Preservation of p-Poincaré inequality for large p under sphericalization and flattening

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Abstract

Li and Shanmugalingam showed in [20] that annularly quasiconvex metric spaces endowed with a doubling measure preserve the property of supporting a p-Poincaré inequality under the sphericalization and flattening procedures. Because natural examples such as the real line or a broad class of metric trees are not annularly quasiconvex, our aim in the present paper is to study under weaker hypothesis on the metric space, the preservation of p-Poincaré inequalites under those conformal deformations for sufficiently large p. We propose similar hypothesis to the ones used in [9], where the preservation of ∞-Poincaré inequality has been studied under the assumption of radially star-like quasiconvexity (for sphericalization) and meridian-like quasiconvexity (for flattening). To finish, using the sphericalization procedure, we exhibit an example of a Cheeger differentiability space whose blow up at a point is not a PI space.

Keywords: Sphericalization, flattening, doubling, Poincaré inequality, quasiconvexity, annular quasiconvexity.

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1 Introduction

One of the cornerstones in the development of first order calculus in the metric measure setting has been the concept of metric space equipped with a doubling measure and supporting a Poincaré inequality. If a space or domain supports a Poincaré inequality, many fruitful geometric properties can be deduced, including the existence of non-trivial differentiable structures. Therefore, it is valuable to explore which metric spaces enjoy such properties. For a general introduction to the subject one can look at [2], [12], [13] or [15].

A common way to construct new metric spaces from old ones is to use conformal deformations. This means to construct a new metric space, which is homeomorphic to the original one, by endowing the old space with a new density function. In order to preserve some certain geometric properties, the measure also plays an important role and should be altered in a similar way. A natural problem is to study the preservation of the doubling property and the Poincaré inequality under these deformations. In the present paper, two types of conformal deformations are considered: sphericalization and flattening.

Sphericalization and flattening are dual transformations in the sense that if one starts from a bounded metric space, then performs a flattening transformation followed by a sphericalization transformation, then the resulting metric space is biLipschitz equivalent to the original space. Furthermore, starting from an unbounded metric space, the performance of sphericalization followed by a flattening transformation leads to a metric space that is biLipschitz equivalent to the original.

The idea of sphericalization and flattening was first considered in the paper of Buckley and Balogh [1] and further studied in [4] and [16]. Within the paper, two types of conformal deformation were introduced in order to generalize the stereographic projection between the Riemann sphere and the complex plane. Their motivation comes from comparing quasihyperbolic metric of a domain (which are considered in Bonk, Heinonen and Koskela [3]) with two types of metric, the length metric and the sphericalized metric on the domain.

The preservation of p-Poincaré inequality under these conformal deformations for the case $p < \infty$ was first studied in Li and Shanmugalingam [20], assuming that the original space is annularly quasiconvex. By a result in Korte [19], spaces supporting a p-Poincaré inequality for sufficiently small $p \geq 1$ are necessarily annularly quasiconvex. It was shown in [20] that the property of annular quasiconvexity cannot be removed in their results regarding preservation of the property of supporting a p-Poincaré inequality, and the authors of [20] pose whether the assumption of annular quasiconvexity is necessary for preserving a p-Poincaré inequality for sufficiently large p.

At this point, it is important to highlight the role that the exponent p plays in p-Poincaré inequalities. The larger the p, the weaker the inequality and the limiting case, the ∞-Poincaré inequality, would be the weakest. We refer the interested reader to [11] for several examples of spaces supporting a p-Poincaré inequality for some but not all values of p in the range $[1, \infty]$. In [9], the preservation of quasiconvexity and ∞-Poincaré inequality has been studied under a weaker assumption, namely, *radially star-like quasiconvexity* and *meridian-like quasiconvexity*. The motivation for introducing these new definitions comes from the fact that there are simple examples that are not annularly quasiconvex but still support a Poincaré inequality, as the real line $\mathbb{R}$ or $\mathbb{S}^1$ when endowed with the length metric. The definition of such properties are inspired by the paper [3], where the authors considered the duality of uniform domains and Gromov hyperbolic spaces and use the concept of *rough star-likeness*. While these new notions are too weak to be used in the setting considered in the current paper, here we consider metric spaces that have a modification of these weak notions which still yield the preservation of the property of supporting a p-Poincaré inequality for sufficiently large p. The notions we consider in this paper are still weaker than the annular quasiconvexity considered in [20].

The different nature of p-Poincaré inequality for finite p versus ∞-Poincaré inequality makes that the techniques used in [20] differ from the ones used in [9]. In [20] a version of chaining arguments found in [6] are used. In [9], which considers the case $p = \infty$, a purely geometric characterization of ∞-Poincaré inequality, proved in [10] and based on a stronger version of quasiconvexity is used instead.

For a metric space supporting a doubling measure there are two exponents related to the doubling measure, the relative upper bound exponent t and the relative lower bound exponent s with $t \leq s$ in general. In the present paper, we improve part of the results in [20], namely, the preservation of p-Poincaré inequality under sphericalization and flattening for $p > s$, under the weaker assumptions of (a refinement of) radially star-like quasiconvexity, for sphericalization, and (a refinement of) meridian-like quasiconvexity, for flattening. On the other hand, it is well-known that Ahlfors Q-regular spaces that support a a p-Poincaré inequality for some $1 \leq p < Q$ are annularly quasiconvex when $Q > 1$ (see [19]). Notice that in this case $t = s = Q$ and therefore, Ahlfors Q-regular spaces preserve the p-Poincaré inequality for $p > 1$ under sphericalization and flattening procedures for the weaker assumptions that we propose.

On the other hand, it is an open question (see for example [8]) whether a blow-up of a differentiability space (in the sense of Cheeger) must be a *PI space*, that is, a metric space with a doubling measure and a p-Poincaré inequality for some $p < \infty$. To finish we exhibit in Example 5.1, using the sphericalization procedure, that this is not always the case.

The paper is organized as follows: in Section 2, basic notations and definitions will be introduced; in Section 3, preservation of p-Poincaré inequality for $p > s$ under sphericalization for (a refinement of) radially star-like quasiconvex spaces) will be proved (see Theorem 3.1). In Section 4 preservation of p-Poincaré inequality for $p > s$ under flattening for (a refinement of) meridian-like quasiconvex spaces) will be presented (see Theorem 4.1). Last section, Section 5, shows an example of a differentiability space whose blow-up is not a PI space.

2 Notation and preliminaries

In this section we gather the key notions, definitions and notations that will be used throughout the paper.

2.1 Curves in metric spaces

Let (X, d) be a metric space. We denote open balls centered at $x \in X$ and of radius $r > 0$ by $B(x, r) := \{y \in X : d(x, y) < r\}$ and closed balls by $\overline{B}(x, r) := \{y \in X : d(x, y) \leq r\}$. For $\lambda > 0$, λB denotes the ball concentric with B (with respect to a predetermined center) but with radius λ-times the radius of B. For $0 < r < R$, $A(a, r, R)$ denotes the *annulus* $A(a, r, R) := \overline{B}(a, R) \setminus B(a, r)$.

Given a continuous map (also known as *curve*) $\gamma : I \to X$, where $I = [a, b]$ for some $a, b \in \mathbb{R}$ with $a < b$, we denote the *length* of γ with respect to the metric

d by

$$\ell_d(\gamma) := \sup \sum_{k=0}^{n-1} d(\gamma(t_k), \gamma(t_{k+1})),$$

where the supremum is taken over all partitions $a = t_0 < t_1 < \cdots < t_n = b$ of the interval $[a, b]$. A curve γ is *rectifiable* if $\ell_d(\gamma) < \infty$. We simply write $\ell(\gamma)$ if the metric is clear from the context. Given two points $x, y \in X$, γ_{xy} denotes a curve connecting x to y.

For a rectifiable curve $\gamma\colon [a, b] \to X$, let $s_\gamma\colon [a, b] \to [0, \ell(\gamma)]$ be the associated *length function*. That is, $s_\gamma(t) = \ell(\gamma|[a, t])$. There exists a unique (1-Lipschitz continuous) map $\gamma_s\colon [0, \ell(\gamma)] \to X$ such that $\gamma = \gamma_s \circ s_\gamma$. The curve γ_s is called the *arc length parametrization* of γ. If γ is a rectifiable curve in X, the line integral over γ of a Borel function $\rho\colon X \to [0, \infty]$ is defined by

$$\int_\gamma \rho ds := \int_0^{\ell(\gamma)} (\rho \circ \gamma_s)(t)dt. \tag{2.1}$$

A metric space (X, d) is said to be C-*quasiconvex* if there exists $C \geq 1$ such that for every pair of points x and y there exists a rectifiable curve γ with $\ell_d(\gamma) \leq Cd(x, y)$. A related notion to quasiconvexity is that of annular quasiconvexity, a notion introduced in [19] and has been further used for example in [4], [12] and [16]. We say that X is A-*annularly quasiconvex* with respect to a base point $a \in X$ if there exists $A \geq 1$ such that for every $r > 0$, and for each pair of points $x, y \in A(a, r/2, r)$ there is a curve γ_{xy} connecting x to y inside the annulus $A(a, r/A, Ar)$ with $\ell_d(\gamma) \leq Ad(x, y)$. We say that X is *annularly quasiconvex* if there exists $A \geq 1$ such that X is A-annularly quasiconvex for every $a \in X$.

2.2 Metric Measure spaces

A metric space endowed with a Borel measure μ is called a metric measure space, that is, (X, d, μ) will denote a *metric measure space*. We say that the measure μ is *doubling* if balls have finite positive measure and there is a constant $C_\mu \geq 1$ such that

$$\mu(2B) \leq C_\mu \mu(B) \tag{2.2}$$

for all balls B.

Condition (2.2) implies that there are constants $C > 0$ and $s > 0$, depending only on C_μ, such that

$$\frac{\mu(B(x,r))}{\mu(B(y,R))} \geq C\left(\frac{r}{R}\right)^s \tag{2.3}$$

whenever $0 < r \leq R$ and $x \in B(y,R)$. See [12] for a proof of this. In this case we also say that X has a *relative lower volume decay* of order $s > 0$.

If the measure is doubling and the space is connected, then there exists an exponent $t > 0$ and constant $C > 0$ such that

$$\frac{\mu(B(x,r))}{\mu(B(y,R))} \leq C\left(\frac{r}{R}\right)^t \tag{2.4}$$

for $0 < r \leq R \leq \dim X/2$ and $x \in B(y,R)$. In general, we have $s \geq t$, and we say X has a *relative upper volume decay* of order $t > 0$.

2.3 First-order calculus in metric measure spaces

Given a real-valued function u in a metric space X, a Borel function $g\colon X \to [0,\infty]$ is an *upper gradient* of u if

$$|u(x) - u(y)| \leq \int_\gamma g ds,$$

for each rectifiable curve γ connecting x to y in X.

Given $1 \leq p < \infty$, we say that (X,d,μ) supports a *p-Poincaré inequality* if each ball in X has finite and positive measure and there are constants $C, \lambda > 0$ such that for every open ball B in X, for every measurable function u on B, and for every upper gradient g of u we have

$$\frac{1}{\mu(B)}\int_B |u - u_B| d\mu \leq C\mathrm{rad}(\lambda B)\Big(\frac{1}{\mu(\lambda B)}\int_{\lambda B} g^p d\mu\Big)^{1/p}.$$

Here, for arbitrary $A \subset X$ with $0 < \mu(A) < \infty$ we write $u_A = \frac{1}{\mu(A)}\int_A u\, d\mu$.

The ∞-Poincaré inequality can be seen when we replace the L^p norm of the right hand side by L^∞ norm.

The following result due to Keith [17] states that to verify a p-Poincaré inequality it suffices to verify the inequality for Lipschitz functions and their continuous upper gradients.

Lemma 2.5. **[17, Theorem 2]** *Let $p \geq 1$ and let (X,d,μ) be a complete metric measure space with μ doubling. Then the following conditions are quantitatively equivalent:*

(*a*) (X, d, μ) *admits the* p*-Poincaré inequality for all measurable functions and their upper gradients.*

(*b*) (X, d, μ) *admits the* p*-Poincaré inequality for all compactly supported Lipschitz functions and their compactly supported Lipschitz upper gradients.*

By the work of Cheeger [7], metric measure spaces endowed with a doubling measure and supporting a p-Poincaré inequality for $p < \infty$ have a very rich infinitesimal "linear" structure that allows to state the Rademacher differentiability theorem in this context.

The interested reader can find in [15] a discussion of the recent advances in the field of analysis on metric measure spaces, including those in [10] and [20] (see [15, Chapter 14]).

2.4 Sphericalization and flattening

The concept of sphericalization and flattening are natural analogs of the stereographic projection between the Riemann sphere and the complex plane. As pointed out in the Introduction, they were introduced by Balogh and Buckley in [1] and further studied in [4] and [16].

For an unbounded locally compact metric space X, we denote its one-point compactification $\dot{X} = X \cup \{\infty\}$.

Definition 2.6 (Sphericalization)**.** Given a complete unbounded metric space (X, d) and a base point $a \in X$, we consider the following density function $d_a : \dot{X} \times \dot{X} \to [0, \infty)$ given by

$$d_a(x, y) = \begin{cases} \frac{d(x,y)}{[1+d(x,a)][1+d(y,a)]} & \text{if } x, y \in X, \\ \frac{1}{1+d(x,a)} & \text{if } x \in X,\ y = \infty, \\ 0 & \text{if } x = \infty = y. \end{cases} \tag{2.7}$$

Although d_a is not a metric since it is possible to violate the triangular inequality, there exists a metric $\hat{d}_a$ on $\dot{X}$ whose metric topology agrees with the topology of $\dot{X}$ and satisfying

$$\frac{1}{4} d_a(x, y) \leq \hat{d}_a(x, y) \leq d_a(x, y) \tag{2.8}$$

for all $x, y \in \dot{X}$ (see [5, Lemma 2.2]).

The metric space $(\dot{X}, \hat{d}_a)$ is said to be the *sphericalization* of (X, d). Since there is no closed form formula for $\hat{d}_a$, for convenience we will use d_a in defining balls in $\dot{X}$. Balls in $\dot{X}$, with respect to d_a, will be denoted $B_a = B_a(x, r)$, while the balls in X, with respect to the original metric d, will be denoted $B = B(x, r)$. Notice that the density function d_a we use satisfies the condition of *sphericalizing function* $g(t) = (1+t)^{-2}$ as in [1]. Furthermore, observe that $\operatorname{diam}(\dot{X}) = 1$.

The operation of flattening, which is dual to the procedure of sphericalization, can be defined analogously. In the flattening procedure, we begin with a bounded metric space and remove a point to construct an unbounded metric space.

Definition 2.9 (Flattening)**.** Given a complete bounded metric space (X, d) with a base point $c \in X$, we consider the metric space $X^c = X \setminus \{c\}$, with a density function $d^c : X^c \times X^c \to [0, \infty)$ defined by

$$d^c(x, y) = \frac{d(x, y)}{d(x, c)d(y, c)} \quad \text{if } \ x, y \in X^c.$$

Just as in the case of sphericalization, the density function d^c is not a metric, but by [4, Lemma 3.2], we have a metric space $(X^c, \bar{d})$ associated to d^c with

$$\frac{1}{4} d^c(x, y) \le \bar{d}(x, y) \le d^c(x, y)$$

for all $x, y \in X^c$.

The metric space $(X^c, \bar{d})$ is said to be a *flattening* of (X, d). Balls in X^c, with respect to the metric d^c, will be denoted $B^c(x, r)$, while the balls in X, with respect to the metric d, will be denoted as usual by $B(x, r)$.

In the sequel, it will be also useful to know how a curve and its corresponding length change under the sphericalization and flattening processes. Let γ be a rectifiable curve in a rectifiably connected unbounded metric space X. Under sphericalization γ corresponds to $\dot{\gamma} \colon [0, \ell(\gamma)] \to \dot{X}$ defined by $\dot{\gamma}(t) = \gamma_s(t)$, where γ_s is the arc-length parametrization of γ with respect to the original metric d. By an abuse of notation we will denote the corresponding curve in $\dot{X}$ by γ as well. One can check (see [1, Proposition 2.6]) that γ is rectifiable with respect to the metric $\widehat{d_a}$ if it is rectifiable with respect to the original metric d.

Then length $\ell_{d_a}(\gamma)$ of γ with respect to "the metric" d_a is is given by the formula

$$\ell_{d_a}(\gamma) = \int_0^{\ell(\gamma)} \frac{1}{[1 + d(\gamma_s(t), a)]^2}\, ds(t)$$

whereas the formula for the length $\ell_{d^c}(\gamma)$ of γ with respect to "the metric" d^c is given by

$$\ell_{d^c}(\gamma) = \int_0^{\ell(\gamma)} \frac{1}{d(\gamma_s(t), c)^2}\, ds(t)$$

In the next lemma we explain how upper gradients are transformed under sphericalization. Note that a function that is Lipschitz continuous on X will be locally Lipschitz continuous on $\dot{X} \setminus \{\infty\}$, and a function that is Lipschitz continuous on $\dot{X}$ is necessarily Lipschitz continuous on X.

Lemma 2.10. [20, Lemma 3.3.1] *Suppose that u is a Lipschitz function on $\dot{X}$. If g is an upper gradient of u in X, then the function $\hat{g}$ given by*

$$\hat{g}(x) = g(x)(1 + d(x, a))^2 \tag{2.11}$$

and extended by setting $\hat{g}(\infty) = 0$ is an upper gradient of u in $\dot{X}$. Furthermore, if h is an upper gradient of a function v in $\dot{X}$, then the function $\check{h}$ given by

$$\check{h}(x) = \frac{h(x)}{(1 + d(x, a))^2} \tag{2.12}$$

is an upper gradient of v in X.

The current work focuses on the preservation of Poincaré inequalities in the setting of metric *measure* spaces under sphericalization and flattening, so we also need to transform the measure on X in a manner compatible with the change in the metric.

Definition 2.13. Suppose (X, d) is proper space equipped with a Borel-regular measure μ such that the measures of non-empty open bounded sets are positive and finite. We consider the *spherical measure* μ_a defined on $\dot{X}$ as follows. For $A \subset \dot{X}$, the measure $\mu_a(A)$ is given by

$$\mu_a(A) = \int_{A \setminus \{\infty\}} \frac{1}{\mu(B(a, 1 + d(z, a)))^2} d\mu(z).$$

We next define the transformation μ^c of the measure μ under flattening. In this case, X is a bounded metric space equipped with a Borel-regular measure μ.

Definition 2.14. The *flattened measure* μ^c corresponding to (X^c, d^c) is given by

$$\mu^c(A) = \int_A \frac{1}{\mu(B(c, d(c, z)))^2} d\mu(z),$$

whenever $A \subset X^c$ is a Borel set.

As shown in [1], the metric space resulting from flattening with respect to the point $\{\infty\}$ the (bounded) sphericalized space $(\dot{X}, \hat{d}_a)$ is bi-Lipschitz equivalent to the (unbounded) space (X, d), making sphericalization and flattening dual transformations. The following lemma, due to N. Shanmugalingam, shows the corresponding result for measures. We are grateful to N. Shanmugalingam for allowing us to include the result here.

Lemma 2.15. *Let X be a connected, unbounded, complete metric measure space and μ be a doubling measure on X. Let $a \in X$ and $X_a = X \cup \{\infty\}$ be the sphericalization of X with respect to the base point a, and X_a^∞ be the flattening of X_a with respect to the base point ∞. Then $\mu_a^\infty \approx \mu$, that is, there is a constant $C > 0$ such that for* all $x \in X$,

$$\frac{1}{C} d\mu(x) \leq d\mu_a^\infty(x) \leq C\, d\mu(x),$$

and μ, μ_a^∞ are mutually absolutely continuous.

Proof. The fact that μ and μ_a^∞ are absolutely continuous with respect to each other is clear from the definitions of μ_a and μ_a^∞.

Note that

$$\begin{aligned} d\mu_a^\infty(x) &= \frac{d\mu_a(x)}{\mu_a(B_a(\infty, d_a(x,\infty)))^2} \\ &= \frac{d\mu(x)}{\mu_a(B_a(\infty, d_a(x,\infty)))^2\, \mu(B(a, 1+d(x,a)))^2}. \end{aligned}$$

Thus we consider $\mu_a(B_a(\infty, d_a(x,\infty)))$. Observe that $y \in B_a(\infty, d_a(x,\infty))$ if and only if $d_a(y,\infty) < d_a(x,\infty)$, that is, $d(y,a) > d(x,a)$. It follows that

$$B_a(\infty, d_a(x,\infty)) = X \setminus \overline{B}(a, d(x,a)).$$

The rest of the proof is divided into two cases.

Case 1: $d(x,a) > 1/C$. In this case, note that for $y \in X \setminus B(a, d(x,a))$ we have that $1 + d(y,a) \approx d(y,a)$ and hence by the doubling property of μ, we also have $\mu(B(a, 1+d(y,a))) \approx \mu(B(a, d(y,a)))$. For non-negative integers j we set

$B_j = B(a, 2^j d(x,a))$. Then by the doubling property of μ,

$$\begin{aligned}\mu_a(B_a(\infty, d_a(\infty, x))) &\approx \int_{X\setminus \overline{B}(a,d(x,a))} \frac{1}{\mu(B(a,d(y,a)))^2}\, d\mu(y)\\ &= \sum_{j=0}^{\infty} \int_{B_{j+1}\setminus \overline{B}_j} \frac{1}{\mu(B(a,d(y,a)))^2}\, d\mu(y)\\ &\approx \sum_{j=0}^{\infty} \frac{\mu(B_{j+1}\setminus \overline{B}_j)}{\mu(B_j)^2}.\end{aligned}$$

By (2.3) and (2.4), there are positive constants t, s (which are independent of j, x) such that $\mu(B_{j+1}\setminus \overline{B}_j) \approx \mu(B_j)$ and

$$\frac{2^{sj}}{C} \le \frac{\mu(B_j)}{\mu(B_0)} \le C\, 2^{tj}. \tag{2.16}$$

Using this, we obtain

$$\mu_a(B_a(\infty, d_a(\infty, x))) \approx \sum_{j=0}^{\infty} \frac{1}{\mu(B_j)},$$

with

$$\frac{1}{C\mu(B_0)} \sum_{j=0}^{\infty} 2^{-sj} \le \sum_{j=0}^{\infty} \frac{1}{\mu(B_j)} \le \frac{C}{\mu(B_0)} \sum_{j=0}^{\infty} 2^{-tj}.$$

It follows from the assumption $d(x,a) > 1/C$ that

$$\mu_a(B_a(\infty, d_a(x,\infty))) \approx \frac{1}{\mu(B_0)} = \frac{1}{\mu(B(a,d(x,a)))} \approx \frac{1}{\mu(B(a,1+d(x,a)))},$$

that is, $d\mu_a^\infty(x) \approx d\mu(x)$ when $d(x,a) > 1/C$.

Case 2: $d(x,a) \le 1/C$. In this case we have $1 + d(x,a) \approx 1$, and so by the doubling property of μ,

$$\mu(B(a, 1+d(x,a))) \approx \mu(B(a,1)).$$

For non-negative integers j we now choose $B_j = B(a, 2^j)$. Then

$$\begin{aligned}\mu_a(B_a(\infty, d_a(x,\infty))) &= \int_{X\setminus \overline{B}(a,d(x,a))} \frac{1}{\mu(B(a,1+d(y,a)))^2}\, d\mu(y)\\ &\approx \int_{B(a,1)\setminus \overline{B}(a,d(x,a))} \frac{1}{\mu(B(a,1))^2}\, d\mu(y)\\ &\qquad + \sum_{j=0}^{\infty} \int_{B_{j+1}\setminus \overline{B}_j} \frac{1}{\mu(B(a,1+d(y,a)))^2}\, d\mu(y).\end{aligned}$$

Since $d(x,a) \leq 1/C$, we have

$$\mu(B(a,1) \setminus \overline{B}(a, d(x,a))) \approx \mu(B(a,1)),$$

and for $y \in B_{j+1} \setminus B_j$ we also have that

$$\mu(B(a, 1 + d(y,a))) \approx \mu(B_j).$$

Hence

$$\mu_a(B_a(\infty, d_a(x,\infty))) \approx \frac{1}{\mu(B(a,1))} + \sum_{j=0}^{\infty} \frac{\mu(B_{j+1} \setminus \overline{B}_j)}{\mu(B_j)^2}.$$

An application of (2.16) to the above now yields

$$\mu_a(B_a(\infty, d_a(x,\infty))) \approx \frac{1}{\mu(B(a,1))} \approx \mu(B(a, 1 + d(x,a))).$$

It now follows that $d\mu_a^\infty(x) \approx d\mu(x)$ even when $d(x,a) \leq 1/C$.

This completes the proof of the lemma. □

2.5 Radially star-like quasiconvex spaces and meridian-like quasiconvex spaces

The notions of radially star-like and meridian-like quasiconvexity were introduced in [9] to investigate the preservation of ∞-Poincaré inequality under the transformations of sphericalization and flattening. In order to deduce the preservation of p-Poincaré inequality for a finite p, we need to consider a refinement version of these properties named *refinement of radially star-like quasiconvexity* (in the case of sphericalization) and *refinement of meridian-like quasiconvexity* (in the case of flattening).

Definition 2.17. A metric space is a *refinement of K-radially star-like quasiconvex* with respect to the base point $a \in X$ if there exist a constant $K \geq 1$, a fixed radius $r_0 > 0$, $N_0 \in \mathbb{N}$ and a collection of base-point quasiconvex rays $\beta_1, \beta_2, \cdots, \beta_{N_0}$ connecting a to ∞ such that for every $r > r_0$ and $x \in A(a, r/2, r)$ there exists $z \in \beta_i$ for some $i \in \{1, 2, \cdots N_0\}$ and a quasiconvex curve $\gamma_{xz} \subset A(a, r/K, Kr)$ connecting x to z such that

$$\ell(\gamma_{xz}) \leq K d(a,z).$$

Here we say that a ray $\gamma : [0, \infty) \to X$ with $\gamma(0) = a$ is *base-point quasiconvex* if for each $z \in |\gamma|$, $\ell(\gamma_{az}) \leq Cd(a, z)$, where γ_{az} is the subcurve of γ connecting a to z.

In the definition of *radially star-like quasiconvexity* introduced in [9] we connect every point (in the corresponding annulus) to a base-point quasiconvex ray (depending on the point) by a controlled quasiconvex curve. Notice that if (X, d) is a connected complete locally compact metric space which is quasiconvex and annularly quasiconvex with respect to a point $a \in X$, then (X, d) is a refinement of K-radially star-like quasiconvex with $N_0 = 1$. See [9, Lemma 3.3].

Definition 2.18. A (bounded) metric space is a *refinement of K-meridian-like quasiconvex* with respect to a base point $c \in X$, if there exists a constant $K \geq 1$, a fixed radius $r_0 > 0$, a point $a \in X$ with $4d(a, c) \geq \mathrm{diam}(X)$, and a collection of double base-point quasiconvex curves $\beta_1, \beta_2, \cdots, \beta_{N_0}$ with respect to base points a and c, and connecting a to c such that for every $x \in A(c, r/2, r)$ and $r_0 \geq r$, there exists $z \in \beta_i$ for some $i \in \{1, 2, \cdots N_0\}$ and a quasiconvex curve $\gamma_{xz} \subset A(c, r/K, Kr)$ connecting x to z such that

$$\ell(\gamma_{xz}) \leq Kd(x, c).$$

By *double base-point quasiconvex curve* we mean that for any $z \in |\gamma_{ac}|$, $\ell(\gamma_{cz}) \leq Cd(c, z)$ and $\ell(\gamma_{az}) \leq Cd(a, z)$. Here γ_{az} and γ_{cz} denote the subcurves of γ_{ac} with end points a and z and c and z respectively.

In a similar fashion to the definition of radially star-like quasiconvexity, the definition of *meridian-like quasiconvexity* introduced in [9] requires for every point x (in the corresponding annulus) the existence of a double base-point quasiconvex curve (depending on x) and a controlled quasiconvex curve connecting x to the base-point quasiconvex curve. In general there may be a need for infinitely many such double base-point quasiconvex curves, thus the above notion is a refinement of the one from [9], where a fixed finite number of such curves serve all the points in X.

Remark 2.19. The idea is to choose the point $a \in X$ (in Definition 2.18) in $A(c, R/2, R)$ where $R = \sup_{z \in X} d(c, z)$. If this is the case, $2d(a, c) \geq \sup_z d(c, z) \geq \mathrm{diam}(X)/2$. Additionally, when $0 < r \ll R$ and $x \in B(c, r)$, we have $d(x, a) \approx$

$d(a,c)$. Indeed, for $x \in B(c,r)$, we have that

$$2d(a,c) > d(a,c) + d(x,c) \geq d(a,x) \geq d(a,c) - d(x,c) \geq d(a,c) - r \approx d(a,c).$$

Notice that if (X,d) is a bounded connected complete locally compact metric space which is annularly quasiconvex with respect to a point $c \in X$, then (X,d) is a refinement of K-meridian-like quasiconvex with respect to c. See [9, Lemma 4.3].

Remark 2.20. It is possible to show that sphericalization of unbounded spaces having a refinement of radially star-like quasiconvexity property will result in a bounded space endowed with a refinement of meridian-like quasiconvexity, and vice versa. In fact, these two concepts are dual to each other via the dual transformations of sphericalization and flattening. The idea of proof is essentially the same as the proof of [9, Lemma 4.6, Lemma 4.7].

It seems to be unnatural to require that we need only finitely many base-point quasiconvex rays (or double base-point quasiconvex curves), especially when we assume the doubling property of the metric space. Since the assumption of doubling measure ensures that the number of balls of radius $r/2$ covering the balls $B(x,r)$ is controlled by the doubling constant, the refinements seem to be redundant. However, we need for the proofs a decomposition of the metric space in a good order for each annulus that so far we are only able to obtain under the additional refinement conditions. See (3.8) and (4.3) for the technical details.

Unless otherwise stated, the letter C denotes various positive constants whose exact values are not important for the purposes of this paper, and its value might change even within a line.

3 Preservation of p-Poincaré inequality for $p > s$ under sphericalization

Li and Shanmugalingam proved in [20, Theorem 3.3.5] the preservation of p-Poincaré inequality ($1 \leq p < \infty$) under sphericalization for annular quasiconvex spaces. In what follows we show the preservation of p-Poincaré inequality under sphericalization for p sufficiently large for metric spaces satisfying the refinement of radially star-like quasiconvexity (see Definition 2.17). Metric spaces that are not annular quasiconvex but are radially star-like quasiconvex are for example the real line, the Euclidean infinite strip $\mathbb{R} \times [-1, 1]$ or some classes of metric trees.

Theorem 3.1. *Let (X, d, μ) be a complete unbounded metric space with a doubling measure μ so that (X, d, μ) supports a p-Poincaré inequality for some $p > s$, where s is the exponent of relative lower volume decay associated to μ as in* (2.3)*. Let $a \in X$ be a base point in X, and assume (X, d) is a refined K-radially star-like quasicovex with respect to a for some $K \geq 1$. Then $(\dot{X}, d_a, \mu_a)$ also supports a p-Poincaré inequality.*

Remark 3.2. Notice that we need $p > s$, which is associated to the exponent s related to the original measure μ rather than the spherical measure μ_a from (2.3). See Example 3.16 below.

Proof. We need to verify p-Poincaré inequality for balls $B_a(x, r)$ with $x \in \dot{X}$ and $r > 0$. We divide the proof into three different cases: balls far away from ∞ (whose behavior is similar to the balls in the original metric), balls centered at ∞, and more general balls. We assume $0 < r < 1/(10\lambda K^2)$, where λ is the scaling constant involved in the Poincaré inequality and K is the constant in the refinement of radially star-like quasiconvex property, because balls with radius $r \geq 1/(10\lambda K^2)$ can be compared to balls centered at ∞ with radius 1, that is, balls that are equal to $\dot{X}$. Indeed, we will prove the Poincaré inequality for balls centered at ∞ in Case 2 without restricting the radius r in that case.

Let $u \in \mathrm{Lip}(\dot{X})$ and let g be an upper gradient of u in X with respect to the original metric d.

Case 1: $d_a(x, \infty) \geq 8\lambda r$. We choose a positive integer $k_0 \geq 3$ so that

$$2^{k_0}\lambda r \leq d_a(x, \infty) = \frac{1}{1 + d(x, a)} \leq 2^{k_0+1}\lambda r. \tag{3.3}$$

Then $1/(2^{k_0+1}\lambda r) \leq 1 + d(x, a) \leq 1/(2^{k_0}\lambda r)$. If $y \in X$ such that $d_a(x, y) < r$, then

$$d(x, y) < r(1 + d(x, a))(1 + d(y, a)) \leq \frac{1 + d(y, a)}{2^{k_0}\lambda} \leq \frac{1 + d(x, a) + d(x, y)}{2^{k_0}\lambda},$$

and so because $\lambda \geq 1$,

$$d(x, y) \leq \frac{1 + d(x, a)}{2^{k_0}\lambda - 1} \leq \frac{1}{2^{2k_0-1}\lambda^2 r},$$

that is, $B_a(x,r) \subset B(x, 2^{1-2k_0}\lambda^{-2}/r)$. Furthermore, if $z \in B(x, 2^{-2k_0-3}\lambda^{-2}/r)$, then by (3.3),

$$d(x,z) < \frac{1}{2^{2k_0+3}\lambda^2 r} \le \frac{1}{2^{k_0+2}\lambda}(1+d(x,a))$$

and

$$1+d(z,a) \ge 1+d(x,a)-d(x,z) \ge \frac{1}{2^{k_0+1}\lambda r} - \frac{1}{2^{2k_0+3}\lambda^2 r} > \frac{1}{2^{k_0+2}\lambda r}.$$

Combining the above two estimates, we obtain

$$d(x,z) < r\,(1+d(x,a))(1+d(z,a)),$$

that is, $B(x, 2^{-2k_0-3}\lambda^{-2}/r) \subset B_a(x,r)$. Thus we have

$$B(x, \tfrac{1}{2^{2k_0+3}\lambda^2 r}) \subset B_a(x,r) \subset B(x, \tfrac{1}{2^{2k_0-1}\lambda^2 r}).$$

We simplify notation by setting

$$B_s = B(x, \tfrac{1}{2^{2k_0+3}\lambda^2 r}),\ B_l = B(x, \tfrac{1}{2^{2k_0-1}\lambda^2 r}).$$

Then we have $B_s \subset B_a(x,r) \subset B_l = 16B_s$. Notice that B_s and B_l are balls with respect to the original metric d, while $B_a(x,r)$ represents the ball with respect to the metric d_a. Note that when $z \in \lambda B_l$,

$$1+d(z,a) \le 1+d(x,a)+d(z,x) < \frac{1}{2^{k_0}\lambda r} + \frac{1}{2^{2k_0-1}\lambda r} \le \frac{2}{2^{k_0}\lambda r} \le 4(1+d(x,a)).$$

Since $k_0 \ge 3$,

$$1+d(z,a) \ge 1+d(x,a)-d(z,x) > \frac{1}{2^{k_0+1}\lambda r} - \frac{1}{2^{2k_0-1}\lambda r} \ge \frac{1}{2^{k_0+2}\lambda r} \ge \frac{1+d(x,a)}{4}.$$

Hence for $z \in \lambda B_l$ we have

$$\frac{1}{2^{k_0+3}\lambda r} \le \frac{1+d(x,a)}{4} \le 1+d(z,a) \le 4(1+d(x,a)) \le \frac{1}{2^{k_0-2}\lambda r}. \tag{3.4}$$

It follows from the above estimates and the doubling property of μ that for $z \in \lambda B_l$,

$$C^{-1}\frac{d\mu(z)}{\mu(B(a,1/(2^{k_0}r)))^2} \le d\mu_a(z) = \frac{d\mu(z)}{\mu(B(a,1+d(a,z))^2} \le C\,\frac{d\mu(z)}{\mu(B(a,1/(2^{k_0}r)))^2}. \tag{3.5}$$

It follows that

$$\mu_a(B_a(x,r)) \le C\,\frac{\mu(B_l)}{\mu(B(a,1/(2^{k_0}r)))^2} \le C\,C_\mu^4 \frac{\mu(B_s)}{\mu(B(a,1/(2^{k_0}r)))^2},$$

and

$$\mu_a(B_a(x,r)) \ge C^{-1}\,\frac{\mu(B_s)}{\mu(B(a,1/(2^{k_0}r)))^2},$$

from which we obtain

$$\frac{1}{C}\,\frac{\mu(B_s)}{\mu(B(a,1/(2^{k_0}r)))^2} \le \mu_a(B_a(x,r)) \le C\,\frac{\mu(B_s)}{\mu(B(a,1/(2^{k_0}r)))^2}. \tag{3.6}$$

From (3.4) again and Lemma 2.10, for $z \in \lambda B_l$ we also get

$$\frac{1}{C}\,\frac{1}{2^{2k_0}r^2}\,g(z) \le \hat{g}(z) = g(z)(1+d(a,z))^2 \le C\,\frac{1}{2^{2k_0}r^2}\,g(z). \tag{3.7}$$

Now, by applying (3.6), (3.5), and the p-Poincaré inequality of (X,d,μ) in order, we obtain

$$\begin{aligned}
\fint_{B_a(x,r)} |u-u_{B_a(x,r)}|\,d\mu_a &\le 2 \fint_{B_a(x,r)} |u-u_{B_l}|\,d\mu_a \\
&\le \frac{C\,\mu(B(a,1/(2^{k_0}r)))^2}{\mu(B_s)} \int_{B_l} |u-u_{B_l}|\,d\mu_a \\
&\le \frac{C\,\mu(B(a,1/(2^{k_0}r)))^2}{\mu(B_l)} \int_{B_l} |u-u_{B_l}|\,d\mu_a \\
&\le C \fint_{B_l} |u-u_{B_l}|\,d\mu \\
&\le C\,\frac{1}{2^{2k_0-1}\lambda^2 r} \left(\fint_{\lambda B_l} g^p\,d\mu\right)^{1/p}.
\end{aligned}$$

In the above, $u_{B_l} = \mu(B_l)^{-1}\int_{B_l} u\,d\mu$ is the *un-sphericalized* average of u on B_l. Now by applying (3.5) again as well as (3.7), we obtain the inequality

$$\fint_{B_a(x,r)} |u-u_{B_a(x,r)}|\,d\mu_a \le \frac{C\,r}{2^{-1}\lambda^2}\left(\frac{1}{\mu_a(B_a(x,r))}\int_{\lambda B_l} \hat{g}^p\,d\mu_a\right)^{1/p}.$$

From (3.4) and the definition of B_l, if $z \in \lambda B_l$ we have

$$d_a(x,z) = \frac{d(x,z)}{(1+d(x,a))(1+d(z,a))} \le C\,\frac{1}{2^{2k_0-1}\lambda^2 r}\,2^{2k_0}\lambda^2 r^2 \le C\,r.$$

That is, $\lambda B_l \subset CB_a(x,r)$. Hence by the doubling property of μ_a (proved in Subsection 3.2),

$$\fint_{B_a(x,r)} |u - u_{B_a(x,r)}|\, d\mu_a \le \frac{C\,r}{2^{-1}\lambda^2} \left(\fint_{CB_a(x,r)} \hat{g}^p\, d\mu_a \right)^{1/p},$$

which is the p-Poincaré inequality on $B_a(x,r)$ as desired.

Case 2: $x = \infty$ and $0 < r < 1/(10\lambda K^2)$. As mentioned in Remark 2.20, because X has the refinement of radially star-like quasiconvexity, $\dot{X}$ equipped with d_a has the refinement of meridian-like quasiconvexity property. Therefore we can write the ball $B_a(\infty, r)$ as a finite union of measurable sets, namely $B_a(\infty,r) = \bigcup_{i=1}^{N_0} (S_i \cap B_a(\infty,r))$,

$$S_i := \bigcup_{R > r_0} \Big\{ x \in A(a, R/2, R) \,:\, \exists z \in \beta_i \text{ and quasiconvex curve } \gamma \subset A(a, R/K, KR) \\ \text{with end points } x, z \text{ and } \ell(\gamma_{xz}) \le K d(a,x) \Big\}. \tag{3.8}$$

Note that each S_i is open because of the quasiconvexity of X, and hence is measurable. Here, $\beta_1, \cdots, \beta_{N_0}$ are the curves referred to in Definition 2.17. Observe that the intersection of two sets S_i and S_j, $i \ne j$ could possibly be nonempty. For $i \in \{1, 2, \cdots N_0\}$ there exists $z_i \in \beta_i$ with $3r/4 \le d_a(z_i, \infty) \le r$. Let $\rho = \frac{r}{20\lambda K^2}$. Observe that $B_i := B_a(z_i, \rho) \subset \frac{1}{3K\lambda} B_a(\infty, 6K\lambda r)$ and that $KB_a(\infty, 6K\lambda r) \subset 70K^3\lambda B_i$. By the doubling property of μ_a (see [20, Proposition 3.2.3]) we also have $\mu_a(B_i) \approx \mu_a(S_i \cap B_a(\infty, r))$. Following the same argument as in [20, Case 2, Theorem 3.3.5], we see that

$$\fint_{S_i \cap B_a(\infty,r)} |u - u_{B_i}| d\mu_a \le Cr \Big(\fint_{\lambda B_i} \hat{g}^p d\mu_a \Big)^{1/p}. \tag{3.9}$$

Observe that for a fixed $i_0 \in \{1, 2, \cdots N_0\}$,

$$\begin{aligned}\int_{B_a(\infty,r)} |u - u_{B_a(\infty,r)}| d\mu_a \leq\ & 2\int_{B_a(\infty,r)} |u - u_{B_{i_0}}| d\mu_a \leq 2\sum_{i=1}^{N_0} \int_{S_i \cap B_a(\infty,r)} |u - u_{B_{i_0}}| d\mu_a \\ &\leq 2\sum_{i=1}^{N_0} \int_{S_i \cap B_a(\infty,r)} (|u - u_{B_i}| + |u_{B_i} - u_{B_{i_0}}|) d\mu_a \\ &\leq 2\sum_{i=1}^{N_0} \int_{S_i \cap B_a(\infty,r)} |u - u_{B_i}| d\mu_a + 2\sum_{i=1}^{N_0} \mu_a(S_i \cap B_a(\infty, r))|u_{B_i} - u_{B_{i_0}}|.\end{aligned} \tag{3.10}$$

Notice that the first summand of the last inequality can be estimated by using (3.9) as follows:

$$\begin{aligned}\sum_{i=1}^{N_0} \int_{S_i \cap B_a(\infty,r)} |u - u_{B_i}| d\mu_a &\leq Cr \sum_{i=1}^{N_0} \mu_a(B_i)\Big(\fint_{\lambda B_i} \hat{g}^p d\mu_a\Big)^{1/p} \\ &\approx C\, N_0\, \mu_a(B_a(\infty, r)) r \Big(\fint_{\lambda B_a(\infty,r)} \hat{g}^p d\mu_a\Big)^{1/p}.\end{aligned} \tag{3.11}$$

The second summand of the last inequality in (3.10) can be estimated by using the point $x = \infty$:

$$\sum_{i=1}^{N_0} \mu_a(S_i \cap B_a(x, r))|u_{B_i} - u_{B_{i_0}}| \leq \sum_{i=1}^{N_0} \mu_a(S_i \cap B_a(\infty, 2r))(|u_{B_i} - u(\infty)| + |u(\infty) - u_{B_{i_0}}|). \tag{3.12}$$

Now, fix $1 \leq i \leq N_0$.

Denote $z_{i,0} = z_i$ and $B_{i,0} = B_i$. We can choose a sequence of points $z_{i,j} \in \beta_i$ by induction to estimate $|u_{B_i} - u(\infty)|$. Suppose $z_{i,j-1}$ has been chosen, with $z_{i,j-1} \in A(\infty, 2^{-l_{j-1}-1}r, 2^{-l_{j-1}}r)$ (with respect to the metric d_a), where l_{j-1} is an integer depending only on j. We can find a point $z_{i,j}$ in the subcurve of β_i connecting $z_{i,j-1}$ to ∞, denoted by $\beta_{\infty z_{i,j-1}}$, such that the length of the subcuve $\gamma_{i,j}$ of $\beta_{\infty z_{i,j-1}}$ with end points $z_{i,j-1}$ and $z_{i,j}$ satisfies $2^{-l_{j-1}-1}\rho \leq \ell_{d_a}(\gamma_{i,j}) \leq 2^{-l_{j-1}}\rho$. Since $d_a(z_{i,j-1}, \infty) \geq 2^{-l_{j-1}-1}r \geq 2^{-l_{j-1}}\rho \geq \ell_{d_a}(\gamma_{i,j})$, such $z_{i,j}$ always exists. Once $z_{i,j}$ has been chosen, we can choose $z_{i,j+1}$ in the subcurve of β_i connecting $z_{i,j}$ to ∞ satsfying $2^{-l_j-1}\rho \leq \ell(\gamma_{i,j+1}) \leq 2^{-l_j}\rho$, where $\gamma_{i,j+1}$ can be defined as before. Therefore, we have chosen a sequence of points $z_{i,j} \in \beta_i$.

We now need to prove that

$$\lim_{j\to\infty} d_a(z_{i,j}, \infty) = 0.$$

Let $N_l := \{j \in \mathbb{N} : l_j = l\}$. We first need to show that for every $l \geq 0$, we have $\#(N_l) \leq M(K,\lambda)$. Let $s_l = \min j \in N_l$. By the base-point quasiconvexity of β_i with respect to base point ∞, we have

$$\begin{aligned}\#(N_l)2^{-l-1}\rho &= \sum_{j\in N_l} 2^{-l_j-1}\rho \leq \sum_{j=s_l}^{\infty} 2^{-l_j-1}\rho \leq \sum_{j=s_l}^{\infty} \ell_{d_a}(\gamma_{i,j+1}) \\ &\leq \ell_{d_a}(\beta_{\infty z_{i,s_l}}) \leq C d_a(z_{i,s_l},\infty) \\ &\leq 2^{-l_{s_l}} r = 2^{-l} r,\end{aligned} \tag{3.13}$$

so $\#(N_l) \leq M$ for some $M = M(K,\lambda)$. Hence, for each $l \geq 0$, there exists $j \in \mathbb{N}$ so that when $j \geq Ml$, we have $l_j \geq l$, and so it follows that $\lim_{j\to\infty} d_a(z_{i,j},\infty) \leq \lim_{j\to\infty} 2^{-l_j} r = 0$.

Then we can take a collection of sphericalized balls $B_{i,j} = B_a(z_{i,j}, 2^{-l_j}\rho)$ to estimate $|u_{B_i} - u(\infty)|$. Notice that $\mathrm{rad}(B_{i,j})$ tends to zero when j approach to ∞. Then we can obtain the estimate as follows:

$$\begin{aligned}|u_{B_i} - u(\infty)| &\leq \sum_{j=0}^{\infty} |u_{B_{i,j}} - u_{B_{i,j+1}}| \leq 4\sum_{j=0}^{\infty} \fint_{2B_{i,j}} |u - u_{2B_{i,j}}| d\mu_a \\ &\leq C\sum_{j=0}^{\infty} \frac{\mathrm{rad}_a(2B_{i,j})}{\mu_a(2B_{i,j})^{1/p}} \left(\int_{6\lambda K^2 B_{i,j}} \hat{g}^p d\mu_a\right)^{1/p} \\ &\leq C\left(\sum_{j=0}^{\infty} (\frac{\mathrm{rad}_a(2B_{i,j})}{\mu_a(2B_{i,j})^{1/p}})^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}} \left(\sum_{j=0}^{\infty}\int_{6\lambda K^2 B_{i,j}} \hat{g}^p d\mu_a\right)^{1/p} \\ &\leq C\left(\sum_{j=0}^{\infty} (\frac{\mathrm{rad}_a(B_{i,j})}{\mu_a(B_{i,j})^{1/p}})^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}} \left(M\int_{6\lambda K^2 (S_i\cap B_a(x,r))} \hat{g}^p d\mu_a\right)^{1/p}.\end{aligned} \tag{3.14}$$

where in the third line we have used Hölder inequality and in the second line we have applied Poincaré inequality for balls B_i^k, which satisfies the hypothesis of Case 1.

On the other hand, we need to estimate the quantity $(\mathrm{rad}_a(B_{i,j}))/(\mu_a(B_{i,j}))$. Since $\mathrm{rad}_a(B_{i,j}) = 2^{-l_j}\rho$ and $d_a(z_{i,j},\infty) \geq 2^{-l_j-1}r$, by (3.6), we have

$$\mu_a(B_{i,j}) \approx \frac{\mu(B(z_{i,j}, C2^{l_j}/\rho))}{\mu(B(a, C2^{l_j}/\rho))^2}$$

and

$$\mu_a(B_i) \approx \frac{\mu(B(z_i, c/\rho))}{\mu(B(a,c/\rho))^2}.$$

Therefore by (2.3), we have

$$\frac{\mu_a(B_{i,j})}{\mu_a(B_i)} \approx \frac{\mu(B(z_{i,j}, c2^{l_j}/\rho))\mu(B(a, c/\rho))^2}{\mu(B(z_i, c/\rho))\mu(B(a, c2^{l_j}/\rho))^2} \approx \frac{\mu(B(a, c/\rho))}{\mu(B(a, c2^{l_j}/\rho))}$$
$$\geq C^{-1}(\frac{c/\rho}{c2^{l_j}/\rho})^s$$
$$\approx C(\frac{2^{-l_j}\rho}{\rho})^s,$$

where the last two inequalities follow from an argument similar to that of (2.3). Therefore, we obtain the inequality

$$\frac{(2^{-l_j}\rho)^{s/p}}{\mu_a(B_{i,j})^{1/p}} \leq C\frac{\rho^{s/p}}{(\mu_a(B_a(z_i, \rho)))^{1/p}}.$$

Then we obtain the upper bound of the first term in the last inequality of (3.14), which is

$$\begin{aligned}\left(\sum_{j=0}^{\infty}\left(\frac{\mathrm{rad}_a(B_{i,j})}{\mu_a(B_{i,j})^{1/p}}\right)^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}} &= \left(\sum_{j=0}^{\infty}\left(\frac{(2^{-l_j}\rho)^{s/p}(2^{-l_j}\rho)^{1-s/p}}{\mu_a(B_{i,j})^{1/p}}\right)^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}} \\ &\leq \left(\sum_{j=0}^{\infty}\left(\frac{\rho^{s/p}(2^{-l_j}\rho)^{1-s/p}}{\mu_a(B_i)^{1/p}}\right)^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}} \\ &= \frac{\rho}{(\mu_a(B_i))^{1/p}}\left(\sum_{j=0}^{\infty}2^{-l_j\frac{p-s}{p-1}}\right)^{\frac{p-1}{p}}.\end{aligned} \tag{3.15}$$

Notice from the argument of (3.13) and the subsequent paragraph that for each $k \in \mathbb{N}$, there are at most M number of j with $l_j = k$. So the quantity $\sum_{j=0}^{\infty} 2^{-l_j\frac{p-s}{p-1}}$ is finite. Combining (3.14) and (3.15) we obtain that

$$|u_{B_i} - u(\infty)| \leq Cr\left(\fint_{6\lambda K^2(S_i \cap B_a(x,r))} \hat{g}^p d\mu_a\right)^{1/p}.$$

Combine (3.10),(3.11), (3.12) and the inequality above, we can obtain that

$$\fint_{B_a(\infty)} |u - u_{B_a(\infty,r)}| d\mu_a \leq Cr\left(\fint_{6\lambda K^2 B_a(\infty,r)} \hat{g}^p d\mu_a\right)^{1/p}.$$

Case 3: $d_a(x,\infty) < 8\lambda r$. In this case we use the conclusion of Case 2 above as an aid, since $B_a(x,r) \subset B_a(\infty, 16\lambda r)$, $B_a(\infty, 96A\lambda^2 r) \subset B_a(x, 105A\lambda^2 r)$ and the ball $B_a(\infty, 16\lambda r)$ satisfies the hypothesis of Case 2. Hence by the doubling property of μ_a,

$$\begin{aligned}
\fint_{B_a(x,r)} |u - u_{B_a(x,r)}| \, d\mu_a &\le 2 \fint_{B_a(x,r)} |u - u_{B_a(\infty,16\lambda r)}| \, d\mu_a \\
&\le C \fint_{B_a(\infty,16\lambda r)} |u - u_{B_a(\infty,16\lambda r)}| \, d\mu_a \\
&\le Cr \left(\fint_{96A\lambda^2 B_a(\infty,r)} \hat{g}^p d\mu_a \right)^{1/p} \\
&\le Cr \left(\fint_{B_a(x,105A\lambda^2 r)} \hat{g}^p d\mu_a \right)^{1/p}.
\end{aligned}$$

□

The following example was considered in [20] and shows that the previous theorem is not true for $p \le s$.

Example 3.16. Let X be the 2-dimensional Euclidean strip $\mathbb{R} \times [-1, 1]$ equipped with the Euclidean metric and the weighted measure $d\mu(x) = \max\{1, |x|^2\} d\mathscr{L}^2(x)$, which is clearly a radially star-like quasiconvex space. By [14, Corollary 15.35] the measure μ is p-admissible in $\mathbb{R}^2$ for any $p > 1$, which means that μ is doubling and $(\mathbb{R}^2, |\cdot|, \mu)$ supports a p-Poincaré inequality for any $p > 1$. In particular $(X, |\cdot|, \mu)$ supports a 2-Poincaré inequality. The sphericalized space with respect to the base point $a = (0, 0)$ is the region trapped between two tangential circles in the sphere. In particular, the boundary of such a region is a quadratic cusp and one can check that $\mu_a = \mathscr{L}^2$. Therefore (X_a, d_a, μ_a) supports a p-Poincaré inequality for any $p > 3$. Observe that in this case the exponent of relative lower volume decay associated to μ is $s = 3$.

The example [9, Example 3.14] provides a metric measure space endowed with a doubling measure, which is not radially star-like quasiconvex, supporting an ∞-Poincaré inequality but whose sphericalized space fails to support an ∞-Poincaré inequality. We can therefore conclude that Theorem 3.1 is no longer true if the hypothesis of refinement of radially star-like quasiconvexity is removed.

4 Preservation of p-Poincaré inequality for $p > s$ under flattening

In this section we show the preservation of p-Poincaré inequality under flattening for p sufficiently large for metric spaces satisfying the refinement of meridian-like quasiconvexity (see Definition 2.18).

Theorem 4.1. *Let (X, d, μ) be a bounded complete metric space endowed with a doubling measure μ and supporting a p-Poincaré inequality for some $p > s$, where s is the exponent of relative lower volume decay associated to μ as in* (2.3)*. Let $c \in X$ be a base point on X, and assume (X, d) is a refinement of K-meridian-like quasiconvex with respect to the base point c for some $K \geq 1$. Then (X^c, d^c, μ^c) also supports a p-Poincaré inequality.*

Remark 4.2. Notice that we only require $p > s$, where s is associated to the original measure μ rather than the flattened measure μ^c.

Proof. Let $u \in \mathrm{Lip}(X^c)$ and g be an upper gradient of u in X with respect to the metric (X, d). We split the proof into three cases depending on the quantity $\lambda r d(x, c)$.

Case 1: $6\lambda r d(x, c) \leq 1/2$. As it was shown in [20, Proposition 4.1.1],

$$B(x, 2rd(x, c)^2/3) \subset B^c(x, r) \subset B(x, 2r\lambda d(x, c)^2) \subset B^c(x, 6\lambda r).$$

Furthermore, from that argument, we also obtain that $\frac{2}{3}d(x, c) < d(y, c) < 2d(x, c)$ whenever $y \in B^c(x, 6\lambda r)$. Hence we obtain

$$\mu^c(B^c(x, kr)) = \int_{B^c(x,kr)} \frac{d\mu(y)}{\mu\left(B(c, d(y, c))\right)^2} \approx \frac{\mu(B^c(x, kr))}{\mu\left(B(c, d(x, c))\right)^2},$$

whenever $0 < k \leq 6\lambda$. In addition, for $y \in 6\lambda B^c(x, r)$ the upper gradient

$$\bar{g}(y) = g(y)(d(y, c))^2 \approx g(y)(d(x, c)^2).$$

Therefore, by the doubling property of μ we obtain a Poincaré inequality on $B^c(x,r)$ when $6\lambda r d(x,c) \le 1/2$ as follows:

$$\begin{aligned}
\fint_{B^c(x,r)} |u - u_{B^c(x,r)}| d\mu^c \le & 2 \fint_{B^c(x,r)} |u - u_{B(x,2rd(x,c)^2)}| d\mu^c \\
\le & \frac{C}{\mu^c(B^c(x,r))} \int_{B^c(x,r)} \frac{|u - u_{B(x,2rd(x,c)^2)}|}{\mu\left(B(c,d(x,c))\right)^2} \, d\mu \\
\le & \frac{C}{\mu(B^c(x,r))} \int_{B^c(x,r)} |u - u_{B(x,2rd(x,c)^2)}| \, d\mu \\
\le & \frac{C}{\mu(B(x,2rd(x,c)^2/3))} \int_{B(x,2rd(x,c)^2)} |u - u_{B(x,2rd(x,c)^2)}| \, d\mu.
\end{aligned}$$

Now applying the p-Poincaré inequality valid for X, we obtain

$$\begin{aligned}
\fint_{B^c(x,r)} |u - u_{B^c(x,r)}| d\mu^c \le & C r \, d(x,c)^2 \left(\fint_{B(x,2r\lambda d(x,c)^2)} g^p \, d\mu \right)^{1/p} \\
\le & C r \left(\int_{B(x,2r\lambda d(x,c)^2)} \bar{g}^p(y) \frac{\mu(B(c,d(x,c)))^2}{\mu(B^c(x,6\lambda r))} \, d\mu^c(y) \right)^{1/p} \\
\le & C r \left(\fint_{B^c(x,6\lambda r)} \bar{g}^p \, d\mu^c \right)^{1/p}
\end{aligned}$$

as desired. This completes the proof of p-Poincaré inequality for balls $B^c(x,r)$ when $6\lambda r d(x,c) < 1/2$.

Case 2: $\lambda r d(x,c) \ge 4\lambda$. According to Case 2 of [20, Proposition 4.1.1], we can see that

$$X \setminus \bar{B}(c, 2/r) \subset B^c(x,r) \subset X \setminus \bar{B}(c, 2/(3r)).$$

Let $\beta_1, \cdots, \beta_{N_0}$ be the double base-point quasiconvex curves guaranteed by the refinement of meridian-like quasiconvexity of X. For $i = 1, \cdots, N_0$ let

$$\begin{aligned}
S_i := \bigcup_{r \le r_0} \Big\{ & x \in X : \text{ if } x \in A(c, r/2, r) \text{ with } r \le r_0 \ \exists\, z \in \beta_i \text{ and curve} \\
& \gamma_{xz} \subset A(c, r/K, Kr) \text{ connecting } x \text{ to } z \text{ with } \ell(\gamma_{xz}) < K d(x,c) \Big\}.
\end{aligned} \tag{4.3}$$

We can split the ball $B^c(x,r)$ into a finite number of measurable sets $B^c(x,r) = \bigcup_{i=1}^{N_0} (S_i \cap B^c(x,r))$. Observe that the intersection of two sets S_i and S_j, $i \neq j$

could possibly be non empty. For each $i = 1, \cdots, N_0$ we have $\beta_i \subset S_i$, and by the connectedness of β_i we can find $z_i \in \beta_i$ such that $rd(z_i, c) = 4$. Let i be such that $S_i \cap B^c(x,r)$ is non-empty, and set $\rho = r/(96\lambda K)$ and $B_i = B^c(z_i, \rho)$. We will now show that $B^c(z_i, r/(96\lambda K)) \subset B^c(x,r)$.

Notice that the radius of the ball $B_i = B^c(z_i, \rho)$ satisfies the hypothesis of Case 1, that is, $6\lambda\rho d(z_i, c) = 6\lambda d(z_i, c) r/(96\lambda K) \le 1/2$. Then we have

$$B_i = B^c\left(z_i, \frac{r}{96\lambda K}\right) \subset B\left(z_i, \frac{rd(z_i,c)^2}{48K}\right) = B\left(z_i, \frac{d(z_i,c)}{12K}\right) = B\left(z_i, \frac{1}{3Kr}\right).$$

If $y \in X \setminus B^c(x,r)$, then

$$4d(y,c) \le rd(x,c)d(y,c) \le d(x,y) \le d(x,c) + d(y,c).$$

It follows that for such y, we have $[rd(y,c) - 1]d(x,c) \le d(y,c)$, and hence by $rd(x,c) \ge 4$, we see that

$$\frac{4\,[rd(y,c) - 1]}{r} \le d(y,c),$$

i.e., $d(y,c) \le 4/(3r) \le 2/r$, and so $y \in \overline{B}(c, 2/r)$. Hence $X \setminus B^c(x,r) \subset \overline{B}(c,2/r)$. Thus we have $X \setminus \overline{B}(c,2/r) \subset B^c(x,r)$. Because $d(z_i,c) = 4/r$, we have $B(z_i, \frac{1}{3Kr}) \cap B(c, 2/r)$ is empty, and so $B_i \subset X \setminus \bar{B}(c,2/r) \subset B^c(x,r)$ (it also shows that for each $i = 1, \cdots, N_0$ we have $B^c(x,r) \cap S_i$ is non-empty).

Since (X,d) is a refinement of K-meridian-like quasiconvex with respect to a base point c, it follows that given $x \in B(z_i, \frac{d(z_i,c)}{12K})$, there exists a quasiconvex curve in $A(c, d(z_i,c)/K, Kd(z_i,c))$ connecting x and z_i, so $x \in S_i$. Hence, we have $B_i \subset S_i \cap B^c(x,r) \subset B^c(x,r)$, and by the doubling property of X^c (see [20, Proposition 4.2.1]), it follows that $\mu^c(B_i) \approx \mu^c(S_i \cap B^c(x,r))$.

Following the same argument as Case 2 of [20, Theorem 4.3.3] we see that

$$\fint_{S_i \cap B^c(x,r)} |u - u_{B_i}| d\mu^c \le Cr\Big(\fint_{6\lambda K B^c(x,r)} \bar{g}^p d\mu^c\Big)^{1/p}. \tag{4.4}$$

Next observe that

$$\begin{aligned}\int_{B^c(x,r)} |u - u_{B^c(x,r)}| d\mu^c \leq & 2 \int_{B^c(x,r)} |u - u_{B_1}| d\mu^c \leq 2 \sum_{i=1}^{N_0} \int_{S_i \cap B^c(x,r)} |u - u_{B_1}| d\mu^c \\ \leq & \sum_{i=1}^{N_0} \int_{S_i \cap B^c(x,r)} (|u - u_{B_i}| + |u_{B_i} - u_{B_1}|) d\mu^c \\ \leq & \sum_{i=1}^{N_0} \int_{S_i \cap B^c(x,r)} |u - u_{B_i}| + \sum_{i=1}^{N_0} \mu^c(S_i \cap B^c(x,r)) |u_{B_i} - u_{B_1}|.\end{aligned} \tag{4.5}$$

Notice that we can estimate the first summand of the last inequality by using (4.4), so we only need to estimate the second summand.

Since $d(a,c) \approx \sup_{z \in X} d(z,c)$, there exists $l \geq 0$ with $2^l d(z_i,c) \leq d(a,c) < 2^{l+1} d(z_i,c)$. In what follows, denote $z_{i,0} = z_i$, $z_{i,M_i} = a$ where M_i will be shown to be bounded in the next paragraph. Then similar to Case 2 of Theorem 3.1, we can construct a collection of points $z_{i,k}$, where $k = 0, 1, 2 \ldots, M_i$ from β_i by induction. Suppose $z_{i,k-1}$ has been chosen. Then denote $\beta_{z_{i,k-1}a}$ to be the subcurve of β_i connecting $z_{i,k-1}$ to a, and $z_{i,k-1} \in A(c, 2^{l_k - 1} d(z_i,c), 2^{l_k} d(z_i,c))$(with respect to the metric d), where l_k is an integer depending only on k. We can find a point $z_{i,k} \in \beta_{z_{i,k-1}a}$ such that the length of the subcurve $\gamma_{i,k}$ of β_i connecting $z_{i,k-1}$ to $z_{i,k}$ satisfies $2^{-l_{k-1}-1}\rho \leq \ell^c(\gamma_{i,k}) \leq 2^{-l_{k-1}}\rho$.

Let $N_s = \{j \leq M_i : l_j = s\}$. We first need to show that for each $s \leq l$, $\#(N_s) \leq M$ for $M = M(K, \lambda)$. Let $j_s = \min j \in N_s$. Because (X,d) is a refinement of meridian-like quasiconvex, β_i is a base-point quasicovex ray with respect to the point a and the metric d^c and so we have

$$\#(N_s) 2^{-s-1} \rho = \sum_{j \in N_s} 2^{-l_j} \rho \leq \sum_{j \in N_s} \ell^c(\gamma_{i,j}) \leq \sum_{j=j_s}^{M_l} \ell^c(\gamma_{i,j}) = \ell^c(\beta_{z_{i,j_s}a}) \leq C d^c(a, z_{i,j_s}).$$

Since $d^c(a, z_{i,j_s}) = \frac{d(a, z_{i,j_s})}{d(a,c) d(z_{i,j_s}, c)}$ and $d(a, z_{i,j_s}) \leq d(a,c) + d(z_{i,j_s}, c) \leq 2d(a,c)$, so

$$d^c(a, z_{i,j_s}) \leq \frac{2}{d(z_{i,j_s}, c)} \leq \frac{2}{2^{l_{j_s}-1} d(z_i,c)} = 2^{-s} 16r, \text{ for } rd(z_i,c) = 4.$$

Therefore, we have $\#(N_s) \leq M$ and we have $M_i \leq \sum_{s=0}^{l} \#(N_s) \leq M(l+1)$.

Set $\rho_k = 2^{-j_k}\rho$. Then we can construct a collection of flattened balls $B_{i,k} = B^c(z_{i,k}, \rho_k)$ in order to estimate the second summand in (4.5). Note that

$$|u_{B_i} - u_{B_1}| \leq |u_{B_i} - u_{B_{i,M_i}}| + |u_{B_{i,M_i}} - u_{B_1}|. \tag{4.6}$$

Without loss of generality, it suffices to only estimate $|u_{B_{i,M_i}} - u_{B_i}|$. To estimate $|u_{B_{i,M_i}} - u_{B_i}|$, notice that $u_{B_{i,M_i}} = u_{B_{1,M_1}}$ for $i = 1, 2, \dots, N_0$.

Then we have

$$
\begin{aligned}
|u_{B_i} - u_{B_{i,M_i}}| &\le \sum_{k=1}^{M_i} |u_{B_{i,k}} - u_{B_{i,k+1}}| \le 2\sum_{k=1}^{M_i} \fint_{2B_{i,k}} |u - u_{B_{i,k}}| d\mu^c \\
&\le C \sum_{k=1}^{M_i} \frac{\mathrm{rad}(2B_{i,k})}{\mu^c(2B_{i,k})^{1/p}} \Big(\int_{6\lambda K B_{i,k}} \bar{g}^p d\mu^c \Big)^{1/p} \\
&\le C \Big(\sum_{k=1}^{M_i} \big(\frac{\mathrm{rad}(2B_{i,k})}{\mu^c(2B_{i,k})^{1/p}} \big)^{p/(p-1)} \Big)^{(p-1)/p} \Big(\int_{6\lambda K B_{j,k}} \bar{g}^p d\mu^c \Big)^{1/p} \\
&\le C \Big(\sum_{k=1}^{M_i} \big(\frac{\mathrm{rad}(2B_{i,k})}{\mu^c(2B_{i,k})^{1/p}} \big)^{p/(p-1)} \Big)^{(p-1)/p} \Big(C' \int_{6\lambda K S_i} \bar{g}^p d\mu^c \Big)^{1/p},
\end{aligned} \tag{4.7}
$$

where we have used Hölder inequality in the third line and the fact that by the doubling assumption on μ, μ^c is also doubling. In the second line, we have applied Poincaré inequality for the balls $B_{i,k}$ which satisfy the hypothesis of Case 1. Indeed, recall that $B_{i,k} = B^c(z_{i,k}, \rho_k)$, and $2^{j_k} d(z_i, c) \le d(z_{j,k}, c) \le 2^{j_k+1} d(z_i, c)$, $\rho_k = 2^{-j_k}\rho = 2^{-j_k} r/(96\lambda K)$, so $d(z_{j,k}, c)\rho_j \le 2d(z_i, c) r/(96\lambda K) = 1/(12\lambda K)$.

Now, according to Case 1, since $6\lambda \mathrm{rad}(B_{i,k}) d(z_{i,k}, c) \le 1/2$, then we have

$$
\begin{aligned}
\mu^c(B^c(z_{i,k}, \rho_k)) &= \int_{B^c(z_{i,k},\rho_k)} \frac{d\mu(y)}{\mu(B(c, d(y,c)))^2} \approx \frac{\mu(B^c(z_{i,k}, \rho_k))}{\mu(B(c, d(z_{i,k}, c)))^2} \\
&\approx \frac{\mu(B(z_{i,k}, \rho_k d(z_{i,k}, c)^2))}{\mu(B(c, d(z_{i,k}, c)))^2}.
\end{aligned} \tag{4.8}
$$

Notice that $\rho_k d(z_{i,k}, c) \approx \rho d(z_i, c) = 1/(12\lambda K) \le 1/2$. By the doubling property, of μ we have

$$
C_\mu \ge \frac{\mu(B(z_{i,k}, \rho_k d(z_{i,k}, c)^2))}{\mu(B(c, d(z_{i,k}, c)))} \ge C \frac{\big(\frac{d(z_{i,k},c)}{24\lambda K}\big)^s}{d(z_{i,k}, c)^s} \ge \frac{1}{C}.
$$

Therefore, from the estimate above and (4.8) we can induce that

$$
\begin{aligned}
\frac{\mu^c(B^c(z_{i,k}, \rho_k))}{\mu^c(B^c(z_i, \rho))} &\approx \frac{\mu(B(z_{i,k}, \rho_k d(z_{i,k}, c)^2))}{\mu(B(c, d(z_{i,k}, c)))^2} \frac{\mu(B(c, d(z_i, c)))^2}{\mu(B(z_i, \rho d(z_i, c)^2))} \\
&\approx \frac{\mu(B(c, d(z_i, c)))}{\mu(B(c, d(z_{i,k}, c)))} \ge C \Big(\frac{d(z_i, c)}{d(z_{i,k}, c)} \Big)^s \approx C \Big(\frac{(4/r)}{2^{j_k}(4/r)} \Big)^s.
\end{aligned}
$$

Hence, we can get

$$\frac{(2^{-j_k}r)^{s/p}}{\mu^c(B_{i,k})^{1/p}} \le C\frac{r^{s/p}}{\mu^c(B^c(z_i,\rho))^{1/p}}.$$

From this estimate together with (4.7), we obtain

$$\begin{aligned}
\Big(\sum_{k=1}^{M_i}\big(\frac{\operatorname{rad}(2B_{i,k})}{\mu^c(2B_{i,k})^{1/p}}\big)^{p/(p-1)}\Big)^{(p-1)/p} \le & C\Big(\sum_{k=1}^{M_i}\big(\frac{(2^{-j_k}r)^{s/p}(2^{-j_k}r)^{1-s/p}}{(\mu^c(2B_{i,k})^{1/p})}\big)^{p/(p-1)}\Big)^{(p-1)/p}\\
\le & C\Big(\sum_{k=1}^{M_i}\big(\frac{r^{s/p}(2^{-j_k}r)^{1-s/p}}{(\mu^c(B_i)^{1/p})}\big)^{p/(p-1)}\Big)^{(p-1)/p}\\
\le & C\Big(\sum_{k=1}^{M_i}2^{-j_k(p-s)/(p-1)}\Big)^{(p-1)/p}\frac{r}{\mu^c(B_i)^{1/p}}.
\end{aligned} \tag{4.9}$$

From (4.9), we can go back to (4.7), then we can derive that

$$|u_{B_i} - u_{B_{i,M_i}}| \le C\frac{r}{\mu^c(B_i)^{1/p}}\Big(\int_{6\lambda KS_i}\bar{g}^p d\mu^c\Big)^{1/p}. \tag{4.10}$$

Combining with (4.5), (4.6) and (4.10), we have proved the Case 2.

Case 3. The proof of this case is similar to case 3 of [20, Theorem 4.3.3].

For $1/4 \le \lambda r d(x,c) \le 4\lambda$, we combine the outcome of Case 2 above to obtain

$$\begin{aligned}
\fint_{B^c(x,r)}|u-u_{B^c(x,r)}|d\mu^c \le 2\fint_{B^c(x,r)}|u-u_{B^c(x,8r)}|d\mu^c \le & C\fint_{B^c(x,8r)}|u-u_{B^c(x,8r)}|d\mu^c\\
\le & Cr\left(\fint_{48A\lambda B^c(x,r)}\bar{g}^p d\mu^c\right)^{1/p}.
\end{aligned}$$

Here we used the fact that $B^c(x,8r)$ satisfies the hypothesis of Case 2.

By combining the above three cases we have proved the theorem.

□

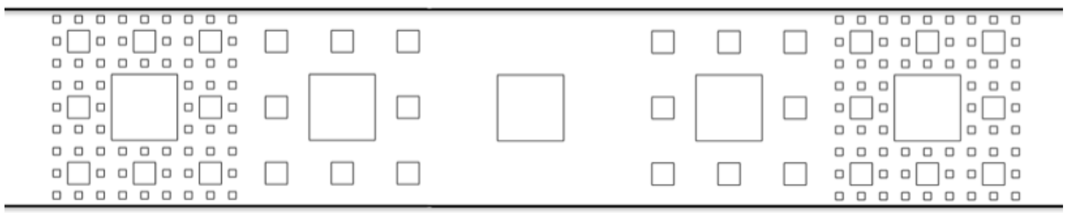
The example [9, Example 4.12] gives a metric measure space endowed with a doubling measure, which is not meridian-like quasiconvexity, supporting an ∞-Poincaré inequality but whose flattened space fails to support an ∞-Poincaré inequality. Therefore we cannot dispense of the hypothesis of refinement of meridian-like quasiconvexity in Theorem 4.1.

5 Blow-up of a differentiability space: an example

As far as we know, it is an open question whether a blow-up of a differentiability space must be a PI space, that is, a metric space endowed with a doubling measure an a p-Poincaré inequality for some $p < \infty$. See for example [8].

The following example is a modification of [11, Example 2] and shows that this is not always the case.

Example 5.1. Let $Q = [0,1] \times [0,1] \subset \mathbb{R}^2$ be the unit square.



First we divide Q into nine equal squares of side-length $1/3$ and remove the central (open) one. We define the set Q_1 to be the union of the 8 remaining squares. Repeating this procedure on each of the 8 squares making up Q_1 we obtain the set Q_2, a union of 8^2 squares, each of side-length $1/3^2$. Iterating this process we get a sequence of sets Q_j consisting of 8^j squares of side-length $1/3^j$. Because Q_j has positive area for each j, we can define a probability measure μ_j concentrated on Q_j obtained by renormalizing the Lebesgue measure (restricted to Q_j) to have measure one. We now consider the following metric measure space:

$$X = ... \cup (Q_3 + (-2,0)) \cup (Q_2 + (-1,0)) \cup Q_1 \cup (Q_2 + (1,0)) \cup \cdots$$

endowed with the measure

$$\mu = \sum_{j=-1}^{\infty} \chi_{Q_{|j-1|}+(j,0)} \cdot \mu_{j-1} + \sum_{j=1}^{\infty} \chi_{Q_j+(j-1,0)} \cdot \mu_j,$$

and with the Euclidean metric restricted to X. In the previous formula, $Q_j + (j-1,0)$ is the set obtained by translating Q_j in the direction parallel to the x-axis by $j-1$ units and μ_j is the measure given by

$$\mu_j = (9/8)^j \mathscr{L}^2|_{Q_j+(j-1,0)} \quad \text{for } j \in 1,2,\cdots,$$

and

$$\mu_j = (9/8)^{|j|}\, \mathscr{L}^2|_{Q_{|j|}+(j+1,0)} \quad \text{for } j \in \cdots -3, -2.$$

It can be directly verified that the measure μ is doubling on X. As shown in [11], the space (X, d, μ) supports an ∞-Poincaré inequality but does nos support any p-Poincaré inequality for finite p. This space, being a countable union of spaces with a Euclidean differentiable structure, is a metric differentiability space in the sense of Cheeger. By [20, Theorem 3.13], the sphericalization $(\dot{X}, d_a, \mu_a)$ also supports an ∞-Poincaré inequality and a metric differentiable structure (given via the sphericalization of the metric differentiable structure of (X, d, μ)). See [7] or [8] for the relevant definitions. On the other hand, the blow up of $(\dot{X}, d_a, \mu_a)$ at the point $x = \infty$ does not support a p-Poincaré inequality for any $p < \infty$ (because (X, d, μ) does not).

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