ASYMPTOTIC BEHAVIOR OF THE RIEMANNIAN HEISENBERG GROUP AND ITS HOROBOUNDARY

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ABSTRACT. The paper is devoted to the large scale geometry of the Heisenberg group $\mathbb{H}$ equipped with left-invariant Riemannian distances. We prove that two such distances have bounded difference if and only if they are asymptotic, i.e., their ratio goes to one, at infinity. Moreover, we show that for every left-invariant Riemannian distance d on $\mathbb{H}$ there is a unique subRiemanniann metric d' for which $d - d'$ goes to zero at infinity, and we estimate the rate of convergence. As a first immediate consequence we get that the Riemannian Heisenberg group is at bounded distance from its asymptotic cone. The second consequence, which was our aim, is the explicit description of the horoboundary of the Riemannian Heisenberg group.

CONTENTS

1. Introduction

In large-scale geometry, various notions of a space at infinity have received special interest for differently capturing the asymptotic geometric behavior. Two main examples of spaces at infinity are the asymptotic cone and the horoboundary. The description of the *asymptotic cone* for finitely generated groups is a crucial step in the algebraic characterization of groups of polynomial growth, [Mil68; Tit72; Gro81b; Wol68; Pan83; BLD13]. The notion of *horoboundary* has been formulated by Gromov [Gro81a], inspired by the seminal work of Busemann on the theory of parallels on geodesic spaces [Bus55]. The horoboundary has a fully satisfying visual description in the framework of $CAT(0)$-spaces and of Gromov-hyperbolic spaces, [Gro87; Bal95; BH99]. It plays a major role in the study of dynamics and rigidity of negatively curved spaces, [Had98; Mos73; Gro87; Sul79; Pat76; Rob03]. The visual-boundary description breaks down for non-simply connected manifolds [DPS12] and when the curvature has variable sign, as we will make evident for the Riemannian Heisenberg group.

This paper contributes to the study of the asymptotic geometry of the simplest non-Abelian nilpotent group: the Heisenberg group. The asymptotic cone of the Heisenberg group equipped with a left-invariant Riemannian metric d_R is the Heisenberg group equipped with a Carnot-Carathéodory distance d_{CC}, see [Pan83] and also [BLD13]. Our contribution is a finer analysis of the asymptotic comparison of d_R and d_{CC}. This leads to the explicit knowledge of the (Riemannian) horoboundary. We remark that the Heisenberg group is not hyperbolic, hence one does not consider its visual boundary.

We recall the definition of horoboundary. Let (X, d) be a metric space. We consider the space of continuous real functions $\mathscr{C}(X)$ endowed with the topology of uniform convergence on compact sets. We denote by $\mathscr{C}(X)/\mathbb{R}$ the quotient with respect to the subspace of constant functions. The embedding $x \mapsto d(x, \cdot)$ induces an embedding $X \hookrightarrow \mathscr{C}(X)/\mathbb{R}$. The horoboundary of X is defined as $\partial_h X := \bar{X} \setminus X \subset \mathscr{C}(X)/\mathbb{R}$. See Section 5 for a detailed exposition.

The horoboundary of finite dimensional normed vector spaces has been investigated for normed spaces, [Wal07], for Hilbert geometries, [Wal08], and for infinite graphs [WW06]. For non-simply connected, negatively curved manifolds it has been studied in [DPS12]. Nicas and Klein computed the horoboundary of the Heisenberg group when endowed with the Korany metric in [KN09], and with the metric d_{CC} in [KN10].

We will show that the horoboundary of the Heisenberg group endowed with a left-invariant Riemannian metric d_R coincides with the second case studied by Nicas and Klein, see Corollary 1.3. This will be an immediate consequence of our main result Theorem 1.2, which

implies that the difference $d_R - d_{CC}$ converges to zero as the distances diverge.

1.1. **Detailed results.** The Heisenberg group $\mathbb{H}$ is the simply connected Lie group whose Lie algebra $\mathfrak{h}$ is generated by three vectors X, Y, Z with only non-zero relation $[X, Y] = Z$. A left-invariant Riemannian metric d on $\mathbb{H}$ is determined by a scalar product g on $\mathfrak{h}$; a left-invariant *strictly* subRiemannian metric d is induced by a bracket generating plane $V \subset \mathfrak{h}$ and a scalar product g on V (see Section 2 for detailed exposition). In both cases we say that d is subRiemannian with *horizontal space* (V, g), where $\dim V$ is either 2 or 3; subRiemannian metrics are also called Carnot-Carathéodory.

We are interested in the asymptotic comparison between these distances. Given two left-invariant subRiemannian metrics d and d' on $\mathbb{H}$, we deal with three asymptotic behaviors, in ascending order of strength, each of which defines an equivalence relation among subRiemannian distances:

(i) $\lim_{d(p,q) \to \infty} \frac{d(p,q)}{d'(p,q)} = 1$;

(ii) There exists $c > 0$ such that $|d(p,q) - d'(p,q)| < c$, for all p, q;

(iii) $\lim_{d(p,q) \to \infty} |d(p,q) - d'(p,q)| = 0$.

A first example of the implication $(i) \Rightarrow (ii)$ was proved by Burago in [Bur94] for $\mathbb{Z}^n$-invariant metrics d on $\mathbb{R}^n$, by showing that d and the associated stable norm stay at bounded distance from each other. This result has been extended quantitatively for $\mathbb{Z}^n$-invariant metrics on geodesic metric spaces in [CS14]. Gromov and Burago asked for other interesting cases where the same implication holds. Another well-known case where (i) is equivalent to (ii) is that of hyperbolic groups. Beyond Abelian and hyperbolic groups, Krat proved the equivalence for word metrics on the discrete Heisenberg group $\mathbb{H}(\mathbb{Z})$, [Kra99]. For general subFinsler metrics on Carnot groups it has been proven in [BLD13], following [Sto98], that (i) is equivalent to the fact that the projections onto $\mathbb{H}/[\mathbb{H}, \mathbb{H}]$ of the corresponding unit balls coincide, see (c) below. Our first result shows that this last condition is equivalent to each one of (i) and (ii) in the case of the Heisenberg group endowed with subRiemannian metrics.

Theorem 1.1. *Let d and d' be two left-invariant subRiemannian distances on $\mathbb{H}$ whose horizontal spaces are (V, g) and (V', g') respectively. Let $\pi : \mathfrak{h} \to \mathfrak{h}/[\mathfrak{h}, \mathfrak{h}]$ be the quotient projection and $\hat{\pi} : \mathbb{H} \to \mathfrak{h}/[\mathfrak{h}, \mathfrak{h}]$ the corresponding group morphism.*

Then the following assertions are equivalent:

(a) there exists $c > 0$ such that $|d - d'| < c$;

(b) $\frac{d(p,q)}{d'(p,q)} \to 1$ when $d(p,q) \to \infty$;

(c) $\hat{\pi}(\{p \in \mathbb{H} : d(0,p) \le R\}) = \hat{\pi}(\{p \in \mathbb{H} : d'(0,p) \le R\})$, for all $R > 0$, here 0 denotes the neutral element of $\mathbb{H}$;

(d) $\pi \left(\{v \in V : g(v,v) \leq 1\}\right) = \pi \left(\{v' \in V' : g'(v',v') \leq 1\}\right)$;

(e) there exists a scalar product $\bar{g}$ on $\mathfrak{h}/[\mathfrak{h},\mathfrak{h}]$ such that both
$$\pi|_V : (V,g) \to (\mathfrak{h}/[\mathfrak{h},\mathfrak{h}], \bar{g}) \quad and \quad \pi|_{V'} : (V',g') \to (\mathfrak{h}/[\mathfrak{h},\mathfrak{h}], \bar{g})$$
are submetries.

Next, we prove that in every class of the equivalence relation (iii) there is exactly one strictly subRiemannian metric. To every left-invariant subRiemannian metric d we define the *associated subRiemannian metric* d' as follows. If d is Riemannian defined by a scalar product g on $\mathfrak{h}$, then d' is the strictly subRiemannian metric for which the horizontal space V is g-orthogonal to $[\mathfrak{h},\mathfrak{h}]$ and the scalar product is $g|_V$. If d is strictly subRiemannian, then $d' = d$.

Theorem 1.2. *Let d and d' be two left-invariant subRiemannian distances on $\mathbb{H}$. Their associated subRiemannian metrics are the same if and only if*

(1)
$$\lim_{d(p,q) \to \infty} |d(p,q) - d'(p,q)| = 0.$$

Moreover, if (1) holds, then there is $C > 0$ such that

(2)
$$|d(p,q) - d'(p,q)| \leq \frac{C}{d(p,q)}, \qquad \forall p,q \in \mathbb{H}.$$

We remark that the estimate (2) in Theorem 1.2 is sharp, as we will show in Remark 4.2.

The above result can be interpreted in terms of asymptotic cones. Namely, if d is a left-invariant Riemannian metric on $\mathbb{H}$ and d' is the associated subRiemannian metric, then $(\mathbb{H}, d')$ is the asymptotic cone of $(\mathbb{H}, d)$. For the analogous result in arbitrary nilpotent groups see [Pan83]. By Theorem 1.2, more is true: $(\mathbb{H}, d)$ is at bounded distance from $(\mathbb{H}, d')$. Notice that this consequence cannot be deduced by the similar results for discrete subgroups of the Heisenberg group in [Kra99] and [DS14], because the word metric is only quasi-isometric to the Riemannian one. Moreover, we remark that there are examples of nilpotent groups of step two that are not at bounded distance from their asymptotic cone, see [BLD13].

We now focus on the horoboundary. As a consequence of Theorem 1.2 and of the results of Klein-Nikas [KN10], we get :

Corollary 1.3. *If d_R is a left-invariant Riemannian metric on $\mathbb{H}$ with associated metric d_{CC}, then the horoboundary of $(\mathbb{H}, d_R)$ coincides with the horoboundary of $(\mathbb{H}, d_{CC})$; hence, it is homeomorphic to a 2-dimensional closed disk $\bar{D}^2$.*

More precisely, let g be the scalar product of d_R on $\mathfrak{h}$ and $W \subset \mathfrak{h}$ the orthogonal plane to $[\mathfrak{h},\mathfrak{h}]$. Define the norm $\|w\| := \sqrt{g(w,w)}$ on W. Fix a orthonormal basis (X,Y) for W and set $Z := [X,Y] \in [\mathfrak{h},\mathfrak{h}]$, so that (X,Y,Z) is a basis of $\mathfrak{h}$. We identify $\mathfrak{h} \simeq \mathbb{H}$ via the exponential

map, which is a global diffeomorphism. So, we write $p = w + zZ$ with $w \in W$ and $z \in \mathbb{R}$ for any point $p \in \mathbb{H}$. A diverging sequence of points $\{p_n\}_{n \in \mathbb{N}} \subset \mathbb{H}$, where $p_n = w_n + z_n Z$, diverges:

    (1) *vertically*, if there exists $M < \infty$ such that $\|w_n\| < M$ for all n;
    (2) *non-vertically with quadratic rate* $\nu \in [-\infty, +\infty]$, if w_n diverges and[1] $\lim_{n \to \infty} \frac{z_n}{4\|w_n\|^2} = -\nu$.

Then, according to [KN10] (see Corollary 5.6, 5.9 and 5.13 therein), we deduce the following description of the Riemannian horofunctions:

(v): a vertically diverging sequence $p_n = w_n + z_n Z$ converges to a horofunction h if and only if $w_n \to w_\infty$, and in this case

$$h(w + zZ) = \|w_\infty\| - \|w_\infty - w\|;$$

(nv): a non-vertically diverging sequence $p_n = w_n + z_n Z$ with quadratic rate ν converges to a horofunction h if and only if $\frac{w_n}{\|w_n\|} \to \hat{w}$, and then

$$h(w + zZ) = g(R_\vartheta(-\hat{w}), w)$$

where R_ϑ is the anti-clockwise rotation in W of angle $\vartheta = \mu^{-1}(\nu)$, and $\mu : [-\pi, \pi] \to \overline{\mathbb{R}}$ is the extended Gaveau function

$$\mu(\vartheta) := \frac{\vartheta - \sin\vartheta\cos\vartheta}{\sin^2(\vartheta)}.$$

Moreover, all the horofunctions of $(\mathbb{H}, d_R)$ are of type **(v)** or **(nv)**, by Theorem 5.16 in [KN10]; it is also clear that neither is of both types.

In section 5 we will also determine the Busemann points of $\partial_h(\mathbb{H}, d_R)$, that is those horofunctions obtained by points diverging along quasi-geodesics (see Definition 5.1). We obtain, as in the subRiemannian case:

Corollary 1.4. *The Busemann points of $(\mathbb{H}, d_R)$ are the horofunctions of type (nv) and can be identified to the boundary of the disk $\bar{D}^2$.*

The paper is organized as follows. In Section 2 we introduce the main objects and their basic properties. In Section 3 we estimate the difference between any two strictly subRiemannian left-invariant distances on $\mathbb{H}$. In Section 4 we compare any Riemannian left-invariant distance on $\mathbb{H}$ and its associated distance. At the end of the section we shall prove Theorems 1.1 and 1.2. In Section 5 we concentrate on the horofunctions and we prove Corollaries 1.3 and 1.4. Appendix A is devoted to the explicit description of subRiemannian geodesics.

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[1] From the paper [KN10], there is an extra 4 and a change of sign due to our different choice of coordinates.

2. Preliminaries

2.1. **Definitions.** The *first Heisenberg group* $\mathbb{H}$ is the connected, simply connected Lie group associated to the Heisenberg Lie algebra $\mathfrak{h}$. The *Heisenberg Lie algebra* $\mathfrak{h}$ is the only three dimensional nilpotent Lie algebra that is not commutative. It can be proven that, for any two linearly independent vectors $X, Y \in \mathfrak{h} \setminus [\mathfrak{h}, \mathfrak{h}]$, the triple $(X, Y, [X, Y])$ is a basis of $\mathfrak{h}$ and $[X, [X, Y]] = [Y, [X, Y]] = 0$.

We denote by $\omega_{\mathbb{H}} : T\mathbb{H} \to \mathfrak{h}$ the left-invariant Maurer-Cartan form. Namely, denoting by 0 the neutral element of $\mathbb{H}$ and identifying $\mathfrak{h}$ with $T_0\mathbb{H}$, we have $\omega_{\mathbb{H}}(v) := \mathrm{d}L_p^{-1}v$ for $v \in T_p\mathbb{H}$, where L_p is the left translation by p.

Let $\pi : \mathfrak{h} \to \mathfrak{h}/[\mathfrak{h}, \mathfrak{h}]$ be the quotient projection. Notice that $\mathfrak{h}/[\mathfrak{h}, \mathfrak{h}]$ is a commutative 2-dimensional Lie algebra. So the map π induces a Lie group epimorphism $\hat{\pi} : \mathbb{H} \to \mathfrak{h}/[\mathfrak{h}, \mathfrak{h}] \simeq \mathbb{H}/[\mathbb{H}, \mathbb{H}]$.

2.2. **SubRiemannian metrics in $\mathbb{H}$.** Let $V \subset \mathfrak{h}$ be a bracket generating subspace. We have only two cases: either $V = \mathfrak{h}$ or V is a plane and $\mathfrak{h} = V \oplus [\mathfrak{h}, \mathfrak{h}]$. In both cases the restriction of the projection $\pi|_V : V \to \mathfrak{h}/[\mathfrak{h}, \mathfrak{h}]$ is surjective.

Let g be a scalar product on V and set the corresponding norm $\|v\| := \sqrt{g(v, v)}$ for $v \in V$.

An absolutely continuous curve $\gamma : [0, 1] \to \mathbb{H}$ is said *horizontal* if $\omega_{\mathbb{H}}(\gamma'(t)) \in V$ for almost every t. For a horizontal curve we have the *length*

$$\ell(\gamma) := \int_0^1 \|\omega_{\mathbb{H}}(\gamma'(t))\| \, \mathrm{d}t.$$

A *subRiemannian metric* d is hence defined as

$$d(p, q) := \inf \left\{ \ell(\gamma) : \gamma \text{ horizontal curve from } p \text{ to } q \right\}.$$

SubRiemannian distances on $\mathbb{H}$ are complete, geodesic, and left-invariant. They are either Riemannian, when $V = \mathfrak{h}$, or *strictly subRiemannian*, when $\dim V = 2$. The pair (V, g) is called the *horizontal space of* d.

Since $\pi|_V : V \to \mathfrak{h}/[\mathfrak{h}, \mathfrak{h}]$ is surjective, it induces a norm $\| \cdot \|$ on $\mathfrak{h}/[\mathfrak{h}, \mathfrak{h}]$ such that $\pi : (V, \| \cdot \|) \to (\mathfrak{h}/[\mathfrak{h}, \mathfrak{h}], \| \cdot \|)$ is an submetry, i.e., for all $w \in \mathfrak{h}/[\mathfrak{h}, \mathfrak{h}]$ it holds $\|\pi(w)\| = \inf\{\|v\| : \pi(v) = w\}$. Here we use the same notation for norms on V and on $\mathfrak{h}/[\mathfrak{h}, \mathfrak{h}]$, because there will be no possibility of confusion. The norm on $\mathfrak{h}/[\mathfrak{h}, \mathfrak{h}]$ is characterized by

$$(3) \qquad \pi\left(\{v \in V : \|v\| \leq R\}\right) = \{w \in \mathfrak{h}/[\mathfrak{h}, \mathfrak{h}] : \|w\| \leq R\},$$

for all $R > 0$.

Proposition 2.1. *Let d be subRiemannian metric on $\mathbb{H}$ with horizontal space (V, g). Then for all $R > 0$*

$$\pi\left(\{v \in V : \|v\| \leq R\}\right) = \hat{\pi}\left(\{p \in \mathbb{H} : d(0, p) \leq R\}\right).$$

In particular, $\hat{\pi} : (\mathbb{H}, d) \to (\mathfrak{h}/[\mathfrak{h}, \mathfrak{h}], \|\cdot - \cdot\|)$ is a submetry, i.e., for all $v, w \in \mathfrak{h}/[\mathfrak{h}, \mathfrak{h}]$

$$\|v - w\| = \inf\{d(p, q) : \hat{\pi}(p) = v, \ \hat{\pi}(q) = w\}.$$

Proof. $\subseteq$ Let $v \in V$ with $\|v\| \le R$. Set $\gamma(t) := \exp(tv)$. Then $\gamma : [0, 1] \to \mathbb{H}$ is a horizontal curve with $d(0, \exp(v)) \le \ell(\gamma) = \|v\| \le R$. Since $\hat{\pi}(\exp(v)) = \pi(v)$, then we have proven this inclusion.

$\supseteq$ Let $p \in \mathbb{H}$ with $d(0, p) \le R$ and let $\gamma : [0, T] \to \mathbb{H}$ be a d-length-minimizing curve from 0 to p parametrized by arc-length, so $T = d(0, p)$. Then $\hat{\pi} \circ \gamma : [0, T] \to \mathfrak{h}/[\mathfrak{h}, \mathfrak{h}]$ is a curve from 0 to $\pi(p)$ and

$$\|\hat{\pi}(p)\| \le \int_0^T \|(\hat{\pi} \circ \gamma)'(t)\| \, \mathrm{d}t$$

$$= \int_0^T \|\pi \circ \omega_{\mathbb{H}}(\gamma'(t))\| \, \mathrm{d}t$$

$$\le \int_0^T \|\omega_{\mathbb{H}}(\gamma'(t))\| \, \mathrm{d}t$$

$$= \ell(\gamma) = d(0, p).$$

In the first equality we used the fact that $\hat{\pi}$ is a morphism of Lie groups and its differential is π, i.e., $\omega_{\mathbb{H}/[\mathbb{H},\mathbb{H}]} \circ \mathrm{d}\hat{\pi} = \pi \circ \omega_{\mathbb{H}}$, where $\omega_{\mathbb{H}/[\mathbb{H},\mathbb{H}]}$ is the Mauer-Cartan form of $\mathbb{H}/[\mathbb{H}, \mathbb{H}]$. $\square$

Proposition 2.2. *Let d, d' be two subRiemannian metrics on $\mathbb{H}$ such that*

$$\lim_{p \to \infty} \frac{d(0, p)}{d'(0, p)} = 1.$$

Then

(4) $\qquad \hat{\pi}(\{p \in \mathbb{H} : d(0, p) \le R\}) = \hat{\pi}(\{p \in \mathbb{H} : d'(0, p) \le R\}).$

Proof. Let $\|\cdot\|$ and $\|\cdot\|'$ be the norms on $\mathfrak{h}/[\mathfrak{h}, \mathfrak{h}]$ induced by d and d', respectively. We will show that

(5) $$\lim_{v \to \infty} \frac{\|v\|}{\|v\|'} = 1,$$

which easily implies $\|\cdot\| = \|\cdot\|'$ and (4) by (3) and Proposition 2.1.

Since both maps $\hat{\pi} : (\mathbb{H}, d) \to (\mathfrak{h}/[\mathfrak{h}, \mathfrak{h}], \|\cdot\|)$ and $\hat{\pi} : (\mathbb{H}, d') \to (\mathfrak{h}/[\mathfrak{h}, \mathfrak{h}], \|\cdot\|')$ are submetries, for every $v \in \mathfrak{h}/[\mathfrak{h}, \mathfrak{h}]$ there are $p_v, p'_v \in \mathbb{H}$ such that $\hat{\pi}(p_v) = \hat{\pi}(p'_v) = v$, $\|v\| = d(0, p_v)$ and $\|v\|' = d'(0, p'_v)$.

Moreover it holds $\|v\|' \le d'(0, p_v)$ and $\|v\| \le d(0, p'_v)$, again because $\hat{\pi}$ is a submetry in both cases. Therefore

$$\frac{d(0, p_v)}{d'(0, p_v)} \le \frac{\|v\|}{\|v\|'} \le \frac{d(0, p'_v)}{d'(0, p'_v)}$$

Finally, if $v \to \infty$, then both $d(0, p_v)$ and $d(0, p'_v)$ go to infinity as well. The relation (5) is thus proven. $\square$

2.3. **Balayage area and lifting of curves.** Let $V \subset \mathfrak{h}$ be a two-dimensional subspace with $V \cap [\mathfrak{h}, \mathfrak{h}] = \{0\}$. Then $[\mathfrak{h}, \mathfrak{h}] = [V, V]$, i.e., V is bracket generating. Moreover, $\pi|_V : V \to \mathfrak{h}/[\mathfrak{h}, \mathfrak{h}]$ is an isomorphism.

If $\rho : [0, T] \to \mathfrak{h}/[\mathfrak{h}, \mathfrak{h}]$ is a curve with $\rho(0) = 0$, then there is a unique $\tilde{\rho} : [0, T] \to \mathbb{H}$ such that

$$\begin{cases} \tilde{\rho}(0) = 0, \\ \omega_{\mathbb{H}}(\tilde{\rho}'(t)) = \pi|_V^{-1}(\rho(t)'). \end{cases}$$

Since $(\pi \circ \tilde{\rho})' = \rho'$, then $\pi \circ \tilde{\rho} = \rho$. So, $\tilde{\rho}$ is called the *lift* of ρ.

The previous ODE system that defines $\tilde{\rho}$ can be easily integrated. Let $X, Y \in V$ be a basis, set $Z := [X, Y]$, so that (X, Y, Z) is a basis of $\mathfrak{h}$. Let $(x, y, z) = \exp(xX + yY + zZ)$ be the exponential coordinates on $\mathbb{H}$ defined by (X, Y, Z). Using the Backer-Campbell-Hausdorff formula, one shows that X, Y, Z induce the following left-invariant vector fields on $\mathbb{H}$:

$$\hat{X} = \partial_x - \frac{y}{2}\partial_z, \qquad \hat{Y} = \partial_y + \frac{x}{2}\partial_z, \qquad \hat{Z} = \partial_z.$$

Thanks to these vector fields, we can describe the Maurer-Cartan form as

$$\omega_{\mathbb{H}}(a\hat{X} + b\hat{Y} + c\hat{Z}) = aX + bY + cZ.$$

The lift of ρ is hence given by the ODE

$$\begin{cases} \tilde{\rho}_1' = \rho_1', \\ \tilde{\rho}_2' = \rho_2', \\ \tilde{\rho}_3' = \frac{1}{2}\left(\rho_1\rho_2' - \rho_2\rho_1'\right). \end{cases}$$

Take the coordinates (x, y) on $\mathfrak{h}/[\mathfrak{h}, \mathfrak{h}]$ given by the basis $(\pi(X), \pi(Y))$ and define the *balayage area* of a curve $\rho : [0, T] \to \mathfrak{h}/[\mathfrak{h}, \mathfrak{h}]$ as

$$(6) \qquad \mathscr{A}(\rho) = \frac{1}{2} \int_\rho (x\,\mathrm{d}y - y\,\mathrm{d}x).$$

If $\rho(0) = 0$, then the balayage area of ρ corresponds to the signed area enclosed between the curve ρ and the line passing through 0 and $\rho(T)$.

It follows that

$$\tilde{\rho}(t) = \left(\rho_1(t), \rho_2(t), \mathscr{A}(\rho|_0^t)\right).$$

In an implicit form we can write

$$(7) \qquad \tilde{\rho}(t) = \exp\left((\pi|_V)^{-1}(\rho(t)) + \mathscr{A}(\rho|_0^t)Z\right).$$

Notice that the lift $\tilde{\rho}$ of a curve ρ depends on the choice of V. Moreover, both the area and the Balayage area in $\mathfrak{h}/[\mathfrak{h}, \mathfrak{h}]$ depend on the choice of the basis (X, Y). Nevertheless, once a plane $V \subset \mathfrak{h}$ is fixed, the lift $\tilde{\rho}$ does not depend on the choice of the basis X, Y.

If g is a scalar product on V and d is the corresponding strictly subRiemannian metric, the balayage area gives a characterization of d-length-minimizing curves. Let $\bar{g}$ be the scalar product on $\mathfrak{h}/[\mathfrak{h}, \mathfrak{h}]$

induced by g. Then the d-length of a curve $\tilde{\rho} : [0, T] \to \mathbb{H}$ equals the length of $\rho = \pi \circ \tilde{\rho}$.

Therefore, given $p = (x, y, z) \in \mathbb{H}$, we have

$$d(0, p) = \inf \left\{ \ell(\rho) : \rho : [0, 1] \to \mathfrak{h}/[\mathfrak{h}, \mathfrak{h}], \rho(0) = 0, \rho(1) = \hat{\pi}(p), \mathscr{A}(\rho) = z \right\}.$$

This express the so-called *Dido's problem* in the plane, and the solutions are arc of circles. It degenerates into a line if $z = 0$. We can summarize the last discussion in the following result.

Lemma 2.3. *A curve* $\tilde{\rho} : [0, 1] \to \mathbb{H}$ *is d-length-minimizing from 0 to* $p = (x, y, z)$ *if and only if* $\rho := \hat{\pi} \circ \tilde{\rho}$ *is an arc of a circle from 0 to $\hat{\pi}(p)$ with* $\mathscr{A}(\rho) = z$.

3. Comparison between strictly subRiemannian metrics

The present section is devoted to comparing strictly subRiemannian distances. For such distances, Proposition 3.1 gives the only non-trivial implication in Theorem 1.1. The general case will follow from Proposition 4.1.

Proposition 3.1. *Let d and d' be two strictly subRiemannian metrics on $\mathbb{H}$ with horizontal spaces (V, g) and (V', g'), respectively. Suppose that there exists a scalar product $\bar{g}$ on $\mathfrak{h}/[\mathfrak{h}, \mathfrak{h}]$ such that both*

$$\pi|_V : (V, g) \to (\mathfrak{h}/[\mathfrak{h}, \mathfrak{h}], \bar{g}) \quad and \quad \pi|_{V'} : (V', g') \to (\mathfrak{h}/[\mathfrak{h}, \mathfrak{h}], \bar{g})$$

are submetries.

Then

$$(8) \qquad\qquad \sup_{p \in \mathbb{H}} |d(0, p) - d'(0, p)| < \infty.$$

Moreover, if $d \neq d'$, then

$$(9) \qquad\qquad \limsup_{p \to \infty} |d(0, p) - d'(0, p)| > 0.$$

In the proof we will give the exact value of the supremum in (8). Indeed, by (11) and (12), we get $\sup_{p \in \mathbb{H}} |d(0, p) - d'(0, p)| = 2|h|$, where h is defined below.

For (8) we will first prove that two of such subRiemannian distances are one the conjugate of the other and then we apply Lemma 3.2.

For (9) we will give a sequence $p_n \to \infty$ and a constant $c > 0$ such that $|d(0, p_n) - d'(0, p_n)| > c$ for all $n \in \mathbb{N}$.

3.1. **Proof of** (8)**.** Since $\dim V = \dim V' = 2$, then $\pi|_V$ and $\pi|_{V'}$ are isomorphisms. Therefore by the assumption they are isometries onto $(\mathfrak{h}/[\mathfrak{h}, \mathfrak{h}], \bar{g})$.

Let $X \in V \cap V'$ be with $g(X, X) = 1$. Then $g'(X, X) = 1$ as well.

Let $Y \in V$ be orthogonal to X with $g(Y, Y) = 1$. Then $Z := [X, Y] \neq 0$ and (X, Y, Z) is a basis of $\mathfrak{h}$.

Let $Y' := \pi|_{V'}^{-1}(\pi(Y)) \in V'$. Then $g'(Y', Y') = 1$ and $g'(X, Y') = 0$. Moreover, there is $h \in \mathbb{R}$ such that $Y' = Y + hZ$. In particular, $[X, Y'] = Z$.

Using the formula $\mathrm{Ad}_{\exp(hX)}(v) = e^{\mathrm{ad}_h X} v = v + h[X, v]$, we notice that

$$(10) \qquad \begin{cases} \mathrm{Ad}_{\exp(hX)}(X) = X, \\ \mathrm{Ad}_{\exp(hX)}(Y) = Y', \\ \mathrm{Ad}_{\exp(hX)}(Z) = Z. \end{cases}$$

In particular $\mathrm{Ad}_{\exp(hX)}|_V : (V, g) \to (V', g')$ is an isometry.

Therefore, the conjugation

$$C_{\exp(hX)}(p) := \exp(hX) \cdot p \cdot \exp(hX)^{-1}$$

is an isometry $C_{\exp(hX)} : (\mathbb{H}, d) \to (\mathbb{H}, d')$.

We can now use the following Lemma 3.2 and get

$$(11) \qquad \sup_{p \in \mathbb{H}} |d(0, p) - d'(0, p)| \leq 2|h|.$$

Lemma 3.2. *Let G be a group with neutral element e and let d, d' be two left-invariant distances on G.*

If there is $g \in G$ such that for all $p \in G$

$$d'(e, p) = d(e, gpg^{-1}),$$

then for all $p \in G$

$$|d(e, p) - d'(e, p)| \leq 2 \max\{d(e, g), d'(e, g)\}.$$

Proof. Note that, since d is left invariant, then for all $a, b \in G$ we have $d(e, ab) \leq d(e, a) + d(e, b)$ and $d(e, a) = d(e, a^{-1})$.

Hence $d(e, p) = d(e, g^{-1}gpg^{-1}g) \leq d(e, g^{-1}) + d(e, gpg^{-1}) + d(e, g) = 2d(e, g) + d'(e, p)$. The other inequality follows by symmetry. $\qquad\square$

3.2. **Proof of** (9)**.** We keep the same notation of the previous subsection. Up to switching V with V', we can assume $h > 0$.

Let (x, y, z) be the exponential coordinates on $\mathbb{H}$ induced by the basis (X, Y, Z) of $\mathfrak{h}$, i.e., $(x, y, z) = \exp(xX + yY + zZ) \in \mathbb{H}$. Similary, on $\mathfrak{h}/[\mathfrak{h}, \mathfrak{h}]$ we have coordinates $(x, y) = x\pi(X) + y\pi(Y)$.

For $R > 0$, define

$$p_R := \left(0, 2R, \frac{\pi R^2}{2} + 2hR\right).$$

We will show that

$$(12) \qquad \lim_{R \to \infty} d(0, p_R) - d'(0, p_R) = 2h.$$

Fix $R > 0$. Let $\gamma : [0, T] \to \mathbb{H}$ be a d'-minimizing curve from 0 to p_R. Then $\hat{\pi} \circ \gamma : [0, T] \to \mathfrak{h}/[\mathfrak{h}, \mathfrak{h}]$ is half circle of center $(0, R)$ and radius

R. The balayage area of $\hat{\pi} \circ \gamma$ is

$$\mathscr{A}(\hat{\pi} \circ \gamma) = \frac{\pi R^2}{2}.$$

Let $\eta : [0, T] \to \mathbb{H}$ be the d-length-minimizing curve from 0 to p_R. Then $\hat{\pi} \circ \eta : [0, T] \to \mathfrak{h}/[\mathfrak{h}, \mathfrak{h}]$ is an arc of a circle of radius S_R whose balayage area is

$$(13) \qquad \mathscr{A}(\hat{\pi} \circ \eta) = \frac{\pi R^2}{2} + 2Rh = \mathscr{A}(\hat{\pi} \circ \gamma) + 2Rh.$$

It is clear that $S_R > R$ and that the circle of $\hat{\pi} \circ \eta$ has center (μ_R, R) for some $\mu_R > 0$. So we have

$$S_R^2 = R^2 + \mu_R^2.$$

It is also clear from the picture that

$$(14) \qquad \frac{\pi S_R^2}{2} + 2R\mu_R \leq \mathscr{A}(\hat{\pi} \circ \eta) \leq \frac{\pi S_R^2}{2} + 2S_R\mu_R.$$

Now, let's look at the lengths. First of all, notice that $\ell_{d'}(\gamma) = \ell(\hat{\pi} \circ \gamma)$ and $\ell_d(\eta) = \ell(\hat{\pi} \circ \eta)$. For one curve we have

$$\ell(\hat{\pi} \circ \gamma) = \pi R,$$

for the other we have the estimate

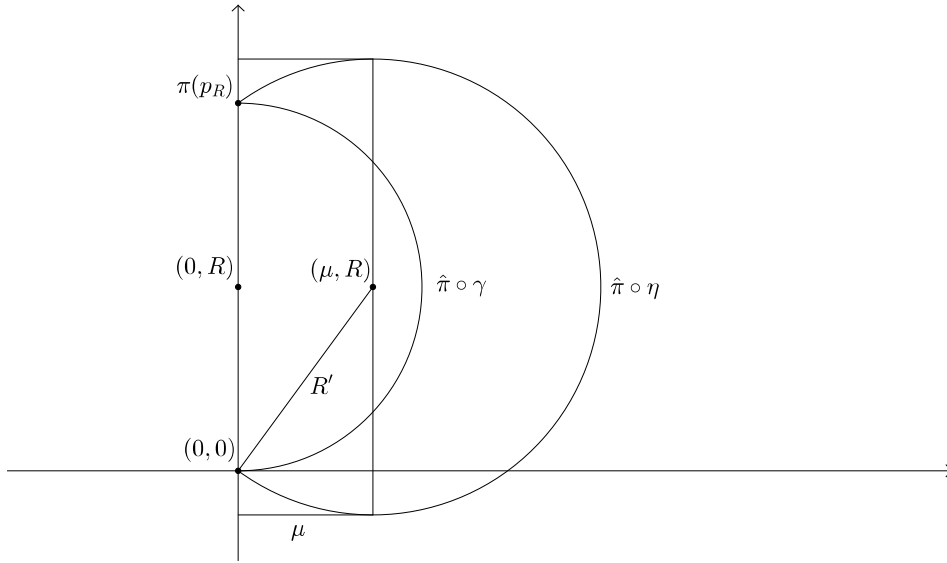$$\pi S_R + 2\mu_R \leq \ell(\hat{\pi} \circ \eta),$$

FIGURE 1. Curves in $\mathfrak{h}/[\mathfrak{h}, \mathfrak{h}]$ for the proof of (9).

which is clear from the picture. Hence

$$
\liminf_{R\to\infty} d(0,p) - d'(0,p) = \liminf_{R\to\infty} \ell(\hat{\pi}\circ\eta) - \ell(\hat{\pi}\circ\gamma)
$$
$$
\geq \lim_{R\to\infty} \pi S_R + 2\mu_R - \pi R
$$
$$
= \lim_{R\to\infty} \pi(S_R - R) + 2\mu_R.
$$

We claim that

$$
(15) \qquad \lim_{R\to\infty} \pi(S_R - R) + 2\mu_R = 2h.
$$

Let us start by checking that,

$$
(16) \qquad \mu_R < h.
$$

Indeed, from the first inequality of (14) together with (13) it follows

$$
\frac{\pi S_R^2}{2} + 2R\mu_R \leq \frac{\pi R^2}{2} + 2hR.
$$

Since $S_R > R$, then

$$
0 < \frac{\pi S_R^2}{2} - \frac{\pi R^2}{2} \leq 2R(h - \mu_R),
$$

i.e., the inequality (16).

From the second inequality of (14) together with (13) we get

$$
\frac{\pi R^2}{2} + 2Rh \leq \frac{\pi S_R^2}{2} + 2S_R\mu_R.
$$

Using the facts $\mu_R \leq h$ and $S_R \leq R + \mu_R \leq R + h$, from this last inequality one gets

$$
0 \leq 2(h - \mu_R) \leq (S_R - R)\frac{1}{R}\left(\frac{\pi}{2}(S_R + R) + 2\mu_R\right)
$$
$$
(17) \qquad\qquad \leq (S_R - R)\left(\pi + \frac{h}{R}(\frac{\pi}{2} + 2)\right)
$$

Moreover, since $h^2 \geq \mu_R^2 = S_R^2 - R^2 = (S_R - R)(S_R + R)$, we also have

$$
(18) \qquad \lim_{R\to\infty}(S_R - R) = 0.
$$

Finally, from (18) and (17) we obtain (15), as claimed. This completes the proof of (12) and of Proposition 3.1.

4. COMPARISON BETWEEN RIEMANNIAN AND STRICTLY SUBRIEMANNIAN METRICS

Let d_R be a Riemannian metric on $\mathbb{H}$ with horizontal space $(\mathfrak{h}, g)$.

Let $V \subset \mathfrak{h}$ be the plane orthogonal to $[\mathfrak{h}, \mathfrak{h}]$ and let d_{CC} be the strictly subRiemannian metric on $\mathbb{H}$ with horizontal space $(V, g|_V)$.

Fix a basis (X, Y, Z) for $\mathfrak{h}$ such that (X, Y) is an orthonormal basis of $(V, g|_V)$ and $Z = [X, Y]$. The matrix representation of g with respect to (X, Y, Z) is

$$g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \zeta^2 \end{pmatrix}$$

where $\zeta > 0$.

Let d_{CC} be the strictly subRiemannian metric on $\mathbb{H}$ with horizontal space $(V, g|_V)$.

Our aim in this section is to prove the following proposition.

Proposition 4.1. *If $d_{CC}(0, p)$ is large enough, then:*

$$(19) \qquad 0 \le d_{CC}(0, p) - d_R(0, p) \le \frac{4\pi^2}{\zeta^2} \frac{1}{d_{CC}(0, p) - \frac{2^{3/2\pi}}{\zeta}}.$$

In particular it holds

$$(20) \qquad \lim_{p \to \infty} |d_{CC}(0, p) - d_R(0, p)| = 0.$$

For the proof of this statement, we need to know length-minimizing curves for d_R and d_{CC}, and a few properties of those, see the exposition in the Appendix A.

Proof. Let (x, y, z) be the exponential coordinates on $\mathbb{H}$ induced by the basis (X, Y, Z) of $\mathfrak{h}$, i.e., $(x, y, z) = \exp(xX + yY + zZ) \in \mathbb{H}$. Fix $p = (p_1, p_2, p_3) \in \mathbb{H}$.

Notice that both d_R and d_{CC} are generated as length metrics using the same length measure ℓ, with the difference that d_R minimizes the length among all the curves, while d_{CC} takes into account only the curves tangent to V. This implies that

$$\forall p, q \in \mathbb{H} \qquad d_{CC}(p, q) \ge d_R(p, q),$$

therefore we get the first inequality in (19). We need to prove the second inequality of (19).

If $p \in \{z = 0\}$, then $d_{CC}(0, p) = d_R(0, p)$ by Corollary A.7, and the thesis is true.

Suppose $p \notin \{z = 0\}$ and let $\gamma : [0, T] \to \mathbb{H}$ be a d_R-length minimizing curve from $0 = \gamma(0)$ to $p = \gamma(T)$. Since $p \notin \{z = 0\}$ and since we supposed p to be far away enough, then by Corollary A.5 we can parametrize γ in such a way that γ is exactly in the form expressed in TYPE II in Proposition A.2 for some $k > 0$ and $\theta \in \mathbb{R}$.

By Corollary A.4 it holds

$$(21) \qquad kT \le 2\pi.$$

Moreover, by Corollary A.8

$$(22) \qquad d_R(0, p) = \|\omega_{\mathbb{H}}(\gamma')\| \cdot T = \sqrt{1 + \frac{k^2}{\zeta^2}} \cdot T.$$
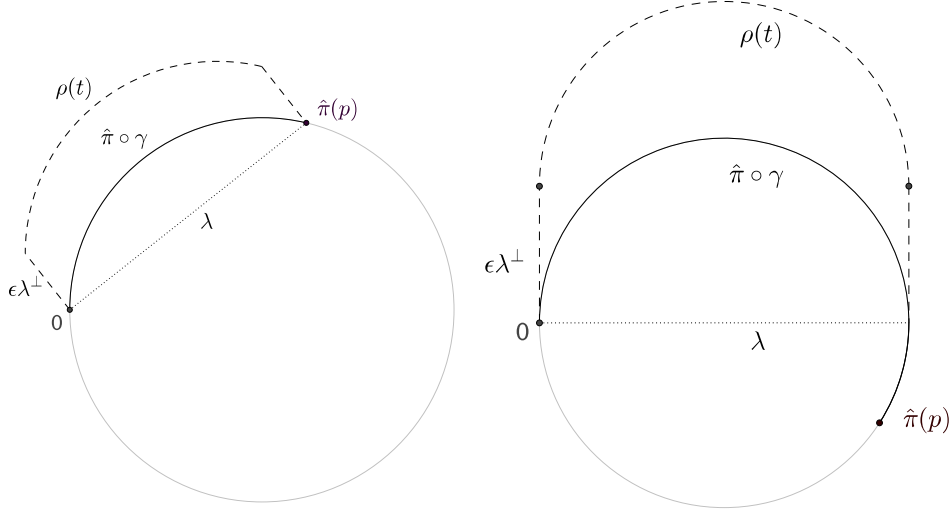
FIGURE 2. Curves for Case 1 and Case 2.

Let $\eta : [0, T] \to \mathbb{H}$ be the d_{CC}-length-minimizing curve corresponding to γ as shown in Corollary A.8. Then we know that $d_{CC}(0, \eta(T)) = \ell(\eta) = T$, and

$$(23) \qquad p = \gamma(T) = \eta(T) + (0, 0, \frac{kT}{\zeta^2}).$$

Hence by Corollary A.8 and (21)

$$d_{CC}(0, p) \leq d_{CC}(0, \eta(T)) + d_{CC}(\eta(T), \gamma(T)) \leq T + \frac{2^{3/2}\pi}{\zeta},$$

i.e.,

$$(24) \qquad \frac{1}{T} \leq \frac{1}{d_{CC}(0, p) - \frac{2^{3/2}\pi}{\zeta}}.$$

Since η is a d_{CC}-rectifiable curve, then $\eta(T)_3 = \mathscr{A}(\hat{\pi} \circ \eta)$, where $\eta(T)_3$ is the third coordinate of the point in the exponential coordinates. Since $\hat{\pi} \circ \gamma = \hat{\pi} \circ \eta$, then we have by (23)

$$(25) \qquad p_3 = \mathscr{A}(\hat{\pi} \circ \gamma) + \frac{kT}{\zeta^2}.$$

Notice that $\hat{\pi} \circ \gamma$ is an arc of a circle in $\mathfrak{h}/[\mathfrak{h}, \mathfrak{h}]$ of radius $\frac{1}{k}$, see Proposition A.2.

Now we want to define a horizontal curve $\tilde{\rho} : [-\epsilon, T + \epsilon] \to \mathbb{H}$, where $\epsilon > 0$ has to be chosen, such that $\tilde{\rho}(-\epsilon) = 0$ and $\tilde{\rho}(T + \epsilon) = p$. We first define a curve $\rho : [-\epsilon, T + \epsilon] \to \mathfrak{h}/[\mathfrak{h}, \mathfrak{h}]$ and then take its lift to $\mathbb{H}$.

For the definition of ρ we follow two different strategies for two different cases:

CASE 1. Suppose that $\hat{\pi} \circ \gamma$ doesn't cover the half of the circle, i.e., $T \leq \frac{\pi}{k}$. Set $\lambda = \hat{\pi}(p) \in \mathfrak{h}/[\mathfrak{h}, \mathfrak{h}]$. Then T is smaller than the circle of diameter $\|\lambda\|$, i.e.,

$$\|\lambda\| \geq \frac{T}{\pi}. \tag{26}$$

Let $\lambda^\perp \in \mathfrak{h}/[\mathfrak{h}, \mathfrak{h}]$ be the unitary vector perpendicular to λ and forming an angle smaller than $\pi/2$ with the arc $\hat{\pi} \circ \gamma$. Let $\epsilon > 0$ such that

$$\epsilon \cdot \|\lambda\| = \frac{kT}{\zeta^2}. \tag{27}$$

Now, define $\rho : [-\epsilon, T + \epsilon] \to \mathfrak{h}/[\mathfrak{h}, \mathfrak{h}]$ as

$$\rho(t) = \begin{cases} (t + \epsilon)\lambda^\perp & \text{for } -\epsilon \leq t \leq 0 \\ \epsilon\lambda^\perp + \hat{\pi} \circ \gamma(t) & \text{for } 0 \leq t \leq T \\ \epsilon\lambda^\perp + \hat{\pi} \circ \gamma(T) - (t - T)\lambda^\perp & \text{for } T \leq t \leq T + \epsilon \end{cases}$$

Notice that

$$\mathscr{A}(\rho) = \mathscr{A}(\hat{\pi} \circ \gamma) + \epsilon \cdot \|\lambda\| = \mathscr{A}(\hat{\pi} \circ \gamma) + \frac{kT}{\zeta^2} \overset{(25)}{=} p_3$$

and that $\rho(T + \epsilon) = \hat{\pi} \circ \gamma(T) = \hat{\pi}(p)$. Then the horizontal lift $\tilde{\rho} : [-\epsilon, T + \epsilon] \to \mathbb{H}$ of ρ is a d_{CC}-rectifiable curve from 0 to p.

CASE 2. Suppose that $\hat{\pi} \circ \gamma$ covers more than half of the circle. Let $\lambda \in \mathfrak{h}/[\mathfrak{h}, \mathfrak{h}]$ be the diameter of the circle that containing 0. Since T is shorter than the whole circle, then

$$\|\lambda\| \geq \frac{T}{\pi}. \tag{28}$$

Let $\lambda^\perp$ be the unitary vector perpendicular to λ and forming an angle smaller than $\pi/2$ with the arc $\hat{\pi} \circ \gamma$. Let $\epsilon > 0$ be such that

$$\epsilon \cdot \|\lambda\| = \frac{kT}{\zeta^2}. \tag{29}$$

Now, define $\rho : [-\epsilon, T + \epsilon] \to \mathfrak{h}/[\mathfrak{h}, \mathfrak{h}]$ as

$$\rho(t) = \begin{cases} (t + \epsilon)\lambda^\perp & \text{for } -\epsilon \leq t \leq 0 \\ \epsilon\lambda^\perp + \hat{\pi} \circ \gamma(t) & \text{for } 0 \leq t \leq \frac{\pi\|\lambda\|}{2} \\ \epsilon\lambda^\perp + \lambda - (t - \frac{\pi\|\lambda\|}{2})\lambda^\perp & \text{for } \frac{\pi\|\lambda\|}{2} \leq t \leq \frac{\pi\|\lambda\|}{2} + \epsilon \\ \hat{\pi} \circ \gamma(t - \epsilon) & \text{for } \frac{\pi\|\lambda\|}{2} + \epsilon \leq t \leq T + \epsilon \end{cases}$$

where we used the fact $\lambda = \hat{\pi} \circ \gamma(\frac{\pi\|\lambda\|}{2})$. Notice that

$$\mathscr{A}(\rho) = \mathscr{A}(\hat{\pi} \circ \gamma) + \epsilon \cdot \|\lambda\| = \mathscr{A}(\hat{\pi} \circ \gamma) + \frac{kT}{\zeta^2} = p_3.$$

Then the horizontal lift $\tilde{\rho} : [-\epsilon, T + \epsilon] \to \mathbb{H}$ of ρ is a d_{CC}-rectifiable curve from 0 to p.

In both cases $\tilde{\rho}$ is a horizontal curve from 0 to p of length

(30) $$\ell(\tilde{\rho}) = T + 2\epsilon$$

Moreover, from (26) and (27) (respectively (28) and (29)) we get

(31) $$\epsilon = \frac{kT}{\zeta^2\|\lambda\|} \leq \frac{kT}{\zeta^2}\frac{\pi}{T} \overset{(21)}{\leq} \frac{2\pi}{\zeta^2}\frac{\pi}{T} = \frac{2\pi^2}{\zeta^2 T}$$

Finally using in order (22), (30), (31), (24)

$$d_{CC}(0,p) - d_R(0,p) \leq \ell(\tilde{\rho}) - \sqrt{1 + \frac{k^2}{\zeta^2}} \cdot T \leq$$

$$\leq T + 2\epsilon - \sqrt{1 + \frac{k^2}{\zeta^2}} \cdot T \leq 2\epsilon \leq 2\frac{2\pi^2}{\zeta^2 T} \leq \frac{4\pi^2}{\zeta^2}\frac{1}{d_{CC}(0,p) - \frac{2^{3/2}\pi}{\zeta}}.$$

$\square$

Remark 4.2. The inequality (2) is sharp. Indeed, for $z \to \infty$, we have the asymptotic equivalence

(32) $$d_{CC}(0,(0,0,z)) - d_R(0,(0,0,z)) \sim \frac{4\pi^2}{\zeta^2}\frac{1}{d_{CC}(0,(0,0,z))}.$$

Proof of (32). We claim that, for $z > 0$ large enough,

(33) $$d_R(0,(0,0,z)) = 2\sqrt{\pi}\sqrt{z - \frac{\pi}{\zeta^2}}.$$

Let $\gamma : [0,T] \to \mathbb{H}$ be a d_R-length-minimizing curve from 0 to $(0,0,z)$. Since z is large, we assume that γ is of (TYPE II), see Proposition A.2, for some $k > 0$ and $\theta = 0$. Since the end point is on the Z axis, we have

(34) $$kT = 2\pi$$

and $z = \frac{T}{2k} + \frac{kT}{\zeta^2}$, from which follows

(35) $$T^2 = 4\pi\left(z - \frac{2\pi}{\zeta^2}\right).$$

We know also the length of γ (see Corollary A.8) and so we get

$$d_R(0,(0,0,z)) = \ell(\gamma) = T\|\omega_{\mathbb{H}}(\gamma')\| = T\sqrt{1 + \frac{k^2}{\zeta^2}} = \sqrt{T^2 + \frac{4\pi^2}{\zeta^2}}$$

$$= \sqrt{4\pi\left(z - \frac{2\pi}{\zeta^2}\right) + \frac{4\pi^2}{\zeta^2}} = 2\sqrt{\pi}\sqrt{z - \frac{\pi}{\zeta^2}}.$$

Claim (33) is proved. From Corollary A.6 we get $d_{CC}(0,(0,0,z)) = 2\sqrt{\pi}\sqrt{z}$ and

$$d_{CC}(0,(0,0,z)) - d_R(0,(0,0,z)) = 2\sqrt{\pi}\left(\sqrt{z} - \sqrt{z - \frac{\pi}{\zeta^2}}\right)$$

$$= \frac{2\sqrt{\pi}}{\sqrt{z}}\frac{\frac{\pi}{\zeta^2}}{1 + \sqrt{1 - \frac{\pi}{\zeta^2 z}}} = \frac{1}{2\sqrt{\pi}\sqrt{z}}\frac{4\pi^2}{\zeta^2}\frac{1}{1 + \sqrt{1 - \frac{\pi}{\zeta^2 z}}}.$$

$\square$

We are now ready to give the proof of the main theorems:

Proof of Theorem 1.1. The implication $(a) \Rightarrow (b)$ is trivial. The implication $(b) \Rightarrow (c)$ is proven in Proposition 2.2. The equivalence $(c) \Leftrightarrow (d)$ follows from Proposition 2.1. The assertion (e) is a restatement of (d). For $(d) \Rightarrow (a)$ one uses Proposition 4.1 in order to reduce to the case when both d and d' are strictly subRiemannian and then one applies Proposition 3.1. $\square$

Proof of Theorem 1.2. This is a consequence of Propositions 4.1 and of the sharpness result (9) of Proposition 3.1. $\square$

5. THE HOROBOUNDARY

Let (X,d) be a geodesic space and $\mathscr{C}(X)$ the space of continuous functions $X \to \mathbb{R}$ endowed with the topology of the uniform convergence on compact sets. The map $\iota : X \hookrightarrow \mathscr{C}(X)$, $(\iota(x))(y) := d(x,y)$, is an embedding, i.e., a homeomorphism onto its image.

Let $\mathscr{C}(X)/\mathbb{R}$ be the topological quotient of $\mathscr{C}(X)$ with kernel the constant functions, i.e., for every $f, g \in \mathscr{C}(X)$ we set the equivalence relation $f \sim g \Leftrightarrow f - g$ is constant.

Then the map $\hat{\iota} : X \hookrightarrow \mathscr{C}(X)/\mathbb{R}$ is still an embedding. Indeed, since the map $\mathscr{C}(X) \to \mathscr{C}(X)/\mathbb{R}$ is continuous and open, we only need to show that $\hat{\iota}$ is injective: if $x, x' \in X$ are such that $\iota(x) - \iota(x')$ is constant, then one takes $z \in Z$ such that $d(x,z) = d(x',z)$, which exists because (X,d) is a geodesic space, and checks that

$$d(x,x') = \iota(x)(x') - \iota(x')(x') = \iota(x)(z) - \iota(x')(z) = 0.$$

Define the *horoboundary of* (X,d) as

$$\partial_h X := cl(\hat{\iota}(X)) \setminus \hat{\iota}(X) \subset \mathscr{C}(X)/\mathbb{R},$$

where $cl(\hat{\iota}(X))$ is the topological closure.

Another description of the horoboundary is possible. Fix $o \in X$ and set

$$\mathscr{C}(X)_o := \{f \in \mathscr{C}(X) : f(o) = 0\}.$$

Then the restriction of the quotient projection $\mathscr{C}(X)_o \to \mathscr{C}(X)/\mathbb{R}$ is an isomorphism of topological vector spaces. Indeed, one easily checks that it is both injective and surjective, and that its inverse map is

$[f] \mapsto f - f(o)$, where $[f] \in \mathscr{C}(X)/\mathbb{R}$ is the class of equivalence of $f \in \mathscr{C}(X)$.

Hence, we can identify $\partial_h X$ with a subset of $\mathscr{C}(X)_o$. More explicitly: $f \in \mathscr{C}(X)_o$ belongs to $\partial_h X$ if and only if there is a sequence $p_n \in X$ such that $p_n \to \infty$ (i.e., for every compact $K \subset X$ there is $N \in \mathbb{N}$ such that $p_n \notin K$ for all $n > N$) and the sequence of functions $f_n \in \mathscr{C}(X)_o$,

$$(36) \qquad f_n(x) := d(p_n, x) - d(p_n, o),$$

converge uniformly on compact sets to f.

Proof of Corollary 1.3. Let us first remark that if d, d' are two geodesic distances on X and

$$(37) \qquad \lim_{d(p,q)+d'(p,q)\to\infty} |d'(p,q) - d(p,q)| = 0.$$

then

$$\partial_h(X, d') = \partial_h(X, d).$$

Indeed, first of all the space $\mathscr{C}(X)_o$ depends only on the topology of X. Moreover, if $f \in \partial_h(X, d)$, let $p_n \in X$ be a sequence as in (36) and set $f'_n(x) := d'(p_n, x) - d'(p_n, o)$. Then

$$|f'_n(x) - f_n(x)| \leq |d'(p_n, x) - d(p_n, x)| + |d'(p_n, o) - d(p_n, o)|,$$

and as a consequence of (37) we get $f'_n \to f$ uniformly on compact sets. This shows $\partial_h(X, d) \subset \partial_h(X, d')$. The other inclusion follows by the simmetry of (37) in d and d'.

Now, if d_R and d_{CC} are distances on $\mathbb{H}$ like in Corollary 1.3, then (37) is easily satisfied thanks to Theorem 1.2, and therefore $\partial_h(\mathbb{H}, d_R) = \partial_h(\mathbb{H}, d_R)$ if the Riemannian metric d_R and the subRiemannian metric d_{CC} are compatible. The conclusion follows from [KN10]. $\qquad \square$

The Busemann points in the boundary $\partial_h(X, d)$ are usually defined as the horofunctions associated to sequences of points (p_n) diverging along rays or "almost geodesic rays". However, in literature there are different definitions of almost geodesic rays, according to the generality of the metric space (X, d) under consideration ([Haa96], [Rie02], [DPS12]). A map $\gamma : I = [0, +\infty) \to (X, d)$ into a complete length space is called

- a *quasi-ray*, if the length excess

$$\Delta_N(\gamma) = \sup_{t,s\in[N,+\infty]} \ell(\gamma; t, s) - d(\gamma(t), \gamma(s))$$

tends to zero for $N \to +\infty$;
- an *almost geodesic ray*, if

$$\Theta_N(\gamma) = \sup_{t,s\in[N,+\infty]} d(\gamma(t), \gamma(s)) + d(\gamma(s), \gamma(0)) - t$$

tends to zero for $N \to +\infty$.

(Notice that the second definition depends on the parametrization, while the first one is intrinsic). We will use here a notion of Busemann points which is more general than both of them:

Definition 5.1. A diverging sequence of points (p_n) in (X, d) is said to diverge *almost straightly* if for all $\epsilon > 0$ there exists L such that for every $n \geq m \geq L$ we have

$$(38) \qquad d(p_L, p_m) + d(p_m, p_n) - d(p_L, p_n) < \epsilon$$

It is easy to verify that points diverging along a quasi-ray or along an almost-geodesic ray diverge almost straightly. We then define a *Busemann point* as a horofunction f which is the limit of a sequence $f_n(x) = d(p_n, x) - d(p_n, o)$, for points (p_n) diverging almost straightly.

To prove Corollary 1.4, we need the following

Lemma 5.2. *Let (X, d) be a boundedly compact, geodesic space, $o \in X$ and $\{p_n\}_{n \in \mathbb{N}} \subset X$ a sequence of points diverging almost straightly. Then:*

(i) the sequence $f_n(x) = d(p_n, x) - d(p_n, o)$ converges uniformly on compacts to a horofunction f;

(ii) $\lim_{n \to \infty} f(p_n) + d(o, p_n) = 0$.

Proof. Since the 1-Lipschitz functions f_n are uniformly bounded on compact sets and (X, d) is boundedly compact, then the family $\{f_n\}_{n \in \mathbb{N}}$ is pre-compact with respect to the uniform convergence on compact sets. Hence, if we prove that there is a unique accumulation point, then we obtain that the whole sequence $\{f_n\}_{n \in \mathbb{N}}$ converges.

So, let $g, g' \in \mathscr{C}^0(X)$ and let $\{f_{n_k}\}_{k \in \mathbb{N}}$ and $\{f_{n'_k}\}_{k \in \mathbb{N}}$ be two sub-sequences of $\{f_n\}_{n \in \mathbb{N}}$ such that $f_{n_k} \to g$ and $f_{n'_k} \to g'$ uniformly on compact sets. We claim

$$(39) \qquad \forall \epsilon > 0 \; \exists R_\epsilon \in \mathbb{R} \; \forall x \in X \quad |g'(x) + R_\epsilon - g(x)| \leq \epsilon.$$

Let $\epsilon > 0$. Let $L \in \mathbb{N}$ be such that (38) holds. Define

$$g_L(x) := \lim_{k \to \infty} d(p_{n_k}, x) - d(p_{n_k}, p_L) = g(x) - g(p_L)$$

$$g'_L(x) := \lim_{k \to \infty} d(p_{n'_k}, x) - d(p_{n'_k}, p_L) = g'(x) - g'(p_L).$$

Then for $n_i \geq n'_j \geq L$, we get

$$d(p_{n_i}, x) - d(p_{n_i}, p_L) - d(p_{n'_j}, x) + d(p_{n'_j}, p_L)$$
$$\leq d(p_{n_i}, p_{n'_j}) - d(p_{n_i}, p_L) + d(p_{n'_j}, p_L) \leq \epsilon.$$

By taking the limit $i \to \infty$ and $j \to \infty$, we obtain for all $x \in X$

$$g_L(x) - g'_L(x) \leq \epsilon.$$

Thanks to the symmetry of the argument, also $g'_L(x) - g_L(x) \leq \epsilon$ holds. Therefore for all $x \in X$

$$\epsilon \geq |g'_L(x) - g_L(x)| = |g'(x) - g(x) + g(p_L) - g'(p_L)|.$$

Setting $R_\epsilon = g(p_L) - g'(p_L)$, we conclude the proof of claim (39).

It is now easy to conclude from (39). Indeed, taking $x = o$, we have $|R_\epsilon| \le \epsilon$, therefore for all $\epsilon > 0$ and for all $x \in X$ $|g(x) - g'(x)| \le 2\epsilon$, i.e., $g = g'$. This completes the proof of (i).

To prove assertion (ii), fix $\epsilon > 0$ and let $L \in \mathbb{N}$ be as above. Then we have for all $n \ge m \ge L$

$$
\begin{aligned}
0 \ &\le\ d(p_m, p_n) - d(p_n, o) + d(p_m, o) \\
&=\ d(p_m, p_n) + d(p_L, p_m) - d(p_L, p_n) \\
&\qquad\qquad + d(p_L, p_n) - d(p_n, o) - d(p_L, p_m) + d(p_m, o) \\
&\le\ \epsilon + d(p_L, p_n) - d(p_n, o) - d(p_L, p_m) + d(p_m, o).
\end{aligned}
$$

Taking the limit $n \to \infty$, we obtain the estimate

$$
\begin{aligned}
0 \le \liminf_{m \to \infty}\ &f(p_m) + d(p_m, o) \\
&\le \limsup_{m \to \infty} f(p_m) + d(p_m, o) \le \epsilon + f(p_L) - f(p_L) = \epsilon.
\end{aligned}
$$

Since $\epsilon > 0$ is arbitrary, then (ii) holds true. $\qquad\square$

Then, the proof of Corollary 1.4 runs similarly to Theorem 6.5 of [KN10].

Proof of Corollary 1.4. The horofunctions of type **(nv)** clearly are Busemann points, as they are limit, in particular, of the Riemannian geodesic rays which are the horizontal half-lines issued from the origin and which are always minimizing, see Proposition A.2 and Corollary A.3 in the Appendix. On the other hand, consider a horofunction of type **(v)**, $h_u = (v, z) = |u| - |u - v|$, for $u \in W$. Assume that there exists an almost straightly diverging sequence of points $p_n = v_n + z_n Z$ converging to h_u. By Lemma 5.2 (ii), we deduce that

$$
\lim_{n \to \infty} f_u(p_n) + d_R(o, p_n) = \lim_{n \to \infty} |u| - |u - v_n| + d_R(o, p_n) = 0,
$$

hence $\{v_n\}_{n \in \mathbb{N}}$ is necessarily an unbounded sequence. By Corollary 1.3 and the following description of horofunctions, it follows that h_u should be of type **(nv)**, a contradiction. $\qquad\square$

5.0.1. *Concluding remarks.* The Riemannian Heisenberg group shows a number of counterintuitive features which is worth to stress:

(i) in view of Corollary 1.3, all Riemannian metrics on $\mathbb{H}$ with the same associated distance have the same Busemann functions, though they are not necessarily isometric (in contrast, notice that all strictly subRiemannian metrics on $\mathbb{H}$ are isometric). However, this is not surprising, because all left-invariant Riemannian metrics on $\mathbb{H}$ are homothetic.

(ii) there exist diverging sequences of points $\{p_n\}_{n \in \mathbb{N}}$ that *visually converge* to a limit direction v (that is, the minimizing geodesics γ_n from o to p_n tend to a limit, minimizing geodesic γ_v with initial direction v), but whose associated limit point $h_{\{p_n\}}$ is not given by the limit

point $\gamma_v(+\infty)$ of γ_v. This happens for all vertically divergent sequences $\{p_n\}_{n\in\mathbb{N}}$, as the limit geodesic γ_v is horizontal in this case (see Proposition A.2 and Corollary A.4 in the Appendix).

(iii) there exist diverging trajectories $\{p_n\}_{n\in\mathbb{N}}$, $\{q_n\}_{n\in\mathbb{N}}$ staying at bounded distance from each other, but defining different limit points (e.g., vertically diverging sequences of points with different limit horofunctions).

(iv) it is not true that, for a cocompact group of isometries G of $(\mathbb{H}, d_R)$, the limit set of G (which is the set of accumulation points of an orbit Gx_0 in $\partial(\mathbb{H}, d_R)$) equals the whole Gromov boundary; for instance, the discrete Heisenberg group $G = \mathbb{H}(\mathbb{Z})$, has a limit set equal to the set of all Busemann points, plus a discrete subset of the interior of the disk boundary $\bar{D}^2$. Also, the limit set may depend on the choice of the base point $x_0 \in \mathbb{H}$.

(v) The functions appearing in $(\mathbf{nv})$ coincide with the Busemann functions of a Euclidean plane in the direction $R_\vartheta(\hat{v}_\infty)$; that is, the horofunction $h(v, z)$ associated to a diverging sequence $P_n = (v_n, z_n)$ of $(\mathbb{H}, d_R)$ is obtained just by dropping the vertical component z of the argument, and then applying to v the usual Euclidean Busemann function in the direction which is opposite to the limit direction of the v_n's, rotated by an angle ϑ depending on the quadratic rate of divergence of the sequence (ϑ is zero for points diverging sub-quadratically, and $\vartheta = \pm\pi$ when the divergence is sup-quadratical).

These properties mark a remarkable difference with the theory of nonpositively curved, simply connected spaces.

APPENDIX A. LENGTH-MINIMIZING CURVES FOR d_{CC} AND d_R

In the Heisenberg group, locally length-minimizing curves are smooth solutions of an Hamiltonian system both in the Riemannian and in the subRiemannian case. Locally length-minimizing curves are also called *geodesics*.

Let d_R be a Riemannian metric on $\mathbb{H}$ with horizontal space $(\mathfrak{h}, g)$. Let $V \subset \mathfrak{h}$ be the plane orthogonal to $[\mathfrak{h}, \mathfrak{h}]$ and let d_{CC} be the strictly subRiemannian metric on $\mathbb{H}$ with horizontal space $(V, g|_V)$.

Fix a basis (X, Y, Z) for $\mathfrak{h}$ such that (X, Y) is an orthonormal basis of $(V, g|_V)$ and $Z = [X, Y]$. Set $\zeta = \sqrt{g(Z, Z)}$.

Let d_{CC} be the strictly subRiemannian metric on $\mathbb{H}$ with horizontal space $(V, g|_V)$.

The basis (X, Y, Z) induces the exponential coordinates (x, y, z) on $\mathbb{H}$, i.e., $(x, y, z) = \exp(xX + yY + zZ)$. We will work in this coordinate system.

The Riemannian and subRiemannian length-minimizing curves are known and we recall their parametrization in the following two propositions.

Proposition A.1 (subRiemannian geodesics)**.** *All the non-constant locally length-minimizing curves of d_{CC} starting from 0 and parametrized by arc-length are the following: given $k \in \mathbb{R} \setminus \{0\}$ and $\theta \in \mathbb{R}$*

(TYPE I) *The horizontal lines $t \mapsto (t \cos\theta, t \sin\theta, 0)$;*
(TYPE II) *The curves $t \mapsto (x(t), y(t), z(t))$ given by*

$$\begin{cases} x(t) = \dfrac{1}{k} \left(\cos\theta(\cos(kt) - 1) - \sin\theta\sin(kt) \right) \\[2mm] y(t) = \dfrac{1}{k} \left(\sin\theta(\cos(kt) - 1) + \cos\theta\sin(kt) \right) \\[2mm] z(t) = \dfrac{1}{2k}t - \dfrac{1}{2k^2}\sin(kt) \end{cases}$$

Here the derivative at $t = 0$ is $(-\sin\theta, \cos\theta, 0)$.

Proposition A.2 (Riemannian geodesics)**.** *All non-constant locally length-minimizing curves of d_R parametrized by a multiple of arc-length and starting from 0 are the following: given $k \in \mathbb{R} \setminus \{0\}$ and $\theta \in \mathbb{R}$*

(TYPE 0) *The vertical line $t \mapsto (0, 0, t)$;*
(TYPE I) *The horizontal lines $t \mapsto (t \cos\theta, t \sin\theta, 0)$;*
(TYPE II) *The curves $t \mapsto (x(t), y(t), z(t))$ given by*

$$\begin{cases} x(t) = \dfrac{1}{k} \left(\cos\theta(\cos(kt) - 1) - \sin\theta\sin(kt) \right) \\[2mm] y(t) = \dfrac{1}{k} \left(\sin\theta(\cos(kt) - 1) + \cos\theta\sin(kt) \right) \\[2mm] z(t) = \dfrac{1}{2k}t - \dfrac{1}{2k^2}\sin(kt) + \dfrac{k}{\zeta^2}t \end{cases}$$

Here the derivative at $t = 0$ is $(-\sin\theta, \cos\theta, \frac{k}{\zeta^2})$, which has Riemannian length $\sqrt{1 + \frac{k^2}{\zeta^2}}$.

The expression of geodesics helps us to prove the following facts.

Corollary A.3. *The horizontal lines of* TYPE I *are globally d_R- and d_{CC}-length-minimizing curves.*

Corollary A.4. *Both d_R- and locally d_{CC}-length-minimizing curves γ of* TYPE II *are not minimizing from 0 to $\gamma(t)$ if $|t| > \frac{2\pi}{k}$.*

Proof. This statement depends on the fact that, if we fix $k \in \mathbb{R} \setminus \{0\}$, then for all θ the corresponding length-minimizing curves $\gamma_{k,\theta}$ of TYPE II meet each other at the point $\gamma_{k,\theta}(2\pi/k) = (0, 0, \frac{2\pi}{k^2})$ or $\gamma_{k,\theta}(2\pi/k) = (0, 0, \frac{2\pi}{k^2} + \frac{2\pi}{\zeta^2})$. □

Corollary A.5. *The locally d_R-length-minimizing curve γ of* TYPE 0, *$t \mapsto (0, 0, t)$, is not minimizing from 0 to $\gamma(t)$ for $|t| > \frac{2\pi}{\zeta}$.*

Proof. For $k > 0$ let γ_k be the d_R-length-minimizing curve of TYPE II with this k and $\theta = 0$. Then $(\gamma_k)_3(\frac{2\pi}{k}) = \frac{\pi}{k^2} + \frac{2\pi}{\zeta^2}$. Letting $k \to \infty$ we obtain $\hat{z} := \frac{2\pi}{\zeta^2}$. This means that for every $\epsilon > 0$ there is $z \le \hat{z} + \epsilon$ and $k > 0$ such that $\gamma_k(\frac{2\pi}{k}) = (0, 0, z)$. Therefore $t \mapsto (0, 0, t)$ cannot be minimizing after z, and therefore after $\hat{z}$. $\square$

Corollary A.6. *If $p = (x, y, p_3)$ and $q = (x, y, q_3)$, then*
$$d_{CC}(p, q) = 2\sqrt{\pi} \cdot \sqrt{|p_3 - q_3|}.$$

Proof. First suppose $p = 0$: we have to prove that $d_{CC}(0, (0, 0, z)) = 2\sqrt{\pi}\sqrt{|z|}$. This is done by looking at the length-minimizing curves: they comes from complete circle of perimeter $2\pi R = d$ and area $\pi R^2 = |z|$, so that $d_{CC}(0, (0, 0, z)) = 2\pi\sqrt{\frac{|z|}{\pi}} = 2\sqrt{\pi}\sqrt{|z|}$.

The general case follows from the left-invariance of d_{CC}:
$$d_{CC}((x, y, p_3), (x, y, q_3)) = d_{CC}(0, (x, y, p_3)^{-1}(x, y, q_3))$$
$$= d_{CC}(0, (0, 0, q_3 - p_3)).$$
$\square$

Corollary A.7. *If $p \in \{z = 0\}$, then*
$$d_{CC}(0, p) = d_R(0, p)$$

Corollary A.8. *d_R- and d_{CC}-length-minimizing curves of* TYPE II *are in bijection with the following rule: If $\eta : [0, T] \to \mathbb{H}$ is a d_{CC}-length-minimizing curve of* TYPE II, *then*
$$\gamma(t) = \eta(t) + \left(0, 0, \frac{kt}{\zeta^2}\right)$$
is d_R-length-minimizing of TYPE II, *where $k \in \mathbb{R}$ is given by η. Moreover, it holds*
$$\|\omega_{\mathbb{H}}(\gamma')\|^2 = 1 + \frac{k^2}{\zeta^2}$$
and
$$d_{CC}(\gamma(t), \eta(t)) = 2\sqrt{\pi}\sqrt{\frac{kt}{\zeta^2}}$$

Proof. All the statements come directly from the expression of the geodesics. Notice that a d_{CC}-length-minimizing curve η of TYPE II is parametrized by arc-length, i.e., $\|\omega_{\mathbb{H}}(\eta')\| \equiv 1$.

On the other hand, the corresponding d_R-length-minimizing curve γ has derivative $\omega_{\mathbb{H}}(\gamma') = \omega_{\mathbb{H}}(\eta') + \frac{k}{\zeta^2}Z$, where $\omega_{\mathbb{H}}(\eta')$ is orthogonal to Z. Hence $\|\omega_{\mathbb{H}}(\gamma')\|^2 = 1 + \frac{k^2}{\zeta^2}$ $\square$

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