 allow S_2S_1 to be a well defined tempered distribution. The next Theorem states that such cancellations also occur in the composition of a maximal function with a singular integral operator. The classical maximal function is too 'rough' for such composition, so one considers the smooth maximal function.

DEFINITION 15.1. *Given a family of functions* $\{\rho^\nu\}_\nu \subset L^\infty_c(\mathbb{R}^N)$, *for every function* $u \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ *we define the* $\{\rho^\nu\}$*-maximal function of* u *as*

$$M_{\{\rho^\nu\}}(u)(x) = \sup_\nu \sup_{\varepsilon>0} \left| \int_{\mathbb{R}^N} \rho^\nu_\varepsilon(x-y)u(y)dy \right| = \sup_\nu \sup_{\varepsilon>0} \left|(\rho^\nu_\varepsilon * u)(x)\right|.$$

In the case when u *is a measure, we take a compactly supported family* $\{\rho^\nu\}_\nu \subset C^\infty_c(\mathbb{R}^N)$ *and define in the distributional sense*

$$M_{\{\rho^\nu\}}(u)(x) = \sup_\nu \sup_{\varepsilon>0} \left|\langle u, \rho^\nu_\varepsilon(x-.)\rangle_{\mathcal{D}',\mathcal{D}}\right|.$$

The smooth averages and the absence of the absolute value within the integral allows cancellations that take place in the composition of $M_{\{\rho^\nu\}}$ with operator S. This plays together with the cancellations in the singular kernel K, giving rise to a bounded composition operator $M_{\{\rho^\nu\}}S : L^1 \to M^1$.

THEOREM 15.2. *Let* K *be a singular kernel of fundamental type, and let* $Su = K * u$, *for every* $u \in L^2(\mathbb{R}^N)$. *Let* $\{\rho^\nu\}_\nu \subset C^\infty_c(\mathbb{R}^N)$ *be a family of kernels such that*

$$\operatorname{spt} \rho^\nu \subset B_1 \qquad \text{and} \qquad ||\rho^\nu||_{L^1(\mathbb{R}^N)} \leq Q_1 \text{ for every } \nu. \tag{5.18}$$

Then we have the following estimates.

(1) (a) There exists a constant C_N, *depending on the dimension* N *only, such that*

$$|||M_{\rho^\nu}(Su)|||_{M^1(\mathbb{R}^N)} \leq C_N Q_1(C_0 + C_1 + ||\widehat{K}||_\infty)||u||_{L^1(\mathbb{R}^N)} \tag{5.19}$$

[1] For vector fields which are only locally bounded, an extension of the DiPerna Lions theory for local flows was considered in [2].

for every $u \in L^1 \cap L^2(\mathbb{R}^N)$.

(b) *The estimate* (5.19) *holds also for all* $u \in \mathcal{M}(\mathbb{R}^N)$, *where* Su *is defined as a distribution.*

(2) *If* $Q_2 = \sup_\nu ||\rho^\nu||_{L^\infty(\mathbb{R}^N)}$ *is finite, then there exists* C_N, *such that*

$$||M_{\rho^\nu}(Su)||_{L^2(\mathbb{R}^N)} \leq C_N Q_2 ||\widehat{K}||_\infty ||u||_{L^2(\mathbb{R}^N)}. \tag{5.20}$$

Proof. The proof will rely on Lemma 6.6. The goal is to prove that the composition $(\rho^\nu_\varepsilon * K)(x)$ satisfies the assumptions (1)-(5) and has a decay at infinity comparable to an approximation to the identity in Lemma 5.8.

Step 1. (Definition of the operator $\Delta^\nu_\varepsilon(x)$.)

We define a convolution operator, which, up to bounded factor, is the same as $M_{\rho^\nu}S$. Fix a radial function $\chi \in C^\infty(\mathbb{R}^N), 0 \leq \chi \leq 1$, such that $\chi(x) = 0$ for $|x| \leq 1/2$, $\chi(x) = 1$ for $|x| \geq 1$, and $||\nabla \chi||_\infty \leq 3$. Let

$$\Delta^\nu_\varepsilon(x) = [K(\varepsilon \cdot) * \rho^\nu]\left(\frac{x}{\varepsilon}\right) - \left(\int_{\mathbb{R}^N} \rho^\nu(y)dy\right)\chi\left(\frac{x}{\varepsilon}\right)K(x) \tag{5.21}$$

Notice that

$$\Delta^\nu_\varepsilon = \rho^\nu_\varepsilon * K + C\chi\left(\frac{\cdot}{\varepsilon}\right)K,$$

where the first term on the right-hand side is the 'almost' the composition $M_{\rho^\nu}S$, and the second term is the product of K and a smooth function, and C is some constant less than Q_1. We show that the operators Δ^ν_ε and $\chi\left(\frac{\cdot}{\varepsilon}\right)K$ are of weak type (1,1) and strong type (2,2). Observe that by the regularity assumption on ρ^ν there is some constant Q_2 such that for every $\varepsilon > 0$ and every ν,

$$||(\varepsilon^N K(\varepsilon \cdot)) * \rho^\nu||_{L^\infty(\mathbb{R}^N)} \leq Q_3.$$

By (5.21) and the definition of χ, we have that for every $\varepsilon > 0$ and ν, $\Delta^\nu_\varepsilon \in C_b(\mathbb{R}^N)$, and

$$|\Delta^\nu_\varepsilon(x)| \leq \frac{Q_3}{\varepsilon^N} + \frac{Q_1 C_0 2^N}{\varepsilon^N} \tag{5.22}$$

for every $x \in \mathbb{R}^N$. For $|x| > 2\varepsilon$, (5.21) becomes

$$\Delta^\nu_\varepsilon(x) = \int_{\mathbb{R}^N} [K(y) - K(x)]\rho^\nu_\varepsilon(x-y)dy.$$

For every $t \in [0,1]$,

$$|ty + (1-t)x| \geq |x| - |y-x| \geq |x| - \varepsilon \geq |x| - |x|/2 = |x|/2.$$

Then for every $|x| > 2\varepsilon$,

$$\begin{aligned}
|\Delta^\nu_\varepsilon(x)| &\leq \int_{\mathbb{R}^N}\int_0^1 \frac{d}{dt}K(ty+(1-t)x)\rho^\nu_\varepsilon(x-y)dtdy && (5.23)\\
&\leq \int_{\mathbb{R}^N}\int_0^1 \nabla K(ty+(1-t)x)|y-x|\rho^\nu_\varepsilon(x-y)dtdy\\
&\leq \int_{\mathbb{R}^N}\int_0^1 \frac{C_1}{|ty+(1-t)x|^{N+1}}|y-x|\rho^\nu_\varepsilon(x-y)dtdy\\
&\leq \int_{\mathbb{R}^N} \frac{C_1|y-x|}{(|x|/2)^{N+1}}\rho^\nu_\varepsilon(x-y)dy \leq \frac{2^{N+1}C_1Q_1\varepsilon}{|x|^{N+1}}. && (5.24)
\end{aligned}$$

Putting together (5.22) and (5.23), we get a constant C_N such that

$$|\Delta^\nu_\varepsilon(x)| \leq C_N \frac{Q_3 + Q_1(C_0 + C_1)}{\varepsilon^N\left(1 + \left(\frac{|x|}{\varepsilon}\right)^{N+1}\right)}.$$

Clearly $\Delta_\varepsilon^\nu(x) \in L^1 \cap L^\infty(\mathbb{R}^N)$. Applying Lemma 5.8 yields that for any $u \in L^1_{\text{loc}}(\mathbb{R}^N)$,

$$\sup_\nu \sup_{\varepsilon>0} |(\Delta_\varepsilon^\nu * u)(x)| \leq C_N(Q_3 + Q_1(C_0 + C_1))Mu(x), \qquad \forall x \in \mathbb{R}^N. \tag{5.25}$$

This immediately implies the estimates

$$\begin{aligned} &|||\sup_\nu \sup_{\varepsilon>0} |\Delta_\varepsilon^\nu * u|||_{M^1(\mathbb{R}^N)} \leq C_N(Q_3 + Q_1(C_0 + C_1))||u||_{L^1(\mathbb{R}^N)}, &&\text{for every } u \in L^1(\mathbb{R}^N), \\ &||\sup_\nu \sup_{\varepsilon>0} |\Delta_\varepsilon^\nu * u|||_{L^2(\mathbb{R}^N)} \leq C_N(Q_3 + Q_1(C_0 + C_1))||u||_{L^2(\mathbb{R}^N)}, &&\text{for every } u \in L^2(\mathbb{R}^N), \end{aligned} \tag{5.26}$$

so that Δ_ε^ν is a bounded operator on $L^1 \to M^1$ and $L^2 \to L^2$.

Step 2. (Definition of the operator T_+.)
We have from (5.21) that

$$(\rho_\varepsilon^\nu * K)(x) = \Delta_\varepsilon^\nu(x) + \left(\int_{\mathbb{R}^N} \rho^\nu(y)dy\right) \chi\left(\frac{x}{\varepsilon}\right) K(x). \tag{5.27}$$

Since $\rho_\varepsilon^\nu * K \in L^2$, Plancherel's identity implies the associativity property

$$\rho_\varepsilon^\nu * (Su) = (\rho_\varepsilon^\nu * K) * u \qquad \text{for every } u \in L^2(\mathbb{R}^N). \tag{5.28}$$

For every $u \in L^2(\mathbb{R}^N)$, we have the following characterization

$$M_{\rho_\nu}(Su) = \sup_\nu \sup_{\varepsilon>0} |\rho_\varepsilon^\nu * (Su)| = \sup_\nu \sup_{\varepsilon>0} |(\rho_\varepsilon^\nu * K) * u|,$$

which is the operator in (5.27). By step 1 and since $|\int \rho^\nu(y)dy| \leq Q_1$, it remains to study the operator

$$T_+(u) = \sup_{\varepsilon>0} \left|\left(\chi\left(\frac{\cdot}{\varepsilon}\right) K\right) * u\right| \qquad \text{for } u \in L^2(\mathbb{R}^N). \tag{5.29}$$

We apply the Interpolation Lemma 6.6 to this operator T_+. All assumptions except for (4) and (5) are obvious. We postpone the proof for (4) and (4). We obtain constants $P_1 = C_N(C_0 + C_1)$ and $P_2 = C_N(C_0 + C_1 + ||\widehat{K}||_\infty)$, such that

$$|||T_+(u)|||_{M^1(\mathbb{R}^N)} \leq P_2 ||u||_{L^1(\mathbb{R}^N)} \qquad \text{for every } u \in L^1 \cap L^2(\mathbb{R}^N). \tag{5.30}$$

Combining this with (5.26) proves the first statement of the theorem.
For $1(b)$, suppose that $u \in \mathcal{M}(\mathbb{R}^N)$. Let ζ_n be the standard mollifier, and denote with $u_n = \zeta_n * u$. Now $u_n \in L^1 \cap L^2(\mathbb{R}^N)$, so that $1(a)$ applies. Observe that $Su_n \to Su$ in $\mathcal{S}'(\mathbb{R}^N)$. Then for fixed ε, ν, x, $(\rho_\varepsilon^\nu * (Su_n))(x) \to (\rho_\varepsilon^\nu * (Su))(x)$ as $n \to \infty$.
This implies that, for every $\lambda > 0$,

$$\mathbb{1}\left\{\sup_\nu \sup_{\varepsilon>0} |\rho_\varepsilon^\nu * (Su)| > \lambda\right\} \leq \liminf_{n\to\infty} \mathbb{1}\left\{\sup_\nu \sup_{\varepsilon>0} |\rho_\varepsilon^\nu * (Su_n)| > \lambda\right\}$$

We conclude using Fatou's Lemma.
(2) follows from the inequality $|\rho^\nu(x)| \leq Q_3 \mathbb{1}_{B_1(x)}$ for a.e. $x \in \mathbb{R}^N$, so that for all $u \in L^2(\mathbb{R}^N)$,

$$M_{\rho^\nu}(u)(x) \leq Q_3 \mathcal{L}^N(B_1) Mu(x).$$

Combining this with the inequality $||Su|||_{L^2} \leq ||\widehat{K}||_\infty ||u||_{L^2}$ gives the result.
Step 3. Here we verify assumption (5) of Lemma 6.6. We need to check that for any $u \in L^2(\mathbb{R}^N)$ satisfying $\operatorname{spt} u \subset \bar{B}_R(x_0)$ and $\int_{\mathbb{R}^N} u = 0$, there holds

$$\int\limits_{|x-x_0|>2R} T_+(u)dx \leq P_1 ||u||_{L^1(\mathbb{R}^N)}. \tag{5.31}$$

Since $\operatorname{spt} u \subset \bar{B}_R(x_0)$ and $\int_{\mathbb{R}^N} u = 0$ we can write for any $x \notin B_{2R}(x_0)$,

$$\begin{aligned}\left(\left(\chi\left(\frac{\cdot}{\varepsilon}\right)K\right)*u\right)(x) &= \int\limits_{|y-x_0|\le R}\left[\chi\left(\frac{x-y}{\varepsilon}\right)K(x-y)-\chi\left(\frac{x-x_0}{\varepsilon}\right)K(x-x_0)\right]u(y)dy\\ &= \int\limits_{|y-x_0|\le R}\chi\left(\frac{x-y}{\varepsilon}\right)\left[K(x-y)-K(x-x_0)\right]u(y)dy\\ &+ \int\limits_{|y-x_0|\le R}\left[\chi\left(\frac{x-y}{\varepsilon}\right)-\chi\left(\frac{x-x_0}{\varepsilon}\right)\right]K(x-x_0)u(y)dy\\ &= I+II.\end{aligned}$$

For any $s \in [0,1]$,

$$|x-x_0+s(x_0-y)| \ge |x-x_0| - |x_0-y| \ge |x-x_0| - R \ge |x-x_0| - |x-x_0|/2 = |x-x_0|/2. \tag{5.32}$$

For I we can estimate the variation as

$$\begin{aligned}|K(x-y)-K(x-x_0)| &\le \int_0^1 |\nabla K(s(x-y)+(1-s)(x-x_0))||y-x_0|ds\\ &\le \int_0^1 \frac{C_1|y-x_0|}{|x-x_0+s(x_0-y)|^{N+1}}ds \le \frac{2^{N+1}C_1R}{|x-x_0|^{N+1}}.\end{aligned}$$

Then I has the upper bound

$$\left|\int\limits_{|y-x_0|\le R}\chi\left(\frac{x-y}{\varepsilon}\right)\left[K(x-y)-K(x-x_0)\right]u(y)dy\right| \tag{5.33}$$

$$\le \int\limits_{|y-x_0|\le R}\frac{2^{N+1}C_1R}{|x-x_0|^{N+1}}|u(y)|dy = \frac{2^{N+1}C_1R}{|x-x_0|^{N+1}}||u||_{L^1(\mathbb{R}^N)}. \tag{5.34}$$

Observe that

$$\left|\chi\left(\frac{x-y}{\varepsilon}\right)-\chi\left(\frac{x-x_0}{\varepsilon}\right)\right| \le ||\nabla\chi||_\infty\frac{R}{\varepsilon}, \tag{5.35}$$

and the variation of χ vanishes whenever

$$\left|\frac{x-y}{\varepsilon}\right| > 1 \quad \text{and} \quad \left|\frac{x-x_0}{\varepsilon}\right| > 1.$$

Now since $|x-x_0| \le 2|x-y|$, whenever $|x-y| \le \varepsilon$ or $|x-x_0| \le \varepsilon$, we have $|x-x_0|/2 \le \varepsilon$. This improves the bound in (5.35),

$$\left|\chi\left(\frac{x-y}{\varepsilon}\right)-\chi\left(\frac{x-x_0}{\varepsilon}\right)\right| \le ||\nabla\chi||_\infty\frac{R}{|x-x_0|}. \tag{5.36}$$

It follows that

$$\int\limits_{|y-x_0|\le R}\left[\chi\left(\frac{x-y}{\varepsilon}\right)-\chi\left(\frac{x-x_0}{\varepsilon}\right)\right]K(x-x_0)u(y)dy \tag{5.37}$$

$$\le \int\limits_{|y-x_0|\le R}\frac{6R}{|x-x_0|}\frac{C_0}{|x-x_0|^N}|u(y)|dy = \frac{6C_0R}{|x-x_0|^{N+1}}||u||_{L^1(\mathbb{R}^N)} \tag{5.38}$$

This combined with the estimate for I yields

$$T_+(u)(x) = I + II \leq (6C_0 + 2^{N+1}C_1)\frac{R}{|x-x_0|^{N+1}}||u||_{L^1(\mathbb{R}^N)},$$

for every x such that $|x-x_0|> 2R$. Integrating over this set $|x-x_0|> 2R$ gives (5.31) with $P_1 = C_N(C_0+C_1)$.

Step 4. We lastly verify assumption (4) of Lemma 6.6.
Fix a nonnegative convolution kernel $\tilde{\rho} \in C_c^\infty(B_1)$ with $\int_{\mathbb{R}^N} \tilde{\rho} = 1$. We define $\tilde{\Delta}_\varepsilon$ as in (5.21), with $\tilde{\rho}$ instead of ρ^ν. The inequality (5.25) holds, for all $u \in L^2(\mathbb{R}^N)$,

$$\sup_{\varepsilon>0}\left|(\tilde{\Delta}_\varepsilon * u)(x)\right| \leq C_N(C_0 + C_1 + ||\widehat{K}||_\infty)Mu(x) \tag{5.39}$$

Moreover, we have also

$$\sup_{\varepsilon>0}|(\tilde{\rho}_\varepsilon * Su)(x)| \leq C_N M(Su)(x). \tag{5.40}$$

Step 2 also implies

$$T_+(u) = \sup_{\varepsilon>0}\left|\left(\chi\left(\frac{\cdot}{\varepsilon}\right)K\right) * u\right| \leq \sup_{\varepsilon>0}\left|\tilde{\Delta}_\varepsilon * u\right| + \sup_{\varepsilon>0}|\tilde{\rho}_\varepsilon * Su|\,.$$

Combining this with (5.39) and (5.40), and the strong estimate on L^2 from Propositions 5.6 and 6.4 gives finally

$$||T_+(u)||_{L^2(\mathbb{R}^N)}\leq C_N(C_0 + C_1 + ||\widehat{K}||_\infty)||u||_{L^2(\mathbb{R}^N)},$$

which implies assumption (4) of Lemma 6.6 with $P_2 = C_N(C_0 + C_1 + ||\widehat{K}||_\infty)$.

□

16. Stability for vector fields whose gradient is the singular integral of an L^1 function

We review an extension of the estimates in section 14, performed under the assumption that the gradient of b is no longer in L^p (or even L^1), but which is the singular integral of an L^1 function. In order to obtain well-posedness results for the regular Lagrangian flow, we require growth conditions on the vector field as well as the gradient. We first make precise the regularity setting under which the integrals in 14.1 make sense, when the vector field is not globally bounded. Rather than truncating the integrals over B_r, one should integrate only over bounded trajectories of the flow, which we define as the sublevel G_λ. Let

$$G_\lambda = \{x \in \mathbb{R}^N : |X(s,x)|\leq \lambda \text{ for every } s \in [t,T]\}\,. \tag{5.41}$$

In order to ensure that the complement of the sublevel (the superlevel of the flow) does not grow too large, we impose the following growth condition on b.

(R1) $b(t,x)$ can be decomposed as

$$\frac{b(t,x)}{1+|x|} = \tilde{b}_1(t,x) + \tilde{b}_2(t,x)\,,$$

with

$$\tilde{b}_1 \in L^1((0,T);L^1(\mathbb{R}^N)) \qquad \text{and} \qquad \tilde{b}_2 \in L^1((0,T);L^\infty(\mathbb{R}^N))\,.$$

When the vector field is not globally bounded, the associated flow $X(t,\cdot)$ is not locally integrable in $\mathbb{R}^2$. Thus we describe a formulation in the renormalized sense of the ODE that makes sense under a relaxed growth condition. Given a vector field satisfying **(R1)**, we formalize the notion of regular Lagrangian flows with a logarithmic summability, which are Lagrangian flows in a renormalized sense.

DEFINITION 16.1 (Regular Lagrangian flow). *If b is a vector field satisfying* **(R1)**, *then for fixed $t \in [0,T)$, a map*

$$X \in C([t,T]_s; L^0_{\text{loc}}(\mathbb{R}^N)) \cap \mathcal{B}([t,T]_s; \log L_{\text{loc}}(\mathbb{R}^N))$$

is a regular Lagrangian flow in the renormalized sense relative to b starting at t if we have the following:

(1) *The equation*

$$\partial_s(\beta(X(s,x)))= \beta'(X(s,x))b(s,X(s,x)) \tag{5.42}$$

holds in $\mathcal{D}'((t,T)\times\mathbb{R}^N)$, for every function $\beta \in C^1(\mathbb{R}^N;\mathbb{R})$ that satisfies $|\beta(z)|\leq C(1+\log(1+|z|))$ and $|\beta'(z)|\leq \frac{C}{1+|z|}$ for all $z \in \mathbb{R}^N$,

(2) *$X(t,x) = x$ for $\mathcal{L}^N$-a.e $x \in \mathbb{R}^N$,*

(3) *There exists a constant $L \geq 0$ such that $\int_{\mathbb{R}^N} \varphi(X(s,x))dx \leq L\int_{\mathbb{R}^N} \varphi(x)dx$ for all measurable $\varphi : \mathbb{R}^N \to [0,\infty)$.*

REMARK 16.2. *Note that* **(R1)** *enables the right-hand side of (5.42) to be in $L^1((t,T);L^1_{\text{loc}}(\mathbb{R}^N))$. Since we do not assume global boundedness of b, $X(s,\cdot)$ is not locally integrable in $\mathbb{R}^N$. The $\log L_{\text{loc}}(\mathbb{R}^N)$ bound comes from integrating (5.42) in s.*

We remark that by now this is the usual definition of flows for weakly differentiable vector fields satisfying the general growth condition **(R1)**. The renormalization setting has been introduced and exploited in [**25**, 5] in the Sobolev and BV settings.

The following lemma gives an estimate for the decay of the superlevels of a regular Lagrangian flow.

LEMMA 16.3. *Let $b : (0,T)\times\mathbb{R}^N \to \mathbb{R}^N$ be a vector field satisfying* **(R1)** *and let $X : [t,T]\times\mathbb{R}^N \to \mathbb{R}^N$ be a regular Lagrangian flow relative to b starting at time t, with compressibility constant L. Then for all $r, \lambda > 0$*

$$\mathcal{L}^N(B_r \setminus G_\lambda) \leq g(r,\lambda)\,,$$

where the function g depends only on L, $\|\tilde{b}_1\|_{L^1((0,T);L^1(\mathbb{R}^N))}$ and $\|\tilde{b}_2\|_{L^1((0,T);L^\infty(\mathbb{R}^N))}$ and satisfies $g(r,\lambda)\downarrow 0$ for r fixed and $\lambda \uparrow \infty$.

PROOF. The result follows from the bound

$$\int_{B_r} \log\left(\frac{1+|X(s,x)|}{1+r}\right) dr \leq L||\tilde{b}_1||_{L^2(\mathbb{R}^N)}+\mathcal{L}^N(B_r)||\tilde{b}_2||_{L^2(\mathbb{R}^N)}, \tag{5.43}$$

for any $r > 0$. We omit the full proof, since we perform a similar estimate in section 27.2. □

A second regularity assumption is that Db has the representation **(R2)**:

(R2)

$$\partial_j b = \sum_{k=1}^m S_{jk} g_{jk} \quad \text{in } \mathcal{D}'((0,T)\times\mathbb{R}^N),$$

where S_{jk} are singular integral operators of fundamental type in $\mathbb{R}^N$ and the functions g_{jk} are in $L^1((0,T)\times\mathbb{R}^N)$ for every $j = 1,...N$ and $k = 1,...,m$.

We will additionally assume that

(R3)

$$b \in L^p_{\text{loc}}([0,T]\times\mathbb{R}^N) \qquad \text{for some } p > 1.$$

16.1. Estimate of difference quotients. We recall that in Section 14 in order to estimate the quantity $\Phi_\delta(t)$ in 14.1 an estimate of the difference quotients of b is given by the maximal function of Db. For this maximal function to be integrable, we require that $Db \in L^p_{\text{loc}}$. This not the case when Db is merely integrable, or a measure. However, it turns out that an analogous lemma as 5.9 holds for L^1 functions whose derivatives are singular integrals of measures or L^1 functions.

PROPOSITION 16.4. *Let $f \in L^1_{\text{loc}}(\mathbb{R}^N)$ and assume that for every $j = 1, \dots, d$ we have*

$$\partial_j f = \sum_{k=1}^{m} R_{jk} g_{jk} \tag{case b}$$

in the sense of distributions, where R_{jk} are singular integral operators of fundamental type in $\mathbb{R}^N$ and $g_{jk} \in \mathcal{M}(\mathbb{R}^N)$ for $j = 1, \dots, d$ and $k = 1, \dots, m$, and $R_{jk} g_{jk}$ is defined in the sense of tempered distributions. Then there exists a nonnegative function $V \in M^1(\mathbb{R}^n)$ and an $\mathcal{L}^N$-negligible set $\mathcal{N} \subset \mathbb{R}^N$ such that for every $x, y \in \mathbb{R}^d \setminus \mathcal{N}$ there holds

$$|f(x) - f(y)| \leq |x - y| \Big(V(x) + V(y) \Big),$$

where V is given by

$$V := \mathcal{V}(R, g) = \sum_{j=1}^{N} \sum_{k=1}^{m} M_{\{\Upsilon^{\xi,j},\, \xi \in \mathbb{S}^{d-1}\}}(R_{jk} g_{jk}) \tag{5.44}$$

and $\Upsilon^{\xi,j}$, for $\xi \in \mathbb{S}^{N-1}$ and $j = 1, \dots, N$, is a family of smooth functions explicitly constructed in the course of the proof.

PROOF. We omit the proof, since we will prove a similar Proposition in section 17. □

REMARK 16.5. *Theorem 15.2 implies that the operator $g \mapsto \mathcal{V}(R, g)$ is bounded $L^2 \to L^2$ and $\mathcal{M} \to M^1$.*

It is by using this result that the following stability theorem is obtained in [16]. The idea is to consider a functional $\Phi_\delta(t)$ in the same spirit as section 14, with the added difficulty that the operator controlling the difference quotients of b is no longer integrable. For this issue, the interpolation estimate in Lemma 5.10 will be useful.

THEOREM 16.6. *Let b and $\bar{b}$ be two vector fields satisfying assumption* **(R1)**, *and assume that b also satisfies assumptions* **(R2)** *and* **(R3)**. *Fix $t \in [0, T)$ and let X and $\bar{X}$ be regular Lagrangian flows starting at time t associated to b and $\bar{b}$ respectively, with compressibility constants L and $\bar{L}$. Then the following holds. For every $\gamma > 0$ and $r > 0$ and for every $\eta > 0$ there exist $\lambda > 0$ and $C_{\gamma,r,\eta} > 0$ such that*

$$\mathcal{L}^N \left(B_r \cap \{ |X(s, \cdot) - \bar{X}(s, \cdot)| > \gamma \} \right) \leq C_{\gamma,r,\eta} ||b - \bar{b}||_{L^1((0,T) \times B_\lambda)} + \eta$$

for all $s \in [t, T]$. The constants λ and $C_{\gamma,r,\eta}$ also depend on:

- *The equi-integrability in $L^1((0,T); L^1(\mathbb{R}^N))$ of the functions g_{jk} associated to b as in* **(R2)**,
- *The norms of the singular integral operators $S^{\cdot i}_{jk}$, associated to b as in* **(R2)** *(i.e. the constants $C_0 + C_1 + ||\hat{K}||_\infty$,*
- *The norm in $L^p((0,T) \times B_\lambda)$ of b,*
- *The $L^1((0,T); L^1(\mathbb{R}^N)) + L^1((0,T); L^\infty(\mathbb{R}^N))$ norms of the decompositions of b and $\bar{b}$ as in* **(R1)**,
- *The compressibility constants L and $\bar{L}$.*

In order to improve the readability of the following (many) estimates, we will use the notation "$\lesssim$" to denote an estimate up to a constant only depending on absolute constants and on the bounds assumed in Theorem 16.6, and the notation "$\lesssim_\lambda$" to mean that the constant could also depend on the truncation parameter λ.

PROOF. For any $\delta, \lambda < 0$, $s \in [t, T]$, let

$$\Phi_\delta(s) = \int\limits_{B_r \cap G_\lambda \cap \bar{G}_\lambda} \log\left(1 + \frac{|X(s,x) - \bar{X}(s,x)|}{\delta}\right) dx. \tag{5.45}$$

where G_λ and $\bar{G}_\lambda$ are the sublevels of X and $\bar{X}$. Following the line of (5.4) and because of Remark 14.11 we have

$$\begin{aligned}\Phi'_\delta(s) &\leq \frac{\bar{L}}{\delta}||b(s,\cdot) - \bar{b}(s,\cdot)||_{L^1(t,\tau)\times B_\lambda)} \\ &+ \int\limits_{B_r \cap G_\lambda \cap \bar{G}_\lambda} \min\left\{\frac{|b(s,X(s,x))|+|b(s,\bar{X}(s,x))|}{\delta}, \frac{|b(s,X(s,x)) - b(s,\bar{X}(s,x))|}{|X(s,x) - \bar{X}(s,x)|}\right\} dx.\end{aligned}$$

Integrating over $s \in (t, \tau)$ and applying Proposition 16.4 for almost every s, we have existence of a function $\mathcal{V}(S, g) := V \in M^1(\mathbb{R}^N)$ (defined as in (5.44)) so that

$$\begin{aligned}\Phi_\delta(\tau) &\leq \frac{\bar{L}}{\delta}||b - \bar{b}||_{L^1(t,\tau)\times B_\lambda)} \\ &+ \int_t^\tau \int\limits_{B_r \cap G_\lambda \cap \bar{G}_\lambda} \min\left\{\frac{|b(s,X(s,x))|+|b(s,\bar{X}(s,x))|}{\delta}, V(s,X(s,x)) + V(s,\bar{X}(s,x))\right\} dxds.\end{aligned}$$

Fix $\varepsilon > 0$. We apply Lemma 5.11 to the finite family $g_{jk} \in L^1((0,T) \times \mathbb{R}^N)$. This gives a constant C_ε and a set of finite measure A_ε such that for each $j = 1, ..., N$ and $k = 1, ..., m$,

$$g_{jk}(s,x) = g^1_{jk}(s,x) + g^2_{jk}(s,x),$$

with

$$||g^1_{jk}||_{L^1((0,T)\times\mathbb{R}^N)} \leq \varepsilon, \qquad \operatorname{spt}(g^2_{jk}) \subset A_\varepsilon, \qquad ||g^2_{jk}||_{L^2((0,T)\times\mathbb{R}^N)} \leq C_\varepsilon. \tag{5.46}$$

Then we exploit sub-additivity of V to get

$$V = \mathcal{V}(S,g) = \mathcal{V}(S, g^1 + g^2) \leq \mathcal{V}(S, g^1) + \mathcal{V}(S, g^2) = V^1 + V^2.$$

Plugging this into the integral gives

$$\Phi_\delta(\tau) \leq \frac{\bar{L}}{\delta}||b - \bar{b}||_{L^1((t,\tau)\times B_\lambda)} \tag{5.47}$$

$$+ \int_t^\tau \int\limits_{B_r \cap G_\lambda \cap \bar{G}_\lambda} \min\left\{\frac{|b(s,X(s,x))|+|b(s,\bar{X}(s,x))|}{\delta}, V^1(s,X(s,x)) + V^1(s,\bar{X}(s,x))\right\} dxds \tag{5.48}$$

$$+ \int_t^\tau \int\limits_{B_r \cap G_\lambda \cap \bar{G}_\lambda} \min\left\{\frac{|b(s,X(s,x))|+|b(s,\bar{X}(s,x))|}{\delta}, V^2(s,X(s,x)) + V^2(s,\bar{X}(s,x))\right\} dxds \tag{5.49}$$

$$= \frac{\bar{L}}{\delta}||b - \bar{b}||_{L^1((t,\tau)\times B_\lambda)} + I_1 + I_2. \tag{5.50}$$

We can disregard the first element in the minimum, change variable and estimate the second integral by

$$I_2 \leq (L + \bar{L}) \int_t^\tau ds \int\limits_{B_r \cap G_\lambda \cap \bar{G}_\lambda} V^2(s,x)dx \leq (L + \bar{L})[(\tau - t)\mathcal{L}^N(B_\lambda)]^{1/2}||V^2||_{L^2(t,\tau)\times\mathbb{R}^N)} \tag{5.51}$$

$$\leq (L + \bar{L})P_2[(\tau - t)\mathcal{L}^N(B_\lambda)]^{1/2}||g^2||_{L^2(t,\tau)\times\mathbb{R}^N)} \lesssim_{\lambda,\tau,L,\bar{L}} C_\varepsilon, \tag{5.52}$$

where in the last line we have applied Theorem 15.2 to the operator V^2. Applying Theorem 15.2 to V^1 and using the inequality in Remark 5.4 we get

$$|||V^1|||_{M^1((t,\tau)\times\mathbb{R}^N)} \leq P_1||g^1||_{L^1((t,\tau)\times\mathbb{R}^N)} \lesssim \varepsilon. \tag{5.53}$$

We apply now the Interpolation Lemma 5.10 to the function

$$\varphi(s,x) = \min\left\{\frac{|b(s,X(s,x))|+|b(s,\bar{X}(s,x))|}{\delta}, V^1(s,X(s,x)) + V^1(s,\bar{X}(s,x))\right\}$$

to estimate I_1. After changing variable $X(s,x) \mapsto x$, we get

$$I_1 = ||\varphi||_{L^1((t,\tau)\times(B_r\cap G_\lambda\cap\bar{G}_\lambda))} \lesssim_{N,r,p,L,\bar{L}} ||g^1||_{L^1((t,\tau)\times\mathbb{R}^N)}\left[1+\log\left(\frac{||b||_{L^p((t,\tau)\times B_\lambda)}}{||g^1||_{L^1((0,T)\times\mathbb{R}^N)}\delta}\right)\right]. \tag{5.54}$$

Plugging this into (5.47) and using (5.53) we deduce that

$$\Phi_\delta(\tau) \lesssim_{\lambda,\tau,N,p,L,\bar{L}} \frac{\bar{L}}{\delta}||b-\bar{b}||_{L^1((t,\tau)\times B_\lambda)} + C_\varepsilon + \varepsilon\left[1+\log\left(\frac{||b||_{L^p((t,\tau)\times B_\lambda)}}{\varepsilon\delta}\right)\right]. \tag{5.55}$$

Arguing as in (5.5), we can derive the upper bound

$$\mathcal{L}^N(B_r\cap\{|X(\tau,x)-\bar{X}(\tau,x)|>\gamma\}) \leq \frac{\Phi_\delta(\tau)}{\log\left(1+\frac{\gamma}{\delta}\right)} + \mathcal{L}^N(B_r\setminus G_\lambda) + \mathcal{L}^N(B_r\setminus\bar{G}_\lambda)\,. \tag{5.56}$$

Combining this with (5.55) we obtain

$$\begin{aligned}\mathcal{L}^N(B_r\cap\{|X(\tau,x)-\bar{X}(\tau,x)|>\gamma\}) &\lesssim_{N,p,r,L,\bar{L}} \frac{\bar{L}}{\delta\log(1+\frac{\gamma}{\delta})}||b-\bar{b}||_{L^1((t,\tau)\times B_\lambda)} \\ &+ \frac{C_\varepsilon}{\log(1+\frac{\gamma}{\delta})} + \frac{\varepsilon\log\left(\frac{||b||_{L^p((t,\tau)\times B_\lambda)}}{\varepsilon\delta}\right)}{\log(1+\frac{\gamma}{\delta})} + \mathcal{L}^N(B_r\setminus G_\lambda) + \mathcal{L}^N(B_r\setminus\bar{G}_\lambda).\end{aligned} \tag{5.57}$$

We fix $\eta > 0$. To conclude we choose $\lambda > 0$ large so that by Lemma 16.3 the last two terms are smaller than $\eta/2$. Then choosing $\varepsilon > 0$ small enough so that the third term is bounded, we conclude by choosing $\delta > 0$ small enough so that the second term is bounded. This fixes

$$C_{\gamma,r,\eta} = \frac{\bar{L}}{\delta(1+\gamma/\delta)}.$$

□

17. Anisotropic vector fields

We now consider the following splitting of space variables. We write $\mathbb{R}^N = \mathbb{R}^{n_1}_x \times \mathbb{R}^{n_2}_y$, and split the vector field $b = (b_1, b_2)$. We consider the case in which D_1b_2 is the tensor product of an $L^p_{\text{loc}}(\mathbb{R}^{n_2})$ function with a singular integral (in $\mathbb{R}^{n_1}$) of a measure, while D_1b_1, D_2b_1 and D_2b_2 are tensor product of an $L^p_{\text{loc}}(\mathbb{R}^{n_2})$ function with singular integrals (in $\mathbb{R}^{n_1}$) of integrable functions:

$$Db = \begin{pmatrix}(S_x * L^1_x)L^p_{\text{loc,y}} & (S_x * L^1_x)L^p_{\text{loc,y}} \\ (S_x * \mathcal{M}_x)L^p_{\text{loc,y}} & (S_x * L^1_x)L^p_{\text{loc,y}}\end{pmatrix} \tag{5.58}$$

where $p > 1$. Compared to [16], we are able to consider a situation in which some entries of the differential matrix Db are measures. (From a PDE point of view, related contexts have been considered in [42, 43].

The idea of the anisotropic functional in the spirit of (5.4) will be to weight differently the two (groups of) directions, according to the different degrees of regularity. In our context, this can be done by considering, instead of (5.45), a functional depending on *two* parameters δ_1 and δ_2, with $\delta_1 \le \delta_2$, namely

$$\Phi_{\delta_1,\delta_2}(s) = \int \log\left(1 + \left|\left(\frac{|X_1(s,x) - \bar{X}_1(s,x)|}{\delta_1}, \frac{|X_2(s,x) - \bar{X}_2(s,x)|}{\delta_2}\right)\right|\right) dx\,. \tag{5.59}$$

Following the same strategy as before (estimate of the difference quotients and interpolation in the minimum in (5.45)), we derive the following bound, which replaces (5.55) in this context:

$$\Phi_{\delta_1,\delta_2}(s) \lesssim \left[\frac{\delta_1}{\delta_2}\|D_1 b_2\|_{\mathcal{M}} + \frac{\delta_2}{\delta_1}\|D_2 b_1\|_{L^1} + \|D_1 b_1\|_{L^1} + \|D_2 b_2\|_{L^1}\right] \log\left(\frac{1}{\delta_2}\right)\,.$$

We need to gain some "smallness" in criterion (5.7). Observe that $\|D_2 b_1\|_{L^1}$, $\|D_1 b_1\|_{L^1}$ and $\|D_2 b_2\|_{L^1}$ can be assumed to be small, by the same equi-integrability argument as in [16]. This is however not the case for $\|D_1 b_2\|_{\mathcal{M}}$. But we can exploit the presence of the coefficient δ_1/δ_2 multiplying this term: both δ_1 and δ_2 have to be sent to zero, but we can do this with $\delta_1 \ll \delta_2$.

One relevant technical point in the proof is the estimate for the anisotropic difference quotients showing up when differentiating (5.59). We need an estimate of the form:

$$|f(x) - f(y)| \lesssim \left|\left(\frac{x_1 - y_1}{\delta_1}, \frac{x_2 - y_2}{\delta_2}\right)\right| \Big[U(x) + U(y)\Big]\,. \tag{5.60}$$

This is complicated by the fact that, as in the classical case, one expects to use a maximal function in x_1 and x_2 in order to estimate the difference quotients, but however this would not match (in terms of persistence of cancellations) with the presence of a singular integral in the variable x_1 only. This is resolved in Section 19 by the use of tensor products of maximal functions, and will result in the proof of (5.60) together with a bound of the form

$$\|U\| \le \delta_1 \|D_1 f\| + \delta_2 \|D_2 f\|\,.$$

Another technical issue is that a smooth isotropic maximal function cannot be composed with the 'dilated' singular integral in x and y variables since the persistence of cancellations fails: the operator norm blows up like $(\delta_2/\delta_1)^{N-1}$. Our estimate can however reconcile with the $W^{1,1}(\mathbb{R}^N)$ case since the delta distribution does not see this dilation. This is the plan how to obtain the proof of our main Theorem 20.1. As recalled in Section 20 we obtain as a corollary of Theorem 20.1 existence, uniqueness, stability (with an effective rate) and compactness for regular Lagrangian flows, and well-posedness for Lagrangian solutions to the continuity and transport equations.

18. Regularity assumptions and the anisotropic functional

We wish to consider a regularity setting of the vector field $b(t,x)$ in which the (weak) regularity has a different character with respect to different directions in space. We split $\mathbb{R}^N$ as $\mathbb{R}^N = \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ with variables $x_1 \in \mathbb{R}^{n_1}$ and $x_2 \in \mathbb{R}^{n_2}$. We denote by $D_1 = D_{x_1}$ the derivative with respect to the first n_1 variables x_1, and by $D_2 = D_{x_2}$ the derivative with respect to the last n_2 variables x_2. Accordingly, we denote $b = (b_1, b_2)(s, x_1, x_2)$. For $X(s, x_1, x_2)$ a regular Lagrangian flow associated to b we denote $X = (X_1, X_2)(s, x_1, x_2)$.

We are going to assume that $D_1 b_2$ is "less regular" than $D_1 b_1, D_2 b_1, D_2 b_2$: the derivative $D_1 b_2$ is a singular integral of a measure, whereas the other derivatives are singular integrals of L^1 functions. This is made precise as follows:

(R2a) Assume that

$$Db = \begin{pmatrix} D_1 b_1 & D_2 b_1 \\ D_1 b_2 & D_2 b_2 \end{pmatrix} = \begin{pmatrix} \gamma^1 S^1 \mathfrak{p} & \gamma^2 S^2 \mathfrak{q} \\ \gamma^3 S^3 \mathfrak{m} & \gamma^4 S^4 \mathfrak{r} \end{pmatrix}\,, \tag{5.61}$$

where the sub-matrices have the representation

$i, j \in \{1, \ldots, n_1\}$:

$$(D_1 b_1)^i_j = \sum_{k=1}^{m} \gamma^{1i}_{jk}(s, x_2) S^{1i}_{jk} \mathfrak{p}^i_{jk}(s, x_1)$$

$i \in \{1, \ldots, n_1\}, \quad j \in \{n_1 + 1, \ldots, n_2\}$:

$$(D_2 b_1)^i_j = \sum_{k=1}^{m} \gamma^{2i}_{jk}(s, x_2) S^{2i}_{jk} \mathfrak{q}^i_{jk}(s, x_1)$$

$i \in \{n_1 + 1, \ldots, n_2\}, \quad j \in \{1, \ldots, n_1\}$:

$$(D_1 b_2)^i_j = \sum_{k=1}^{m} \gamma^{3i}_{jk}(s, x_2) S^{3i}_{jk} \mathfrak{m}^i_{jk}(s, x_1)$$

$i \in \{n_1 + 1, \ldots, n_2\}, \quad j \in \{n_1 + 1, \ldots, n_2\}$:

$$(D_2 b_2)^i_j = \sum_{k=1}^{m} \gamma^{4i}_{jk}(s, x_2) S^{4i}_{jk} \mathfrak{r}^i_{jk}(s, x_1)\,.$$

In the above assumptions we have that:

- $S^{1i}_{jk}, S^{2i}_{jk}, S^{3i}_{jk}, S^{4i}_{jk}$ are singular integral operators associated to singular kernels of fundamental type in $\mathbb{R}^{n_1}$,
- the functions $\mathfrak{p}^i_{jk}$, $\mathfrak{q}^i_{jk}$, $\mathfrak{r}^i_{jk}$ belong to $L^1((0,T); L^1(\mathbb{R}^{n_1}))$,
- $\mathfrak{m}^i_{jk} \in L^1((0,T); \mathcal{M}(\mathbb{R}^{n_1}))$,
- the functions $\gamma^{1,i}_{jk}, \gamma^{2,i}_{jk}, \gamma^{3,i}_{jk}, \gamma^{4,i}_{jk}$ belong to $L^\infty((0,T); L^q(\mathbb{R}^{n_2}))$ for some $q > 1$.

We have denoted by $L^1((0,T); \mathcal{M}(\mathbb{R}^{n_1}))$ the space of all functions $t \mapsto \mu(t, \cdot)$ taking values in the space $\mathcal{M}(\mathbb{R}^{n_1})$ of finite signed measures on $\mathbb{R}^{n_1}$ such that

$$\int_0^T \|\mu(t, \cdot)\|_{\mathcal{M}(\mathbb{R}^{n_1})}\, dt < \infty\,.$$

REMARK 18.1. *The assumption on the functions $\gamma^{1,i}_{jk}, \gamma^{2,i}_{jk}, \gamma^{3,i}_{jk}, \gamma^{4,i}_{jk}$ could be relaxed to $L^\infty((0,T); L^q_{\mathrm{loc}}(\mathbb{R}^{n_2}))$. This would require the use of a localized maximal function as in section 14.*

The alternative assumption to **(R2a)** is the following. Rather than considering singular integral operators on $\mathbb{R}^N$, we consider a vector field for which $D_1 b_2$ is a measure on $\mathbb{R}^N$, and for which all other derivatives belong to $L^1(\mathbb{R}^N)$. This would correspond to the case of a vector field $b = (b_1, b_2)$ such that b_2 is BV in x_1 and $W^{1,1}$ in x_2, and b_1 is $W^{1,1}$ in both x_1 and x_2. Here we may discard the tensor product of functions on $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ in favor of L^1 functions on $\mathbb{R}^N$ if we only consider the singular integral operator on $L^1(\mathbb{R}^N)$ given by a Dirac delta distribution. It will become clear later why one cannot generalize the Dirac delta to more general singular integral operators when we formally derive norm estimates for anisotropic dilations of such operators.

(R2b) Assume that

$$Db = \begin{pmatrix} D_1 b_1 & D_2 b_1 \\ D_1 b_2 & D_2 b_2 \end{pmatrix} = \begin{pmatrix} \mathfrak{p} & \mathfrak{q} \\ \mathfrak{m} & \mathfrak{r} \end{pmatrix}, \tag{5.62}$$

where the sub-matrices have the representation

$i, j \in \{1, \ldots, n_1\}$:

$$(D_1 b_1)^i_j = \sum_{k=1}^{m} \mathfrak{p}^i_{jk}(s, x_1, x_2)$$

$i \in \{1, \ldots, n_1\}, \quad j \in \{n_1 + 1, \ldots, n_2\}$:

$$(D_2 b_1)^i_j = \sum_{k=1}^{m} \mathfrak{q}^i_{jk}(s, x_1, x_2)$$

$i \in \{n_1 + 1, \ldots, n_2\}, \quad j \in \{1, \ldots, n_1\}$:

$$(D_1 b_2)^i_j = \sum_{k=1}^{m} \mathfrak{m}^i_{jk}(s, x_1, x_2)$$

$i \in \{n_1 + 1, \ldots, n_2\}, \quad j \in \{n_1 + 1, \ldots, n_2\}$:

$$(D_2 b_2)^i_j = \sum_{k=1}^{m} \mathfrak{r}^i_{jk}(s, x_1, x_2)\,.$$

In the above assumptions we have that:
- the functions $\mathfrak{p}^i_{jk}$, $\mathfrak{q}^i_{jk}$, $\mathfrak{r}^i_{jk}$ belong to $L^1((0,T);L^1(\mathbb{R}^N))$,
- $\mathfrak{m}^i_{jk} \in L^1((0,T);\mathcal{M}(\mathbb{R}^N))$.

We will additionally need to assume property **(R3)** as before.

As mentioned in the Introduction, the proof of our main result will exploit an *anisotropic* functional (already provisionally introduced in (5.59)), which extends the functional (5.45) to the regularity setting under investigation. Let A be the constant $N \times N$ matrix

$$A = \mathrm{Diag}\,(\delta_1, \dots, \delta_1, \delta_2, \dots, \delta_2)\,. \tag{5.63}$$

A acts on vectors in $\mathbb{R}^N$ by a dilation of a factor δ_1 on the first n_1 coordinates, and of a factor δ_2 on the last n_2 coordinates: $A(x_1,x_2) = (\delta_1 x_1, \delta_2 x_2)$.

Given $X(t,x_1,x_2)$ and $\bar X(t,x_1,x_2)$ regular Lagrangian flows associated to b and $\bar b$ respectively, we denote by G_λ and $\bar G_\lambda$ the sublevels of X and $\bar X$ defined as in (5.41). The proof of our main theorem (see Theorem 20.1) is based on the study of the following anisotropic functional:

$$\Phi_{\delta_1,\delta_2}(s) = \int\limits_{B_r \cap G_\lambda \cap \bar G_\lambda} \log\left(1 + \left|A^{-1}\left[X(s,x_1,x_2) - \bar X(s,x_1,x_2)\right]\right|\right)\,dx\,. \tag{5.64}$$

19. Estimates of anisotropic difference quotients

In this section we wish to generalize the classical estimate in Lemma 5.9 for the difference quotients of a BV function, into an analogous "anisotropic" setting for vector fields in the regularity setting of **(R2a) or (R2b)**. This will be a key tool in order to estimate the functional (5.64).

In the following three subsections we prove similar estimates in the anisotropic context.

19.1. Split regularity: the isotropic estimate. Given $\{\gamma^\nu(x_1)\}_\nu \subset C^\infty_c(\mathbb{R}^{n_1})$, $\{\rho^\sigma(x_2)\}_\sigma \subset C^\infty_c(\mathbb{R}^{n_2})$ and $u \in \mathcal{S}'(\mathbb{R}^N)$ we define

$$M_{\{\gamma^\nu\otimes\rho^\sigma\}}u(x) = \sup_{\varepsilon>0}\sup_{\nu,\sigma}|(\gamma^\nu(x_1)\rho^\sigma(x_2))_\varepsilon * u(x)| = \sup_{\varepsilon>0}\sup_{\nu,\sigma}\left|\left(\frac{1}{\varepsilon^N}\gamma^\nu\left(\frac{x_1}{\varepsilon^{n_1}}\right)\rho^\sigma\left(\frac{x_2}{\varepsilon^{n_2}}\right)\right) * u(x)\right|\,. \tag{5.65}$$

We first of all prove an isotropic estimate in a regularity context related to **(R2a)**, in contrast to case b in Proposition 16.4.

Lemma 19.1. *Let $f : \mathbb{R}^N \to \mathbb{R}$ be a function such that for each $j = 1, \dots, N$ we have*

$$\partial_j f = \sum_{k=1}^m (R_{jk} g_{jk})(x_1)\gamma_{jk}(x_2)\,, \tag{case a}$$

where R_{jk} are singular integrals of fundamental type in $\mathbb{R}^{n_1}$, $g_{jk} \in \mathcal{M}(\mathbb{R}^{n_1})$ and $\gamma_{jk} \in L^q(\mathbb{R}^{n_2})$, for some $q > 1$.

Then there exists a nonnegative function $V : \mathbb{R}^N \to [0,\infty)$ and an $\mathcal{L}^N$-negligible set $\mathcal{N} \subset \mathbb{R}^N$ such that for every $x, y \in \mathbb{R}^N \setminus \mathcal{N}$

$$|f(x) - f(y)| \leq |x-y|\Big(V(x) + V(y)\Big)\,.$$

The function V is given by

$$V := \mathcal{V}(R,\gamma,g) = \sum_{j=1}^N \sum_{k=1}^m M_{\{\Upsilon^{\xi,j}\otimes\bar\Upsilon^{\xi,j}\}}(\gamma_{jk}R_{jk}g_{jk})\,, \tag{case a}$$

for suitable smooth compactly supported functions $\Upsilon^{\xi,j}$ and $\bar\Upsilon^{\xi,j}$, which will be introduced in the proof.

PROOF. We adapt the proof of Proposition 16.4 to the current regularity setting. The difficulty is that a smooth maximal function in $\mathbb{R}^N$ composed with the singular kernel on $\mathbb{R}^{n_1}$ does not enjoy suitable bounds, and so we use a tensor product of smooth functions, as in (5.65).

Let $w = (w_1, w_2) \in \mathbb{R}^N$, and let $\{e_j\}_j$ be the standard basis for $\mathbb{R}^N$. We denote $\{w_1\}^j = (w_1, 1, \ldots, 1)\cdot e_j$ and $\{w_2\}^j = (1, \ldots, 1, w_2) \cdot e_j$. Define the families of functions

$$\begin{cases} \Upsilon^{\xi,j}(w_1) = h^1\left(\frac{\xi_1}{2} - w_1\right)\{w_1\}^j \\ \\ \bar{\Upsilon}^{\xi,j}(w_2) = h^2\left(\frac{\xi_2}{2} - w_2\right)\{w_2\}^j\,, \end{cases}$$

where $h^i \in C_c^\infty(\mathbb{R}^{n_i})$ with $\int_{\mathbb{R}^{n_i}} h^i dx_i = 1$ and $\xi \in \mathbb{S}^{N-1}$. Let $h_r = \frac{1}{r^N} h^1(\frac{\cdot}{r})h^2(\frac{\cdot}{r})$, set $r = |x-y|$, and write

$$f(x) - f(y) = \int_{\mathbb{R}^N} h_r\left(z - \frac{x+y}{2}\right)(f(x) - f(z))dz + \int_{\mathbb{R}^N} h_r\left(z - \frac{x+y}{2}\right)(f(z) - f(y))dz\,.$$

We assume that f, γ_{jk} and g_{jk} are smooth and compute the following:

$$\begin{aligned} &\int\limits_{\mathbb{R}^N} h_r\left(z - \frac{x+y}{2}\right)(f(x) - f(z))dz \\ &= -\sum_{j=1}^N \int\limits_{\mathbb{R}^N} \int_0^1 h_r\left(z - \frac{x+y}{2}\right)\partial_j f(x + t(z-x))(z\cdot e_j - x\cdot e_j)\,dtdz\,. \end{aligned}$$

After the change of variable $-t(z-x) \mapsto w$ we get

$$\begin{aligned} &= \sum_{j=1}^N \int\limits_{\mathbb{R}^N} \int_0^1 h_r\left(\frac{x-y}{2} - \frac{w}{t}\right)\partial_j f(x-w)\frac{w\cdot e_j}{t^{N+1}}\,dtdw \\ &= r\sum_{j=1}^N \int\limits_{\mathbb{R}^N} \int_0^1 \frac{1}{t^N} h_r\left(\frac{x-y}{2} - \frac{w}{t}\right)\frac{w\cdot e_j}{tr}\partial_j f(x-w)\,dtdw \\ &= r\sum_{j=1}^N \sum_{k=1}^m \int_0^1 \Bigg[\int\limits_{\mathbb{R}^{n_1}} \frac{1}{t^{n_1}} h_r^1\left(\frac{x_1-y_1}{2} - \frac{w_1}{t}\right)\left\{\frac{w_1}{tr}\right\}^j R_{jk}g_{jk}(x_1 - w_1)\,dw_1 \\ &\qquad\qquad \times \int\limits_{\mathbb{R}^{n_2}} \frac{1}{t^{n_2}} h_r^2\left(\frac{x_2-y_2}{2} - \frac{w_2}{t}\right)\left\{\frac{w_2}{tr}\right\}^j \gamma_{jk}(x_2 - w_2)\,dw_2\Bigg]dt \\ &= r\sum_{j=1}^N \sum_{k=1}^m \int_0^1 \left[\frac{1}{t^{n_1}} h_r^1\left(\frac{x_1-y_1}{2} - \frac{w_1}{t}\right)\left\{\frac{w_1}{tr}\right\}^j \underset{w_1}{\ast} R_{jk}g_{jk}(w_1)\right](x_1) \\ &\qquad\qquad \times \left[\frac{1}{t^{n_2}} h_r^2\left(\frac{x_2-y_2}{2} - \frac{w_2}{t}\right)\left\{\frac{w_2}{tr}\right\}^j \underset{w_2}{\ast} \gamma_{jk}(w_2)\right](x_2)\,dt\,. \end{aligned}$$

Denoting $\Upsilon_\varepsilon^{\xi,j}(w_1) = \frac{1}{\varepsilon^{n_1}}\Upsilon^{\xi,j}\left(\frac{w_1}{\varepsilon}\right)$ and $\bar{\Upsilon}_\varepsilon^{\xi,j}(w_2) = \frac{1}{\varepsilon^{n_2}}\bar{\Upsilon}^{\xi,j}\left(\frac{w_2}{\varepsilon}\right)$, this expression equals

$$r\sum_{j=1}^N \sum_{k=1}^m \int_0^1 [\Upsilon_{tr}^{\frac{x-y}{|x-y|},j} \underset{1}{\ast} R_{jk}g_{jk}]\,(x_1)\;[\bar{\Upsilon}_{tr}^{\frac{x-y}{|x-y|},j} \underset{2}{\ast} \gamma_{jk}]\,(x_2)\,dt,$$

and so

$$
\begin{aligned}
&\left|\int_{\mathbb{R}^N} h_r\left(z-\frac{x+y}{2}\right)(f(x)-f(z))dz\right| \\
&\leq |x-y|\sum_{j=1}^{N}\sum_{k=1}^{m}\int_0^1 |[\Upsilon_{tr}^{\frac{x_1-y_1}{|x-y|},j} \underset{1}{*} R_{jk}g_{jk}]\,(x_1)\;[\bar{\Upsilon}_{tr}^{\frac{x_2-y_2}{|x-y|}} \underset{2}{*} \gamma_{jk}]\,(x_2)|\,dt \\
&\leq |x-y|\sum_{j=1}^{N}\sum_{k=1}^{m}\int_0^1 \sup_{\varepsilon>0}\sup_{\xi} |[\Upsilon_{\varepsilon}^{\xi,j} \underset{1}{*} R_{jk}g_{jk}]\,(x_1)\;[\bar{\Upsilon}_{\varepsilon}^{\xi,j} \underset{2}{*} \gamma_{jk}]\,(x_2)|\,dt \\
&= |x-y|\sum_{j=1}^{N}\sum_{k=1}^{m} M_{\{\Upsilon^{\xi,j}\otimes\bar{\Upsilon}^{\xi,j}\}}(\gamma_{jk}R_{jk}g_{jk})(x) = |x-y|V(x)\,.
\end{aligned}
$$

This proves the statement in the smooth case. By a similar approximation argument as in [16], we conclude this holds for functions of the type in (case a). □

19.2. Split regularity: the anisotropic estimate. We now modify Lemma 19.1 to obtain an estimate in which distances are measured "anisotropically" through the matrix A defined in (5.63). In the next lemma we will use the following notation:

$$\check{g}_{ij}(x_1) = g_{jk}(\delta_1 x_1)\,, \qquad \check{\gamma}_{ij}(x_2) = \gamma_{ij}(\delta_2 x_2)\,,$$

where with $g_{jk}(\delta_1 x_1)$ we denote the measure on $\mathbb{R}^{n_1}$ defined through

$$\langle g_{jk}(\delta_1 x_1), \varphi(x_1)\rangle = \delta_1^{-n_1}\langle g_{ij}(y_1), \varphi(y_1/\delta_1)\rangle\,, \qquad \varphi\in C_c^\infty(\mathbb{R}^{n_1})\,.$$

Moreover, $R_{jk}^{\delta_1}$ denotes the singular integral operator in $\mathbb{R}^{n_1}$ associated to the kernel $K_{jk}^{\delta_1}$, where

$$K_{jk}^{\delta_1}(x_1) = \delta_1^{n_1}K_{ij}(\delta_1 x_1)\,. \tag{5.66}$$

Lemma 19.2. *Let $f:\mathbb{R}^N\to\mathbb{R}$ be a function in $L^1_{\mathrm{loc}}(\mathbb{R}^N)$ such that for each $j=1,\dots,N$ we have that $\partial_j f$ is as in (case a), or (case b) with $R(x)=\delta(x)$. Let A be the matrix defined in (5.63). Then there exists a nonnegative function $U:\mathbb{R}^N\to[0,\infty)$, such that for $\mathcal{L}^N$-a.e. $x,y\in\mathbb{R}^N$,*

$$|f(x)-f(y)|\leq |A^{-1}[x-y]|\Big(U(x)+U(y)\Big)\,,$$

where (with the notation above)

$$
\begin{cases}
U(x) = \mathcal{U}(R,\gamma,g)(x) = \displaystyle\sum_{j=1}^{N}\sum_{k=1}^{m}[M_{\{\Upsilon^{\xi,j}\otimes\bar{\Upsilon}^{\xi,j}\}}(R_{jk}^{\delta_1}\check{g}_{jk}\check{\gamma}_{jk}A_{jj})](A^{-1}x) & \textit{in (case a)},\\
U(x) = \mathcal{U}(g)(x) = \displaystyle\sum_{j=1}^{N}\sum_{k=1}^{m}[M_{\Upsilon^{\xi,j}}(g_{jk}(A\cdot))A_{jj})](A^{-1}x) & \textit{in (case b)}.
\end{cases}
$$

Proof. Define the following rescaled vector field. For each $z\in\mathbb{R}^N$, define

$$\check{f}(z) = f(Az)\,.$$

Now $D\check{f}$ is related to Df by the following:

$$\partial_j\check{f}(z) = \partial_j f(Az)A_{jj} = \begin{cases}\displaystyle\sum_{k=1}^{m}\gamma_{jk}(\delta_2 z_2)R_{jk}g_{jk}(\delta_1 z_1)A_{jj} & \textit{in (case a)},\\ \displaystyle\sum_{k=1}^{m} g(Az)A_{jj} & \textit{in (case b)}.\end{cases} \tag{5.67}$$

We now apply Lemma 19.1. This gives the existence of a function $V \in M^1_{\mathrm{loc}}(\mathbb{R}^N)$ to estimate the difference quotient of $\check{f}$:

$$|\check{f}(z) - \check{f}(w)| \leq |z-w|(V(z)+V(w))\,, \tag{5.68}$$

with V given by

$$V(z) = \begin{cases} \mathcal{V}(R,\gamma,g) = \displaystyle\sum_{j=1}^{N}\sum_{k=1}^{m} M_{\{\Upsilon^{\xi,j}\otimes\bar{\Upsilon}^{\xi,j}\}}\Big(\gamma_{jk}(\delta_2 z_2)R_{jk}g_{jk}(\delta_1 z_1)\Big)A_{jj} & \textit{in (case a)}, \\ \mathcal{V}(\delta,g) = \displaystyle\sum_{j=1}^{N}\sum_{k=1}^{m} M_{\{\Upsilon^{\xi,j}\}}(g_{jk}(A\cdot))A_{jj}) & \textit{in (case b)}. \end{cases} \tag{5.69}$$

With a change of variable we can verify that

$$(R_{jk}g_{jk})(\delta_1 z_1) = (R^{\delta_1}_{jk}\check{g}_{jk})(z_1)\,.$$

Thus we can rewrite $\mathcal{V}(R,\gamma,g)$ as

$$V(z) = \sum_{j=1}^{N}\sum_{k=1}^{m}[M_{\{\Upsilon^{\xi,j}\otimes\bar{\Upsilon}^{\xi,j}\}}(R^{\delta_1}_{jk}\check{g}_{jk}\check{\gamma}_{jk}A_{jj})](z)\,. \tag{5.70}$$

By letting $U(x) = V(A^{-1}x)$ the proof is concluded. $\square$

REMARK 19.3. *In order to treat case b, when $R(x)$ is a singular integral operator on $\mathbb{R}^N$, one should consider the function*

$$U(x) = \mathcal{U}(R,g)(x) = \sum_{j=1}^{N}\sum_{k=1}^{m}[M_{\{\Upsilon^{\xi,j}\}}R^{A}_{jk}(g_{jk}(A\cdot))A_{jj})](A^{-1}x)\,,$$

where R^A_{ij} is the singular integral operator corresponding to the kernel

$$K^A_{ij}(x) = |\det A|\, K_{ij}(Ax)$$

and A is the diagonal matrix defined in (5.63). This would however give a more singular estimate in Lemma 19.4 below, since the dilation A in both variables x_1, x_2 'stretches' the norm of the associated operator R^A_{ij} by a factor $\left(\frac{\delta_2}{\delta_1}\right)^{N-1}$ and would therefore be useless for the proof of Theorem 20.1.

On the other hand it is possible to treat the case $R_{ij} = \delta$ in (case b), since the Dirac delta "does not see the dilation".

19.3. Split regularity: operator bounds. We finally establish suitable estimates on the norms of the operators defined in Lemma 19.2.

LEMMA 19.4 (case a). *Let $\mathcal{U}(R,\gamma,g)$ be as in Lemma 19.2, case a. Then for any $1<p<\infty$ we have*

$$|||\mathcal{U}(R,\gamma,g)|||_{M^1(\Omega_r)} \leq C_{r,p,m}\left(\delta_1\sum_{j=1}^{n_1}\sum_{k=1}^{m}||\gamma_{jk}||_{L^p(\mathbb{R}^{n_2})}||g_{jk}||_{\mathcal{M}(\mathbb{R}^{n_1})} + \delta_2\sum_{j=n_1+1}^{N}\sum_{k=1}^{m}||\gamma_{jk}||_{L^p(\mathbb{R}^{n_2})}||g_{jk}||_{\mathcal{M}(\mathbb{R}^{n_1})}\right),$$

where $\Omega_r = B^1_r \times B^2_r \subset \mathbb{R}^{n_1}\times\mathbb{R}^{n_2}$, and

$$||\mathcal{U}(R,\gamma,g)||_{L^p(\mathbb{R}^N)} \leq C_p\left(\delta_1\sum_{j=1}^{n_1}\sum_{k=1}^{m}||\gamma_{jk}||_{L^p(\mathbb{R}^{n_2})}||g_{jk}||_{L^p(\mathbb{R}^{n_1})} + \delta_2\sum_{j=n_1+1}^{N}\sum_{k=1}^{m}||\gamma_{jk}||_{L^p(\mathbb{R}^{n_2})}||g_{jk}||_{L^p(\mathbb{R}^{n_1})}\right).$$

The constants $C_{r,p,m}$ and C_p also depends on the singular integral operators R_{jk} in (case a) and on the space dimension. The first constant $C_{r,p,m}$ also depend on the integer m in (case a).

PROOF. Let us start with the estimate in M^1. We define $\check{B}^1_r = B^1_{r/\delta_1}$, $\check{B}^2_r = B^2_{r/\delta_2}$ and $\check{\Omega}_r = \check{B}^1_r \times \check{B}^2_r$. Consider first the measure of the superlevels of $U(x)$: changing variable via the linear transformation $z = A^{-1}x$ we obtain

$$\begin{aligned}\mathcal{L}^N(\{x \in \Omega_r : |U(x)| > \lambda\}) &= \mathcal{L}^N(\{x \in \Omega_r : |V(A^{-1}x)| > \lambda\}) \\ &= \delta_1^{n_1}\delta_2^{n_2}\mathcal{L}^N(\{z \in \check{\Omega}_r : |V(z)| > \lambda\}),\end{aligned}$$

where V is as before given by

$$\begin{aligned}V(z) = {}& \delta_1 \sum_{j=1}^{n_1}\sum_{k=1}^{m}[M_{\{\Upsilon^{\xi,j}\otimes\bar{\Upsilon}^{\xi,j}\}}(R^{\delta_1}_{jk}\check{g}_{jk}\check{\gamma}_{jk})](z) \\ & + \delta_2 \sum_{j=n_1+1}^{N}\sum_{k=1}^{m}[M_{\{\Upsilon^{\xi,j}\otimes\bar{\Upsilon}^{\xi,j}\}}(R^{\delta_1}_{jk}\check{g}_{jk}\check{\gamma}_{jk})](z)\end{aligned} \tag{5.71}$$

(compare with (5.70) and split the sum for $1 \le j \le n_1$ and $n_1 + 1 \le j \le n_1 + n_2$).
Remembering that $|||f(x_1,x_2)|||_{M^1_{x_1x_2}} \le \left\| |||f(x_1,x_2)|||_{M^1_{x_1}} \right\|_{L^1_{x_2}}$ we estimate for fixed $j = 1,\dots,N$ as follows:

$$\begin{aligned}&\delta_1^{n_1}\delta_2^{n_2}\left|\left|\left|\sum_{k=1}^{m}[M_{\{\Upsilon^{\xi,j}\otimes\bar{\Upsilon}^{\xi,j}\}}(R^{\delta_1}_{jk}\check{g}_{jk}\check{\gamma}_{jk})](z)\right|\right|\right|_{M^1(\check{\Omega}_r)} \\ &\le C_m\,\delta_1^{n_1}\delta_2^{n_2}\sum_{k=1}^{m}\left|\left|\left|M_{\{\Upsilon^{\xi,j}\}}(R^{\delta_1}_{jk}\check{g}_{jk})\right|\right|\right|_{M^1(\check{B}^1_r)}\left\|M_{\{\bar{\Upsilon}^{\xi,j}\}}\check{\gamma}_{jk}\right\|_{L^1(\check{B}^2_r)} \\ &\le C_m[\mathcal{L}^{n_2}(\check{B}^2_r)]^{1-1/p}\,\delta_1^{n_1}\delta_2^{n_2}\sum_{k=1}^{m}\|\check{g}_{jk}\|_{\mathcal{M}(\mathbb{R}^{n_1})}\left\|M_{\{\bar{\Upsilon}^{\xi,j}\}}\check{\gamma}_{jk}\right\|_{L^p(\check{B}^2_r)} \\ &\le C_{r,p,m}\,\delta_2^{-n_2+n_2/p}\,\delta_1^{n_1}\delta_2^{n_2}\sum_{k=1}^{m}\|\check{g}_{jk}\|_{\mathcal{M}(\mathbb{R}^{n_1})}\|\check{\gamma}_{jk}\|_{L^p(\mathbb{R}^{n_2})} \\ &= C_{r,p,m}\sum_{k=1}^{m}\|g_{jk}\|_{\mathcal{M}(\mathbb{R}^{n_1})}\|\gamma_{jk}\|_{L^p(\mathbb{R}^{n_2})}\,.\end{aligned}$$

In the above chain of inequalities we have used the fact that the norm of $R^{\delta_1}_{jk}$ as singular integral operator coincides with the norm of R_{jk} as singular integral operator.

Recalling (5.71) we immediately obtain the first inequality claimed in the lemma. The second one follows with a similar argument, using the continuity of the operator

$$\check{g}_{jk} \mapsto R^{\delta_1}_{jk}\check{g}_{jk}$$

from $L^p(\mathbb{R}^{n_1})$ into itself. $\square$

LEMMA 19.5 (case b). *Let $\mathcal{U}(R,g)(x) = U(x)$ be defined as in Lemma 19.2, (case b), with $R(x) = \delta(x)$. Then, up to a constant $C = C(N,m,P_1,P_2)$, we have the estimates*

$$|||\mathcal{U}(g)|||_{M^1(\mathbb{R}^N)} \le C\left(\delta_1\sum_{j=1}^{n_1}\sum_{k=1}^{m}||g_{jk}||_{\mathcal{M}(\mathbb{R}^N)} + \delta_2\sum_{j=n_1+1}^{N}\sum_{k=1}^{m}||g_{jk}||_{\mathcal{M}(\mathbb{R}^N)}\right)$$

for $g_{jk} \in \mathcal{M}$, and

$$||\mathcal{U}(g)||_{L^p(\mathbb{R}^N)} \le C\left(\delta_1\sum_{j=1}^{n_1}\sum_{k=1}^{m}||g_{jk}||_{L^p(\mathbb{R}^N)} + \delta_2\sum_{j=n_1+1}^{N}\sum_{k=1}^{m}||g_{jk}||_{L^p(\mathbb{R}^N)}\right)$$

for $g_{jk} \in L^p, p > 1$.

PROOF. The proof is similar to Lemma 19.4, except that we estimate

$$\mathcal{L}^N(\{x \in \mathbb{R}^N : |U(x)| > \lambda\}) = \delta_1^{n_1}\delta_2^{n_2}\mathcal{L}^N(\{z \in \mathbb{R}^N : |V(z)| > \lambda\}),$$

with V given by

$$V(z) = \delta_1 \sum_{j=1}^{n_1}\sum_{k=1}^{m}[M_{\{\Upsilon^{\xi,j}\}}(g_{jk}(A\cdot))](z) + \delta_2 \sum_{j=n_1+1}^{N}\sum_{k=1}^{m}[M_{\{\Upsilon^{\xi,j}\}}(g_{jk}(A\cdot))](z). \tag{5.72}$$

With a change of variable it is easy to see that

$$\delta_1^{n_1}\delta_2^{n_2}\left|\left|\left|\sum_{k=1}^{m} M_{\{\Upsilon^{\xi,j}\}}(g_{jk}(A\cdot))\right|\right|\right|_{M^1(\mathbb{R}^N)} \leq C_m \sum_{k=1}^{m}||g_{jk}||_{\mathcal{M}(\mathbb{R}^N)}, \tag{5.73}$$

which yields the first statement of the lemma. The second is obvious by continuity of the operator

$$g_{jk} \mapsto M_{\{\Upsilon^{\xi,j}\}}g_{jk}$$

from $L^p(\mathbb{R}^N)$ into itself. □

20. The fundamental estimate for flows: main theorem and corollaries

Our main theorem is the following:

THEOREM 20.1 (case a and b). *Let b and $\bar{b}$ be two vector fields satisfying assumption **(R1)**, and assume that b also satisfies assumption **(R2a)** or **(R2b)**, as well as **(R3)**. Fix $t \in [0,T)$ and let X and $\bar{X}$ be regular Lagrangian flows starting at time t associated to b and $\bar{b}$ respectively, with compressibility constants L and $\bar{L}$. Then the following holds. For every $\gamma > 0$ and $r > 0$ and for every $\eta > 0$ there exist $\lambda > 0$ and $C_{\gamma,r,\eta} > 0$ such that*

$$\mathcal{L}^n\left(B_r \cap \{|X(s,\cdot) - \bar{X}(s,\cdot)| > \gamma\}\right) \leq C_{\gamma,r,\eta}||b - \bar{b}||_{L^1((0,T)\times B_\lambda)} + \eta$$

for all $s \in [t,T]$. The constants λ and $C_{\gamma,r,\eta}$ also depend on:

- *The equi-integrability in $L^1((0,T);L^1(\mathbb{R}^{n_1}))$ of $\mathfrak{p},\mathfrak{q},\mathfrak{r}$ in case of **(R2a)**, the equi-integrability in $L^1((0,T)\times\mathbb{R}^N)$ of the functions $\mathfrak{p},\mathfrak{q},\mathfrak{r}$ associated to b in case of **(R2b)**, as well as $||m||_{L^1(\mathcal{M})}$, (where $\mathfrak{p}$, $\mathfrak{q}$, $\mathfrak{r}$ and $\mathfrak{m}$ are associated to b as in **(R2a)-(R2b)**)*
- *The norms of the singular integral operators $S_{jk}^{\cdot i}$, as well as the norms in $L^\infty((0,T);L^q(\mathbb{R}^{n_2}))$ of $\gamma_{jk}^{\cdot i}$ (associated to b as in **(R2a)**),*
- *The norm in $L^p((0,T)\times B_\lambda)$ of b,*
- *The $L^1((0,T);L^1(\mathbb{R}^N)) + L^1((0,T);L^\infty(\mathbb{R}^N))$ norms of the decompositions of b and $\bar{b}$ as in **(R1)**,*
- *The compressibility constants L and $\bar{L}$.*

From this fundamental estimate, the various corollaries regarding the well posedness of the regular Lagrangian flow and of Lagrangian solutions to the continuity and transport equations follow with the same proofs as in Sections 6 and 7 in [16]. This will be discussed in section 21.3. In particular, we obtain:

- Uniqueness of the regular Lagrangian flow associated to a vector field satisfying **(R1)**, **(R2a)** or **(R2b)** and **(R3)**,
- Stability (with an explicit rate) for a sequence X_n of regular Lagrangian flows associated to vector fields b_n, that converge in $L^1_{\mathrm{loc}}([0,T]\times\mathbb{R}^N)$ to a vector field satisfying **(R1)**, **(R2a)** or **(R2b)** and **(R3)**, under the assumption that the decompositions of b_n in **(R1)** and the compressibility constants of X_n satisfy uniform bounds,
- Compactness for a sequence X_n of regular Lagrangian flows associated to vector fields b_n satisfying **(R1)**, **(R2a)** or **(R2b)** and **(R3)** with suitable uniform bounds,

- Existence of a regular Lagrangian flow associated to a vector field satisfying **(R1)**, **(R2a)** or **(R2b)** and **(R3)** and such that $[\operatorname{div} b]^- \in L^1((0,T);L^\infty(\mathbb{R}^N))$,
- If a vector field b satisfies **(R1)**, **(R2a)** or **(R2b)** and **(R3)** and $\operatorname{div} b \in L^1((0,T);L^\infty(\mathbb{R}^N))$, then there exists a unique forward and backward regular Lagrangian flow associated to b, which satisfies the usual group property, and the Jacobian of the flow is well defined,
- Lagrangian solutions to the continuity and transport equations with a vector field b satisfying **(R1)**, **(R2)** or **(R2b)** and **(R3)** and $\operatorname{div} b \in L^1((0,T);L^\infty(\mathbb{R}^N))$ are well defined and stable.

REMARK 20.2. *We remark that it is unclear whether an approximate differentiability of the flow holds as in Proposition 14.6. This is due to the fact that an estimate on the quantity $A_p(R,X)$ introduced in section 14 requires a 'double' maximal function of the derivative, or more precisely a composition of the form $M_{\{\Upsilon^{\xi,j}\}}[M_{\{\Upsilon^{\xi,j}\}}(Db)]$, which does not enjoy bounds in L^1 (or even M^1) in our regularity setting.*

21. Proof of the fundamental estimate (Theorem 20.1)

The proof of Theorem 20.1 makes use of the integral functional

$$\Phi_{\delta_1,\delta_2}(s) = \int\limits_{B_r\cap G_\lambda\cap\bar{G}_\lambda} \log\left(1+\left|A^{-1}\left[X(s,x_1,x_2)-\bar{X}(s,x_1,x_2)\right]\right|\right)\,dx$$

already defined in (5.64). In the following proof we assume $\delta_1 \le \delta_2$.

We will again use the notation "$\lesssim$" to denote an estimate also up to a constant only depending on the bounds assumed in Theorem 20.1. We will however write explicitly the norm of the measure $\mathfrak{m}$, in order to make the reader aware of its role in the estimates.

21.1. Proof of Theorem 20.1 in case a.

Step 1: Differentiating Φ_{δ_1,δ_2}. We start by differentiating the integral functional with respect to time:

$$\Phi'_{\delta_1,\delta_2}(s) \le \int\limits_{B_r\cap G_\lambda\cap\bar{G}_\lambda} \frac{|A^{-1}[b(s,X(s,x_1,x_2))-\bar{b}(s,\bar{X}(s,x_1,x_2))]|}{1+|A^{-1}[X(s,x_1,x_2)-\bar{X}(s,x_1,x_2)]|}dx\,.$$

For simplicity, we drop the notation $X(s,x_1,x_2)$, setting $X(s,x_1,x_2)=X$ and $\bar{X}(s,x_1,x_2)=\bar{X}$. We estimate

$$\Phi'_{\delta_1,\delta_2}(s) \le \int\limits_{B_r\cap G_\lambda\cap\bar{G}_\lambda} |A^{-1}[b(s,\bar{X})-\bar{b}(s,\bar{X})]|dx + \int\limits_{B_r\cap G_\lambda\cap\bar{G}_\lambda} \frac{|A^{-1}[b(s,X)-b(s,\bar{X})]|}{1+|A^{-1}[X-\bar{X}]|}dx\,.$$

After a change in variable along the flow $\bar{X}$ in the first integral, and noting that $\delta_1\le\delta_2$, we further obtain

$$\begin{aligned}\Phi'_{\delta_1,\delta_2}(s) \le{}& \frac{\bar{L}}{\delta_1}\|b(s,\cdot)-\bar{b}(s,\cdot)\|_{L^1(B_\lambda)} \\ &+ \int\limits_{B_r\cap G_\lambda\cap\bar{G}_\lambda} \min\left\{|A^{-1}[b(s,X)-b(s,\bar{X})]|,\frac{|A^{-1}[b(s,X)-b(s,\bar{X})]|}{|A^{-1}[X-\bar{X}]|}\right\}dx\,.\end{aligned}\tag{5.74}$$

Step 2: Decomposing the minimum. We consider the second element of the minimum. We have

$$A^{-1}[b(s,X)-b(s,\bar{X})] = \left(\frac{b_1(s,X)-b_1(s,\bar{X})}{\delta_1},\frac{b_2(s,X)-b_2(s,\bar{X})}{\delta_2}\right),$$

and therefore

$$\frac{|A^{-1}[b(s,X)-b(s,\bar{X})]|}{|A^{-1}[X-\bar{X}]|} \lesssim \frac{1}{\delta_1}\frac{|b_1(s,X)-b_1(s,\bar{X})|}{|A^{-1}[X-\bar{X}]|} + \frac{1}{\delta_2}\frac{|b_2(s,X)-b_2(s,\bar{X})|}{|A^{-1}[X-\bar{X}]|}\,.\tag{5.75}$$

Step 3: Definition of the functions $U_{\mathfrak{p}}$, $U_{\mathfrak{q}}$, $U_{\mathfrak{m}}$ and $U_{\mathfrak{r}}$. We aim at estimating the difference quotients in (5.75). We apply Lemma 19.2 and (with a slight extension of the notation) we obtain that

$$\frac{|b_1(s,x)-b_1(s,\bar{x})|}{|A^{-1}[x-\bar{x}]|} \leq \mathcal{U}(S^1,S^2,\gamma^1,\gamma^2,\mathfrak{p},\mathfrak{q})(x)+\mathcal{U}(S^1,S^2,\gamma^1,\gamma^2,\mathfrak{p},\mathfrak{q})(\bar{x}) =: U_{\mathfrak{p},\mathfrak{q}}(x)+U_{\mathfrak{p},\mathfrak{q}}(\bar{x})$$

and

$$\frac{|b_2(s,x)-b_2(s,\bar{x})|}{|A^{-1}[x-\bar{x}]|} \leq \mathcal{U}(S^3,S^4,\gamma^3,\gamma^4,\mathfrak{m},\mathfrak{r})(x)+\mathcal{U}(S^3,S^4,\gamma^3,\gamma^4,\mathfrak{m},\mathfrak{r})(\bar{x}) =: U_{\mathfrak{m},\mathfrak{r}}(x)+U_{\mathfrak{m},\mathfrak{r}}(\bar{x})$$

for a.e. x and $\bar{x} \in \mathbb{R}^N$ and $s \in [t,T]$.

It is immediate from the definition of the operator $\mathcal{U}$ that it is subadditive in its entries. Therefore we can further estimate

$$U_{\mathfrak{p},\mathfrak{q}}(x) = \mathcal{U}(S^1,S^2,\gamma^1,\gamma^2,\mathfrak{p},\mathfrak{q})(x) \leq \mathcal{U}(S^1,\gamma^1,\mathfrak{p})(x)+\mathcal{U}(S^2,\gamma^2,\mathfrak{q})(x) =: U_{\mathfrak{p}}(x)+U_{\mathfrak{q}}(x)$$

and

$$U_{\mathfrak{m},\mathfrak{r}}(x) = \mathcal{U}(S^3,S^4,\gamma^3,\gamma^4,\mathfrak{m},\mathfrak{r})(x) \leq \mathcal{U}(S^3,\gamma^3,\mathfrak{m})(x)+\mathcal{U}(S^4,\gamma^4,\mathfrak{r})(x) =: U_{\mathfrak{m}}(x)+U_{\mathfrak{r}}(x)$$

for a.e. $x \in \mathbb{R}^N$, implying that

$$\frac{|b_1(s,x)-b_1(s,\bar{x})|}{|A^{-1}[x-\bar{x}]|} \leq U_{\mathfrak{p}}(x)+U_{\mathfrak{q}}(x)+U_{\mathfrak{p}}(\bar{x})+U_{\mathfrak{q}}(\bar{x}) \tag{5.76}$$

and

$$\frac{|b_2(s,x)-b_2(s,\bar{x})|}{|A^{-1}[x-\bar{x}]|} \leq U_{\mathfrak{m}}(x)+U_{\mathfrak{r}}(x)+U_{\mathfrak{m}}(\bar{x})+U_{\mathfrak{r}}(\bar{x}) \tag{5.77}$$

for a.e. x and $\bar{x} \in \mathbb{R}^N$ and $s \in [t,T]$.

Step 4. Splitting of the quotient. Let $\Omega = (t,\tau)\times B_r \cap G_\lambda \cap \bar{G}_\lambda \subset \mathbb{R}^{N+1}$. We return to the estimate in (5.74) of Step 1. For any $\tau \in [t,T]$ we integrate this expression over $s \in (t,\tau)$, and recall (5.75) to get

$$\begin{aligned}
\Phi_{\delta_1,\delta_2}(\tau) &\lesssim \frac{\bar{L}}{\delta_1}||b(s,\cdot)-\bar{b}(s,\cdot)||_{L^1((t,\tau)\times B_\lambda)} \\
&\quad + \int_\Omega \min\left\{|A^{-1}[b(s,X)-b(s,\bar{X})]|, \frac{1}{\delta_1}\frac{|b_1(s,X)-b_1(s,\bar{X})|}{|A^{-1}[X-\bar{X}]|}+\frac{1}{\delta_2}\frac{|b_2(s,X)-b_2(s,\bar{X})|}{|A^{-1}[X-\bar{X}]|}\right\} dxds \\
&= \frac{\bar{L}}{\delta_1}||b(s,\cdot)-\bar{b}(s,\cdot)||_{L^1((t,\tau)\times B_\lambda)}+\widetilde{\Phi}_{\delta_1,\delta_2}(\tau)\,.
\end{aligned} \tag{5.78}$$

We analyze the term $\widetilde{\Phi}_{\delta_1,\delta_2}(\tau)$. Using the estimates in (5.76) and (5.77) in Step 3, we can write

$$\begin{aligned}
\widetilde{\Phi}_{\delta_1,\delta_2}(\tau) &\lesssim \int_\Omega \min\left\{|A^{-1}[b(s,X)-b(s,\bar{X})]|, \frac{1}{\delta_1}\frac{|b_1(s,X)-b_1(s,\bar{X})|}{|A^{-1}[X-\bar{X}]|}\right\} dxds \\
&\quad + \int_\Omega \min\left\{|A^{-1}[b(s,X)-b(s,\bar{X})]|, \frac{1}{\delta_2}\frac{|b_2(s,X)-b_2(s,\bar{X})|}{|A^{-1}[X-\bar{X}]|}\right\} dxds \\
&\leq \int_\Omega \min\left\{|A^{-1}[b(s,X)-b(s,\bar{X})]|, \frac{1}{\delta_1}\left((U_{\mathfrak{p}}+U_{\mathfrak{q}})(s,X)+(U_{\mathfrak{p}}+U_{\mathfrak{q}})(s,\bar{X})\right)\right\} dxds \\
&\quad + \int_\Omega \min\left\{|A^{-1}[b(s,X)-b(s,\bar{X})]|, \frac{1}{\delta_2}\left((U_{\mathfrak{m}}+U_{\mathfrak{r}})(s,X)+((U_{\mathfrak{m}}+U_{\mathfrak{r}}))(s,\bar{X})\right)\right\} dxds\,.
\end{aligned} \tag{5.79}$$

Step 5. Decomposition of the functions $U_{\mathfrak{p}}$, $U_{\mathfrak{q}}$ and $U_{\mathfrak{r}}$. We further decompose the functions $U_{\mathfrak{p}}$, $U_{\mathfrak{q}}$ and $U_{\mathfrak{r}}$ exploiting the equi-integrability of $\mathfrak{p}$, $\mathfrak{q}$ and $\mathfrak{r}$.

We apply the equi-integrability Lemma 5.11 in $L^1 + L^q$, with the same $1 < q \leq \infty$ as in the assumption on the functions γ in **(R2)**. Given $\varepsilon > 0$, we find $C_\varepsilon > 0$, a Borel set $A_\varepsilon \subset (0,T) \times \mathbb{R}^{n_1}$ with finite measure and decompositions

$$\begin{aligned} \mathfrak{p}^i_{jk} &= (\mathfrak{p}^i_{jk})^1 + (\mathfrak{p}^i_{jk})^2 =: \mathfrak{p}^1 + \mathfrak{p}^2 , \\ \mathfrak{q}^i_{jk} &= (\mathfrak{q}^i_{jk})^1 + (\mathfrak{q}^i_{jk})^2 =: \mathfrak{q}^1 + \mathfrak{q}^2 \end{aligned}$$

and

$$\mathfrak{r}^i_{jk} = (\mathfrak{r}^i_{jk})^1 + (\mathfrak{r}^i_{jk})^2 =: \mathfrak{r}^1 + \mathfrak{r}^2 ,$$

so that

$$\begin{gathered} \|\mathfrak{p}^1\|_{L^1((0,T)\times\mathbb{R}^{n_1})} \leq \varepsilon , \quad \|\mathfrak{q}^1\|_{L^1((0,T)\times\mathbb{R}^{n_1})} \leq \varepsilon , \quad \|\mathfrak{r}^1\|_{L^1((0,T)\times\mathbb{R}^{n_1})} \leq \varepsilon , \\ \|\mathfrak{p}^2\|_{L^q((0,T)\times\mathbb{R}^{n_1})} \leq C_\varepsilon , \quad \|\mathfrak{q}^2\|_{L^q((0,T)\times\mathbb{R}^{n_1})} \leq C_\varepsilon , \quad \|\mathfrak{r}^2\|_{L^q((0,T)\times\mathbb{R}^{n_1})} \leq C_\varepsilon , \end{gathered}$$

and

$$\text{spt } (\mathfrak{p}^2) \subset A_\varepsilon , \quad \text{spt } (\mathfrak{q}^2) \subset A_\varepsilon , \quad \text{spt } (\mathfrak{r}^2) \subset A_\varepsilon .$$

We then decompose the functions $U_{\mathfrak{p}}$, $U_{\mathfrak{q}}$ and $U_{\mathfrak{r}}$ from Step 3 as

$$\begin{aligned} U_{\mathfrak{p}} &= \mathcal{U}(S^1, \gamma^1, \mathfrak{p}) \leq \mathcal{U}(S^1, \gamma^1, \mathfrak{p}^1) + \mathcal{U}(S^1, \gamma^1, \mathfrak{p}^2) =: U^1_{\mathfrak{p}} + U^2_{\mathfrak{p}} , \\ U_{\mathfrak{q}} &= \mathcal{U}(S^2, \gamma^2, \mathfrak{q}) \leq \mathcal{U}(S^2, \gamma^2, \mathfrak{q}^1) + \mathcal{U}(S^2, \gamma^2, \mathfrak{q}^2) =: U^1_{\mathfrak{q}} + U^2_{\mathfrak{q}} \end{aligned}$$

and

$$U_{\mathfrak{r}} = \mathcal{U}(S^4, \gamma^4, \mathfrak{r}) \leq \mathcal{U}(S^4, \gamma^4, \mathfrak{r}^1) + \mathcal{U}(S^4, \gamma^4, \mathfrak{r}^2) =: U^1_{\mathfrak{r}} + U^2_{\mathfrak{r}} .$$

Applying Lemma 19.4 to $U^1_{\mathfrak{p}}$ and $U^2_{\mathfrak{p}}$ we get

$$\begin{aligned} |||U^1_{\mathfrak{p}}|||_{M^1((0,T)\times B_\lambda)} &\lesssim_\lambda \delta_1 ||\gamma^1||_{L^\infty((0,T);L^q(\mathbb{R}^{n_2}))} ||\mathfrak{p}^1||_{L^1((0,T)\times\mathbb{R}^{n_1}))} \lesssim_\lambda \delta_1 \varepsilon , \\ \|U^2_{\mathfrak{p}}\|_{L^q((0,T)\times B_\lambda)} &\lesssim \delta_1 ||\gamma^1||_{L^\infty((0,T);L^q(\mathbb{R}^{n_2}))} ||\mathfrak{p}^2||_{L^q((0,T)\times\mathbb{R}^{n_1})} \lesssim \delta_1 C_\varepsilon . \end{aligned} \tag{5.80}$$

We have a similar estimate for $U_{\mathfrak{q}}$ and $U_{\mathfrak{r}}$:

$$\begin{aligned} |||U^1_{\mathfrak{q}}|||_{M^1((0,T)\times B_\lambda)} &\lesssim_\lambda \delta_2 \varepsilon , \qquad |||U^1_{\mathfrak{r}}|||_{M^1((0,T)\times B_\lambda)} \lesssim_\lambda \delta_2 \varepsilon , \\ \|U^2_{\mathfrak{q}}\|_{L^q((0,T)\times B_\lambda)} &\lesssim \delta_2 C_\varepsilon , \qquad \|U^2_{\mathfrak{r}}\|_{L^q((0,T)\times B_\lambda)} \lesssim \delta_2 C_\varepsilon . \end{aligned} \tag{5.81}$$

Note that we cannot apply such a decomposition to $U_{\mathfrak{m}}$, since it is defined as the operator $\mathcal{U}$ acting on a measure rather than integrable function. We only have the bound

$$|||U_{\mathfrak{m}}|||_{M^1((0,T)\times B_\lambda)} \lesssim_\lambda \delta_1 ||\mathfrak{m}||_{L^1((0,T);\mathcal{M}(\mathbb{R}^{n_1}))} . \tag{5.82}$$

We further split the minima according to this decomposition:

$$\begin{aligned} \widetilde{\Phi}_{\delta_1,\delta_2}(\tau) \lesssim & \int_\Omega \min\left\{|A^{-1}[b(s,X) - b(s,\bar{X})]|, \frac{1}{\delta_2}(U_{\mathfrak{m}}(s,X) + U_{\mathfrak{m}}(s,\bar{X}))\right\} dxds \\ & + \int_\Omega \min\left\{|A^{-1}[b(s,X) - b(s,\bar{X})]|, \frac{1}{\delta_2}(U^1_{\mathfrak{r}}(s,X) + U^1_{\mathfrak{r}}(s,\bar{X}))\right\} dxds \\ & + \int_\Omega \min\left\{|A^{-1}[b(s,X) - b(s,\bar{X})]|, \frac{1}{\delta_2}(U^2_{\mathfrak{r}}(s,X) + U^2_{\mathfrak{r}}(s,\bar{X}))\right\} dxds \\ & + \int_\Omega \min\left\{|A^{-1}[b(s,X) - b(s,\bar{X})]|, \frac{1}{\delta_1}((U^1_{\mathfrak{p}} + U^1_{\mathfrak{q}})(s,X) + (U^1_{\mathfrak{p}} + U^1_{\mathfrak{q}})(s,\bar{X}))\right\} dxds \\ & + \int_\Omega \min\left\{|A^{-1}[b(s,X) - b(s,\bar{X})]|, \frac{1}{\delta_1}((U^2_{\mathfrak{p}} + U^2_{\mathfrak{q}})(s,X) + (U^2_{\mathfrak{p}} + U^2_{\mathfrak{q}})(s,\bar{X}))\right\} dxds \\ = & \int_\Omega \varphi_1(s,X,\bar{X}) + \int_\Omega \varphi_2(s,X,\bar{X}) + \int_\Omega \varphi_3(s,X,\bar{X}) + \int_\Omega \varphi_4(s,X,\bar{X}) + \int_\Omega \varphi_5(s,X,\bar{X}) . \end{aligned} \tag{5.83}$$

Step 6. Estimating the functions φ_j. Let $\Omega' = (t,\tau) \times B_\lambda \subset \mathbb{R}^{N+1}$. We estimate the first element of each minima in L^p: changing variables along the flows we obtain

$$\|\varphi_j(s,X,\bar{X})\|_{L^p(\Omega)} \leq \frac{L^{1/p}+\bar{L}^{1/p}}{\delta_1}\|b\|_{L^p(\Omega')} \lesssim \frac{1}{\delta_1} \tag{5.84}$$

for every $j = 1,\dots,5$.

We now consider the second elements of the minima. Let us start with φ_1. Changing variable along the flows and using (5.82) we obtain

$$\begin{aligned}|||\varphi_1(s,X,\bar{X})|||_{M^1(\Omega)} &\leq \frac{1}{\delta_2}\left|\left|\left|U_{\mathfrak{m}}(s,X)+U_{\mathfrak{m}}(s,\bar{X})\right|\right|\right|_{M^1(\Omega)} \\ &\lesssim \frac{1}{\delta_2}|||U_{\mathfrak{m}}|||_{M^1(\Omega')} \lesssim_\lambda \frac{\delta_1}{\delta_2}||\mathfrak{m}||_{L^1((0,T);\mathcal{M}(\mathbb{R}^{n_1}))}\,.\end{aligned} \tag{5.85}$$

Consider φ_2. Using (5.81) we obtain

$$\begin{aligned}|||\varphi_2(s,X,\bar{X})|||_{M^1(\Omega)} &\leq \frac{1}{\delta_2}|||U^1_{\mathfrak{r}}(s,X)+U^1_{\mathfrak{r}}(s,\bar{X})|||_{M^1(\Omega)} \\ &\lesssim \frac{1}{\delta_2}|||U^1_{\mathfrak{r}}|||_{M^1(\Omega')} \lesssim_\lambda \varepsilon\,.\end{aligned} \tag{5.86}$$

For φ_3 and φ_5 we neglect the first element of the minimum, since we have directly an estimate on the $L^1(\Omega)$ norm. Using (5.81) we obtain

$$\begin{aligned}\|\varphi_3(s,X,\bar{X})\|_{L^1(\Omega)} &\leq \frac{1}{\delta_2}\|U^2_{\mathfrak{r}}(s,X)+U^2_{\mathfrak{r}}(s,\bar{X})\|_{L^1(\Omega)} \\ &\lesssim \frac{1}{\delta_2}||U^2_{\mathfrak{r}}||_{L^1(\Omega')} \lesssim_\lambda C_\varepsilon\,.\end{aligned} \tag{5.87}$$

Similarly, using (5.80) and (5.81), we estimate φ_5 as follows:

$$\begin{aligned}||\varphi_5(s,X,\bar{X})||_{L^1(\Omega)} &\leq \frac{1}{\delta_1}||(U^2_{\mathfrak{p}}+U^2_{\mathfrak{q}})(s,X)+(U^2_{\mathfrak{p}}+U^2_{\mathfrak{q}})(s,\bar{X})||_{L^1(\Omega)} \\ &\lesssim \frac{1}{\delta_1}||(U^2_{\mathfrak{p}}+U^2_{\mathfrak{q}})||_{L^1(\Omega')} \lesssim_\lambda \frac{\delta_2}{\delta_1}C_\varepsilon\,.\end{aligned} \tag{5.88}$$

Finally, using (5.80) and (5.81), we estimate φ_4:

$$\begin{aligned}|||\varphi_4(s,X,\bar{X})|||_{M^1(\Omega)} &\leq \frac{1}{\delta_1}|||(U^1_{\mathfrak{p}}+U^1_{\mathfrak{q}})(s,X)+(U^1_{\mathfrak{p}}+U^1_{\mathfrak{q}})(s,\bar{X})|||_{M^1(\Omega)} \\ &\lesssim \frac{1}{\delta_1}|||(U^1_{\mathfrak{p}}+U^1_{\mathfrak{q}})|||_{M^1(\Omega')} \\ &\lesssim_\lambda \frac{\delta_1\varepsilon+\delta_2\varepsilon}{\delta_1} \lesssim_\lambda \frac{\delta_2}{\delta_1}\varepsilon\,.\end{aligned} \tag{5.89}$$

Step 7. Interpolation. We now apply the Interpolation Lemma 5.10 to estimate the $L^1(\Omega)$ norms of φ_1, φ_2 and φ_4.

Using (5.84) and (5.85) we obtain

$$\|\varphi_1(s,X,\bar{X})\|_{L^1(\Omega)} \lesssim_\lambda \frac{\delta_1}{\delta_2}\|\mathfrak{m}\|\left[1+\log\left(\frac{\delta_2}{\delta_1^2\|\mathfrak{m}\|}\right)\right]\,. \tag{5.90}$$

Proceeding similarly and using (5.84), (5.86) and (5.89) we obtain

$$||\varphi_2(s,X,\bar{X})||_{L^1(\Omega)} \lesssim_\lambda \varepsilon\left[1+\log\left(\frac{1}{\delta_1\varepsilon}\right)\right] \tag{5.91}$$

and

$$||\varphi_4(s,X,\bar X)||_{L^1(\Omega)} \lesssim_\lambda \frac{\delta_2}{\delta_1}\varepsilon\left[1+\log\left(\frac{1}{\delta_2\varepsilon}\right)\right]. \tag{5.92}$$

Finally, we sum all the terms in (5.83). Using (5.90), (5.91), (5.87), (5.92) and (5.88), and setting $\dfrac{\delta_1}{\delta_2}=\alpha$, we get:

$$\begin{aligned}\Phi_{\delta_1,\delta_2}(\tau) \lesssim_\lambda\ & \frac{1}{\delta_1}||b(s,\cdot)-\bar b(s,\cdot)||_{L^1(B_\lambda\times(t,\tau))}\\ & + \alpha\|\mathfrak{m}\|\left[1+\log\left(\frac{1}{\delta_1\alpha\|\mathfrak{m}\|}\right)\right] + \varepsilon\left[1+\log\left(\frac{1}{\delta_1\varepsilon}\right)\right] + C_\varepsilon\\ & + \frac{\varepsilon}{\alpha}\left[1+\log\left(\frac{1}{\delta_2\varepsilon}\right)\right] + \frac{1}{\alpha}C_\varepsilon\,.\end{aligned} \tag{5.93}$$

Step 8. The final estimate. By definition of Φ_{δ_1,δ_2}, given $\gamma>0$ we estimate

$$\begin{aligned}\Phi_{\delta_1,\delta_2}(\tau) &\geq \int_{B_r\cap\{|X(\tau,x)-\bar X(\tau,x)|>\gamma\}\cap G_\lambda\cap\bar G_\lambda} \log\left(1+\frac{\gamma}{\delta_2}\right)dx\\ &= \log\left(1+\frac{\gamma}{\delta_2}\right)\mathcal{L}^N\Big(B_r\cap\{|X(\tau,x)-\bar X(\tau,x)|>\gamma\}\cap G_\lambda\cap\bar G_\lambda\Big)\,.\end{aligned} \tag{5.94}$$

This implies that

$$\mathcal{L}^N(B_r\cap\{|X(\tau,x)-\bar X(\tau,x)|>\gamma\}) \leq \frac{\Phi_{\delta_1,\delta_2}(\tau)}{\log\left(1+\frac{\gamma}{\delta_2}\right)} + \mathcal{L}^N(B_r\setminus G_\lambda) + \mathcal{L}^N(B_r\setminus\bar G_\lambda)\,. \tag{5.95}$$

Combining (5.93) and (5.95) we obtain

$$\begin{aligned}&\mathcal{L}^N(B_r\cap\{|X(\tau,x)-\bar X(\tau,x)|>\gamma\})\\ &\leq\ C_\lambda\left\{\frac{\frac{1}{\delta_1}\|b-\bar b\|_{L^1}}{\log\left(1+\frac{\gamma}{\delta_2}\right)} + \frac{\alpha\|\mathfrak{m}\|\left[1+\log\left(\frac{1}{\delta_1\alpha\|\mathfrak{m}\|}\right)\right]}{\log\left(1+\frac{\gamma}{\delta_2}\right)} + \frac{\varepsilon\left[1+\log\left(\frac{1}{\delta_1\varepsilon}\right)\right]}{\log\left(1+\frac{\gamma}{\delta_2}\right)}\right.\\ &\qquad \left. + \frac{\frac{\varepsilon}{\alpha}\left[1+\log\left(\frac{1}{\delta_2\varepsilon}\right)\right]}{\log\left(1+\frac{\gamma}{\delta_2}\right)} + \frac{\frac{1}{\alpha}C_\varepsilon}{\log\left(1+\frac{\gamma}{\delta_2}\right)} + \frac{C_\varepsilon}{\log\left(1+\frac{\gamma}{\delta_2}\right)}\right\}\\ &\quad + \mathcal{L}^N(B_r\setminus G_\lambda) + \mathcal{L}^N(B_r\setminus\bar G_\lambda)\\ &=: 1)+2)+3)+4)+5)+6)+7)+8)\,.\end{aligned} \tag{5.96}$$

Fix $\eta>0$. By Lemma 16.3, we can choose $\lambda>0$ large enough so that $7)+8)\leq 2\eta/7$. Choose α small enough so that $2)\leq\eta/7$. Then choose $\varepsilon<\alpha^2$ small enough so that $3)+4)\leq 2\eta/7$, since these terms are uniformly bounded as $\delta_1,\delta_2\to 0$ and for all $\varepsilon>0$.

Now λ and ε (and therefore C_ε) are fixed. Also α is fixed, but δ_1 and δ_2 are free to be chosen so long as the ratio equals α. Hence, we now choose δ_2 small enough, in particular depending on C_ε, so that $5)+6)\leq 2\eta/7$. This fixes all parameters.

Setting

$$C_{\gamma,r,\eta} = \frac{C_\lambda}{\delta_1\log(1+\frac{\gamma}{\delta_2})}$$

we have proved our statement. $\square$

21.2. Proof of Theorem 20.1 in case b. The proof of the theorem in case of regularity setting **(R2b)** follows a similar argument.

Step 3: Definition of the functions $U_{\mathfrak{p}}$, $U_{\mathfrak{q}}$, $U_{\mathfrak{m}}$ and $U_{\mathfrak{r}}$. We now define the operator $\mathcal{U}$ according to Proposition 16.4. In this case we obtain that

$$\begin{aligned}\frac{|b_1(s,x)-b_1(s,\bar{x})|}{|A^{-1}[x-\bar{x}]|} &\leq \mathcal{U}(\mathfrak{p},\mathfrak{q})(x)+\mathcal{U}(\mathfrak{p},\mathfrak{q})(\bar{x}) =: U_{\mathfrak{p},\mathfrak{q}}(x)+U_{\mathfrak{p},\mathfrak{q}}(\bar{x})\\ &\leq U_{\mathfrak{p}}(x)+U_{\mathfrak{q}}(x)+U_{\mathfrak{p}}(\bar{x})+U_{\mathfrak{q}}(\bar{x})\end{aligned}$$

and

$$\begin{aligned}\frac{|b_2(s,x)-b_2(s,\bar{x})|}{|A^{-1}[x-\bar{x}]|} &\leq \mathcal{U}(\mathfrak{m},\mathfrak{r})(x)+\mathcal{U}(\mathfrak{m},\mathfrak{r})(\bar{x}) =: U_{\mathfrak{m},\mathfrak{r}}(x)+U_{\mathfrak{m},\mathfrak{r}}(\bar{x})\\ &\leq U_{\mathfrak{m}}(x)+U_{\mathfrak{m}}(x)+U_{\mathfrak{r}}(\bar{x})+U_{\mathfrak{r}}(\bar{x}),\end{aligned}$$

for a.e. x and $\bar{x}\in\mathbb{R}^N$ and $s\in[t,T]$. This leads to the same estimate for $\widetilde{\Phi}_{\delta_1,\delta_2}(\tau)$ with a replacement of the terms $U_{\mathfrak{m}},U_{\mathfrak{p}},U_{\mathfrak{q}},U_{\mathfrak{r}}$, so that Step 4 works identically.

Step 5. Decomposition of the functions $U_{\mathfrak{p}}$, $U_{\mathfrak{q}}$ and $U_{\mathfrak{r}}$. We further decompose the functions $U_{\mathfrak{p}}$, $U_{\mathfrak{q}}$ and $U_{\mathfrak{r}}$ exploiting the equi-integrability of $\mathfrak{p}$, $\mathfrak{q}$ and $\mathfrak{r}$, this time on $\mathbb{R}^N$. We apply the equi-integrability Lemma 5.11 in $L^1(\mathbb{R}^N)+L^2(\mathbb{R}^N)$. Given $\varepsilon>0$, we find $C_\varepsilon>0$, a Borel set $A_\varepsilon\subset(0,T)\times\mathbb{R}^N$ with finite measure and decompositions to obtain $\mathfrak{p}^1,\mathfrak{p}^2,\mathfrak{q}^1,\mathfrak{q}^2,\mathfrak{r}^1,\mathfrak{r}^2$ such that

$$\begin{gathered}\|\mathfrak{p}^1\|_{L^1((0,T)\times\mathbb{R}^N)}\leq\varepsilon\,,\quad \|\mathfrak{q}^1\|_{L^1((0,T)\times\mathbb{R}^N)}\leq\varepsilon\,,\quad \|\mathfrak{r}^1\|_{L^1((0,T)\times\mathbb{R}^N)}\leq\varepsilon\,,\\ \|\mathfrak{p}^2\|_{L^2((0,T)\times\mathbb{R}^N)}\leq C_\varepsilon\,,\quad \|\mathfrak{q}^2\|_{L^2((0,T)\times\mathbb{R}^N)}\leq C_\varepsilon\,,\quad \|\mathfrak{r}^2\|_{L^2((0,T)\times\mathbb{R}^N)}\leq C_\varepsilon\,,\end{gathered}$$

and

$$\operatorname{spt}(\mathfrak{p}^2)\subset A_\varepsilon\,,\quad \operatorname{spt}(\mathfrak{q}^2)\subset A_\varepsilon\,,\quad \operatorname{spt}(\mathfrak{r}^2)\subset A_\varepsilon\,.$$

Applying Lemma 19.5 to $U^1_{\mathfrak{p}}$ and $U^2_{\mathfrak{p}}$ we get

$$\begin{aligned}|||U^1_{\mathfrak{p}}|||_{M^1((0,T)\times B_\lambda)} &\lesssim \delta_1||\mathfrak{p}^1||_{L^1((0,T)\times\mathbb{R}^N))}\lesssim \delta_1\varepsilon\,,\\ \|U^2_{\mathfrak{p}}\|_{L^2((0,T)\times B_\lambda)} &\lesssim \delta_1||\mathfrak{p}^2||_{L^2((0,T)\times\mathbb{R}^N)}\lesssim \delta_1 C_\varepsilon\,.\end{aligned}\tag{5.97}$$

The lemma gives a similar estimate for $U_{\mathfrak{q}}$ and $U_{\mathfrak{r}}$:

$$\begin{aligned}|||U^1_{\mathfrak{q}}|||_{M^1((0,T)\times B_\lambda)} &\lesssim \delta_2\varepsilon\,, & |||U^1_{\mathfrak{r}}|||_{M^1((0,T)\times B_\lambda)} &\lesssim \delta_2\varepsilon\,,\\ \|U^2_{\mathfrak{q}}\|_{L^2((0,T)\times B_\lambda)} &\lesssim \delta_2 C_\varepsilon\,, & \|U^2_{\mathfrak{r}}\|_{L^2((0,T)\times B_\lambda)} &\lesssim \delta_2 C_\varepsilon\,,\end{aligned}\tag{5.98}$$

as well as the bound

$$|||U_{\mathfrak{m}}|||_{M^1((0,T)\times B_\lambda)}\lesssim_\lambda \delta_1||\mathfrak{m}||_{L^1((0,T);\mathcal{M}(\mathbb{R}^N))}\,.\tag{5.99}$$

We then decompose the functions $U_{\mathfrak{p}}$, $U_{\mathfrak{q}}$ and $U_{\mathfrak{r}}$ from Step 3 using this modification to get

$$\begin{aligned}U_{\mathfrak{p}} &= \mathcal{U}(\mathfrak{p})\leq\mathcal{U}(\mathfrak{p}^1)+\mathcal{U}(\mathfrak{p}^2) =: U^1_{\mathfrak{p}}+U^2_{\mathfrak{p}}\,,\\ U_{\mathfrak{q}} &= \mathcal{U}(\mathfrak{q})\leq\mathcal{U}(\mathfrak{q}^1)+\mathcal{U}(\mathfrak{q}^2) =: U^1_{\mathfrak{q}}+U^2_{\mathfrak{q}}\end{aligned}$$

and

$$U_{\mathfrak{r}} = \mathcal{U}(\mathfrak{r})\leq\mathcal{U}(\mathfrak{r}^1)+\mathcal{U}(\mathfrak{r}^2) =: U^1_{\mathfrak{r}}+U^2_{\mathfrak{r}}\,.$$

Splitting the minima according to this decomposition gives (5.83).

Step 6. Estimating the functions φ_j. Let $\Omega' = (t,\tau) \times B_\lambda \subset \mathbb{R}^{N+1}$. Changing variable along the flows and using (5.99) we obtain estimates identical to (5.85)-(5.89), minus the dependence on λ. Applying Interpolation lemma 5.10 gives the same final estimate as in Step 7, so that we may conclude in the same way. □

REMARK 21.1. *Following the proof of Theorem 20.1, one may consider a weakened assumption to* (**R1**). *Let* **(R1a)** *denote the following: for all regular Lagrangian flows $X : [t,T] \times \mathbb{R}^N \to \mathbb{R}^N$ relative to b starting at time t with compression constant L, and for all $r, \lambda > 0$,*

$$\mathcal{L}^N(B_r \setminus G_\lambda) \leq g(r,\lambda), \quad \text{with } g(r,\lambda) \to 0 \text{ as } \lambda \to \infty \text{ at fixed } r, \tag{R1a}$$

where G_λ denotes the sublevel of the flow X, defined in (5.41)*. This assumption is implied by* **(R1)** *thanks to Lemma 16.3 and alone is sufficient to conclude the estimate.*

21.3. Lagrangian solutions to the linear transport equation. In the next chapter we apply the result of sections 16 and 17 to proving existence of Lagrangian solutions the Euler and Vlasov Poisson equations. These are defined as the superposition of the initial data with the regular Lagrangian flow associated to the vector field b, which in case of the Euler equations is within the regularity setting **(R2)**, and in case of the Vlasov equation is within the setting **(R2a)**. The compactness results in the previous sections dictate that such Lagrangian solutions are well-defined almost everywhere and stable. They are in particular renormalized solutions in the sense of definition 8.1. It was proved in [16] for general vector fields with bounded divergence satisfying **(R1)-(R3)** that there exists a unique *forward-backward* regular Lagrangian flow in the sense of Definition 16.1. A consequence of Theorem 20.1 is that compactness, existence, and uniqueness of forward-backward Lagrangian flows associated to vector fields with bounded divergence satisfying **(R1), (R2a)-(R2b),(R3)** follow, with little modification in the proofs. We remark that for such vector fields it is not possible to exclude non-uniqueness of renormalized or distributional solutions. It may happen that several weak solutions exist, with only one associated to a Lagrangian flow. However this special class of solutions associated to flows is stable under approximation and gives rise to existence of both Lagrangian and renormalized weak solutions to the non-linear Euler and Vlasov Poisson equations, in particular with L^1 data.
We can define the Lagrangian solution to (2.2) by the push-forward of the initial datum via the flow (5.100). This is of course equal to the classical solution in the case of smooth data. Because of the compactness results in Theorem 20.1, these solutions are well defined and stable. One then obtains an analog of Theorem 8.2 for Lagrangian solutions.

DEFINITION 21.2 (Lagrangian solution to the linear transport equation). *Assume that b satisfies* **(R1), (R2), (R3)** *or* **(R1a), (R2a) or (R2b), (R3)** *and* $\operatorname{div} b = 0$. *Let X be the forward-backward regular Lagrangian flow associated to b. For $u^0 \in L^1(\mathbb{R}^N)$, define the Lagrangian solution to the transport equation* (2.2) *by*

$$u(t,x) = u^0(X(0,t,x)). \tag{5.100}$$

COROLLARY 21.3 (Stability). *Let b_n and b be divergence free vector fields satisfying* **(R1), (R2a) or (R2b), (R3)** *with uniform bounds such that $b_n \to b$ in $L^1_{\mathrm{loc}}([0,T] \times \mathbb{R}^N)$. Let u_n denote the Lagrangian solutions to* (2.2) *with coefficient b_n and datum u_n^0. Let $1 \leq q < \infty$. Then*

(1) If $u_n^0 \rightharpoonup u^0$ in $L^q(\mathbb{R}^N) - w$, then $u_n \rightharpoonup u$ in $C([0,T]; L^q(\mathbb{R}^N) - w)$.
(2) If $u_n^0 \to u^0$ in $L^q(\mathbb{R}^N) - s$, then $u_n \to u$ in $C([0,T]; L^q(\mathbb{R}^N) - s)$.

PROOF. See [16]. □

CHAPTER 6

The Euler equation with L^1 vorticity

In this chapter we study the existence of infinite kinetic energy solutions associated to initial vorticities belonging to $L^1(\mathbb{R}^2)$. The results from Theorem 16.6 hold for vector fields in the general regularity setting **(R1)-(R3)**, which in particular encompasses the properties **(I)-(III)** of the velocity from chapter 3, section 3, when $\omega \in L^\infty(L^1)$. Existence of a Lagrangian flow associated to the velocity is a consequence. However, there is difficulty in identifying suitable notions of the weak formulation. Due to the absence of (even local) kinetic energy bounds, the velocity formulation (3.1) cannot be given the usual distributional meaning (see Definition 10.1). Though, a *symmetrized velocity formulation* can be introduced, see Definition 22.1.

For vorticities $\omega \in L^\infty([0,T]; L^1(\mathbb{R}^2))$ the decomposition

$$v = K_1 * \omega + K_2 * \omega, \tag{6.1}$$

where $K_1 = K\,\mathbf{1}_{B_1(0)} \in L^1(\mathbb{R}^2)$ and $K_2 = K\,\mathbf{1}_{B_1(0)^c} \in L^\infty(\mathbb{R}^2)$, gives immediately with Young's inequality that $v \in L^\infty([0,T]; L^1(\mathbb{R}^2)) + L^\infty([0,T]; L^\infty(\mathbb{R}^2))$. A direct distributional formulation is not available even for the vorticity formulation (3.3) in such a context, since the factors in the product $v\,\omega$ are not summable enough to define a locally integrable product. In order to circumvent this issue, one can consider three alternate formulations of weak solutions for the vorticity equation:

(1) *Renormalized* solutions [25], defined by the requirement that $\beta(\omega)$ is a distributional solution of (3.3) in the sense of distributions for a suitable class of functions β:

$$\begin{cases} \partial_t(\beta(\omega)) + \operatorname{div}(v\beta(\omega)) = 0 \\ \beta(\omega(0,\cdot)) = \beta(\omega^0), \end{cases} \tag{6.2}$$

(2) *Symmetrized vorticity* solutions [24, 63], defined by exploiting the antisymmetry of the Biot-Savart kernel K, so that multiplying (3.3) by a test function ϕ and integrating give the formulation

$$\int_0^T \int_{\mathbb{R}^2} \phi_t(t,x)\omega(t,x)\,dxdt + \int_0^T \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} H_\phi(t,x,y)\omega(t,y)\omega(t,x)\,dydxdt + \int_{\mathbb{R}^2} \phi(0,x)\omega^0(x)\,dx = 0, \tag{6.3}$$

where H_ϕ is a suitable bounded function,

(3) *Lagrangian* solutions, i.e. solutions ω transported by a suitable flow associated to the velocity v, to be precisely defined in the sequel.

We will address the question whether initial vorticities in L^1 give rise to weak solutions which are transported by flows. The key point of our strategy relies on a compactness property (under bounds that are natural in our setting) for regular Lagrangian flows. The novelty of this approach, in contrast to [24, 45, 34, 69], is that it entirely relies on the Lagrangian formulation, and therefore proves existence of solutions which are naturally associated to flows. In this setting we also allow for velocities with locally infinite kinetic energy.

A uniform L^1 bound on the vorticities is still sufficient to guarantee the L^1_{loc} convergence of the smoothed velocities, it is generally insufficient for the strong convergence in L^2_{loc}, as discussed in chapter 3. However, concentration-cancellation may still occur, for instance if the vorticity is a measure with distinguished sign

[24]. For L^1 vorticities with compact support, without necessarily distinguished sign, and initial velocities with locally finite kinetic energy, the propagation of the equi-integrability guarantees concentration-cancellations [69]. However, these solutions may not be Lagrangian, since a limiting flow may not exist.

A stability estimate for flows associated to velocity fields with gradient given by the singular integral of an L^1 function was derived in chapter 5. Our regularity setting falls under this theory. From such a theory it follows that Lagrangian flows associated to velocities whose curl are equi-integrable are strongly precompact, and thus stable under approximation, so that the limit flow solves the ODE that involves the limit velocity. We will therefore conclude that vorticities in L^1 are stable under approximation, in the sense that if ω_n^0 converge strongly in L^1 to ω^0, then (up to a subsequence) the solutions ω_n of the corresponding vorticity formulation converge strongly in L^1 to a *Lagrangian* solution ω. Additionally, even for weakly convergent initial vorticities, the flow always converges strongly.

A classical difficulty in proving strong compactness is related to time oscillations. Indeed, when dealing with velocity formulations, the strong compactness in space follows from the L^1 bound on the vorticity, but the compactness in time relies on bounds on $\partial_t v_n$ in $L_t^\infty(\mathcal{D}'_x)$ in order for Aubin-Lions' lemma to apply. Without the assumption $v \in L^2_{\mathrm{loc}}$, we do not have such regularity in time of v and we cannot apply Aubin-Lions' lemma. We thus propose a refinement of the stability estimates in [16] so that weak time convergence of the velocities is still sufficient for the stability of regular Lagrangian flows. We nevertheless prove a posteriori the strong compactness of v in time and space.

22. Weak solutions

We summarize the vorticity and velocity formulations available in our setting when the vorticity is only L^1 summable, as opposed to (3.20) and (3.19).

22.1. Symmetrized velocity solutions. In order to deal with solutions with locally infinite kinetic energy we can propose a weaker formulation than the one in Definition 3.14. It is in the same spirit as the symmetrized vorticity formulation (6.3). Using the identity $\operatorname{div}(v \otimes v) = v \cdot \nabla v = \omega\, v^\perp + \nabla \frac{|v|^2}{2}$, that is valid when $\operatorname{div} v = 0$, we can formally rewrite (3.1) as

$$\partial_t v + \omega v^\perp + \nabla p' = 0, \tag{6.4}$$

where $p' = p + \frac{|v|^2}{2}$. This modified pressure p' can be eliminated by taking suitable test functions as in (**??**). With this form (6.4) we can observe that only the quantities $v_1 v_2$ and $v_1^2 - v_2^2$ need to be in L^1, since we can write $\omega v^\perp = \operatorname{div}(v \otimes v - (|v|^2/2)\mathrm{I}d)$, and the entries of the matrix $v \otimes v - (|v|^2/2)\mathrm{I}d$ are just these two scalars $v_1 v_2$ and $v_1^2 - v_2^2$. However, without such assumptions, we observe that the term $\omega v^\perp$ has a priori no pointwise meaning when ω only belongs to L^p for some $p < 4/3$, since in such a case ω and v would not have conjugate summabilities. Nevertheless, with the only assumption $\omega \in L^1$, that yields $v \in M^2$ (but $v \notin L^2_{\mathrm{loc}}$ in general), we can give a meaning in distribution sense to this term by exploiting the symmetrization technique analog to that in [24, 69], that uses the antisymmetry property $K(-x) = -K(x)$.

Let $\phi \in C_c^1([0,T)\times\mathbb{R}^2,\mathbb{R}^2)$. Then using the Biot-Savart law we can write

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}^2} (\omega v^\perp)(t,x)\cdot\phi(t,x)\,dxdt \\
&= \int_0^T \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \omega(t,x)\omega(t,y)K(x-y)^\perp\cdot\phi(t,x)\,dxdydt \\
&= -\int_0^T \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \omega(t,y)\omega(t,x)K(x-y)^\perp\cdot\phi(t,y)\,dxdydt \\
&= \frac{1}{2}\int_0^T \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \omega(t,x)\omega(t,y)K(x-y)^\perp\cdot(\phi(t,x)-\phi(t,y))\,dxdydt \\
&= \int_0^T \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \omega(t,x)\omega(t,y)\bar{H}_\phi(t,x,y)\,dxdydt,
\end{aligned} \tag{6.5}$$

where $\bar{H}_\phi(t,x,y)$ is the function on $[0,T)\times\mathbb{R}^2\times\mathbb{R}^2$ given by

$$\bar{H}_\phi(t,x,y) = \frac{1}{2}K(x-y)^\perp\cdot(\phi(t,x)-\phi(t,y)). \tag{6.6}$$

For $\phi \in C_c^1([0,T)\times\mathbb{R}^2,\mathbb{R}^2)$ we have that $\bar{H}_\phi$ is a bounded function, continuous outside the diagonal, that tends to zero at infinity. Indeed we have

$$|\bar{H}_\phi(t,x,y)| \leq \frac{1}{4\pi}\mathrm{Lip}(\phi(t,\cdot)). \tag{6.7}$$

Thus for vorticities belonging to $L^\infty((0,T);L^1(\mathbb{R}^2))$, the last integral in (6.5) is well-defined. This motivates the next definition of weak solutions.

Definition 22.1 (Symmetrized velocity formulation). *Let $(\omega^0,v^0)\in L^1(\mathbb{R}^2)\times M^2(\mathbb{R}^2)$, with $\omega^0 = \operatorname{curl} v^0$. We say that the couple (ω,v) is a symmetrized velocity solution of (3.1) in $[0,T)$ with initial datum (ω^0,v^0), if*

(1) $\omega\in L^\infty((0,T);L^1(\mathbb{R}^2))$,

(2) the velocity field v is given by the convolution in (3.4),

(3) for all test functions $\phi\in C_c^1([0,T)\times\mathbb{R}^2,\mathbb{R}^2)$ with $\operatorname{div}\phi = 0$, we have

$$\int_0^T\int_{\mathbb{R}^2}\partial_t\phi\cdot v\,dxdt - \int_0^T\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\bar{H}_\phi(t,x,y)\omega(t,x)\omega(t,y)\,dxdydt + \int_{\mathbb{R}^2}\phi(0,x)\cdot v^0(x)\,dx = 0, \tag{6.8}$$

where $\bar{H}_\phi$ is the function on $[0,T)\times\mathbb{R}^2\times\mathbb{R}^2$ given by (6.6).

22.2. Three formulations of the vorticity equation. According to the introduction, we now define three notions of solution to the vorticity formulation (3.3) when the vorticity is only L^1 summable. Since we do not assume $v^0\in L^2_{\mathrm{loc}}(\mathbb{R}^2)$, we deal with velocities that belong to $M^2(\mathbb{R}^2)$, a consequence of the Hardy-Littlewood inequality (3.10).

Definition 22.2 (Renormalized solutions). *Let $(\omega^0,v^0)\in L^1(\mathbb{R}^2)\times M^2(\mathbb{R}^2)$ with $\omega^0 = \operatorname{curl} v^0$. We say the couple (ω,v) is a renormalized solution to (3.3) with initial data (ω^0,v^0), if*

(1) $\omega\in L^\infty((0,T);L^1(\mathbb{R}^2))$,

(2) the velocity field v is given by the convolution in (3.4),

(3) for every nonlinearity $\beta \in C^1(\mathbb{R})$ with β bounded, we have that

$$\begin{cases} \partial_t(\beta(\omega)) + \operatorname{div}(\beta(\omega)v) = 0, \\ \beta(\omega)(0,\cdot) = \beta(\omega^0) \end{cases} \tag{6.9}$$

hold in the sense of distributions.

For smooth solutions this is equivalent to the classical notion of solution (as can be seen by multiplying the equation by $\beta'(\omega)$ and applying the chain rule.) This formulation derives from the classical DiPerna-Lions [25] framework for transport equations.

Definition 22.3 (Symmetrized vorticity formulation). *As mentioned in the introduction, the symmetrization technique for the term* $\operatorname{div}(\omega v)$ *provides a second formulation of the vorticity equation. Let* $\phi \in C_c^2([0,T)\times\mathbb{R}^2)$. *Computations as in* (6.5) *give*

$$\int_0^T \int_{\mathbb{R}^2} \operatorname{div}(\omega v)(t,x)\phi(t,x)\,dxdt = \int_0^T \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} H_\phi(t,x,y)\omega(t,x)\omega(t,y)\,dxdydt, \tag{6.10}$$

with

$$H_\phi(t,x,y) = -\frac{1}{2}K(x-y)\cdot(\nabla\phi(t,x) - \nabla\phi(t,y)). \tag{6.11}$$

We say that (ω, v) *is a symmetrized vorticity solution to* (3.3) *if (1), (2) above are satisfied and if for all test functions* $\phi \in C_c^2([0,T)\times\mathbb{R}^2)$ *there holds*

$$\int_0^T \int_{\mathbb{R}^2} \partial_t\phi(t,x)\omega(t,x)\,dxdt - \int_0^T \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} H_\phi(t,x,y)\omega(t,x)\omega(t,y)\,dxdydt + \int_{\mathbb{R}^2}\phi(0,x)\omega^0(x)\,dx = 0. \tag{6.12}$$

Proposition 22.4. *We have the following equivalence of notions of solutions to the Euler system.*

(1) Symmetrized velocity solutions (Definition 22.1) are symmetrized vorticity solutions (Definition 22.3), and conversely.
(2) If (ω, v) *is such that* $v \in L^\infty((0,T); L^2_{\mathrm{loc}}(\mathbb{R}^2))$, *then it is a symmetrized velocity solution if and only if it is a weak velocity solution (Definition 3.14).*

Proof. For (1), taking a test function of the form $-\nabla^\perp\phi$ in (6.8) we see that a solution to the symmetrized velocity formulation is also a solution to the symmetrized vorticity formulation, indeed one has $\bar H_{-\nabla^\perp\phi} = H_\phi$. The converse is also true since all functions $\bar\phi \in C_c^2([0,T)\times\mathbb{R}^2,\mathbb{R}^2)$ with $\operatorname{div}\bar\phi = 0$ can be written $\bar\phi = -\nabla^\perp\phi$ for some $\phi \in C_c^2([0,T)\times\mathbb{R}^2)$. For $\bar\phi$ only C^1 one just approximates it by a C^2 function. It follows that Definitions 22.1 and 22.3 are indeed equivalent. Finally, the statement (2) follows from the next lemma. ☐

Lemma 22.5. *Let* $\omega \in L^1(\mathbb{R}^2)$, *define* $v = K * \omega$ *with* K *the Biot-Savart kernel* (3.4), *and assume that* $v \in L^2_{\mathrm{loc}}(\mathbb{R}^2)$. *Then for all* $\phi \in C_c^1(\mathbb{R}^2,\mathbb{R}^2)$ *with* $\operatorname{div}\phi = 0$, *we have*

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \bar H_\phi(x,y)\omega(x)\omega(y)\,dxdy = -\int_{\mathbb{R}^2} \nabla\phi(x) : (v(x)\otimes v(x))\,dx \tag{6.13}$$

where $\bar H_\phi$ *is given by* (6.6).

Proof. For smooth ω and v, the formula is just the weak form of the already mentioned identity $\operatorname{div}(v\otimes v) = \omega\, v^\perp + \nabla\frac{|v|^2}{2}$, taking into account the computation (6.5). The general case follows easily by smoothing ω and v by a regularizing kernel and passing to the limit. ☐

22.3. Lagrangian solutions. We describe a third class of weak solutions which are transported by a Lagrangian flow in the renormalized sense of definition (16.1). With this notion of flow, we can define a third class of solutions. We will consider regular Lagrangian flows X as in Definition 16.1, except that now $s \in [0,T]$ instead of $s \in [t,T]$ (the forward-backward flow in [16]), with compression constant L independent of $t \in [0,T]$. We define in accordance with [16] a Lagrangian solution (ω, v) to the Euler equations by

$$\omega(t,x) = \omega^0\Big(X(s=0,t,x)\Big), \qquad \text{for all } t \in [0,T]. \tag{6.14}$$

DEFINITION 22.6 (Lagrangian solution). *Let $(\omega^0, v^0) \in L^1(\mathbb{R}^2) \times M^2(\mathbb{R}^2)$ with $\omega^0 = \operatorname{curl} v^0$. We say the couple (ω, v) is a Langrangian solution to (3.3) in $[0,T)$ with initial data (ω^0, v^0), if*

(1) $\omega \in L^\infty([0,T); L^1(\mathbb{R}^2))$,
(2) the associated velocity field v is given by the convolution in (3.4),
(3) for all $t \in [0,T)$, ω is given by the formula in (6.14), where X is a regular Lagrangian flow associated to v.

REMARK 22.7. *Theorem 16.6 gives stability and may be used to prove compactness of Lagrangian flows, in case when v_n is an approximating sequence of velocities converging strongly to v in L^1_{loc}. The theory of stability and compactness of forward-backward flows applies to the regularity setting* **(I)-(III)***, thus the formula (6.14) is well defined for arbitrary $\omega^0 \in L^1(\mathbb{R}^2)$. In particular such solutions ω belong to $C([0,T]; L^1(\mathbb{R}^2))$, and are also renormalized.*

23. Compactness of Lagrangian flows associated to infinite kinetic energy velocities

In Theorem 10.5, strong L^1_{loc} convergence of smoothed velocities was guaranteed for initial data v^0 belonging to $L^2_{\text{loc}}(\mathbb{R}^2)$. In order to allow for solutions with infinite kinetic energy, we bypass this assumption and use the weaker M^2 estimate arising in (3.10). We show later that an a priori *weak* convergence of the velocities v_n is sufficient to obtain stability of the associated flows. Given equi-integrable initial data, it follows a posteriori by Corollary 21.3 that the velocities converge strongly in L^1_{loc} as well.

To the reader's convenience we expand in the following Theorem 23.3 a remark from [22] proving that weak convergence is sufficient for stability, and adapt the proof from the setting of Sobolev regularity to the regularity given by **(R2)**. We begin with two lemmas, the first arising from standard analysis.

LEMMA 23.1. *Let K be the Biot-Savart kernel in (3.4), and denote by $\tau_h K(x) = K(x+h)$. Then*

$$\|\tau_h K(x) - K(x)\|_{L^p(\mathbb{R}^2)} \leq c|h|^\alpha$$

for some $\alpha > 0$ depending on p, and for any $1 < p < 2$. In particular, the linear mapping $T : L^1(\mathbb{R}^2) \to L^1_{\text{loc}}(\mathbb{R}^2)$ defined by $T : g \mapsto K * g$ is a compact operator. Let K be the Biot-Savart kernel (3.4), and denote by $\tau_h K(x) = K(x+h)$. Then for any $1 < p < 2$ and all $h \in \mathbb{R}^2$ one has

$$\|\tau_h K - K\|_{L^p(\mathbb{R}^2)} \leq c_p|h|^\alpha \tag{6.15}$$

with $\alpha = 2/p - 1 > 0$. In particular, the linear mapping $L^1(\mathbb{R}^2) \to L^1_{\text{loc}}(\mathbb{R}^2)$ defined by $g \mapsto K * g$ is a compact operator.

PROOF. See Lemma 29.1 in chapter 7 for the proof of the first inequality. Let g_n be a bounded sequence in $L^1(\mathbb{R}^2)$. For any $1 < p < 2$ we can estimate

$$\begin{aligned} \|\tau_h T g_n - T g_n\|_{L^p(\mathbb{R}^2)} &= \|(\tau_h K(x) - K(x)) * g_n(x)\|_{L^p(\mathbb{R}^2)} \\ &\leq c\,\|\tau_h K - K\|_{L^p(\mathbb{R}^2)}\,\|g_n\|_{L^1(\mathbb{R}^2))} \\ &\leq c\|g_n\|_{L^1(\mathbb{R}^2))}|h|^\alpha. \end{aligned} \tag{6.16}$$

Thus $\tau_h T g_n - T g_n$ tends to zero uniformly in n as $h \to 0$. Applying Riesz-Fréchet-Kolmogorov's Theorem gives a subsequence of $T g_n$ converging strongly in $L^1_{\text{loc}}(\mathbb{R}^2)$. ☐

The second lemma states that given classical flows associated to Lipschitz vector fields, weak convergence of the vector fields suffices for the associated flows to converge uniformly.

LEMMA 23.2. *Let v_n be a sequence of smooth vector fields uniformly bounded in $L^\infty([0,T]\times\mathbb{R}^2)$ with $\nabla_x v_n$ uniformly bounded in $L^\infty([0,T]\times\mathbb{R}^2)$. Assume that there exists $v\in L^\infty([0,T]\times\mathbb{R}^2)$ with $\nabla_x v$ uniformly bounded in $L^\infty([0,T]\times\mathbb{R}^2)$ such that $v_n \rightharpoonup^* v$ in $L^\infty([0,T]\times\mathbb{R}^2)-w*$. Let $X_n(s,t,x)$ and $X(s,t,x)$ be Lagrangian flows (in the DiPerna-Lions sense) associated to v_n and v. Then $X_n(s,t,x)\to X(s,t,x)$ in $L^\infty([0,T]^2;L^\infty_{\rm loc}(\mathbb{R}^2))$.*

PROOF. It follows from the uniform bounds on v_n and $\nabla_x v_n$ that $\nabla_x X_n$ is uniformly bounded on $[0,T]^2\times\mathbb{R}^2$, since

$$|\nabla_x X_n(s,t,x)|\leq \exp\left(T\|\nabla v_n\|_{L^\infty([0,T]\times\mathbb{R}^2)}\right).$$

As for the Lipschitz regularity in s and t, the ODE for X_n implies that $\partial_s X_n$ is uniformly bounded. The equation

$$\partial_t X_n + v_n\cdot\nabla X_n = 0 \qquad \text{in } \mathcal{D}'((0,T)\times\mathbb{R}^2) \tag{6.17}$$

implies also that $\partial_t X_n$ is bounded. Thus up to modifying X_n on a Lebesgue negligible set, X_n is uniformly bounded in $\mathrm{Lip}([0,T]^2\times\mathbb{R}^2)$. By Arzelà-Ascoli's theorem there exists $Y(s,t,x)\in \mathrm{Lip}([0,T]^2\times\mathbb{R}^2)$ such that up to a subsequence $X_n(s,t,x)\to Y(s,t,x)$ locally uniformly in $[0,T]^2\times\mathbb{R}^2$. Using the identity $v_n\cdot\nabla(X_n)=\mathrm{div}\,(X_n\otimes v_n)-X_n\,\mathrm{div}\,v_n$, it follows immediately from the uniform convergence of X_n and the weak convergence of v_n that we can pass to the limit in (6.17), so that by the uniqueness property of v we must have $Y=X$. □

Lemmas 23.1 and 23.2, together with Theorem 16.6, yield the following stability result for Lagrangian flows, which states that weak convergence of the velocity fields implies that the associated flows converge strongly anyway.

THEOREM 23.3. *Let (v_n) be a sequence of divergence free velocity fields uniformly bounded in $L^\infty([0,T);M^2(\mathbb{R}^2))$. Assume that $v_n\rightharpoonup v$ in $\mathcal{D}'((0,T)\times\mathbb{R}^2)$, where $v(t,x)$ is a divergence free velocity field. Assume additionally that $\mathrm{curl}\,v_n$ is equi-integrable in $L^1([0,T)\times\mathbb{R}^2)$. Let X_n be a sequence of regular Lagrangian flows associated to v_n, and let X be a regular Lagrangian flow associated to v. Then X_n converges locally in measure to X, uniformly in s and t.*

PROOF. The assumptions imply that $\omega_n\equiv\mathrm{curl}\,v_n$, $\omega\equiv\mathrm{curl}\,v\in L^1((0,T)\times\mathbb{R}^2)$, $v_n=K*\omega_n$, $v=K*\omega$, thus the conditions **(R1)**,**(R2)**,**(R3)** are satisfied for v_n and v, justifying the existence and uniqueness of X_n and X. We regularize v_n and v with respect to the spatial variable. Take $\rho\in C_c^\infty(\mathbb{R}^2)$ be the standard mollifier with $\mathrm{spt}(\rho)\subset B_1$. Denote by $\rho_\varepsilon(x)=\varepsilon^{-2}\rho(x/\varepsilon)$, and define

$$v_n^\varepsilon = v_n \underset{x}{*} \rho_\varepsilon, \qquad v^\varepsilon = v\underset{x}{*}\rho_\varepsilon.$$

Let X_n^ε and X^ε denote the DiPerna-Lions flows associated to v_n^ε and v^ε respectively, as in Lemma 23.2. Since v_n^ε and v^ε also satisfy **(R1)**,**(R2)**,**(R3)**, it is easy to see that X_n^ε and X^ε are also the regular Lagrangian flows in the sense of Definition 5.42. Then we write

$$\begin{aligned} X_n - X &= (X_n - X_n^\varepsilon) + (X_n^\varepsilon - X^\varepsilon) + (X^\varepsilon - X)\\ &\equiv I + II + III. \end{aligned} \tag{6.18}$$

By Theorem 16.6 the term III tends to zero locally in measure, uniformly in s,t, as $\varepsilon\to0$. For I, applying also Theorem 16.6, which is possible because the ω_n are uniformly equi-integrable, gives that for all $\gamma>0$, $r>0$, $\eta>0$, there exist $\lambda>0$ and $C>0$ such that

$$\mathcal{L}^2(B_r\cap\{|X_n^\varepsilon(s,t,\cdot)-X_n(s,t,\cdot)|>\gamma\})\leq C\|v_n^\varepsilon-v_n\|_{L^1((0,T)\times B_\lambda)}+\eta, \tag{6.19}$$

for all $s,t \in [0,T]^2$. Using Minkowski's inequality and applying Lemma 29.1, we estimate

$$\begin{aligned}
\|v_n^\varepsilon - v_n\|_{L^1((0,T);L^p(\mathbb{R}^2))} &= \int_0^T \left[\int_{\mathbb{R}^2} \left| \int_{B_\varepsilon} [v_n(t,x-y) - v_n(t,x)\rho_\varepsilon(y)dy \right|^p dx \right]^{1/p} dt \\
&\leq \int_0^T \int_{B_\varepsilon} \left[\int_{\mathbb{R}^2} |v_n(t,x-y) - v_n(t,x)|^p dx \right]^{1/p} |\rho_\varepsilon(y)| dy\, dt \\
&\leq c_p \|\omega_n\|_{L^1((0,T);L^1(\mathbb{R}^2))} \int_{B_\varepsilon} |y|^\alpha |\rho_\varepsilon(y)| dy \\
&\leq C\,\varepsilon^\alpha.
\end{aligned}$$

Thus the first term in the right-hand side of (6.19) tends to zero as $\varepsilon \to 0$, uniformly in n. We deduce that the terms I and III can be made arbitrarily small independently of n, for a suitable choice of ε. Once such ε is chosen, we observe that we can apply Lemma 23.2 to the vector fields v_n^ε and v^ε. We deduce that $X_n^\varepsilon \to X^\varepsilon$ locally uniformly in s,t,x, as $n \to \infty$, which concludes the proof of the Proposition. □

24. Existence of Lagrangian solutions

We now apply the compactness results for Lagrangian flows derived in the previous section to derive compactness and existence of Lagrangian solutions to the Euler equations.

THEOREM 24.1 (Compactness of Lagrangian solutions). *Let $(\omega_n, v_n) \in L^\infty([0,T];L^1(\mathbb{R}^2)) \times L^\infty([0,T];M^2(\mathbb{R}^2))$ be a sequence of Lagrangian solutions to the Euler equations associated to uniformly in n equi-integrable initial vorticity data ω_n^0. Let X_n denote the regular Lagrangian flows associated to v_n. Then, up to the extraction of a subsequence, there exists $(\omega, v) \in L^\infty([0,T];L^1(\mathbb{R}^2)) \times L^\infty([0,T];M^2(\mathbb{R}^2))$ such that*

(1) $X_n \to X$ locally in measure, uniformly in time, and X is a regular Lagrangian flow associated to v.

In addition,

(2) If $\omega_n^0 \rightharpoonup \omega^0$ weakly in $L^1(\mathbb{R}^2)$, then $\omega_n \rightharpoonup \omega$ in $C([0,T];L^1(\mathbb{R}^2) - w)$.
(3) If $\omega_n^0 \to \omega^0$ strongly in $L^1(\mathbb{R}^2)$, then $\omega_n \to \omega$ in $C([0,T];L^1(\mathbb{R}^2) - s)$.
(4) $v_n \to v$ strongly in $C([0,T];L^1_{\mathrm{loc}}(\mathbb{R}^2))$, where v is given by the convolution in (3.4) for a.e. $(t,x) \in [0,T] \times \mathbb{R}^2$.

Moreover, (ω, v) is a Lagrangian solution to (3.3).

PROOF. The incompressibility of X_n implies that $\omega_n(t,\cdot)$ inherit the equi-integrability of ω_n^0. Moreover, v_n is equi-bounded in $L^\infty([0,T];M^2(\mathbb{R}^2))$ and after passing to a subsequence $v_n \rightharpoonup^* v$ in $L^\infty([0,T];L^1_{\mathrm{loc}}(\mathbb{R}^2))-w*$. Applying Theorem 23.3 to the vector fields v_n gives existence of a forward-backward regular Lagrangian flow $X(s,t,x)$ associated to v such that $X_n \to X$ locally in measure and uniformly in time. Indeed, from Fatou's lemma applied to X_n with $L = \liminf L_n = 1$ we deduce that X has compressibility constant 1. Arguing as in Lemma 6.3 of [16], one can pass to the limit in

$$\partial_s \left(\beta(X_n(s,x))\right) = \beta'(X_n(s,x)) \cdot v_n(s, X_n(s,x)) \qquad \text{in } \mathcal{D}'((0,T)\times\mathbb{R}^2),$$

for every test function $\beta \in C^1(\mathbb{R}^N;\mathbb{R})$ with β' bounded.

It follows from weak convergence of ω_n^0 that ω is given by the formula in (6.14) and (ω, v) is a Lagrangian solution to (3.3) with initial datum (ω^0, v^0). Applying Corollary 21.3 gives (2) and (3). Eventually, applying Lemma 23.1 to the map $\omega_n \mapsto K * \omega_n$, we obtain that v_n is strongly precompact in $C([0,T];L^1_{\mathrm{loc}}(\mathbb{R}^2))$. It also follows that v is given by the convolution in (3.4). □

Corollary 24.2 (Existence). *Let* $(\omega^0, v^0) \in L^1(\mathbb{R}^2) \times M^2(\mathbb{R}^2)$ *with* $\operatorname{div} v^0 = 0$ *and* $\omega^0 = \operatorname{curl} v^0$. *Then there exists a Lagrangian solution* $(\omega, v) \in L^\infty([0,T]; L^1(\mathbb{R}^2)) \times L^\infty([0,T]; M^2(\mathbb{R}^2))$ *to* (3.3).

Proof. Let $\rho(x) \in C_c^\infty(\mathbb{R}^2)$ be the standard mollifier. Let $\omega_n^0 = \rho_n * \omega^0$ and set $v_n^0 = K * \omega_n^0$. Then $\omega_n^0 \to \omega^0$ in $L^1(\mathbb{R}^2)$ and for each n there exists a unique smooth solution (v_n, ω_n) to

$$\begin{cases} \partial_t \omega + \operatorname{div}(v\,\omega) = 0 \\ \omega(0,\cdot) = \omega_n^0(x) \\ v = K * \omega. \end{cases}$$

Thus for each n there exists a regular Lagrangian flow $X_n(s,t,x)$, associated to v_n, such that $\omega_n(t,x) = \omega_n^0(X_n(0,t,x))$. Applying Theorem 24.1, we obtain that up to subsequences X_n converges to X locally in measure, uniformly in time, and (ω_n, v_n) converges strongly in $C([0,T]; L^1(\mathbb{R}^2)) \times C([0,T]; L^1_{\text{loc}}(\mathbb{R}^2))$ to a Lagrangian solution (ω, v). $\square$

25. Existence of renormalized and symmetrized solutions

We have proven indirectly in Theorem 24.1 the strong compactness of smooth approximations. These give rise to Lagrangian, renormalized, and symmetrized solutions. While compactness of Lagrangian solutions imply the existence of symmetrized and renormalized solutions, it is generally not true that a Lagrangian solution (ω, v) is also a symmetrized solution. We therefore define solutions which are associated to Lagrangian flows as well as satisfying the formulation in (6.8). The following conditions are satisfied, for instance, by a smooth solution to (3.3).

Definition 25.1 (Lagrangian symmetrized velocity solution). *Let* $(\omega^0, v^0) \in L^1(\mathbb{R}^2) \times M^2(\mathbb{R}^2)$ *with* $\omega^0 = \operatorname{curl} v^0$. *We say the couple* (ω, v) *is a Langrangian symmetrized velocity solution to* (3.3) *in* $[0,T)$ *with initial data* (ω^0, v^0), *if*

(1) $\omega \in L^\infty([0,T); L^1(\mathbb{R}^2))$,
(2) the associated velocity field v *is given by the convolution in* (3.4),
(3) for all $t \in [0,T)$, ω *is given by the formula in* (6.14), *where* X *is a regular Lagrangian flow associated to* v, *and*
(4) (ω, v) *satisfy the formula in* (6.8) *where* $\bar{H}_\phi$ *is given by* (6.11).

We have the following corollary.

Corollary 25.2. *Let* (ω_n, v_n) *be a sequence of Lagrangian symmetrized velocity solutions of* (3.3) *and let* ω_n *have uniformly in* n *equi-integrable initial data* ω_n^0. *Then up to subsequences,* $v_n(t,x) \to v(t,x)$ *strongly in* $C([0,T]; L^1_{\text{loc}}(\mathbb{R}^2))$, *where* v *is given by the convolution in* (3.4) *for a.e.* $(t,x) \in [0,T] \times \mathbb{R}^2$, *and*

(1) If $\omega_n^0 \rightharpoonup \omega^0$ *weakly in* $L^1(\mathbb{R}^2)$, *then* $\omega_n \rightharpoonup \omega$ *in* $C([0,T]; L^1(\mathbb{R}^2) - w)$.
(2) If $\omega_n^0 \to \omega^0$ *strongly in* $L^1(\mathbb{R}^2)$, *then* $\omega_n \to \omega$ *in* $C([0,T]; L^1(\mathbb{R}^2) - s)$.

Moreover, (ω, v) *is a Lagrangian symmetrized velocity solution.*

Proof. Let X_n denote the flows associated to v_n. Convergence of v_n, (1) and (2) follow from Theorem 24.1. Moreover, X_n converges to a regular Lagrangian flow X associated to v, and $\omega(t,x) = \omega^0(X(0,t,x))$ is a Lagrangian solution to (3.3) with initial datum ω^0. Then (ω, v) is also symmetrized, since we may pass to the limit in the equation (6.8) for (ω_n, v_n). The linear terms clearly converge, and convergence of the nonlinear term follows by boundedness of $\bar{H}_\phi$ and convergence of ω_n in $C([0,T]; L^1(\mathbb{R}^2) - w)$. $\square$

Theorem 23.3 guarantees that the strong L^1_{loc} convergence of v_n is not necessary to prove existence of renormalized or symmetrized (vorticity and velocity) solutions: indeed, the compactness of the flows suffices for the strong convergence of the approximating vorticities and therefore for passing to the limit in the various formulations. We now show that the compactness of Lagrangian solutions implies stability of weak solutions in all five senses defined in section 22.

THEOREM 25.3 (Existence of renormalized, symmetrized vorticity, symmetrized velocity, and velocity solutions). *Let* $(\omega^0, v^0) \in L^1(\mathbb{R}^2) \times M^2(\mathbb{R}^2)$ *with* $\omega^0 = \operatorname{curl} v^0$ *and* $\operatorname{div} v^0 = 0$. *Then there exists a couple* $(\omega, v) \in L^\infty([0,T); L^1(\mathbb{R}^2)) \times L^\infty([0,T); M^2(\mathbb{R}^2))$ *such that* (ω, v) *is a Lagrangian solution, and:*

(1) (ω, v) *is a symmetrized velocity solution to* (3.1), *or symmetrized vorticity solution to* (3.3),
(2) (ω, v) *is a renormalized solution to* (3.3),
(3) *Under the additional assumption* $(\omega^0, v^0) \in L^1(\mathbb{R}^2) \times L^2_{\text{loc}}(\mathbb{R}^2)$, (ω, v) *is a solution to the velocity formulation* (10.1).

PROOF. **Part 1.** This follows from Corollary 25.2 after mollifying (ω^0, v^0). Extracting a subsequence, the convergence of ω_n and v_n from Theorem 24.1 implies also that one can pass to the limit in the equations (6.8) and (6.12).

Part 2. To check that (ω, v) is renormalized, we have to pass to the limit in

$$\int_0^T \int_{\mathbb{R}^2} \phi_t(t,x)\beta(\omega_n(t,x))\,dxdt + \int_0^T \int_{\mathbb{R}^2} v_n(t,x)\beta(\omega_n(t,x)) \cdot \nabla\phi(t,x)\,dxdt + \int_{\mathbb{R}^2} \phi(x,0)\beta(\omega_n^0(x))\,dx = 0.$$

Convergence of the first and last terms follows by Theorem 24.1. Applying Dominated Convergence we have that $\beta(\omega_n) \to \beta(\omega)$ strongly in $L^p_{\text{loc}}([0,T]\times\mathbb{R}^2)$ for any $1 \le p < \infty$. Then since $v_n \to v$ in $L^\infty([0,T]; L^1(\mathbb{R}^2))$, we deduce that the product $v_n\beta(\omega_n) \rightharpoonup v\beta(\omega)$ weakly in $L^1_{\text{loc}}([0,T]\times\mathbb{R}^2)$.

Part 3. When $v_0 \in L^2_{\text{loc}}$, the convergence $\omega_n \to \omega$ in $L^1([0,T]\times\mathbb{R}^2)$ ensures that we can pass to the limit in the weak velocity formulation (10.1), arguing as in Theorem 10.8. $\square$

CHAPTER 7

The Vlasov Poisson equation with L^1 density

In this chapter we study existence of solutions to the Vlasov Poisson equation, in the weak sense when the data belongs to L^1. Similar to the existence result of chapter 3, we prove that the solutions are Lagrangian, and associated to the flows of anisotropic vector fields considered in chapter 5, section 17. The weak solutions in [**46, 47, 48**], where the distribution function is a measure, do not have well-defined characteristics. We extend the existence result of [**30**] to initial data in L^1 with finite energy (in the repulsive case $\omega = +1$), avoiding the $L\log^+ L$ assumption.

We apply the results of chapter 5, section 17 to prove existence and stability of global Lagrangian solutions to the repulsive Vlasov-Poisson system with only integrable initial distribution function with finite energy. These solutions have a well-defined Lagrangian flow. We will need a priori estimate on the smallness of the superlevels of the flow in three dimensions in order to control the characteristics, analogous to Lemma 16.3. Our existence result is Theorem 29.4. It involves a well-defined flow. In this context we prove stability results with strongly or weakly convergent initial distribution function. The flow is proved to converge strongly anyway.

26. Regularity of the force field for L^1 densities

We recall results from section 12 to give estimates for the regularity and growth of the electric field $E(t,x)$ as defined in (4.3) Let $\rho(t,x) \in L^\infty((0,T);L^1(\mathbb{R}^N))$. We denote by

$$b(t,x,v) = (b_1,b_2)(t,x,v) = (v, E(t,x)) = \Big(v, -\omega\nabla_x(-\Delta_x)^{-1}(\rho(t,x)-\rho_b(x))\Big) \tag{7.1}$$

the associated vector field on $(0,T)\times\mathbb{R}^N\times\mathbb{R}^N$.

26.1. Local integrability. For L^1 densities, we have the weak estimates from the Hardy-Littlewood-Sobolev inequality:

$$|||E|||_{L^\infty((0,T);M^{\frac{N}{N-1}}(\mathbb{R}^N))} \leq c_N ||\rho-\rho_b||_{L^\infty((0,T);L^1(\mathbb{R}^N))}, \tag{7.2}$$

and using the inclusion $M^{\frac{N}{N-1}}(\mathbb{R}^N) \subset L^p_{\text{loc}}(\mathbb{R}^N)$ for $1 \leq p < \frac{N}{N-1}$ we conclude that $b \in L^\infty((0,T); L^p_{\text{loc}}(\mathbb{R}^N_x \times \mathbb{R}^N_v))$ for any $1 \leq p < \frac{N}{N-1}$, since $v \in L^p_{\text{loc}}(\mathbb{R}^N_x\times\mathbb{R}^N_v)$ for any p. Then b satisfies **(R3)**.

26.2. Spatial regularity. Recall that the differential matrix of the vector field is given by (4.19) and falls under the assumptions of **(R2a)**: we have by (4.15)

$$(D_xE)_{ij} \equiv \partial_{x_j}E_i = -\omega\partial^2_{x_ix_j}((-\Delta_x)^{-1}(\rho-\rho_b)) \quad \text{for } 1 \leq i,j \leq N, \tag{7.3}$$

so that (each component of) D_xE is a singular integral of an $L^\infty((0,T);L^1(\mathbb{R}^N_x))$ function.

26.3. Time regularity. According to (4.12), $\partial_t E$ belongs to $L^\infty((0,T);\mathcal{S}'(\mathbb{R}^N))$, and is a singular integral of the current $L^\infty((0,T);L^1(\mathbb{R}^N_x))$ function J defined by (4.8).

We wish to extend the notion of characteristics discussed in Lemma 13.1 to non-smooth solutions. We let $\mathbb{R}^{2N}_z = \mathbb{R}^N_x \times \mathbb{R}^N_v$, and we denote by $Z(t,x,v) = (X,V)(t,x,v)$ the regular Lagrangian flow in definition (16.1) associated to b.

DEFINITION 26.1. *Define the sublevel of the regular Lagrangian flow Z as the set*

$$G_\lambda = \{z \in \mathbb{R}^{2N} : |Z(s,z)| \leq \lambda \text{ for almost all } s \in [t,T]\}. \tag{7.4}$$

27. Control of superlevels

In order to apply Theorem 20.1, we need to satisfy **(R1a)**. Therefore, we seek an upper bound on the size of $B_r \setminus G_\lambda$.

27.1. The case of low space dimension. Observe that Lemma 16.3 allows us to control the superlevels of b in 1 or 2 dimensions.

PROPOSITION 27.1. *Let b be the vector field in (4.15), with $E \in L^\infty((0,T);L^2(\mathbb{R}^N))$. For $N=2$ or $N=1$, b satisfies* **(R1)**, *hence also* **(R1a)**.

PROOF. It is clear that

$$\frac{v}{1+|x|+|v|} \in L^\infty_t(L^\infty_{x,v}), \tag{7.5}$$

and

$$\frac{E(t,x)}{1+|x|+|v|} = \frac{E(t,x)}{1+|x|+|v|}\mathbb{1}_{|v|\leq|E(t,x)|} + \frac{E(t,x)}{1+|x|+|v|}\mathbb{1}_{|v|>|E(t,x)|} \equiv \tilde{E}_1 + \tilde{E}_2. \tag{7.6}$$

Clearly $\tilde{E}_2 \in L^\infty_t(L^\infty_{x,v})$, and if $N=2$, $\tilde{E}_1 \in L^\infty((0,T);L^1_{x,v})$ since

$$\begin{aligned} &\iint\limits_{\mathbb{R}^2\times\mathbb{R}^2} \frac{|E(t,x)|}{1+|x|+|v|}\mathbb{1}_{|v|\leq|E(t,x)|}\,dxdv \\ &\leq \int\limits_{\mathbb{R}^2} |E(t,x)| \Bigg(\int\limits_{|v|\leq|E(t,x)|} \frac{1}{|v|}dv \Bigg) dx = 2\pi \int\limits_{\mathbb{R}^2} |E(t,x)|^2 dx. \end{aligned} \tag{7.7}$$

In the case $N=1$, we have directly that $E(t,x)/(1+|v|) \in L^\infty((0,T);L^2_{x,v})$. □

27.2. The case of three space dimensions. The condition **(R1)** being not satisfied in 3 dimensions (the above computation would require $E \in L^1_t(L^3_x)$), we need an estimate on $|Z|$ in order to control the superlevels. For getting this we integrate in space a function growing slower at infinity than $\log(1+|Z|)$ (this corresponding to the case **(R1)**).

PROPOSITION 27.2. *Let b be as in (4.15) with $N=3$, $E \in L^\infty((0,T);L^2(\mathbb{R}^N))$, satisfying (4.12) with $J \in L^\infty((0,T);L^1(\mathbb{R}^N))$. Furthermore, assume that $\omega = +1$, $\rho \geq 0$, and $\rho_b \in L^1 \cap L^p(\mathbb{R}^3)$ for some $p > 3/2$. Then* **(R1a)** *holds, where the function g depends only on L, T, $||E||_{L^\infty_t(L^2_x)}$, $||J||_{L^\infty_t(L^1_x)}$, $\|(-\Delta)^{-1}\rho_b\|_{L^\infty}$, and one has $g(r,\lambda) \to 0$ as $\lambda \to \infty$ at fixed r.*

PROOF. Step 1.) Let $Z : [t,T]\times\mathbb{R}^3\times\mathbb{R}^3 \to \mathbb{R}^3\times\mathbb{R}^3$ be a regular Lagrangian flow relative to b starting at time t, with compression constant L and sublevel G_λ. Denoting $Z=(X,V)$, we have the ODEs

$$\begin{cases} \dot{X}(s,x,v) = V(s,x,v), \\ \dot{V}(s,x,v) = E(s,X(s,x,v)). \end{cases} \tag{7.8}$$

Recalling that $E = -\nabla_x U$, one has

$$\begin{aligned} \partial_s \frac{|V(s,x,v)|^2}{2} &= V(s,x,v)\cdot\partial_s V(s,x,v) = E(s,X(s,x,v))\cdot\partial_s X(s,x,v) \\ &= -\partial_s[U(s,X(s,x,v))] + \partial_t U(s,X(s,x,v)). \end{aligned} \tag{7.9}$$

This computation is indeed related to the form of the Hamiltonian for (4.1), $\mathcal{H} = |v|^2/2 + U(t,x)$.

We are going to bound the superlevels of $V(s,x,v)$. We claim that

$$\iint_{B_r} \sup_{s\in[t,T]} \Big(1+\log\Big(1+|V(s,x,v)|^2/2\Big)\Big)^{\alpha} dxdv \le A, \tag{7.10}$$

where $0<\alpha<1/3$, and for some constant A depending on L, T, r, α, and on the norms $||E||_{L_t^\infty(L_x^2)}$, $||J||_{L_t^\infty(L_x^1)}$, $\|(-\Delta)^{-1}\rho_b\|_{L^\infty}$. Assume for the moment that this holds. From the lower bound

$$\begin{aligned}&\iint_{B_r} \sup_{s\in[t,T]} \Big(1+\log\Big(1+|V(s,x,v)|^2/2\Big)\Big)^{\alpha} dxdv\\ &\ge \mathcal{L}^6(B_r\setminus \widetilde{G}_\lambda)(1+\log(1+\lambda^2/2))^{\alpha},\end{aligned} \tag{7.11}$$

with $\widetilde{G}_\lambda$ the sublevel of V, we get that

$$\mathcal{L}^6(B_r\setminus \widetilde{G}_\lambda) \le \frac{A}{(1+\log(1+\lambda^2/2))^{\alpha}}. \tag{7.12}$$

Next, we remark that by the first equation in (7.8), whenever $(x,v)\in\widetilde{G}_\lambda$ one has $|X(s,x,v)|\le |x|+|s-t|\lambda$, and $|Z(s,x,v)|\le |x|+(1+T)\lambda$. Thus for $\lambda>r$, one has $B_r\setminus G_\lambda \subset B_r\setminus \widetilde{G}_{(\lambda-r)/(1+T)}$, which enables to conclude the proposition (for $\lambda\le r$ we can just bound $\mathcal{L}^6(B_r\setminus G_\lambda)$ by $\mathcal{L}^6(B_r)$).
Step 2.) By Step 1, it is enough to prove that we have a decomposition

$$\left(1+\log\left(1+\frac{|V(s,\cdot,\cdot)|^2}{2}\right)\right)^{\alpha} \le f_1+f_2 \in L^1(\mathbb{R}^3_x\times\mathbb{R}^3_v)+L^\infty(\mathbb{R}^3_x\times\mathbb{R}^3_v), \tag{7.13}$$

for $(x,v)\in B_r$, where f_1, f_2 are independent of $s\in[t,T]$. Let

$$\beta(y) = (1+\log(1+y))^{\alpha}, \quad \text{for } y\ge 0. \tag{7.14}$$

Then

$$\begin{aligned}\beta'(y) &= \frac{\alpha(1+\log(1+y))^{\alpha-1}}{1+y},\\ 0< -\beta''(y) &\le \frac{(1+\log(1+y))^{\alpha-1}}{(1+y)^2}.\end{aligned} \tag{7.15}$$

Using (7.9), we compute

$$\begin{aligned}&\partial_s\left[\beta\left(\frac{|V(s,x,v)|^2}{2}\right)\right]\\ =&\Big(-\partial_s[U(s,X(s,x,v))]+\partial_t U(s,X(s,x,v))\Big)\beta'\left(\frac{|V(s,x,v)|^2}{2}\right)\\ =&-\partial_s\left[U(s,X(s,x,v))\beta'\left(\frac{|V(s,x,v)|^2}{2}\right)\right]\\ &+U(s,X(s,x,v))\beta''\left(\frac{|V(s,x,v)|^2}{2}\right)V(s,x,v)\cdot E(s,X(s,x,v))\\ &+\partial_t U(s,X(s,x,v))\beta'\left(\frac{|V(s,x,v)|^2}{2}\right).\end{aligned} \tag{7.16}$$

Thus, integrating between t and s,

$$
\begin{aligned}
&\left(1+\log\left(1+\frac{|V(s,x,v)|^2}{2}\right)\right)^{\alpha}\\
&= -\frac{\alpha U(s,X(s,x,v))}{\left(1+\frac{|V(s,x,v)|^2}{2}\right)\left(1+\log\left(1+\frac{|V(s,x,v)|^2}{2}\right)\right)^{1-\alpha}}\\
&+\frac{\alpha U(t,x)}{\left(1+\frac{|v|^2}{2}\right)\left(1+\log\left(1+\frac{|v|^2}{2}\right)\right)^{1-\alpha}}+\left(1+\log\left(1+\frac{|v|^2}{2}\right)\right)^{\alpha}\\
&+\int_t^s\left\{U(\tau,X(\tau,x,v))V(\tau,x,v)\cdot E(\tau,X(\tau,x,v))\beta''\left(\frac{|V(\tau,x,v)|^2}{2}\right)\right.\\
&\left.+\partial_t U(\tau,X(\tau,x,v))\beta'\left(\frac{|V(\tau,x,v)|^2}{2}\right)\right\}d\tau.
\end{aligned}
\tag{7.17}
$$

Step 3.) Since $E(t,\cdot)\in L^2(\mathbb{R}^3)$, we have by the Sobolev embedding that $U(t,\cdot)\in L^6(\mathbb{R}^3)$. Thus clearly

$$
\frac{U(t,x)}{1+\frac{|v|^2}{2}}\in L^6(\mathbb{R}^3_x\times\mathbb{R}^3_v)\subset L^1(\mathbb{R}^3_x\times\mathbb{R}^3_v)+L^\infty(\mathbb{R}^3_x\times\mathbb{R}^3_v). \tag{7.18}
$$

Next, since $\omega=+1$ and $\rho\geq 0$, one has $U=U_\rho-U_{\rho_b}$, with $U_\rho=\frac{1}{4\pi|x|}*\rho\geq 0$. Thus $U\geq-\|U_{\rho_b}\|_{L^\infty}$. Thus the first three terms in the expansion (7.17) are upper bounded in $L^1(\mathbb{R}^3_x\times\mathbb{R}^3_v)+L^\infty(\mathbb{R}^3_x\times\mathbb{R}^3_v)$. It remains to estimate the integral. We can bound it by $\Phi_1+\Phi_2$, with

$$
\Phi_1:=\int_t^T\left|U(\tau,X(\tau,x,v))V(\tau,x,v)\cdot E(\tau,X(\tau,x,v))\beta''\left(\frac{|V(\tau,x,v)|^2}{2}\right)\right|d\tau, \tag{7.19}
$$

$$
\Phi_2:=\int_t^T\left|\partial_t U(\tau,X(\tau,x,v))\beta'\left(\frac{|V(\tau,x,v)|^2}{2}\right)\right|d\tau. \tag{7.20}
$$

Note that Φ_1,Φ_2 are independent of s. We estimate Φ_1 in $L^{3/2}(\mathbb{R}^3_x\times\mathbb{R}^3_v)\subset L^1(\mathbb{R}^3_x\times\mathbb{R}^3_v)+L^\infty(\mathbb{R}^3_x\times\mathbb{R}^3_v)$. Passing the $L^{3/2}$ norm under the integral and changing $(X(\tau,x,v),V(\tau,x,v))$ to (x,v), this gives (up to a factor L)

$$
\begin{aligned}
&\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\left|\frac{U(\tau,x)v\cdot E(\tau,x)}{\left(1+\frac{|v|^2}{2}\right)^2\left(1+\log\left(1+\frac{|v|^2}{2}\right)\right)^{1-\alpha}}\right|^{3/2}dxdv\\
&\leq\int_{\mathbb{R}^3}|U(\tau,x)E(\tau,x)|^{3/2}dx\int_{\mathbb{R}^3}\frac{2^{3/4}\,dv}{\left(1+\frac{|v|^2}{2}\right)^{9/4}\left(1+\log\left(1+\frac{|v|^2}{2}\right)\right)^{3(1-\alpha)/2}}\\
&\leq c||U(\tau,\cdot)||^{3/2}_{L^6(\mathbb{R}^3)}||E(\tau,\cdot)||^{3/2}_{L^2(\mathbb{R}^3)}.
\end{aligned}
$$

Thus $\Phi_1\in L^{3/2}(\mathbb{R}^3_x\times\mathbb{R}^3_v)$.

Step 4.) For Φ_2, we notice that E satisfies (4.12) and $E=-\nabla_x U$, thus

$$
\partial_t U=-\omega(-\Delta_x)^{-1}\mathrm{div}_x J. \tag{7.21}
$$

Since $J(\tau,\cdot)\in L^1(\mathbb{R}^3_x)$, we deduce by the Hardy Littlewood Sobolev inequality that

$$
|||\partial_t U(\tau,\cdot)|||_{M^{3/2}(\mathbb{R}^3)}\leq c||J(\tau,\cdot)||_{L^1(\mathbb{R}^3)}. \tag{7.22}
$$

Therefore, we estimate

$$
\begin{aligned}
&\left|\left|\left|\frac{\partial_t U(\tau, X(\tau,x,v))}{\left(1+\frac{|V(\tau,x,v)|^2}{2}\right)\left(1+\log\left(1+\frac{|V(\tau,x,v)|^2}{2}\right)\right)^{1-\alpha}}\right|\right|\right|_{M^{3/2}(\mathbb{R}^3_x\times\mathbb{R}^3_v)} \\
&\le L^{2/3}\left|\left|\left|\frac{\partial_t U(\tau, x)}{\left(1+\frac{|v|^2}{2}\right)\left(1+\log\left(1+\frac{|v|^2}{2}\right)\right)^{1-\alpha}}\right|\right|\right|_{M^{3/2}(\mathbb{R}^3_x\times\mathbb{R}^3_v)} \\
&\le L^{2/3}\left|\left|\left|\frac{\partial_t U(\tau, x)}{\left(1+\frac{|v|^2}{2}\right)\left(1+\log\left(1+\frac{|v|^2}{2}\right)\right)^{1-\alpha}}\right|\right|\right|_{L^{3/2}(\mathbb{R}^3_v;M^{3/2}(\mathbb{R}^3_x))} \\
&\le L^{2/3}|||\partial_t U(\tau,\cdot)|||_{M^{3/2}(\mathbb{R}^3_x)}\left(\int_{\mathbb{R}^3}\frac{dv}{\left(1+\frac{|v|^2}{2}\right)^{3/2}\left(1+\log\left(1+\frac{|v|^2}{2}\right)\right)^{3(1-\alpha)/2}}\right)^{2/3} \\
&\le c||J(\tau,\cdot)||_{L^1(\mathbb{R}^3)},
\end{aligned}
\tag{7.23}
$$

where the last integral is convergent since $3(1-\alpha)/2 > 1$. From the inclusion $M^{3/2}(\mathbb{R}^3_x\times\mathbb{R}^3_v)\subset L^1(\mathbb{R}^3_x\times\mathbb{R}^3_v)+L^\infty(\mathbb{R}^3_x\times\mathbb{R}^3_v)$, and integrating (7.20) over B_r, we get (7.13) as desired. □

28. Renormalized solutions and Lagrangian solutions to the VP system

We recall the different notions of weak solutions for the Vlasov-Poisson system. We shall always assume that $1\le N\le 3$, and we consider an initial datum $f^0\in L^1(\mathbb{R}^N_x\times\mathbb{R}^N_v)$, $f^0\ge 0$. We introduce first renormalized solutions, following [**30**, **31**].

DEFINITION 28.1. *We say that $f\in L^\infty((0,T);L^1(\mathbb{R}^N_x\times\mathbb{R}^N_v))$, $f\ge 0$, is a solution to the Vlasov equation (4.1) in the renormalized sense if for all test functions $\beta\in C^1([0,\infty))$ with β bounded, we have that*

$$
\partial_t\beta(f)+v\cdot\nabla_x\beta(f)+\operatorname{div}_v\Big(E(t,x)\beta(f)\Big)=0,
\tag{7.24}
$$

in $\mathcal{D}'((0,T)\times\mathbb{R}^N_x\times\mathbb{R}^N_v)$.

We next introduce the notion of Lagrangian solutions.

DEFINITION 28.2. *Let be given a vector field $b(t,x,v)=(v,E(t,x))$ as in (4.15) for some $\rho\in L^\infty((0,T);L^1(\mathbb{R}^N))$, $\rho\ge 0$, and $\rho_b\in L^1(\mathbb{R}^N)$. We assume that $E\in L^\infty((0,T);L^2(\mathbb{R}^N))$, and that (4.12) holds with $J\in L^\infty((0,T);L^1(\mathbb{R}^N))$. We assume furthermore that either $N=1$ or 2, or $N=3$ and $\omega=+1$, $\rho_b\in L^p(\mathbb{R}^3)$ for some $p>3/2$. We consider regular Lagrangian flows Z as in definition 16.1, except that now $s\in[0,T]$ instead of $s\in[t,T]$ (forward-backward flow), and with compression constant L independent of $t\in[0,T]$. According to subsections 26.1, 26.2, Proposition 27.1, Proposition 27.2, the vector field b satisfies assumptions* **(R1a), (R2a), (R3)**. *Therefore, Theorem 20.1 yields the existence and uniqueness of the forward-backward regular Lagrangian flow $Z=(X,V)$, with compression constant 1. We can thus define in accordance with* [**16**] *a Lagrangian solution f to the Vlasov equation (4.1) by*

$$
f(t,x,v)=f^0\Big(X(s=0,t,x,v),V(s=0,t,x,v)\Big),\qquad \textit{for all } t\in[0,T],
\tag{7.25}
$$

for arbitrary $f^0\in L^1(\mathbb{R}^N_x\times\mathbb{R}^N_v)$. It verifies in particular $f\in C([0,T];L^1(\mathbb{R}^N_x\times\mathbb{R}^N_v))$, and it is indeed also a renormalized solution.

DEFINITION 28.3. *We define a Lagrangian solution to the Vlasov-Poisson system as a couple (f, E) such that*

(1) $f \in C([0,T]; L^1(\mathbb{R}_x^N \times \mathbb{R}_v^N))$, $f \geq 0$, $|v|^2 f \in L^\infty((0,T); L^1(\mathbb{R}_x^N \times \mathbb{R}_v^N))$,
(2) $E(t,x)$ *is given by the convolution* (4.6) *with* $\rho(t,x) = \int f(t,x,v)dv$, $\rho_b \in L^1(\mathbb{R}^N)$, $\rho_b \geq 0$ *(and if* $N = 3$, $\omega = +1$, $\rho_b \in L^p(\mathbb{R}^3)$ *for some* $p > 3/2$*),*
(3) $E \in L^\infty((0,T); L^2(\mathbb{R}_x^N))$,
(4) *The relation* (4.12) *holds with* $J(t,x) = \int v f(t,x,v)dv$,
(5) f *is a Lagrangian solution to the Vlasov equation, in the sense of* (7.25).

29. Existence of Lagrangian solutions

29.1. Compactness. In this subsection we prove two compactness results, Theorems 29.2 and 29.3, for families of Lagrangian solutions to the Vlasov-Poisson system, with strongly or weakly convergent initial data.

LEMMA 29.1. *Let* $g(x) = \frac{x}{|x|^N}$ *for* $x \in \mathbb{R}^N$, *and denote by* $\tau_h g(x) = g(x+h)$. *Then for any* $1 < p < \frac{N}{N-1}$,

$$\|\tau_h g(x) - g(x)\|_{L^p(\mathbb{R}^N)} \leq c|h|^\alpha, \tag{7.26}$$

with $\alpha = 1 - N + N/p > 0$, *and where* c *depends on* N, p.

PROOF. Fix $h \in \mathbb{R}^N$, $h \neq 0$. For $|x| > 2|h|$, we have for all $0 \leq \theta \leq 1$, $|x + \theta h| \geq |x| - \theta|h| > |x|/2$, thus we have

$$\begin{aligned} |\tau_h g(x) - g(x)| &\leq |h| \sup_{0\leq\theta\leq1} |\nabla g(x+\theta h)| \\ &\leq |h| \sup_{0\leq\theta\leq1} \frac{c_N}{|x+\theta h|^N} \leq c_N \frac{|h|}{|x|^N}. \end{aligned} \tag{7.27}$$

Then we estimate

$$\int\limits_{|x|>2|h|} \frac{|h|^p}{|x|^{Np}} dx = c_N |h|^p \int\limits_{2|h|}^{\infty} r^{N-1-Np} dr = c_N |h|^p \frac{(2|h|)^{N-Np}}{Np - N} = c_{N,p} |h|^{p-Np+N}.$$

Next, for $|x| \leq 2|h|$, we write

$$|\tau_h g(x) - g(x)| \leq \frac{1}{|x+h|^{N-1}} + \frac{1}{|x|^{N-1}}, \tag{7.28}$$

and clearly

$$\begin{aligned} &\int\limits_{|x|\leq 2|h|} \left(\frac{1}{|x+h|^{(N-1)p}} + \frac{1}{|x|^{(N-1)p}} \right) dx \\ &\leq 2 \int\limits_{|y|\leq 3|h|} \frac{dy}{|y|^{(N-1)p}} = c_N \int\limits_0^{3|h|} r^{N-Np+p-1} dr = c_{N,p} |h|^{N-Np+p}, \end{aligned} \tag{7.29}$$

since the last integral is convergent for $p < \frac{N}{N-1}$. ☐

THEOREM 29.2. *Let* (f_n, E_n) *be a sequence of Lagrangian solutions to the Vlasov-Poisson system satisfying*

$$f_n^0 \to f^0 \text{ in } L^1(\mathbb{R}_x^N \times \mathbb{R}_v^N), \tag{7.30}$$

and

$$\iint |v|^2 f_n(t,x,v) dx dv + \int |E_n(t,x)|^2 dx \leq C, \qquad \text{for all } t \in [0,T]. \tag{7.31}$$

Then, up to a subsequence f_n converges strongly in $C([0,T];L^1(\mathbb{R}^N_x \times \mathbb{R}^N_v))$ to f, E_n converges in $C([0,T];L^1_{\mathrm{loc}}(\mathbb{R}^N))$ to E, and (f,E) is a Lagrangian solution to the Vlasov-Poisson system with initial datum f^0. Moreover, the regular forward-backward Lagrangian flow $Z_n(s,t,x,v)$ converges to $Z(s,t,x,v)$ locally in measure in $\mathbb{R}^N \times \mathbb{R}^N$, uniformly in $s,t \in [0,T]$.

PROOF. **Step 1.** (Equi-integrability)
Because of (7.25) and (7.30) we have

$$\|f_n(t,\cdot,\cdot)\|_{L^1(\mathbb{R}^N\times\mathbb{R}^N)} = \|f_n^0\|_{L^1(\mathbb{R}^N\times\mathbb{R}^N)} \le M. \tag{7.32}$$

Then because of the bounds (7.32), (7.31), and applying Propositions 27.1, 27.2, one has for any $r>0$

$$\mathcal{L}^{2N}\{(x,v)\in B_r : \sup_{0\le s\le T}|Z_n(s,t,x,v)| > \gamma\} \to 0, \text{ as } \gamma\to\infty, \text{uniformly in } t,n. \tag{7.33}$$

Since the sequence f_n^0 is uniformly equi-integrable, and since Z_n is measure-preserving, we have by (7.25) and (7.33) that $f_n(t,\cdot)$ is equi-integrable, uniformly in t,n. Consequently, $\rho_n(t,\cdot)$ is also equi-integrable, uniformly in t,n. Using the bound (7.31), $vf_n(t,\cdot)$ is also equi-integrable, and therefore $J_n(t,\cdot)$ is also equi-integrable, uniformly in t,n.

Step 2. (Spatial compactness of the field)
In order to prove that $E_n \to E$ in $L^1_{\mathrm{loc}}((0,T)\times\mathbb{R}^N_x)$, we first look at the compactness in x. Denote by $\tau_h E_n(t,x) = E_n(t,x+h)$. Then using (4.6),

$$\begin{aligned}&||\tau_h E_n(t,\cdot) - E_n(t,\cdot)||_{L^p(\mathbb{R}^N)}\\ \le\, & c\left\|\tau_h\frac{x}{|x|^N} - \frac{x}{|x|^N}\right\|_{L^p(\mathbb{R}^N)} ||\rho_n(t,\cdot)-\rho_b(\cdot)||_{L^1(\mathbb{R}^N)}.\end{aligned} \tag{7.34}$$

Thus according to Lemma 29.1, we get for any $1<p<\frac{N}{N-1}$ that

$$||E_n(t,x+h)-E_n(t,x)||_{L^p(\mathbb{R}^N)} \to 0, \qquad \text{as } h\to 0, \text{ uniformly in } t,n. \tag{7.35}$$

Step 3. (Time continuity of E_n)
To prove compactness in time, we check continuity. Denote by $\langle\cdot,\cdot\rangle$ the L^2 inner product. Let $\varphi\in H^s(\mathbb{R}^N)$ with $s>N/2$, and define ϕ_n as the mapping $\phi_n : t\mapsto \langle E_n(t,\cdot),\varphi\rangle$. Then

$$\lim_{\tau\to 0}\frac{\phi_n(t+\tau)-\phi_n(t)}{\tau} = \lim_{\tau\to0}\left\langle \frac{E_n(t+\tau,\cdot)-E_n(t,\cdot)}{\tau},\varphi\right\rangle = \partial_t\langle E_n(t,\cdot),\varphi\rangle. \tag{7.36}$$

We have from (4.12) that $\partial_t E_n$ is uniformly bounded in $L^\infty((0,T);H^{-s}(\mathbb{R}^N_x))$, since

$$\begin{aligned}\langle\partial_t E_n,\varphi\rangle_{H^{-s},H^s} &= \langle Sj_n(t,\cdot),\varphi\rangle_{H^{-s},H^s}\\ &\le ||j_n(t)||_{L^1(\mathbb{R}^N)}||\tilde{S}\varphi||_{L^\infty(\mathbb{R}^N)}\\ &\le C||\tilde{S}\varphi||_{H^s(\mathbb{R}^N)} \le C,\end{aligned}$$

so that $\partial_t\langle E_n(t,\cdot),\varphi\rangle \in L^\infty(0,T)$, for every $\varphi\in H^s(\mathbb{R}^N)$, and the above bracket in (7.36) is well defined. It is clear that this estimate is uniform in time, hence $\frac{d\phi_n}{dt}$ exists almost everywhere in $[0,T]$. Fix n. Then for a.e. $t\in[0,T]$, we have

$$\sup_{\varphi\in H^s(\mathbb{R}^N),||\varphi||_{H^s}\le 1}|\langle E_n(t+\tau,\cdot),\varphi\rangle - \langle E_n(t,\cdot),\varphi\rangle| \le \tau\sup_{0<t\le T}|\langle\partial_t E_n(t,\cdot),\varphi\rangle| \le C\tau. \tag{7.37}$$

Since this estimate is uniform in n, we get

$$\sup_n \sup_{\varphi\in H^s(\mathbb{R}^N),||\varphi||_{H^s}\le 1}|\langle E_n(t+\tau,\cdot),\varphi\rangle - \langle E_n(t,\cdot),\varphi\rangle| \to 0, \qquad \text{as } \tau\to 0, \tag{7.38}$$

uniformly in time. We proceed with the following localization argument. Fix $B_r \Subset \mathbb{R}^N$, and let $\chi(x) \in C_c^\infty(B_{r/2})$ be a cutoff function. (7.38) implies that

$$\sup_n \sup_{\varphi\in H^s(\mathbb{R}^N),||\varphi||_{H^s}\le 1} \|\langle \chi(\cdot)E_n(t+\tau,\cdot)-\chi(\cdot)E_n(t,\cdot),\varphi\rangle\|_{L^1((0,T))}\to 0, \qquad \text{as } \tau\to 0. \tag{7.39}$$

Step 4. (Compactness in time)
Using step 2, we check the local time translations of E_n. Let $\xi \in C_c^\infty(B_r)$ be the standard mollifier and set $\xi_\varepsilon = \varepsilon^{-n}\xi(x/\varepsilon)$ with $\int \xi_\varepsilon = 1, ||\xi||_{H^s(\mathbb{R}^N)}\le c$, and denote by

$$E^\varepsilon_{n,\chi} = (\chi E_n) *_x \xi_\varepsilon = \langle \chi(\cdot)E_n(t,\cdot), \xi_\varepsilon(x-\cdot)\rangle.$$

We have $\operatorname{spt} E^\varepsilon_{n,\chi} \subset \operatorname{spt}\chi + \operatorname{spt}\xi_\varepsilon \subset B_{\frac{r}{2}+\varepsilon}$, and $||\xi_\varepsilon(x-\cdot)||_{H^s(\mathbb{R}^N)}\le \varepsilon^{-s}||\xi||_{H^s(\mathbb{R}^N)}$. For fixed ε, the following uniform in n and x convergence holds:

$$\begin{aligned} &||E^\varepsilon_{n,\chi}(t+\tau,x)-E^\varepsilon_{n,\chi}(t,x)||_{L^1((0,T))}\\ \le&\sup_n\|\langle \chi(\cdot)E_n(t+\tau,\cdot)-\chi(\cdot)E_n(t,\cdot),\xi_\varepsilon(x-\cdot)\rangle\|_{L^1((0,T))}\\ \le&\varepsilon^{-s}c\sup_n\|\langle \chi(\cdot)E_n(t+\tau,\cdot)-\chi(\cdot)E_n(t,\cdot),\varepsilon^s||\xi||^{-1}_{H^s}\xi_\varepsilon(x-\cdot)\rangle\|_{L^1((0,T))}\to 0, \qquad \text{as } \tau\to 0. \end{aligned} \tag{7.40}$$

Next we write

$$\chi(x)E_n(t,x) - E^\varepsilon_{n,\chi}(t,x) = \int_{\mathbb{R}^N}[\chi(x)E_n(t,x)-\chi(x-y)E_n(t,x-y)]\xi_\varepsilon(y)dy.$$

Using step 1 we may estimate this as

$$\begin{aligned} ||\chi E_n - E^\varepsilon_{n,\chi}||_{L^1((0,T);L^2(\mathbb{R}^N))}\le &\sup_{|y|<\varepsilon}\big\{||\chi||_{L^\infty}||E_n(t,x)-E_n(t,x-y)||_{L^1((0,T);L^2(\mathbb{R}^N))}\\ &+||E_n(t,x)||_{L^1((0,T);L^2(\mathbb{R}^N))}||\chi(x)-\chi(x-y)||_{L^2(\mathbb{R}^N)}\big\}\le\eta, \end{aligned} \tag{7.41}$$

for some ε sufficiently small.

Step 5. Let $\eta>0$. First choose $\varepsilon>0$, and then $\tau>0$ sufficiently small so that by (7.40) and (7.41), we have

$$\begin{aligned} &||\chi(x)E_n(t+\tau,x)-\chi(x)E_n(t,x)||_{L^1((0,T)\times\mathbb{R}^N)}\\ &\le 2||\chi(x)E_n(t,x)-E^\varepsilon_{n,\chi}(t,x)||_{L^1((0,T);L^1(\mathbb{R}^N))}+||E^\varepsilon_{n,\chi}(t+\tau,x)-E^\varepsilon_{n,\chi}(t,x)||_{L^1((0,T);L^1(\mathbb{R}^N))}\\ &\le 3\eta. \end{aligned}$$

Combining this with step 2 and applying Riesz-Fréchet-Kolmogorov, we conclude that up to a subsequence $E_n\to E$ strongly in $L^1((0,T);L^1_{\rm loc}(\mathbb{R}^N_x))$.
Step 6. (Convergence of the flow)
Because of the bound (7.31), one has $E\in L^\infty((0,T),L^2(\mathbb{R}^N))$. Also, using the uniform bounds on ρ_n, J_n in $L^\infty((0,T);L^1(\mathbb{R}^N))$ and the uniform equi-integrability obtained in Step 1, one has up to a subsequence $\rho_n\to\rho$, $J_n\to J$ in the sense of distributions, with $\rho, J\in L^\infty((0,T);L^1(\mathbb{R}^N))$. We can pass to the limit in (4.6) and (4.12). Therefore, $b=(v,E)$ satisfies the assumptions **(R1a), (R2a), (R3)** and Definition 28.2 applies. According to Theorem 20.1, since (7.33) holds, we deduce the convergence of Z_n to Z locally in measure in $\mathbb{R}^N_x\times\mathbb{R}^N_v$, uniformly with respect to $s,t\in[0,T]$, where Z is the regular forward-backward Lagrangian flow associated to b.
Step 7. (Convergence of f)
Using the convergence (7.30), we can apply Corollary 21.3, and we conclude that $f_n\to f$ in $C([0,T];L^1(\mathbb{R}^N_x\times\mathbb{R}^N_v))$, where f is the Lagrangian solution to the Vlasov equation with coefficient b and initial datum f^0. It follows that $\rho_n\to\rho=\int f dv$ in $C([0,T];L^1(\mathbb{R}^N_x))$. By lower semi-continuity, we get from (7.31) that

$$\iint|v|^2f(t,x,v)dxdv+\int|E(t,x)|^2dx\le C, \qquad \text{for all } t\in[0,T]. \tag{7.42}$$

The bound (7.31) gives also that $J_n \to J = \int v f dv$ in $C([0,T];L^1(\mathbb{R}^N_x))$. Therefore, (f,E) is a Lagrangian solution to the Vlasov-Poisson system. Using (4.6), we get that $E_n \to E$ in $C([0,T];L^1_{\mathrm{loc}}(\mathbb{R}^N))$, which concludes the proof. □

THEOREM 29.3. *Let (f_n,E_n) be a sequence of Lagrangian solutions to the Vlasov-Poisson system satisfying*

$$f_n^0 \rightharpoonup f^0 \text{ weakly in } L^1(\mathbb{R}^N_x \times \mathbb{R}^N_v), \tag{7.43}$$

and the bound (7.31). Then, up to a subsequence f_n converges in $C([0,T];\mathrm{weak}-L^1(\mathbb{R}^N_x \times \mathbb{R}^N_v))$ to f, E_n converges in $C([0,T];L^1_{loc}(\mathbb{R}^N))$ to E, and (f,E) is a Lagrangian solution to the Vlasov-Poisson system with initial datum f^0. Moreover, the regular forward-backward Lagrangian flow $Z_n(s,t,x,v)$ converges to $Z(s,t,x,v)$ locally in measure in $\mathbb{R}^N \times \mathbb{R}^N$, uniformly in $s,t \in [0,T]$.

PROOF. It is the same as that of Theorem 29.2, except the last step 7. Instead we apply Corollary 21.3 and conclude that $f_n \rightharpoonup f$ in $C([0,T];\mathrm{weak}-L^1(\mathbb{R}^N_x \times \mathbb{R}^N_v))$, where f is the Lagrangian solution to the Vlasov equation with coefficient b and initial datum f^0. It follows that $\rho_n \rightharpoonup \rho = \int f dv$ in $C([0,T];\mathrm{weak}-L^1(\mathbb{R}^N_x))$. By lower semi-continuity, we get again from (7.31) the energy bound (7.42). The bound (7.31) also enables to conclude that $J_n \rightharpoonup J = \int v f dv$ in $C([0,T];\mathrm{weak}-L^1(\mathbb{R}^N_x))$. Therefore, (f,E) is a Lagrangian solution to the Vlasov-Poisson system. Using (4.6) and the compactness estimate (7.35), we get that $E_n \to E$ in $C([0,T];L^1_{\mathrm{loc}}(\mathbb{R}^N))$, which concludes the proof. □

29.2. Existence. We conclude this section by the existence of Lagrangian solutions to the Vlasov-Poisson system for initial datum in L^1 with finite energy, in the repulsive case.

THEOREM 29.4. *Let $N = 1, 2$ or 3, and let $f^0 \in L^1(\mathbb{R}^N_x \times \mathbb{R}^N_v)$, $f^0 \geq 0$. Define ρ^0 and E^0 by*

$$\rho^0(x) = \int f^0(x,v)dv, \qquad E^0(x) = \frac{\omega}{|S^{N-1}|}\frac{x}{|x|^N} * (\rho^0(x) - \rho_b(x)), \tag{7.44}$$

with $\omega = +1$ (repulsive case), $\rho_b \in L^1(\mathbb{R}^N)$, $\rho_b \geq 0$, and in the case $N = 3$ $\rho_b \in L^p(\mathbb{R}^3)$ for some $p > 3/2$. Assume that the initial energy is finite,

$$\iint |v|^2 f^0(x,v)dxdv + \int |E^0(x)|^2 dx < \infty. \tag{7.45}$$

Then there exists a Lagrangian solution (f,E) to the Vlasov-Poisson system defined for all time, having f^0 as initial datum, and satisfying for all $t \geq 0$

$$\iint |v|^2 f(t,x,v)dxdv + \int |E(t,x)|^2 dx \leq \iint |v|^2 f^0(x,v)dxdv + \int |E^0(x)|^2 dx. \tag{7.46}$$

PROOF. We use the classical way of getting global weak solutions to the Vlasov-Poisson system, i.e. we approximate the initial datum f^0 by a sequence of smooth data $f_n^0 \geq 0$ with compact support. We approximate also ρ_b by smooth $\rho_b^n \geq 0$ with compact support (with $\int(\rho_n^0 - \rho_b^n)dx = 0$ if $N = 1, 2$). It is possible to do that with the upper bounds

$$\begin{aligned} &\limsup_{n\to\infty} \iint |v|^2 f_n^0(x,v)dxdv \leq \iint |v|^2 f^0(x,v)dxdv, \\ &\limsup_{n\to\infty} \int |E_n^0(x)|^2 dx \leq \int |E^0(x)|^2 dx. \end{aligned} \tag{7.47}$$

Then, for each n, there exists a smooth classical solution (f_n,E_n) with initial datum f_n^0, to the Vlasov-Poisson system, defined for all time $t \geq 0$. Note that we can alternatively consider a regularized Vlasov-Poisson system with energy identity, as in [18]. Since $\omega = +1$, the conservation of energy (4.14) gives for all $t \geq 0$,

$$\iint |v|^2 f_n(t,x,v)dxdv + \int |E_n(t,x)|^2 dx = \iint |v|^2 f_n^0(x,v)dxdv + \int |E_n^0(x)|^2 dx. \tag{7.48}$$

The couple (f_n, E_n) is in particular a Lagrangian solution to the Vlasov-Poisson system, for all intervals $[0, T]$. We can therefore apply Theorem 29.2. Extracting a diagonal subsequence, we get the convergence of (f_n, E_n) to (f, E) as stated in Theorem 29.2, where (f, E) is a Lagrangian solution to the Vlasov-Poisson system defined for all time, with f^0 as initial datum. The bound (7.42), together with (7.47), gives (7.46). $\square$

Let us end with a remark on measure densities. In step 6 of Theorem 29.2 we do not require the assumption that the densities are equi-integrable. When considering a sequence of solutions to the Vlasov-Poisson system, if we require only that $D_1 b_n^2$ converges in the sense of distributions to $D_1 b^2 = S(\rho - \rho_b)$, for some measure $\rho \in \mathcal{M}(\mathbb{R}^N)$, then Theorem 20.1 still applies. If ρ_n is uniformly bounded in $L^1((0,T); \mathcal{M}(\mathbb{R}^N))$, and $b_n \to b$ strongly in $L^1((0,T); L^1_{\mathrm{loc}}(\mathbb{R}^N_x \times \mathbb{R}^N_v))$ with b satisfying (5.62), we conclude that $Z_n \to Z$ strongly, where Z is the regular Lagrangian flow associated to b. However, we are not able to define the push forward (7.25) of a measure f^0. This prevents from applying fully the context of section 5.59 to the Vlasov-Poisson system with measure data.

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