Hölder continuity results for a class of functionals with non standard growth

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SUNTO: In questo lavoro si provano risultati di regolarità per minimi di funzionali scalari $\int f(x,u,Du)$ a crescita non-standard di tipo $p(x)$, cioè:

$$L^{-1}|z|^{p(x)} \le f(x,s,z) \le L(1+|z|^{p(x)}) \ .$$

Si considerano per la funzione esponente $p(x)>1$ ipotesi di regolarità ottimali.

ABSTRACT: We prove regularity results for real valued minimizers of the integral functional $\int f(x,u,Du)$ under non-standard growth conditions of $p(x)$-type, i.e.

$$L^{-1}|z|^{p(x)} \le f(x,s,z) \le L(1+|z|^{p(x)})$$

under sharp assumptions on the continuous function $p(x)>1$.

1. – Introduction

The aim of this paper is the study of the regularity properties of local minimizers of integral functionals of the type

$$\mathcal{F}(u,\Omega) := \int_\Omega f(x,u(x),Du(x))dx \ , \tag{1.1}$$

where Ω is a bounded open set of $\mathbb{R}^n$, $f : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is a Carathéodory function and $u \in W^{1,1}_{\mathrm{loc}}(\Omega,\mathbb{R})$. The regularity theory for minimizers was successfully carried out under the assumption of p-growth

$$L^{-1}|z|^p \le f(x,s,z) \le L(1+|z|^p) \ , \qquad p>1$$

and under natural assumptions of convexity or quasiconvexity of f (see for example [G], [Ev], [AF1], [AF2]). At the end of the eighties some articles considering the more flexible (p,q)-growth

$$L^{-1}|z|^p \le f(x,s,z) \le L(1+|z|^q) \ , \qquad q>p>1$$

were published, after the pioneering papers of Marcellini (see [M1] - [M3], and [ELM] with the references therein). Despite the considerable number of publications devoted to the issue, for this type of functionals a general theory is still lacking. A borderline case between standard and non-standard growth is the so called $p(x)$-growth

$$L^{-1}|z|^{p(x)} \leq f(x, s, z) \leq L(1 + |z|^{p(x)}) \tag{1.2}$$

a prominent model functional being:

$$\int_{\Omega} |Du|^{p(x)} \, dx \, . \tag{1.3}$$

Such types of energies owe their importance to the fact that several models (also non variational) coming from Mathematical Physics are built using a variable growth exponent. For instance, Rajagopal and Růžička (for more details see [RR], [R1], [R2], [D], [AM3] and [AM4]) elaborated a model for electrorheological fluids, which are special non-Newtonian fluids characterized by their ability to change very quickly their mechanical properties in presence of an electromagnetic field $\mathbf{E}(x)$. Later, a model for fluids showing a similar dependence on the temperature was elaborated by Zhikov ([Z2]). In a different setting, (see [Z1]) the differential system modelling the so called "thermistor problem" includes equations like

$$-\mathrm{div}(p(x)|Du|^{p(x)-2}Du) = 0 \, .$$

On the other hand, functionals like the one in (1.3) have been studied also from a functional spaces theorical point of view since they motivate the introduction of certain related function spaces with interesting features (see, for instance, [ER1], [ER2], [F]).
For such functionals a regularity theory was recently developed ([AF2], [Z1], [FZ], [CM], [AM1], [AM2], [MM]) obtaining some optimal regularity results for local minimizers of integrals functionals of the type

$$\mathcal{F}_0(u, \Omega) := \int_{\Omega} f(x, Du(x))dx$$

with the Lagrangian $f(x, z)$ satisfying a $p(x)$ growth assumption as in (1.2).
In this article we extend the results in [AM1] to more general functionals of the type in (1.1), including model examples like:

$$\int_{\Omega} a(x, u(x))|Du|^{p(x)} \, dx \, , \tag{1.4}$$

and, more generally:

$$\int_{\Omega} a(x, u(x))f(x, Du) \, dx \, , \tag{1.5}$$

where $f(x,z)$ is as in (1.2) and $a(x,u)$ is a continuous function of its arguments. Our results can be shortly summarized as follows: if the exponent $p(x)$ has modulus of continuity ω_1, satisfying the following assumption:

$$\lim_{R\to 0} \omega_1(R) \log\left(\frac{1}{R}\right) = \lambda \ , \tag{1.6}$$

then $u \in C^{0,\alpha}_{\mathrm{loc}}(\Omega)$ where $\alpha \equiv \alpha(\lambda)$ is such that:

$$\lim_{\lambda\to 0} \alpha(\lambda) = 1 \ .$$

Clearly, if

$$\lim_{R\to 0} \omega_1(R) \log\left(\frac{1}{R}\right) = 0 \ , \tag{1.7}$$

it turns out that $u \in C^{0,\alpha}_{\mathrm{loc}}(\Omega)$ for each $\alpha < 1$. Moreover if both $p(x)$ and $a(x,u)$ are Hölder continuous, then Du is Hölder continuous too. It is worth stressing that the previous results are optimal, in the sense that if the condition (1.6) fails for each λ, then, as shown by mean of a counterexample by Zhikov, (see [Z1]), local minimizers fail to be, in general, locally Hölder continuous. In this respect our result is therefore sharp. In a second step, assuming higher regularity both on $p(x)$ and $a(x,u)$ (i.e.: Hölder continuity) we prove the Hölder continuity of the gradient Du itself. Since the Hölder continuity of the gradient is the maximal regularity expected even when $p(x)$ is constant (compare [Ur], where the scalar case is treated; the vectorial case has been subsequently studied in [Uh]; see also [FM] for the case of non standard growth conditions) also this result is the best possible.
Finally, let us comment on some technical aspects of the paper. We are dealing with very general convex Lagrangians of the type $f(x,u,Du)$. Indeed our functionals will be of the type:

$$\int |Du|^{p(x)} + g(x,u,Du)\ dx \tag{1.8}$$

where g is a Carathéodory function, convex with respect to variable z, such that:

$$0 < g(x,u,z) \leq (1 + |z|^{p(x)}) \ .$$

In particular such functions are not C^2 and fail to be even differentiable at each point. Therefore, such Lagrangians f are convex but fail to be smooth and depend explicitly on the variable $u \in \mathbb{R}$; so when proving our results we have to adopt a refined freezing, variational argument based on the Ekeland variational principle and combine it with the arguments developed in the paper [AM1]. This is due to the fact that, in order to overcome the lack of smoothness of the function f, an involved approximation procedure is required. In turn this leads to consider a sequence of approximating functionals whose (approximating) minimizers do converge to a certain limit function. For such minimizers, uniform regularity estimates are found. Now, since the functional we consider is not, in general, convex (due to the u dependence of the function f) uniqueness of minimizers, and therefore the convergence of the approximating minimizers to the original minimizer, is not a priori guaranteed. To overcome this obstruction, the above mentioned Ekeland principle turns out to be the appropriate tool, ensuring that the constructed approximating minimizers converge to the original one. The regularity of the original minimizer is then obtained passing to the limit the uniform estimates found for the approximating ones. We like to remark that such a technique has been successfully adopted for functionals with standard p-growth in the paper [CFP] (see also [CP], [FH]), and its application in our setting arises a certain number of technical problems, especially when dealing with the estimates.

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2. – Notation and statements

In the sequel Ω will denote an open bounded domain in $\mathbb{R}^n$ and $B(x,R)$ the open ball $\{y \in \mathbb{R}^n : |x-y| < R\}$. If u is an integrable function defined on $B(x,R)$, we will set

$$(u)_{x,R} = \fint_{B(x,R)} u(x)dx = \frac{1}{\omega_n R^n} \int_{B(x,R)} u(x)dx \ ,$$

where ω_n is the Lebesgue measure of $B(0,1)$. We shall also adopt the convention of writing B_R and $(u)_R$ instead of $B(x,R)$ and $(u)_{x,R}$ respectively, when the center will not be relevant or it is clear from the context; moreover, unless otherwise stated, all balls considered will have the same center. Finally the letter c will freely denote a constant, not necessarily the same in any two occurrences, while only the relevant dependences will be highlighted.
The Carathéodory function $f : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ will be supposed to satisfy a growth condition of the following type:

$$L^{-1}|z|^{p(x)} \leq f(x,u,z) \leq L(1+|z|^{p(x)}) \tag{2.1}$$

for all $x \in \Omega$, $u \in \mathbb{R}$, $z \in \mathbb{R}^n$, where $p : \Omega \to (1,+\infty)$ is a continuous function and $L \geq 1$. Next, we will set

$$\mathcal{F}(u,\mathcal{A}) := \int_{\mathcal{A}} f(x,u(x),Du(x))dx$$

for all $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ and for all $\mathcal{A} \subset \Omega$.
With this type of non-standard growth, we adopt the following notion of local minimizer and local Q-minimizer:

Definition 2.1. *We say that a function* $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ *is a local minimizer of the functional* $\mathcal{F}$ *if* $|Du(x)|^{p(x)} \in L^1_{\mathrm{loc}}(\Omega)$ *and*

$$\int_{\mathrm{spt}\,\varphi} f(x,u(x),Du(x))dx \leq \int_{\mathrm{spt}\,\varphi} f(x,u(x)+\varphi(x),Du(x)+D\varphi(x))dx$$

for all $\varphi \in W^{1,1}_0(\Omega)$ *with compact support in* Ω.

Definition 2.2. *We say that a function* $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ *is a local* Q*-minimizer of the functional* $\mathcal{F}$ *with* $Q \geq 1$ *if for all* $v \in W^{1,1}_{\mathrm{loc}}(\Omega)$ *we have*

$$\mathcal{F}(u,K) \leq Q\mathcal{F}(v,K) \; ,$$

where we set $K =: \mathrm{spt}(u-v) \subset\subset \Omega$.

We shall consider the following growth, ellipticity and continuity conditions:

$$L^{-1}(\mu^2+|z|^2)^{p(x)/2} \leq f(x,u,z) \leq L(\mu^2+|z|^2)^{p(x)/2} \; , \tag{H1}$$

$$\begin{aligned}&\int_{Q_1} [f(x_0,u_0,z_0+D\varphi(x)) - f(x_0,u_0,z_0)]dx \\ &\qquad \geq L^{-1}\int_{Q_1} (\mu^2+|z_0|^2+|D\varphi(x)|^2)^{(p(x_0)-2)/2}|D\varphi(x)|^2 dx\end{aligned} \tag{H2}$$

for some $0 \leq \mu \leq 1$, for all $z_0 \in \mathbb{R}^n$, $u_0 \in \mathbb{R}$, $x_0 \in \Omega$, $\varphi \in \mathcal{C}^\infty_0(Q_1)$, where $Q_1 = (0,1)^n$,

$$\begin{aligned}&|f(x,u,z) - f(x_0,u,z)| \\ &\leq \; L\omega_1(|x-x_0|)\Big[\Big(\mu^2+|z|^2\Big)^{p(x)/2} + \Big(\mu^2+|z|^2\Big)^{p(x_0)/2}\Big]\Big[1+|\log(\mu^2+|z|^2)|\Big]\end{aligned} \tag{H3}$$

for all $z \in \mathbb{R}^n$, $u \in \mathbb{R}$, x and $x_0 \in \Omega$, where $L \geq 1$. Here $\omega_1 : \mathbb{R}^+ \to \mathbb{R}^+$ is a nondecreasing continuous function, vanishing at zero, which represents the modulus of continuity of p :

$$|p(x)-p(y)| \leq \omega_1(|x-y|). \tag{H4}$$

We will always assume that ω_1 satisfies the following condition:

$$\limsup_{R\to 0} \omega_1(R)\log\left(\frac{1}{R}\right) < +\infty \; ; \tag{2.2}$$

thus in particular, without loss of generality, we may assume that

$$\omega_1(R) \leq L|\log R|^{-1} \tag{2.3}$$

for all $R < 1$.

We shall also consider the following continuity condition with respect to u :

$$|f(x,u,z)-f(x,u_0,z)| \leq L\omega_2(|u-u_0|)(\mu^2+|z|^2)^{p(x)/2} \ . \tag{H5}$$

for any $u, u_0 \in \mathbb{R}$. As usual, without loss of generality, we shall suppose that ω_2 is a concave, bounded and, hence, subadditive function.

Remark. Following [FFM] it is possible to prove that a functional satisfying the previous assumptions can be written in the form (1.8), with g described as in the introduction.

No differentiability is assumed on f with respect to x or with respect to z. Since all our results are local in nature, without loss of generality we shall suppose that

$$1 < \gamma_1 \leq p(x) \leq \gamma_2 \qquad \forall x \in \Omega \ ,$$

and

$$\int_\Omega |Du(x)|^{p(x)} dx < +\infty \ . \tag{2.4}$$

Our main result is contained in the following:

Theorem 2.3. *Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ be a local minimizer of the functional* (1.1), *where f is a continuous function satisfying* (H1)-(H5). *Moreover suppose that*

$$\lim_{R\to 0} \omega_1(R) \log\left(\frac{1}{R}\right) + \omega_2(R) = 0 \ . \tag{2.5}$$

Then $u \in \mathcal{C}^{0,\alpha}_{\mathrm{loc}}(\Omega)$, for all $0 < \alpha < 1$.

After the proof of the previous results we shall make some remarks leading to the following more precise statement:

Theorem 2.4. *Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ be a local minimizer of the functional* (1.1), *where f is a continuous function satisfying* (H1)-(H5). *Then there exists a nonincreasing function:*

$$\alpha : \mathbb{R}^+ \to (0,1) \ , \qquad \lim_{s\to 0} \alpha(s) = 1$$

such that if

$$\lim_{R\to 0} \omega_1(R) \log\left(\frac{1}{R}\right) + \omega_2(R) \leq \lambda \ , \tag{2.6}$$

then $u \in \mathcal{C}^{0,\alpha(\lambda)}_{\mathrm{loc}}(\Omega)$.

Clearly, Theorem 2.3 is then a consequence of Theorem 2.4, taking $\lambda = 0$.
In the case when both the functions f and $p(x)$ are smoother, we recover the classical $\mathcal{C}^{1,\alpha}$ regularity of local minimizers:

Theorem 2.5. *Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ be a local minimizer of the functional* (1.1), *where f is a continuous function satisfying* (H1)-(H5). *Moreover suppose that $\omega_1(R) + \omega_2(R) \leq LR^\alpha$ for some $0 < \alpha \leq 1$ and for all $R \leq 1$. Suppose also that f is of class $\mathcal{C}^2$ with respect to the variable z in $\Omega \times \mathbb{R} \times (\mathbb{R}^n \setminus \{0\})$, with $D^2 f$ satisfying*

$$L^{-1}(\mu^2 + |z|^2)^{(p(x)-2)/2}|\lambda|^2 \leq D^2 f(x,u,z)\lambda \otimes \lambda \leq L(\mu^2 + |z|^2)^{(p(x)-2)/2}|\lambda|^2$$

for all $\lambda \in \mathbb{R}^n$. Then Du is locally Hölder continuous in Ω.

3. – Preliminary results

Before proving our main theorems, we need some preliminary results and establish some basic notation. In the following we shall consider varying balls, always having the same center when not differently specified. Moreover, by c (or similar symbols) we denote a constant, that may vary from line to line, while only the important connections will be highlighted. If $B_{4R} \equiv B(x_c, 4R)$ we shall set:

$$p_{1,x_c}(R) := \min_{x \in \overline{B_{4R}}} p(x) \ , \quad p_{2,x_c}(R) := \max_{x \in \overline{B_{4R}}} p(x) \ . \tag{3.1}$$

When it will be clear from the context we shall omit to indicate the dependence on x_c just denoting

$$p_1 \equiv p_{1,x_c} \qquad\qquad p_2 \equiv p_{2,x_c} \ .$$

The following is a higher integrability result which is due, in its original version, to Zhikov, and which we adapt to functionals of type (1.1).

Theorem 3.1. *Let $\mathcal{O}$ be an open subset of Ω, let $u \in W^{1,1}_{\mathrm{loc}}(\mathcal{O})$ be a local minimizer of the functional* (1.1) *with $f : \mathcal{O} \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ satisfying* (H1) *and with the function $p(x)$ satisfying* (H4) *and* (2.2). *Moreover suppose that*

$$\int_{\mathcal{O}} |Du(x)|^{p(x)} dx \leq M_1$$

for some constant M_1. Then, there exist two positive constants c_0, δ depending on $\gamma_1, \gamma_2, L, M_1$, such that, if $B_R \subset\subset \mathcal{O}$, then

$$\left(\fint_{B_{R/2}} |Du(x)|^{p(x)(1+\delta)} dx \right)^{1/(1+\delta)} \leq c_0 \fint_{B_R} |Du(x)|^{p(x)} dx + c_0 \ . \tag{3.2}$$

Proof. *First step:* let $R/2 \leq t < s \leq R \leq 1$, and let $\eta \in C^\infty_0(B_R)$ be a cut-off function such that $0 \leq \eta \leq 1$, $\eta \equiv 0$ outside B_s, $\eta \equiv 1$ on B_t, $|D\eta| \leq 2(s-t)^{-1}$. Moreover we set $\varphi(x) = \eta(x)(u(x) - (u)_R)$ and let $g = u - \varphi$. We remark that $g = u$ on ∂B_s while on B_t we have $g = (u)_R$, consequently $Dg = 0$ on B_t.

Hence, using the fact that u is a local minimizer, we may write

$$\begin{aligned}
&\int_{B_t} |Du(x)|^{p(x)} dx \\
&\leq L \int_{B_s} f(x, u(x), Du(x)) dx \\
&\leq L \int_{B_s} f(x, g(x), Dg(x)) dx \\
&\leq L^2 \int_{B_s} \left(1 + |Dg(x)|^{p(x)}\right) dx \\
&\leq L^2 \int_{B_s \setminus B_t} [(1-\eta(x))|Du(x)| + |u(x) - (u)_R||D\eta(x)|]^{p(x)} dx + \bar{c} \\
&\leq \hat{c} \int_{B_s \setminus B_t} |Du(x)|^{p(x)} dx + \tilde{c} \int_{B_s} \left| \frac{u(x) - (u)_R}{s - t} \right|^{p(x)} dx + \bar{c} \\
&\leq \hat{c} \int_{B_s \setminus B_t} |Du(x)|^{p(x)} dx + \tilde{c} \frac{1}{|s-t|^{p_2}} \int_{B_R} |u(x) - (u)_R|^{p(x)} dx + \bar{c} \,,
\end{aligned}$$

where $\hat{c} = L^2 2^{\gamma_2 - 1}$, $\tilde{c} = L^2 2^{2\gamma_2 - 1}$, $\bar{c} = L^2 |B_R|$. Now adding the quantity (i.e.: "filling the hole")

$$\hat{c} \int_{B_t} |Du(x)|^{p(x)} dx$$

to the first and the last term of the previous chain of inequalities and dividing by $\hat{c} + 1$, we get

$$\int_{B_t} |Du(x)|^{p(x)} dx \leq \vartheta_1 \int_{B_s} |Du(x)|^{p(x)} dx + \tilde{d} \frac{1}{|s-t|^{p_2}} \int_{B_R} |u(x) - (u)_R|^{p(x)} dx + \bar{d} \,,$$

where

$$\vartheta_1 = \frac{\hat{c}}{\hat{c}+1} < 1 \,, \qquad \tilde{d} = \frac{L^2 2^{2\gamma_2 - 1}}{L^2 2^{\gamma_2 - 1} + 1} \,, \qquad \bar{d} = \frac{L^2 |B_R|}{L^2 2^{\gamma_2 - 1} + 1} \,.$$

Now we can apply [G], Lemma 6.1 with the choices

$$Z(t) = \int_{B_t} |Du(x)|^{p(x)} dx \,,$$

$$A = \tilde{d} \int_{B_R} |u(x) - (u)_R|^{p(x)} dx, \;\; B = \bar{d}, \;\; C = 0, \;\; \alpha = p_2, \;\; \beta = 0, \;\; \rho = \frac{R}{2} \,,$$

obtaining

$$\begin{aligned}
\int_{B_{R/2}} |Du(x)|^{p(x)} dx &\leq c \left[(R/2)^{-p_2} \tilde{d} \int_{B_R} |u(x) - (u)_R|^{p(x)} dx + \bar{d} \right] \\
&\leq c R^{p_1 - p_2} \int_{B_R} \left| \frac{u(x) - (u)_R}{R} \right|^{p(x)} dx + c R^n \\
&\leq c R^{-\omega_1(8R)} \int_{B_R} \left| \frac{u(x) - (u)_R}{R} \right|^{p(x)} dx + c R^n
\end{aligned}$$

$$
\begin{aligned}
&\le c \exp(8L) \int_{B_R} \left| \frac{u(x)-(u)_R}{R} \right|^{p(x)} dx + cR^n \\
&\le c \int_{B_R} \left| \frac{u(x)-(u)_R}{R} \right|^{p(x)} dx + cR^n \,,
\end{aligned}
$$

where in the fourth inequality we used (2.3) and c is a constant depending only on γ_1, γ_2, L.
According to the previous facts, we find that

$$
\fint_{B_{R/2}} |Du(x)|^{p(x)} dx \le c \fint_{B_R} \left| \frac{u(x)-(u)_R}{R} \right|^{p(x)} dx + c\,. \tag{3.3}
$$

Second step: we fix $\vartheta = \min\left\{ \sqrt{\frac{n+1}{n}}, \gamma_1 \right\}$ and we take $R < R_0/16$ where R_0 is small enough to have $\omega_1(8R_0) \le \vartheta - 1$. It is easy to see that

$$
1 \le \frac{p_2 \vartheta}{p_1} \le \vartheta^2 \le \frac{n+1}{n}\,.
$$

From the standard Sobolev-Poincaré inequality for a ball with $q = \frac{p_1}{\vartheta} \ge 1$, $t = \frac{p_2\vartheta}{p_1}$, we get

$$
\begin{aligned}
&\fint_{B_R} \left| \frac{u(x)-(u)_R}{R} \right|^{p(x)} dx \\
&\le 1 + \fint_{B_R} \left| \frac{u(x)-(u)_R}{R} \right|^{p_2} dx \\
&\le 1 + c \left(\fint_{B_R} |Du(x)|^{\frac{p_1}{\vartheta}} dx \right)^{\frac{(p_2-p_1)\vartheta}{p_1}} \left(\fint_{B_R} |Du(x)|^{\frac{p_1}{\vartheta}} dx \right)^{\vartheta} \\
&\le 1 + c \left(\int_{B_R} (1+|Du(x)|^{p(x)}) dx \right)^{\frac{(p_2-p_1)\vartheta}{p_1}} R^{\frac{-(p_2-p_1)\vartheta n}{p_1}} \left(\fint_{B_R} |Du(x)|^{\frac{p_1}{\vartheta}} dx \right)^{\vartheta} \\
&\le c(M_1) \left(\fint_{B_R} |Du(x)|^{\frac{p_1}{\vartheta}} dx \right)^{\vartheta} + c\,,
\end{aligned}
$$

where in the third inequality we use the fact that $\frac{p_1}{\vartheta} \le \frac{p(x)}{\vartheta} \le p(x)$ and in the last one we use again the fact that, by (2.3), $R^{\frac{-(p_2-p_1)\vartheta n}{p_1}}$ is bounded. So, by the second step

$$
\fint_{B_R} \left| \frac{u(x)-(u)_R}{R} \right|^{p(x)} dx \le c \left(\fint_{B_R} |Du(x)|^{\frac{p(x)}{\vartheta}} dx \right)^{\vartheta} + c\,. \tag{3.4}
$$

Third step: from (3.3) and (3.4) we obtain

$$\fint_{B_{R/2}} |Du(x)|^{p(x)}dx \leq c\left(\fint_{B_R} |Du(x)|^{\frac{p(x)}{\vartheta}}dx\right)^{\vartheta} + c\,.$$

Let us observe that the previous reverse Hölder estimate follows only for those radii $R < R_0/16$, so we recall the version of Gehring's lemma that can be found, for instance, in [S] and we can finish the proof. The desired dependence of the constant follows again looking at the statement in [S]. $\square$

Corollary 3.2 (Caccioppoli inequality). *Suppose that the function $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ is a local minimizer of the functional (1.1), with f satisfying (2.1) and (2.3), and let $B_R \subset\subset \Omega$. Then*

$$\fint_{B_{R/2}} |Du(x)|^{p(x)}dx \leq c \fint_{B_R} \left|\frac{u(x)-(u)_R}{R}\right|^{p(x)} dx + c\,,$$

where c depends only on γ_1, γ_2, L.

Proof. It follows from the first step of the previous proof, formula (3.3). $\square$

Before going on, we need to prove some propositions. In the following we shall consider balls $B_R \subset\subset \Omega$ and functions u, such that:

$$u \in W^{1,q}(B_R) \qquad\qquad q > 1.$$

This is a technical assumption that will be always satisfied with a suitable choice of the function u and of the exponent q, when applying the propositions below in the next section.

Proposition 3.3. *Let $g : \mathbb{R}^n \to \mathbb{R}$ be a function of class $\mathcal{C}^2$ satisfying*

$$L^{-1}(\mu^2 + |z|^2)^{q/2} \leq g(z) \leq L(\mu^2 + |z|^2)^{q/2}\,, \tag{H1c}$$

$$\begin{aligned} &\int_{Q_1} [g(z_0 + D\varphi(x)) - g(z_0)]dx \\ &\geq L^{-1} \int_{Q_1} (\mu^2 + |z_0|^2 + |D\varphi(x)|^2)^{(q-2)/2}|D\varphi(x)|^2 dx \end{aligned} \tag{H2c}$$

with $L > 1$, where q is a constant such that $\gamma_1 \leq q \leq \gamma_2$, and $\mu > 0$. Let $u \in W^{1,q}(B_R)$, $B_R \subset\subset \Omega$ and let $\tilde{v} \in u + W^{1,q}_0(\Omega)$ be a minimizer of the functional

$$\begin{aligned} \mathcal{H}(w, B_R) &:= \int_{B_R} g(Dw(x))dx + \vartheta_0 \int_{B_R} |Dw - Dv_0|dx \\ &:= \mathcal{G}_0 + \vartheta_0 \int_{B_R} |Dw - Dv_0|dx \end{aligned}$$

in the Dirichlet class $u + W^{1,q}_0(B_R)$, where $\vartheta_0 \geq 0$ and $v_0 \in u + W^{1,q}_0(B_R)$ is a fixed function. Then for all $\beta > 0$ and for all $A_0 > 0$ we have

$$\int_{B_\rho} |D\tilde{v}(x)|^q dx \leq c\left(\frac{\rho}{R}\right)^n \int_{B_R} (\mu^2 + |D\tilde{v}(x)|^2)^{q/2}dx$$

$$
\begin{aligned}
&+ c\vartheta_0 \int_{B_R} |Du(x) - D\tilde{v}(x)| dx + cR^n \vartheta_0{}^{\frac{q}{q-1}} \left[\frac{1}{A_0}\right]^{\frac{q\beta}{q-1}} \\
&+ c[A_0]^{q\beta} \int_{B_R} (1 + |Du(x)|^q) dx ,
\end{aligned}
$$

for any $0 < \rho < R$*, where* $c \equiv c(\gamma_1, \gamma_2, n)$ *is independent of* v_0*,* $\tilde{v}$*,* u*,* q*,* μ *and* R*.*

Proof. Let $v \in W^{1,q}(B_R)$ be a local minimizer of the functional $w \mapsto \int_{B_R} g(Dw(x))dx$ in the Dirichlet class $u + W_0^{1,q}(B_R)$. We remark that the function $g(z)$ satisfies the assumptions of [AM1], Theorem 3.2 and $\gamma_1 \le q \le \gamma_2$, so comparing v and $\tilde{v}$ in B_R we have, for any $0 < \rho < R$

$$
\int_{B_\rho} (\mu^2 + |Dv(x)|^2)^{q/2} dx \le c \left(\frac{\rho}{R}\right)^n \int_{B_R} (\mu^2 + |D\tilde{v}(x)|^2)^{q/2} dx ,
$$

where $c \equiv c(\gamma_1, \gamma_2, n)$. Now, arguing in a standard way (see again [AM1], [CFP]), it is easy to see that

$$
\begin{aligned}
&\int_{B_\rho} (\mu^2 + |D\tilde{v}(x)|^2)^{q/2} dx \\
&\le c \left(\frac{\rho}{R}\right)^n \int_{B_R} (\mu^2 + |D\tilde{v}(x)|^2)^{q/2} dx \\
&\quad + c \int_{B_R} (\mu^2 + |D\tilde{v}(x)|^2 + |Dv(x)|^2)^{(q-2)/2} |D\tilde{v}(x) - Dv(x)|^2 dx
\end{aligned} \tag{3.5}
$$

and that (since in our case we are assuming $\mu > 0$):

$$
\mathcal{G}_0(\tilde{v}) - \mathcal{G}_0(v) \ge c^{-1} \int_{B_R} (\mu^2 + |D\tilde{v}(x)|^2 + |Dv(x)|^2)^{(q-2)/2} |D\tilde{v}(x) - Dv(x)|^2 dx . \tag{3.6}
$$

Again we remark that c depends only on L, γ_1, γ_2. On the other hand, using the minimality of $\tilde{v}$ and triangular inequality in the second estimate, we deduce

$$
\begin{aligned}
&\mathcal{G}_0(\tilde{v}) - \mathcal{G}_0(v) \\
&\le \mathcal{H}(\tilde{v}) - \mathcal{H}(v) + \vartheta_0 \int_{B_R} |D\tilde{v}(x) - Dv(x)| dx \\
&\quad + \vartheta_0 \int_{B_R} |Dv(x) - Du(x)| dx - \vartheta_0 \int_{B_R} |Dv(x) - Du(x)| dx \\
&\le \vartheta_0 \int_{B_R} |Du(x) - D\tilde{v}(x)| dx + \int_{B_R} \left\{ \vartheta_0 \left[\frac{1}{A_0}\right]^\beta \right\} \left\{ |Dv(x) - Du(x)| \, [A_0]^\beta \right\} dx \\
&\le \vartheta_0 \int_{B_R} |Du(x) - D\tilde{v}(x)| dx + cR^n \vartheta_0{}^{\frac{q}{q-1}} \left[\frac{1}{A_0}\right]^{\frac{q\beta}{q-1}}
\end{aligned}
$$

$$+ c[A_0]^{q\beta} \int_{B_R} (1 + |Du(x)|^q) dx$$

for all $\beta > 0$ and all $A_0 > 0$. Connecting the last inequality to (3.5) and (3.6) we get the thesis. $\square$

The previous result, as the following one, are technical preliminaries that will be needed later. Now, our next task is to derive a "non smooth" version of the previous proposition. Let us start with a simple smoothing result.

Lemma 3.4. *Let $h(z) : \mathbb{R}^n \to \mathbb{R}$ be a continuous function satisfying* (H1c) *and* (H2c) *where q is a constant such that $\gamma_1 \leq q \leq \gamma_2$ and $\mu \geq 0$, and let $(G_m)_{m\in\mathbb{N}}$ be a sequence of continuous functions defined by:*

$$G_m(z) := \int_{B(0,1)} \varphi(y) h\left(z + \frac{y}{m}\right) dy \ ,$$

where $\varphi : B(0,1) \to [0,1]$ is a positive and symmetric mollifier. Then for any $m \in \mathbb{N}$ the function G_m satisfies (H1c) *and* (H2c) *with L replaced by $8^{\gamma_2}L$ and μ^2 replaced by $\mu^2 + \frac{1}{m^2}$.*

Proof. It easily follows from [FF]. $\square$

Proposition 3.5. *Let $h(z) : \mathbb{R}^n \to \mathbb{R}$ be a continuous function satisfying* (H1c) *and* (H2c) *where q is a constant such that $\gamma_1 \leq q \leq \gamma_2$, and $\mu \geq 0$; for all $u \in W^{1,q}(\Omega)$ let $v_0 \in u + W^{1,q}_0(B_R)$ be a minimizer of the functional*

$$\mathcal{H}(w, B_R) := \int_{B_R} h(Dw(x))dx + \vartheta_0 \int_{B_R} |Dw - Dv_0|dx$$

in the Dirichlet class $u + W^{1,q}_0(B_R)$, where $\vartheta_0 \geq 0$. Then for all $\beta > 0$ and all $A_0 > 0$ we have

$$\begin{aligned}\int_{B_\rho} |Dv_0(x)|^q dx \leq c\left(\frac{\rho}{R}\right)^n &\int_{B_R} (\mu^2 + |Dv_0(x)|^2)^{q/2} dx \\ &+ c\vartheta_0 \int_{B_R} |Du(x) - Dv_0(x)|dx + cR^n \vartheta_0{}^{\frac{q}{q-1}} \left[\frac{1}{A_0}\right]^{\frac{q\beta}{q-1}} \\ &+ c[A_0]^{q\beta} \int_{B_R} (1 + |Du(x)|^q) dx \ ,\end{aligned}$$

for any $0 < \rho < R$, where $c \equiv c(\gamma_1, \gamma_2, n)$ is independent of v_0, u and R.

Proof. The proof of this proposition can be obtained following a standard approximation argument (see [FF], [CFP]). We confine ourselves to sketch it. We define $v_m \in u + W^{1,q}_0(B_R)$ as the unique minimizer of the functional

$$\mathcal{H}_m(w, B_R) := \int_{B_R} G_m(Dw(x))dx + \vartheta_0 \int_{B_R} |Dw - Dv_0|dx$$

in the Dirichlet class $u+W_0^{1,q}(B_R)$. Using a standard coercivity argument and the strict convexity of the functional $\mathcal{H}$, it turns out that, up to subsequences, v_m weakly converges to u in $W^{1,q}(B_R)$ and the estimate stated follows passing to the limit the corresponding ones of Proposition 3.3, valid, uniformly, for each v_m. $\square$

Finally, we recall the main result from [FZ]:

Theorem 3.6. *Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ be a local minimizer of the functional* (1.1), *where f is a continuous function satisfying* (H1) *and with the function $p(x)$ satisfying* (H4) *and* (2.2). *Then there exists an exponent $\gamma \equiv \gamma(n, p(x), L) \in (0,1)$ such that any local minimizer of the functional* (1.1) *is in $C^{0,\gamma}_{\mathrm{loc}}(\Omega)$.*

4. – Proof of Theorems 2.3 and 2.4.

We give the proof of Theorem 2.3, the proof of Theorem 2.4 being just a straightforward consequence of the arguments developed for the first one.

Setting of the quantities.

From now on, since we are going to prove local regularity results, we shall assume that our minimizer u is globally Hölder continuous, that is:

$$|u(x) - u(y)| \leq [u]_\gamma |x-y|^\gamma \tag{4.1}$$

for all $x, y \in \Omega$.
We start the proof of the main theorems by fixing some important quantities. We start applying Theorem 3.1 in order to get a higher integrability exponent for the gradient Du , $\delta > 0$. Obviously we can replace at will the exponent δ with smaller constants; so we choose δ such that $\delta < \min\{\gamma_1 - 1, \frac{\gamma}{1-\gamma}\}$, where γ is the Hölder continuity exponent coming from Theorem 3.6. Therefore the exponent δ will depend upon the quantities $\gamma_1, \gamma_2, L, M_1$, where (see (2.4))

$$M_1 := L^2 \int_\Omega (1 + |Du(x)|^2)^{p(x)/2} dx \ . \tag{4.2}$$

Let $0 < R_0 < 1$ (that will be used as a radius) such that $\omega_1(8R_0) \leq \delta/4$, where δ is the higher integrability exponent. Observe that since $\delta \equiv \delta(n, \gamma_1, \gamma_2, M_1, L)$ then also the radius R_0 will depend on the same quantities.
In the following $R > 0$ will always denote a radius such that $16R < R_0 \leq 1$; therefore we shall always take balls $B_R \equiv B(x_c, R) \subset\subset \Omega$ with R satisfying $16R < R_0 \leq 1$. For such a ball we shall set

$$p_1(R) \equiv p_{1,x_c}(R) := \min_{x \in \overline{B_{4R}}} p(x) \ , \quad p_2(R) \equiv p_{2,x_c}(R) := \max_{x \in \overline{B_{4R}}} p(x) \ . \tag{4.3}$$

This choice implies that

$$p_2(1 + \delta/4) \leq p(x)(1 + \delta/4 + \omega_1(8R)) \leq p(x)(1+\delta) \qquad \text{in } B_{4R} \ , \tag{4.4}$$

and also that

$$p(x) \geq \gamma_1 > \delta + 1 > 1 + \delta/4 \ . \tag{4.5}$$

Finally we set

$$p_m := \max_{\overline{B_{R_0}}} p(x) \ .$$

With such a choice, (4.4) and the higher integrability result given by Theorem 3.1 allow us to say that:

$$\begin{aligned}
\int_{B_{R_0/4}} |Du(x)|^{p_m} dx &\leq \int_{B_{R_0/4}} |Du(x)|^{p(x)(1+\delta)} dx + cR_0^n \\
&\leq cR_0^n \left(\fint_{B_{R_0}} (|Du(x)|^{p(x)} + 1) dx \right)^{1+\delta} \\
&\leq cR_0^{-n\delta} \left(\int_{B_{R_0}} (|Du(x)|^{p(x)} + 1) dx \right)^{1+\delta} :\leq M_2 \ .
\end{aligned} \tag{4.6}$$

In the last inequality, we use the previous (4.2) and the fact that $R_0 \equiv R_0(n, \gamma_1, \gamma_2, M_1, L)$ (since it is determined only after δ) to deduce that the constant M_2 depends only on $L, \gamma_1, \gamma_2, \| |Du|^{p(x)} \|_{L^1(\Omega)}$; we may suppose, without loss of generality, that $M_2 \geq M_1$.
Let $B(x_c, 4R) \equiv B_{4R} \subset\subset B_{R_0/4}$ be not necessarily concentric with B_{R_0}; from now on, when not differently specified, all the balls considered, except B_{R_0}, will have the same center x_c.

Freezing.

We first remark that by Theorem 3.1 and by (4.4) we are able to deduce that $u \in W^{1,p_2(1+\delta/4)}(B_{4R})$.
Let $x_0 \in \overline{B_{4R}}$ such that

$$p(x_0) \equiv p_{2,x_c}(R) := \max_{x \in \overline{B_{4R}}} p(x) \ .$$

For any $x \in B_{4R}$, $z \in \mathbb{R}^n$ we set

$$h(z) := f(x_0, (u)_R, z) \ ,$$

$$\mathcal{G}_0(w, B_R) := \int_{B_R} h(Dw(x)) dx = \int_{B_R} f(x_0, (u)_R, Dw(x)) dx \ , \tag{4.7}$$

since we are freezing the function f at the point $(x_0, (u)_R)$, let us remark again that the center of the ball B_R is x_c, which in general it is different form x_0. Let v be the local minimizer of $\mathcal{G}_0$ in the Dirichlet class $u + W_0^{1,1}(B_R)$. We observe that the function $h(z) := f(x_0, (u)_R, z)$ satisfies the assumption of [AM1], Lemma 3.1 with $p = p_2$, $\gamma_1 \leq p_2 \leq \gamma_2$. So, by the minimality of v, it follows that there exist two constants c and $\varepsilon \in (0, \delta/4)$ both depending on γ_1, γ_2, L and independent of R and v, such that

$$\begin{aligned} &\left(\fint_{B_R} |Dv(x)|^{p_2(1+\varepsilon)} dx \right)^{1/(1+\varepsilon)} \\ &\leq c \fint_{B_R} |Dv(x)|^{p_2} dx + c \left(\fint_{B_{2R}} |Du(x)|^{p_2(1+\delta/4)} dx \right)^{1/(1+\delta/4)}, \end{aligned} \tag{4.8}$$

$$\int_{B_R} |Dv(x)|^{p_2} dx \leq c \int_{B_R} (1 + |Du(x)|^{p_2})\, dx. \tag{4.9}$$

Since u is a local minimizer of the functional (1.1), we obtain

$$\begin{aligned} \mathcal{G}_0(u) \leq \mathcal{G}_0(v) &+ \int_{B_R} f(x, v(x), Dv(x)) dx - \int_{B_R} f(x, u(x), Dv(x)) dx \\ &+ \int_{B_R} f(x, u(x), Dv(x)) dx - \int_{B_R} f(x_0, u(x), Dv(x)) dx \\ &+ \int_{B_R} f(x_0, u(x), Dv(x)) dx - \int_{B_R} f(x_0, (u)_R, Dv(x)) dx \\ &+ \int_{B_R} f(x_0, (u)_R, Du(x)) dx - \int_{B_R} f(x_0, u(x), Du(x)) dx \\ &+ \int_{B_R} f(x_0, u(x), Du(x)) dx - \int_{B_R} f(x, u(x), Du(x)) dx \\ = \mathcal{G}_0(v) &+ I + II + III + IV + V\ . \end{aligned} \tag{4.10}$$

Bounds for the quantities $I, II, ..., V$.

First of all we estimate I

$$\begin{aligned} I &\leq L \int_{B_R} \omega_2(|v(x) - u(x)|)(\mu^2 + |Dv(x)|^2)^{p(x)/2} dx \\ &\leq L \int_{B_R} \omega_2(|v(x) - u(x)|)(\mu^2 + |Dv(x)|^2)^{p_2/2} dx \\ &\quad + L \int_{B_R} \omega_2(|v(x) - u(x)|) dx =: A + B\ . \end{aligned}$$

Let $r = p_2(1+\varepsilon) \in (p_2, p_2(1+\delta/4))$ the higher integrability exponent given by [CFP], Lemma 2.7. Using Hölder inequality with exponents $\frac{r}{p_2}$ and $\left(\frac{r}{p_2}\right)' = \frac{r}{r-p_2}$ and the fact that ω_2 is bounded, we deduce that

$$\begin{aligned} A &\leq c \left[\int_{B_R} (\mu^2 + |Dv(x)|^2)^{\frac{r}{2}} dx \right]^{\frac{p_2}{r}} \left[\int_{B_R} \omega_2^{\frac{r}{r-p_2}} (|v(x) - u(x)|) dx \right]^{\frac{r-p_2}{r}} \\ &\leq cR^n \left[\fint_{B_R} \omega_2(|v(x) - u(x)|) dx \right]^{\frac{r-p_2}{r}} \end{aligned}$$

$$+\, c\left(\int_{B_R} |Dv(x)|^r dx\right)^{\frac{p_2}{r}} \left[\int_{B_R} \omega_2^{\frac{r}{r-p_2}}(|v(x)-u(x)|)dx\right]^{\frac{r-p_2}{r}} =: C+D\ ,$$

where $c \equiv c(\gamma_1, \gamma_2, L, n)$. Using the concavity of ω_2 we estimate:

$$C = cR^n \left[\fint_{B_R} \omega_2(|v(x)-u(x)|)dx\right]^{\frac{r-p_2}{r}} \leq c\,\omega_2^{\sigma}\left(\fint_{B_R} (|v(x)-u(x)|)dx\right) R^n\ ,$$

where we set $\sigma = \frac{r-p_2}{r} = \frac{\varepsilon}{1+\varepsilon}$. Further using (4.8), (4.9), (4.4), by Theorem 3.1 and arguing as before, we obtain

$$\begin{aligned}
D &\leq cR^n\left[\fint_{B_R} |Dv(x)|^{p_2}dx + \left(\fint_{B_{2R}} |Du(x)|^{p_2(1+\delta/4)}dx\right)^{\frac{1}{1+\delta/4}}\right] \\
&\qquad\qquad \times \left[\omega_2^{\sigma}\left(\fint_{B_R} |v(x)-u(x)|dx\right)\right] \\
&\leq c\left[\int_{B_R} (1+|Du(x)|^{p_2})dx\right. \\
&\qquad \left. + R^n \fint_{B_{2R}} \left(1+|Du(x)|^{p(x)(1+\delta/4+\omega_1(8R))}dx\right)^{\frac{1}{1+\delta/4}}\right] \\
&\qquad\qquad \times \left[\omega_2^{\sigma}\left(\fint_{B_R} |v(x)-u(x)|dx\right)\right] \\
&\leq c\left[\int_{B_R} (1+|Du(x)|^{p_2})dx + R^n\left[\left(\fint_{B_{4R}} (1+|Du(x)|^{p(x)})dx\right)^{\frac{(1+\delta/4+\omega_1(8R))}{1+\delta/4}}\right]\right] \\
&\qquad\qquad \times \left[\omega_2^{\sigma}\left(\fint_{B_R} |v(x)-u(x)|dx\right)\right] \\
&\leq c\left[\int_{B_R} (1+|Du(x)|^{p_2})dx + R^{-n\frac{\omega_1(8R)}{1+\delta/4}}\left(\int_{B_{4R}} (1+|Du(x)|^{p(x)})dx\right)^{\frac{\omega_1(8R)}{1+\delta/4}}\right. \\
&\qquad\qquad \left.\times \int_{B_{4R}} (1+|Du(x)|^{p_2})dx\right]\left[\omega_2^{\sigma}\left(\fint_{B_R} |v(x)-u(x)|dx\right)\right] \\
&\leq c\left[\int_{B_{4R}} (1+|Du(x)|^{p_2})dx\right]\left[\omega_2^{\sigma}\left(\fint_{B_R} |v(x)-u(x)|dx\right)\right]\ ,
\end{aligned}$$

since $R^{-n\frac{\omega_1(8R)}{1+\delta/4}}$ is bounded (argue as in the first step of Theorem 3.1). Moreover c depends only on $L, \gamma_1, \gamma_2, M_1$. On the other hand, again using the boundedness and the concavity of ω_2

$$B \leq cR^n\omega_2^{\sigma}\left(\fint_{B_R} |v(x)-u(x)|dx\right)\ ,$$

where again, $c \equiv c(\gamma_1, \gamma_2, n, L)$.
Combining the previous facts and using Poincaré inequality we have

$$
\begin{aligned}
I \leq & c\left[\int_{B_{4R}} (1+|Du(x)|^{p_2})dx\right]\omega_2^{\sigma}\left(\fint_{B_R} |v(x)-u(x)|dx\right) \\
\leq & c\|1+|Du|\|^{p_2}_{L^{p_2}(B_{4R})}\omega_2^{\sigma}\left(R\fint_{B_R} |Dv(x)-Du(x)|dx\right) \\
\leq & c\|1+|Du|\|^{p_2}_{L^{p_2}(B_{4R})}\omega_2^{\sigma}\left[\left(R^{p_2}\fint_{B_R} |Dv(x)-Du(x)|^{p_2}dx\right)^{1/p_2}\right] \\
\leq & c\|1+|Du|\|^{p_2}_{L^{p_2}(B_{4R})}\omega_2^{\sigma}\left[\left(R^{p_2}\fint_{B_R} (1+|Du(x)|^{p_2})dx\right)^{1/p_2}\right] \\
\leq & c\|1+|Du|\|^{p_2}_{L^{p_2}(B_{4R})}\omega_2^{\sigma}\left[\left(R^{p_2}\fint_{B_R} (1+|Du(x)|^{p(x)(1+\delta)})dx\right)^{1/p_2}\right] ,
\end{aligned}
$$

where in the last inequality we used (4.5). By Theorem 3.6, $u \in \mathcal{C}^{0,\gamma}(\Omega)$; we set $[u]_\gamma$ to be the Hölder constant of u in Ω and recall that, by our choice, it follows that $\delta < \frac{\gamma}{1-\gamma}$. We set $\tilde{m} := \gamma + \gamma\delta - \delta$ and we remark that $0 < \tilde{m} < 1$. So first using Theorem 3.1 and then Caccioppoli inequality we get

$$
\begin{aligned}
& \omega_2^{\sigma}\left[\left(R^{p_2}\fint_{B_R} (1+|Du(x)|^{p(x)(1+\delta)})dx\right)^{1/p_2}\right] \\
\leq & c\omega_2^{\sigma}\left[R\left(\fint_{B_R} (1+|Du(x)|^{p(x)})dx\right)^{(1+\delta)/p_2}\right] \\
\leq & c\omega_2^{\sigma}\left[R\left(\fint_{B_{4R}}\left(1+\left|\frac{u(x)-(u)_{4R}}{R}\right|^{p_2}\right)dx\right)^{(1+\delta)/p_2}\right] \\
\leq & c\omega_2^{\sigma}\left[\left(R^{p_2}\left[\fint_{B_{4R}}\left(1+\frac{[u]_\gamma^{p_2}R^{p_2\gamma}}{R^{p_2}}\right)dx\right]^{(1+\delta)}\right)^{1/p_2}\right] \\
= & c\omega_2^{\sigma}[(R^{p_2}+[u]_\gamma^{p_2(1+\delta)}R^{p_2[1+\gamma+\gamma\delta-1-\delta]})^{1/p_2}] \\
\leq & c\omega_2^{\sigma}(R^{\tilde{m}}) .
\end{aligned}
$$

So, finally

$$
I \leq c\omega_2^{\sigma}(R^{\tilde{m}})\int_{B_{4R}} (1+|Du(x)|^{p_2})dx ,
$$

where $c \equiv c(\gamma_1, \gamma_2, L, n, M_1)$.
Now we proceed estimating the remaining terms starting by III. We can use (H5) and (4.9) and again the fact that u is Hölder continuous (see (4.1)):

$$
III \leq L\int_{B_R} \omega_2(|u(x)-(u)_R|)\left(\mu^2+|Dv(x)|^2\right)^{p(x)/2} dx
$$

$$\leq c\omega_2(R^\gamma)\int_{B_R}(1+|Du(x)|^{p_2})\,dx\,.$$

In a similar way we get the estimate of IV :

$$\begin{aligned} IV &\leq L\int_{B_R}\omega_2(|u(x)-(u)_R|)\left(\mu^2+|Du(x)|^2\right)^{p(x)/2}dx \\ &\leq c\omega_2(R^\gamma)\int_{B_R}(1+|Du(x)|^{p_2})\,dx\,. \end{aligned}$$

We stress that the constants (denoted by c) found in the previous inequalities depend on $(\gamma_1,\gamma_2,n,L,M_1)$ also via $[u]_\gamma$ (see again Theorem 3.6).
To get the estimates of II and V we can argue exactly as in [AM1] but using (4.6) and our higher integrability Theorem 3.1. We obtain

$$II \leq\; c\omega_1(R)\log\left(\frac{1}{R}\right)\int_{B_{4R}}|Du(x)|^{p_2}dx + c\omega_1(R)R^n\,,$$

$$V \leq\; c\omega_1(R)\log\left(\frac{1}{R}\right)\int_{B_{2R}}|Du(x)|^{p_2}dx + c\omega_1(R)R^n\,,$$

where the constant c now depends also upon M_2.
Collecting the previous bounds and summing up we get (keeping into account that $\omega_2(R^\gamma)\leq c\omega_2^\sigma(R^{\tilde m})$):

$$\begin{aligned} &I+II+III+IV+V \\ &\leq c\left[\omega_1(R)\log\left(\frac{1}{R}\right)+\omega_2^\sigma(R^{\tilde m}))\right]\int_{B_{4R}}(1+|Du(x)|^{p_2})dx\,. \end{aligned} \tag{4.11}$$

Applying Ekeland variational principle.

We set for simplicity

$$F(R):=\omega_1(R)\log\left(\frac{1}{R}\right)+\omega_2^\sigma(R^{\tilde m})\,.$$

The assumption (2.5) allows us to say that

$$\lim_{R\to 0}F(R)=0\,.$$

Now, by the minimality of v, from (4.10) and (4.11), we obtain

$$\mathcal{G}_0(u)\leq \inf_{u+W_0^{1,1}(B_R)}\mathcal{G}_0+H(R)\,,$$

where we set

$$H(R):=cF(R)\int_{B_{4R}}(1+|Du(x)|^{p_2})dx\,.$$

We apply Theorem 1 from [Ek] ("Ekeland variational principle"). Let $V = u + W_0^{1,1}(B_R)$ equipped with the distance

$$d(w_1, w_2) := H^{-\frac{1}{p_2}} R^{-n\frac{p_2-1}{p_2}} \int_{B_R} |Dw_1(x) - Dw_2(x)| dx \; .$$

It is easy to see that the functional $\mathcal{G}_0$ is lower semicontinuous with respect to the topology induced by the distance d. Then by [Ek], Theorem 1 it follows that there exists $v_0 \in u + W_0^{1,1}(B_R)$ such that

$$(i) \int_{B_R} |Du(x) - Dv_0(x)| dx \leq [H(R)]^{\frac{1}{p_2}} R^{n\frac{p_2-1}{p_2}} \; ,$$

$$(ii)\; \mathcal{G}_0(v_0) \leq \mathcal{G}_0(u) \; ,$$

(iii) v_0 is a local minimizer of the functional

$$w \mapsto \mathcal{G}_0(w) + \left[\frac{H(R)}{R^n}\right]^{\frac{p_2-1}{p_2}} \int_{B_R} |Dw - Dv_0| dx \; .$$

By the minimality of v_0 we have that for every $\varphi \in W_0^{1,p_2}(B_R)$:

$$\begin{aligned}
\mathcal{G}_0(v_0, B_R) &\leq \mathcal{G}_0(v_0 + \varphi, B_R) + \left[\frac{H(R)}{R^n}\right]^{\frac{p_2-1}{p_2}} \int_{B_R} |Dv_0(x) + D\varphi(x) - Dv_0(x)| dx \\
&\leq \mathcal{G}_0(v_0 + \varphi, B_R) + \frac{1}{2L} \int_{B_R} |Dv_0(x) + D\varphi(x)|^{p_2} dx \\
&\quad + \frac{1}{2L} \int_{B_R} |Dv_0(x)|^{p_2} dx + cH(R) \; ,
\end{aligned}$$

Using growth assumptions (2.1) it follows in a simple way that

$$\int_{B_R} |Dv_0(x)|^{p_2} dx \leq c \int_{B_R} |Dv_0(x) + D\varphi(x)|^{p_2} dx + c(H(R) + R^n) \; ,$$

with $c \equiv c(\gamma_1, \gamma_2, n, L)$. This means that v_0 is a Q-minimizer of the functional

$$w \mapsto \int_{B_R} \left(|Dw|^{p_2} + \frac{H(R)}{R^n} + 1 \right) dx \; ,$$

where $Q \equiv Q(\gamma_1, \gamma_2, n, L) > 1$. Observe that the dependence upon M_1 and M_2 is incorporated in $H(R)$. Then it is easy to see that (see [G], Theorem 6.7) there exists an exponent of higher integrability $s \in (p_2, p_2(1 + \delta/4))$ and a constant $c > 0$ such that

$$\left(\fint_{B_{R/2}} |Dv_0(x)|^s dx \right)^{p_2/s} \leq c \fint_{B_R} |Dv_0(x)|^{p_2} dx + c \left(1 + \frac{H(R)}{R^n} \right) \; .$$

On the other hand from the growth assumption (2.1) and from property (ii) :

$$L^{-1}\int_{B_R}|Dv_0(x)|^{p_2}dx \le \mathcal{G}_0(v_0)\le \mathcal{G}_0(u)\le L\int_{B_R}(1+|Du(x)|^{p_2})\,dx\ ,$$

so

$$\left(\fint_{B_{R/2}}|Dv_0(x)|^s dx\right)^{p_2/s}\le c\fint_{B_{4R}}(1+|Du(x)|^{p_2})dx\ . \tag{4.12}$$

Comparison and conclusion.

We apply Proposition 3.5 with the following choices: $h(z):=f(x_0,(u)_R,z)$, $q=p_2$, $A_0=F(R)$, $\vartheta_0=\left[\frac{H(R)}{R^n}\right]^{\frac{p_2-1}{p_2}}$ and

$$\mathcal{H}(w,B_R)=\mathcal{G}_0(w)+\left[\frac{H(R)}{R^n}\right]^{\frac{p_2-1}{p_2}}\int_{B_R}|Dw-Dv_0|dx\ .$$

Then, by property (i) we have for every $\beta>0$

$$\begin{aligned}
&\int_{B_\rho}|Dv_0(x)|^{p_2}dx\\
&\le c\left(\frac{\rho}{R}\right)^n\int_{B_R}(\mu^2+|Dv_0(x)|^2)^{\frac{p_2}{2}}dx+c[F(R)]^{p_2\beta}\int_{B_R}(1+|Du(x)|^{p_2})dx\\
&\quad+c\left[\frac{H(R)}{R^n}\right]^{\frac{p_2-1}{p_2}}[H(R)]^{\frac{1}{p_2}}R^{n\frac{p_2-1}{p_2}}+cR^n\left[\frac{H(R)}{R^n}\right]\left[\frac{1}{F(R)}\right]^{\frac{p_2\beta}{p_2-1}}\\
&\le c\left(\frac{\rho}{R}\right)^n\int_{B_R}(\mu^2+|Du(x)|^2)^{\frac{p_2}{2}}dx+cH(R)+cH(R)[F(R)]^{\frac{p_2\beta}{1-p_2}}\\
&\quad+c[F(R)]^{p_2\beta}\int_{B_R}(1+|Du(x)|^{p_2})dx\ ,
\end{aligned}$$

for any $0<\rho<R$. We choose $\beta>0$ such that

$$\frac{\gamma_1-1}{{\gamma_2}^2}<\frac{p_2-1}{{p_2}^2}<\beta<\frac{p_2-1}{p_2}<\frac{\gamma_2-1}{\gamma_1}\ .$$

Combining the previous facts, we easily get

$$\begin{aligned}
\int_{B_\rho}|Dv_0(x)|^{p_2}dx \le& c\left(\frac{\rho}{R}\right)^n\int_{B_R}(\mu^2+|Du(x)|^2)^{\frac{p_2}{2}}dx\\
&+c[F(R)]^{p_2\beta}\int_{B_{4R}}(1+|Du(x)|^{p_2})dx\ ,
\end{aligned} \tag{4.13}$$

with $c\equiv c(\gamma_1,\gamma_2,n,L,M_1,M_2)$ and for any $0<\rho<R$. Now we use (4.13) obtaining

$$\int_{B_\rho}|Du(x)|^{p_2}dx\le c\int_{B_\rho}|Dv_0(x)|^{p_2}dx+c\int_{B_\rho}|Du(x)-Dv_0(x)|^{p_2}dx$$

$$
\begin{aligned}
&\le c\left[\left(\frac{\rho}{R}\right)^n + [F(R)]^{p_2\beta}\right]\int_{B_{4R}} |Du(x)|^{p_2}dx \\
&\qquad + cR^n + c\int_{B_{R/2}} |Du(x) - Dv_0(x)|^{p_2}dx\ .
\end{aligned}
\tag{4.14}
$$

again for any $0 < \rho < R$. In order to complete the proof, we have to estimate of the last term in the previous formula. We are going to do this by (4.12), (4.4), (2.3) and Theorem 3.1. We choose $\theta \in (0,1)$ such that $\theta/s + 1 - \theta = 1/p_2$; then, recalling that $s \in (p_2, p_2(1+\delta/4))$, we have that

$$
\begin{aligned}
&\int_{B_{R/2}} |Du(x) - Dv_0(x)|^{p_2}dx \\
\le\ & cR^n\left(\fint_{B_{R/2}} |Du(x) - Dv_0(x)|^s dx\right)^{\frac{\theta p_2}{s}} \left(\fint_{B_{R/2}} |Du(x) - Dv_0(x)|dx\right)^{(1-\theta)p_2} \\
\le\ & cR^n[H(R)^{\frac{1}{p_2}}R^{-\frac{n}{p_2}}]^{(1-\theta)p_2} \\
&\qquad \times\left[\left(\fint_{B_{R/2}} |Du(x)|^s dx\right)^{\frac{\theta p_2}{s}} + \left(\fint_{B_{R/2}} |Dv_0(x)|^s dx\right)^{\frac{\theta p_2}{s}}\right] \\
\le\ & cR^{n\theta}[H(R)]^{(1-\theta)} \\
&\qquad \times\left[\left(\fint_{B_{R/2}} |Du(x)|^{p_2(1+\delta/4)}dx\right)^{\frac{\theta}{1+\delta/4}} + \left(\fint_{B_{4R}} (1 + |Du(x)|^{p_2})dx\right)^{\theta}\right] \\
\le\ & cR^{n\theta}[H(R)]^{(1-\theta)}\left[\left(\fint_{B_{R/2}} (1 + |Du(x)|^{p(x)(1+\delta/4+\omega_1(8R))})dx\right)^{\frac{\theta}{1+\delta/4}}\right. \\
&\qquad \left. + \left(\fint_{B_{4R}} (1 + |Du(x)|^{p_2})dx\right)^{\theta}\right] \\
\le\ & cR^{n\theta}[H(R)]^{(1-\theta)}\left[\left(\fint_{B_R} (1 + |Du(x)|^{p(x)})dx\right)^{\frac{\theta(1+\delta/4+\omega_1(8R))}{1+\delta/4}}\right. \\
&\qquad \left. + \left(\fint_{B_{4R}} (1 + |Du(x)|^{p_2})dx\right)^{\theta}\right] \\
\le\ & c(M_1)R^{n\theta}[H(R)]^{(1-\theta)}\left[R^{-n\frac{\theta\omega_1(8R)}{1+\delta/4}}\left(\fint_{B_R} (1 + |Du(x)|^{p_2})dx\right)^{\theta}\right. \\
&\qquad \left. + \left(\fint_{B_{4R}} (1 + |Du(x)|^{p_2})dx\right)^{\theta}\right] \\
\le\ & c(L)\left(\int_{B_{4R}} (1 + |Du(x)|^{p_2})dx\right)^{\theta}[H(R)]^{(1-\theta)}
\end{aligned}
$$

$$\leq c[F(R)]^{(1-\theta)} \int_{B_{4R}} (1 + |Du(x)|^{p_2}) dx .$$

In the previous estimate the constant depends on $(\gamma_1, \gamma_2, n, L, M_1, M_2)$ while we remark that we used (2.3) to bound $R^{-n\frac{\theta\omega_1(8R)}{1+\delta/4}} \leq c$. We can now insert this estimate in (4.14) and get

$$\int_{B_\rho} |Du(x)|^{p_2} dx \leq c \left[\left(\frac{\rho}{R}\right)^n + [F(R)]^{(1-\theta)} + [F(R)]^{p_2\beta} \right] \int_{B_{4R}} |Du(x)|^{p_2} dx + cR^n .$$

for any $0 < \rho < R$. We set $W(R) := [F(R)]^{(1-\theta)} + [F(R)]^{p_2\beta}$; from our assumptions it is clear that

$$\lim_{R\to 0} W(R) = 0 .$$

Therefore, since the function

$$R \to p_2(R)$$

is non-decreasing, we may estimate:

$$\int_{B_\rho} |Du(x)|^{p_2(\rho)} dx \leq c \left[\left(\frac{\rho}{R}\right)^n + W(R) \right] \int_{B_{4R}} |Du(x)|^{p_2(R)} dx + cR^n ,$$

for any $0 < \rho < R$, where c depends only on $\gamma_1, \gamma_2, n, L, M_1, M_2$. At this point the conclusion come arguing as in the last part of the proof of [AM1], Proposition 3.1; so fixing $0 < \tau < n$, by [AM1], Lemma 3.2 if we take $R_1 > 0$ depending only on $\gamma_1, \gamma_2, L, M_1, M_2, \omega_1, \omega_2, \tau$, such that $W(R) \leq \varepsilon_0$ whenever $0 < R < 16R_1$, we may conclude, since $p_2(\rho)$ is nondecreasing with respect to ρ,

$$\int_{B_\rho} |Du(x)|^{p_2(\rho)} dx \leq c(M_2)\rho^{n-\tau}$$

whenever $0 < \rho < R_1$, a fact that we may assume without loss of generality. On the other hand $\gamma_1 \leq p_2(\rho)$; so that

$$\int_{B_\rho} |Du(x)|^{\gamma_1} dx \leq c(M_2)\rho^{n-\tau}$$

for any $0 < \rho < R_1$. At this point the thesis of the theorem follows from an integral characterization of Hölder continuous functions due to Campanato (see [G], chapter 2, section 3) together with a standard covering argument. □

Proof of Theorem 2.4. The proof of this theorem can be achieved following Remark 3.3 from [AM1] observing that, fixed the Hölder continuity exponent α, in order to apply the iteration lemma as Proposition 3.1 from [AM1], the assumption (2.5) is only used to establish that, for a constant $\lambda \equiv \lambda(n, p(x), L, \alpha) > 0$ it follows there exists $R_1 \equiv R_1(n, p(x), L, \alpha)$ such that:

$$\lim_{R\to 0} \omega_1(R) \log\left(\frac{1}{R}\right) + \omega_2(R) \leq \lambda ,$$

that is exactly (2.6). □

5. – Proof of Theorem 2.5.

Let f be as in the assumptions of the theorem. For any $u \in W^{1,p(x_0)}(B(x_c,R))$, the problem

$$\min\left\{\int_{B(x_c,R)} f(x_0,(u)_R,Dw)dx : w \in u + W_0^{1,p(x_0)}(B(x_c,R))\right\} \tag{5.1}$$

has a unique solution that we will denote with v. Using [Ma], estimates (2.4) and (2.5), we can easily obtain

$$\begin{aligned}
&\fint_{B(x_c,\rho)} |Dv(x)-(Dv)_{x_c,\rho}|^{p(x_0)}dx \\
&= \fint_{B(x_c,\rho)}\left|\fint_{B(x_c,\rho)}(Dv(x)-Dv(y))\,dy\right|^{p(x_0)}dx \\
&\leq \left[\sup_{x,y\in B(x_c,\rho)}|Dv(x)-Dv(y)|\right]^{p(x_0)} \\
&\leq \left[c\left(\frac{\rho}{R}\right)^{\beta}\sup_{B_{R/2}}|Dv|\right]^{p(x_0)} \\
&\leq c\left(\frac{\rho}{R}\right)^{\beta p(x_0)}\fint_{B(x_c,R)}(1+|Dv(x)|^{p(x_0)})dx\ ,
\end{aligned} \tag{5.2}$$

where $\rho \leq R/2,\ c>0,\ 0<\beta<1$ and both c and β depend only on γ_1,γ_2,L. We consider the ball $B(x_c,4R) \subset\subset B_{R_0/4}$; from now on, when not differently specified, all the balls considered will have the same center x_c. We set $p_2 := \max_{\overline{B_{4R}}} p(x) \equiv p_2(R)$. Let $\tau = \frac{\alpha\xi\beta}{2(n+\beta)}$, where we fix

$$\xi := \min\left\{\frac{1}{4},\frac{\tilde{m}\sigma}{2}\right\}\ .$$

where σ and $\tilde{m}$ are as in the proof of Theorem 2.3. Arguing as in the previous section we get that there exists R_1 and a constant c, both only dependent on $L,\gamma_1,\gamma_2,\alpha$ and $\||Du|^{p(x)}\|_{L^1(\Omega)}$, such that, whenever $0<R<R_1$, we obtain

$$\int_{B_R}|Du(x)|^{p_2(R)}dx \leq cR^{n-\tau}\ . \tag{5.3}$$

Let now R be such that $4R<R_1$, take $x_0 \in \overline{B_{4R}}$ such that $p(x_0)=p_2$ and let $v \in u+W_0^{1,p_2}(B_R)$ be the solution of the previous problem (5.1) . Working in a standard way and recalling the definitions of the function $h(z)$ and of the functional $\mathcal{G}_0$ given in (4.7), we get

$$\begin{aligned}
&\mathcal{G}_0(u)-\mathcal{G}_0(v) \\
&=\int_{B_R}\langle Dh(Dv(x)),Du(x)-Dv(x)\rangle dx \qquad [=0]
\end{aligned}$$

$$+\int_{B_R} dx \int_0^1 (1-t) D^2 h(tDu(x)+(1-t)Dv(x))(Du(x)-Dv(x))$$

$$\otimes (Du(x)-Dv(x))dt$$

$$\geq \nu \int_{B_R} dx \int_0^1 (1-t)(\mu^2+|tDu(x)+(1-t)Dv(x)|^2)^{(p_2-2)/2} \tag{5.4}$$

$$\times |Du(x)-Dv(x)|^2 dt$$

$$\geq c^{-1} \int_{B_R} (\mu^2+|Du(x)|^2+|Dv(x)|^2)^{(p_2-2)/2} |Du(x)-Dv(x)|^2 dx \, .$$

We remark (see [SZ]) that the second integral in the first equality may have a singularity when

$$tDu(x)+(1-t)Dv(x)=0 \, , \tag{5.5}$$

but this may happen at most for one value of t. On the other hand $D^2h(p)$ is a positive defined form for $p \neq 0$, so it is not difficult to see that this identity is also valid in the exceptional case in which (5.5) is satisfied for a certain t_0. For example one can erase an interval $(t_0-\varepsilon, t_0+\varepsilon)$ from the integration domain, get the result of the integral and then let $\varepsilon \to 0$. So estimates (5.4) are also valid in the case of functions f of class $\mathcal{C}^2$ with respect to the variable z in the domain $\Omega \times \mathbb{R} \times (\mathbb{R}^n \setminus \{0\})$, while all the other estimates in this section are still valid without differentiability assumptions on f; hence we can prove Theorem 2.5 without approximation arguments.
Arguing as in the previous section, we get

$$\mathcal{G}_0(u) \leq \mathcal{G}_0(v) + c\left[\omega_2^{\sigma}[R^{\tilde{m}}] + \omega_1(R)\log\left(\frac{1}{R}\right)\right] \int_{B_{4R}} (1+|Du(x)|^{p_2}) dx \, .$$

Now, using the assumptions and by the previous definition of ξ, we get

$$\int_{B_R} (\mu^2+|Du(x)|^2+|Dv(x)|^2)^{(p_2-2)/2} |Du(x)-Dv(x)|^2 dx$$

$$\leq c R^{2\alpha\xi} \int_{B_{4R}} (1+|Du(x)|^{p_2}) dx \, .$$

On the other hand, it is not difficult to get the following estimate:

$$\int_{B_R} |Du(x)-Dv(x)|^{p_2(R)} dx \leq c R^{\alpha\xi} \int_{B_{4R}} (1+|Du(x)|^{p_2}) dx \, ; \tag{5.6}$$

in the case $p \geq 2$, the previous inequality is obvious, while in the case $p \leq 2$ we can rapidly deduce it by Hölder inequality (see [AM1], pag.138), the minimality of v and the bounds for f.

Finally, we recall that we choose $4R < R_1$ and so we can use (5.2), (5.3), (5.6), the minimality of v and the fact that the map $R \mapsto p_2(R)$ is nondecreasing, to get

$$
\begin{aligned}
&\int_{B_\rho} |Du - (Du)_\rho|^{p_2} dx \\
&\leq \int_{B_\rho} |Du - (Dv)_\rho|^{p_2} dx \\
&\leq c\rho^n \fint_{B_\rho} |Dv - (Dv)_\rho|^{p_2} dx + c \int_{B_R} |Du(x) - Dv(x)|^{p_2} dx \\
&\leq c\left(\frac{\rho}{R}\right)^{\beta p_2(R)} \rho^n \fint_{B_R} (1 + |Du|^{p_2(R)}) dx \\
&\quad + cR^{\alpha\xi} \int_{B_{4R}} (1 + |Du(x)|^{p_2}) dx \\
&\leq c\left(\frac{\rho}{R}\right)^{\beta p_2(R)} \rho^n + \left(\frac{\rho}{R}\right)^{\beta p_2(R)} \left(\frac{\rho}{R}\right)^n R^{n-\tau} + cR^{\alpha\xi}[R^n + R^{n-\tau}] \\
&\leq c\rho^{n+\beta} R^{-\beta-\tau} + cR^{\alpha\xi} R^{n-\tau} \,.
\end{aligned}
$$

Now we chose $\rho = \frac{1}{2}R^{1+\theta}$ with $\theta = (\alpha\xi)/(n+\beta)$. If we write again the last term only with ρ, we get that the exponent of the two term of the sum are equal and so by the previous choice of τ, they are equal to $n + \lambda$ with $\lambda = (\alpha\xi\beta)/2(n + \beta + \alpha\xi)$; from the choice of α, β, ξ we easily get that $\lambda \geq \lambda_0 > 0$ for some λ_0 dependent only on L, γ_1, γ_2. From the previous chain of inequalities, again by the integral characterization of Hölder continuous functions due to Campanato and the usual covering argument, we get that Du is Hölder continuous. This finishes the proof. $\square$

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