ON THE SHAPE OF CAPILLARITY DROPLETS IN A CONTAINER

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Abstract. We provide a quantitative description of global minimizers of the Gauss free energy for a liquid droplet bounded in a container in the small volume regime.

1. Introduction

Our aim is to provide a quantitative description of capillarity droplets in a bounded container. We work in the classical setting of capillarity theory based on the minimization of Gauss free energy under a volume constraint. In this framework, denoting by A and E two open bounded sets with Lipschitz boundary in $\mathbb{R}^n$ with $E \subset A$ (so that E is the region occupied by a liquid droplet inside a container A), one looks for volume-constrained global/local minimizers or stable/stationary points of the energy

$$\mathcal{F}_{A,\sigma}(E) + \int_E g(x)\, dx\,,$$

where $g : A \to \mathbb{R}$ is a bounded potential energy density and where $\mathcal{F}_{A,\sigma}$ is the surface tension energy of the droplet,

$$\mathcal{F}_{A,\sigma}(E) = \mathcal{H}^{n-1}(A \cap \partial E) + \int_{\partial A \cap \partial E} \sigma\, d\mathcal{H}^{n-1}\,. \tag{1.1}$$

Here $\mathcal{H}^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure in $\mathbb{R}^n$, so that $\mathcal{H}^{n-1}(A \cap \partial E)$ accounts for the surface tension energy of liquid/air internal interface of the droplet, while $\sigma : \partial A \to (-1, 1)$ is a given function, modeling the relative adhesion coefficient between the liquid droplet and the solid walls of the container. In this way, $\int_{\partial A \cap \partial E} \sigma\, d\mathcal{H}^{n-1}$ accounts for the total surface tension energy of the liquid/solid boundary interface. Typically, one considers liquid droplets under the action of gravity, a situation that corresponds to taking $n = 3$ and $g(x) = c\, x_n$ for a positive constant c.

There is a rich variational theory concerning the functional $\mathcal{F}_{A,\sigma}$. A portion of the classical literature on the problem assumes the existence and the smoothness of minimizers, and starting from these assumptions moves to their qualitative description. An excellent overview on this family of results is provided by the book of Finn [Fin86]. Existence theories have to be formulated in the setting of Geometric Measure Theory, and a particularly suitable framework is that of sets of finite perimeter and functions of bounded variation, see, e.g. [Mag12, Chapter 19]. Since the boundary ∂E of a minimizer of finite perimeter E will just be a countable union of compact subsets of C^1-hypersurfaces, then one faces the problem of showing additional regularity for minimizers, which is a crucial step for understanding the relation of the mathematical model with the physical world.

The regularity issue can be trivialized by exploiting symmetries. For example, in the sessile and pendant liquid droplet problems it can be proved that minimizers are rotationally symmetric, a property which easily implies smoothness; see for example [Gon77]. In the case of a generic container one cannot exploit symmetry to simplify the regularity problem. This regularity problem has then drawn the attention of several authors [Tay78, CF85, GJ86, Grü87a, Grü87b, Grü87c, Luc87, CM07, DPM14, DPM15].

For example, if A has boundary of class $C^{1,1}$, g is bounded and σ is a Lipschitz function, then for each volume-constrained local minimizer E of $\mathcal{F}_{A,\alpha}$ there exists a closed subset Σ of $M = \overline{A \cap \partial E}$ such that $M \setminus \Sigma$ is a $C^{1,\alpha}$-hypersurface with boundary (for every $\alpha \in (0,1)$). Moreover, the set of boundary points $\mathrm{bd}\,(M \setminus \Sigma)$ of $M \setminus \Sigma$ is contained in ∂A, and if ν_G denotes the outer unit normal to $G \subset \mathbb{R}^n$, then *Young's law* holds, i.e.

$$\nu_A(x) \cdot \nu_E(x) = \sigma(x)\,, \qquad \forall x \in \mathrm{bd}\,(M \setminus \Sigma)\,.$$

Finally, Σ is empty if $n \le 3$, it is discrete if $n = 4$ and it has Hausdorff dimension less than $n-4$ otherwise. This result was proved in the case $n = 3$ by Taylor [Tay78], and in higher dimension by Luckhaus [Luc87]. Luckhaus' result was unawares rediscovered as [DPM15, Corollary 1.4] in a study on Young's law for anisotropic surface energies. (We also notice that in the above papers the regularity result is obtained with $\alpha = 1/2$, although one can improve this to every $\alpha < 1$ by boundary elliptic regularity.)

Capillarity phenomena are characterized by the dominance of the surface tension energy on the bulk/potential energy term. This is typically the case when the volume parameter m is suitably small with respect to the various data of the problem and then the surface energy is order $m^{(n-1)/n} >> m$, while the potential energy is of order m. In the case where there is no container, the problem of describing global minimizers in the small volume regime has been addressed in [FM11]. There it is shown that if g is a locally bounded potential energy density with the property that $g(x) \to \infty$ as $|x| \to \infty$ (in particular, the gravitational potential is not allowed in this simplified model), then there exists $m_0 = m_0(n, g) > 0$ such that every minimizer E_m in

$$\inf \Big\{ P(E) + \int_E g(x)\,dx : |E| = m \Big\}\,,$$

with $m \le m_0$ is connected and satisfies, for some $y_m \in \mathbb{R}^n$,

$$\Big| \Big(\frac{E_m - y_m}{m^{1/n}} \Big) \Delta \frac{B}{\omega_n^{1/n}} \Big| \le C(n,g)\, m^{1/2n}\,, \qquad \mathrm{hd}\Big(\frac{\partial E_m - y_m}{m^{1/n}}, \frac{\partial B}{\omega_n^{1/n}} \Big) \le C(n,g)\, m^{1/n^2}\,, \tag{1.2}$$

where $B = \{x \in \mathbb{R}^n : |x| < 1\}$, $\omega_n = |B|$, and $\mathrm{hd}(X, Y)$ denotes the Hausdorff distance between $X, Y \subset \mathbb{R}^n$, see (1.18) below. Moreover, if $g \in C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^n)$ for some $\alpha > 0$, then ∂E_m is a C^2-hypersurface whose second fundamental form $\nabla \nu_{E_m}(x)$ at $x \in T_x \partial E_m$, after a suitable rescaling, is uniformly close to a suitable multiple of the identity tensor on $T_x \partial E_m$, with the quantitative estimate

$$\max_{x \in \partial E_m} \| m^{1/n}\, \nabla \nu_{E_m}(x) - c(n)\, \mathrm{Id}_{T_x \partial E_m} \| \le C(n,g)\, m^{2/(n+2)}\,. \tag{1.3}$$

In particular, E_m is convex. What is crucial here is the quantitative aspect of (1.2) and (1.3). We are not only asserting a proximity result, we are also quantifying the distance (measured in various ways) from being a ball. In passing, let us also mention that this kind of analysis has been recently extended to the case of local minimizers, and even of stationary points, in [CM15].

Returning to the case where a container A is present, our goal here is to quantitatively describe the shape of global minimizers E_m of

$$\gamma(m) = \inf \Big\{ \mathcal{F}_{A,\sigma}(E) + \int_E g(x)\,dx : E \subset A\,, |E| = m \Big\}\,, \tag{1.4}$$

in the small volume regime. When g and σ vanish identically, then (2.1) reduces to the well-known relative isoperimetric problem in A, and global minimizers in (2.1) are called *isoperimetric regions in A*. In this case Fall [Fal10] has shown that isoperimetric regions converge, as $m \to 0^+$, to boundary points of A where the mean curvature of ∂A achieves its maximum. When A is a convex polytope (and again g and σ vanish identically) then an analogous result was obtained by Ritoré and Vernadakis, who showed that isoperimetric regions converge to vertices of ∂A with smallest solid angle [RV15].

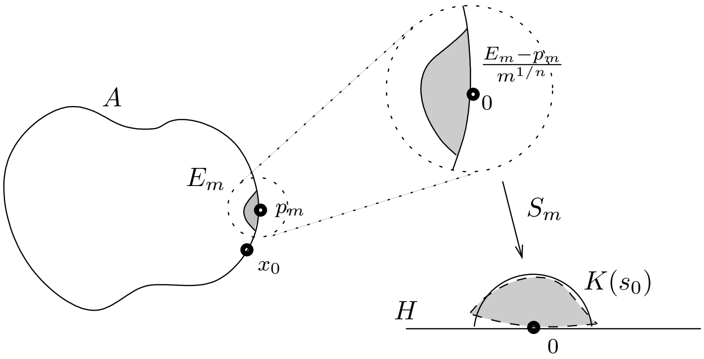

FIGURE 1. The situation in Theorem 1.1. A global minimizer E_m is confined in a ball of radius $O(m^{1/n})$ centered at a point $p_m \in \partial A$ such that $\sigma(p_m) - \sigma_0 = O(m^{1/n})$. In particular, if $\sigma_0 = \sigma(x_0)$ for a unique $x_0 \in \partial\Omega$, then p_m has to converge to x_0 with a velocity that depends on how fast σ detaches from its minimum value at x_0. After a blow-up at p_m to the scale $m^{1/n}$, and after a suitable rigid motion S_m, the global minimizer is Hausdorff close to the ideal droplet $K(\sigma_0)$, and the part of its boundary interior to A is actually $C^{1,\alpha}$-diffeomorphic (for every $\alpha \in (0,1)$) to the spherical cap $H \cap \partial K(\sigma_0)$.

We thus expect, also the general case where σ and g are non-trivial, that E_m should have small diameter and that it should concentrate around a boundary point of A. Specifically, (global) energy minimization should favor those points in ∂A where σ achieves its minimum value

$$\sigma_0 = \min_{\partial A} \sigma \in (-1,1)\,,$$

and correspondingly E_m should be contained in a ball of radius $O(m^{1/n})$ and center $p_m \in \partial A$ such that $\sigma(p_m) \to \sigma_0$ as $m \to 0^+$. In particular, in the small volume regime and if σ has a strict minimum point on ∂A, then neither the potential energy nor the mean curvature of ∂A should play a role in determining the asymptotic position of global minimizers.

Next, assuming this concentration at the boundary to happen, we would expect the blow-ups

$$\frac{E_m - p_m}{m^{1/n}}$$

to converge (as $m \to 0$) to minimizers in the sessile droplet problem with no gravity and constant adhesion coefficient σ_0. To be more precise, let us fix a reference half-space H to be

$$H = \{x \in \mathbb{R}^n : x_n > 0\}\,.$$

and given $\tau \in (-1,1)$ let us consider the variational problem

$$\psi(\tau) = \inf\Big\{\mathcal{F}_{H,\tau}(F) : F \subset H\,, |F| = 1\Big\}\,. \tag{1.5}$$

If we set

$$S(\tau) = \{x \in B : x_n > -\tau\}\,, \qquad K(\tau) = \tau\, e_n + \frac{S(\tau)}{|S(\tau)|^{1/n}}\,, \tag{1.6}$$

then $\{z + K(\tau) : z \in \partial H\}$ is the family of all minimizers in (1.5), see e.g. [Mag12, Theorem 19.21]. We thus expect $(E_m - p_m)/m^{1/n}$ to converge in some sense to $K(\sigma_0)$. Our main result, which is illustrated in Figure 1, proves that this converge happens in a global $C^{1,\alpha}$-sense, and it also quantifies the rate of convergence in Hausdorff distance.

Theorem 1.1. *Let $n \ge 2$, A be a bounded open connected set with boundary of class $C^{1,1}$, let $\sigma \in \mathrm{Lip}(\partial A)$ with $-1 < \sigma(x) < 1$ for every $x \in \partial A$, and let $g \in \mathrm{Lip}(A)$. Then there exist positive constants C_0 and m_0 depending on A, σ and g with the following property.*

If E_m is a minimizer in (1.4) with $m \le m_0$, then there exists $p_m \in \partial A$ such that

$$E_m \subset B_{p_m, C_0\, m^{1/n}} \qquad 0 \le \sigma(p_m) - \sigma_0 \le C_0\, m^{1/n}\,, \tag{1.7}$$

and, for a linear isometry S_m of $\mathbb{R}^n$ and with $M_m = \overline{A \cap \partial E_m}$ and $M_0 = \overline{H \cap \partial K(\sigma_0)}$,

$$\mathrm{hd}\Big(S_m\Big(\frac{M_m - p_m}{m^{1/n}}\Big), M_0\Big) \le C_0\, m^{1/2\,n^2}\,. \tag{1.8}$$

Moreover, M_m is a $C^{1,\alpha}$-hypersurface with boundary for every $\alpha \in (0,1)$, the set of boundary points $\mathrm{bd}\,(M_m)$ is contained in ∂A, and there exist a $C^{1,\alpha}$-diffeomorphism $f_m : M_0 \to M_m$ and $y_m \in \partial A$ such that

$$\inf\Big\{\|f_m - \big(y_m + m^{1/n} S\big)\|_{C^1(M_0)} : S : \mathbb{R}^n \to \mathbb{R}^n \text{ is an isometry}\Big\} = o(m^{1/n})\,, \tag{1.9}$$

as $m \to 0^+$, where this limit relation depends on A, σ and g only, but not on the family of minimizers E_m.

Remark 1.2. In the course of the proof of Theorem 1.1 we shall prove that

$$\gamma(m) = \psi(\sigma_0)\, m^{(n-1)/n}\Big(1 + O(m^{1/n})\Big)\,, \tag{1.10}$$

see Remark 4.2. When g and σ are identically zero this formula was proved by Bayle and Rosales [BR05], and the coefficient in front of $O(m^{1/n})$ was identified by Fall in [Fal10], thus leading to the above mentioned criterion that, in the small volume regime, isoperimetric regions converge to boundary points of maximal mean curvature.

We now describe the proof of Theorem 1.1. An initial difficulty is excluding that minimizers break down into smaller droplets, or that they take elongated shapes with comparatively larger diameter than volume. This issue is partially addressed by a grid argument (see in particular step four in the proof of Lemma 3.1 below) where it is shown the existence of points $x_m \in A$ such that

$$E_m \subset B_{x_m, C\, m^{1/2n}}\,. \tag{1.11}$$

(Here $B_{x,r} = x + B_r$ is the ball of center x and radius r in $\mathbb{R}^n$.) Although the possibility of replacing $x_m \in A$ by $y_m \in \partial A$ with $\sigma(y_m) - \sigma_0 \le C_0\, m^{1/2n}$ follows from (1.11) by a direct variational argument, we are not able, at this stage of the proof, to improve the diameter estimate from the order $m^{1/2n}$ to the natural order $m^{1/n}$. (In other words, our droplet, rescaled by a factor $m^{1/n}$ so to bring it to unit volume, could still look like a very elongated ellipsoid.) The inclusion (1.11), with $y_m \in \partial A$ in place of $x_m \in A$, is however sufficient to use the boundary diffeomorphisms of A to map back E_m from the container into our reference half-space H. More precisely (see Notation 3.2 below) there exist positive constants s_0 and r_0 such that for every $y \in \partial A$ we can find an open set $U_y \subset \mathbb{R}^n$ with $B_{s_0} \subset U_y$, and a $C^{1,1}$-diffeomorphism $\phi_y : U_y \to B_{y,2r_0}$ with $\phi_y(0) = y$ and $\nabla\phi_y(0)$ an orientation preserving isometry, such that

$$\phi_y(U_y \cap H) = B_{y,2r_0} \cap A\,, \qquad \phi_y(U_y \cap \partial H) = B_{y,2r_0} \cap \partial A\,.$$

Thanks to (1.11), we can thus consider the deformed and rescaled shapes

$$F_m = \frac{\phi_{y_m}^{-1}(E_m)}{\lambda_m}\,, \qquad \lambda_m = |\phi_{y_m}^{-1}(E_m)|^{1/n} = m^{1/n}\Big(1 + O(m^{1/2n})\Big)\,.$$

Clearly $F_m \subset H$ and $|F_m| = 1$, and moreover by exploiting the minimality of E_m and the fact that $\sigma(y_m) \to \sigma_0$ as $m \to 0^+$, one can see that

$$\mathcal{F}_{H,\sigma_0}(F_m) - \psi(\sigma_0) \le C\, m^{1/n}\,, \tag{1.12}$$

so that F_m is asymptotically optimal in (1.5) with $\tau = \sigma_0$. We can thus exploit a qualitative stability theorem (see Proposition 2.1-(ii)) to show the existence of $z_m \in \partial H$ such that

$$\lim_{m\to 0} |(F_m - z_m)\Delta K(\sigma_0)| = 0\,. \tag{1.13}$$

The arguments described so far are contained in Lemma 3.1 below. The next step in our analysis is Lemma 4.1, where we show that the existence of positive constants Λ and ρ_0, and of elliptic functionals $\mathbf{\Psi}_m$ such that each F_m is a (Λ, ρ_0)-minimizer of the corresponding $\mathbf{\Psi}_m$, i.e.

$$\mathbf{\Psi}_m(F_m; W) \le \mathbf{\Psi}_m(F; W) + \Lambda\, |F\Delta F_m|\,, \tag{1.14}$$

whenever $F \subset H$, $F_m\Delta F \subset\subset W$ for some open set W with $\mathrm{diam}(W) < 2\rho_0$; see Definition 2.2 and Definition 2.3 for the terminology and notation used here. Since we can show that each Ψ_m is λ-elliptic on a $3\rho_0$-neighborhood of F_m (for some $\lambda \ge 1$ independent of m), the minimality inequality (1.14) implies uniform volume density estimates at boundary points of each F_m. In turn, this information allows one to improve (1.13) into

$$\lim_{m\to 0} \mathrm{hd}\big(F_m - z_m, K(\sigma_0)\big) = 0\,, \tag{1.15}$$

so that $\mathrm{diam}(F_m) \le C$, and thus the natural diameter estimate $\mathrm{diam}(E_m) \le C\, m^{1/n}$.

The next step in our analysis is to notice that if we set

$$\Phi(\nu) = \sup\big\{x\cdot\nu : \nu \in S(\sigma_0)\big\}\,, \qquad \nu \in S^{n-1}\,,$$

then, in the terminology of [FM91], $S(\sigma_0)$ is the Wulff shape associated to Φ and

$$\mathcal{F}_{H,\sigma_0}(F) \ge \mathbf{\Phi}(F) = \int_{\partial^* F} \Phi(\nu_F)\, d\mathcal{H}^{n-1}\,, \qquad \forall F \subset H\,,$$
$$\mathcal{F}_{H,\sigma_0}(K(\sigma_0)) = \mathbf{\Phi}(K(\sigma_0)) = \psi(\sigma_0)\,,$$

where $K(\sigma_0)$ is just a translation of the unit volume rescaling of $S(\sigma_0)$. In particular, by (1.12) we have $\mathbf{\Phi}(F_m) - \mathbf{\Phi}(K(\sigma_0)) \le C\, m^{1/2n}$, and then by the quantitative Wulff inequality from [FMP10] we infer the existence of $w_m \in \mathbb{R}^n$ such that

$$|(F_m - w_m)\Delta K(\sigma_0)| \le C\, m^{1/2n}\,. \tag{1.16}$$

We now exploit a simple geometric argument from [FI13] together with the inclusion

$$\frac{K(\sigma_0)}{2} \subset F_m - z_m$$

(which follows immediately from (1.15)) to conclude that one may as well take $w_m \cdot e_n = 0$, and thus set $w_m = z_m \in \partial H$ in (1.16). By combining this fact with the uniform volume density estimates for F_m, we are able to quantify (1.15) and obtain

$$\mathrm{hd}\big(F_m - z_m, K(\sigma_0)\big) \le C\, m^{1/2n^2}\,. \tag{1.17}$$

In order to complete the proof of Theorem 1.1 we are thus left to construct the diffeomorphisms between M_0 and M_m, and to rewrite (1.17) in terms of $(E_m - y_m)/m^{1/n}$. We comment here only on the former task, which is achieved by combining the boundary regularity theorem from [DPM15] with a tool for constructing almost-normal diffeomorphsims between manifolds with boundary which was recently presented in [CLM14]. The corresponding diffeomorphisms enjoy a quite rigid structure, which should allow one to quantify more explicitly the rate of convergence in (1.9). We leave this task for future investigations.

We now describe the organization of our paper. Section 2 is focused on the sessile droplet problem with no gravity, see (1.5). We first discuss some stability properties, see Proposition 2.1, and then we present an improved convergence theorem for sequence of uniform almost-minimizers of $\mathcal{F}_{H,\tau}$ converging in volume to $K(\tau)$, Theorem 2.4. In the latter result, convergence in volume is improved to C^1-convergence, in the sense that we extract from the almost-minimality condition the existence of $C^{1,\alpha}$-diffeomorphisms between the interior interfaces M_0 and M_m converging in C^1 to the identity map. In section 3 we begin the proof of Theorem 1.1. In particular, in Lemma 3.1, we obtain (1.11) (with $y_m \in \partial A$ in place of x_m) and prove (1.13) along the lines described above. The proof of Theorem 1.1 is then concluded in section 4, first by proving the uniform almost-minimality of the sets F_m (see Lemma 4.1), and then by wrapping-up the various information collected up to that point into a final discussion. We conclude this introduction by gathering the basic notation used in the paper.

Sets in $\mathbb{R}^n$: Given $x \in \mathbb{R}^n$ and $r > 0$, the ball of center x and radius r is denoted by $B_{x,r} = \{y \in \mathbb{R}^n : |x - y| < r\}$, and we set $B_r = B_{0,r}$, $B = B_1 = B_{0,1}$. Given $X, Y \subset \mathbb{R}^n$, the Hausdorff distance between X and Y is defined as

$$\mathrm{hd}(X,Y) = \max\Big\{\sup_{x\in X}\mathrm{dist}(x,Y), \sup_{x\in Y}\mathrm{dist}(x,X)\Big\}\,, \tag{1.18}$$

while $I_\rho(X) = \{x \in \mathbb{R}^n : \mathrm{dist}(x, X) < \rho\}$ denotes the ρ-neighborhood of X, $\rho > 0$.

Manifolds in $\mathbb{R}^n$: If $M \subset \mathbb{R}^n$ is a k-dimensional manifold with boundary, $1 \le k \le n-1$, then we denote by $\mathrm{int}\,(M)$ and $\mathrm{bd}\,(M)$ its interior and boundary points respectively, by ν_M^{co} the outer unit normal to $\mathrm{bd}\,(M)$ in M, and we set

$$[M]_\rho = M \setminus I_\rho(\mathrm{bd}\,(M))\,, \qquad \rho > 0\,.$$

Sets of finite perimeter: Given a Borel set $E \subset \mathbb{R}^n$ of locally finite perimeter in $\mathbb{R}^n$, we denote by $\partial^* E$ and ν_E the reduced boundary and the measure-theoretic outer unit normal of E. We have

$$\overline{\partial^* E} = \Big\{x \in \mathbb{R}^n : 0 < |B_{x,r} \cap E| < |B_{x,r}|\ \forall r > 0\Big\} \subset \partial E\,,$$

and, up to modifying E by a set of volume zero, one can always achieve

$$\overline{\partial^* E} = \partial E\,,$$

see [Mag12, Proposition 12.19]. We shall always assume that the sets of finite perimeter under consideration have been modified in this way.

Acknowledgment: This work was supported by the NSF Grants DMS-1265910 and DMS-1361122.

2. Some properties of droplets in half-spaces

In this section we discuss some basic properties of the variational problem

$$\psi(\tau) = \inf\big\{\mathcal{F}_{H,\tau}(F) : F \subset H\,, |F| = 1\big\}\,, \tag{2.1}$$

where $H = \{x \in \mathbb{R}^n : x_n > 0\}$ and $\tau \in (-1, 1)$. Let us recall from the introduction that if we set

$$S(\tau) = \{x \in B : x_n > -\tau\}\,, \tag{2.2}$$

then the unique minimizer $K(\tau)$ in (2.1) with horizontal barycenter at the origin is given by the formula

$$K(\tau) = \tau\, e_n + \frac{S(\tau)}{|S(\tau)|^{1/n}}\,. \tag{2.3}$$

In other words, F is a minimizer in (2.1) if and only if $F = z + K(\tau)$ for some $z \in \partial H$. In section 2.1 we discuss some stability properties of (2.1), while section 2.2 contains an improved convergence theorem towards the ideal droplet $K(\tau)$.

2.1. **Stability properties.** The following proposition collects the properties of problem (2.1) that we shall need in the study of (1.4). Property (i) consists just in the monotonicity of ψ, while property (ii) is a qualitative stability statement. In property (iii) we exploit the main result of [FMP10] to quantify stability under a technical containment assumption. This containment assumption is not needed, and indeed it could be eliminated by mimicking the arguments in [FI13]. However, this more general result is not needed here.

Proposition 2.1. *(i) One has $\psi'(\tau) > 0$ for every $\tau \in (-1, 1)$.*

(ii) If $\{F_h\}_{h\in\mathbb{N}}$ is a sequence of subsets of H with $|F_h| = 1$ for every $h \in \mathbb{N}$ and

$$\lim_{h\to\infty} \mathcal{F}_{H,\tau}(F_h) = \psi(\tau)\,,$$

then there exists $\{z_h\}_{h\in\mathbb{N}} \subset \partial H$ such that, up to extracting subsequences, one has

$$\lim_{h\to\infty} |(F_h + z_h)\Delta K(\tau)| = 0\,.$$

(iii) There exist positive constants $\varepsilon(n,\tau)$ and $c(n,\tau)$ with the following property: If $F \subset H$ with $|F| = 1$ and

$$\frac{K(\tau)}{2} \subset F\,, \qquad \mathcal{F}_{H,\tau}(F) \le (1 + \varepsilon(n,\tau))\,\psi(\tau)\,, \tag{2.4}$$

then

$$\mathcal{F}_{H,\tau}(F) - \psi(\tau) \ge c(n,\tau) \inf_{z\in\partial H} |(F - z)\Delta K(\tau)|^2\,. \tag{2.5}$$

Proof of Proposition 2.1-(i). Given $\tau \in (-1, 1)$, let

$$V(\tau) = |S(\tau)|\,, \qquad A(\tau) = P(S(\tau); \{x_n > -\tau\})\,, \qquad A_0(\tau) = P(S(\tau); \{x_n = -\tau\})\,.$$

Since $\nu_B \cdot (-e_n) = \tau$ along $\{x_n = -\tau\} \cap \partial S(\tau)$, by (2.3) we find that

$$\psi(\tau) = \mathcal{F}_{H,\tau}(K(\tau)) = \frac{A(\tau) + \tau\, A_0(\tau)}{V(\tau)^{(n-1)/n}}\,.$$

We now notice that, if ω_k denotes the volume of the unit sphere in $\mathbb{R}^k$, then

$$V(\tau) = \int_{-\tau}^{1} \omega_{n-1}(1-\rho^2)^{(n-1)/2} d\rho\,, \quad A(\tau) = \int_{-\tau}^{1} (n-1)\omega_{n-1}(1-\rho^2)^{(n-3)/2} d\rho\,,$$

while $A_0(\tau) = \omega_{n-1}(1-\tau^2)^{(n-1)/2}$. On noticing that

$$V' = A_0\,, \qquad (A + \tau A_0)' = (n-1)\frac{A_0}{1-\tau^2} + A_0 - (n-1)\frac{\tau^2 A_0}{1-\tau^2} = n\,A_0\,,$$

we find that

$$\psi' = \frac{1}{V^{2-(2/n)}}\Big(V^{1-(1/n)} n\,A_0 - \frac{n-1}{n}\,V^{-1/n}\,A_0\,(A + \tau A_0)\Big) = \frac{A_0}{n\,V^{2-(1/n)}}\,\varphi\,,$$

where $\varphi = n^2\,V - (n-1)(A + \tau\,A_0)$. By the divergence theorem, $A(0) = n\,V(0)$, where $V(0) = \omega_n/2$, so that $\varphi(0) = n\,\omega_n/2$. Moreover, $\varphi' = n^2 A_0 - (n-1)nA_0 = nA_0$, thus

$$\varphi(\tau) = n\Big(\frac{\omega_n}{2} + \int_0^\tau A_0\Big) = n\Big(\frac{\omega_n}{2} + \mathrm{sign}(\tau)\,\Big|\Big\{x \in B : 0 < x_n < |\tau|\Big\}\Big|\Big) > 0\,,$$

for every $\tau \in (-1, 1)$. This proves that $\psi' > 0$ on $(-1, 1)$. $\square$

We now discuss statement (ii). As usual the issue is ensuring compactness in volume. We solve this problem by combining slicing with isoperimetry to prove that one can always reduce to consider sequences of sets F_h with uniformly bounded diameters, which therefore are compact in volume modulo horizontal translations. The main modifications with respect to the case of the standard isoperimetric problem, corresponding formally to $\tau = 1$, (see, for example, [FMP08, Lemma 5.1], which in turn was inspired by [FM91, Theorem 3.1]) are found in the case $\tau < 0$.

Before entering into the proof, it is convenient to introduce some notation and terminology in analogy with [FMP08]. Given $F \subset H$ and $\tau \in (-1,1)$ we define the *deficit* of F (relatively to the variational problem (2.1)) as

$$\delta_\tau(F) = \frac{\mathcal{F}_{H,\tau}(F)}{\psi(\tau)\,|F|^{(n-1)/n}} - 1\,.$$

For every $\lambda > 0$ we have $\delta_\tau(\lambda F) = \delta_\tau(F) \ge 0$. Moreover, by [Mag12, Theorem 19.21], we have that $\delta_\tau(F) = 0$ if and only if $|F\Delta(z + r\,K(\tau))| = 0$ for some $z \in \partial H$ and $r > 0$. Correspondingly we define the *asymmetry index* of F (again, relatively to problem (2.1)) as

$$\alpha_\tau(F) = \inf\Big\{\frac{|F\Delta(z + r\,K(\tau))|}{|F|} : r^n = |F|\,, z \in \partial H\Big\}\,.$$

With this terminology in force, statement (ii) is equivalent in saying that if $\delta_\tau(F_h) \to 0$, then $\alpha_\tau(F_h) \to 0$. The key point in the proof will thus be obtaining (2.8) and (2.25) below.

Proof of Proposition 2.1-(ii). Step one: As a preparatory remark, we show that if $G, F \subset \mathbb{R}^n$ with

$$|G| \le |F|\,, \qquad |F\Delta G| \le \varepsilon\,|F|, \qquad \mathcal{F}_{H,\tau}(G) \le \mathcal{F}_{H,\tau}(F) + (\varepsilon\,|F|)^{(n-1)/n}\,, \tag{2.6}$$

for some $\varepsilon \in (0,1)$, then

$$|\alpha_\tau(F) - \alpha_\tau(G)| \le 3\,\varepsilon, \qquad \delta_\tau(G) \le \frac{\delta_\tau(F) + (1 + \psi(\tau)^{-1})\,\varepsilon^{(n-1)/n}}{(1-\varepsilon)^{(n-1)/n}}\,. \tag{2.7}$$

Since the volume of the symmetric difference defines a distance on subsets of $\mathbb{R}^n$, we easily find that $||F|\alpha_\tau(F) - |G|\alpha_\tau(G)| \le |F\Delta G|$. Hence, by $\alpha_\tau(G) < 2$,

$$|F||\alpha_\tau(F) - \alpha_\tau(G)| \le |F\Delta G| + ||F| - |G||\,\alpha_\tau(G) \le (1 + \alpha_\tau(G))\,|F\Delta G| \le 3\,|F\Delta G|\,,$$

and the first estimate in (2.7) follows. Next we notice that

$$\begin{aligned}
|G|^{(n-1)/n}\delta_\tau(G) - |F|^{(n-1)/n}\delta_\tau(F) \;&\le\; \frac{\mathcal{F}_\tau(G) - \mathcal{F}_\tau(F)}{\psi(\tau)} + |F|^{(n-1)/n} - |G|^{(n-1)/n}\\
&\le\; \frac{(\varepsilon|F|)^{(n-1)/n}}{\psi(\tau)} + ||F| - |G||^{(n-1)/n}\\
&\le\; \Big(1 + \frac{1}{\psi(\tau)}\Big)\,(\varepsilon|F|)^{(n-1)/n}\,,
\end{aligned}$$

and we conclude the proof by $|G| \ge (1-\varepsilon)|F|$.

Step two: We claim that if $F \subset H$ with $\delta_\tau(F) \le \delta_0(n,\tau) < 1$, then there exists $G \subset H$ with

$$|\alpha_\tau(G) - \alpha_\tau(F)| \le C(n,\tau)\,\delta_\tau(F)\,, \quad \delta_\tau(G) \le C(n,\tau)\,\delta_\tau(F)\,, \quad \sup_{x,y\in G}\frac{|x_1 - y_1|}{|F|^{1/n}} \le C(n,\tau)\,, \tag{2.8}$$

where $\delta_0(n,\tau)$ and $C(n,\tau)$ are suitable positive constants. Without loss of generality, we may assume that $|F| = 1$ and $F = F^{(1)}$, the set of points of density 1 of F. Thus it suffices to prove

$$\sup_{x,y\in G}|x_1 - y_1| \le C(n,\tau)\,, \tag{2.9}$$

together with

$$|F\Delta G| \le C(n,\tau)\,\delta_\tau(F)^{n/(n-1)}\,, \qquad \mathcal{F}_{H,\tau}(G) \le \mathcal{F}_{H,\tau}(F) + C(n,\tau)\,\delta_\tau(F)\,, \tag{2.10}$$

which take the place of the first two inequalities in (2.8) by step one. Let us set

$$v(t) = |F \cap \{x_1 < t\}|\,, \qquad s(t) = \mathcal{H}^{n-1}(F \cap \{x_1 = t\})\,, \tag{2.11}$$

so that $v(t)$ is absolutely continuous on $\mathbb{R}$ with $v'(t) = s(t)$ for a.e. $t \in \mathbb{R}$ thanks to Fubini's theorem. We notice that for every $t \in \mathbb{R}$ one has

$$\begin{aligned} s(t) = \mathcal{H}^{n-1}(F \cap \{x_1 = t\}) \le P(F; H \cap \{x_1 < t\})\,, \\ P(F; \{x_1 < t\} \cap \partial H) \le P(F; H \cap \{x_1 < t\})\,. \end{aligned} \tag{2.12}$$

Indeed, by [Mag12, Equation (16.7)] and $F = F^{(1)}$ one finds that

$$\begin{aligned} \partial^*(F \cap \{x_1 < t\}) = &\big(H \cap \{x_1 < t\} \cap \partial^* F\big) \\ &\cup \big(\{x_1 < t\} \cap \partial H \cap \partial^* F\big) \\ &\cup \big(F \cap \{x_1 = t\}\big) \\ &\cup \big(\partial^* F \cap \{x_1 = t\} \cap \{\nu_F = e_1\}\big) \end{aligned} \tag{2.13}$$

where the identity holds up to $\mathcal{H}^{n-1}$-negligible sets, and where the elements of the union are disjoint. Since the outer unit normal to $F \cap \{x_1 < t\}$ coincides with ν_F on the first set on the right-hand side, with $-e_n$ on the second one, and with e_1 on the third and the fourth one, the first inequality in (2.12) follows by applying the divergence theorem on $F \cap \{x_1 < t\}$ to the constant vector field $f(x) = e_1$,

$$\begin{aligned} 0 \;&=\; \int_{H\cap\{x_1<t\}\cap\partial^* F} e_1 \cdot \nu_F + \mathcal{H}^{n-1}(\{x_1 = t\} \cap F) + \mathcal{H}^{n-1}(\{x_1 = t\} \cap \partial^* F \cup \{\nu_F = e_1\}) \\ &\ge\; \int_{H\cap\{x_1<t\}\cap\partial^* F} e_1 \cdot \nu_F + \mathcal{H}^{n-1}(\{x_1 = t\} \cap F)\,, \end{aligned}$$

while the second inequality follows by using the vector field $f(x) = e_n$,

$$0 = \int_{H\cap\{x_1<t\}\cap\partial^* F} e_n \cdot \nu_F - P(F; \{x_1 < t\} \cap \partial H)\,.$$

The actual proof requires a truncation of the vector field $f(x)$. We omit the details and refer to [Mag12, Proposition 19.22] for a complete exposition of an identical argument.

With (2.12) at hand, we now prove the existence of $G \subset F$ such that (2.10) and (2.9) hold. For $t \in \mathbb{R}$ we set

$$F_t^- = F \cap \{x_1 < t\}\,, \qquad F_t^+ = F \cap \{x_1 > t\}\,.$$

By (2.13), and by an analogous formula for F_t^+, one has

$$\mathcal{F}_{H,\tau}(F_t^+) + \mathcal{F}_{H,\tau}(F_t^-) \le \mathcal{F}_{H,\tau}(F) + 2\,s(t)\,, \qquad \forall t \in \mathbb{R}\,.$$

(The inequality sign depends on the possibility that $\mathcal{H}^{n-1}(\partial^* F \cap \{x_1 = t\}) > 0$.) By definition of $\psi(\tau)$ we have $\mathcal{F}_{H,\tau}(F_t^\pm) \ge \psi(\tau)|F_t^\pm|^{(n-1)/n}$, so that

$$\begin{aligned} 2\,s(t) + \psi(\tau)\,\delta_\tau(F) \;&=\; 2\,s(t) + \mathcal{F}_{H,\tau}(F) - \psi(\tau) \ge \mathcal{F}_{H,\tau}(F_t^+) + \mathcal{F}_{H,\tau}(F_t^-) - \psi(\tau) \\ &\ge\; \psi(\tau)\,\Psi(v(t))\,, \qquad \forall t \in \mathbb{R}\,, \end{aligned}$$

where $\Psi(\gamma) = (1-\gamma)^{(n-1)/n} + \gamma^{(n-1)/n} - 1$, $\gamma \in [0,1]$. We rearrange this inequality as

$$s(t) \ge \frac{\psi(\tau)}{2}\Big(\Psi(v(t)) - \delta_\tau(F)\Big)\,, \qquad \forall t \in \mathbb{R}\,. \tag{2.14}$$

Let us notice that we can find $\kappa(n) > 0$ such that

$$\Psi(\gamma) \ge \kappa(n)\,\min\{\gamma, 1-\gamma\}^{(n-1)/n}\,, \qquad \forall \gamma \in [0,1]\,. \tag{2.15}$$

We now use the assumption that $\delta_\tau(F) \le \delta_0(n,\tau)$ to ensure that, if we set

$$t_1 = \inf\Big\{t \in \mathbb{R} : \frac{\kappa(n)\,v(t)^{(n-1)/n}}{2} \ge \delta_\tau(F)\Big\}\,, \tag{2.16}$$

then $t_1 \in \mathbb{R}$. Since v is increasing, we have $\kappa(n)\, v(t)^{(n-1)/n} \ge 2\delta_\tau(F)$ for every $t > t_1$, and thus we can apply (2.14) and (2.15) to find that if $t > t_1$ with $v(t) \le 1/2$, then

$$s(t) \ge \frac{\psi(\tau)}{2}\Big(\kappa(n)\, v(t)^{(n-1)/n} - \delta_\tau(F)\Big) \ge \frac{\psi(\tau)\kappa(n)}{4}\; v(t)^{(n-1)/n}\,; \tag{2.17}$$

similarly, if we define t_2 by

$$t_2 = \sup\Big\{t \in \mathbb{R} : \frac{\kappa(n)\,(1-v(t))^{(n-1)/n}}{2} \ge \delta_\tau(F)\Big\}\,, \tag{2.18}$$

then $t_2 \in \mathbb{R}$ with $\kappa(n)\,(1-v(t))^{(n-1)/n} \ge 2\delta_\tau(F)$ for every $t < t_2$, and again by (2.14) and (2.15)

$$s(t) \ge \frac{\psi(\tau)}{2}\Big(\kappa(n)\,(1-v(t))^{(n-1)/n} - \delta_\tau(F)\Big) \ge \frac{\psi(\tau)\kappa(n)}{4}\;(1-v(t))^{(n-1)/n}\,, \tag{2.19}$$

whenever $t < t_2$ with $v(t) \ge 1/2$. In conclusion,

$$s(t) \ge \frac{\psi(\tau)\kappa(n)}{4}\; \min\{v(t)\,, 1-v(t)\}^{(n-1)/n}\,, \qquad \forall\, t \in (t_1, t_2)\,, \tag{2.20}$$

so that, by taking $s = v'$ a.e. on $\mathbb{R}$ into account,

$$\frac{\psi(\tau)\,\kappa(n)(t_2-t_1)}{4} \le \int_{t_1}^{t_2} \frac{v'}{\min\{v, 1-v\}^{(n-1)/n}} \le \int_0^1 \frac{d\gamma}{\min\{\gamma, 1-\gamma\}^{(n-1)/n}} = C(n)\,,$$

that is $t_2 - t_1 \le C(n,\tau)$. Hence, the set

$$G' = F \cap \{t_1 < x_1 < t_2\}$$

has directional diameter along the x_1-axis bounded by $C(n,\tau)$, with

$$|F \setminus G'| = v(t_1) + (1 - v(t_2)) \le 2\Big(\frac{2\,\delta_\tau(F)}{\kappa(n)}\Big)^{n/(n-1)}\,. \tag{2.21}$$

We now split the argument depending on the sign of τ.

When $\tau \ge 0$ we conclude the proof of (2.10) and (2.9) by setting $G = G'$. Indeed, with this choice, (2.9) is immediate from $t_2 - t_1 \le C(n,\tau)$, while the first bound in (2.10) follows from (2.21). Moreover, by (2.12),

$$\begin{aligned}\mathcal{F}_{H,\tau}(F) - \mathcal{F}_{H,\tau}(G) \;\;\ge\;\; & P(F; H \cap \{x_1 < t_1\}) - s(t_1) + \tau\, P(F; \partial H \cap \{x_1 < t_1\}) \\ & + P(F; H \cap \{x_1 > t_2\}) - s(t_2) + \tau\, P(F; \partial H \cap \{x_2 > t_1\})\,,\end{aligned} \tag{2.22}$$

where $P(F; H \cap \{x_1 < t_1\}) \ge s(t_1)$ by (2.12), and similarly $P(F; H \cap \{x_1 > t_2\}) \ge s(t_2)$ (and where the first inequality sign depends on the possibility that $\mathcal{H}^{n-1}(\partial^* F \cap \{x_1 = t_i\}) > 0$ for $i = 1, 2$.) Thus $\tau \ge 0$ implies $\mathcal{F}_{H,\tau}(F) \ge \mathcal{F}_{H,\tau}(G)$, and the second bound in (2.10).

When $\tau < 0$ we can fix the argument by cutting F at nearby levels of t such that $s(t)$ is sufficiently small in terms of deficit. To make this precise, let us consider the sets

$$\begin{aligned} I_1 &= \Big\{t \in \mathbb{R} : v(t) \le \frac{1}{2}\,, \quad s(t) < \frac{\psi(\tau)\delta_\tau(F)}{2}\Big\}\,, \\ I_2 &= \Big\{t \in \mathbb{R} : v(t) \ge \frac{1}{2}\,, \quad s(t) < \frac{\psi(\tau)\delta_\tau(F)}{2}\Big\}\,, \end{aligned}$$

and define

$$t_1^* = \begin{cases} \inf\{t \in \mathbb{R} : v(t) > 0\}\,, & \text{if } I_1 = \emptyset \\ \sup I_1\,, & \text{if } I_1 \ne \emptyset \end{cases} \qquad t_2^* = \begin{cases} \sup\{t \in \mathbb{R} : v(t) > 0\}\,, & \text{if } I_2 = \emptyset \\ \inf I_2\,, & \text{if } I_2 \ne \emptyset \end{cases}\,.$$

We first note that

$$\operatorname{diam}(\{0 < v < 1/2\}) \le C(n,\tau)\,, \qquad \text{if } I_1 = \emptyset\,, \tag{2.23}$$

$$\operatorname{diam}(\{1/2 < v < 1\}) \le C(n,\tau)\,, \qquad \text{if } I_2 = \emptyset\,; \tag{2.24}$$

indeed, if for example $I_1 = \emptyset$, then by $s = v'$ a.e. on $\mathbb{R}$ we find

$$\frac{1}{2} \ge \int_{\{0<v<1/2\}} s(t)\,dt \ge \mathrm{diam}(\{0 < v < 1/2\})\,\frac{\psi(\tau)\delta_\tau(F)}{2}\,.$$

Next we remark that if $I_1 \ne \emptyset$, then $t_1^* \in \mathbb{R}$ and actually $t_1^* \le t_1$: indeed, by the same argument leading to (2.17) one can deduce that

$$s(t) \ge \frac{\psi(\tau)}{2}\,\delta_\tau(F)\,, \qquad \text{for every } t > t_1 \text{ with } v(t) \le \frac{1}{2}\,.$$

Similarly, by arguing as in the proof of (2.19), we see that if $I_2 \ne \emptyset$, then $t_2^* \in \mathbb{R}$ with $t_2^* \ge t_2$, as

$$s(t) \ge \frac{\psi(\tau)}{2}\,\delta_\tau(F)\,, \qquad \text{for every } t < t_2 \text{ with } v(t) \ge \frac{1}{2}\,.$$

Finally, we notice that if $I_1 \ne \emptyset$ or $I_2 \ne \emptyset$, then one has, respectively,

$$t_1 - t_1^* \le C(n,\tau)\,, \qquad t_2^* - t_2 \le C(n,\tau)\,.$$

To prove the first relation, notice that for every $t \in (t_1^*, t_1)$ one has $\psi(\tau)\delta_\tau(F) \le 2\,s(t)$, and thus $s = v'$ a.e. on $\mathbb{R}$ and $v(t_1) = (2\delta_\tau(F)/\kappa(n))^{n/(n-1)}$ give

$$t_1 - t_1^* \le \frac{2}{\psi(\tau)\delta_\tau(F)} \int_{t_1^*}^{t_1} v' \le \frac{2\,v(t_1)}{\psi(\tau)\delta_\tau(F)} \le C(n,\tau)\,\delta_\tau(F)^{1/(n-1)}\,;$$

similarly, for every $t \in (t_2, t_2^*)$ one has $\psi(\tau)\delta_\tau(F) \le 2\,s(t)$, and thus $s = v'$ a.e. on $\mathbb{R}$ and $v(t_2) \ge 1 - (2\delta_\tau(F)/\kappa(n))^{n/(n-1)}$ give

$$t_2^* - t_2 \le \frac{2\,(v(t_2^*) - v(t_2))}{\psi(\tau)\delta_\tau(F)} = \frac{2\,[(1 - v(t_2)) - (1 - v(t_2^*))]}{\psi(\tau)\delta_\tau(F)} \le C(n,\tau)\,\delta_\tau(F)^{1/(n-1)}\,;$$

With these remarks at hand we finally set

$$t_1^{**}$$

to be $\inf\{v > 0\} - 1$ if $I_1 = \emptyset$ or to be any $t < t_1^*$ with

$$s(t_1^{**}) < \frac{\psi(\tau)\delta_\tau(F)}{2}\,, \qquad t_1^* - t_1^{**} \le 1\,,$$

in case $I_1 \ne \emptyset$; we set

$$t_2^{**}$$

to be $\sup\{v > 0\} + 1$ if $I_2 = \emptyset$ or to be any $t > t_2^*$ with

$$s(t_2^{**}) < \frac{\psi(\tau)\delta_\tau(F)}{2}\,, \qquad t_2^{**} - t_2^* \le 1\,,$$

in case $I_2 \ne \emptyset$; and finally define G by taking

$$G = F \cap \{t_1^{**} < x_1 < t_2^{**}\}\,.$$

Notice that the above remarks show that it must be

$$t_2^{**} - t_1^{**} \le C(n,\tau)\,, \qquad \max\Big\{s(t_1^{**}), s(t_2^{**})\Big\} \le \frac{\psi(\tau)\delta_\tau(F)}{2}\,,$$

so that (2.9) holds. Since $G' \subset G = F \cap \{t_1 < x_1 < t_2\}$ we have

$$|F \setminus G| \le |F \setminus G'| \le C(n,\tau)\,\delta_\tau(F)^{n/(n-1)}\,,$$

and thus the first bound in (2.10) holds. At the same time if we write down (2.22) with t_i^{**} in place of t_i and exploit the inequalities $|\tau|\,P(F;\partial H \cap \{x_1 < t_i^{**}\}) < P(F; H \cap \{x_1 < t_i\})$ (recall the second inequality in (2.12)), then we find

$$\mathcal{F}_{H,\tau}(F) - \mathcal{F}_{H,\tau}(G) \ge -s(t_1^{**}) - s(t_2^{**}) \ge -\psi(\tau)\,\delta_\tau(F)\,,$$

which is the second bound in (2.10). This completes step two.

Step three: We show that if $F \subset H$ with $\delta_\tau(F) \le \delta_0(n,\tau)$, then there exists $G \subset H$ with

$$|\alpha_\tau(G) - \alpha_\tau(F)| \le C(n,\tau)\,\delta_\tau(F)\,, \quad \delta_\tau(G) \le C(n,\tau)\,\delta_\tau(F)\,, \quad \sup_{x\in G}\frac{x_n}{|F|^{1/n}} \le C(n,\tau)\,. \tag{2.25}$$

Once again, we can prove this with $|F| = 1$ and $F = F^{(1)}$. Similarly to step two we set

$$v(t) = |F \cap \{x_n < t\}|\,, \quad s(t) = \mathcal{H}^{n-1}(F \cap \{x_n = t\})\,, \quad p(t) = \mathcal{H}^{n-2}(\partial^* F \cap \{x_n = t\})\,,$$

and notice that, thanks to $F = F^{(1)}$ and [Mag12, Equation (16.7)], for every $t \in \mathbb{R}$ one has

$$\mathcal{F}_{H,\tau}(F_t^+) + \mathcal{F}_{H,\tau}(F_t^-) \le \mathcal{F}_{H,\tau}(F) + 2\,s(t)\,,$$

where now $F_t^+ = F \cap \{x_n > t\}$ and $F_t^- = F \cap \{x_n < t\}$, and where the inequality sign holds since it may happen that $\mathcal{H}^{n-1}(\partial^* F \cap \{x_n = t\}) > 0$ for some (but at most for countably many) values of t. By setting, similarly to what done in (2.16) and (2.18),

$$t_1 = \inf\Big\{t > 0 : \frac{\kappa(n)\,v(t)^{(n-1)/n}}{2} \ge \delta_\tau(F)\Big\}\,, \ t_2 = \sup\Big\{t > 0 : \frac{\kappa(n)\,(1-v(t))^{(n-1)/n}}{2} \ge \delta_\tau(F)\Big\}\,,$$

and by repeating the same arguments we find once again $t_2 - t_1 \le C(n,\tau)$. The step will be then be completed, once again thanks to step one, by setting $G = F \cap \{x_1 < t_2\}$ and by showing that $t_1 \le C(n,\tau)$. We shall actually prove that $t_1 \le C(n,\tau)\,\delta_\tau(F)^{n/(n-1)}$. To this end, we first notice that for a.e. $t \in (0,t_1)$ and by definition of $\psi(1)$,

$$\begin{aligned}
-\tau\,P(F;\partial H) + \psi(\tau)\,\delta_\tau(F) &= P(F;H) - \psi(\tau) \ge P(F;\{x_n > t\}) - \psi(\tau)\\
&= P(F \cap \{x_n > t\}) - \psi(\tau) - s(t)\\
&\ge \psi(1)\,|F \cap \{x_n > t\}|^{(n-1)/n} - \psi(\tau) - s(t)\\
&\ge \psi(1) - \psi(\tau) - C(n,\tau)\,v(t_1) - s(t)\,.
\end{aligned}$$

If we integrate this inequality on $(0,t_1)$ and take into account that $\int_0^{t_1} s(t)\,dt = v(t_1) \le C(n,\tau)\,\delta_\tau(F)^{n/(n-1)}$, then we find, provided $\delta_0(n,\tau)$ is small enough,

$$\begin{aligned}
-\tau\,P(F;\partial H)\,t_1 &\ge (\psi(1) - \psi(\tau))\,t_1 - C(n,\tau)\,\delta_\tau(F)\,t_1 - C(n,\tau)\,\delta_\tau(F)^{n/(n-1)}\\
&\ge \frac{\psi(1) - \psi(\tau)}{2}\,t_1 - C(n,\tau)\,\delta_\tau(F)^{n/(n-1)}\,.
\end{aligned}$$

When $\tau \ge 0$, then the left-hand side of this last estimate is negative, and thus we obtain $t_1 \le C(n,\tau)\ \delta_\tau(F)^{n/(n-1)}$, as claimed. Assuming from now on that $\tau \le 0$, we rewrite the above estimate as

$$P(F;\partial H) \ge c(n,\tau) - C(n,\tau)\,\frac{\delta_\tau(F)^{n/(n-1)}}{t_1}\,. \tag{2.26}$$

Now, by exploiting the divergence theorem we easily see that, up to possibly excluding countably many values of t, $s(t) \to P(F;\partial H)$ as $t \to 0^+$. By combining this fact with [Mag12, Equation 19.57] we see that

$$P(F;\partial H) \le P(F \cap \{x_n < t\};H) = P(F;\{0 < x_n < t\}) + s(t)\,, \qquad \text{for a.e. } t > 0\,,$$

and thus $\tau\,P(F;\partial H) \ge \tau\,P(F:\{0 < x_n < t\}) + \tau\,s(t)$. In particular, for a.e. $t \in (0,t_1)$,

$$\begin{aligned}
\psi(\tau)\,\delta_\tau(F) &= P(F;\{x_n > t\}) + P(F;\{t > x_n > 0\}) + \tau\,s(0) - \psi(\tau)\\
&\ge P(F;\{x_n > t\}) + (1+\tau)\,P(F;\{t > x_n > 0\}) + \tau\,s(t) - \psi(\tau)\\
&= \mathcal{F}_{H,\tau}(F \cap \{x_n > t\} - t\,e_n) - \psi(\tau) + (1+\tau)\,P(F;\{t > x_n > 0\})\\
&\ge \psi(\tau)\,|F \cap \{x_n > t\}|^{(n-1)/n} - \psi(\tau) + (1+\tau)\,P(F;\{t > x_n > 0\})\\
&\ge \psi(\tau)\,(1 - v(t_1))^{(n-1)/n} - \psi(\tau) + (1+\tau)\,P(F;\{t > x_n > 0\})\\
&\ge -C(n,\tau)\,\delta_\tau(F)^{n/(n-1)} + (1+\tau)\,P(F;\{t > x_n > 0\})\,.
\end{aligned} \tag{2.27}$$

Now, by a minor modification of [Mag12, Theorem 18.11], we find that

$$P(F;\{t>x_n>0\})\ge\int_0^t\sqrt{s'(t)^2+p(t)^2}\,dt\ge\int_0^t|s'(t)|\,dt\ge P(F;\partial H)-s(t)\,,$$

(where we have exploited again $s(t)\to P(F;\partial H)$ as $t\to 0^+$) so that, integrating over $(0,t_1)$ and taking (2.26) into account

$$\int_0^{t_1}P(F;\{t>x_n>0\})\,dt\ge t_1\,P(F;\partial H)-v(t_1)\ge c(n,\tau)\,t_1-C(n,\tau)\,\delta_\tau(F)^{n/(n-1)}\,.$$

By first integrating (2.27) over $(0,t_1)$ and by then plugging this last inequality, we find that

$$C(n,\tau)\,t_1\,\delta_\tau(F)\ge(1+\tau)\Big(c(n,\tau)\,t_1-C(n,\tau)\,\delta_\tau(F)^{n/(n-1)}\Big)\,,$$

which, for $\delta_0(n,\tau)$ small enough, implies $t_1\le C(n,\tau)\,\delta_\tau(F)^{n/(n-1)}$.

Step four: We finally prove the statement. Arguing by contradiction, we assume the existence of $\eta>0$ and of a sequence of sets $F_h\subset H$ such that $\delta_\tau(F_h)\to 0$ but $\alpha_\tau(F_h)\ge\eta$. Without loss of generality, we can assume that $|F_h|=1$. By step two and step three we can find sets $G_h\subset H$ such that $\mathrm{diam}(G_h)\le C(n,\tau)$, $\sup_{x\in G_h}|x_n|\le C(n,\tau)$, $|F_h\Delta G_h|\to 0$, $\mathcal{F}_{H,\tau}(G_h)\to\psi(\tau)$ and $\alpha_\tau(G_h)\ge\eta/2$. By [Mag12, Equation (19.58)] we have $\mathcal{F}_{H,\tau}(G_h)\ge(1+\tau)\,P(G_h)/2$, so that $\mathcal{F}_{H,\tau}(G_h)\to\psi(\tau)$ implies $P(G_h)\le C(n,\tau)$. This bound, together with $\mathrm{diam}(G_h)\le C(n,\tau)$ and $\sup_{x\in G_h}|x_n|\le C(n,\tau)$, implies that up to extracting subsequences and up to horizontal translations, $|G_h\Delta G|\to 0$ for some $G\subset H$. Clearly $\alpha_\tau(G)\ge\eta/2$, however, by lower semicontinuity of $\mathcal{F}_{h,\tau}$ (see [Mag12, Proposition 19.27]) it must be

$$\mathcal{F}_{H,\tau}(G)\le\liminf_{h\to\infty}\mathcal{F}_{H,\tau}(G_h)=\psi(\tau)\,,$$

where $\mathcal{F}_{H,\tau}(G)\ge\psi(\tau)$ as $|F_h\Delta G_h|\to 0$ implies $|G|=1$. In conclusion, $\mathcal{F}_{H,\tau}(G)=\psi(\tau)$ with $|G|=1$, and thus $G=z+K(\tau)$ for some $z\in\partial H$. But then $\alpha_\tau(G)=0$, a contradiction. $\square$

We conclude this section by showing the validity of property (iii) in Proposition 2.1. To this end, it is convenient to define $\Phi:S^{n-1}\to(0,\infty)$ by setting

$$\Phi(\nu)=\sup\Big\{x\cdot\nu:x\in S(\tau)\Big\}=\sup\Big\{x\cdot\nu:|x|<1\,,x_n>-\tau\Big\}\,,\qquad\nu\in S^{n-1}\,,$$

and then consider a corresponding anisotropic perimeter functional $\mathbf{\Phi}$ defined by setting

$$\mathbf{\Phi}(F)=\int_{\partial^*F}\Phi(\nu_F(x))\,d\mathcal{H}^{n-1}_x\in[0,\infty]\,,$$

whenever F is of locally finite perimeter in $\mathbb{R}^n$. It is immediate to check that

$$\Phi(-e_n)=-\tau\,,\qquad\Phi(\nu)=1\text{ if }\nu\cdot e_n>-\tau\,,\tag{2.28}$$

so that

$$\mathbf{\Phi}(K(\tau))=P(K(\tau);H)+\tau\,P(K(\tau);\partial H)=\psi(\tau)\,,\tag{2.29}$$

while

$$\mathbf{\Phi}(F)=\int_{H\cap\partial^*F}\Phi(\nu_F)\,d\mathcal{H}^{n-1}+\tau\,P(F;\partial H)\le\mathcal{F}_{H,\tau}(F)\,,\qquad\forall F\subset H\,,\tag{2.30}$$

where of course we have used that $F\subset H$ implies $\nu_F=-e_n$ for $\mathcal{H}^{n-1}$-a.e. $x\in\partial^*F\cap\partial H$ (see, e.g., [Mag12, Exercise 16.6]) as well as that $\Phi\le 1$ on S^{n-1}.

Proof of Proposition 2.1-(iii). By (2.29), (2.30) and the main result in [FMP10] we have that if $F \subset H$ with $|F| = 1$, then for some $w \in \mathbb{R}^n$

$$\mathcal{F}_{H,\tau}(F) - \psi(\tau) \ge \mathbf{\Phi}(F) - \mathbf{\Phi}(K(\tau)) \ge c_0(n)\, \mathbf{\Phi}(K(\tau))\, |F\Delta(w + K(\tau))|^2\,, \tag{2.31}$$

for some positive constant $c_0(n)$. Of course $w = z + t\, e_n$ where $z \in \partial H$ and $t \in \mathbb{R}$. If $t < 0$, then by $F \subset H$ we obtain

$$\begin{aligned} |F\Delta(w + K(\tau))| &\ge |(w + K(\tau)) \setminus F| \ge |(w + K(\tau)) \setminus H| = |(t\, e_n + K(\tau)) \setminus H| \\ &= |K(\tau) \cap \{0 < x_n < -t\}|\,. \end{aligned} \tag{2.32}$$

By combining (2.31), (2.32), and (2.4) we find that

$$\varepsilon(n, \tau) \ge c_0(n)\, |K(\tau) \cap \{0 < x_n < -t\}|^2\,,$$

so that if $\varepsilon(n, \tau)/c_0(n) \le (|K(\tau)|/2)^2$ then $|t| \le t_0(n, \tau) < 1$ and thus $|K(\tau) \cap \{0 < x_n < -t\}| \ge c(n, \tau)\, |t|$; by combining this last inequality with (2.32) and (2.31) we thus conclude that if $t < 0$, then

$$\mathcal{F}_{H,\tau}(F) - \psi(\tau) \ge c(n, \tau)\, |t|^2\,. \tag{2.33}$$

We now prove that (2.33) holds even when $t > 0$. In this case we use $w + K(\tau) \subset \{x_n > t\}$ and the inclusion $K(\tau)/2 \subset F$ in (2.4) to deduce that

$$|F\Delta(w + K(\tau))| \ge |F \setminus (w + K(\tau))| \ge |F \setminus \{x_n > t\}| \ge \Big|\frac{K(\tau)}{2} \cap \{x_n \le t\}\Big|\,.$$

By (2.4), (2.31) and the last inequality, if $\varepsilon(n, \tau)$ is small enough then we are able to infer that $t \le t_0(n, \tau) < 1/2$ and then to exploit an elementary lower bound of the form

$$|(K(\tau)/2) \cap \{x_n \le t\}| \ge c(n, \tau)\, t\,,$$

for $t \in (0, t_0(n, \tau)) \subset\subset (0, 1/2)$, to conclude that (2.33) holds. This said, by combining the bound

$$\begin{aligned} |(z + K(\tau))\Delta F| &\le |(w + K(\tau))\Delta F| + |(z + K(\tau))\Delta(w + K(\tau))| \\ &\le |(w + K(\tau))\Delta F| + C(n, \tau)\, |t| \end{aligned}$$

with (2.31) and (2.33) we conclude the proof of (2.5). □

2.2. **An improved convergence theorem.** Thorough this section, we set $H = \{x_n > 0\}$ and $K = K(\tau)$ for some fixed $\tau \in (-1, 1)$, see (1.6). Our main result, Theorem 2.4 below, consists in showing that if F_h is sequence of almost-minimizing sets in H which converges to K in volume, then $M_h = \overline{H \cap \partial F_h}$ is a $C^{1,\alpha}$-hypersurface with boundary for h large enough, and there exist $C^{1,\alpha}$-diffeomorphisms f_h between M_h and $M_0 = \overline{H \cap \partial K}$ such that $f_h \to \mathrm{Id}$ in C^1 and enjoys certain precise structure properties. In order to formulate this result in rigorous terms we need to set some definitions.

Definition 2.2 (Elliptic integrands)**.** Given an open set Ω one says that Φ is an *elliptic integrand on* Ω if $\Phi : \overline{\Omega} \times \mathbb{R}^n \to [0, \infty]$ is lower semicontinuous with $\Phi(x, \cdot)$ convex and one-homogeneous on $\mathbb{R}^n$ for every $x \in \overline{\Omega}$. If F is of locally finite perimeter in Ω and $W \subset \Omega$ is a Borel set, then we define

$$\mathbf{\Phi}(F; W) = \int_{W \cap \partial^* F} \Phi(x, \nu_F(x))\, d\mathcal{H}^{n-1}(x) \in [0, \infty]\,. \tag{2.34}$$

Given $\lambda \ge 1$ and $\ell \ge 0$ we let $\boldsymbol{\mathcal{E}}(\Omega, \lambda, \ell)$ denote the family of those elliptic integrands Φ in Ω such that $\Phi(x, \cdot) \in C^{2,1}(S^{n-1})$ for every $x \in \overline{\Omega}$ and such that for every $x, x' \in \overline{\Omega}$, $\nu, \nu' \in S^{n-1}$, one has

$$\frac{1}{\lambda} \le \Phi(x, \nu) \le \lambda\,,$$

$$|\Phi(x, \nu) - \Phi(x', \nu)| + |\nabla\Phi(x, \nu) - \nabla\Phi(x', \nu)| \le \ell |x - x'|\,,$$

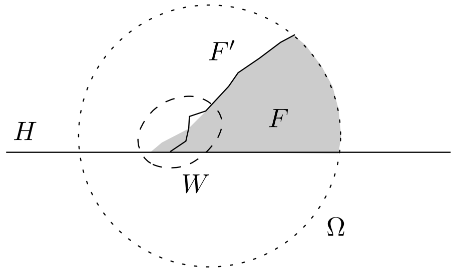

FIGURE 2. A competitor F' of F in (2.35) can have a different trace along ∂H, but must agree with F outside of an open set $W \subset\subset \Omega$ with small diameter. The boundary of F' and W are depicted, respectively, as a black line and as a dashed line.

$$|\nabla\Phi(x,\nu)| + \|\nabla^2\Phi(x,\nu)\| + \frac{\|\nabla^2\Phi(x,\nu) - \nabla^2\Phi(x,\nu')\|}{|\nu-\nu'|} \le \lambda\,,$$
$$\big|\nabla^2\Phi(x,\nu)[\tau,\tau]\big| \ge \frac{|\tau|^2}{\lambda}\,, \qquad \forall \tau \in \nu^\perp\,,$$

where $\nabla\Phi$ and $\nabla^2\Phi$ are taken with respect to the ν-variable, and with $\nu^\perp = \{y \in \mathbb{R}^n : y \cdot \nu = 0\}$.

The following minimality condition is tailored to the description of capillarity problems, in the sense that one considers subsets F of an half-space which minimize an elliptic integrand with respect to local perturbations which are allowed to freely modify $\partial F \cap \partial H$. In other words, we impose a Dirichlet condition inside of H, and a Neumann/free-boundary condition on ∂H; see Figure 2.

Definition 2.3 ((Λ,ρ_0)-minimizers)**.** Let Ω be an open set in $\mathbb{R}^n$, H be an half-space in $\mathbb{R}^n$, and let $\Phi \in \boldsymbol{\mathcal{E}}(\Omega\cap H,\lambda,\ell)$. Given $\Lambda \ge 0$ and $\rho_0 > 0$, a set $F \subset H$ of locally finite perimeter in Ω is a (Λ,ρ_0)-minimizer of $\boldsymbol{\Phi}$ in (Ω, H) if

$$\boldsymbol{\Phi}(F; H\cap W) \le \boldsymbol{\Phi}(F'; H \cap W) + \Lambda\,|F\Delta F'|\,, \tag{2.35}$$

whenever $F' \subset H$ is such that $F\Delta F' \subset\subset W$ with $W \subset\subset \Omega$ open and such that $\mathrm{diam} W < 2\,\rho_0$.

We are now ready to state the following theorem, which is the main result of this section.

Theorem 2.4. *Let $H = \{x_n > 0\}$ and $K = K(\tau)$ for some fixed $\tau \in (-1,1)$. Given $\lambda \ge 1$, $\ell, \Lambda \ge 0$, $\rho_0 > 0$, and $\alpha \in (0,1)$ there exists C_α depending on n, τ, λ, ℓ, Λ, ρ_0 and α with the following property.*

If Ω is an open set such that $K \subset\subset \Omega$, $\{\Phi_h\}_{h\in\mathbb{N}} \subset \boldsymbol{\mathcal{E}}(\Omega,\lambda,\ell)$, and, for each $h \in \mathbb{N}$, F_h is a (Λ,ρ_0)-minimizer of $\boldsymbol{\Phi}_h$ in (Ω,H) such that $|F_h\Delta K| \to 0$ as $h \to \infty$, then $M_h = \overline{H \cap \partial F_h}$ is a compact connected orientable $C^{1,\alpha}$-hypersurface with boundary for every $\alpha \in (0,1)$ and for h large enough, and there exists a diffeomorphism $f_h : M_0 \to M_h$, $M_0 = \overline{H\cap\partial K}$ such that

$$\|f_h\|_{C^{1,\alpha}(M_0)} \le C_\alpha\,, \qquad \lim_{h\to\infty} \|f_h - \mathrm{Id}\|_{C^1(M_0)} = 0\,.$$

Moreover, there exists $\varepsilon_h \to 0$ as $h \to \infty$ such that if we set

$$u_h = (f_h - \mathrm{Id}) - [\nu_K \cdot (f_h - \mathrm{Id})]\nu_K$$

for the tangential displacement of f_h, then

$$\mathrm{spt}\, u_h \subset M_0 \cap \{x_n < \varepsilon_h\}\,, \qquad \lim_{h\to\infty} \|u_h\|_{C^1(M_0)} = 0\,.$$

Remark 2.5. Theorem 2.4 works as well if we just assume that M_0 is a $C^{2,1}$-hypersurface with boundary such that $\mathrm{bd}\,(M_0) \subset \partial H$. We have decided to focus on the case relevant to the capillarity problem, where M_0 is a spherical cap, just for the sake of definiteness. Considering a sequence of elliptic energies Φ_h is necessary in view of the application of this result to sequences of $m_h^{1/n}$-blow-ups of global minimizers in (1.4) as $m_h \to 0^+$.

The rest of the section is devoted to the proof of Theorem 2.4. The following lemma allows us to exploit [CLM14, Theorem 3.5] for constructing the diffeomorphisms f_h. Let us recall that if $M \subset \mathbb{R}^n$ is a k-dimensional manifold with boundary, then we denote by $\mathrm{int}\,(M)$ and $\mathrm{bd}\,(M)$ its interior and boundary points respectively, by ν_M^{co} the outer unit normal to $\mathrm{bd}\,(M)$ in M, and we set $[M]_\rho = M \setminus I_\rho(\mathrm{bd}\,(M))$ for every $\rho > 0$.

Lemma 2.6. *Under the assumptions of Theorem 2.4, there exist positive constants $\mu_0 \in (0,1)$ and L (depending on n, τ, λ, ℓ, Λ, and ρ_0) and L_α (also depending on $\alpha \in (0,1)$) such that, for h large enough and for every $\alpha \in (0,1)$, M_h is a compact connected orientable $C^{1,\alpha}$-hypersurface with boundary with*

$$\mathrm{bd}\,(M_h) \neq \emptyset\,, \qquad \lim_{h\to\infty} \mathrm{hd}(M_h, M_0) = 0\,.$$

Furthermore, if we set $\nu_{M_h} = \nu_{F_h}$, then

$$\begin{cases} |\nu_{M_h}(x) - \nu_{M_h}(y)| \le L\,|x-y|\,, \\ |\nu_{M_h}(x)\cdot(y-x)| \le L\,|x-y|^2\,, \end{cases} \qquad \forall x, y \in M_h\,. \tag{2.36}$$

In addition, the following holds:

(i) if $n = 2$, then $\mathrm{bd}\,(M_0) = \{p_0, q_0\}$ and $\mathrm{bd}\,(M_h) = \{p_h, q_h\}$, with

$$\lim_{h\to\infty} |p_h - p_0| + |q_h - q_0| + |\nu_{M_h}^{co}(p_h) - \nu_{M_0}^{co}(p_0)| + |\nu_{M_h}^{co}(q_h) - \nu_{M_0}^{co}(q_0)| = 0\,;$$

if $n \ge 3$, then there exist $C^{1,\alpha}$-diffeomorphisms $f_{0,h}$ between M_h and M_0 such that

$$\begin{aligned} \|f_{0,h}\|_{C^{1,\alpha}(\mathrm{bd}\,(M_0))} &\le L_\alpha\,, \\ \lim_{h\to\infty} \|f_{0,h} - \mathrm{Id}\|_{C^1(\mathrm{bd}\,(M_0))} &= 0\,, \\ \lim_{h\to\infty} \|\nu_{M_h}(f_{0,h}) - \nu_{M_0}\|_{C^1(\mathrm{bd}\,(M_0))} &= 0\,, \\ \lim_{h\to\infty} \|\nu_{M_h}^{co}(f_{0,h}) - \nu_{M_0}^{co}\|_{C^1(\mathrm{bd}\,(M_0))} &= 0\,; \end{aligned}$$

where, by definition, $\nu_{M_0} = \nu_K$.

(ii) for every $\rho < \mu_0^2$ there exist $h(\rho) \in \mathbb{N}$ and $\{\psi_h\}_{h \ge h(\rho)} \subset C^{1,\alpha}([M_0]_\rho)$ such that

$$\begin{aligned} &[M_h]_{3\rho} \subset (\mathrm{Id} + \psi_h\,\nu_{M_0})([M_0]_\rho) \subset M_h\,, \qquad \forall h \ge h(\rho)\,, \\ &\sup_{h\ge h(\rho)} \|\psi_h\|_{C^{1,\alpha}([M_0]_\rho)} \le L_\alpha\,, \qquad \lim_{h\to\infty} \|\psi_h\|_{C^1([M_0]_\rho)} = 0\,. \end{aligned} \tag{2.37}$$

Proof of Theorem 2.4. By combining Lemma 2.6 with [CLM14, Theorem 3.5] we find that for every $\mu \in (0,\mu_0)$ there exist $h(\mu) \in \mathbb{N}$ and, for each $h \ge h(\mu)$, a $C^{1,\alpha}$-diffeomorphisms f_h between M_0 and M_h such that

$$\begin{aligned} &f_h = f_{0,h} \text{ on } \mathrm{bd}\,(M_0)\,, \qquad f_h = \mathrm{Id} + \psi_h\,\nu_{M_0} \text{ on } [M_0]_\mu\,, \\ &\sup_{h \ge h(\mu)} \|f_h\|_{C^{1,\alpha}(M_0)} \le C_\alpha\,, \qquad \lim_{h\to\infty} \|f_h - \mathrm{Id}\|_{C^1(M_0)} = 0\,, \\ &\|u_h\|_{C^1(M_0)} \le \frac{C_\alpha}{\mu} \begin{cases} \|(f_{0,h} - \mathrm{Id})\cdot\nu_{M_0}^{co}\|_{C^0(\mathrm{bd}\,(M_0))}\,, & \text{if } n = 2\,, \\ \|f_{0,h} - \mathrm{Id}\|_{C^1(\mathrm{bd}\,(M_0))}\,, & \text{if } n \ge 3\,, \end{cases} \end{aligned} \tag{2.38}$$

where, in the case $n = 2$, $f_{0,h}$ is defined by the relations $f_{0,h}(p_0) = p_h$, $f_{0,h}(q_0) = q_h$. Of course (2.38) implies the conclusions of Theorem 2.4. □

We now focus on the proof of Lemma 2.6. The proof is based on the regularity theory for (Λ, ρ_0)-minimizers of elliptic integrands as discussed in [Mag12, Part III] and in [DPM15]. Given the regularity theorems, the argument is rather standard and so we limit ourselves to describe the main points.

Proof of Lemma 2.6. One deduces that $\mathrm{hd}(M_h, M_0) \to 0$ by $|F_h \Delta K| \to 0$ and by the uniform density estimates satisfied by the sets F_h (recall that the functionals $\mathbf{\Phi}_h$ are uniformly elliptic), see for example [DPM15, Theorem 2.9]. Let us now set

$$\mathrm{Reg}(M_h) = \left\{ x \in M_h : \begin{array}{l} \text{there exists } r_x > 0 \text{ such that } M_h \cap B_{x,r_x} \\ \text{is a } C^1\text{-manifold with boundary s.t. } \mathrm{bd}\,(M_h) \subset \partial H \end{array} \right\},$$

for the regular part of M_h and $\Sigma(M_h) = M_h \setminus \mathrm{Reg}(M_h)$ for its singular set. For $x \in \Omega$ and $r < \mathrm{dist}(x, \partial\Omega)$, define the *spherical excess of F_h at the point x, at scale r, relative to H* as

$$\mathbf{exc}^H(F_h, x, r) = \inf \left\{ \frac{1}{r^{n-1}} \int_{B_{x,r} \cap H \cap \partial^* F_h} \frac{|\nu_{F_h} - \nu|^2}{2} \, d\mathcal{H}^{n-1} : \nu \in S^{n-1} \right\}.$$

By (for example) [DPM15, Theorem 3.1] in the case $x \in M_h \cap \partial H$, and [DS02] in the case $x \in M_h \cap H$, there exists $\varepsilon_0 = \varepsilon_0(n, \lambda) > 0$ such that

$$\Sigma(M_h) = \left\{ x \in M_h : \liminf_{r \to 0^+} \mathbf{exc}^H(F_h, x, r) \ge \varepsilon_0 \right\}. \tag{2.39}$$

This characterization of the singular set allows to deduce easily that for every $\tau > 0$ one has $\Sigma(M_h) \subset I_\tau(\Sigma(M_0))$ provided h is large enough. In our case M_0 is just a spherical cap, and so we have $\Sigma(M_0) = \emptyset$. In particular, M_h is a $C^{1,\alpha}$-hypersurface with boundary for every $\alpha \in (0,1)$.

In fact one has more precise information. We first describe the argument qualitatively. Consider for example a point $x \in M_0 \cap H$, and fix $r > 0$ such that $\mathrm{dist}(x, \partial H) > r$ and $\mathbf{exc}^H(K, x, r) < \varepsilon_0/2$. The Hausdorff convergence of M_h to M_0, and the fact that almost-minimizers converging in volume also converge in perimeter, implies the existence of $x_h \in M_h \cap H$ such that $x_h \to x$ and $\mathbf{exc}^H(F_h, x_h, r) < \varepsilon_0$. One can thus apply the ε-regularity criterion to K and to F_h to find that they are both epigraphs of $C^{1,\alpha}$-functions v_h and v defined on a same $(n-1)$-dimensional disk or radius $c_0\, r$, with $v_h \to v$ in C^1 and $c_0 = c_0(n, \alpha)$. By patching this local graphicality property on a uniform scale one come to prove conclusion (ii) in the lemma. This kind of argument is described in great detail in [CLM14, Theorem 4.1, Lemma 4.3, Lemma 4.4, Theorem 4.12] and so we do not further discuss this point.

We now exploit the same argument at the boundary. Given $x \in \partial H$, $r > 0$ and $\nu \in S^{n-1}$ with $\nu \cdot \nu_H = 0$, set

$$\mathbf{D}^\nu_{x,r} = \left\{ y \in H : |(y - x) \cdot \nu| = 0\,, |y - x| < r \right\}, \qquad \mathbf{C}^\nu_{x,r} = \left\{ y + t\nu : y \in \mathbf{D}^\nu_{x,r}, |t| < r \right\}.$$

Let us fix $x \in M_0 \cap \partial H$, and consider $r_x > 0$ such that $\mathbf{exc}^H(K, x, r_x) < \varepsilon_0/2$. By exploiting a *boundary* ε-regularity criterion [DPM15, Theorem 3.1], we come to prove that for every $x \in M_0 \cap \partial H$ there exist $\nu_x \in S^{n-1}$ with $\nu_x \cdot \nu_H = 0$ and functions $v_h, v \in C^{1,\alpha}(\mathbf{D}^{\nu_x}_{x,c_0\, r_x})$ such that $v_h \to v$ in $C^1(\mathbf{D}^{\nu_x}_{x,c_0\, r_x})$ with

$$\mathbf{C}^{\nu_x}_{x,c_0\, r_x} \cap F_h = \left\{ y \in \mathbf{C}^{\nu_x}_{x,c_0\, r_x} : y \cdot \nu_x > v_h\big(y - (y \cdot \nu_x)\nu_x\big) \right\},$$

and with an analogous relation between K and v. In particular,

$$\begin{aligned} \mathbf{C}^{\nu_x}_{x,c_0\, r_x} \cap M_h \cap \partial H &= \left\{ z + v_h(z)\, \nu_x : z \in \overline{\mathbf{D}^{\nu_x}_{x,c_0\, r_x}} \cap \partial H \right\}, \\ \mathbf{C}^{\nu_x}_{x,c_0\, r_x} \cap M_0 \cap \partial H &= \left\{ z + v(z)\, \nu_x : z \in \overline{\mathbf{D}^{\nu_x}_{x,c_0\, r_x}} \cap \partial H \right\}. \end{aligned} \tag{2.40}$$

Let us now cover the $(n-2)$-dimensional sphere $M_0 \cap \partial H$ by finitely many cylinders satisfying (2.40). By exploiting the fact that $v_h \to v$ in C^1 (see [CLM14, Lemma 4.3] for the details of such a construction) we can show that the normal projection over $M_0 \cap \partial H$ defines a $C^{1,\alpha}$-diffeomorphism between $M_h \cap \partial H$ and $M_0 \cap \partial H$. Denoting by $f_{0,h}$ the inverse of this map, we complete the proof of conclusion (i).

Summarizing we have proved the validity of conclusions (i) and (ii). The fact that M_h is compact and connected is also easily inferred by covering a neighborhood of M_0 by finitely many cylinders of graphicality for both M_h and M_0 and by recalling that $\mathrm{hd}(M_h, M_0) \to 0$. Let us finally consider the vector fields $\nu_{M_h} = \nu_{F_h}$ and ν_K, and notice that

$$\begin{cases} |\nu_K(x) - \nu_K(y)| \le L\,|x-y|\,, \\ |\nu_K(x)\cdot(y-x)| \le L\,|x-y|^2\,, \end{cases} \qquad \forall x,y \in K\,, \tag{2.41}$$

for a suitably large constant L. By arguing as in [Mag12, Theorem 26.6] we see that if $x_h \in M_h$ and $x_h \to x$, then $x \in M_0$ and $\nu_{M_h}(x_h) \to \nu_K(x)$; by exploiting this fact and (2.41) we easily deduce (2.36). $\square$

3. Convergence in volume to the ideal droplet

Throughout this section A denotes an open bounded connected set with boundary of class $C^{1,1}$, while $\sigma : \partial A \to (-1,1)$ and $g : A \to [0,\infty)$ are Lipschitz functions. The minimum value on ∂A of σ is denoted by σ_0,

$$\sigma_0 = \min_{\partial A} \sigma\,.$$

and we denote by $K = K(\sigma_0)$ the reference unit volume droplet associated to $\mathcal{F}_{H,\sigma_0}$ as in (1.6), where $H = \{x \in \mathbb{R}^n : x_n > 0\}$. The goal of this section is showing that minimizers E_m in the variational problem (1.4), which we recall was defined by

$$\gamma(m) = \inf\Big\{\mathcal{F}_{A,\sigma}(E) + \int_E g(x)\,dx : E \subset A\,, |E| = m\Big\}\,,$$

are such that $\psi_m(E_m)/|\psi_m(E_m)|^{1/n} \to K$ in volume as $m \to 0^+$. Here the maps ψ_m are defined on neighborhoods (of uniformly positive diameter) of E_m and converge in $C^{1,1}$ to the identity map.

Lemma 3.1. *If A is an open bounded connected set of class $C^{1,1}$, $\sigma \in \mathrm{Lip}(\partial A; (-1,1))$ and $g \in \mathrm{Lip}(A)$ with $g \ge 0$, then there exist positive constants C_0 and m_0 (depending on A, σ and g) such that if E_m is a minimizer in* (1.4) *with $m < m_0$, then there exists $y_m \in \partial A$ such that*

$$E_m \subset B_{y_m, C_0\, m^{1/2n}} \qquad 0 \le \sigma(y_m) - \sigma_0 \le C_0\, m^{1/2n}\,. \tag{3.1}$$

Moreover, for every $\varepsilon > 0$ there exists $m_\varepsilon \le m_0$ (depending on A, σ, g and ε) such that

$$\inf_{z \in \partial H} |F_m \Delta (z+K)| < \varepsilon\,, \qquad \text{where} \qquad F_m = \frac{\phi_{y_m}^{-1}(E_m)}{|\phi_{y_m}^{-1}(E_m)|^{1/n}}\,, \tag{3.2}$$

where ϕ_{y_m} is defined as in Notation 3.2 below.

Notation 3.2. (See Figure 3.) By compactness of ∂A, we can find $r_0, s_0 > 0$ (depending on A) so that for every $x \in \partial A$ there exist an open neighborhood U_x of $0 \in \mathbb{R}^n$ with $\overline{B}_{s_0} \subset U_x$ and a $C^{1,1}$-diffeomorphism $\phi_x : U_x \to B_{x,2r_0}$ such that $\phi_x(0) = x$, $\nabla\phi_x(0)$ is an orientation preserving isometry, and

$$\phi_x(U_x \cap H) = B_{x,2r_0} \cap A\,, \qquad \phi_x(U_x \cap \partial H) = B_{x,2r_0} \cap \partial A\,. \tag{3.3}$$

Thanks to the fact that $L = \nabla\phi_x(0)$ is an orientation preserving isometry one has $L^{-1} = L^*$, $\det L = 1$, $\mathrm{cof}\,(L) = [\det(L)L^{-1}]^* = L$, and thus the Jacobian of $\nabla\phi_x(0)$ and its tangential Jacobian on ant hyperplane $\nu^\perp$ corresponding to $\nu \in S^{n-1}$ are given by

$$J\phi_x(0) = |\det \nabla\phi_x(0)| = 1\,, \qquad J^{\nu^\perp}\phi_x(0) = |\mathrm{cof}\,(\nabla\phi_x(0))\nu| = 1\,. \tag{3.4}$$

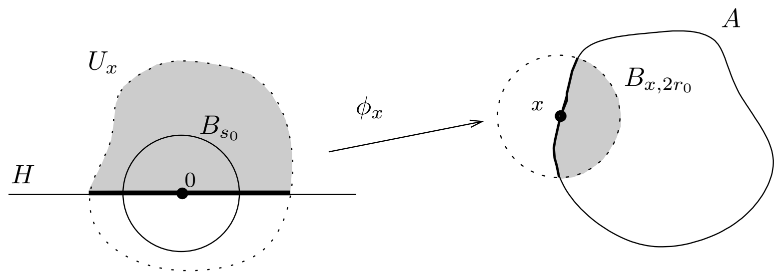

FIGURE 3. A summary of Notation 3.2. The map ϕ_x is such that $\phi_x(0) = x$. The constants r_0 and s_0 are independent of $x \in \partial A$.

In particular, there exists a constant C_1 depending only on A such that

$$\|\phi_x\|_{C^{1,1}(U_x)} + \|\phi_x^{-1}\|_{C^{1,1}(B_{x,2\,r_0})} \le C_1\,, \qquad \begin{cases} \|J\phi_x - 1\|_{C^0(B_s)} \le C_1\, s\,, \\ \|J^\Sigma\phi_x - 1\|_{C^0(\Sigma\cap B_s)} \le C_1\, s\,, \end{cases} \tag{3.5}$$

for every $s < s_0$ and for every $(n-1)$-rectifiable set Σ in $\mathbb{R}^n$. We can also assume $C_1\, s_0$ as small as needed depending on A. For example, we can certainly entail $J\phi_x \ge 1/2$ on B_{s_0}.

Remark 3.3. We recall that $O(m^{1/2n})$ in (3.1) is not optimal, and that it will be improved to $O(m^{1/n})$ in the next section. By being able of immediately obtain the latter information we could simplify some technicalities in Lemma 4.1 below. However, this limitation seems an unavoidable consequence of the grid argument used in step five below.

Proof of Lemma 3.1. In the course of the argument, C denotes a generic constant whose value depends on A, g and σ. The values r_0 and s_0 introduced in Notation 3.2 as A-dependent constants will be further decreased depending on A, g and σ.

Step one: We show that

$$\gamma(m) \le \psi(\sigma_0)\, m^{(n-1)/n}\Big(1 + C\, m^{1/n}\Big)\,, \qquad \forall m < m_0\,. \tag{3.6}$$

To this end, given $m < m_0$ it suffices to construct $E \subset A$ such that

$$|E| = m\,, \qquad \mathcal{F}_{A,\sigma}(E) \le \psi(\sigma_0)\, m^{(n-1)/n}(1 + Cm^{1/n})\,.$$

Let us fix $x \in \partial A$ such that $\sigma(x) = \sigma_0 = \min_{\partial A}\sigma$, and correspondingly set $U = U_x$ and $\phi = \phi_x$. We can find $t_0 > 0$ (depending on A and σ_0) such that $K_t = t\,K \subset B_{s_0}$ for every $t < t_0$, and thus, by (3.3),

$$E(t) = \phi(K_t) \subset B_{x,2\,r_0} \cap A\,, \qquad \forall t \in (0, t_0)\,.$$

By the area formula $|E(t)| = \int_{K_t} J\phi$, and since $J\phi \ge 1/2$ on B_{s_0} by (3.5), we find that $t \in (0,t_0) \mapsto |E(t)|$ is strictly increasing. In particular, we can find m_0 such that

$$(0, m_0) = \{|E(t)| : t \in (0, t_0)\}\,,$$

and for every $m < m_0$ there exists a unique $t(m) < t_0$ such that $m = |E(t(m))|$. We notice that $t(m) \le C\, m^{1/n}$: indeed, since $K_t \subset B_{C\,t}$ and $|K_t| = t^n$ for every $t > 0$, by (3.5) we find

$$m = |E(t(m))| = \int_{K_{t(m)}} J\phi \ge (1 - C\, t(m))\, |K_{t(m)}| = (1 - C\, t(m))\, t(m)^n \ge \frac{t(m)^n}{2}\,,$$

where the last inequality follows up to further decreasing the value of t_0 (depending on A and σ_0). By the area formula,

$$\mathcal{F}_{A,\sigma}(E(t)) + \int_{E(t)} g = \int_{H\cap\partial K_t} J^{\partial K_t}\phi\, d\mathcal{H}^{n-1} + \int_{\partial H\cap\partial K_t} \sigma(\phi)\, J^{\partial K_t}\phi\, d\mathcal{H}^{n-1} + \int_{K_t} g(\phi)\, J\phi\,,$$

so that (3.5), $\mathcal{F}_{H,\sigma_0}(K_t) = t^{n-1}\,\psi(\sigma_0)$ for every $t > 0$, and

$$\|\sigma \circ \phi_x - \sigma(x)\|_{C^0(\partial H \cap B_s)} \le C\,s\,, \qquad \forall s < s_0\,. \tag{3.7}$$

give us

$$\begin{aligned}\mathcal{F}_{A,\sigma}(E(t)) + \int_{E(t)} g &\le (1 + C\,t)\,P(K_t, H) + (\sigma_0 + C\,t)\,\mathcal{H}^{n-1}(\partial H \cap \partial K_t) + C\,|K_t| \\ &= \mathcal{F}_{H,\sigma_0}(K_t) + C\,t\,P(K_t) + C\,t^n \le \psi(\sigma_0)\,t^{n-1} + C\,t^n\,,\end{aligned}$$

which combined with $t(m) \le C\,m^{1/n}$ leads to (3.6).

Step two: We show that, if $E \subset A$ with $\mathrm{diam}(E) < r_0$, then

$$\mathcal{F}_{A,\sigma}(E) \ge \psi(\sigma_0)\,|E|^{(n-1)/n}\,(1 - C\,\mathrm{diam}(E))\,. \tag{3.8}$$

Indeed let $r = \mathrm{diam}(E)$. If $E \subset B_{x,r}$ for some $x \in A$ with $\mathrm{dist}(x, \partial A) > r$, then $\mathcal{F}_{\sigma,A}(E) = P(E)$, and thus

$$\mathcal{F}_{\sigma,A}(E) = P(E) \ge n\omega_n^{1/n}\,|E|^{(n-1)/n} \ge \psi(\sigma_0)\,|E|^{(n-1)/n}\,, \tag{3.9}$$

where we have applied the isoperimetric inequality and the fact that $n\omega_n^{1/n} = \psi(1) \ge \psi(\sigma_0)$ thanks to Proposition 2.1-(i). Since (3.9) implies (3.8), and since $E \subset A$ with $\mathrm{diam}(E) < r$, we are left to consider the case when

$$E \subset B_{x,2r}\,, \qquad \text{for some } x \in \partial A\,.$$

Set $U = U_x$ and $\phi = \phi_x$. Since $r < r_0$ it makes sense to define $F = \phi^{-1}(E)$. Since $F \subset B_{C\,r}$, by the area formula and by (3.5) one has

$$\begin{aligned}\mathcal{F}_{A,\sigma}(E) &= \int_{H \cap \partial^* F} J^{\partial^* F}\,\phi\,d\mathcal{H}^{n-1} + \int_{\partial H \cap \partial^* F} \sigma(\phi)\,J^{\partial^* F}\,\phi\,d\mathcal{H}^{n-1} \\ &\ge (1 - C\,r)\,P(F; H) + (\sigma(x) - C\,r)\,\mathcal{H}^{n-1}(\partial H \cap \partial^* F) \ge \mathcal{F}_{H,\sigma_0}(F) - C\,r\,P(F)\,.\end{aligned}$$

By [Mag12, Proposition 19.22] we have $\mathcal{F}_{H,\sigma_0}(F) \ge (1 + \sigma_0)P(F)/2$, thus

$$\mathcal{F}_{A,\sigma}(E) \ge (1 - C\,r)\,\mathcal{F}_{H,\sigma_0}(F) \ge (1 - C\,r)\,\psi(\sigma_0)\,|F|^{(n-1)/n}\,.$$

Finally, since $|E| = \int_F J\phi$ implies $|F|^{(n-1)/n} \ge |E|^{(n-1)/n}(1 - C\,r)$, one finds (3.8).

Step three: We prove that if $m < m_0$, then

$$P(E_m) \le C\,m^{(n-1)/n}\,. \tag{3.10}$$

We are going to deduce this from (3.6) and the general inequality

$$P(E) \le C\,\mathcal{F}_{A,\sigma}(E)\,, \qquad \forall E \subset A\,, |E| = m < m_0\,. \tag{3.11}$$

Let us prove (3.11). This is obvious if $\sigma_0 > 0$, as in this case,

$$P(E) = P(E; A) + P(E; \partial A) \le \sigma_0^{-1}\big(P(E; A) + \sigma_0\,P(E; \partial A)\big) \le \sigma_0^{-1}\,\mathcal{F}_{A,\sigma}(E)\,.$$

Let us now assume that $\sigma_0 \le 0$, and consider the relative isoperimetric problems

$$\begin{aligned}\lambda_A(m) &= \inf\Big\{\frac{P(E; A)}{\mathcal{H}^{n-1}(\partial^* E \cap \partial A)} : |E| = m, E \subset A\Big\}\,, \\ \mu_A &= \inf\Big\{\frac{P(E; A)}{|E|^{n/(n-1)}} : |E| \le \frac{|A|}{2},\, E \subset A\Big\}\,,\end{aligned}$$

so that $\mu_A > 0$ (as A is a connected bounded open set with Lipschitz boundary). Let us first show that for all $\tau > 0$ there is m_1 (depending on A and τ) such that

$$\lambda_A(m) \ge \frac{1}{1 + \tau}\,, \qquad \forall m < m_1\,. \tag{3.12}$$

Indeed, since ∂A is of class $C^{1,1}$, one can construct $T\in\mathrm{Lip}(\mathbb{R}^n;\mathbb{R}^n)$ with $T=-\nu_A$ on ∂A and $|T|\le 1$ on $\mathbb{R}^n$. By the divergence theorem, if E is a competitor in $\lambda_A(m)$, then

$$\mathcal{H}^{n-1}(\partial^*E\cap\partial A)\le P(E;A)+\Big|\int_E\mathrm{div}\,T\Big|\le P(E;A)+\mathrm{Lip}(T)\,m\le\Big(1+\frac{\mathrm{Lip}(T)\,m^{1/n}}{\mu_A}\Big)\,P(E;A)\,,$$

so that (3.12) follows immediately. Let us now consider $\tau>0$ and $a>0$ such that

$$(1+\tau)\,a\,\sigma_0=-(a+\sigma_0-1)\,. \tag{3.13}$$

(The reason for this choice will become apparent in a moment.) Notice that (3.13) is equivalent to requiring

$$a=\frac{1-\sigma_0}{1+(1+\tau)\sigma_0}>0\,,$$

which is certainly possible (by $\sigma_0>-1$) as soon as τ is small enough depending on σ_0. Let $m_0\le m_1$ for $m_1=m_1(A,\tau)$. By (3.12), if $E\subset A$ and $|E|=m<m_0$, then

$$\begin{aligned}
P(E)&=P(E;A)+(1-\sigma_0)\mathcal{H}^{n-1}(\partial^*E\cap\partial A)+\sigma_0\mathcal{H}^{n-1}(\partial^*E\cap\partial A)\\
&=\mathcal{F}_{A,\sigma_0}(E)+(a-(a+\sigma_0-1))\mathcal{H}^{n-1}(\partial^*E\cap\partial A)\\
&\le\mathcal{F}_{A,\sigma_0}(E)+\frac{a}{\lambda_A(m)}\,P(E;A)-(a+\sigma_0-1)\mathcal{H}^{n-1}(\partial^*E\cap\partial A)\\
&\le\mathcal{F}_{A,\sigma_0}(E)+(1+\tau)a\,P(E;A)-(a+\sigma_0-1)\mathcal{H}^{n-1}(\partial^*E\cap\partial A)\\
&=(1+(1+\tau)a)\mathcal{F}_{A,\sigma_0}(E)\le(1+(1+\tau)\,a)\,\mathcal{F}_{A,\sigma}(E)\,,
\end{aligned}$$

where in the identity on the last line we have used (3.13). This completes the proof of (3.11).

Step four: With the goal in mind of bounding the diameter of E_m in terms of m, we now estimate the normalized volume error one makes in boxing E_m into a (properly centered) cube Q_r of side length r. More precisely, we show the existence of positive constants C_2 and $r_1\le r_0$ (depending on A, σ and g) such that if $m<m_0$ and $r<r_1$, then

$$\frac{|E_m\setminus Q_r|}{m}\le C_2\,\Big(\frac{m^{1/n}}{r}+r\Big)^{n/(n-1)} \tag{3.14}$$

for a cube Q_r of side length r. Indeed, by applying [FM12, Lemma 5.1] to $E_m^{(1)}$ (the set of points of density one of E_m in $\mathbb{R}^n$), we find that for every $r>0$ there is a partition of (Lebesgue almost all of) $\mathbb{R}^n$ into a family $\mathcal{Q}$ of open parallel cubes with side length r such that

$$\frac{|E_m|}{r}=\frac{|E_m^{(1)}|}{r}\ge\frac{1}{2n}\sum_{Q\in\mathcal{Q}}\mathcal{H}^{n-1}(E_m^{(1)}\cap\partial Q)\,, \tag{3.15}$$

and it is actually clear from the proof of that lemma that these cubes can be chosen so that

$$\mathcal{H}^{n-1}(\partial Q\cap\partial A)=\mathcal{H}^{n-1}(\partial Q\cap\partial^*E_m)=0\qquad\text{for every }Q\in\mathcal{Q}\,. \tag{3.16}$$

If r_1 is small enough in terms of A, σ and g and C is the constant appearing on the left-hand side of (3.8) (see step two), then we can entail $(1-C\,r)>0$ with $(1-C\,r)^{-1}\le 1+2\,C\,r$ for $r<r_1$. Up to also requiring that $r_1<r_0/\sqrt{n}$, by applying (3.8) to $E=E_m\cap Q$ we find that

$$\psi(\sigma_0)\,|E_m\cap Q|^{(n-1)/n}\le(1+2\,C\,r)\,\mathcal{F}_{A,\sigma}(E_m\cap Q)\,. \tag{3.17}$$

By [Mag12, Equation (16.4)], $E_m^{(1)}\cap A=E_m^{(1)}$, $\mathcal{H}^{n-1}(\partial^*Q\Delta\partial Q)=0$, and (3.16), we have

$$P(E_m\cap Q;A)=P(E_m;Q\cap A)+\mathcal{H}^{n-1}(\partial Q\cap E_m^{(1)})\,,\qquad\forall Q\in\mathcal{Q}\,,$$

and thus, taking also into account the $\mathcal{H}^{n-1}$-equivalence of the sets $\partial A \cap \partial^*(E_m \cap Q)$ and $Q \cap \partial A \cap \partial^* E_m$ (which follows by (3.16)),

$$\mathcal{F}_{A,\sigma}(E_m \cap Q) = P(E_m; Q \cap A) + \int_{Q \cap \partial A \cap \partial^* E_m} \sigma + \mathcal{H}^{n-1}(\partial Q \cap E_m^{(1)})\,.$$

By this last identity, by adding up over Q in (3.17), and by (3.15) one finds

$$\begin{aligned}
\psi(\sigma_0) \sum_{Q \in \mathcal{Q}} |E_m \cap Q|^{(n-1)/n} &\le (1 + Cr) \sum_{Q \in \mathcal{Q}} \mathcal{F}_{A,\sigma}(E_m \cap Q)\\
&= (1 + Cr)\Big(\mathcal{F}_{A,\sigma}(E_m) + \sum_{Q \in \mathcal{Q}} \mathcal{H}^{n-1}(E_m^{(1)} \cap \partial Q)\Big)\\
&\le (1 + Cr)\Big(\gamma(m) - \int_{E_m} g + 2n\,\frac{m}{r}\Big)\\
&\le (1 + C\,r)\Big(\gamma(m) + C\,\frac{m}{r}\Big)\,.
\end{aligned}$$

By step one, see (3.6),

$$\psi(\sigma_0) \sum_{Q \in \mathcal{Q}} |E_m \cap Q|^{(n-1)/n} \le C\,\frac{m}{r} + (1 + C\,r)\,\psi(\sigma_0)\,m^{(n-1)/n}\,(1 + C\,m^{1/n})\,,$$

so that, first dividing both sides by $m^{(n-1)/n}$, and then subtracting $\psi(\sigma_0)$, we have

$$\begin{aligned}
\psi(\sigma_0)\Big(\sum_{Q \in \mathcal{Q}} \Big(\frac{|E_m \cap Q|}{m}\Big)^{(n-1)/n} - 1\Big) &\le C\,\frac{m^{1/n}}{r} + C\,(r + m^{1/n})\\
&\le C\,\Big(\frac{m^{1/n}}{r} + r\Big)\,.
\end{aligned} \tag{3.18}$$

Now let $\{Q_i\}_{i=1}^N = \{Q \in \mathcal{Q} : |E_m \cap Q| > 0\}$. (Notice that $N < \infty$ as only finitely many cubes from $\mathcal{Q}$ can intersect A, thus E_m.) We can order these cubes so that

$$\Big|E_m \cap \bigcup_{i=1}^{M-1} Q_i\Big| < \frac{|E_m|}{2}\,, \qquad \Big|E_m \cap \bigcup_{i=M+1}^{N} Q_i\Big| < \frac{|E_m|}{2}\,.$$

By concavity,

$$\sum_{i=1}^N \Big(\frac{|E_m \cap Q_i|}{m}\Big)^{(n-1)/n} \ge \Big(\sum_{i=1}^{M-1} \frac{|E_m \cap Q_i|}{m}\Big)^{(n-1)/n} + \Big(\sum_{i=M}^{N} \frac{|E_m \cap Q_i|}{m}\Big)^{(n-1)/n}\,,$$

and since $t^{(n-1)/n} + (1-t)^{(n-1)/n} - 1 \ge c(n)\,t^{(n-1)/n}$ for every $t \in [0, 1/2]$, (3.18) gives

$$\Big(\sum_{i=1}^{M-1} \frac{|E_m \cap Q_i|}{m}\Big)^{(n-1)/n} \le C\,\Big(\frac{m^{1/n}}{r} + r\Big)\,,$$

and, by an analogous argument,

$$\Big(\sum_{i=M+1}^{N} \frac{|E_m \cap Q_i|}{m}\Big)^{(n-1)/n} \le C\,\Big(\frac{m^{1/n}}{r} + r\Big)\,.$$

In conclusion,

$$\frac{|E_m \setminus Q_M|}{m} = \sum_{i \ne M} \frac{|E_m \cap Q_i|}{m} \le C\,\Big(\frac{m^{1/n}}{r} + r\Big)^{n/(n-1)}\,,$$

and (3.14) is proved.

Step five: From now on we shall always assume that

$$m_0 \le r_1^{2n}\,. \tag{3.19}$$

with r_1 as in the previous step. We claim that for every $m < m_0$ there exists $x_m \in A$ such that

$$\big|E_m \setminus B_{x_m,C(n)m^{1/2n}}\big| \le C_3\, m^{1+[1/2(n-1)]}\,, \tag{3.20}$$

where $C_3 = C_3(A,\sigma,g)$. To prove this we apply step four with $r = m^{1/2n}$ (as we can as $r < r_1$ by (3.19)) to find a cube Q_r of side length r such that $Q_r \cap A \ne \emptyset$ and

$$|E_m \setminus Q_r| \le C_2\, m\,\Big(r + \frac{m^{1/n}}{r}\Big)^{n/(n-1)} \le C_3\, m^{1+[1/2(n-1)]}\,.$$

Since $Q_r \cap A \ne \emptyset$, there exists $x_m \in A$ such that $Q_r \subset B_{x_m,C(n)\,r}$, and thus (3.20) is proved.

Step six: We prove if $m < m_0$ and if x_m is defined as in step five, then

$$E_m \subset B_{x_m,C\,m^{1/2n}}\,, \tag{3.21}$$

for a constant $C = C(A,\sigma,g)$. To prove our assertion it suffices to show that given a sequence $m_h \to 0^+$, then, possibly up to extracting subsequences, one has

$$E_h \subset B_{x_h,C\,m_h^{1/2n}}\,, \qquad \text{where} \quad E_h = E_{m_h}\,, \quad x_h = x_{m_h}\,, \tag{3.22}$$

for some constant C. Up to subsequences we may assume that $x_h \to x_0$ for some $x_0 \in \overline{A}$. We now distinguish two cases.

Case one, $x_0 \in A$: We set, for $C(n)$ as in the left-hand side of (3.20),

$$I = \Big(C(n)\, m_h^{1/2n}, \frac{\mathrm{dist}(x_0,\partial A)}{4}\Big)\,,$$

and, for every $r \in I$, we let

$$E_h^r = E_h \cap B_{x_h,r}\,, \qquad u_h(r) = \frac{|E_h \setminus B_{x_h,r}|}{m_h}\,, \qquad \lambda_h(r) = \Big(\frac{m_h}{|E_h^r|}\Big)^{1/n}\,. \tag{3.23}$$

If $\hat E_h^r$ denotes the dilation of E_h^r by a factor $\lambda_h(r)$ with respect to the point x_0, then

$$|\hat E_h^r| = m_h\,, \qquad \hat E_h^r \subset B_{x_0,\lambda_h(r)\,\mathrm{dist}(x_0,\partial A)/2} \qquad \forall r \in I\,, \tag{3.24}$$

provided h is large enough. Indeed the inclusion follows by noticing that, for h large,

$$E_h^r \subset B_{x_h,r} \subset B_{x_0,r+|x_h-x_0|} \subset B_{x_0,\mathrm{dist}(x_0,\partial A)/2}\,.$$

Now, since $u_h(r)$ is decreasing in r, by (3.20) we find

$$u_h(r) \le u_h(C(n)\, m_h^{1/2n}) \le C_3\, m_h^{1/2(n-1)}\,, \qquad \forall r \in I\,, \tag{3.25}$$

so that, for h large enough,

$$\lambda_h(r) = \frac{1}{(1-u_h(r))^{1/n}} \le 1 + C\,u_h(r) \le 1 + C\, m_h^{1/2(n-1)} \le 2\,, \tag{3.26}$$

and thus

$$\hat E_h^r \subset B_{x_0,\mathrm{dist}(x_0,\partial A)} \subset A\,, \qquad \forall r \in I\,. \tag{3.27}$$

By (3.24) and (3.27) we can exploit the minimality of E_h to deduce

$$\mathcal{F}_{A,\sigma}(E_h) + \int_{E_h} g \le \mathcal{F}_{A,\sigma}(\hat E_h^r) + \int_{\hat E_h^r} g = P(\hat E_h^r) + \int_{\hat E_h^r} g\,. \tag{3.28}$$

We notice that by $g \ge 0$, $\hat E_h^r = x_0 + \lambda_h(r)(E_h^r - x_0)$, and (3.26)

$$\int_{\hat E_h^r} g - \int_{E_h} g \;\;\le\;\; \lambda_h(r)^n \int_{E_h^r} g(x_0 + \lambda_h(y - x_0))\,dy - \int_{E_h^r} g$$

$$
\begin{aligned}
&\le C\Big(\|g\|_{C^0(A)}\, m_h\, u_h(r) + \int_{E_h^r} g(x_0+\lambda_h(y-x_0)) - g(y)\,dy\Big)\\
&\le C\Big(\|g\|_{C^0(A)}\, m_h\, u_h(r) + \mathrm{Lip}(g)\,|\lambda_h(r)-1| \int_{E_h^r} |y-x_0|\,dy\Big)\\
&\le C\, m_h\, u_h(r)\,. \qquad (3.29)
\end{aligned}
$$

At the same time, since $\mathcal{H}^{n-1}(E_h\cap\partial B_{x_h,r}) = -m_h\,u_h'(r)$ for a.e. $r>0$, we have

$$
\begin{aligned}
P(\hat E_h^r) &= \lambda_h(r)^{n-1}\,P(E_h\cap B_{x_h,r}) \le (1+C\,u_h(r))\big(P(E_h;B_{x_h,r}) + m_h\,|u_h'(r)|\big)\\
&\le P(E_h;B_{x_h,r}) + C\,m_h\,\big(|u_h'(r)| + m_h^{-1/n}\,u_h(r)\big)\,, \qquad (3.30)
\end{aligned}
$$

where we have also used the fact that $P(E_h;B_{x_h,r})\le P(E_h)\le C\,m_h^{(n-1)/n}$. By combining (3.28), (3.29) and (3.30) we find that for a.e. $r\in I$,

$$
P(E_h;A) + \int_{\partial A\cap\partial^* E_h}\sigma \le P(E_h;B_{x_h,r}) + C\,m_h\,\big(|u_h'(r)| + m_h^{-1/n}\,u_h(r)\big)\,,
$$

that is

$$
C\,m_h\,\big(|u_h'(r)| + m_h^{-1/n}\,u_h(r)\big) \ge P(E_h;A\setminus B_{x_h,r}) + \int_{\partial A\cap\partial^* E_h}\sigma\,,
$$

for a.e. $r\in I$. By adding up $\mathcal{H}^{n-1}(E_h\cap\partial B_{x_h,r}) = m_h\,|u_h'(r)|$ to both sides and using that $B_{x_h,r}\cap\partial A=\emptyset$ implies $\partial A\cap\partial^*E_h = \partial A\cap\partial^*(E_h\setminus B_{x_h,r})$, we conclude

$$
\begin{aligned}
C\,m_h\,\big(|u_h'(r)| + m_h^{-1/n}\,u_h(r)\big) &\ge P(E_h\setminus B_{x_h,r};A) + \int_{\partial A\cap\partial^* E_h}\sigma = \mathcal{F}_{A,\sigma}(E_h\setminus B_{x_h,r}) \quad (3.31)\\
&\ge C^{-1}\,P(E_h\setminus B_{x_h,r}) \ge C^{-1}\,n\omega_n^{1/n}\,m_h^{(n-1)/n}\,u_h(r)^{(n-1)/n}\,,
\end{aligned}
$$

for a.e. $r\in I$, where in the last line we have used (3.11) and the isoperimetric inequality. Thanks to (3.25), provided h is large enough, one has

$$
C\,u_h(r) \le \frac{n\omega_n^{1/n}}{2}\,u_h(r)^{(n-1)/n}\,,\qquad \forall r\in I\,,
$$

so that, for a.e. $r\in I$,

$$
C\,m_h\,|u_h'(r)| \ge \frac{n\omega_n^{1/n}}{2}\,m_h^{(n-1)/n}\,u_h(r)^{(n-1)/n}\,,\quad \text{i.e.}\quad [(m_h u_h)^{1/n}]'(r) \le -C\,.
$$

By integrating this inequality over $(C(n)\,m_h^{1/2n}, r)$ and by (3.25), one finds that for every $r\in I$

$$
\begin{aligned}
(m_h\,u_h(r))^{1/n} &\le (m_h u_h(C(n)m_h^{1/2n}))^{1/n} - C\big(r - C(n)m_h^{1/2n}\big)\\
&\le (C_3\,m_h^{1+[1/2(n-1)]})^{1/n} - C\big(r - C(n)m_h^{1/2n}\big)\,.
\end{aligned}
$$

For a suitably large value of C_* we find that

$$
u_h(r_*) = 0\,,\qquad \text{if}\qquad r_* = C(n)\,m_h^{1/2n} + C_*(m_h^{1+[1/2(n-1)]})^{1/n}\,,
$$

where $r_*\in I$ provided h is large enough in terms of C_*. Since $r_*\le C\,m_h^{1/2n}$ this completes the proof of (3.22) in case one.

Case two, $x_0\in\partial A$: Let r_0, s_0, $U=U_{x_0}$ and $\phi=\phi_{x_0}$ be as in (3.3). For a constant $N\ge 2$ to be determined later on in terms of n and A, let us consider $L>0$ (depending on N, n and A) such that

$$
B_{x_0,r_0/L}\subset\phi(B_{s_0/N})\,. \qquad (3.32)
$$

As in case one, we define $E_h^r = E_h \cap B_{x_h,r}$ and $u_h(r) = |E_h \setminus B_{x_h,r}|/m_h$ for every $r \in I$, where now – with $C(n)$ as in the left-hand side of (3.20) – we take

$$I = \Big(C(n) m_h^{1/2n}, \frac{r_0}{2L}\Big).$$

For h large enough and $r \in I$, by (3.32) we have

$$E_h^r \subset B_{x_h,r} \cap A \subset B_{x_0,r_0/L} \cap A \subset \phi(B_{s_0/N}) \cap A,$$

so that it makes sense to consider

$$\phi^{-1}(E_h^r) \subset B_{s_0/N} \cap H \subset B_{s_0/2} \cap H. \tag{3.33}$$

We claim that for every h large enough and for every $r \in I$ there exists $\lambda_h(r) \in (1,2)$ such that

$$\hat{E}_h^r = \phi\big(\lambda_h(r)\,\phi^{-1}(E_h^r)\big)$$

satisfies

$$|\hat{E}_h^r| = m_h, \qquad \hat{E}_h^r \subset A, \qquad 1 \le \lambda_h(r) \le 1 + C\,u_h(r). \tag{3.34}$$

Indeed, by (3.33) we find

$$\lambda\phi^{-1}(E_h^r) \subset H \cap B_{\lambda s_0/2} \subset H \cap B_{s_0} \subset H \cap U, \qquad \forall \lambda < 2, \tag{3.35}$$

so that $\phi(\lambda\,\phi^{-1}(E_h^r))$ is defined for every $\lambda < 2$. If $w_h^r(\lambda) = |\phi(\lambda\,\phi^{-1}(E_h^r))|$, $\lambda < 2$, then we have

$$w_h^r(1) = |E_h^r| = m_h(1 - u_h(r)) \le m_h,$$

while at the same time, if $1 < \lambda < 2$ then by (3.5) and (3.35) we find

$$\begin{aligned} w_h^r(\lambda) &= \int_{\lambda\,\phi^{-1}(E_h^r)} J\phi \ge (1 - C_1 s_0)\lambda^n |\phi^{-1}(E_h^r)| = (1 - C_1 s_0)\lambda^n \int_{E_h^r} J\phi^{-1}(x)\,dx \\ &= (1 - C_1 s_0)\lambda^n \int_{E_h^r} \frac{dx}{J\phi(\phi^{-1}(x))} \ge \frac{1 - C_1 s_0}{1 + C_1 s_0}\,\lambda^n\,|E_h^r|, \end{aligned} \tag{3.36}$$

that is, by (3.25) (which still holds for every $r \in I$ even with the new definition of I)

$$\frac{w_h^r(\lambda)}{m_h} \ge \frac{1 - C_1 s_0}{1 + C_1 s_0}\,(1 - u_h(r))\,\lambda^n \ge \frac{1 - C_1 s_0}{1 + C_1 s_0}\,\big(1 - C_3\,m_h^{1/2n}\big)\,\lambda^n.$$

In particular, up to further decreasing the value of s_0 and for every h large enough, one can always find $\lambda_h^*(r) \in (1,2)$ such that

$$w_h^r(\lambda_h^*(r)) > m_h \ge w_h^r(1).$$

Since $w_h^r(\lambda) = \lambda^n \int_{\phi^{-1}(E_h^r)} J\phi(\lambda z)\,dz$ is a continuous function, we find $\lambda_h(r) \in [1,2)$ such that

$$m_h = w_h^r(\lambda_h(r)) = \lambda_h(r)^n \int_{\phi^{-1}(E_h^r)} J\phi(\lambda_h(r)z)\,dz = \lambda_h(r)^n \int_{E_h^r} J\phi(\lambda_h(r)\phi^{-1}(y))\,J\phi^{-1}(y)\,dy$$

so that

$$m_h\,u_h(r) = m_h - |E_h^r| = \int_{E_h^r} v(\lambda_h(r), y)\,dy. \tag{3.37}$$

where we have set

$$v(\lambda, y) = \lambda^n\,J\phi(\lambda\phi^{-1}(y))\,J\phi^{-1}(y) - 1, \qquad y \in B_{x_0,2\,r_0} \cap A, \lambda \in [1,2).$$

Now, if $y \in E_h^r$, then $|\phi^{-1}(y)| \le s_0/N$ by (3.33), so that for every $\lambda \in [1,2)$ and $y \in E_h^r$ we find

$$\begin{aligned} v(\lambda, y) &\ge \lambda^n - 1 + \lambda^n\Big(J\phi(\lambda\phi^{-1}(y)) - J\phi(\phi^{-1}(y))\Big)\,J\phi^{-1}(y) \\ &\ge n\,(\lambda - 1) - 2^n\,\Big|J\phi(\lambda\phi^{-1}(y)) - J\phi(\phi^{-1}(y))\Big|\,J\phi^{-1}(y) \\ &\ge n\,(\lambda - 1) - C\,(\lambda - 1)\,|\phi^{-1}(y)| \ge \Big(n - C_4\,\frac{s_0}{N}\Big)\,(\lambda - 1), \end{aligned}$$

where $C_4 = C_4(n, A)$. Provided we pick N suitably large in terms of n and A we thus find

$$v(\lambda, y) \geq \lambda - 1\,, \qquad \forall y \in E_h^r\,, \lambda \in (1,2)\,,$$

and thus deduce from (3.37) and $|E_h^r| \geq m_h/2$ that if h is large enough, then

$$\lambda_h(r) \leq 1 + C\,u_h(r)\,, \qquad \forall r \in I\,.$$

This completes the proof of (3.34). By (3.34) and the minimality of E_h we obtain

$$\mathcal{F}_{A,\sigma}(E_h) + \int_{E_h} g \leq \mathcal{F}_{A,\sigma}(\hat{E}_h^r) + \int_{\hat{E}_h^r} g\,. \tag{3.38}$$

On the one hand

$$\begin{aligned}
\int_{\hat{E}_h^r} g - \int_{E_h} g &\leq \int_{E_h^r} \Big(g\big(\phi(\lambda_h(r)\phi^{-1}(y))\big)\,\big(v(\lambda_h(r), y) + 1\big) - g(y) \Big)\,dy \\
&\leq C \int_{E_h^r} \Big(g\big(\phi(\lambda_h(r)\phi^{-1}(y))\big) - g(y) \Big)\,dy + \int_{E_h^r} g(y)\,v(\lambda_h(r), y)\,dy \\
&\leq C\,\mathrm{Lip}(g) \int_{E_h^r} \Big| \phi(\lambda_h(r)\phi^{-1}(y))) - y \Big|\,dy + \|g\|_{C^0(A)} \int_{E_h^r} v(\lambda_h(r), y)\,dy \\
&\leq C\,(\lambda_h(r) - 1)\,|E_h^r| \leq C\,m_h\,u_h(r)\,;
\end{aligned}$$

on the other hand, by repeatedly applying the area formula and by [Mag12, Proposition 17.1]

$$P(\hat{E}_h^r; A) = \lambda_h(r)^{n-1} \int_{A \cap \partial^* E_h^r} \big(J^{\lambda_h(r)\,\phi^{-1}(\partial^* E_h^r)}\phi\big)(\lambda_h(r)\phi^{-1}(x))\,J^{\partial^* E_h^r}\phi(x)\,d\mathcal{H}^{n-1}(x)\,,$$

and since for every $(n-1)$-rectifiable set $\Sigma \subset B_{x_0, 2r_0} \cap A$ we have

$$\big(J^{\phi^{-1}(\Sigma)}\phi\big)(\phi^{-1}(x)) J^{\Sigma}\phi(x) = 1\,, \qquad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x \in \Sigma\,,$$

by $\lambda_h(r) \leq 1 + C\,u_h(r)$ we find

$$P(\hat{E}_h^r; A) \leq (1 + C\,u_h(r))\,P(E_h^r; A)\,, \qquad \forall r \in I\,,$$

and similarly, since σ is a Lipschitz function,

$$\int_{\partial A \cap \partial^* \hat{E}_h^r} \sigma \leq (1 + C\,u_h(r)) \int_{\partial A \cap \partial^* E_h^r} \sigma\,, \qquad \forall r \in I\,.$$

By combining these estimates with (3.38) we thus find

$$P(E_h; A) + \int_{\partial A \cap \partial^* E_h} \sigma \leq (1 + C\,u_h(r)) \Big(P(E_h^r; A) + \int_{\partial A \cap \partial^* E_h^r} \sigma \Big) + C\,m_h\,u_h(r)\,,$$

and thus, rearranging terms and for a.e. $r \in I$,

$$P(E_h; A \setminus B_{x_h, r}) + \int_{(\partial A \cap \partial^* E_h) \setminus B_{x_h, r}} \sigma \leq C\Big(m_h\,|u_h'(r)| + u_h(r)\,\mathcal{F}_{A,\sigma}(E_h^r) + m_h\,u_h(r) \Big)\,. \tag{3.39}$$

Since $\mathcal{F}_{A,\sigma} \leq P$ on any subset of A and since perimeter decreased under intersection with convex sets, we have

$$\mathcal{F}_{A,\sigma}(E_h^r) \leq P(E_h^r) \leq P(E_h) \leq C\,m_h^{(n-1)/n}\,, \tag{3.40}$$

where in the last inequality we have used (3.10). By combining (3.39) and (3.40), and by adding up $m_h\,|u_h'(r)| = \mathcal{H}^{n-1}(E_h \cap \partial B_{x_h, r})$ to both sides of the resulting inequality, we eventually get

$$\mathcal{F}_{A,\sigma}(E_h \setminus B_{x_h, r}) \leq C\,m_h\,\big(|u_h'(r)| + m_h^{-1/n}\,u_h(r)\big)\,,$$

for a.e. $r \in I$, which is analogous to (3.31). From here we conclude by arguing exactly as in case one. This completes the proof of (3.21).

Step seven: We prove (3.1). We first notice that it must be $B_{x_m,C\,m^{1/2n}}\cap\partial A\neq\emptyset$, for otherwise by (3.21) we would get

$$\mathcal{F}_{A,\sigma}(E_m)=P(E_m)=n\,\omega_n^{1/n}\,m^{(n-1)/n}=\Big(\psi(\sigma_0)+\delta\Big)\,m^{(n-1)/n}\,,$$

for some positive δ independent of m, thus contradicting (3.6): in particular,

$$E_m\subset B_{y_m,C\,m^{1/2n}}\qquad\text{for some } y_m\in\partial A\,. \tag{3.41}$$

This proves the first part of (3.1). We now prove that $\sigma(y_m)-\sigma_0\le C\,m^{1/2n}$. Let us set

$$\sigma_1=\max_{\partial A}\sigma\,,\qquad c_0=\inf\{\psi'(\tau):\tau\in[\sigma_0,\sigma_1]\}\,,$$

so that $c_0>0$ as $[\sigma_0,\sigma_1]\subset\subset(-1,1)$ and ψ' is continuous with $\psi'>0$ on $(-1,1)$ thanks to Proposition 2.1-(i). In particular,

$$\psi(\tau)\ge\psi(\sigma_0)+c_0\,(\tau-\sigma_0)\,,\qquad\forall\tau\in[\sigma_0,\sigma_1]\,,$$

and since $\sigma(y_m)\in[\sigma_0,\sigma_1]$, by (3.6) and (3.8) (which we can apply thanks to (3.41)), we conclude that we have

$$\begin{aligned}(1+C\,m^{1/n})\,\psi(\sigma_0)\ge\frac{\mathcal{F}_{A,\sigma}(E_m)}{m^{(n-1)/n}}&\ge(1-C\,m^{1/2n})\,\psi(\sigma(y_m))\\&\ge(1-C\,m^{1/2n})\Big(\psi(\sigma_0)+c_0\,(\sigma(y_m)-\sigma_0)\Big)\,.\end{aligned}\tag{3.42}$$

This completes the proof of (3.1).

Step eight: We prove (3.2). Thanks to (3.1) it makes sense to define

$$F_m=\frac{\phi_{y_m}^{-1}(E_m)}{|\phi_{y_m}^{-1}(E_m)|^{1/n}}\,.$$

Thanks to Proposition 2.1-(ii), it is enough to fix $m_h\to0^+$, set

$$E_h=E_{m_h}\,,\qquad F_h=F_{m_h}\,,\qquad y_h=y_{m_h}\,,\qquad\phi_h=\phi_{y_h}\,,$$

and show that

$$\lim_{h\to\infty}\mathcal{F}_{H,\sigma_0}(F_h)=\psi(\sigma_0)\,. \tag{3.43}$$

We first notice that

$$\Big|\frac{|\phi_h^{-1}(E_h)|}{m_h}-1\Big|\le\frac{1}{m_h}\int_{E_h}|J\phi_h^{-1}-1|\le\frac{C}{m_h}\int_{E_h}|y-y_h|\,dy\le C\,m_h^{1/2n}\,, \tag{3.44}$$

where we have used $J\phi_h^{-1}(y_h)=1$ and (3.41). Similarly, by the area formula on rectifiable sets, one sees that

$$\begin{aligned}\big|P(\phi_h^{-1}(E_h);H)-P(E_h;A)\big|&\le\int_{A\cap\partial^*E_h}|J^{\partial^*E_h}\phi_h^{-1}-1|\\&\le C\int_{A\cap\partial^*E_h}|y-y_h|\,d\mathcal{H}^{n-1}(y)\le C\,P(E_h;A)\,m_h^{1/2n}\end{aligned}$$

and, again by (3.1),

$$\begin{aligned}\Big|\sigma_0\,P(\phi_h^{-1}(E_h);\partial H)-\int_{\partial A\cap\partial^*E_h}\sigma\Big|&\le\int_{\partial A\cap\partial^*E_h}|\sigma_0\,J^{\partial^*E_h}\phi_h^{-1}-\sigma|\\&\le C\int_{\partial A\cap\partial^*E_h}\big(|y-y_h|+\sigma(y_h)-\sigma_0\big)\,d\mathcal{H}^{n-1}(y)\\&\le C\,P(E_h;\partial A)\,m_h^{1/2n}\,,\end{aligned}$$

so that, thanks to (3.10)

$$|\mathcal{F}_{A,\sigma}(E_h)-\mathcal{F}_{H,\sigma_0}(\phi_h^{-1}(E_h))|\le C\,P(E_h)\,m_h^{1/2n}\le C\,m_h^{(n-1)/n}\,m_h^{1/2n}\,.$$

By combining this last estimate with (3.44) and (3.6) we thus find that

$$\begin{aligned}\psi(\sigma_0)&\le&\mathcal{F}_{H,\sigma_0}(F_h)=|\phi_h^{-1}(E_h)|^{(1-n)/n}\,\mathcal{F}_{H,\sigma_0}(\phi_h^{-1}(E_h))\\&\le&m_h^{(1-n)/n}(1+C\,m_h^{1/2n})\left(\psi(\sigma_0)\,m_h^{(n-1)/n}\,(1+C\,m^{1/n})+C\,m_h^{(n-1)/n}\,m_h^{1/2n}\right)\\&\le&(1+C\,m_h^{1/2n})\psi(\sigma_0)\,,\end{aligned}$$

so that (3.43) follows. □

4. $C^{1,\alpha}$-convergence to the ideal droplet

We now conclude the analysis started in Lemma 3.1. Let us recall that so far we have proved the existence of $m_0>0$ such that if E_m is a minimizer in the variational problem

$$\gamma(m)=\inf\left\{\mathcal{F}_{A,\sigma}(E)+\int_E g(x)\,dx:E\subset A\,,|E|=m\right\},$$

(introduced in (1.4)) with $m<m_0$, then for some $y_m\in\partial A$ and setting

$$\phi_m=\phi_{y_m}\,,\qquad\lambda_m=|\phi_m^{-1}(E_m)|^{1/n}\,,\qquad F_m=\frac{\phi_m^{-1}(E_m)}{\lambda_m}\,,$$

one has

$$\begin{aligned}E_m\subset B_{y_m,C\,m^{1/2n}}\,,&\qquad 0\le\sigma(y_m)-\sigma_0\le C\,m^{1/2n}\,,\\\left|\frac{\lambda_m}{m^{1/n}}-1\right|\le C\,m^{1/2n}\,,&\qquad\lim_{m\to0^+}|(F_m-z_m)\Delta K|=0\,,\end{aligned}\tag{4.1}$$

where $z_m\in\partial H$, $K=K(\sigma_0)$, $\sigma_0=\min_{\partial A}\sigma$; see (3.1), (3.2) and (3.44). Here, as it was set in Notation 3.2, ϕ_m is a $C^{1,1}$-diffeomorphism between $U_m=U_{y_m}$ and $B_{y_m,2\,r_0}$ such that $\phi_m(0)=y_m$, $\nabla\phi_m(0)$ is a linear isometry of $\mathbb{R}^n$, $B_{s_0}\subset\subset U_m$, and $\phi_m(U_m\cap H)=B_{y_m,2r_0}\cap A$, where r_0 and s_0 are positive constant depending on A, g and σ (whose value will be further decreased in the course of the proof). Moreover, we notice that, as a consequence of (3.5), one has that

$$\|\phi_m\|_{C^{1,1}(U_m)}\le C\,,\qquad\|J\phi_m-1\|_{C^0(B_s)}\le C\,s\qquad\forall s<s_0\,,\tag{4.2}$$

for C depending on A only. With this situation in mind, we now improve the convergence in volume of F_m-z_m to K into $C^{1,\alpha}$-convergence. Taking into account Theorem 2.4 it will suffice to show that the sets F_m satisfy uniform almost-minimality conditions with respect to uniformly elliptic functionals.

Lemma 4.1. *Under the assumptions of Lemma 3.1, and with the notation introduced at beginning of this section, there exists $\lambda\ge1$, $\ell,\Lambda\ge0$ and $C,\rho_0>0$ (depending on A, σ and g) and elliptic integrands*

$$\Psi_m\in\boldsymbol{\mathcal{E}}(B_{C/m^{1/2n}},\lambda,\ell)$$

such that, for every $m<m_0$, F_m is a (Λ,ρ_0)-minimizer of $\boldsymbol{\Psi}_m$ in $(B_{C/m^{1/2n}},H)$, where

$$I_{3\rho_0}(F_m)\subset\subset B_{C/m^{1/2n}}\,.$$

Proof. Let us set

$$G_m=\phi_m^{-1}(E_m)\,,\qquad\Omega_m=\frac{U_m}{\lambda_m}\,,\qquad F_m=\frac{G_m}{\lambda_m}\,.$$

By (4.1) and $\phi_m(0)=y_m$ we have $G_m\subset B_{C\,m^{1/2n}}$, and thus

$$F_m\subset B_{C\,m^{1/2n}/\lambda_m}\subset B_{C/m^{1/2n}}\,,\qquad\forall m<m_0\,,\tag{4.3}$$

where in the last inclusion we have used (4.1) again. If we define, for $x \in U_m$, $y \in U_m \cap \partial H$, and $\nu \in S^{n-1}$,

$$\Phi_m(x,\nu) = |\mathrm{cof}\,\nabla\phi_m(x)\,\nu|\,, \qquad \sigma_m(y) = \sigma(\phi_m(y))\,\Phi_m(y,e_n)\,, \qquad g_m(x) = g(\phi_m(x))\,J\phi_m(x)\,,$$

then G_m is a minimizer in the variational problem

$$\inf\Big\{\mathbf{\Phi}_m(G;H) + \int_{\partial^* G\cap\partial H}\sigma_m + \int_G g_m : G \subset H\cap U_m\,, \int_G J\phi_m = m\Big\}\,. \tag{4.4}$$

Indeed, if G is a competitor in (4.4), then $E = \phi_m(G)$ satisfies $E \subset A$ and $|E| = m$, and, by the area formula,

$$\mathcal{F}_{A,\sigma}(E) + \int_E g = P(E;A) + \int_{\partial A\cap\partial^* E}\sigma + \int_E g = \mathbf{\Phi}_m(G;H) + \int_{\partial H\cap\partial^* G}\sigma_m + \int_G g_m\,.$$

Similarly, if we set, for $x \in \Omega_m$, $y \in \Omega_m \cap \partial H$, and $\nu \in S^{n-1}$,

$$\hat\Phi_m(x,\nu) = \Phi_m(\lambda_m\,x,\nu)\,, \qquad \hat\sigma_m(y) = \sigma_m(\lambda_m\,y)\,, \qquad \hat g_m(x) = g_m(\lambda_m\,x)\,,$$

then $F = G/\lambda_m \subset \Omega_m \cap H$ if and only if $G = \lambda_m\,F \subset U_m \cap H$, with

$$\mathbf{\Phi}_m(G;H) + \int_{\partial^* G\cap\partial H}\sigma_m + \int_G g_m = \lambda_m^{n-1}\Big(\hat{\mathbf{\Phi}}_m(F;H) + \int_{\partial^* F\cap\partial H}\hat\sigma_m\Big) + \lambda_m^n\int_F \hat g_m\,.$$

Hence the fact that G_m is a minimizer in (4.4) implies that F_m is a minimizer in

$$\inf\Big\{\hat{\mathbf{\Phi}}_m(F;H) + \int_{\partial^* F\cap\partial H}\hat\sigma_m + \lambda_m\int_F \hat g_m : F \subset H\cap\Omega_m\,, \int_F \psi_m = \frac{m}{\lambda_m^n}\Big\}\,, \tag{4.5}$$

provided one sets

$$\psi_m(y) = J\Phi_m(\lambda_m\,y) = |\det\nabla\phi_m(\lambda_m\,y)|\,, \qquad \forall y \in \Omega_m\,.$$

It is useful to notice that by (3.5) and $\lambda_m\,F_m = G_m \subset B_{C\,m^{1/2n}}$ (recall (4.1) and (4.3))

$$\|\psi_m - 1\|_{L^\infty(F_m)} \le C\,m^{1/2n}\,, \qquad \|\nabla\psi_m\|_{L^\infty(F_m)} \le C\,\lambda_m, \qquad \|\psi_m\|_{C^{1,1}(\Omega_m)} \le C\,, \tag{4.6}$$

for a constant C depending on A only.

We now want to exploit the minimality of F_m in (4.5) to show the following uniform almost minimality property of F_m: for every $F \subset H$ such that $F\Delta F_m \subset\subset B_{z,2\,\rho_0} \subset\subset \Omega_m$ for some $z \in H$, one has

$$\mathbf{\Psi}_m(F_m;H) \le \mathbf{\Psi}_m(F;H) + \Lambda\,|F\Delta F_m|\,,$$

for some $\Psi_m \in \boldsymbol{\mathcal{E}}(B_{C/m^{1/2n}},\lambda,\ell)$. Of course the difficulty here is that such a competitor F may fail to belong to the competition class in (4.5). However, as F is close to F_m, then $\int_F \psi_m$ should be close to $\int_{F_m}\psi_m$. We should be possible to slightly modify F into a new competitor F' with $\int_{F'}\psi_m = \int_{F_m}\psi_m$, while keeping track of the change in surface energy. This is what we do in the next argument, which is loosely based on [Alm76], see also [Mag12, Section 29.6].

For a value of ε_0 to be chosen later depending on K (thus on σ_0) only, let us now decrease the value of m_0 so to entail

$$|(F_m - z_m)\Delta K| < \varepsilon_0\,, \qquad \forall m < m_0\,, \tag{4.7}$$

where $z_m \in \partial H$ as in (4.1). Let us fix $F \subset H$ such that $F\Delta F_m \subset\subset B_{z,2\,\rho_0} \subset\subset \Omega_m$ for some $z \in H$, where ρ_0 is also to be properly chosen. We pick $x, y \in H \cap \partial K$ (independently from F_m) and fix $\tau_0 > 0$ so that

$$\overline{B}_{x,\tau_0} \cap \overline{B}_{y,\tau_0} = \emptyset\,, \qquad B_{x,\tau_0} \cup B_{y,\tau_0} \subset\subset B_{s_0/2\,m_0^{1/n}} \subset\subset H \cap \Omega_m \quad \forall m < m_0\,.$$

(The last condition follows from the fact that $B_{s_0} \subset\subset U_m$, so that we can entail $K \subset\subset B_{s_0/\lambda_m} \subset\subset \Omega_m$ for every $m < m_0$, and thanks to the fact that $\lambda_m/m^{1/n} \to 1$.) Notice that, up to decreasing the value of ρ_0, either $\overline{B}_{z-z_m,2\,\rho_0}\cap\overline{B}_{x,\tau_0}=\emptyset$, or $\overline{B}_{z-z_m,2\,\rho_0}\cap\overline{B}_{y,\tau_0}=\emptyset$. Without loss of generality we assume that

$$\overline{B}_{z-z_m,2\,\rho_0}\cap\overline{B}_{x,\tau_0}=\emptyset\,, \tag{4.8}$$

and then we fix $T\in C^\infty_c(B_{x,\tau_0};\mathbb{R}^n)$ with

$$1=\int_{\partial K}T\cdot\nu_K=\int_K \mathrm{div}\,T\,.$$

(The existence of T follows from $x\in\partial K$, of course.) Correspondingly, we can find $t_0>0$ (depending on τ_0 and $\|T\|_{C^1(\mathbb{R}^n)}$, thus on K, thus on σ_0) such that for each $|t|<t_0$ the map $f_t=\mathrm{Id}+t\,T$ defines a diffeomorphism of $\mathbb{R}^n$ with $\{f_t\ne\mathrm{Id}\}\subset\subset B_{x,\tau_0}$. Now let us consider

$$\psi_m^*(w)=\psi_m(w+z_m)=J\Phi_m(\lambda_m\,(w+z_m))\,,\qquad w\in\Omega_m-z_m\,,$$

and define a smooth function $\gamma:(-t_0,t_0)\to\mathbb{R}$ by setting

$$\gamma(t)=\int_{f_t(F_m-z_m)}\psi_m^*-\int_{F_m-z_m}\psi_m^*=t\int_{F_m-z_m}\mathrm{div}\,(\psi_m^*T)+O(t^2)\,.$$

Here, thanks to (4.6), $|O(t^2)|\le C\,t^2$ for a constant C depending on $\|T\|_{C^1(\mathbb{R}^n)}$ and $\|\psi_m^*\|_{C^{1,1}(F_m-z_m)}$, thus on A and σ_0, only. Also,

$$\begin{aligned}\Big|1-\int_{F_m-z_m}\mathrm{div}\,(\psi_m^*T)\Big| &\le \Big|\int_K\mathrm{div}\,T-\int_{F_m-z_m}\psi_m^*\mathrm{div}\,T+T\cdot\nabla\psi_m^*\Big|\\ &\le \|T\|_{C^1(\mathbb{R}^n)}\Big(|K\Delta(F_m-z_m)|+\int_{F_m-z_m}|\psi_m^*-1|+|\nabla\psi_m^*|\Big)\\ &\le \|T\|_{C^1(\mathbb{R}^n)}\big(\varepsilon_0+m^{1/2n}+\lambda_m\big)\,,\end{aligned}$$

where we have used both (4.6) and (4.7). In particular, by (4.1), if we further decrease the values of t_0, m_0 and ε_0 depending on $\|T\|_{C^1(\mathbb{R}^n)}$, then we find

$$\gamma(0)=0\,,\qquad\gamma'(0)\ge\frac12\,,\qquad\mathrm{Lip}(\gamma';(-t_0,t_0))\le C\,. \tag{4.9}$$

By (4.9), up to further decreasing the value of t_0, we can find $\eta_0>0$ (depending on A, g and σ) such that γ^{-1} is well-defined on $(-2\eta_0,2\eta_0)$ with

$$|\gamma^{-1}(v)|\le C\,|v|\,,\qquad\text{for every }|v|\le2\eta_0\,. \tag{4.10}$$

Recalling that $F\subset H$ with $F\Delta F_m\subset\subset B_{z,2\,\rho_0}\subset\subset\Omega_m$, we set

$$v=\int_{F_m-z_m}\psi_m^*-\int_{F-z_m}\psi_m^*\,.$$

Up to further decreasing the value of ρ_0 we find

$$|v|\le\|\psi_m\|_{C^0(\Omega_m)}\,|F\Delta F_m|\le C\,\omega_n\,\rho_0^n<2\,\eta_0\,, \tag{4.11}$$

so that by (4.10) we can compute $\gamma^{-1}(v)$ and correspondingly define $F'\subset H$ by letting

$$F'-z_m=\Big((F-z_m)\cap B_{z-z_m,2\,\rho_0}\Big)\cup\Big(f_{\gamma^{-1}(v)}(F_m-z_m)\setminus B_{z-z_m,2\,\rho_0}\Big)\,.$$

Notice that, by construction, F' and F_m are equal on $H\setminus(B_{z,2\rho_0}\cup B_{x+z_m,\tau_0})$, so that

$$\begin{aligned}\int_{F'}\psi_m-\int_{F_m}\psi_m &= \int_{F\cap B_{z,2\rho_0}}\psi_m-\int_{F_m\cap B_{z,2\rho_0}}\psi_m\\ &\quad+\int_{f_{\gamma^{-1}(v)}(F_m-z_m)\cap B_{x,\tau_0}}\psi_m^*-\int_{(F_m-z_m)\cap B_{x,\tau_0}}\psi_m^*\end{aligned}$$

$$
\begin{aligned}
&= \int_F \psi_m - \int_{F_m} \psi_m + \int_{f_{\gamma^{-1}(v)}(F_m - z_m)} \psi_m^* - \int_{F_m - z_m} \psi_m^* \\
&= \int_F \psi_m - \int_{F_m} \psi_m + \gamma(\gamma^{-1}(v)) = 0 \,.
\end{aligned}
$$

Hence, F' is a competitor for (4.5), and since $B_{x,\tau_0} \subset\subset H$ (thus $B_{x+z_m,\tau_0} \subset\subset H$ too) by comparing F_m to F' we find

$$
\begin{aligned}
&\hat{\mathbf{\Phi}}_m(F_m; B_{x+z_m,\tau_0} \cup (B_{z,2\rho_0} \cap H)) + \int_{B_{z,2\rho_0} \cap \partial^* F_m \cap \partial H} \hat{\sigma}_m \\
\le\ & \hat{\mathbf{\Phi}}_m(F'; B_{x+z_m,\tau_0} \cup (B_{z,2\rho_0} \cap H)) + \int_{B_{z,2\rho_0} \cap \partial^* F \cap \partial H} \hat{\sigma}_m + \lambda_m \|\hat{g}_m\|_{C^0(\Omega_m)} |F' \Delta F_m| \,,
\end{aligned}
\tag{4.12}
$$

where in writing the second term on the second line we have taken into account that $\partial^* F$ and $\partial^* F'$ are equal on $B_{z,2\rho_0} \cap \partial H$. In order to exploit (4.12) it is useful to show that

$$
|(F' \Delta F_m) \cap B_{x+z_m,\tau_0}| + |\hat{\mathbf{\Phi}}_m(F_m; B_{x+z_m,\tau_0}) - \hat{\mathbf{\Phi}}_m(F'; B_{x+z_m,\tau_0})| \le C\, |F \Delta F_m| \,. \tag{4.13}
$$

To begin with, by [Mag12, Lemma 17.9]

$$
\begin{aligned}
|(F' \Delta F_m) \cap B_{x+z_m,\tau_0}| &\le \left| f_{\gamma^{-1}(v)}(F_m - z_m) \Delta (F_m - z_m) \right| \\
&\le C\, |\gamma^{-1}(v)|\, P(F_m - z_m; B_{x,\tau_0}) \le C\, |v|\, P(F_m; H) \,;
\end{aligned}
$$

since $P(F_m; H) = \lambda_m^{1-n}\, P(G_m; H) \le C\, m^{(1-n)/n}\, P(E_m)$, (3.10) gives $P(F_m; H) \le C$, and thus by (4.6) and the definition of v

$$
\begin{aligned}
|(F' \Delta F_m) \cap B_{x+z_m,\tau_0}| &\le C \left| \int_{F_m} \psi_m - \int_F \psi_m \right| \le C\, \|\psi_m\|_{C^0(\Omega_m)}\, |F_m \Delta F| \\
&\le C\, |F_m \Delta F| \,.
\end{aligned}
\tag{4.14}
$$

This proves part of (4.13). To complete the proof of (4.13), given $p \in \partial^* F_m$, let $\{\tau_i(p)\}_{i=1}^{n-1}$ denote an orthonormal basis of the approximate tangent space of $\partial^* F_m$ at p, chosen so that $\nu_{F_m}(p) = \wedge_{i=1}^{n-1} \tau_i(p)$. Since $\hat{\Phi}_m$ has finite Lipschitz constant on $\Omega_m \times \mathbb{R}^n$, one has

$$
\left| \hat{\Phi}_m\Big(f_t(p), \bigwedge_{i=1}^{n-1} df_t(p)[\tau_i(p)] \Big) - \hat{\Phi}_m(p, \nu_{F_m}(p)) \right| \le C\, |t| \,,
$$

for every $|t| < t_0$. By the area formula, setting $t = \gamma^{-1}(v)$ for the sake of brevity, we have

$$
\begin{aligned}
&|\hat{\mathbf{\Phi}}_m(F_m; B_{x+z_m,\tau_0}) - \hat{\mathbf{\Phi}}_m(F'; B_{x+z_m,\tau_0})| \\
\le\ & \int_{B_{x+z_m,\tau_0} \cap \partial^* F_m} \left| \hat{\Phi}_m\Big(f_t(p), \bigwedge_{i=1}^{n-1} df_t(p)[\tau_i(p)] \Big) - \hat{\Phi}_m(p, \nu_{F_m}(p)) \right| d\mathcal{H}^{n-1}(p) \\
\le\ & C\, |\gamma^{-1}(v)|\, P(F_m; B_{x+z_m,\tau_0}) \le C\, |F \Delta F_m| \,,
\end{aligned}
\tag{4.15}
$$

where in the last inequality we have argued as in the proof of (4.14). This proves (4.13), which combined with (4.12) gives us

$$
\hat{\mathbf{\Phi}}_m(F_m; H) + \int_{\partial^* F_m \cap \partial H} \hat{\sigma}_m \le \hat{\mathbf{\Phi}}_m(F; H) + \int_{\partial^* F \cap \partial H} \hat{\sigma}_m + C\, |F \Delta F_m| \,. \tag{4.16}
$$

Finally, let $\sigma_m^* : \mathbb{R}^n \to \mathbb{R}$ be a Lipschitz function such that $\sigma_m^* = \hat{\sigma}_m$ on ∂H and

$$
\mathrm{Lip}(\sigma_m^*) = \mathrm{Lip}(\hat{\sigma}_m; \partial H) \,. \tag{4.17}
$$

(There is a huge freedom in the choice of σ_m^* and we shall exploit it later.) By the divergence theorem

$$\begin{aligned}\int_{\partial^* F\cap\partial H}\hat\sigma_m &= \int_{\partial^* F\cap\partial H}(-e_n\,\sigma_m^*)\cdot(-e_n) = \int_{\partial^* F\cap H}\sigma_m^*\,e_n\cdot\nu_F + \int_F \mathrm{div}\,(-e_n\sigma_m^*)\\ &= \int_{H\cap\partial F}\sigma_m^*\,e_n\cdot\nu_F + \int_F(-e_n)\cdot\nabla\sigma_m^*\,,\end{aligned}$$

so that

$$\hat{\boldsymbol{\Phi}}_m(F;H) + \int_{\partial^* F\cap\partial H}\hat\sigma_m = \int_{H\cap\partial^* F}(\hat\Phi_m(x,\nu_F) + \sigma_m^* e_n\cdot\nu_F) - \int_F e_n\cdot\nabla\sigma_m^*\,.$$

In conclusion, if we set for $x\in\Omega_m$ and $\nu\in\mathbb{R}^n$,

$$\Psi_m(x,\nu) = \hat\Phi_m(x,\nu) + \sigma_m^*(x)\,e_n\cdot\nu = |\mathrm{cof}\nabla\phi_m(\lambda_m x)\nu| + \sigma_m^*(x)\,e_n\cdot\nu\,,\tag{4.18}$$

then (4.16) implies

$$\boldsymbol{\Psi}_m(F_m;H)\le\boldsymbol{\Psi}_m(F;H) + \Lambda\,|F\Delta F_m|\,,\tag{4.19}$$

whenever $F\subset H$ is such that $\mathrm{diam}(F\Delta F_m)<2\rho_0$. We now claim that

$$\Psi_m\in\boldsymbol{\mathcal{E}}(B_{C/m^{1/2n}},\lambda,\ell)\,,\qquad\forall m<m_0\,,$$

and for suitable constants λ and ℓ. It is clear from (4.18) that Ψ_m is lower semicontinuous on $\Omega_m\times\mathbb{R}^n$ with $\Psi_m(x,\cdot)$ convex, one homogeneous and with restriction to S^{n-1} of class $C^{2,1}$ for every fixed $x\in\Omega_m$. Similarly, one easily deduces from (4.2) and (4.17) an upper bound on Ψ_m and Lipschitz-type bounds on $\Psi_m(\cdot,\nu)$, $\nabla\Psi_m(\cdot,\nu)$, $\Psi_m(x,\cdot)$, $\nabla\Psi_m(x,\cdot)$ and $\nabla^2\Psi_m(x,\cdot)$ on $\Omega_m\times\mathbb{R}^n$ (recall that here ∇ and ∇^2 denote derivatives in the ν variable). The restriction of x to $B_{C/m^{1/2n}}$, and a more precise choice of σ_m^*, come to play in order to check that

$$\Psi_m(x,\nu)\ge\frac{1}{\lambda}\,,\qquad\big|\nabla^2\Psi_m(x,\nu)[\tau,\tau]\big|\ge\frac{|\tau|^2}{\lambda}\,,\qquad\forall\tau\in\nu^\perp\,,$$

whenever $x\in B_{C/m^{1/2n}}$ and $\nu\in S^{n-1}$. Indeed, by (3.4) and (4.1) we have

$$|\mathrm{cof}\,(\nabla\phi_m(\lambda_m\,x))\nu|\ge|\mathrm{cof}\,(\nabla\phi_m(0))\nu| - C\,\lambda_m\,|x| = 1 - C\,\lambda_m\,|x|\ge 1 - C\,m^{1/2n}\,,\tag{4.20}$$

for every $x\in B_{C/m^{1/2n}}$ and $\nu\in S^{n-1}$. By an analogous argument

$$|\hat\sigma_m(x)-\sigma_0| = |\sigma(\phi_m(\lambda_m x))|\mathrm{cof}\,(\nabla\phi_m(\lambda_m\,x))e_n| - \sigma(\phi_m(0))|\le C\,\lambda_m\,|x|\le C\,m^{1/2n}\,,\tag{4.21}$$

for every $x\in B_{C/m^{1/2n}}\cap\partial H$. The idea is then the following: having in mind (4.3), we first pick C so large to enforce $I_{3\rho_0}(F_m)\subset\subset B_{C/m^{1/2n}}$ (in this way, all the variations considered in (4.19) are contained in the domain of ellipticity of Ψ_m); next, we define σ_m^* as a Lipschitz-preserving extension to $\mathbb{R}^n$ of the restriction to $B_{C/m^{1/2n}}$ of $\hat\sigma_m$, which is then truncated so to preserve the bounds (4.21), so that

$$|\sigma_m^*(x)-\sigma_0|\le C\,m^{1/2n}\,,\qquad\forall x\in\mathbb{R}^n\,.\tag{4.22}$$

By combining (4.20) and (4.21) we find

$$\Psi_m(x,\nu)\ge 1 - |\sigma_0| - C\,m^{1/2n}\ge\frac{1-|\sigma_0|}{2}\,,$$

for every $x\in B_{C/m^{1/2n}}$ and $\nu\in S^{n-1}$, provided m_0 is small enough. The Hessian bound is even simpler (as it does not involve the adhesion coefficient σ), and so the proof is complete. $\square$

We are now ready to complete the proof of Theorem 1.1.

Proof of Theorem 1.1. Our starting point is given by Lemma 3.1 and Lemma 4.1. The fact that F_m is a (Λ, r_0)-minimizer of $\boldsymbol{\Psi}_m$ in $(B_{C/m^{1/2n}}, H)$ with $I_{3\rho_0}(F_m) \subset\subset B_{C/m^{1/2n}}$ implies the existence of $\rho_1 = \rho_1(\rho_0, \lambda, \Lambda) > 0$ and $\kappa = \kappa(n, \lambda) > 0$ such that if $r \le \rho_1$, then

$$\begin{aligned} |F_m \cap B_{x,r}| &\ge \kappa\, |B_{x,r} \cap H|\,, \qquad \forall x \in \overline{H} \cap \partial F_m\,,\\ |F_m \cap B_{x,r}| &\le (1-\kappa)\, |B_{x,r} \cap H|\,, \qquad \forall x \in \overline{H} \cap \partial (H \setminus F_m)\,; \end{aligned}$$

see, e.g. [DPM15, Lemma 2.8]. In particular, if we set

$$F_m^* = F_m - z_m\,,$$

then for every $r \le \rho_1$

$$\begin{aligned} |F_m^* \cap B_{x,r}| &\ge \kappa\, |B_{x,r} \cap H|\,, \qquad \forall x \in \overline{H} \cap \partial F_m^*\,,\\ |F_m^* \cap B_{x,r}| &\le (1-\kappa)\, |B_{x,r} \cap H|\,, \qquad \forall x \in \overline{H} \cap \partial (H \setminus F_m^*)\,. \end{aligned} \tag{4.23}$$

The density estimates (4.23) combined with the L^1-convergence of F_m^* to K imply the convergence of $M_m^* = \overline{H \cap \partial F_m^*}$ to $M_0 = \overline{H \cap \partial K}$ in Hausdorff distance. Indeed, let $x \in M_m^*$ be such that

$$\mathrm{dist}(x, M_0) = \sup_{y \in M_m^*} \mathrm{dist}(y, M_0) = \beta\,,$$

and let $r = \min\{\beta, \rho_1\}$. Since $M_m^* \subset \overline{H} \cap \partial F_m^*$ and $M_m^* \subset \overline{H} \cap \partial (H \setminus F_m^*)$, we have that both estimates in (4.23) hold at x with $r = \min\{\beta, \rho_1\}$. Since $B_{x,\beta} \cap M_0 = \emptyset$, we have $H \cap B_{x,r} \subset H \setminus K$ or $H \cap B_{x,r} \subset K$: in the first case

$$\kappa |B_{x,r} \cap H| \le |F_m^* \cap B_{x,r}| \le |F_m^* \setminus K| \le \varepsilon_0\,,$$

while in the second case one finds

$$\kappa |B_{x,r} \cap H| \le |(H \cap B_{x,r}) \setminus F_m^*| \le |K \setminus F_m^*| \le \varepsilon_0\,;$$

in both cases, $r^n \le C\, \varepsilon_0$, so that, up to further decreasing the value of ε_0 one has $\beta^n \le C\, |K \Delta F_m^*|$. Since density estimates analogous to (4.23) holds with K in place of F_m (for constants ρ_1 and κ depending on n and σ_0 only) we conclude that

$$\mathrm{hd}(M_0, M_m^*)^n \le C\, |F_m^* \Delta K|\,, \qquad \forall m < m_0\,. \tag{4.24}$$

An important consequence of (4.24) is that provided ε_0 is small enough, then one has

$$\frac{K}{2} \subset F_m^*\,, \qquad \forall m < m_0\,. \tag{4.25}$$

By exploiting this inclusion together with (3.6) we see that F_m^* satisfies (2.4), and thus conclude by means of Proposition 2.1-(iii) that

$$C\, m^{1/n} \ge \mathcal{F}_{H,\sigma}(F_m^*) - \psi(\tau) \ge c(n, \tau) \inf_{z \in \partial H} |(F_m^* - z) \Delta K|^2\,.$$

Since, by definition of z_m,

$$|F_m^* \Delta K| = \inf_{z \in \partial H} |(F_m - z) \Delta K| = \inf_{z \in \partial H} |(F_m^* - z) \Delta K|\,,$$

by also taking (4.24) we have found the quantitative estimates

$$|F_m^* \Delta K| \le C\, m^{1/2n}\,, \qquad \mathrm{hd}(M_0, M_m^*) \le C\, m^{1/2n^2}\,, \qquad \forall m < m_0\,. \tag{4.26}$$

Let us now go back to a simpler consequence of (4.24), namely

$$I_{3\rho_0}(F_m^*) \subset\subset B_C\,. \tag{4.27}$$

By setting $\Psi_m^*(x, \nu) = \Psi_m(x + z_m, \nu)$, we find that each F_m^* is a (Λ, ρ_0)-minimizer of $\boldsymbol{\Psi}_m^*$ in (B_C, H) where $\Psi_m^* \in \boldsymbol{\mathcal{E}}(B_C, \lambda, \ell)$. We are thus in the position to apply Theorem 2.4 to deduce that, for every $m < m_0$, M_m^* is a compact connected orientable $C^{1,\alpha}$-hypersurface with boundary for every $\alpha \in (0,1)$ and there exists a diffeomorphism $f_m^* : M_0 \to M_m^*$ such that

$$\|f_m^*\|_{C^{1,\alpha}(M_0)} \le C_\alpha\,, \qquad \lim_{m\to 0} \|f_m^* - \mathrm{Id}\|_{C^1(M_0)} = 0\,.$$

Notice that, by construction, this last limit relation does not depend on the specific family of minimizers E_m that we are considering, but just on A, g and σ.

We now complete the proof of the theorem. We first notice that (4.27) implies $\mathrm{diam}(F_m) \le C$, and thus, thanks to (4.1),

$$\mathrm{diam}(\phi_m^{-1}(E_m)) \le C\,\lambda_m \le C\, m^{1/n}\,.$$

Since the maps ϕ_m^{-1} are uniformly Lipschitz, we conclude that $\mathrm{diam}(E_m) \le C\, m^{1/n}$. In turn, up to change our choice of $y_m \in \partial A$, we can improve the factor $m^{1/2n}$ in (3.41) into $m^{1/n}$ and repeat the above arguments (starting from step seven in the proof of Lemma 3.1, continuing with the whole proof of Lemma 4.1, and including the current proof up to this point) using the more precise information

$$E_m \subset B_{y_m, C\, m^{1/n}} \tag{4.28}$$

in place of (3.41). In this way we improve $\sigma(y_m) - \sigma_0 \le C\, m^{1/2n}$ to

$$\sigma(y_m) - \sigma_0 \le C\, m^{1/n}\,, \tag{4.29}$$

we improve $|m^{-1/n}\lambda_m - 1| \le C\, m^{1/2n}$ in (4.1) to

$$\Big|\frac{\lambda_m}{m^{1/n}} - 1\Big| \le C\, m^{1/n}\,, \tag{4.30}$$

(by the same argument used in (3.44)) and we replace $F_m \subset B_{C/m^{1/2n}}$ with

$$F_m \subset B_C\,.$$

By combining this last inclusion with (4.25) (which gives $0 \in F_m^* = F_m - z_m$) we find that $|z_m| \le C$, and thus

$$z_m\,\lambda_m \in B_{C\, m^{1/n}} \cap \partial H \subset B_{s_0} \cap \partial H \subset \phi_m^{-1}(B_{2r_0}(y_m) \cap \partial A)\,.$$

In particular there exists $p_m \in \partial A$ such that

$$z_m\,\lambda_m = \phi_m^{-1}(p_m)\,,$$

and by $\phi_m^{-1}(y_m) = 0$ we find $|y_m - p_m| \le C\, m^{1/n}$. In this way (4.28) and (4.29) give us

$$E_m \subset B_{p_m, C\, m^{1/n}}\,, \qquad 0 \le \sigma(p_m) - \sigma_0 \le C\, m^{1/n}\,, \tag{4.31}$$

that is (1.7). We now prove that (1.8) holds with the linear isometry $S_m = \nabla\phi_m^{-1}(y_m)$ (S_m is a linear isometry since $S_m = R_m^{-1}$ for $R_m = \nabla\phi_m(0)$). Indeed, by (4.26) and with the notation $M_m = \overline{A \cap \partial E_m}$, we find

$$C\, m^{1/2n^2} \ge \mathrm{hd}(M_0, M_m^*) = \mathrm{hd}\Big(M_0, \frac{\phi_m^{-1}(M_m) - \phi_m^{-1}(p_m)}{\lambda_m}\Big)\,. \tag{4.32}$$

By

$$\|\phi_m^{-1} - \phi_m^{-1}(p_m) - \nabla\phi_m^{-1}(p_m)(\cdot - p_m)\|_{C^0(B_{p_m, C\, m^{1/n}})} \le C\, m^{2/n}\,,$$

and thanks to (4.30),

$$\mathrm{hd}\Big(\frac{\phi_m^{-1}(M_m) - \phi_m^{-1}(p_m)}{\lambda_m}, \frac{\nabla\phi_m^{-1}(p_m)(M_m - p_m)}{\lambda_m}\Big) \le C\, m^{1/n}\,,$$

so that by (4.32) and by linearity

$$C\, m^{1/2n^2} \ge \mathrm{hd}\Big(M_0, \nabla\phi_m^{-1}(p_m)\Big(\frac{M_m - p_m}{\lambda_m}\Big)\Big)\,.$$

By $|\nabla\phi_m^{-1}(p_m)-S_m|\le C\,m^{1/n}$, (4.27) and (4.30) we find

$$\mathrm{hd}\Big(S_m\Big(\frac{M_m-p_m}{m^{1/n}}\Big),\nabla\phi_m^{-1}(p_m)\Big(\frac{M_m-p_m}{\lambda_m}\Big)\Big)\le C\,m^{1/n}\,,$$

and thus, in conclusion,

$$C\,m^{1/2n^2}\ge \mathrm{hd}\Big(M_0,S_m\Big(\frac{M_m-p_m}{m^{1/n}}\Big)\Big)\,,$$

that is (1.8). We are thus left to prove the existence of the diffeomorphism f_m between M_0 and M_m such that (1.9) holds. To this end, let us define

$$f_m(x)=\phi_m\big(\lambda_m\,(f_m^*(x)+z_m)\big)=\phi_m(\lambda_m\,g_m(x))\,,\qquad x\in M_0\,,$$

where

$$g_m(x)=f_m^*(x)+z_m\,,\qquad x\in M_0\,.$$

By construction, f_m is a $C^{1,\alpha}$-diffeomorphism between M_0 and M_m, so that M_m is a connected $C^{1,\alpha}$-hypersurface with boundary such that $\mathrm{bd}\,(M_m)\subset\partial A$. Now $|z_m|\le C$, while (4.27) gives us $\|f_m^*\|_{C^0(M_0)}\le C$, so that

$$\|g_m\|_{C^0(M_0)}\le C\,. \tag{4.33}$$

If $v_m=R_m\,z_m$ (so that $|v_m|\le C$), then we have

$$\begin{aligned} f_m(x)-\big(y_m+\lambda_m(v_m+R_m\,x)\big)\;&=\;\phi_m\big(\lambda_m\,g_m(x)\big)-\phi_m(0)-\nabla\phi_m(0)[\lambda_m\,g_m(x)]\\ &\quad+\nabla\phi_m(0)[\lambda_m\,(f_m^*(x)-x)]\end{aligned}$$

and thus, by $\|\phi_m\|_{C^{1,1}(U_m)}\le C$,

$$\big\|f_m-\big(y_m+\lambda_m(v_m+R_m\,x)\big)\big\|_{C^0(M_0)}\le C\,\lambda_m\Big(\lambda_m+\|f_m^*-\mathrm{Id}\|_{C^0(M_0)}\Big)\,. \tag{4.34}$$

Similarly, if $\tau\in T_xM_0$, then

$$\nabla^{M_0}f_m(x)[\tau]=\nabla\phi_m\big(\lambda_m\,g_m(x)\big)[\lambda_m\,\nabla f_m^*(x)\tau]\,,$$

so that

$$\begin{aligned}\nabla^{M_0}f_m(x)[\tau]-\lambda_m\,R_m\tau\;&=\;\lambda_m\,\nabla\phi_m\big(\lambda_m\,g_m(x)\big)\,\big[\nabla f_m^*(x)[\tau]-\tau\big]\\ &\quad+\lambda_m\,\Big(\nabla\phi_m\big(\lambda_m\,g_m(x)\big)-\nabla\phi_m(0)\Big)[\tau]\,,\end{aligned}$$

and hence

$$\|\nabla^{M_0}f_m-\lambda_m\,R_m\|_{C^0(M_0)}\le C\,\lambda_m\,\Big(\lambda_m+\|f_m^*-\mathrm{Id}\|_{C^1(M_0)}\Big)\,. \tag{4.35}$$

By (4.30), (4.34) and (4.35) we finally find

$$\big\|f_m-\big(y_m+m^{1/n}(v_m+R_m\,x)\big)\big\|_{C^1(M_0)}\le C\,m^{1/n}\,\Big(m^{1/n}+\|f_m^*-\mathrm{Id}\|_{C^1(M_0)}\Big)=o(m^{1/n})\,,$$

where this limit relation depends on A, σ and g only, but not on the particular family of minimizers E_m under consideration. Since $x\mapsto v_m+R_mx$ is an isometry of $\mathbb{R}^n$ we have completed the proof of (1.9). □

Remark 4.2. In Remark 1.2 we claimed that $\gamma(m)=\psi(\sigma_0)\,m^{(n-1)/n}\,(1+O(m^{1/n}))$. By taking into account the upper bound (3.6), we are left to show that

$$\mathcal{F}_{A,\sigma}(E_m)\ge\psi(\sigma_0)\,m^{(n-1)/n}\,(1+O(m^{1/n}))\,.$$

Going back to the proof of Lemma 3.1, we notice that (3.42) can be improved into

$$\frac{\mathcal{F}_{A,\sigma}(E_m)}{m^{(n-1)/n}}\ge(1-C\,m^{1/1n})\,\psi(\sigma(p_m))\,,$$

since now we are replacing (3.41) with $E_m\subset B_{p_m,C\,m^{1/n}}$. We conclude as $\psi(\sigma(p_m))\ge\psi(\sigma_0)$.

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