GLOBALLY STABLE QUASISTATIC EVOLUTION FOR STRAIN GRADIENT PLASTICITY COUPLED WITH DAMAGE

VITO CRISMALE

Abstract. We consider evolutions for a material model which couples scalar damage with strain gradient plasticity, in small strain assumptions. For strain gradient plasticity, we follow the Gurtin-Anand formulation [19]. The aim of the present model is to account for different phenomena: on the one hand the elastic stiffness reduces and the plastic yield surface shrinks due to material's degradation, on the other hand the dislocation density affects the damage growth. The main result of this paper is the existence of a globally stable quasistatic evolution (in the so-called energetic formulation). Furthermore we study the limit model as the strain gradient terms tend to zero. Under stronger regularity assumptions, we show that the evolutions converge to the ones for the coupled elastoplastic–damage model studied in [8].

Keywords: variational models, quasistatic evolution, energetic solutions, strain gradient plasticity, damage models, incomplete damage, softening.
2010 MSC: 74C05, 74R05, 74G65, 35Q74, 49J45.

Contents

Introduction

Plasticity and damage describe the inelastic behavior of materials in response to applied forces, respectively accounting for permanent deformations and for discontinuities on microscales, both of surface type (microcracks) and of volume type (microvoids). In spite of their different macroscopical implications, the initial causes of the two phenomena are identical, in particular in metals they are originated by movement and accumulation of dislocations (cf. [24, Chapter 7]). Several strain gradient plasticity models have been proposed (see e.g. [1, 6, 14, 21, 18, 19]) in order to provide a description of the interaction among dislocations, and to capture size effects, such as strengthening and strain hardening, caused by these defects in the range 500 nm–50 μm.

In this paper, we present a mathematical model coupling scalar damage with the Gurtin-Anand gradient plasticity in small strain assumptions (for Gurtin-Anand plasticity see the original paper [19], and the mathematical treatment of the model in [30], [16], [15]). The aim of the present formulation is to account for different phenomena occurring in solid mechanics: on the one hand the elastic stiffness reduces and the plastic yield surface shrinks due to

Preprint SISSA 31/2015/MATE.

material's degradation, on the other hand the dislocation density affects the damage growth. The coupling between plasticity and damage is also investigated for instance in [2] and in [8], [9], by models that do not include plastic strain derivatives. The inclusion of such terms in the formulation permits to describe size effects.

We prove the existence of quasistatic evolutions for the present model in the framework of the energetic approach to rate-independent processes (see e.g. [26] for the abstract formulation, [10] and [33] for applications to perfect plasticity, [27] and [36] for damage, [8], and [16]). Moreover we study the asymptotics of these evolutions as the strain gradient terms tend to zero: precisely we show, under stronger regularity assumptions, the convergence to evolutions for the coupled elastoplastic damage model studied in [8].

We now present the strong formulation and next our existence result for the corresponding *energetic solutions*. Let the reference configuration of a given elasto-plastic body be a Lipschitz set $\Omega \subset \mathbb{R}^n$, $n \geq 2$, with $\partial\Omega$ partitioned into $\partial_D\Omega$ and $\partial_N\Omega$. According to the classical theory for isotropic plastic materials in small strain assumptions, the variables

$$u\colon [0,T]\times\overline{\Omega}\to\mathbb{R}^n\,,\quad e\colon[0,T]\times\Omega\to\mathbb{M}^{n\times n}_{sym}\,,\quad p\colon[0,T]\times\Omega\to\mathbb{M}^{n\times n}_D$$

respectively denoting the displacement, the elastic strain and the plastic strain, satisfy for every $t\in[0,T]$ the additive decomposition (we denote the total strain as $\mathrm{E}u=\frac{\nabla u+\nabla u^T}{2}$)

$$\mathrm{E}u(t)=e(t)+p(t)\qquad\text{in }\Omega\,. \tag{sf1a}$$

Moreover, assuming isotropic damage, we employ the variable $\alpha\colon[0,T]\times\Omega\to[0,1]$ for the damage state of the body: here $\alpha(t,x)=1$ stands for no damage and $\alpha(t,x)=0$ for maximal damage in the vicinity of a point $x\in\Omega$ at time t. In this presentation of the strong formulation we consider smooth variables, both is space and in time.

We study the evolution for u, e, p, and α in a time interval $[0,T]$ when the body undergoes an imposed boundary displacement $w(t)\colon\partial_D\Omega\to\mathbb{R}^n$ on $\partial_D\Omega$, namely

$$u(t)=w(t)\qquad\text{on }\partial_D\Omega\,, \tag{sf1b}$$

and volume and surface forces (on $\partial_N\Omega$), whose densites are denoted by $f(t)\colon\Omega\to\mathbb{R}^n$ and $g(t)\colon\partial_N\Omega\to\mathbb{R}^n$.

The starting point, as in the approach of Gurtin and Anand [19], is to consider $\dot e(t)$, $\dot p(t)$, and $\nabla\dot p(t)$ as *independent rate-like kinematical descriptors* with conjugated internal forces $\sigma(t)$, $\sigma^p(t)$, and $\mathbb{K}^p(t)$ such that the (internal) power expenditure within a subdomain $\mathcal{B}\subset\Omega$ at a time t is expressed by

$$\mathcal{W}_{\mathrm{int}}(\mathcal{B},t)=\int_{\mathcal{B}}\sigma(t)\cdot\dot e(t)+\sigma^p(t)\cdot\dot p(t)+\mathbb{K}^p(t)\cdot\nabla\dot p(t)\,\mathrm{d}x\,. \tag{0.1}$$

Then the stress configuration of the system is described by $\sigma(t)$, which is the usual Cauchy stress, by a second order tensor $\sigma^p(t)$ and by a third order tensor $\mathbb{K}^p(t)$. (We denote by "$\cdot$" the scalar product between tensors of the same order, independently of the order.) In [19] a balance between the power of the internal forces (0.1) and the one of the external forces usually considered in gradient plasticity is imposed for every subdomain and every virtual velocity of the fields u, e, p; then the following macroforce and microforce balance conditions are deduced:

$$\begin{cases}-\operatorname{div}\sigma(t)=f(t) & \text{in }\Omega\\ \sigma(t)\nu=g(t) & \text{on }\partial_N\Omega\end{cases} \tag{sf2a}$$

and

$$\sigma^p(t)=\sigma_D(t)+\operatorname{div}\mathbb{K}^p(t)\quad\text{in }\Omega\,, \tag{sf2b}$$

where ν is the outward normal to Ω and we denote the deviatoric part of a matrix A by A_D. Moreover we have that for every subbody $\mathcal{B}$ with outward normal ν, the deviatoric matrix $\mathbb{K}^p\nu$ represents the surface density of microtractions associated to the plastic strain (cf. [18, Sections 9 and 11] for the connection between microtractions and thermodynamic force between dislocations). As in [19, Section 8] we assume null microscopic power expenditure at the boundary, namely

$$\mathbb{K}^p(t)\nu=0\quad\text{on }\partial\Omega\,. \tag{sf2c}$$

The total energy density for our model is

$$\psi = \mu(\alpha)|e_D|^2 + \frac{1}{2}k(\alpha)|\operatorname{tr} e|^2 + \frac{L^2}{2}\mu(\alpha)|\operatorname{curl} p|^2 + \frac{\ell^2}{2}|\nabla\alpha|^2 + d(\alpha)\,,$$

where μ and k are nonincreasing and positive functions giving respectively the elastic shear and the bulk modulus, d is a continuous nonnegative function, and L, ℓ are length scales. We can look at ψ as the sum of two parts: the first three terms correspond to the free energy density proposed by Gurtin and Anand, with the elastic moduli depending on the damage; the part of ψ depending only on α and $\nabla\alpha$ is taken as in [2] and [29], and comes from mechanical considerations in [29]. In particular the density of energy dissipated by the material in the process of damage growth is included in the free energy and the dependence on the damage gradient is quadratic. The assumptions on μ and k imply that the elastic tensor $\mathbb{C}(\alpha)$, defined by

$$\mathbb{C}(\alpha)e := 2\mu(\alpha)e_D + k(\alpha)(\operatorname{tr} e)I\,,$$

is equicoercive and nonincreasing with respect to α; then we consider incomplete damage with softening. The term $\frac{L^2}{2}\mu(\alpha)|\operatorname{curl} p|^2$ is the density of energy stored by the *geometrically necessary dislocations*.

Dislocations are line defects within a crystal structure that are characterized by two vectors: the Burgers vector, b, that measures the slip displacement associated with the line defect, and a unit vector t, that points in the direction of the dislocation line. There are two main types of dislocations: the edge dislocations, where b and t are perpendicular, and the screw dislocations, where the two vectors are parallel. In the most general case the dislocation line lies at an arbitrary angle to its Burgers vector and the dislocation has a mixed edge and screw character. The energy stored per unit length by a dislocation is proportional to $\mu|b|^2$, see e.g. in [22, Section 4.4] and [24, Section 1]. The *macroscopic Burgers tensor* $\operatorname{curl} p$ measures the incompatibility of the field p and, for every unit vector m, $(\operatorname{curl} p)\,m$ is the Burgers vector, measured per unit area, associated with small loops orthogonal to m, namely with those dislocation whose lines pierce the plane with normal m (see [19, Section 3]); then $\operatorname{curl} p$ provides a measure of the dislocation density.

Therefore, in order to minimize $\mu(\alpha)|\operatorname{curl} p|^2$ it is convenient to damage portions of the material with high dislocation density (recall that μ is nondecreasing). Actually this type of interplay between damage and dislocations complies with various models of microcrack formation and coalescence by dislocation pile-up (see e.g. [38], [34], [7], [32]); moreover one can use the length scale L as a parameter tuning the relevance of this term in the process of damage growth.

By the standard assumption that $\sigma := \frac{\partial\psi}{\partial e}$, the constitutive equation for the *effective* Cauchy stress

$$\sigma(t) := \mathbb{C}(\alpha(t))e(t) \tag{sf3a}$$

is derived.

In analogy to [19], we define the energetic higher order stress $\mathbb{K}^p_{\rm en}$ as the symmetric-deviatoric part in the first two components (cf. (1.1)) of the partial derivative of ψ with respect to $\operatorname{curl} p$, namely

$$\mathbb{K}^p_{\rm en}(t)\cdot\nabla A := \mu(\alpha(t))L^2\operatorname{curl} p(t)\cdot\operatorname{curl} A \tag{sf3b}$$

for every $\mathbb{M}^{n\times n}_{sym}$-valued function A, the dissipative higher order stress $\mathbb{K}^p_{\rm diss}$ by

$$\mathbb{K}^p_{\rm diss} := \mathbb{K}^p - \mathbb{K}^p_{\rm en}\,. \tag{sf3c}$$

Moreover, we impose a maximum plastic dissipation principle requiring the constraint condition

$$(\sigma^p(t,x), \mathbb{K}^p_{\rm diss}(t,x)) \in K(\alpha(t,x)) := \left\{(A,\mathbb{B}):\ \frac{|A|^2}{S_1(\alpha(t,x))^2} + \frac{|\mathbb{B}|^2}{l^2 S_2(\alpha(t,x))^2} \le 1\right\}, \tag{sf4}$$

with $K(\alpha(t,x))$ an ellipsoid in the product space of the deviatoric matrices with the third order tensors symmetric-deviatoric in the first two components, and the flow rule

$$(\dot p(t,x), \nabla\dot p(t,x)) \in N_{K(\alpha(t,x))}\left((\sigma^p(t,x), \mathbb{K}^p_{\rm diss}(t,x))\right), \tag{sf5}$$

where $N_E(\xi)$ denotes the normal cone to a convex set E at $\xi \in E$. In other words, if $(\dot p(t,x), \nabla \dot p(t,x)) \neq (0,0)$ then $(\sigma^p(t,x), \mathbb{K}^p_{\text{diss}}(t,x))$ belongs to the boundary of $K(\alpha(t,x))$ and

$$\dot p(t,x) = \lambda(t,x) \frac{\sigma^p(t,x)}{S_1(\alpha(t,x))^2}\,, \quad \nabla \dot p(t,x) = \lambda(t,x) \frac{\mathbb{K}^p_{\text{diss}}(t,x)}{l^2 S_2(\alpha(t,x))^2}$$

for a suitable $\lambda(t,x) > 0$. Here $l > 0$ is a dissipative length scale and S_1, S_2 are nondecreasing positive functions of damage. Notice that we can deal with two different softening-type behaviors corresponding to different directions of the generalized constraint sets, as proposed in [19, Remark in Subsection 6.3] for a generalization of the model with a further internal variable. The three length scales l, ℓ, and L are constitutive parameters of the material.

In order to derive the equation governing the evolution of the damage variable we introduce the total energy $\mathcal{E}$, that is obtained by integrating ψ, and then it reads as

$$\mathcal{E}(\alpha, e, \text{curl}\; p) := \mathcal{Q}_1(\alpha, e) + \mathcal{Q}_2(\alpha, \text{curl}\; p) + \frac{\ell^2}{2}\|\nabla\alpha\|^2_{L^2} + D(\alpha)\,,$$

with

$$\mathcal{Q}_1(\alpha, e) := \frac{1}{2}\int_\Omega \mathbb{C}(\alpha) e\cdot e \,\mathrm{d}x\,,\ \mathcal{Q}_2(\alpha, \text{curl}\, p) := \frac{L^2}{2}\int_\Omega \mu(\alpha)|\text{curl}\, p|^2 \,\mathrm{d}x\,,\ D(\alpha) := \int_\Omega d(\alpha)\,\mathrm{d}x\,.$$

Following e.g. [29], the strong formulation of damage evolution is provided by the Kuhn-Tucker conditions

$$\dot\alpha(t) \leq 0 \quad \text{in } \Omega\,, \tag{sf6a}$$

$$\langle \partial_\alpha \mathcal{E}(\alpha(t), e(t), \text{curl}\, p(t)), \beta - \dot\alpha(t)\rangle \geq 0 \quad \text{for every } \beta \leq 0\,. \tag{sf6b}$$

where $\partial_\alpha\mathcal{E}$ is the Gâteaux derivative of $\mathcal{E}$ with respect to α. The expression above makes sense for β sufficiently regular, see Proposition 5.3 for details. Notice that, under regularity assumptions, (sf6b) implies that

$$\begin{cases} -\partial_\alpha\psi + \ell^2 \triangle\alpha \geq 0 & \text{in } \Omega\,,\\ \frac{\partial\alpha}{\partial\nu} \leq 0 & \text{on } \partial\Omega\,, \end{cases} \quad \text{and} \quad \begin{cases} \big(-\partial_\alpha\psi + \ell^2 \triangle\alpha\big)\,\dot\alpha = 0 & \text{in } \Omega\,,\\ \frac{\partial\alpha}{\partial\nu}\,\dot\alpha = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{0.2}$$

where $\partial_\alpha\psi = \mu'(\alpha)|e_D|^2 + \frac{1}{2}k'(\alpha)|\,\text{tr}\; e|^2 + \frac{L^2}{2}\mu'(\alpha)|\,\text{curl}\; p|^2 + d'(\alpha)$.

The conditions (sf1)–(sf6) constitute the strong formulation of the present model of Gurtin-Anand gradient plasticity coupled with damage. We now give the weak formulation of this model in the sense of [26]: the existence of a corresponding evolution is the main result of the paper.

Recalling (0.1) and (0.2) we get that the energy dissipated on a subbody $\mathcal{B}$, namely the difference between the power expended and the rate of the free energy, is

$$\int_{\mathcal{B}} \sigma^p\cdot\dot p + \mathbb{K}^p_{\text{diss}}\cdot\nabla\dot p \;\mathrm{d}x\,.$$

We have only a plastic term, since the density of the energy dissipated by damage growth is comprised in ψ. By (sf5), the expression above is nonnegative (as expected from thermodynamical considerations) and we are led to define the plastic potential as the relaxation of the functional

$$(\alpha, p) \mapsto \sqrt{S_1(\alpha)^2|p|^2 + l^2 S_2(\alpha)^2|\nabla p|^2}\,.$$

We therefore consider for every $\alpha\in H^1(\Omega)$ and $p\in BV(\Omega;\mathbb{M}^{n\times n}_D)$

$$\mathcal{H}(\alpha, p) := \int_\Omega \sqrt{S_1(\alpha)^2|p|^2 + l^2 S_2(\alpha)^2|\nabla p|^2}\,\mathrm{d}x + l\int_\Omega S_2(\widetilde\alpha)\,\mathrm{d}|\mathrm{D}^s p|\,.$$

with ∇p and $\mathrm{D}^s p$ the absolutely continuous and the singular part of $\mathrm{D}p$ with respect to the Lebesgue measure $\mathcal{L}^n$, and $\widetilde\alpha$ the precise representative of α, which is well defined at $\mathcal{H}^{n-1}$–a.e. $x\in\Omega$.

We remark that the H^1 damage regularization employed here is the one used in engineering (see for instance [2] and [25]). This is an improvement with respect to the elastoplastic-damage models in [8] and [9]: the strong damage regularizations therein (respectively $W^{1,\gamma}$, $\gamma > n$, and H^m, $m > \frac{n}{2}$) permitted us to work with a continuous field α (see also e.g. [23], with H^m

regularization), and therefore to use Reshetnyak's Theorem for the plastic dissipation. Here, in contrast, in order to get the lower semicontinuity of $\mathcal{H}$ we prove an abstract Reshetnyak-type lower semicontinuity theorem (Theorem 3.1) tailored to the discontinuous functions and to the special measures considered. The proof exploits also tools from the theory of capacity.

As in [8] and in [9] the plastic dissipation corresponding to an evolution of α and p in a time interval $[s,t]$ is the *$\mathcal{H}$-variation of p with respect to α* on $[s,t]$, namely

$$\mathcal{V}_{\mathcal{H}}(\alpha,p;s,t):=\sup\Big\{\sum_{j=1}^{N}\mathcal{H}(\alpha(t_j),p(t_j)-p(t_{j-1}))\,:\; s=t_0<t_1<\dots<t_N=t\,,\,N\in\mathbb{N}\Big\}.$$

Thus if w, f, g are absolutely continuous from $[0,T]$ into $H^1(\Omega;\mathbb{R}^n)$, $L^n(\Omega;\mathbb{R}^n)$, $L^n(\partial_N\Omega;\mathbb{R}^n)$, respectively, we define *quasistatic evolution for the Gurtin-Anand model coupled with damage* any function

$$[0,T]\ni t\mapsto(\alpha(t),u(t),e(t),p(t))\in H^1(\Omega;[0,1])\times W^{1,\frac{n}{n-1}}(\Omega;\mathbb{R}^n)\times L^2(\Omega;\mathbb{M}^{n\times n}_{sym})\times BV(\Omega;\mathbb{M}^{n\times n}_D)$$

that satisfies the following conditions:

(qs0) *irreversibility* : for every $x\in\Omega$ the function $[0,T]\ni t\mapsto\alpha(t,x)$ is nonincreasing;

(qs1) *global stability*: $(u(t),e(t),p(t))$ is admissible for the boundary condition $w(t)$ (i.e., its energy is finite and (sf1) hold) for every $t\in[0,T]$ and

$$\mathcal{E}(\alpha(t),e(t),\operatorname{curl}p(t))-\langle\mathcal{L}(t),u(t)\rangle\le\mathcal{E}(\beta,\eta,\operatorname{curl}q)-\langle\mathcal{L}(t),v\rangle+\mathcal{H}(\beta,q-p(t))$$

for every $\beta\le\alpha(t)$ and every triple (v,η,q) admissible for $w(t)$, where

$$\langle\mathcal{L}(t),u\rangle:=\int_\Omega f(t)\cdot u\,\mathrm{d}x+\int_{\partial_N\Omega}g(t)\cdot u\,\mathrm{d}\mathcal{H}^{n-1};$$

(qs2) *energy balance*: the function $t\mapsto p(t)$ from $[0,T]$ into $BV(\Omega;\mathbb{M}^{n\times n}_D)$ has bounded variation and for every $t\in[0,T]$

$$\begin{aligned}&\mathcal{E}(\alpha(t),e(t),\operatorname{curl}p(t))-\langle\mathcal{L}(t),u(t)\rangle+\mathcal{V}_{\mathcal{H}}(\alpha,p;0,t)\\&=\mathcal{E}(\alpha(0),e(0),\operatorname{curl}p(0))-\langle\mathcal{L}(0),u(0)\rangle+\int_0^t\langle\sigma(s),\mathrm{E}\dot w(s)\rangle\,\mathrm{d}s\\&\quad-\int_0^t\langle\dot{\mathcal{L}}(s),u(s)\rangle\,\mathrm{d}s-\int_0^t\langle\mathcal{L}(s),\dot w(s)\rangle\,\mathrm{d}s\,.\end{aligned}$$

Our existence result (Theorem 2.5) for quasistatic evolutions is based on time discretization and on approximation by means of solutions to incremental minimization problems, as common for globally stable quasistatic evolutions. The condition (qs0) corresponds to (sf6a), while (qs1) provides the desired balance equations (sf2) and the constraint condition (sf4). In order to deduce the plastic flow rule (sf5) and the activation condition for damage (sf6b), we have to assume more regularity on α and p to differentiate in time the energy balance (qs2). In particular, to recover the strong formulation from the weak one we have to work with a continuous damage field. Let us also mention that the mathematical treatment of the evolution problem for a model with a damage regularization of the type $W^{1,\gamma}$, $\gamma>1$, instead of H^1, is analogous to the one developed here.

In the last part of the paper we study the limit evolutions as the length scales l and L tend to zero. In [16] it is proven that, in this case, evolutions for the classical Gurtin-Anand formulation converge weakly for every time to evolutions for von Mises perfect plasticity model. We show an analogous convergence of the quasistatic evolutions for the present model to evolutions for the coupled elastoplastic damage model proposed in [8], which corresponds to the perfect plasticity for heterogeneous materials studied in [33] when the damage is constant in time. However, we have to consider a stronger (gradient) damage regularization for the Gurtin-Anand model with damage, since in [8] (and [9]) the space continuity of α is needed. An important difference with respect to the analysis in [16] relies on the fact that we cannot still characterize the global stability in the limit model by the equilibrium conditions for the Cauchy stress and the plastic constraint (cf. [10, Theorem 3.6]). Therefore our proof is different from that in [16]. Indeed we exploit the approximation in a strong sense of every admissibile triple for perfect plasticity with more regular ones that assume the boundary datum in a classical sense; in a forthcoming work M.G. Mora proves such approximation for every Lipschitz domain, here we show it in dimension two, and in higher dimension for a star-shaped domain.

The structure of the paper is the following: in Section 1 we fix the notation and recall some basic facts about the theory of capacity, in Section 2 we introduce the model starting from the mathematical formulation of the classical Gurtin-Anand model provided in [15], we give the definition of quasistatic evolutions, and state the existence result, which is proved in Sections 3 and 4. The connection between strong and energetic formulation of the evolution is studied in Section 5, while Section 6 is devoted to the asymptotic analysis for vanishing strain gradient terms.

1. Notation and preliminaries

We recall in this section the definitions and the main properties of the mathematical objects employed in the paper.

Measures and function spaces. We denote by $\mathcal{L}^n$ the Lebesgue measure on $\mathbb{R}^n$ and by $\mathcal{H}^s$ the s-dimensional Hausdorff measure, for every $s > 0$. Given a locally compact subset B of $\mathbb{R}^n$ and a finite dimensional Hilbert space X, we use the symbol $M_b(B;X)$ for the space of bounded X-valued Radon measures on B, the indication of X being omitted when X=$\mathbb{R}$. This space is endowed with the norm $\|\mu\|_1 := |\mu|(B)$, where $|\mu| \in M_b(B)$ is the total variation of the measure μ. For every $\mu \in M_b(B;X)$ we denote by μ^a and μ^s the absolutely continuous and the singular part of μ with respect to $\mathcal{L}^n$. By the Riesz Representation Theorem, $M_b(B;X)$ can be regarded as the dual of $C_0(B;X)$, the space of continuous functions $\varphi\colon B \to X$ such that $\{|\varphi| \geq \varepsilon\}$ is compact for every $\varepsilon > 0$ (see, e.g., [31, Theorem 6.19]). The weak* topology of $M_b(B;X)$ is defined using this duality. Moreover we say that a sequence $(\mu_k)_k \subset M_b(B;X)$ converges strictly to a bounded Radon measure μ if and only if it converges in the weak* topology and $|\mu_k|(B) \to |\mu|(B)$. We use the symbol $\|\cdot\|_p$ for the L^p norm and $\|\cdot\|_{1,q}$ for the norm of the Sobolev spaces $W^{1,q}$. Notice that if $L^1(B;X)$ is identified with the space of bounded measures μ with $\mu^s = 0$ (considering the density of μ^a with respect to $\mathcal{L}^n$), then $\|\cdot\|_1$ coincides with the induced norm, so that the notation is consistent. Throughout the paper we adopt the brackets $\langle\cdot,\cdot\rangle$ to denote the product between dual spaces, the arrows $\to$, $\rightharpoonup$, and $\overset{*}{\rightharpoonup}$ for the strong, weak, and weak* convergences, respectively, and $\overset{s}{\to}$ for the strict convergence of measures.

Given an open subset U of $\mathbb{R}^n$ the space $BV(U;X)$ is the set of the functions $u \in L^1(U;X)$ whose distributional derivative $\mathrm{D}u$ is a vector-valued bounded Radon measure. This is a Banach space with respect to the norm

$$\|u\|_{BV} := \|u\|_1 + \|\mathrm{D}u\|_1 \,.$$

A sequence $(u_k)_k$ converges to u weakly* in BV if and only if $u_k \to u$ in L^1 and $\mathrm{D}u_k \overset{*}{\rightharpoonup} \mathrm{D}u$ in M_b. We recall that if U is bounded and has Lipschitz boundary then every bounded sequence in $BV(U;X)$ has a weakly* convergent subsequence and $BV(U;X)$ is continuously embedded into $L^q(U;X)$ for every $1 \leq q \leq \frac{n}{n-1}$, the embedding being compact for $1 \leq q < \frac{n}{n-1}$. For the general theory of BV functions we refer to [3].

Capacity. We recall some facts about the theory of capacity, referring to [20] for a complete treatment of the subject. Given an open subset U of $\mathbb{R}^n$ and $1 \leq q < +\infty$, for every $E \subset U$ the q-capacity of E in U is defined by

$$C_q(E,U) := \inf\left\{\int_U |\nabla u|^q \,\mathrm{d}x \colon u \in W^{1,q}_0(U),\, u \geq 1 \text{ a.e. in a neighbourhood of } E\right\}.$$

We shall use the shorter notation $C_q(E)$ when there is no ambiguity on the domain. The q-capacity is indeed a Carathéodory outer measure such that if $1 < q < n$ and $C_q(E) = 0$, then the Hausdorff dimension of E is at most $n - q$. We say that a real valued function u is C_q-quasicontinuous in U if for every $\varepsilon > 0$ there is an open set G such that $C_q(G) < \varepsilon$ and the restriction of u to $U \setminus G$ is continuous. A sequence of real valued functions u_k converges C_q-quasiuniformly in U to u if for every $\varepsilon > 0$ there is an open set G such that $C_q(G) < \varepsilon$

and $u_k \to u$ uniformly in $U \setminus G$. For every $(u_k)_k \subset C(U) \cap W^{1,q}(U)$ that is a Cauchy sequence in $W^{1,q}(U)$, there exist a function $u \in W^{1,q}(U)$ and a subsequence converging locally C_q-quasiuniformly (namely, quasiuniformly in the compact subsets of U) to u. It follows that such a limit u is C_q-quasicontinuous, that $u_k \to u$ pointwise C_q-quasieverywhere in U (that is, pointwise except on a set of C_q-capacity zero), and that every $W^{1,q}$ function admits a quasicontinuous representative uniquely defined up to a C_q-negligible set. For every $u \in W^{1,q}(U)$ its precise representative $\widetilde{u}$, that is defined as the approximate limit of u in the Lebesgue points and takes value zero elsewhere, is a C_q-quasicontinuous representative of u. When $u_k \rightharpoonup u$ in $W^{1,q}(U)$ there exists a subsequence $(u_j)_j$ such that $\widetilde{u_j} \to \widetilde{u}$ in μ-measure, for every μ nonnegative bounded Radon measure that vanishes on all C_q-negligible Borel sets (cf. [5, Proposition 3.5 and Remark 3.4]). These results hold also for vector-valued functions, as one can see considering each component.

Matrices. We denote by $\mathbb{M}^{n\times n}$ (respectively by $\mathbb{M}^{n\times n\times n}$) the space of $n \times n$ real matrices (resp. third order tensors) endowed with the Euclidean scalar product $\xi \cdot \eta := \sum_{i,j} \xi_{ij}\eta_{ij}$ (resp. $\mathbb{A} \cdot \mathbb{B} := \sum_{i,j,k} \mathbb{A}_{ijk}\mathbb{B}_{ijk}$) and with the corresponding Euclidean norm $|\xi| := (\xi \cdot \xi)^{1/2}$. Moreover $\mathbb{M}^{n\times n}_{sym}$ denotes the subspace of symmetrix matrices and $\mathbb{M}^{n\times n}_D$ the subspace of trace free matrices in $\mathbb{M}^{n\times n}_{sym}$. Given $\xi \in \mathbb{M}^{n\times n}_{sym}$, its orthogonal projection on $\mathbb{M}^{n\times n}_D$ is the deviator $\xi_D := \xi - \frac{1}{n}(\mathrm{tr}\,\xi)I$.

The symmetrized gradient of an $\mathbb{R}^n$-valued function $u(x)$ is the $\mathbb{M}^{n\times n}_{sym}$-valued function $\mathrm{E}u(x)$ with components $E_{ij}u := \frac{1}{2}(\mathrm{D}_j u_i + \mathrm{D}_i u_j)$, where D_i denotes the derivative $\frac{\partial}{\partial x_i}$ for $1 \leq i \leq n$.

The gradient, the divergence, and the curl of a $\mathbb{M}^{n\times n}$-valued function $\xi(x) = (\xi_{ij}(x))$ are defined as

$$(\nabla\xi)_{ijk} := \mathrm{D}_k\,\xi_{ij}\,, \quad (\mathrm{div}\ \xi)_i := \sum_j \mathrm{D}_j\,\xi_{ij}\,, \quad (\mathrm{curl}\ \xi)_{ij} := \sum_{p,q} \epsilon_{ipq}\,\mathrm{D}_p\,\xi_{jq}\,,$$

where ϵ_{ipq} is the standard permutation symbol.

We say that a third order tensor $\mathbb{A} = (a_{ijk})$ is *symmetric deviatoric in its first two components*, and we write $\mathbb{A} \in \mathbb{M}^{n\times n\times n}_D$, if

$$a_{ijk} = a_{jik} \quad \text{and} \quad \sum_p a_{ppk} = 0\,. \tag{1.1}$$

The divergence of a $\mathbb{M}^{n\times n\times n}$-valued function $\mathbb{A}(x) = (a_{ijk}(x))$ is the $\mathbb{M}^{n\times n}$-valued function given by

$$(\mathrm{div}\ \mathbb{A})_{ij} := \sum_k \mathrm{D}_k\,a_{ijk}\,.$$

2. Quasistatic evolutions for the Gurtin-Anand model coupled with damage

In this section we introduce the weak formulation of our model, corresponding to the strong formulation described in the Introduction, and we specify the mathematical framework adopted.

The reference configuration. The reference configuration of the elasto-plastic body considered is a bounded, open, and Lipschitz set $\Omega \subset \mathbb{R}^n$, $n \geq 2$, whose boundary is decomposed as

$$\partial\Omega = \partial_D\Omega \cup \partial_N\Omega, \quad \partial_D\Omega \cap \partial_N\Omega = \emptyset\,, \tag{2.1a}$$

$\partial_D\Omega$ being the part of $\partial\Omega$ where the displacement is prescribed, while traction forces are applied on $\partial_N\Omega$. Here $\partial_D\Omega$ and $\partial_N\Omega$ are open (in the relative topology), with the same boundary Γ such that

$$\mathcal{H}^{n-2}(\Gamma) < +\infty\,. \tag{2.1b}$$

The external loading. We consider an evolution up to a time $T > 0$, driven by an absolutely continuous loading: this is given by an imposed boundary displacement (in the sense of trace on $\partial_D\Omega$)

$$w \in AC(0,T; H^1(\Omega;\mathbb{R}^n))\,, \tag{2.2a}$$

and by volume and surface forces (on $\partial_N\Omega$) with densites

$$f \in AC(0,T; L^n(\Omega;\mathbb{R}^n))\,, \quad g \in AC(0,T; L^n(\partial_N\Omega;\mathbb{R}^n))\,. \tag{2.2b}$$

For every $t \in [0,T]$ we define $\mathcal{L}(t)\colon W^{1,\frac{n}{n-1}}(\Omega;\mathbb{R}^n) \to \mathbb{R}$ as

$$\langle \mathcal{L}(t), u\rangle := \int_\Omega f(t)\cdot u \,\mathrm{d}x + \int_{\partial_N\Omega} g(t)\cdot u \,\mathrm{d}\mathcal{H}^{n-1}\,.$$

It is easily seen that $\mathcal{L}(t)$ is linear and continuous on $W^{1,\frac{n}{n-1}}(\Omega;\mathbb{R}^n)$.

Admissible configurations. As usual in linarized plasticity, the variables

$$u\colon [0,T]\times\overline{\Omega} \to \mathbb{R}^n\,, \quad e\colon [0,T]\times\Omega \to \mathbb{M}^{n\times n}_{sym}\,, \quad p\colon [0,T]\times\Omega \to \mathbb{M}^{n\times n}_D\,,$$

denoting the displacement and the elastic and plastic strains respectively, satisfy for every $t\in[0,T]$ the additive strain decomposition

$$\mathrm{E}u(t) = e(t) + p(t) \quad \text{in } \Omega\,,$$

that corresponds to small strain assumptions ($\mathrm{E}u = \frac{\nabla u + \nabla u^T}{2}$ is the linearized strain). Given $\overline{w} \in H^1(\Omega;\mathbb{R}^n)$, an admissible configuration relative to $\overline{w}$ is a triple (u,e,p) such that

$$u \in W^{1,\frac{n}{n-1}}(\Omega;\mathbb{R}^n)\,, \quad e \in L^2(\Omega;\mathbb{M}^{n\times n}_{sym})\,, \quad p \in BV(\Omega;\mathbb{M}^{n\times n}_D)\,, \tag{2.3a}$$

$$\mathrm{E}u = e + p \quad \text{in } \Omega\,, \quad u = \overline{w} \quad \text{on } \partial_D\Omega\,, \quad \mathrm{curl}\, p \in L^2(\Omega;\mathbb{M}^{n\times n}_{sym})\,, \tag{2.3b}$$

the second equality in (2.3b) being in the sense of traces. The set of admissible configurations is then

$$A(\overline{w}) := \{(u,e,p) :\ (2.3)\,\text{hold}\}\,.$$

Notice that if $u\colon \Omega \to \mathbb{R}^n$ measurable, $e \in L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$, $p \in BV(\Omega;\mathbb{M}^{n\times n}_D)$ satisfy (2.3b), then $u \in W^{1,\frac{n}{n-1}}(\Omega;\mathbb{R}^n)$ by properties of BV functions and Korn's inequality.

The damage variable. The damage state of the body is described by a scalar internal variable

$$\alpha\colon [0,T]\times\Omega \to \mathbb{R}\,.$$

We shall see that during the evolution $\alpha(t) \in H^1(\Omega;[0,1])$ for every $t\in[0,T]$, by the expression of our total energy. At a given $x\in\Omega$, as $\alpha(\cdot,x)$ decreases from 1 to 0, the material point x passes from a sound state to a fully damaged one.

The elastic energy. In our formulation the elastic shear and bulk moduli of the body, denoted respectively by μ and k, depend on the damage state α. We assume that they are Lipschitz and nondecreasing functions defined on $\mathbb{R}$ and constant in $\mathbb{R}^-$ with

$$\mu(\alpha) > c > 0, \quad 2\mu(\alpha) + k(\alpha) > c \quad \text{for every } \alpha \in [0,1]. \tag{2.4}$$

This corresponds to say that the stiffness decreases as the damage grows and that an elastic response is present even in the most damaged state. Defining for every $\alpha\in\mathbb{R}$ the elastic tensor $\mathbb{C}(\alpha)$ by

$$\mathbb{C}(\alpha)e := 2\mu(\alpha)e_D + k(\alpha)(\mathrm{tr}\, e)I\,, \tag{2.5}$$

the assumptions above imply that

$$\mathbb{C}\colon \mathbb{R} \to Lin(\mathbb{M}^{n\times n}_{sym};\mathbb{M}^{n\times n}_{sym}) \text{ is Lipschitz and } \mathbb{C}(\mathbb{R}^-) = \{\mathbb{C}(0)\}\,, \tag{2.6a}$$

$$\alpha \mapsto \mathbb{C}(\alpha)\,\xi\cdot\xi \quad \text{is nondecreasing for every } \xi \in \mathbb{M}^{n\times n}_{sym}\,, \tag{2.6b}$$

$$\gamma_1|\xi|^2 \le \mathbb{C}(\alpha)\,\xi\cdot\xi \le \gamma_2|\xi|^2 \quad \text{for every } \alpha\in\mathbb{R}\,,\ \xi \in \mathbb{M}^{n\times n}_{sym} \tag{2.6c}$$

for suitable positive constants γ_1 and γ_2. The elastic energy is

$$\mathcal{Q}_1(\alpha, e) := \frac{1}{2}\int_\Omega \mathbb{C}(\alpha)\,e\cdot e\,\mathrm{d}x\,. \tag{2.7}$$

The energy stored by the dislocations. As explained in [19, Section 3], the macroscopic Burgers tensor curl p measures the incompatibility of the field p and it provides a measure of the dislocation density. Following the approach of Gurtin-Anand, the energy stored by the dislocations is given by

$$\mathcal{Q}_2(\alpha, \operatorname{curl} p) := \frac{L^2}{2}\int_\Omega \mu(\alpha)|\operatorname{curl} p|^2 \,\mathrm{d}x\,,$$

with $L > 0$ a length scale and μ the shear modulus. Notice that, since μ is nondecreasing, in order to minimize $\mu(\alpha)|\operatorname{curl} p|^2$ it is convenient to damage portions of the material with high dislocation density.

Remark 2.1. Let us consider the functionals $\mathcal{Q}_1$ and $\mathcal{Q}_2$: their densities are the functions $(\alpha,\xi) \mapsto \frac{1}{2}\mathbb{C}(\alpha)\,\xi\cdot\xi$ and $(\alpha,\xi) \mapsto \frac{L^2}{2}\mu(\alpha)|\xi|^2$, convex in ξ and continuous. Then the Ioffe-Olach Semicontinuity Theorem gives that $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are lower semicontinuous with respect to the strong convergence of the first variable in $L^1(\Omega)$ and the weak convergence of the second variable in $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$, namely for $i \in \{1,2\}$

$$\mathcal{Q}_i(\alpha,\eta) \le \liminf_{k\to\infty} \mathcal{Q}_i(\alpha_k,\eta_k) \quad \text{for every } \alpha_k \to \alpha \text{ in } L^1(\Omega),\ \eta_k \rightharpoonup \eta \text{ in } L^2(\Omega;\mathbb{M}^{n\times n}_{sym})\,. \tag{2.8}$$

The total energy. The total energy of a quadruple (α,u,e,p) such that $\alpha \in H^1(\Omega)$ and $(u,e,p) \in A(\overline{w})$ for some $\overline{w}$ is given by:

$$\mathcal{E}(\alpha,e,\operatorname{curl} p) := \mathcal{Q}_1(\alpha,e) + \mathcal{Q}_2(\alpha,\operatorname{curl} p) + \frac{\ell^2}{2}\|\nabla\alpha\|_2^2 + D(\alpha)\,,$$

where $\ell > 0$ is an internal length and

$$D(\alpha) := \int_\Omega d(\alpha)\,\mathrm{d}x\,, \tag{2.9a}$$

with

$$d\colon \mathbb{R} \to \mathbb{R}^+ \cup \{0\} \text{ continuous and } d(x) > d(0) \text{ for } x < 0\,. \tag{2.9b}$$

We include in the total energy the function D and a quadratic gradient damage term. This choice is motivated by [29], where an analogous expression of (elastic) strain work is derived for an isotropic material in absence of prestress, under the assumption that the strain work depends also on $\nabla\alpha$, by an expansion up to the second order in the strain and in $\nabla\alpha$. The term $D(\alpha)$ is related to the energy dissipated during the damage growth up to α.

The plastic dissipation. We now introduce a term which accounts for the energy dissipated in the evolution of plasticity. Let us first define the *plastic potential* $\mathcal{H}$ for every $(\alpha,p) \in H^1(\Omega) \times BV(\Omega;\mathbb{M}^{n\times n}_D)$ as

$$\mathcal{H}(\alpha,p) := \int_\Omega \sqrt{S_1(\alpha)^2|p|^2 + l^2 S_2(\alpha)^2|\nabla p|^2}\,\mathrm{d}x + l\int_\Omega S_2(\widetilde{\alpha})\,\mathrm{d}|\mathrm{D}^s p|\,, \tag{2.10}$$

with $\widetilde{\alpha}$ the precise representative of α, which is well defined at $\mathcal{H}^{n-1}$–a.e. $x \in \Omega$ (indeed it is a C_2-quasicontinuous representative of α), and ∇p and $\mathrm{D}^s p$ the absolutely continuous and the singular part of $\mathrm{D}p$ with respect to the Lebesgue measure $\mathcal{L}^n$. We recall that

$$\int_\Omega S_2(\widetilde{\alpha})\,\mathrm{d}|\mathrm{D}^s p| = \int_{J_p} S_2(\widetilde{\alpha})\,|p^+ - p^-|\,\mathrm{d}\mathcal{H}^{n-1} + \int_\Omega S_2(\widetilde{\alpha})\,\mathrm{d}|\mathrm{D}^c p|\,,$$

where J_p is the jump set of p, the functions p^+ and p^- are the approximate upper and lower limit of p, respectively, and $\mathrm{D}^c p$ is the Cantor part of $\mathrm{D}p$ (see [3, Section 3.9]). We assume for $i \in \{1,2\}$

$$S_i\colon \mathbb{R} \to \mathbb{R} \quad \text{bounded, Lipschitz and nondecreasing,} \quad S_i(\alpha) = S_i(0) > 0 \quad \text{for } \alpha < 0\,. \tag{2.11}$$

This definition of $\mathcal{H}$ is a generalization of the one in [16], where $S_1(\alpha) = S_2(\alpha) = S^0_Y > 0$. Notice that for every α in $H^1(\Omega)$ and p_1, p_2 in $BV(\Omega;\mathbb{M}^{n\times n}_D)$

$$\mathcal{H}(\alpha,p_1+p_2) \le \mathcal{H}(\alpha,p_1) + \mathcal{H}(\alpha,p_2)$$

and $\mathcal{H}$ is positively 1-homogeneous in p. Moreover, for every α in $H^1(\Omega)$ and $p \in BV(\Omega;\mathbb{M}^{n\times n}_D)$

$$r\|p\|_{BV} \leq \mathcal{H}(\alpha,p) \leq R\|p\|_{BV}\,, \tag{2.12}$$

where $r := S_1(0) \wedge (lS_2(0))$ and $R := \sup_{\mathbb{R}} S_1 \vee (l\sup_{\mathbb{R}} S_2)$.

Given $\alpha\colon [s,t] \to H^1(\Omega)$ and $p\colon [s,t] \to BV(\Omega;\mathbb{M}^{n\times n}_D)$ evolutions of damage and plastic strain in a time interval $[s,t]$, the *plastic dissipation* corresponding is defined as the *$\mathcal{H}$-variation of p with respect to α* on $[s,t]$, namely

$$\mathcal{V}_{\mathcal{H}}(\alpha,p;s,t) := \sup\Big\{\sum_{j=1}^N \mathcal{H}(\alpha(t_j), p(t_j)-p(t_{j-1}))\,:\; s=t_0<t_1<\dots<t_N=t\,,\; N\in\mathbb{N}\Big\}\,. \tag{2.13}$$

We denote the variation of p on $[s,t]$ by

$$\mathcal{V}(p;s,t) := \sup\Big\{\sum_{j=1}^N \|p(t_j)-p(t_{j-1})\|_{BV}\,:\; s=t_0<t_1<\dots<t_N=t\,,\; N\in\mathbb{N}\Big\}\,.$$

The safe load conditions. Besides the assumptions (2.2), we require that the forces satisfy the following *strong safe load condition*: for every $t\in[0,T]$ there exists $\varrho(t)\in L^n(\Omega;\mathbb{M}^{n\times n}_{sym})$ such that

$$\begin{cases} -\operatorname{div}\,\varrho(t) = f(t) & \text{in } \Omega \\ \varrho(t)\nu = g(t) & \text{on } \partial_N\Omega \end{cases} \tag{2.14a}$$

and there exists $c_0>0$ such that for every $A\in\mathbb{M}^{n\times n}_D$ with $|A|\leq c_0$ we have

$$|A+\varrho_D(t)| \leq S_1(0)\wedge S_2(0) \quad \text{a.e. in } \Omega\,. \tag{2.14b}$$

We also assume that the functions $t\mapsto\varrho(t)$ and $t\mapsto\varrho_D(t)$ are absolutely continuous from $[0,T]$ into $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$ and $L^\infty(\Omega;\mathbb{M}^{n\times n}_D)$ respectively. Notice that the second equality in (2.14a) is well defined in the dual of the space of traces on $\partial_N\Omega$ of $W^{1,\frac{n}{n-1}}(\Omega;\mathbb{R}^n)$ since $\varrho(t)$ and div $\varrho(t)$ are in L^n for every t, and that for every $(u,e,p)\in A(w)$ the representation formula

$$\langle\mathcal{L}(t),u\rangle = -\langle\varrho(t)\nu,w\rangle_{\partial_D\Omega} + \int_\Omega \varrho(t)\cdot e\,\mathrm{d}x + \int_\Omega \varrho_D(t)\cdot p\,\mathrm{d}x \tag{2.15}$$

holds, where $\langle\cdot,\cdot\rangle$ denotes the pairing between $H^{-1/2}(\partial_D\Omega;\mathbb{R}^n)$ and $H^{1/2}(\partial_D\Omega;\mathbb{R}^n)$ (here we use $\mathcal{H}^{n-2}(\Gamma)<\infty$).

Remark 2.2. Adapting the proof of [16, Lemma 4.3], we also have the coercivity estimate

$$\mathcal{H}(\alpha,p) - \int_\Omega \varrho_D(t)\cdot p\,\mathrm{d}x \geq \tfrac{c_0}{2}\|p\|_1 + \min\{l\tfrac{c_0}{2}, lS_2(0)\}\|\mathrm{D}p\|_1\,, \tag{2.16}$$

and then

$$\mathcal{H}(\alpha,p) - \int_\Omega \varrho_D(t)\cdot p\,\mathrm{d}x \geq C(c_0,l,S_2(0))\|p\|_{BV} \tag{2.17}$$

for every $t\in[0,T]$, $\alpha\in H^1(\Omega)$, and $p\in BV(\Omega;\mathbb{M}^{n\times n}_D)$.

Quasistatic evolutions. We are now ready to give the definition of quasistatic evolution for the present model. We define, for given $\alpha\in H^1(\Omega)$ and $\overline{w}\in H^1(\Omega;\mathbb{R}^n)$,

$$\mathcal{A}(\alpha,w) := \{(\beta,u,e,p)\,:\; \beta\in H^1(\Omega),\ \beta\leq\alpha,\ \text{and } (u,e,p)\in A(\overline{w})\}\,. \tag{2.18}$$

Definition 2.3. A *quasistatic evolution for the Gurtin-Anand model coupled with damage* is a function

$$[0,T]\ni t\mapsto(\alpha(t),u(t),e(t),p(t))\in H^1(\Omega;[0,1])\times W^{1,\frac{n}{n-1}}(\Omega;\mathbb{R}^n)\times L^2(\Omega;\mathbb{M}^{n\times n}_{sym})\times BV(\Omega;\mathbb{M}^{n\times n}_D)$$

that satisfies the following conditions:

(qs0) *irreversibility* : for every $x\in\Omega$ the function $[0,T]\ni t\mapsto\alpha(t,x)$ is nonincreasing;

(qs1) *global stability*: for every $t\in[0,T]$ we have $(u(t),e(t),p(t))\in A(w(t))$ and

$$\mathcal{E}(\alpha(t),e(t),\operatorname{curl}\,p(t)) - \langle\mathcal{L}(t),u(t)\rangle \leq \mathcal{E}(\beta,\eta,\operatorname{curl}\,q) - \langle\mathcal{L}(t),v\rangle + \mathcal{H}(\beta,q-p(t))$$

for every $(\beta,v,\eta,q)\in\mathcal{A}(\alpha(t),w(t))$;

(qs2) *energy balance*: the function $t \mapsto p(t)$ from $[0,T]$ into $BV(\Omega;\mathbb{M}^{n\times n}_D)$ has bounded variation and for every $t\in[0,T]$

$$\begin{aligned}&\mathcal{E}(\alpha(t),e(t),\operatorname{curl} p(t))-\langle\mathcal{L}(t),u(t)\rangle+\mathcal{V}_{\mathcal{H}}(\alpha,p;0,t)\\&\quad=\mathcal{E}(\alpha(0),e(0),\operatorname{curl} p(0))-\langle\mathcal{L}(0),u(0)\rangle+\int_0^t\langle\sigma(s),\mathrm{E}\dot w(s)\rangle\,\mathrm{d}s\\&\qquad-\int_0^t\langle\dot{\mathcal{L}}(s),u(s)\rangle\,\mathrm{d}s-\int_0^t\langle\mathcal{L}(s),\dot w(s)\rangle\,\mathrm{d}s\,,\end{aligned}$$

where $\sigma(s):=\mathbb{C}(\alpha(s))e(s)$.

Remark 2.4. We shall prove in Lemma 4.1 that such an evolution is measurable and the integrals in (qs2) are well defined.

We now state the main result of the paper, that will be proved in Sections 3 and 4.

Theorem 2.5 (Existence of quasistatic evolutions)**.** *Assume* (2.1)*,* (2.2)*,* (2.4)*–*(2.7)*,* (2.9)*–*(2.11) *and* (2.14)*, and let* $(\alpha_0,(u_0,e_0,p_0))\in H^1(\Omega;[0,1])\times A(w(0))$ *satisfy the stability condition*

$$\mathcal{E}(\alpha_0,e_0,\operatorname{curl} p_0)-\langle\mathcal{L}(0),u_0\rangle\le\mathcal{E}(\beta,\eta,\operatorname{curl} q)-\langle\mathcal{L}(0),v\rangle+\mathcal{H}(\beta,q-p_0)$$

for every $(\beta,v,\eta,q)\in\mathcal{A}(\alpha_0,w(0))$*. Then there exists a quasistatic evolution for the Gurtin-Anand model coupled with damage* $t\mapsto(\alpha(t),u(t),e(t),p(t))$ *such that* $\alpha(0)=\alpha_0$*,* $u(0)=u_0$*,* $e(0)=e_0$*,* $p(0)=p_0$*.*

3. The minimization problem

This section is focused on the minimization problem employed in the construction of time discrete approximations for a quasistatic evolution. If $\overline{\alpha}\in H^1(\Omega;[0,1])$ and $\overline{p}\in BV(\Omega;\mathbb{M}^{n\times n}_D)$ are the current values of the damage variable and of the plastic strain, and $w\in H^1(\Omega;\mathbb{R}^n)$, $f\in L^n(\Omega;\mathbb{R}^n)$, $g\in L^n(\partial_N\Omega;\mathbb{R}^n)$ are the updated values of the boundary displacement and of the body and surface loads, the updated values of the internal variables α,u,e,p are obtained by solving the problem

$$\operatorname{argmin}\left\{\mathcal{E}(\alpha,e,\operatorname{curl} p)-\langle\mathcal{L},u\rangle+\mathcal{H}(\alpha,p-\overline{p}):\ (\alpha,u,e,p)\in\mathcal{A}(\overline{\alpha},w)\right\},\tag{3.1}$$

where

$$\langle\mathcal{L},u\rangle:=\int_\Omega f\cdot u\,\mathrm{d}x+\int_{\partial_N\Omega}g\cdot u\,\mathrm{d}\mathcal{H}^{n-1}\,.\tag{3.2}$$

First we show the existence of solutions to this problem and their main properties, and afterwards a stability property of the solutions with respect to variations of the data.

The following semicontinuity theorem will be used several times in the following, for instance to prove the existence of solutions to (3.1). Notice that in the case when the energy includes a gradient damage term $\|\nabla\alpha\|_\gamma^\gamma$, with $\gamma>n$ the result follows easily from Reshetnyak's Lower Semicontinuity Theorem (see [8] and [9]). Instead for $\gamma=2$ the proof relies on the specific form of $\mathcal{H}$; in particular we use the fact that $\mathrm{D}p$ is the gradient of a BV function and then it vanishes on sets with dimension lower than $n-1$.

Theorem 3.1. *The plastic potential* $\mathcal{H}$ *defined in* (2.10) *is lower semicontinuous with respect to the weak–*$H^1(\Omega)$ *convergence of* α_k *and the weak*–*$BV(\Omega;\mathbb{M}^{n\times n}_D)$ *convergence of* p_k*, namely*

$$\mathcal{H}(\alpha,p)\le\liminf_{k\to\infty}\mathcal{H}(\alpha_k,p_k)\tag{3.3}$$

for every $\alpha_k\rightharpoonup\alpha$ *in* $H^1(\Omega)$ *and* $p_k\overset{*}{\rightharpoonup}p$ *in* $BV(\Omega;\mathbb{M}^{n\times n}_D)$*.*

Proof. Let $(\alpha_k)_k$ and $(p_k)_k$ be two sequences in $H^1(\Omega)$ and $BV(\Omega;\mathbb{M}^{n\times n}_D)$ such that $\alpha_k\rightharpoonup\alpha$ in $H^1(\Omega)$ and $p_k\overset{*}{\rightharpoonup}p$ in $BV(\Omega;\mathbb{M}^{n\times n}_D)$. We divide the proof into two steps, starting from the case when the functions p_k are uniformly bounded, that is $\|p_k\|_\infty<M$, for a suitable $M>0$.

Step 1 (p_k uniformly bounded). Notice that for $\beta\in H^1(\Omega)\cap L^\infty(\Omega)$ and $q\in BV(\Omega;\mathbb{M}^{n\times n}_D)\cap L^\infty(\Omega;\mathbb{M}^{n\times n}_D)$ we have that $\beta\,q\in BV(\Omega;\mathbb{M}^{n\times n}_D)$ and

$$\mathrm{D}(\beta\,q)=\widetilde{\beta}\,\mathrm{D}q+q\otimes\nabla\beta\qquad\text{in }M_b(\Omega;\mathbb{M}^{n\times n\times n}_D)\,,\tag{3.4}$$

where $\widetilde{\beta}$ is the precise representative of β. Indeed it is well-known that this formula holds for $\beta \in C^1(\Omega)$; thus we can argue by approximation, considering a sequence $(\beta_k)_k \subset C^1(\Omega)$ uniformly bounded in $L^\infty(\Omega)$ such that $\beta_k \to \beta$ in $H^1(\Omega)$. Therefore the total variations $\|\mathrm{D}(\beta_k\, q)\|_1$ are uniformly bounded and then up to a subsequence

$$\mathrm{D}(\beta_k\, q) \overset{*}{\rightharpoonup} \mathrm{D}(\beta\, q) \quad \text{in } M_b(\Omega; \mathbb{M}_D^{n\times n\times n}) .$$

Moreover, up to a further subsequence $\beta_k(x) \to \widetilde{\beta}(x)$ for $|\mathrm{D}q|$–a.e. $x \in \Omega$; then we recover (3.4) by using the fact that $q \in L^\infty(\Omega; \mathbb{M}_D^{n\times n})$ and the Dominated Convergence Theorem for the convergence of the right-hand side.

We now take $q = p_k$, $\beta = S_i(\alpha_k)$, and recall that S_i are bounded and Lipschitz maps (cf. (2.11)). Since $S_i(\alpha_k) \to S_i(\alpha)$ in $L^2(\Omega)$ and the sequences $(S_i(\alpha_k))_k$ are equibounded in $L^\infty(\Omega)$ and in $H^1(\Omega)$, we get that $S_i(\alpha_k) \to S_i(\alpha)$ in $L^r(\Omega)$ for every $r \in [1, +\infty)$ and $S_i(\alpha_k) \rightharpoonup S_i(\alpha)$ in $H^1(\Omega)$, for $i = 1, 2$. In particular

$$S_i(\alpha_k)p_k \to S_i(\alpha)p \quad \text{in } L^1(\Omega; \mathbb{M}_D^{n\times n}) . \tag{3.5}$$

Evaluating (3.4) with $q = p_k$ and $\beta = S_2(\alpha_k)$ we get

$$\mathrm{D}(S_2(\alpha_k)p_k) = S_2(\widetilde{\alpha}_k)\mathrm{D}p_k + p_k \otimes \nabla(S_2(\alpha_k)) \qquad \text{in } M_b(\Omega; \mathbb{M}_D^{n\times n\times n}) .$$

Hence the measures $\mathrm{D}(S_2(\alpha_k)p_k)$ have uniformly bounded total variations, and (3.5) implies that

$$\mathrm{D}(S_2(\alpha_k)p_k) \overset{*}{\rightharpoonup} \mathrm{D}(S_2(\alpha)p) \qquad \text{in } M_b(\Omega; \mathbb{M}_D^{n\times n\times n}) .$$

On the other hand, since $p_k \to p$ in $L^1(\Omega; \mathbb{M}_D^{n\times n})$ and we are assuming the p_k uniformly bounded, then $p_k \to p$ in $L^r(\Omega; \mathbb{M}_D^{n\times n})$ for every $r \in [1, +\infty)$ and

$$p_k \otimes \nabla S_2(\alpha_k) \rightharpoonup p \otimes \nabla S_2(\alpha) \qquad \text{in } L^1(\Omega; \mathbb{M}_D^{n\times n\times n}) .$$

By difference (and (3.4) with $q = p$ and $\beta = \alpha$) we obtain that

$$S_2(\widetilde{\alpha}_k)\mathrm{D}p_k \overset{*}{\rightharpoonup} S_2(\widetilde{\alpha})\mathrm{D}p \quad \text{in } M_b(\Omega; \mathbb{M}_D^{n\times n\times n}) . \tag{3.6}$$

In order to prove (3.3), we observe that by definition $\mathcal{H}$ is the total variation of a convex function of a measure, defined in the sense of [17]; precisely for every $\beta \in H^1(\Omega)$ and $q \in BV(\Omega; \mathbb{M}_D^{n\times n})$

$$\mathcal{H}(\beta, q) = \|\overline{f}\big((S_1(\beta)q, S_2(\widetilde{\beta})\mathrm{D}q)\big)\|_1 ,$$

where

$$\overline{f}(\xi, \mathbb{A}) := \sqrt{|\xi|^2 + l^2|\mathbb{A}|^2} \quad \text{for every } (\xi, \mathbb{A}) \in \mathbb{M}_D^{n\times n} \times \mathbb{M}_D^{n\times n\times n}$$

and $(S_1(\beta)q, S_2(\widetilde{\beta})\mathrm{D}q) \in M_b(\Omega; \mathbb{M}_D^{n\times n} \times \mathbb{M}_D^{n\times n\times n})$ is the product measure of $S_1(\beta)q$ and $S_2(\widetilde{\beta})\mathrm{D}q$. From (3.5) and (3.6) it follows that

$$(S_1(\alpha_k)p_k, S_2(\widetilde{\alpha_k})\mathrm{D}p_k) \overset{*}{\rightharpoonup} (S_1(\alpha)p, S_2(\widetilde{\alpha})\mathrm{D}p) \quad \text{in } M_b(\Omega; \mathbb{M}_D^{n\times n} \times \mathbb{M}_D^{n\times n\times n})$$

and then we get (3.3) by general results on convex functions of measures.

Step 2 (General case). We now approximate the functions p_k with bounded functions, without increasing the total variation of the gradient. For every $x \in \Omega$, $q \in BV(\Omega; \mathbb{M}_D^{n\times n})$, and $R > 1$ we define

$$\varphi_R(q)(x) := \omega_R(|q(x)|)q(x)$$

where $\omega_R \in C^1(\mathbb{R}^+ \cup \{0\}; [0, 1])$ is a nonincreasing map such that

$$\begin{aligned} \omega_R(\varrho) &= 1 \quad \text{for every } \varrho \le R , \\ \omega_R(\varrho) &= 0 \quad \text{for every } \varrho \ge \widehat{R} , \\ \omega_R(\varrho) + \varrho^2(\omega_R'(\varrho))^2 &\le 1 \quad \text{for every } \varrho \ge 0 . \end{aligned}$$

and $\widehat{R}(R)$ is some radius bigger than R. We can take for instance

$$\omega_R(\varrho) = \begin{cases} 1 - \frac{(\varrho - R)^2}{4(R+1)^2} & \text{for } \varrho \in [R, R+1] , \\ 1 - \frac{1}{4(R+1)^2} - \frac{1}{2(R+1)} \ln \frac{\varrho}{R+1} =: g_R(\varrho) & \text{for } \varrho \in [R+1, (R+1)e^{\frac{4(R+1)^2-1}{2(R+1)}}] , \\ 0 & \text{for } \varrho \in [(R+1)e^{\frac{4(R+1)^2-1}{2(R+1)}}, +\infty) . \end{cases}$$

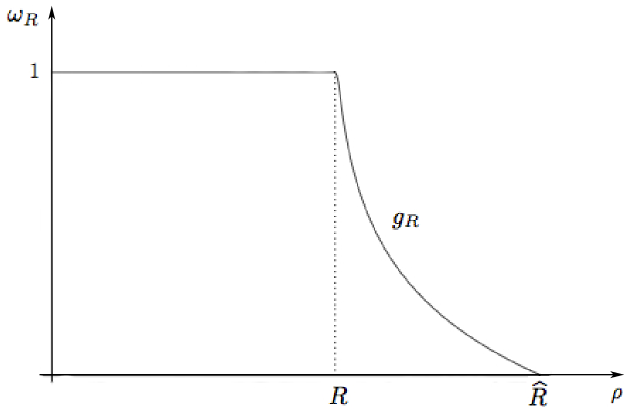

FIGURE 1. The cut-off function ω_R.

The resulting function ω_R has a C^1 discontinuity at $(R+1)e^{\frac{4(R+1)^2-1}{2(R+1)}}$, where g_R vanishes; however we can modify it near the corner to obtain a C^1 function by using a smooth cut-off h_R such that $|h'_R(\varrho)| \leq |g'_R(\varrho)|$ and $h_R(\varrho) + \varrho^2(h'_R(\varrho))^2 \leq 1$.

By construction $|\varphi_R(q)| \leq \widehat{R}$ a.e. in Ω, and we can see that $\varphi_R(q) \in BV(\Omega;\mathbb{M}^{n\times n}_D)$ with

$$|\mathrm{D}\varphi_R(q)| \leq |\mathrm{D}q| \quad \text{in } M_b(\Omega;\mathbb{M}^{n\times n\times n}_D)\,. \tag{3.7}$$

Let us prove (3.7) first in the case $q \in C^1(\Omega;\mathbb{M}^{n\times n}_D)$. Here we see every matrix ξ as a vector in $\mathbb{R}^{n^2}$; then

$$\mathrm{D}_i(\varphi_R(q)_j) = \omega_R(|q|)\mathrm{D}_i q_j + \omega'_R(|q|)\frac{q\cdot \mathrm{D}_i q}{|q|}q_j \quad \text{in } \Omega, \text{ for every } i \in [1,n],\ j\in[1,n^2]$$

which gives

$$\begin{aligned}|\mathrm{D}(\varphi_R(q))|^2 &= (\omega_R(|q|))^2|\mathrm{D}q|^2 + (\omega'_R(|q|))^2\sum_{i=1}^n (q\cdot\mathrm{D}_i q)^2 + 2\frac{\omega_R(|q|)}{|q|}\omega'_R(|q|)\sum_{i=1}^n(q\cdot \mathrm{D}_i q)^2\\ &\leq (\omega_R(|q|))^2|\mathrm{D}q|^2 + (\omega'_R(|q|))^2|q|^2|\mathrm{D}q|^2 \leq |\mathrm{D}q|^2 \quad \text{in } \Omega\,,\end{aligned}$$

by the Cauchy inequality and the fact that ω_R is nonnegative and nondecreasing. Therefore the inequality (3.7) is proved when $q \in C^1(\Omega;\mathbb{M}^{n\times n}_D)$. We now show the general case: since these measures are regular, it is sufficient to prove (3.7) on open sets. Given $q\in BV(\Omega;\mathbb{M}^{n\times n}_D)$ and U an open subset of Ω, by the Anzellotti-Giaquinta Approximation Theorem there exists $(q_k)_k \subset C^1(U;\mathbb{M}^{n\times n}_D)$ such that $q_k \xrightarrow{*} q\restriction_U$ in $BV(U;\mathbb{M}^{n\times n}_D)$ and

$$\|\mathrm{D}q\|_{1,U} = \lim_{k\to\infty}\|\nabla q_k\|_{1,U} = \lim_{k\to\infty}\int_U |\nabla q_k|\,\mathrm{d}x\,;$$

by regularity of ω_R we get that

$$\mathrm{D}(\varphi_R(q_k)) \xrightarrow{*} \mathrm{D}(\varphi_R(q)) \quad \text{in } M_b(U;\mathbb{M}^{n\times n\times n}_D) \tag{3.8}$$

as $k\to\infty$. By semicontinuity of the total variation with respect to weak* convergence the inequality (3.7) is proved for open sets, and this concludes the proof of (3.7).

By (3.8) we have that $\varphi_R(p_k) \xrightarrow{*} \varphi_R(p)$ in $BV(\Omega;\mathbb{M}^{n\times n}_D)$ as $k\to\infty$; then from the Step 1 (recall that $|\varphi_R(p_k)| \leq \widehat{R}$ a.e. in Ω) it follows that

$$\mathcal{H}(\alpha,\varphi_R(p)) \leq \liminf_{k\to\infty}\mathcal{H}(\alpha_k,\varphi_R(p_k)) \quad \text{for every } R>1\,,$$

and we want to pass to the limit as $R\to\infty$. First we prove that for every k

$$\mathcal{H}(\alpha_k,\varphi_R(p_k)) \leq \mathcal{H}(\alpha_k,p_k)\,. \tag{3.9}$$

To this end it is useful to rewrite $\mathcal{H}$ as

$$\mathcal{H}(\beta,q) = \int_\Omega S_2(\widetilde{\beta})\,\mathrm{d}\left|\left(S_1(\beta)S_2(\beta)^{-1}q, l\,\mathrm{D}q\right)\right|\,,$$

where $\big|\big(S_1(\beta)S_2(\beta)^{-1}q, l\,\mathrm{D}q\big)\big|$ is the variation of the product measure

$$(S_1(\beta)S_2(\beta)^{-1}q, l\,\mathrm{D}q) = (S_1(\beta)S_2(\beta)^{-1}q, l\,\nabla q)\mathcal{L}^n + (0, l\,\mathrm{D}^c q) + (0, l\,(q^+ - q^-)\otimes \nu_q \mathcal{H}^{n-1}\lfloor_{J_q})\,.$$

Since by construction $|\varphi_R(q)| \leq |q|$ a.e. in Ω for every $q \in BV(\Omega;\mathbb{M}^{n\times n}_D)$ we get by (3.7) that

$$|(S_1(\beta)S_2(\beta)^{-1}\varphi_R(q), l\,\mathrm{D}(\varphi_R(q)))| \leq |(S_1(\beta)S_2(\beta)^{-1}q, l\,\mathrm{D}q)| \quad \text{in } M_b(\Omega)$$

for every $\beta \in H^1(\Omega)$, $q \in BV(\Omega;\mathbb{M}^{n\times n}_D)$, and $R > 1$. Taking $\beta = \alpha_k$, $q = p_k$, and integrating the positive function $S_2(\widetilde{\alpha}_k)$, we obtain (3.9).

Therefore the proof is completed if we show that

$$\mathcal{H}(\alpha, p) = \lim_{R\to\infty} \mathcal{H}(\alpha, \varphi_R(p))\,. \tag{3.10}$$

The chain rule for BV functions proved in [37] gives in our case

$$\mathrm{D}\varphi_R(p) = \nabla\varphi_R(p)\nabla p\,\mathcal{L}^n + \nabla\varphi_R(\widetilde{p})\,\mathrm{D}^c p + (\varphi_R(p^+) - \varphi_R(p^-))\otimes \nu_p \mathcal{H}^{n-1}\lfloor_{J_p}\,,$$

where $\widetilde{p}(x)$ is the approximate limit of p at any Lebesgue point x, and then

$$\begin{aligned}\mathcal{H}(\alpha, \varphi_R(p)) = &\int_\Omega \sqrt{S_1(\alpha)^2|\varphi_R(p)|^2 + l^2 S_2(\alpha)^2|\nabla(\varphi_R(p))|^2}\,\mathrm{d}x \\ &+ l\int_{\Omega\setminus J_p} S_2(\widetilde{\alpha})\left|\nabla\varphi_R(\widetilde{p})\frac{\mathrm{d}\mathrm{D}^c p}{\mathrm{d}|\mathrm{D}^c p|}\right|\mathrm{d}|\mathrm{D}^c p| + l\int_{J_p} S_2(\widetilde{\alpha})|\varphi_R(p^+) - \varphi_R(p^-)|\,\mathrm{d}\mathcal{H}^{n-1}\,.\end{aligned} \tag{3.11}$$

It is known from the theory of BV functions that $p^+(x)$, $p^-(x) \in \mathbb{R}$ for $\mathcal{H}^{n-1}$–a.e. $x \in \Omega$ and hence $\mathcal{H}^{n-1}$–a.e. $x \in \Omega\setminus J_p$ is a Lebesgue point for p. Since $\omega_R(|x|) = 1$ if $|x| \leq R$, it follows that

$$\lim_{R\to\infty} \varphi_R(p^\pm(x)) = p^\pm(x) \quad \text{for } \mathcal{H}^{n-1}\text{–a.e. } x \in J_p\,,$$

and

$$\lim_{R\to\infty}\left[\nabla\varphi_R(\widetilde{p})\frac{\mathrm{d}\mathrm{D}^c p}{\mathrm{d}|\mathrm{D}^c p|}\right] = \frac{\mathrm{d}\mathrm{D}^c p}{\mathrm{d}|\mathrm{D}^c p|} \quad \text{for } |\mathrm{D}^c p|\text{–a.e. } x \in \Omega\setminus J_p\,.$$

By (3.7) we have that

$$|\nabla(\varphi_R(p))| \leq |\nabla p|\,, \quad \left|\nabla\varphi_R(\widetilde{p})\frac{\mathrm{d}\mathrm{D}^c p}{\mathrm{d}|\mathrm{D}^c p|}\right| \leq 1\,, \quad |\varphi_R(p^+) - \varphi_R(p^-)| \leq |p^+ - p^-|\,.$$

Then we can pass to the limit in (3.11) using the Dominated Convergence Theorem and obtain (3.10). Therefore the proof is concluded. □

Using Theorem 3.1, we can prove the existence of solutions to the minimization problem (3.1) by applying the direct method of the Calculus of Variations.

Lemma 3.2. *Problem* (3.1) *admits a solution, and for every* (α, u, e, p) *solution of* (3.1) *it holds that* $\alpha \in H^1(\Omega;[0,1])$.

Proof. Let

$$(\alpha_k, u_k, e_k, p_k) \in \mathcal{A}(\overline{\alpha}, w)$$

be a minimizing sequence for (3.1); by (2.9b), (2.6a), and (2.11) we can assume $\alpha_k \in H^1(\Omega;[0,1])$ for every k. Since $(0, w, \mathrm{E}w, 0) \in \mathcal{A}(\overline{\alpha}, w)$ and

$$\mathcal{E}(0, \mathrm{E}w, 0) - \langle \mathcal{L}, w\rangle + \mathcal{H}(0, \overline{p}) =: C \in \mathbb{R}$$

we get that $\mathcal{E}(\alpha_k, e_k, \mathrm{curl}\; p_k) - \langle \mathcal{L}, u_k\rangle + \mathcal{H}(\alpha_k, p_k - \overline{p})$ is uniformly bounded in k and

$$\begin{aligned}&\mathcal{E}(\alpha_k, e_k, \mathrm{curl}\; p_k) - \int_\Omega \varrho(t)\cdot e_k\,\mathrm{d}x + \mathcal{H}(\alpha_k, p_k - \overline{p}) - \int_\Omega \varrho_D(t)\cdot(p_k - \overline{p})\,\mathrm{d}x \\ &\leq C + \int_\Omega \varrho_D(t)\cdot\overline{p}\,\mathrm{d}x - \langle \varrho(t)\nu, w\rangle_{\partial_D\Omega}\end{aligned}$$

by the representation formula (2.15). By definition of $\mathcal{E}$ and (2.17) we obtain that

$$\|\nabla\alpha_k\|_2^2 + \|e_k\|_2^2 + \|\mathrm{curl}\,p_k\|_2^2 + \|p_k - \overline{p}\|_{BV} \leq C_1\,,$$

and hence there exist $\alpha \in H^1(\Omega;[0,1])$, $e \in L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$, $p \in BV(\Omega;\mathbb{M}^{n\times n}_D)$ such that up to a subsequence

$$\alpha_k \rightharpoonup \alpha \quad \text{in } H^1(\Omega)\,, \quad e_k \rightharpoonup e \quad \text{in } L^2(\Omega;\mathbb{M}^{n\times n}_{sym})\,, \quad p_k \overset{*}{\rightharpoonup} p \quad \text{in } BV(\Omega;\mathbb{M}^{n\times n}_D)\,.$$

Moreover $\operatorname{curl} p \in L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$ and

$$\operatorname{curl} p_k \rightharpoonup \operatorname{curl} p \quad \text{in } L^2(\Omega;\mathbb{M}^{n\times n}_{sym})\,.$$

Using the embedding $BV(\Omega;\mathbb{M}^{n\times n}_D) \hookrightarrow L^{\frac{n}{n-1}}(\Omega;\mathbb{M}^{n\times n}_D)$ and Korn's inequality it follows easily from $(u_k,e_k,p_k) \in A(w)$ that u_k are uniformly bounded in $W^{\frac{n}{n-1}}(\Omega;\mathbb{R}^n)$: then up to a further subsequence

$$u_k \rightharpoonup u \quad \text{in } W^{\frac{n}{n-1}}(\Omega;\mathbb{R}^n)$$

for a suitable u such that $(u,e,p) \in A(w)$. Collecting the semicontinuity properties (2.8) and (3.3) we get that (α,u,e,p) is a minimizer, and the proof is completed. □

In the very same way of [8, Lemma 3.2], we deduce the remark below from the properties of $\mathcal{H}$.

Remark 3.3. If (α,u,e,p) solves (3.1) then

$$\mathcal{E}(\alpha,e,\operatorname{curl} p) - \langle \mathcal{L},u\rangle \le \mathcal{E}(\widetilde{\alpha},\widetilde{e},\operatorname{curl}\widetilde{p}) - \langle \mathcal{L},\widetilde{u}\rangle + \mathcal{H}(\widetilde{\alpha},\widetilde{p}-p)\,, \tag{3.12}$$

for every $(\widetilde{\alpha},\widetilde{u},\widetilde{e},\widetilde{p}) \in \mathcal{A}(\alpha,w)$.

The following lemma states some differential conditions for a triple (u,e,p) such that (α,u,e,p) satisfies (3.12). We shall make use of these conditions to recover the classical formulation of the model.

Lemma 3.4. *Let (α,u,e,p) satisfy (3.12). Then*

$$\Big|\langle \sigma,\eta\rangle + \langle L^2\mu(\alpha)\operatorname{curl} p,\operatorname{curl} q\rangle - \langle \mathcal{L},v\rangle\Big| \le \mathcal{H}(\alpha,q) \tag{3.13}$$

for every $(v,\eta,q) \in A(0)$, where $\sigma := \mathbb{C}(\alpha)e$. Moreover

$$\begin{cases} -\operatorname{div}\sigma = f & \text{in } \Omega\,,\\ \sigma\nu = g & \text{on } \partial_N\Omega\,. \end{cases} \tag{3.14}$$

Proof. Let us fix $(v,\eta,q) \in A(0)$. Since for every $\varepsilon \in \mathbb{R}$

$$(\alpha, u+\varepsilon v, e+\varepsilon\eta, p+\varepsilon q) \in \mathcal{A}(\alpha,w)\,,$$

from the remark above we have

$$\mathcal{Q}_1(\alpha,e+\varepsilon\eta) + \mathcal{Q}_2(\alpha,\operatorname{curl}(p+\varepsilon q)) + \mathcal{H}(\alpha,\varepsilon q) \ge \mathcal{Q}_1(\alpha,e) + \mathcal{Q}_2(\alpha,\operatorname{curl} p) \quad \text{for every } \varepsilon \in \mathbb{R}\,.$$

Then the positive homogeneity of $\mathcal{H}$ gives that

$$\mathcal{Q}_1(\alpha,e\pm\varepsilon\eta) + \mathcal{Q}_2(\alpha,\operatorname{curl}(p\pm\varepsilon q)) + \varepsilon\mathcal{H}(\alpha,\pm q) \ge \mathcal{Q}_1(\alpha,e) + \mathcal{Q}_2(\alpha,\operatorname{curl} p) \quad \text{for every } \varepsilon \in \mathbb{R}\,.$$

Dividing by ε and passing to the limit as $\varepsilon \to 0$, we recover (3.13).

Choosing in (3.13) $(v,Ev,0)$ for every $v \in C^\infty(\overline{\Omega};\mathbb{R}^n)$ with $v=0$ on $\partial_D\Omega$, we get (3.14). Notice that the normal trace of σ on $\partial\Omega$ is well defined in $H^{-1/2}(\partial\Omega;\mathbb{R}^n)$ since $\sigma \in L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$ with divergence in $L^2(\Omega;\mathbb{R}^n)$. □

The lemma below will permit us to say that when both α and p are continuous at a given time then all the evolution is there continuous. In contrast with [8, Lemma 3.4], here it is not useful to write ω_{12} in terms of $\|\alpha_1-\alpha_2\|_\infty$; indeed we will consider the case when a sequence of functions α_1 tends to a function α_2 weakly in $H^1(\Omega)$, and this does not provide uniform convergence in Ω.

Lemma 3.5. *For $i=1,2$ let $w_i\in H^1(\Omega;\mathbb{R}^n)$, $f_i\in L^n(\Omega;\mathbb{R}^n)$, $g_i\in L^\infty(\partial_N\Omega;\mathbb{R}^n)$, and let $\mathcal{L}_i$ be defined by (3.2) with $f=f_i$ and $g=g_i$. Suppose that (α_i,u_i,e_i,p_i) satisfies (3.12) with data $w=w_i$, $\mathcal{L}=\mathcal{L}_i$, and let*

$$\begin{aligned}\omega_{12}:=&\big\|[\mathbb{C}(\alpha_2)-\mathbb{C}(\alpha_1)]e_1\big\|_2+\big\|(\mu(\alpha_2)-\mu(\alpha_1))\mathrm{curl}\,p_1\big\|_2+\|p_2-p_1\|_{BV}^{1/2}\\&+\|p_2-p_1\|_1+\|f_2-f_1\|_n+\|g_2-g_1\|_{\infty,\partial_N\Omega}+\|\mathrm{E}w_2-\mathrm{E}w_1\|_2\,.\end{aligned}$$

Then there exists a positive constant C depending on L, $\mu(0)$, γ_1, γ_2, R, Ω, $\partial_N\Omega$ such that

$$\begin{aligned}\|e_2-e_1\|_2+\|\mathrm{curl}\,p_2-\mathrm{curl}\,p_1\|_2\le C\,\omega_{12}\,,\\ \|u_2-u_1\|_{1,\frac{n}{n-1}}\le C(\omega_{12}+\|w_2-w_1\|_2)\,.\end{aligned}\tag{3.15}$$

Proof. Let

$$v:=(u_2-w_2)-(u_1-w_1)\,,\quad \eta:=(e_2-\mathrm{E}w_2)-(e_1-\mathrm{E}w_1)\,,\quad q:=p_2-p_1\,.$$

Since $(v,\eta,q)\in A(0)$, by (3.13) we have that

$$\begin{aligned}-\mathcal{H}(\alpha_1,p_2-p_1)\le&\langle\mathbb{C}(\alpha_1)e_1,\eta\rangle+L^2\,\langle\mu(\alpha_1)\,\mathrm{curl}\,p_1,\mathrm{curl}\,(p_2-p_1)\rangle-\langle\mathcal{L}_1,v\rangle\,,\\ &\langle\mathbb{C}(\alpha_2)e_2,\eta\rangle+L^2\,\langle\mu(\alpha_2)\,\mathrm{curl}\,p_2,\mathrm{curl}\,(p_2-p_1)\rangle-\langle\mathcal{L}_2,v\rangle\le\mathcal{H}(\alpha_2,p_2-p_1)\,.\end{aligned}$$

Gathering the inequalities above and using (2.12) we obtain that

$$\begin{aligned}&\langle\mathbb{C}(\alpha_2)(e_2-e_1),\eta)\rangle+L^2\int_\Omega\mu(\alpha_2)|\mathrm{curl}\,(p_2-p_1)|^2\,\mathrm{d}x\\&\le\langle[\mathbb{C}(\alpha_1)-\mathbb{C}(\alpha_2)]e_1,\eta\rangle+L^2\,\langle[\mu(\alpha_1)-\mu(\alpha_2)]\mathrm{curl}\,p_1,\mathrm{curl}\,(p_2-p_1)\rangle+\langle\mathcal{L}_2-\mathcal{L}_1,v\rangle\\&\quad+2R\|p_2-p_1\|_{BV}\,,\end{aligned}$$

and then, by the definition of η,

$$\begin{aligned}&\langle\mathbb{C}(\alpha_2)(e_2-e_1),e_2-e_1\rangle+L^2\int_\Omega\mu(\alpha_2)|\mathrm{curl}\,(p_2-p_1)|^2\,\mathrm{d}x\\&\le\langle\mathbb{C}(\alpha_2)(e_2-e_1),\mathrm{E}w_2-\mathrm{E}w_1\rangle+\langle[\mathbb{C}(\alpha_1)-\mathbb{C}(\alpha_2)]e_1,e_2-e_1+(\mathrm{E}w_1-\mathrm{E}w_2)\rangle\\&\quad+L^2\,\langle[\mu(\alpha_1)-\mu(\alpha_2)]\mathrm{curl}\,p_1,\mathrm{curl}\,(p_2-p_1)\rangle+\langle\mathcal{L}_2-\mathcal{L}_1,v\rangle+2R\|p_2-p_1\|_{BV}\,,\end{aligned}\tag{3.16}$$

Arguing as in the proof of [10, Theorem 3.8] we see that there exists a constant $\widehat{C}$ depending only on Ω and $\partial_N\Omega$ such that

$$|\langle\mathcal{L}_2-\mathcal{L}_1,v\rangle|\le\widehat{C}\big(\|f_2-f_1\|_n+\|g_2-g_1\|_{\infty,\partial_N\Omega}\big)\big(\|e_2-e_1\|_2+\|\mathrm{E}w_2-\mathrm{E}w_1\|_2+\|p_2-p_1\|_1\big)\,.$$

Since

$$\gamma_1\|e_2-e_1\|_2^2+L^2\,\mu(0)\|\mathrm{curl}\,(p_2-p_1)\|_2^2\le\langle\mathbb{C}(\alpha_2)(e_2-e_1),e_2-e_1\rangle+L^2\int_\Omega\mu(\alpha_2)|\mathrm{curl}\,(p_2-p_1)|^2\,\mathrm{d}x\,,$$

we conclude the former of (3.15) from (3.16) using the Cauchy inequality. The latter estimate is easily shown using the compatibility conditions (2.3b) and Korn's Inequality. □

We now prove a stability result for the solutions of (3.12) with respect to the weak convergence of the data.

Theorem 3.6 (Stability of solutions to (3.12))**.** *Let $w_k\in H^1(\Omega;\mathbb{R}^n)$, $\mathcal{L}_k\in\big(W^{\frac{n}{n-1}}(\Omega;\mathbb{R}^n)\big)'$, $\alpha_k\in H^1(\Omega;[0,1])$, and $(u_k,e_k,p_k)\in A(w_k)$ for every k. Assume that $\alpha_k\rightharpoonup\alpha_\infty$ in $H^1(\Omega)$, $u_k\rightharpoonup u_\infty$ in $W^{\frac{n}{n-1}}(\Omega;\mathbb{R}^n)$, $e_k\rightharpoonup e_\infty$ in $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$, $p_k\xrightharpoonup{*}p_\infty$ in $BV(\Omega;\mathbb{M}^{n\times n}_D)$, $w_k\rightharpoonup w_\infty$ in $H^1(\Omega;\mathbb{R}^n)$, $\mathcal{L}_k\rightharpoonup\mathcal{L}$ in $\big(W^{\frac{n}{n-1}}(\Omega;\mathbb{R}^n)\big)'$. Then $(u_\infty,e_\infty,p_\infty)\in A(w_\infty)$. If, in addition,*

$$\mathcal{E}(\alpha_k,e_k,\mathrm{curl}\,p_k)-\langle\mathcal{L}_k,u_k\rangle\le\mathcal{E}(\widehat{\alpha}_k,\widehat{e}_k,\mathrm{curl}\,\widehat{p}_k)-\langle\mathcal{L}_k,\widehat{u}_k\rangle+\mathcal{H}(\widehat{\alpha}_k,\widehat{p}_k-p_k)\tag{3.17}$$

for every k and every $(\widehat{\alpha}_k,\widehat{u}_k,\widehat{e}_k,\widehat{p}_k)\in\mathcal{A}(\alpha_k,w_k)$, then

$$\mathcal{E}(\alpha_\infty,e_\infty,\mathrm{curl}\,p_\infty)-\langle\mathcal{L},u_\infty\rangle\le\mathcal{E}(\alpha,e,\mathrm{curl}\,p)-\langle\mathcal{L},u\rangle+\mathcal{H}(\alpha,p-p_\infty)\tag{3.18}$$

for every $(\alpha,u,e,p)\in\mathcal{A}(\alpha_\infty,w_\infty)$.

Proof. The fact that $(u_\infty, e_\infty, p_\infty) \in A(w_\infty)$ is immediate by the definition of admissible triple and the weak convergences assumed.

Let us now fix $(\alpha, u, e, p) \in \mathcal{A}(\alpha_\infty, w_\infty)$ and test (3.17) by

$$\widehat{\alpha}_k := \alpha \wedge \alpha_k\,, \quad \widehat{u}_k := u - u_\infty + u_k\,, \quad \widehat{e}_k := e - e_\infty + e_k\,, \quad \widehat{p}_k := p - p_\infty + p_k\,.$$

Indeed by assumption $(\widehat{\alpha}_k, \widehat{u}_k, \widehat{e}_k, \widehat{p}_k) \in \mathcal{A}(\alpha_k, w_k)$, and moreover $\widehat{\alpha}_k \rightharpoonup \alpha$ and $\alpha \vee \alpha_k \rightharpoonup \alpha_\infty$ in $H^1(\Omega)$, $\widehat{u}_k \rightharpoonup u$ in $W^{1,\frac{n}{n-1}}(\Omega;\mathbb{R}^n)$, $\widehat{e}_k \rightharpoonup e$ in $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$, $\widehat{p}_k \overset{*}{\rightharpoonup} p$ in $BV(\Omega;\mathbb{M}^{n\times n}_D)$.

Since for every $\alpha \in H^1(\Omega)$ and every $\eta_1, \eta_2 \in L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$ we have that

$$\mathcal{Q}_1(\alpha,\eta_1) - \mathcal{Q}_1(\alpha,\eta_2) = \tfrac{1}{2}\langle \mathbb{C}(\alpha)(\eta_1+\eta_2), \eta_1 - \eta_2\rangle\,, \tag{3.19}$$

$$\mathcal{Q}_2(\alpha,\eta_1) - \mathcal{Q}_2(\alpha,\eta_2) = \tfrac{L^2}{2}\langle \mu(\alpha)(\eta_1+\eta_2), \eta_1 - \eta_2\rangle\,, \tag{3.20}$$

and for every $\alpha, \beta \in H^1(\Omega)$

$$\|\nabla(\alpha\vee\beta)\|_2^2 + \|\nabla(\alpha\wedge\beta)\|_2^2 = \|\nabla\alpha\|_2^2 + \|\nabla\beta\|_2^2\,,$$

then the inequality (3.17) can be rewritten, adding to both sides $-\mathcal{Q}_1(\widehat{\alpha}_k, e_k) - \mathcal{Q}_2(\widehat{\alpha}_k, \operatorname{curl} p_k)$, thus obtaining

$$\begin{aligned}\gamma_k &:= \mathcal{Q}_1(\alpha_k, e_k) - \mathcal{Q}_1(\widehat{\alpha}_k, e_k) + \mathcal{Q}_2(\alpha_k, \operatorname{curl} p_k) - \mathcal{Q}_2(\widehat{\alpha}_k, \operatorname{curl} p_k) + \mathcal{D}(\alpha_k)\\ &\quad + \frac{\ell^2}{2}\|\nabla(\alpha\vee\alpha_k)\|_2^2 - \frac{\ell^2}{2}\|\nabla\alpha\|_2^2\\ &\le \tfrac{1}{2}\langle \mathbb{C}(\widehat{\alpha}_k)(e - e_\infty + 2e_k), e - e_\infty\rangle + \tfrac{L^2}{2}\langle \mu(\widehat{\alpha}_k)\operatorname{curl}(p - p_\infty + 2p_k), \operatorname{curl}(p-p_\infty)\rangle\\ &\quad + \mathcal{D}(\widehat{\alpha}_k) + \mathcal{H}(\widehat{\alpha}_k, p - p_\infty) - \langle \mathcal{L}_k, u - u_\infty\rangle =: \delta_k\,.\end{aligned}$$

Notice that for every $\eta \in L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$

$$\mathcal{Q}_1(\alpha_k,\eta) - \mathcal{Q}_1(\widehat{\alpha}_k,\eta) = \tfrac{1}{2}\langle [\mathbb{C}(\alpha_k) - \mathbb{C}(\widehat{\alpha}_k)]\eta, \eta\rangle\,,$$

$$\mathcal{Q}_2(\alpha_k,\eta) - \mathcal{Q}_2(\widehat{\alpha}_k,\eta) = \tfrac{L^2}{2}\langle (\mu(\alpha_k) - \mu(\widehat{\alpha}_k))\eta, \eta\rangle\,.$$

Moreover $(x,\beta,\xi) \mapsto [\mathbb{C}(\beta) - \mathbb{C}(\beta\wedge\alpha(x))]\xi\cdot\xi$ and $(x,\beta,\xi) \mapsto (\mu(\beta) - \mu(\beta\wedge\alpha(x)))|\xi|^2$ are measurable functions from $\Omega\times\mathbb{R}\times\mathbb{M}^{n\times n}_{sym}$ into $\mathbb{R}^+\cup\{0\}$, continuous in the variable β and convex in ξ. Therefore the Ioffe-Olach Semicontinuity Theorem (cf. [4, Theorem 2.3.1]) implies that

$$\mathcal{Q}_i(\alpha_\infty,\eta_\infty) - \mathcal{Q}_i(\alpha,\eta_\infty) \le \liminf_{k\to\infty}\big[\mathcal{Q}_i(\alpha_k,\eta_k) - \mathcal{Q}_i(\widehat{\alpha}_k,\eta_k)\big]$$

for every $i \in \{1,2\}$ and $\eta_k \rightharpoonup \eta_\infty$ in $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$. Then it follows that

$$\mathcal{E}(\alpha_\infty, e_\infty, \operatorname{curl} p_\infty) - \mathcal{Q}_1(\alpha, e_\infty) - \mathcal{Q}_2(\alpha, \operatorname{curl} p_\infty) - \frac{\ell^2}{2}\|\nabla\alpha\|_2^2 \le \liminf_{k\to\infty}\gamma_k\,. \tag{3.21}$$

On the other hand

$$\lim_{k\to\infty}\delta_k = \mathcal{E}(\alpha, e, \operatorname{curl} p) - \mathcal{Q}_1(\alpha, e_\infty) - \mathcal{Q}_2(\alpha, \operatorname{curl} p_\infty) - \frac{\ell^2}{2}\|\nabla\alpha\|_2^2 + \mathcal{H}(\alpha, p - p_\infty) - \langle \mathcal{L}, u - u_\infty\rangle\,. \tag{3.22}$$

Indeed, since $\widehat{\alpha}_k \rightharpoonup \alpha$ in $H^1(\Omega)$, up to a subsequence $\widetilde{\widehat{\alpha}}_k(x) \to \widetilde{\alpha}(x)$ for $|\mathrm{D}(p-p_\infty)|$–a.e. $x \in \Omega$; therefore, by the Dominated Convergence Theorem

$$\lim_{k\to\infty}\mathcal{H}(\widehat{\alpha}_k, p - p_\infty) = \mathcal{H}(\alpha, p - p_\infty)\,.$$

The convergence of $\mathcal{D}(\widehat{\alpha}_k)$ to $\mathcal{D}(\alpha)$ follows easily from (2.9). Let us consider the first term in δ_k: the simmetry of $\mathbb{C}(\beta)$ for every $\beta \in \mathbb{R}$ gives that

$$\tfrac{1}{2}\langle \mathbb{C}(\widehat{\alpha}_k)(e - e_\infty + 2e_k), e - e_\infty\rangle = \tfrac{1}{2}\langle e - e_\infty + 2e_k, \mathbb{C}(\widehat{\alpha}_k)(e - e_\infty)\rangle\,.$$

Since $\mathbb{C}(\beta)$ is bounded uniformly with respect to $\beta \in \mathbb{R}$ and $\widehat{\alpha}_k \rightharpoonup \alpha$ in $H^1(\Omega)$, by the Dominated Convergence Theorem we get that

$$\mathbb{C}(\widehat{\alpha}_k)(e - e_\infty) \to \mathbb{C}(\alpha)(e - e_\infty) \quad \text{in } L^2(\Omega;\mathbb{M}^{n\times n}_{sym})\,.$$

From the fact that $e_k \rightharpoonup e_\infty$ in $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$ we conclude that

$$\lim_{k\to\infty}\tfrac{1}{2}\langle \mathbb{C}(\widehat{\alpha}_k)(e - e_\infty + 2e_k), e - e_\infty\rangle = \mathcal{Q}_1(\alpha, e) - \mathcal{Q}_1(\alpha, e_\infty)\,,$$

recalling (3.19). In the same way we get that

$$\lim_{k\to\infty} \tfrac{L^2}{2} \langle \mu(\widehat{\alpha}_k)\, \mathrm{curl}\,(p-p_\infty+2p_k), \mathrm{curl}\,(p-p_\infty)\rangle = \mathcal{Q}_2(\alpha, \mathrm{curl}\, p) - \mathcal{Q}_2(\alpha, \mathrm{curl}\, p_\infty)$$

and then we conclude (3.22). Gathering (3.21) and (3.22) we get (3.18) and the proof is completed. ☐

4. Existence of quasistatic evolutions

This section is devoted to the proof of Theorem 2.5, basing on discrete time approximation. First we construct a sequence of approximate evolutions by solving, for the k-th approximant, k incremental problems of the type (3.1) which we have studied in Section 3; then we show that this sequence converges in a suitable sense to a quasistatic evolution for the Gurtin-Anand model coupled with damage. Henceforth we assume the hypoteses of Theorem 2.5, in particular the stability condition on the initial datum $(\alpha_0, u_0, e_0, p_0)$.

Before starting the proof of the existence result, we prove that the integrals in the energy balance (qs2) of Definition 2.3 are well defined. This follows immediately by the following lemma.

Lemma 4.1. *Let (α, u, e, p) be a quasistatic evolution and $\sigma(t) := \mathbb{C}(\alpha(t))e(t)$, according to Definition 2.3. Let $r \in [1,\infty)$. Then the functions $t \mapsto \alpha(t) \in L^r(\Omega)$, $t \mapsto u(t) \in W^{1,\frac{n}{n-1}}(\Omega;\mathbb{R}^n)$, $t \mapsto e(t) \in L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$, and $t \mapsto \sigma(t) \in L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$ are strongly continuous except at most for a countable subset of $[0,T]$, and*

$$(\alpha, u, e, p) \in L^\infty(0,T; H^1(\Omega)\times W^{1,\frac{n}{n-1}}(\Omega;\mathbb{R}^n)\times L^2(\Omega;\mathbb{M}^{n\times n}_{sym})\times BV(\Omega;\mathbb{M}^{n\times n}_D))\,.$$

Proof. By the irreversibility condition and [8, Lemma A.2] it follows that there exists a countable set $E_1 \subset [0,T]$ such that α is continuous at every $t\in[0,T]\setminus E_1$ with respect to the L^r norm, for every $r\in[1,\infty)$. The condition (qs2) gives that $p\in L^\infty(0,T;BV(\Omega;\mathbb{M}^{n\times n}_D))$; then by (qs1), taking $(\beta, v, \eta, q) = (0, w(t), \mathrm{E}w(t), 0)$ for every t, we deduce that $\alpha(t)$, $u(t)$, $e(t)$ are uniformly bounded in $H^1(\Omega)$, $W^{1,\frac{n}{n-1}}(\Omega;\mathbb{R}^n)$, $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$, respectively. Thus for every $t\in[0,T]\setminus E_1$

$$\alpha(s) \rightharpoonup \alpha(t) \quad \text{in } H^1(\Omega), \quad \alpha(s)\to\alpha(t) \quad \text{in } L^r(\Omega) \quad \text{as } s\to t\,. \tag{4.1}$$

Since p has bounded variation into the space $BV(\Omega;\mathbb{M}^{n\times n}_D)$, the set E_2 of its discontinuity points is at most countable. Moreover, by the uniform bound for $\mu(\alpha)$ and $\mathbb{C}(\alpha)$, (4.1), and the Dominated Convergence Theorem it follows that for every $t\in[0,T]\setminus E_1$

$$\mathbb{C}(\alpha(s))e(t)\to\mathbb{C}(\alpha(t))e(t)\,, \quad \mu(\alpha(s))\mathrm{curl}\,p(t)\to\mu(\alpha(t))\mathrm{curl}\,p(t) \quad \text{in } L^2(\Omega;\mathbb{M}^{n\times n}_{sym}) \text{ as } s\to t\,.$$

Then, using Lemma 3.5 (recall that the loading is continuous in time) we obtain that e and u are strongly continuous in $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$ and $W^{1,\frac{n}{n-1}}(\Omega;\mathbb{R}^n)$ at every $t\in[0,T]\setminus E$, with $E = E_1\cup E_2$.

Hence, by (4.1), $\sigma(s)\to\sigma(t)$ in $L^1(\Omega;\mathbb{M}^{n\times n}_{sym})$ as $s\to t$ for every $t\in[0,T]\setminus E$. Since $\mathbb{C}(\alpha)$ is uniformly bounded, and then $|\sigma(s)|\le C|e(s)|$ in Ω, we deduce that this convergence is indeed strong in $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$, applying the Dominated Convergence Theorem. Finally, α is measurable into $H^1(\Omega)$ by the former of (4.1) and the fact that $H^1(\Omega)$ is separable. This concludes the proof. ☐

For every $k\in\mathbb{N}$ we define approximate evolutions $(\alpha_k, u_k, e_k, p_k)$ by induction. Let us set $t_k^i := T\frac{i}{k}$ for $i = 0,\dots,k$ and

$$(\alpha_k^0, u_k^0, e_k^0, p_k^0) := (\alpha_0, u_0, e_0, p_0) \in \mathcal{A}(1, w(0))\,.$$

For $i = 1,\dots,k$ let $(\alpha_k^i, u_k^i, e_k^i, p_k^i)$ be a solution to the incremental problem

$$\mathrm{argmin}\,\{\mathcal{E}(\alpha, e, \mathrm{curl}\ p) - \langle \mathcal{L}_k^i, u\rangle + \mathcal{H}(\alpha, p-p_k^{i-1}) : \, (\alpha, u, e, p)\in\mathcal{A}(\alpha_k^{i-1}, w_k^i)\}\,, \tag{4.2}$$

where $w_k^i := w(t_k^i)$ and $\mathcal{L}_k^i := \mathcal{L}(t_k^i)$. Notice that Lemma 3.2 ensures the existence of solutions. Then we define for $i = 0, \dots, k-1$ and $t \in [t_k^i, t_k^{i+1})$

$$\begin{aligned} \alpha_k(t) := \alpha_k^i\,,\ \ u_k(t) := u_k^i\,,\ \ e_k(t) := e_k^i\,,\ \ p_k(t) := p_k^i\,,\\ \sigma_k(t) := \mathbb{C}(\alpha_k^i)e_k^i\,,\ \ w_k(t) := w_k^i\,,\ \ \mathcal{L}_k(t) := \mathcal{L}_k^i\,, \end{aligned} \tag{4.3}$$

while $(\alpha_k(T), e_k(T), u_k(T), p_k(T)) := (\alpha_k^k, u_k^k, e_k^k, p_k^k)$.

The proposition below gives that these piecewise constant approximants satisfy a discretized version of the stability condition (qs1), a discretized energy inequality, and some a-priori estimates. The proof follows the line of [16, Proposition 6.2], with some modifications due to the presence of the damage variable.

Proposition 4.2. *For every $k \in \mathbb{N}$ the evolution $(\alpha_k, u_k, e_k, p_k)$ defined in (4.3) satisfies the following conditions:*

(qs0)$_k$ *for every $x \in \Omega$ the function $t \in [0,T] \mapsto \alpha_k(t,x)$ is nonincreasing;*

(qs1)$_k$ *for every $t \in [0,T]$ we have $(u_k(t), e_k(t), p_k(t)) \in A(w_k(t))$ and*

$$\mathcal{E}(\alpha_k(t), e_k(t), \operatorname{curl} p_k(t)) - \langle \mathcal{L}_k(t), u_k(t)\rangle \le \mathcal{E}(\beta, \eta, \operatorname{curl} q) - \langle \mathcal{L}_k(t), v\rangle + \mathcal{H}(\beta, q - p_k(t))$$

for every $(\beta, v, \eta, q) \in \mathcal{A}(\alpha_k(t), w_k(t))$;

(qs2)$_k$ *for every $t \in [t_k^i, t_k^{i+1})$*

$$\begin{aligned} \mathcal{E}(\alpha_k(t), e_k(t), \operatorname{curl} p_k(t)) - \langle \mathcal{L}_k(t), u_k(t)\rangle + \mathcal{V}_{\mathcal{H}}(\alpha_k, p_k; 0, t) \le \mathcal{E}(\alpha_0, e_0, \operatorname{curl} p_0) - \langle \mathcal{L}(0), u_0\rangle \\ + \int_0^{t_k^i} \langle \sigma_k(s), E\dot w(s)\rangle\,\mathrm{d}s - \int_0^{t_k^i} \langle \dot{\mathcal{L}}(s), u_k(s)\rangle\,\mathrm{d}s - \int_0^{t_k^i} \langle \mathcal{L}_k(s), \dot w(s)\rangle\,\mathrm{d}s + \delta_k\,, \end{aligned}$$

where $\delta_k \to 0$ as $k \to \infty$.

Moreover there exists a positive constant C independent of k and $t \in [0,T]$ such that

$$\|\alpha_k(t)\|_{1,2} + \|u_k(t)\|_{1,\frac{n}{n-1}} + \|e_k(t)\|_2 + \|\operatorname{curl} p_k(t)\|_2 + \mathcal{V}(p_k; 0, t) \le C\,. \tag{4.4}$$

Proof. The condition (qs0)$_k$ holds since $\alpha_k^i \le \alpha_k^{i-1}$. Moreover $(u_k(t), e_k(t), p_k(t)) \in A(w_k(t))$ for every $t \in [0,T]$, by definition of the approximate evolutions. By (4.2) and Remark 3.3 we get

$$\mathcal{E}(\alpha_k^i, e_k^i, \operatorname{curl} p_k^i) - \langle \mathcal{L}_k^i, u_k^i\rangle \le \mathcal{E}(\beta, e, \operatorname{curl} p) - \langle \mathcal{L}_k^i, u\rangle + \mathcal{H}(\beta, p - p_k^i)$$

for every k, $i = 1, \dots, k$, and $(\beta, u, e, p) \in \mathcal{A}(\alpha_k^i, w_k^i)$, which gives (qs1)$_k$.

In order to prove (qs2)$_k$ let us fix $i \in \{1, \dots, k\}$, $t \in [t_k^{i-1}, t_k^i)$, $u := u_k^{h-1} - w_k^{h-1} + w_k^h$, and $e := e_k^{h-1} - \mathrm{E}w_k^{h-1} + \mathrm{E}w_k^h$ for a given integer h with $1 \le h \le i$. Testing (4.2) for $i = h$ by $(\alpha_k^{h-1}, (u, e, p_k^{h-1})) \in \mathcal{A}(\alpha_k^{h-1}, w_k^h)$ we get

$$\begin{aligned} &\mathcal{E}(\alpha_k^h, e_k^h, \operatorname{curl} p_k^h) - \langle \mathcal{L}_k^h, u_k^h\rangle + \mathcal{H}(\alpha_k^h, p_k^h - p_k^{h-1}) \\ &\quad \le \mathcal{E}(\alpha_k^{h-1}, e_k^{h-1}, \operatorname{curl} p_k^{h-1}) + \mathcal{Q}_1(\alpha_k^{h-1}, \mathrm{E}w_k^h - \mathrm{E}w_k^{h-1}) + \langle \mathbb{C}(\alpha_k^{h-1})e_k^{h-1}, \mathrm{E}w_k^h - \mathrm{E}w_k^{h-1}\rangle \\ &\qquad - \langle \mathcal{L}_k^h, u_k^{h-1} + w_k^h - w_k^{h-1}\rangle \\ &\quad = \mathcal{E}(\alpha_k^{h-1}, e_k^{h-1}, \operatorname{curl} p_k^{h-1}) + \int_{t_k^{h-1}}^{t_k^h} \langle \sigma_k^{h-1}, E\dot w(s)\rangle\,\mathrm{d}s - \langle \mathcal{L}_k^{h-1}, u_k^{h-1}\rangle \\ &\qquad - \int_{t_k^{h-1}}^{t_k^h} \langle \dot{\mathcal{L}}(s), u_k(s)\rangle\,\mathrm{d}s - \int_{t_k^{h-1}}^{t_k^h} \langle \mathcal{L}_k(s), \dot w(s)\rangle\,\mathrm{d}s + \delta_{k,h}\,, \end{aligned} \tag{4.5}$$

where

$$\delta_{k,h} := \mathcal{Q}_1(\alpha_k^{h-1}, \mathrm{E}w_k^h - \mathrm{E}w_k^{h-1}) - \langle \mathcal{L}_k^h - \mathcal{L}_k^{h-1}, w_k^h - w_k^{h-1}\rangle\,.$$

Iterating for $1 \le h \le i$ we deduce (qs2)$_k$, with $\delta_k = \sum_{h=1}^i \delta_{k,h}$. Indeed, since p_k is piecewise constant and continuous from the right, and α_k is nonincreasing, the supremum in the definition of $\mathcal{V}_{\mathcal{H}}$ is attained by the subdivision $(t_k^h)_h$, namely (cf. [8, Lemma A.1])

$$\mathcal{V}_{\mathcal{H}}(\alpha_k, p_k; 0, t) = \sum_{h=1}^{i} \mathcal{H}(\alpha_k^h, p_k^h - p_k^{h-1}) .$$

Moreover, the absolute continuity of the loading (2.2) implies that $\delta_k \to 0$ as $k \to \infty$.

Let us now prove (4.4). By (2.15) we can rewrite the inequality in (4.5) as

$$\begin{aligned} &\mathcal{E}(\alpha_k^h, e_k^h, \mathrm{curl}\, p_k^h) - \int_\Omega \varrho(t_k^h) \cdot e_k^h \,\mathrm{d}x - \int_\Omega \varrho_D(t_k^h) \cdot p_k^h \,\mathrm{d}x + \mathcal{H}(\alpha_k^h, p_k^h - p_k^{h-1}) \\ &\quad\le \mathcal{E}(\alpha_k^{h-1}, e_k^{h-1}, \mathrm{curl}\, p_k^{h-1}) + \mathcal{Q}_1(\alpha_k^{h-1}, \mathrm{E}w_k^h - \mathrm{E}w_k^{h-1}) + \langle \mathbb{C}(\alpha_k^{h-1}) e_k^{h-1}, \mathrm{E}w_k^h - \mathrm{E}w_k^{h-1} \rangle \\ &\quad\quad - \int_\Omega \varrho(t_k^h) \cdot (e_k^{h-1} + \mathrm{E}w_k^h - \mathrm{E}w_k^{h-1}) \,\mathrm{d}x - \int_\Omega \varrho_D(t_k^h) \cdot p_k^{h-1} \,\mathrm{d}x . \end{aligned}$$

By the absolute continuity of w and ϱ

$$\begin{aligned} &\mathcal{E}(\alpha_k^h, e_k^h, \mathrm{curl}\, p_k^h) - \int_\Omega \varrho(t_k^h) \cdot e_k^h \mathrm{d}x + \mathcal{H}(\alpha_k^h, p_k^h - p_k^{h-1}) - \int_\Omega \varrho_D(t_k^h) \cdot (p_k^h - p_k^{h-1}) \,\mathrm{d}x \\ &\quad\le \mathcal{E}(\alpha_k^{h-1}, e_k^{h-1}, \mathrm{curl}\, p_k^{h-1}) - \int_\Omega \varrho(t_k^{h-1}) \cdot e_k^{h-1} \,\mathrm{d}x - \int_{t_k^{h-1}}^{t_k^h} \int_\Omega \dot\varrho(s) \cdot e_k(s) \,\mathrm{d}x \,\mathrm{d}s \\ &\quad\quad - \int_{t_k^{h-1}}^{t_k^h} \langle \varrho(t_k^h), \mathrm{E}\dot w(s) \rangle \,\mathrm{d}s + \int_{t_k^{h-1}}^{t_k^h} \langle \sigma_k(s), \mathrm{E}\dot w(s) \rangle \,\mathrm{d}s + \omega_{k,h} \end{aligned}$$

with $\omega_{k,h} := \mathcal{Q}_1(\alpha_k^{h-1}, \mathrm{E}w_k^h - \mathrm{E}w_k^{h-1}) \to 0$ as $k \to \infty$. Let $t \in [t_k^i, t_k^{i+1})$; summing up for $h = 1, \dots, i$ we get

$$\begin{aligned} &\mathcal{E}(\alpha_k(t), e_k(t), \mathrm{curl}\, p_k(t)) - \int_\Omega \varrho(t_k^i) \cdot e_k(t) \mathrm{d}x + \sum_{h=1}^{i} \Big[\mathcal{H}(\alpha_k^h, p_k^h - p_k^{h-1}) - \int_\Omega \varrho_D(t_k^h) \cdot (p_k^h - p_k^{h-1}) \,\mathrm{d}x \Big] \\ &\quad\le \mathcal{E}(\alpha_0, e_0, \mathrm{curl}\, p_0) - \int_\Omega \varrho(0) \cdot e_0 \,\mathrm{d}x - \int_0^{t_k^h} \int_\Omega \dot\varrho(s) \cdot e_k(s) \,\mathrm{d}x \,\mathrm{d}s - \int_0^{t_k^h} \langle \bar\varrho_k(s), \mathrm{E}\dot w(s) \rangle \,\mathrm{d}s \\ &\quad\quad + \int_0^{t_k^h} \langle \sigma_k(s), E\dot w(s) \rangle \,\mathrm{d}s + \omega_k \end{aligned}$$

with $\bar\varrho_k(s) = \varrho(t_k^j)$ if $s \in (t_k^{j-1}, t_k^j]$ and $\omega_k = \sum_{h=1}^i \omega_{k,h} \to 0$ as $k \to \infty$. By (2.17) we obtain the estimate

$$\sum_{h=1}^{i} \Big[\mathcal{H}(\alpha_k^h, p_k^h - p_k^{h-1}) - \int_\Omega \varrho_D(t_k^h) \cdot (p_k^h - p_k^{h-1}) \,\mathrm{d}x \Big] \ge C(c_0, l, S_2(0)) \mathcal{V}(p_k; 0, t) .$$

Therefore $\|e_k(t)\|_2$ is uniformly bounded in k and t by the hypoteses on $\mathcal{Q}_1$ and the regularity assumptions on the external loading; hence $\alpha_k(t)$, $\mathcal{V}(p_k; 0, t)$, and curl $p_k(t)$ are bounded as well. Finally, also $u_k(t)$ is bounded by Korn's inequality. This concludes the proof. □

The following lemma shows (in the spirit of [10, Theorem 4.7]) that in order to prove that an evolution satisfies Definition 2.3, it is sufficient to verify the irreversibility and the global stability condition (qs0), (qs1), and (qs2) as an inequality.

Lemma 4.3. *Let* $(\alpha, u, e, p)\colon [0,T] \to H^1(\Omega; [0,1]) \times W^{1,\frac{n}{n-1}}(\Omega; \mathbb{R}^n) \times L^2(\Omega; \mathbb{M}^{n\times n}_{sym}) \times BV(\Omega; \mathbb{M}^{n\times n}_D)$ *be such that the conditions* (qs0) *and* (qs1) *of Definition 2.3 hold. Moreover assume that* p *is a function with bounded variation from* $[0,T]$ *into* $BV(\Omega; \mathbb{M}^{n\times n}_D)$ *and that for every* $t \in [0,T]$

$$\begin{aligned} &\mathcal{E}(\alpha(t), e(t), \mathrm{curl}\, p(t)) - \langle \mathcal{L}(t), u(t) \rangle + \mathcal{V}_{\mathcal{H}}(\alpha, p; 0, t) \\ &\quad\le \mathcal{E}(\alpha(0), e(0), \mathrm{curl}\, p(0)) - \langle \mathcal{L}(0), u(0) \rangle + \int_0^t \langle \sigma(s), E\dot w(s) \rangle \,\mathrm{d}s \\ &\quad\quad - \int_0^t \langle \dot{\mathcal{L}}(s), u(s) \rangle \,\mathrm{d}s - \int_0^t \langle \mathcal{L}(s), \dot w(s) \rangle \,\mathrm{d}s , \end{aligned} \tag{4.6}$$

where $\sigma(s) := \mathbb{C}(\alpha(s))e(s)$. *Then* (α, u, e, p) *is a quasistatic evolution for the Gurtin-Anand model coupled with damage.*

Proof. Let us fix $t \in [0,T]$ and let us define $s_k^i := \frac{i}{k}t$ for every $k \in \mathbb{N}$ and $i = 0, 1, \dots, k$. For given k and i we set $u := u(s_k^i) - w(s_k^i) + w(s_k^{i-1})$ and $e := e(s_k^i) - \mathrm{E}w(s_k^i) + \mathrm{E}w(s_k^{i-1})$; from the fact that $(\alpha(s_k^i), u, e, p(s_k^i)) \in \mathcal{A}(\alpha(s_k^{i-1}), w(s_k^{i-1}))$, the global stability condition (qs1) implies

$$\mathcal{E}(\alpha(s_k^{i-1}),\, e(s_k^{i-1}), \operatorname{curl} p(s_k^{i-1})) - \langle \mathcal{L}(s_k^{i-1}), u(s_k^{i-1})\rangle \le \mathcal{E}(\alpha(s_k^i), e(s_k^i), \operatorname{curl} p(s_k^i)) - \langle \mathcal{L}(s_k^{i-1}), u\rangle$$
$$+ \mathcal{Q}_1(\alpha(s_k^i), \mathrm{E}w(s_k^{i-1}) - \mathrm{E}w(s_k^i)) - \langle \sigma(s_k^i), \mathrm{E}w(s_k^i) - \mathrm{E}w(s_k^{i-1})\rangle + \mathcal{H}(\alpha(s_k^i), p(s_k^i) - p(s_k^{i-1}))\,.$$

This inequality can be rewritten as

$$\mathcal{E}(\alpha(s_k^{i-1}),\, e(s_k^{i-1}), \operatorname{curl} p(s_k^{i-1})) - \langle \mathcal{L}(s_k^{i-1}), u(s_k^{i-1})\rangle + \int_{s_k^{i-1}}^{s_k^i} \langle \overline{\sigma}_k(s), E\dot{w}(s)\rangle \mathrm{d}s$$
$$- \int_{s_k^{i-1}}^{s_k^i} \langle \dot{\mathcal{L}}(s), \overline{u}_k(s)\rangle \mathrm{d}s - \int_{s_k^{i-1}}^{s_k^i} \langle \overline{\mathcal{L}}_k(s), \dot{w}(s)\rangle \mathrm{d}s + \overline{\delta}_{k,i}$$
$$\le \mathcal{E}(\alpha(s_k^i), e(s_k^i), \operatorname{curl} p(s_k^i)) - \langle \mathcal{L}(s_k^i), u(s_k^i)\rangle + \mathcal{H}(\alpha(s_k^i), p(s_k^i) - p(s_k^{i-1}))\,,$$

where for $s \in (s_k^{i-1}, s_k^i]$ we define

$$\overline{u}_k(s) := u(s_k^i), \quad \overline{\sigma}_k(s) := \sigma(s_k^i), \quad \overline{\mathcal{L}}_k(s) := \mathcal{L}(s_k^i)$$

and

$$\overline{\delta}_{k,i} := -\mathcal{Q}_1(\alpha(s_k^i), \mathrm{E}w(s_k^{i-1}) - \mathrm{E}w(s_k^i)) - \langle \mathcal{L}(s_k^i) - \mathcal{L}(s_k^{i-1}), w(s_k^i) - w(s_k^{i-1})\rangle\,.$$

Since $\sum_i \mathcal{H}(\alpha(s_k^i), p(s_k^i) - p(s_k^{i-1})) \le \mathcal{V}_{\mathcal{H}}(\alpha, p; 0, t)$, iterating the last inequality for $1 \le i \le k$ we obtain

$$\begin{aligned}&\mathcal{E}(\alpha(0), e(0), \operatorname{curl}\, p(0)) - \langle \mathcal{L}(0), u(0)\rangle + \int_0^t \langle \overline{\sigma}_k(s), E\dot{w}(s)\rangle \mathrm{d}s - \int_0^t \langle \dot{\mathcal{L}}(s), \overline{u}_k(s)\rangle \mathrm{d}s\\ &- \int_0^t \langle \overline{\mathcal{L}}_k(s), \dot{w}(s)\rangle \mathrm{d}s + \overline{\delta}_k \le \mathcal{E}(\alpha(t), e(t), \operatorname{curl}\, p(t)) - \langle \mathcal{L}(t), u(t)\rangle + \mathcal{V}_{\mathcal{H}}(\alpha, p; 0, t)\,,\end{aligned} \tag{4.7}$$

where $\overline{\delta}_k := \sum_{i=1}^k \overline{\delta}_{k,i} \to 0$ as $k \to \infty$. Lemma 4.1 implies that $\overline{\sigma}_k(s) \to \sigma(s)$ in $L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$ and $\overline{u}_k(s) \to u(s)$ in $W^{1,\frac{n}{n-1}}(\Omega; \mathbb{R}^n)$ for a.e. $s \in (0,t)$. Taking into account the continuity in time of the external loading and using the Dominated Convergence Theorem, the inequality (4.7) passes to the limit as $k \to \infty$ and we deduce that

$$\mathcal{E}(\alpha(0), e(0), \operatorname{curl}\, p(0)) - \langle \mathcal{L}(0), u(0)\rangle + \int_0^t \langle \sigma(s), E\dot{w}(s)\rangle\, \mathrm{d}s - \int_0^t \langle \dot{\mathcal{L}}(s), u(s)\rangle\, \mathrm{d}s$$
$$- \int_0^t \langle \mathcal{L}(s), \dot{w}(s)\rangle\, \mathrm{d}s \le \mathcal{E}(\alpha(t), e(t), \operatorname{curl}\, p(t)) - \langle \mathcal{L}(t), u(t)\rangle + \mathcal{V}_{\mathcal{H}}(\alpha, p; 0, t)\,.$$

Then the energy balance (qs2) is proved. □

In the following theorem we prove that the piecewise constant interpolants defined in (4.3) converge in a suitable sense, up to subsequences, to a quasistatic evolution for the Gurtin-Anand model coupled with damage.

Theorem 4.4. *In the hypoteses of Theorem 2.5, for every* $k \in \mathbb{N}$ *let* $(\alpha_k, u_k, e_k, p_k)$ *be the evolution defined in* (4.3)*. Then there exist a subsequence (not relabeled) and a quasistatic evolution* (α, u, e, p) *for the Gurtin-Anand model coupled with damage such that* $(\alpha(0), u(0), e(0), p(0)) =$

$(\alpha_0, u_0, e_0, p_0)$ *and for every* $t \in [0,T]$

$$\alpha_k(t) \to \alpha(t) \qquad \text{in } H^1(\Omega)\,, \tag{4.8a}$$

$$u_k(t) \rightharpoonup u(t) \qquad \text{in } W^{1,\frac{n}{n-1}}(\Omega;\mathbb{R}^n)\,, \tag{4.8b}$$

$$e_k(t) \to e(t) \qquad \text{in } L^2(\Omega;\mathbb{M}^{n\times n}_{sym})\,, \tag{4.8c}$$

$$p_k(t) \rightharpoonup p(t) \qquad \text{in } BV(\Omega;\mathbb{M}^{n\times n}_D)\,, \tag{4.8d}$$

$$\operatorname{curl} p_k(t) \to \operatorname{curl} p(t) \qquad \text{in } L^2(\Omega;\mathbb{M}^{n\times n}_{sym})\,. \tag{4.8e}$$

Proof. Since the functions α_k are nonincreasing in time and $\alpha_k(t,x) \in [0,1]$, we get that the α_k are uniformly bounded in $BV(0,T;L^1(\Omega))$. Moreover, by the a priori estimates (4.4), $\|\alpha_k(t)\|_{1,2} \le C$ for every k and t. Therefore we can apply the generalized version of the classical Helly Theorem given in [11, Helly Theorem] to conclude that there exist a subsequence (not relabeled) and a function $\alpha\colon [0,T] \to H^1(\Omega;[0,1])$ nonincreasing in time such that $\alpha_k(t) \rightharpoonup \alpha(t)$ in $H^1(\Omega)$ for every $t \in [0,T]$. By (4.4) it also follows that $\mathcal{V}(p_k;0,T) \le C$ for every k; then [10, Lemma 7.2] implies the existence of $p \in BV(0,T;BV(\Omega;\mathbb{M}^{n\times n}_D))$ such that the convergence (4.8d) holds up to a subsequence. The uniform bound in $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$ for the $\operatorname{curl} p_k$ gives also that curl $p_k(t) \rightharpoonup$ curl $p(t)$ in $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$.

Let us fix $t \in [0,T]$. The a priori estimates on u_k and e_k imply that there exist two functions $\hat{u} \in W^{1,\frac{n}{n-1}}(\Omega;\mathbb{R}^n)$ and $\hat{e} \in L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$, and an increasing sequence $(k_j)_j$ (possibly depending on t) such that $u_{k_j}(t) \rightharpoonup \hat{u}$ in $W^{1,\frac{n}{n-1}}(\Omega;\mathbb{R}^n)$ and $e_{k_j}(t) \rightharpoonup \hat{e}$ in $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$. By Theorem 3.6, the global stability condition $(\mathrm{qs1})_k$ proved in Proposition 4.2 for the approximate evolutions passes to the limit, so the quadruple $(\alpha(t), \hat{u}, \hat{e}, p(t))$ is a solution to the minimization problem

$$\operatorname{argmin}\, \{\mathcal{E}(\beta,\eta,\operatorname{curl}\, q) - \langle \mathcal{L}(t), v\rangle + \mathcal{H}(\beta, q-p(t)) :\, (\beta,v,\eta,q) \in \mathcal{A}(\alpha(t), w(t))\}\,.$$

In particular $(\hat{u},\hat{e})$ minimizes the functional $(u,e) \mapsto \mathcal{Q}_1(\alpha(t),e) - \langle \mathcal{L}(t), u\rangle$, which is strictly convex in e, on the convex set $K := \{(u,e) :\, (u,e,p(t)) \in A(w(t))\}$. Then $(\hat{u},\hat{e})$ is uniquely determined, using also Korn's inequality; if we define $(u(t),e(t)) := (\hat{u},\hat{e})$, we obtain that (4.8b) holds and that $e_k(t) \rightharpoonup e(t)$ in $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$, without passing to further subsequences.

By construction, the quadruple (α,u,e,p) satisfies the conditions (qs0), (qs1) in Definition 2.3, and $p \in BV(0,T;BV(\Omega;\mathbb{M}^{n\times n}_D))$. By Lemma 4.3, it is enough to show the inequality (4.6) for every $t \in [0,T]$ in order to conclude that (α,u,e,p) is a quasistatic evolution for the Gurtin-Anand model coupled with damage.

Let us then fix $t \in [0,T]$ and consider the discrete inequality $(\mathrm{qs2})_k$ in Proposition 4.2 given by

$$\begin{aligned}\mathcal{E}(\alpha_k(t), e_k(t), \operatorname{curl}\, p_k(t)) - \langle \mathcal{L}_k(t), u_k(t)\rangle + \mathcal{V}_{\mathcal{H}}(\alpha_k, p_k; 0, t) \le \mathcal{E}(\alpha_0, e_0, \operatorname{curl}\, p_0) - \langle \mathcal{L}(0), u_0\rangle\\ + \int_0^{t_k^i} \langle \sigma_k(s), E\dot{w}(s)\rangle\,\mathrm{d}s - \int_0^{t_k^i} \langle \dot{\mathcal{L}}(s), u_k(s)\rangle\,\mathrm{d}s - \int_0^{t_k^i} \langle \mathcal{L}_k(s), \dot{w}(s)\rangle\,\mathrm{d}s + \delta_k\,.\end{aligned}$$

By the approximation properties already shown, the fact that $\mathcal{L}_k(t) \to \mathcal{L}(t)$ strongly in $(W^{1,\frac{n}{n-1}}(\Omega;\mathbb{R}^n))'$, and the Dominated Convergence Theorem, the right-hand side converges to the right-hand side of (qs2) and

$$\langle \mathcal{L}_k(t), u_k(t)\rangle \to \langle \mathcal{L}(t), u(t)\rangle \tag{4.9}$$

as $k \to \infty$. On the other hand, from the lower semicontinuity of $\mathcal{H}$ proved in Lemma 3.3 and the definition of plastic dissipation (2.13) it follows that

$$\mathcal{V}_{\mathcal{H}}(\alpha, p; 0, t) \le \liminf_{k\to\infty} \mathcal{V}_{\mathcal{H}}(\alpha_k, p_k; 0, t)\,. \tag{4.10}$$

Moreover the weak lower semicontinuity of the energetic terms implies that

$$\mathcal{E}(\alpha(t), e(t), \operatorname{curl}\, p(t)) \le \liminf_{k\to\infty} \mathcal{E}(\alpha_k(t), e_k(t), \operatorname{curl}\, p_k(t))\,. \tag{4.11}$$

By (4.9), (4.10), and (4.11), we can pass to the limit in $(\mathrm{qs2})_k$ and deduce (4.6) and the existence result. Furthermore, we obtain the convergence of the total energy and thus, again by lower semicontinuity,

$$\|\alpha(t)\|_{1,2} = \lim_{k\to\infty} \|\alpha_k(t)\|_{1,2}\,,$$
$$\mathcal{Q}_1(\alpha(t), e(t)) = \lim_{k\to\infty} \mathcal{Q}_1(\alpha_k(t), e_k(t))\,,$$
$$\mathcal{Q}_2(\alpha(t), \operatorname{curl} p(t)) = \lim_{k\to\infty} \mathcal{Q}_2(\alpha_k(t), \operatorname{curl} p_k(t))\,,$$

and then (4.8a), (4.8c), (4.8e), by strict convexity. This concludes the proof. □

The main existence result, Theorem 2.5, is now a consequence of the previous theorem.

5. Properties of quasistatic evolutions and classical formulation

In this section we study the connection between the energetic formulation for the Gurtin-Anand model coupled with damage, given in Definition 2.3, and the strong formulation of the model, described in the Introduction. We shall prove that the classical balance equations (sf2) and the constraint condition (sf4) are satisfied during every evolution. Moreover, under additional regularity assumptions, also the flow rule (sf5) holds almost everywhere in space and time, and the evolution of damage is governed by the Kuhn-Tucker type conditions (sf6).

In the following we assume that (α, u, e, p) is a quasistatic evolution for the Gurtin-Anand model coupled with damage, according to Definition 2.3. For every $t \in [0,T]$ let $\mathbb{K}^p_{\mathrm{en}}(t) \in \mathbb{M}^{n\times n\times n}_D$ be given by

$$\mathbb{K}^p_{\mathrm{en}}(t)\cdot \nabla A = \mu(\alpha(t))L^2 \operatorname{curl} p(t)\cdot \operatorname{curl} A \quad \text{for every } \mathbb{M}^{n\times n}_{sym}\text{-valued function } A\,, \tag{5.1}$$

and let $\sigma(t) := \mathbb{C}(\alpha(t))e(t)$.

As in perfect plasticity [10], the balance equations for the Cauchy stress σ easily follow from the global stability condition (qs1), computing the corresponding Euler equation. By Lemma 3.4 we get that for every $t \in [0,T]$ and every $(v, \eta, q) \in A(0)$

$$\Big|\langle \sigma(t), \eta\rangle + \langle L^2\mu(\alpha(t)) \operatorname{curl} p(t), \operatorname{curl} q\rangle - \langle \mathcal{L}(t), v\rangle\Big| \leq \mathcal{H}(\alpha(t), q)\,, \tag{5.2}$$

and then

$$\begin{cases} -\operatorname{div}\, \sigma(t) = f(t) & \text{in } \Omega\,,\\ \sigma(t)\nu = g(t) & \text{on } \partial_N\Omega\,. \end{cases}$$

Following [16], we now characterize the plastic potential $\mathcal{H}$ as the supremum of certain duality products. A similar type of characterization for the plastic potential is given also in perfect plasticity (cf. [33, Corollary 3.8] and [8, equation (2.23)]). In view of the dependence of $\mathcal{H}$ on the damage α, we have to introduce the closed space of measures that vanishes on sets with 2-capacity zero, which was not useful in [16].

Lemma 5.1. *Let us define the closed linear subspace of* $M_b(\Omega;\mathbb{M}^{n\times n\times n}_D)$

$$M^2_b(\Omega;\mathbb{M}^{n\times n\times n}_D) := \{\mu \in M_b(\Omega;\mathbb{M}^{n\times n\times n}_D)\colon \mu(E) = 0 \ \text{ if } C_2(E) = 0\}\,,$$

where we recall that $C_2(E)$ *is the 2-capacity of the set* E, *and let us set for every* $\alpha \in H^1(\Omega)$

$$\begin{aligned}\mathcal{K}_\alpha(\Omega) := \Big\{&(A,\mathbb{B},\mathbb{L}) \in L^\infty(\Omega;\mathbb{M}^{n\times n}_D)\times L^\infty(\Omega;\mathbb{M}^{n\times n\times n}_D)\times (M^2_b(\Omega;\mathbb{M}^{n\times n\times n}_D))'\colon\\ &\frac{|A(x)|^2}{S_1(\alpha(x))^2} + \frac{|\mathbb{B}(x)|^2}{l^2 S_2(\alpha(x))^2} \leq 1 \ \text{a.e. in } \Omega,\ |\langle \mathbb{L},\mu\rangle| \leq l\int_\Omega S_2(\widetilde{\alpha})\,\mathrm{d}|\mu|\ \forall \mu\Big\}\,.\end{aligned} \tag{5.3}$$

Then for every $\alpha \in H^1(\Omega)$ *and* $p \in BV(\Omega;\mathbb{M}^{n\times n}_D)$

$$\mathcal{H}(\alpha, p) = \sup_{(A,\mathbb{B},\mathbb{L})\in\mathcal{K}_\alpha(\Omega)} \langle (A,\mathbb{B},\mathbb{L}), (p, \nabla p, \mathrm{D}^s p)\rangle\,, \tag{5.4}$$

where $\langle (A,\mathbb{B},\mathbb{L}), (p,\nabla p,\mathrm{D}^s p)\rangle := \langle A, p\rangle + \langle \mathbb{B}, \nabla p\rangle + \langle \mathbb{L}, \mathrm{D}^s p\rangle$ *is the duality pairing between* $L^1(\Omega;\mathbb{M}^{n\times n}_D)\times L^1(\Omega;\mathbb{M}^{n\times n\times n}_D)\times M^2_b(\Omega;\mathbb{M}^{n\times n\times n}_D)$ *and its dual space.*

Proof. Let us fix $\alpha \in H^1(\Omega)$ and consider the function
$$\mathcal{F}(\alpha;\cdot,\cdot,\cdot)\colon L^1(\Omega;\mathbb{M}^{n\times n}_D)\times L^1(\Omega;\mathbb{M}^{n\times n\times n}_D)\times M^2_b(\Omega;\mathbb{M}^{n\times n\times n}_D)\to[0,+\infty[$$
defined by
$$\mathcal{F}(\alpha;A',\mathbb{B}',\mathbb{L}')=\int_\Omega\sqrt{S_1(\alpha)^2|A'|^2+l^2S_2(\alpha)^2|\mathbb{B}'|^2}\,\mathrm{d}x+l\int_\Omega S_2(\widetilde{\alpha})\,\mathrm{d}|\mathbb{L}'|\,.$$
This definition is well posed because $\widetilde{\alpha}\in L^\infty(\Omega;\mathbb{L})$ for every $\mathbb{L}\in M^2_b(\Omega;\mathbb{M}^{n\times n\times n}_D)$, and
$$\mathcal{H}(\alpha,p)=\mathcal{F}(\alpha;p,\nabla p,\mathrm{D}^s p)$$
for every $p\in BV(\Omega;\mathbb{M}^{n\times n}_D)$.

Since $\mathcal{F}(\alpha;\cdot,\cdot,\cdot)$ is strongly continuous and convex we have $\mathcal{F}(\alpha;\cdot,\cdot,\cdot)=\mathcal{F}(\alpha;\cdot,\cdot,\cdot)^{**}$, where * is the symbol for the Fenchel transformation. Moreover, using the fact that
$$\xi_1\cdot\xi_2+\zeta_1\cdot\zeta_2=:(\xi_1,\zeta_1)\cdot(\xi_2,\zeta_2)\le\sqrt{\varepsilon^2|\xi_1|^2+\delta^2|\zeta_1|^2}\sqrt{\varepsilon^{-2}|\xi_2|^2+\delta^{-2}|\zeta_2|^2}\tag{5.5}$$
for every $\varepsilon,\delta>0$, ξ_1, $\xi_2\in\mathbb{R}^d$, ζ_1, $\zeta_2\in\mathbb{R}^m$, with the equality if and only if $\xi_1=C\delta^2\xi_2$ and $\zeta_1=C\varepsilon^2\zeta_2$ for any $C>0$, it is not difficult to show that $\mathcal{F}^*(\alpha;\cdot,\cdot,\cdot)$ is the indicator function of the set $\mathcal{K}_\alpha(\Omega)$. Therefore we deduce that $\mathcal{F}(\alpha;\cdot,\cdot,\cdot)$ is the Fenchel transform of the indicator of $\mathcal{K}_\alpha(\Omega)$, that gives (5.4). ☐

We now derive the existence of three higher order stresses conjugated to $p(t)$, $\nabla p(t)$, $\mathrm{D}^s p(t)$ for every t, and prove that they satisfy the constitutive relations and the constraint condition (sf4) in the classical formulation.

Proposition 5.2. *For every $t\in[0,T]$ there exists a triple $(\sigma^p(t),\mathbb{K}^p_{\mathrm{diss}}(t),\mathbb{S}^p(t))\in\mathcal{K}_{\alpha(t)}(\Omega)$ such that, setting $\mathbb{K}^p(t):=\mathbb{K}^p_{\mathrm{en}}(t)+\mathbb{K}^p_{\mathrm{diss}}(t)$, it holds the following*
$$\langle\sigma(t),\eta\rangle+\langle\sigma^p(t),q\rangle+\langle\mathbb{K}^p(t),\nabla q\rangle+\langle\mathbb{S}^p(t),\mathrm{D}^s q\rangle=\langle\mathcal{L}(t),v\rangle\quad\text{for every }(v,\eta,q)\in A(0)\,,\tag{5.6}$$
which implies the balance equations
$$\begin{cases}\sigma^p(t)=\sigma_D(t)+\operatorname{div}\mathbb{K}^p(t) & \text{in }\Omega\,,\\ \mathbb{K}^p(t)\nu=0 & \text{on }\partial\Omega\,.\end{cases}\tag{5.7}$$

Proof. Let us fix $t\in[0,T]$. From the inequality (5.2) we can deduce that the linear functional
$$A(0)\ni(v,\eta,q)\mapsto\langle\sigma(t),\eta\rangle+\langle L^2\mu(\alpha(t))\operatorname{curl}p(t),\operatorname{curl}q\rangle-\langle\mathcal{L}(t),v\rangle\,,$$
depends only on q. Indeed, since $A(0)$ is a linear space, if both (v_1,η_1,q) and (v_2,η_2,q) belong to $A(0)$ we have $(v_1-v_2,\eta_1-\eta_2,0)\in A(0)$ and then $\langle\sigma(t),\eta_1-\eta_2\rangle-\langle\mathcal{L}(t),v_1-v_2\rangle=0$. We can thus consider the linear functional
$$\varphi(q,\nabla q,\mathrm{D}^s q):=\langle\sigma(t),\eta\rangle+\langle L^2\mu(\alpha(t))\operatorname{curl}p(t),\operatorname{curl}q\rangle-\langle\mathcal{L}(t),v\rangle\tag{5.8}$$
defined on the linear subspace of $L^1(\Omega;\mathbb{M}^{n\times n}_D)\times L^1(\Omega;\mathbb{M}^{n\times n\times n}_D)\times M^2_b(\Omega;\mathbb{M}^{n\times n\times n}_D)$
$$X=\{(q,\nabla q,\mathrm{D}^s q)\colon(v,\eta,q)\in A(0)\text{ for some }v\in W^{1,\frac{n}{n-1}}(\Omega;\mathbb{R}^n),\eta\in L^2(\Omega;\mathbb{M}^{n\times n}_{sym})\}\,.$$
By the Hahn-Banach Theorem for seminorms (see [12, Theorem 5.7]), we can extend in a continuous way φ to the whole $L^1(\Omega;\mathbb{M}^{n\times n}_D)\times L^1(\Omega;\mathbb{M}^{n\times n\times n}_D)\times M^2_b(\Omega;\mathbb{M}^{n\times n\times n}_D)$ keeping the constraint condition in (5.2):
$$|\varphi(A,\mathbb{B},\mathbb{L})|\le\int_\Omega\sqrt{S_1(\alpha(t))^2|A|^2+l^2S_2(\alpha(t))^2|\mathbb{B}|^2}\,\mathrm{d}x+l\int_\Omega S_2(\widetilde{\alpha}(t))\,\mathrm{d}|\mathbb{L}|\tag{5.9}$$
for every $(A,\mathbb{B},\mathbb{L})\in L^1(\Omega;\mathbb{M}^{n\times n}_D)\times L^1(\Omega;\mathbb{M}^{n\times n\times n}_D)\times M^2_b(\Omega;\mathbb{M}^{n\times n\times n}_D)$. Since φ is linear and bounded there exist $\sigma^p(t)\in L^\infty(\Omega;\mathbb{M}^{n\times n}_D)$, $\mathbb{K}^p_{\mathrm{diss}}(t)\in L^\infty(\Omega;\mathbb{M}^{n\times n\times n}_D)$, and $\mathbb{S}^p(t)\in(M^2_b(\Omega;\mathbb{M}^{n\times n\times n}_D))'$ such that
$$\varphi(A,\mathbb{B},\mathbb{L})=-\langle\sigma^p(t),A\rangle-\langle\mathbb{K}^p_{\mathrm{diss}}(t),\mathbb{B}\rangle-\langle\mathbb{S}^p(t),\mathbb{L}\rangle\,.$$
Therefore, choosing $(A,\mathbb{B},0)$ and $(0,0,\mathbb{L})$ in (5.9) we get that $(\sigma^p(t),\mathbb{K}^p_{\mathrm{diss}}(t),\mathbb{S}^p(t))\in\mathcal{K}_{\alpha(t)}(\Omega)$ (recall (5.5) and the definition (5.3)). Moreover, by (5.8) it follows that
$$\langle\sigma(t),\eta\rangle+\langle L^2\mu(\alpha(t))\operatorname{curl}p(t),\operatorname{curl}q\rangle-\langle\mathcal{L}(t),v\rangle=-\langle\sigma^p(t),q\rangle-\langle\mathbb{K}^p_{\mathrm{diss}}(t),\nabla q\rangle-\langle\mathbb{S}^p(t),\mathrm{D}^s q\rangle\tag{5.10}$$

for every $(v,\eta,q)\in A(0)$. Hence (5.6) follows recalling the definition of $\mathbb{K}^p_{\rm en}(t)$.

In order to show (5.7) let us consider $q\in C^\infty(\overline{\Omega};\mathbb{M}^{n\times n}_D)$ and choose $(0,-q,q)\in A(0)$ in (5.6). We obtain that

$$-\langle\sigma(t),q\rangle+\langle\sigma^p(t),q\rangle+\langle\mathbb{K}^p(t),\nabla q\rangle=0\,.$$

Since $q(x)\in\mathbb{M}^{n\times n}_D$ for every x, we can replace $\sigma(t)$ by $\sigma_D(t)$ and rewrite the inequality above as

$$\langle\sigma^p(t)-\sigma_D(t),q\rangle+\langle\mathbb{K}^p(t),\nabla q\rangle=0\,. \tag{5.11}$$

The former equation in (5.7) follows immediately; as for the latter, it is enough to integrate by parts, taking into account that the normal trace of $\mathbb{K}^p(t)$ on $\partial\Omega$ is in $H^{-1/2}(\partial\Omega;\mathbb{R}^{n\times n})$ since $\mathbb{K}^p(t)\in L^2(\Omega;\mathbb{M}^{n\times n\times n}_D)$ with divergence in $L^2(\Omega;\mathbb{M}^{n\times n}_D)$ by (5.1), (5.11), and the fact that $(\sigma^p(t),\mathbb{K}^p_{\rm diss}(t),\mathbb{S}^p(t))\in\mathcal{K}_{\alpha(t)}(\Omega)$. Hence (5.7) is proved and the proof is concluded. □

The classical flow rule (sf5) and the Kuhn-Tucker condition for the evolution of the damage can be derived by differentiating the energy balance equation (qs2); therefore some regularity assumptions are needed both on the constitutive coefficients and on the evolution. For instance, if α and p are absolutely continuous from $[0,T]$ respectively into $C(\overline{\Omega})$ and $BV(\Omega;\mathbb{M}^{n\times n}_D)$, then, adapting the argument of [8, Lemma A.4], we have that for every $t\in[0,T]$

$$\mathcal{V}_{\mathcal{H}}(\alpha,p,0,t)=\int_0^t\mathcal{H}(\alpha(s),\dot p(s))\,\mathrm{d}s\,. \tag{5.12}$$

Proposition 5.3 (Kuhn-Tucker conditions and maximum plastic work principle)**.** *Assume that the elastic moduli μ, k in (2.5), and the constitutive functions d, S_1, S_2 are of class C^1. Moreover let α, u, e, p be absolutely continuous from $[0,T]$ into $C(\overline{\Omega})\cap H^1(\Omega)$, $W^{1,\frac{n}{n-1}}(\Omega;\mathbb{R}^n)$, $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$, $BV(\Omega;\mathbb{M}^{n\times n}_D)$, respectively. Then for every $t\in[0,T]$ the functional $C(\overline{\Omega})\cap H^1(\Omega)\ni\beta\mapsto\mathcal{E}(\beta,e(t),\mathrm{curl}\ p(t))$ is differentiable at $\alpha(t)$ with Gâteaux derivative in the direction $\beta\in C(\overline{\Omega})\cap H^1(\Omega)$ given by*

$$\begin{aligned}\langle\partial_\alpha\mathcal{E}(\alpha(t),e(t),\mathrm{curl}\ p(t)),\beta\rangle&=\tfrac12\langle\mathbb{C}'(\alpha(t))\,\beta\,e(t),e(t)\rangle+\tfrac{L^2}{2}\langle\mu'(\alpha(t))\,\beta\,\mathrm{curl}\,p(t),\mathrm{curl}\,p(t)\rangle\\&\quad+\int_\Omega d'(\alpha(t))\,\beta\,\mathrm{d}x+\ell^2\int_\Omega\nabla\alpha(t)\cdot\nabla\beta\,\mathrm{d}x\,.\end{aligned}$$

Moreover

$$\langle\partial_\alpha\mathcal{E}(\alpha(t),e(t),\mathrm{curl}\ p(t)),\beta\rangle\geq0 \tag{5.13}$$

for every $t\in[0,T]$ and every $\beta\in C(\overline{\Omega})\cap H^1(\Omega)$, $\beta\leq0$ in Ω,

$$\langle\partial_\alpha\mathcal{E}(\alpha(t),e(t),\mathrm{curl}\ p(t)),\dot\alpha(t)\rangle=0 \tag{5.14}$$

for a.e. $t\in(0,T)$. Finally, for a.e. $t\in(0,T)$

$$\begin{aligned}\mathcal{H}(\alpha(t),\dot p(t))&=\langle(\sigma^p(t),\mathbb{K}^p_{\rm diss}(t),\mathbb{S}^p(t)),(\dot p(t),\nabla\dot p(t),\mathrm{D}^s\dot p(t))\rangle\\&=\langle\sigma^p(t),\dot p(t)\rangle+\langle\mathbb{K}^p_{\rm diss}(t),\nabla\dot p(t)\rangle+\langle\mathbb{S}^p(t),\mathrm{D}^s\dot p(t)\rangle\,,\end{aligned} \tag{5.15}$$

where $(\sigma^p(t),\mathbb{K}^p_{\rm diss}(t),\mathbb{S}^p(t))\in\mathcal{K}_{\alpha(t)}(\Omega)$ is given by Proposition 5.2.

Proof. The differentiability of $\beta\mapsto\mathcal{E}(\beta,e(t),\mathrm{curl}\ p(t))$ and the expression of its derivative follow from the regularity assumptions on the constitutive functions and on the evolution. Let $t\in[0,T]$ and $\beta\in C(\overline{\Omega})\cap H^1(\Omega)$, $\beta\leq0$ in Ω. For every $h>0$, considering $(\alpha(t)+h\beta,u(t),e(t),p(t))\in\mathcal{A}(\alpha(t),w(t))$ as a test pair in (qs1), we get

$$\frac{\mathcal{E}(\alpha(t)+h\beta,e(t),\mathrm{curl}\,p(t))-\mathcal{E}(\alpha(t),e(t),\mathrm{curl}\,p(t))}{h}\geq0\,.$$

Letting $h\to0$ we obtain (5.13).

Since the evolution is assumed to be absolutely continuous, we can differentiate with respect to t the energy balance (qs2). Recalling (5.12) we get that for a.e. t

$$\begin{aligned}&\langle\partial_\alpha\mathcal{E}(\alpha(t),e(t),\mathrm{curl}\ p(t)),\dot\alpha(t)\rangle+\langle\sigma(t),\dot e(t)\rangle+L^2\langle\mu(\alpha(t))\,\mathrm{curl}\,p(t),\mathrm{curl}\,\dot p(t)\rangle\\&-\langle\mathcal{L}(t),\dot u(t)\rangle+\mathcal{H}(\alpha(t),\dot p(t))=\langle\sigma(t),E\dot w(t)\rangle-\langle\mathcal{L}(t),\dot w(t)\rangle\,.\end{aligned}$$

It is easy to see that $(\dot u(t)-\dot w(t), \dot e(t)-E\dot w(t), \dot p(t)) \in A(0)$, when it exists; thus, using (5.6) (cf. also (5.10)), the previous inequality gives that for a.e. t

$$\begin{aligned} 0 = \mathcal{H}(\alpha(t),\dot p(t)) - (\langle \sigma^p(t), \dot p(t)\rangle + \langle \mathbb{K}^p_{\text{diss}}(t), \nabla \dot p(t)\rangle + \langle \mathbb{S}^p(t), \mathrm{D}^s \dot p(t)\rangle) \\ + \langle \partial_\alpha \mathcal{E}(\alpha(t), e(t), \operatorname{curl} p(t)), \dot\alpha(t)\rangle \,. \end{aligned} \tag{5.16}$$

Since $(\sigma^p(t), \mathbb{K}^p_{\text{diss}}(t), \mathbb{S}^p(t)) \in \mathcal{K}_{\alpha(t)}(\Omega)$, by (5.4) and (5.13) (recall that $\dot\alpha(t) \le 0$ in Ω) the equality (5.16) implies (5.14) and (5.15) for a.e. t. □

We can interpret the equality (5.15) as a maximum plastic work principle, i.e., the supremum in (5.4) is attained on $(\sigma^p(t), \mathbb{K}^p_{\text{diss}}(t), \mathbb{S}^p(t))$. From this we deduce a weak form of the flow rule, expressed by the following conditions.

Corollary 5.4. *Gathering* (5.4) *and* (5.15) *we get that*

$$\langle \sigma^p(t) - A, \dot p(t)\rangle + \langle \mathbb{K}^p_{\text{diss}}(t) - \mathbb{B}, \nabla \dot p(t)\rangle \ge 0 \tag{5.17a}$$

for every $(A,\mathbb{B}) \in L^\infty(\Omega;\mathbb{M}^{n\times n}_D) \times L^\infty(\Omega;\mathbb{M}^{n\times n\times n}_D)$ *with* $\frac{|A(x)|^2}{S_1(\alpha(t,x))^2} + \frac{|\mathbb{B}(x)|^2}{l^2 S_2(\alpha(t,x))^2} \le 1$ *a.e. in* Ω*, and*

$$\langle \mathbb{S}^p(t) - \mathbb{L}, \mathrm{D}^s \dot p(t)\rangle \ge 0 \tag{5.17b}$$

for every $\mathbb{L} \in (M^2_b(\Omega;\mathbb{M}^{n\times n\times n}_D))'$ *such that* $|\langle \mathbb{L}, \mu\rangle| \le l \int_\Omega S_2(\widetilde\alpha(t))\,\mathrm{d}|\mu|$ *for* $\mu \in M^2_b(\Omega;\mathbb{M}^{n\times n\times n}_D)$.

Indeed both (5.4) *and* (5.15) *hold if and only if*

$$\langle \sigma^p(t) - A, \dot p(t)\rangle + \langle \mathbb{K}^p_{\text{diss}}(t) - \mathbb{B}, \nabla \dot p(t)\rangle + \langle \mathbb{S}^p(t) - \mathbb{L}, \mathrm{D}^s \dot p(t)\rangle \ge 0$$

for every $(A,\mathbb{B},\mathbb{L}) \in \mathcal{K}_{\alpha(t)}(\Omega)$.

We are now ready to prove that the classical flow rule (sf5) holds for a.e. (t,x).

Proposition 5.5 (Flow rule)**.** *In the hypoteses of Proposition 5.3, let* $t \in [0,T]$ *such that* $\dot p(t)$ *and* $\nabla \dot p(t)$ *exist and let* $x \in \Omega$ *be a Lebesgue point for* $\sigma^p(t)$, $\mathbb{K}^p_{\text{diss}}(t)$, $\dot p(t)$ *and* $\nabla \dot p(t)$*. Then the condition*

$$\frac{|\sigma^p(t,x)|^2}{S_1(\alpha(t,x))^2} + \frac{|\mathbb{K}^p_{\text{diss}}(t,x)|^2}{l^2 S_2(\alpha(t,x))^2} < 1$$

implies that

$$(\dot p(t,x), \nabla \dot p(t,x)) = (0,0)\,,$$

while if

$$\frac{|\sigma^p(t,x)|^2}{S_1(\alpha(t,x))^2} + \frac{|\mathbb{K}^p_{\text{diss}}(t,x)|^2}{l^2 S_2(\alpha(t,x))^2} = 1$$

we have

$$\dot p(t,x) = \lambda(t,x) \frac{\sigma^p(t,x)}{S_1(\alpha(t,x))^2}\,, \quad \nabla \dot p(t,x) = \lambda(t,x) \frac{|\mathbb{K}^p_{\text{diss}}(t,x)|}{l^2 S_2(\alpha(t,x))^2}$$

with

$$\lambda(t,x) = \sqrt{S_1(\alpha(t,x))^2 |\dot p(t,x)| + l^2 S_2(\alpha(t,x))^2 |\nabla \dot p(t,x)|^2}\,.$$

Proof. Let us fix t and x satisfying the assumption in the statement, and let us define the convex set

$$C_{t,x} := \left\{ (F_0, \mathbb{G}_0) \in \mathbb{M}^{n\times n}_D \times \mathbb{M}^{n\times n\times n}_D \colon \frac{|F_0|^2}{S_1(\alpha(t,x))^2} + \frac{|\mathbb{G}_0|^2}{l^2 S_2(\alpha(t,x))^2} \le 1 \right\}.$$

By assumption $(\sigma^p(t,x), \mathbb{K}^p_{\text{diss}}(t,x)) \in C_{t,x}$. Given $(F_0, \mathbb{G}_0) \in C_{t,x}$ we set

$$(F(z), \mathbb{G}(z)) := \left(F_0 \frac{S_1(\alpha(t,z))}{S_1(\alpha(t,x))}, \mathbb{G}_0 \frac{S_2(\alpha(t,z))}{S_2(\alpha(t,x))} \right) \quad \text{for every } z \in \Omega\,.$$

Since $\alpha(t) \in C(\overline\Omega)$ we get $(F,\mathbb{G}) \in C(\overline\Omega;\mathbb{M}^{n\times n}_D) \times C(\overline\Omega;\mathbb{M}^{n\times n\times n}_D)$; by construction $(F(x),\mathbb{G}(x)) = (F_0,\mathbb{G}_0)$ and $\frac{|F(z)|^2}{S_1(\alpha(t,z))^2} + \frac{|\mathbb{G}(z)|^2}{l^2 S_2(\alpha(t,z))^2} \le 1$ in Ω. We now fix $r>0$ and test (5.17a) by

$$(A_r, \mathbb{G}_r) := \begin{cases} \frac12 (\sigma^p(t) + F, \mathbb{K}^p_{\text{diss}}(t) + \mathbb{G}) & \text{in } B_r(x) \\ (\sigma^p(t), \mathbb{K}^p_{\text{diss}}(t)) & \text{outside } B_r(x) \end{cases}$$

which is an admissible test function by convexity of the constraint set. Hence we obtain that for every $r>0$

$$\frac{1}{r^n}\left[\int_{B_r(x)}(\sigma^p(t)-F)\cdot\dot p(t)\,\mathrm{d}x+\int_{B_r(x)}(\mathbb{K}^p_{\mathrm{diss}}(t)-\mathbb{G})\cdot\nabla\dot p(t)\,\mathrm{d}x\right]\geq 0\,.$$

As $r\to 0$ we get (recall that x is a Lebesgue point for the functions involved)

$$(F_0-\sigma^p(t,x))\cdot\dot p(t,x)+(\mathbb{G}_0-\mathbb{K}^p_{\mathrm{diss}}(t,x))\cdot\nabla\dot p(t,x)\leq 0\,.$$

Since $(F_0,\mathbb{G}_0)$ is arbitrary in $C_{t,x}$, it follows that $(\dot p(t,x),\nabla\dot p(t,x))$ is in the normal cone to $C_{t,x}$ at $(\sigma^p(t,x),\mathbb{K}^p_{\mathrm{diss}}(t,x))$ and this proves the result. □

6. Asymptotic analysis for vanishing strain gradient effects

In this section we study the relation between the Gurtin-Anand model coupled with damage and the coupled elastoplastic damage model proposed in [8].

In [16] it is proven that quasistatic evolutions for the Gurtin-Anand model converge in a suitable sense, as the strain gradient terms vanish, to evolutions for perfectly plastic bodies in the formulation of [10]. Then we expect, when l,L tend to zero, the convergence of quasistatic evolutions in Definition 2.3 to evolutions for perfectly plastic bodies with damage studied in [8]. Indeed the latter model corresponds, when the damage is constant in time, to the perfect plasticity model for heterogeneous materials in [33]. However, while the classical Gurtin-Anand formulation reduces to von Mises perfect plasticity model by setting l and L equal to zero (recall that l is related to the thickness of the plastic shear bands and L to the energy stored by the geometrically necessary dislocations), in the presence of damage the models have two different gradient damage regularizations, because in [8] and [9] the space continuity of α is needed. Thus we start from a coupled gradient plasticity-damage model with a regularizing term $\|\nabla\alpha\|^\gamma_\gamma$, $\gamma>n$, instead of $\|\nabla\alpha\|^2_2$. Moreover, in the model in [8] there is a term related to a fatigue phenomenon, which depends on a parameter λ. For simplicity, we do not consider here the fatigue and thus we take $\lambda=0$.

For technical reasons (see Remark 6.2) we also require that the only loading is the displacement field w applied to the whole of $\partial\Omega$.

Under this assumptions, Theorem 6.1 shows that evolutions for the Gurtin-Anand model coupled with damage converge weakly for every time to evolutions in [8].

For $l_k\to 0$ and $L_k\to 0$, let

$$\mathcal{E}_k(\beta,\eta,\mathrm{curl}\;q):=\mathcal{Q}_1(\beta,\eta)+\tfrac{L_k^2}{2}\int_\Omega\mu(\beta)|\mathrm{curl}\,q|^2\,\mathrm{d}x+\|\nabla\beta\|^\gamma_\gamma+D(\beta)\,,$$

$$\mathcal{H}_k(\beta,q):=\int_\Omega\sqrt{S_1(\beta)^2|q|^2+l_k^2S_2(\beta)^2|\nabla q|^2}\,\mathrm{d}x+l_k\int_\Omega S_2(\widetilde\beta)\,\mathrm{d}|\mathrm{D}^sq|$$

be the total energy and the plastic dissipation of the Gurtin-Anand model coupled with damage for the length scales $l=l_k$, $L=L_k$, $\ell=\sqrt2$. Moreover let

$$t\mapsto(\alpha_k(t),u_k(t),e_k(t),p_k(t))\in W^{1,\gamma}(\Omega;[0,1])\times W^{1,\frac{n}{n-1}}(\Omega;\mathbb{R}^n)\times L^2(\Omega;\mathbb{M}^{n\times n}_{sym})\times BV(\Omega;\mathbb{M}^{n\times n}_D)$$

be a corresponding quasistatic evolution with the prescribed displacement w. Namely the following conditions hold:

(qs0) *irreversibility* : for every $x\in\Omega$ the function $[0,T]\ni t\mapsto\alpha_k(t,x)$ is nonincreasing;

(qs1) *global stability*: for every $t\in[0,T]$ we have $(u_k(t),e_k(t),p_k(t))\in A(w(t))$ and

$$\mathcal{E}_k(\alpha_k(t),e_k(t),\mathrm{curl}\;p_k(t))\leq\mathcal{E}_k(\beta,\eta,\mathrm{curl}\;q)+\mathcal{H}_k(\beta,q-p_k(t))$$

for every $(\beta,v,\eta,q)\in\mathcal{A}(\alpha(t),w(t))$;

(qs2) *energy balance*: the function $t\mapsto p_k(t)$ from $[0,T]$ into $BV(\Omega;\mathbb{M}^{n\times n}_D)$ has bounded variation and for every $t\in[0,T]$

$$\mathcal{E}_k(\alpha_k(t),e_k(t),\mathrm{curl}\;p_k(t))+\mathcal{V}_{\mathcal{H}_k}(\alpha_k,p_k;0,t)$$
$$=\mathcal{E}_k(\alpha_k(0),e_k(0),\mathrm{curl}\;p_k(0))+\int_0^t\langle\sigma_k(s),E\dot w(s)\rangle\,\mathrm{d}s\,,$$

where $\sigma_k(s) := \mathbb{C}(\alpha_k(s))e_k(s)$.

We now recall the notion of globally stable evolution for the coupled elastoplastic-damage model considered in [8], when the parameter λ therein is zero.

The class of admissible configurations for a given boundary datum $w \in H^1(\Omega;\mathbb{R}^n)$ in perfect plasticity is the set

$$A_{\mathrm{pp}}(w) := \{(u,e,p) \in BD(\Omega) \times L^2(\Omega;\mathbb{M}^{n\times n}_{sym}) \times M_b(\overline{\Omega};\mathbb{M}^{n\times n}_D)\colon \\ \mathrm{E}u = e + p \ \text{ in } \Omega,\, p = (w-u)\odot \nu\, \mathcal{H}^{n-1} \ \text{ on } \partial\Omega\}\,,$$

and we define in analogy to (2.18)

$$\mathcal{A}_{\mathrm{pp}}(\alpha,w) := \{(\beta,u,e,p) :\ \beta \in W^{1,\gamma}(\Omega),\, \beta \le \alpha, \text{ and } (u,e,p) \in A_{\mathrm{pp}}(w)\}\,.$$

Here

$$BD(\Omega) := \{u \in L^1(\Omega;\mathbb{R}^n)\colon \mathrm{E}u \in M_b(\Omega;\mathbb{M}^{n\times n}_{sym})\}\,,$$

endowed with the norm

$$\|u\|_{BD} := \|u\|_1 + \|\mathrm{E}u\|_1\,,$$

is the Banach space of functions with bounded deformation on Ω; for its general properties we refer to [35]. Notice that we use the subscripts "pp" (perfect plasticity with damage) to distinguish objects with analogous meaning in the two models, and that the term $w-u$ appearing in the definition of A_{pp} is intended in the sense of traces on $\partial\Omega$.

For every $\beta \in C(\overline{\Omega})$ and $q \in M_b(\overline{\Omega};\mathbb{M}^{n\times n}_D)$ we set

$$\mathcal{H}_{\mathrm{pp}}(\beta,q) := \int_{\overline{\Omega}} S_1(\beta)\,\mathrm{d}|q|\,,$$

in analogy to $\mathcal{H}$. Here we adopt a multiplicative formulation for the constraint sets (indeed we are in von Mises setting). The plastic dissipation $\mathcal{V}_{\mathcal{H}_{\mathrm{pp}}}(\beta,q)$ is defined in the same way of $\mathcal{V}_{\mathcal{H}}$, starting from $\mathcal{H}_{\mathrm{pp}}$, and the total energy is

$$\mathcal{E}_{\mathrm{pp}}(\beta,\eta) := \mathcal{Q}_1(\beta,\eta) + D(\beta) + \|\nabla\beta\|_\gamma^\gamma\,,$$

with $\mathcal{Q}_1$ and D as in (2.7) and (2.9a).

A quasistatic evolution for the coupled perfect plasticity-damage model is a function

$$[0,T] \ni t \mapsto (\alpha(t),u(t),e(t),p(t)) \in W^{1,\gamma}(\Omega;[0,1]) \times BD(\Omega) \times L^2(\Omega;\mathbb{M}^{n\times n}_{sym}) \times M_b(\overline{\Omega};\mathbb{M}^{n\times n}_D)$$

satisfying the following conditions:

(qs0)$_{\mathrm{pp}}$ *irreversibility* : for every $x \in \Omega$ the function $[0,T] \ni t \mapsto \alpha(t,x)$ is nonincreasing;

(qs1)$_{\mathrm{pp}}$ *global stability*: for every $t \in [0,T]$ we have $(u(t),e(t),p(t)) \in A_{\mathrm{pp}}(w(t))$ and

$$\mathcal{E}_{\mathrm{pp}}(\alpha(t),e(t)) \le \mathcal{E}_{\mathrm{pp}}(\beta,\eta) + \mathcal{H}_{\mathrm{pp}}(\beta,q-p(t))$$

for every $(\beta,v,\eta,q) \in \mathcal{A}_{\mathrm{pp}}(\alpha(t),w(t))$;

(qs2)$_{\mathrm{pp}}$ *energy balance*: the function $t \mapsto p(t)$ from $[0,T]$ into $M_b(\overline{\Omega};\mathbb{M}^{n\times n}_D)$ has bounded variation and for every $t \in [0,T]$

$$\mathcal{E}_{\mathrm{pp}}(\alpha(t),e(t)) + \mathcal{V}_{\mathcal{H}_{\mathrm{pp}}}(\alpha,p;0,t) = \mathcal{E}_{\mathrm{pp}}(\alpha(0),e(0)) + \int_0^t \langle \sigma(s), \mathrm{E}\dot{w}(s)\rangle\,\mathrm{d}s\,,$$

where $\sigma(s) := \mathbb{C}(\alpha(s))e(s)$.

Assuming Ω Lipschitz and (2.2a), (2.4), (2.6), (2.9b), and (2.11), it is proven in [8] that for every initial data $(\alpha_0,u_0,e_0,p_0) \in \mathcal{A}_{\mathrm{pp}}(1,w(0))$ such that

$$\mathcal{E}_{\mathrm{pp}}(\alpha_0,e_0) \le \mathcal{E}_{\mathrm{pp}}(\beta,\eta) + \mathcal{H}_{\mathrm{pp}}(\beta,q-p_0)$$

for every $(\beta,v,\eta,q) \in \mathcal{A}_{\mathrm{pp}}(\alpha_0,w(0))$, there exists a quasistatic evolution for the coupled perfect plasticity-damage model (α,u,e,p) such that $(\alpha(0),u(0),e(0),p(0)) = (\alpha_0,u_0,e_0,p_0)$.

Now we consider the limit as $k \to \infty$, assuming for the initial conditions that

$$\begin{aligned} &\alpha_k(0) \rightharpoonup \alpha_0 \quad \text{in } W^{1,\gamma}(\Omega), && u_k(0) \overset{*}{\rightharpoonup} u_0 \quad \text{in } BD(\Omega),\\ &e_k(0) \rightharpoonup e_0 \quad \text{in } L^2(\Omega;\mathbb{M}^{n\times n}_{sym}), && p_k(0) \overset{*}{\rightharpoonup} p_0 \quad \text{in } M_b(\overline{\Omega};\mathbb{M}^{n\times n}_D) \end{aligned} \tag{6.1a}$$

for suitable α_0, e_0, u_0, p_0, and

$$\mathcal{E}_k(\alpha_k(0), e_k(0), \operatorname{curl} p_k(0)) \to \mathcal{E}_{\rm pp}(\alpha_0, e_0)\,. \tag{6.1b}$$

Under this assumption, we can prove the convergence result below.

Theorem 6.1. *Let $\Omega \subset \mathbb{R}^n$ be open, bounded, and Lipschitz; if $n \geq 3$, let Ω be also star-shaped. Assume $\partial_D\Omega = \partial\Omega$,* (2.2a), (2.4), (2.6), (2.9b), *and* (2.11). *Moreover, for $l_k \to 0$ and $L_k \to 0$, let $(\alpha_k, u_k, e_k, p_k)$ be a quasistatic evolution for the Gurtin-Anand model coupled with damage associated with l_k and L_k such that the conditions* (6.1) *hold. Then there exists a quasistatic evolution for the perfect plasticity model coupled with damage (α, u, e, p) with $\alpha(0) = \alpha_0$, $u(0) = u_0$, $e(0) = e_0$, $p(0) = p_0$ such that, up to a subsequence,*

$$\alpha_k(t) \to \alpha(t) \quad \textit{in } W^{1,\gamma}(\Omega)\,, \tag{6.2a}$$

$$u_k(t) \overset{*}{\rightharpoonup} u(t) \quad \textit{in } BD(\Omega)\,, \tag{6.2b}$$

$$e_k(t) \to e(t) \quad \textit{in } L^2(\Omega; \mathbb{M}^{n\times n}_{sym})\,, \tag{6.2c}$$

$$p_k(t) \overset{*}{\rightharpoonup} p(t) \quad \textit{in } M_b(\overline{\Omega}; \mathbb{M}^{n\times n}_D) \tag{6.2d}$$

for every $t \in [0, T]$.

Remark 6.2. An important difference with respect to the analysis in [16] relies on the fact that we cannot still characterize the global stability in the limit evolution by the equilibrium conditions for the Cauchy stress and the plastic constraint (see [10, Theorem 3.6]). This calls for the approximation in a strong sense of admissible triples for perfect plasticity with ones that are admissible for the Gurtin-Anand model. We show this relaxation result in the lemmas below both in the case of dimension two, and in dimension three under the additional assumption that the domain is star shaped. Actually, in the paper [28], M.G. Mora proves the approximation property for every Lipschitz domain; then Theorem 6.1 can be proved for this domains.

Lemma 6.3 (Approximation, $n \geq 3$)**.** *Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$, be open, bounded, star-shaped and Lipschitz. Then for every $(u, e, p) \in A_{\rm pp}(0)$ there exists a sequence of triples $(u_k, e_k, p_k) \in A(0)$ such that*

$$u_k \to u \quad \textit{in } L^1(\Omega; \mathbb{R}^n), \qquad e_k \to e \quad \textit{in } L^2(\Omega; \mathbb{M}^{n\times n}_{sym}), \qquad p_k \overset{s}{\to} p \quad \textit{in } M_b(\overline{\Omega}; \mathbb{M}^{n\times n}_D)\,.$$

Proof. Without loss of generality we can assume that Ω is star-shaped with respect to 0. For an open set $\widetilde{\Omega}$ such that $\overline{\Omega} \subset \widetilde{\Omega}$ let us define

$$\widehat{u} := \begin{cases} u & \text{in } \Omega \\ 0 & \text{in } \widetilde{\Omega} \setminus \Omega \end{cases}, \qquad \widehat{e} := \begin{cases} e & \text{in } \Omega \\ 0 & \text{in } \widetilde{\Omega} \setminus \Omega \end{cases}, \qquad \widehat{p} := \begin{cases} p & \text{in } \overline{\Omega} \\ 0 & \text{in } \widetilde{\Omega} \setminus \overline{\Omega} \end{cases},$$

For k large enough we set

$$\widehat{u}_k(x) := (1 + \tfrac{1}{k})^{-1} u\big((1 + \tfrac{1}{k})x\big)\,, \quad \widehat{e}_k(x) := e\big((1 + \tfrac{1}{k})x\big) \quad \text{for every } x \in \Omega_k := \Omega + B_{\frac{1}{k}}\,,$$

and

$$\widehat{p}_k := E\widehat{u}_k - \widehat{e}_k \quad \text{in } \Omega_k\,.$$

Then it is not difficult to see that

$$\widehat{u}_k(x) = 0 \quad \text{for every } x \in \Omega_k \setminus \big[(1 + \tfrac{1}{k})^{-1}\Omega\big]\,,$$

$$|\widehat{p}_k|(\partial\Omega) = 0\,, \tag{6.3}$$

and that, taking the restriction of $\widehat{u}_k$, $\widehat{e}_k$, $\widehat{p}_k$ to $\overline{\Omega}$, we have

$$\widehat{u}_k \to u \quad \text{in } L^1(\Omega; \mathbb{R}^n), \qquad \widehat{e}_k \to e \quad \text{in } L^2(\Omega; \mathbb{M}^{n\times n}_{sym})\,, \qquad \widehat{p}_k \overset{s}{\to} p \quad \text{in } M_b(\overline{\Omega}; \mathbb{M}^{n\times n}_D)\,.$$

Moreover, if we regularize by convolution for every k with the sequence of mollifiers $(\varrho_{\frac{1}{h}})_{\mathbb{N}\ni h>k}$, we get (taking the restrictions to $\overline{\Omega}$) a sequence of functions

$$(\widehat{u}^h_k, \widehat{e}^h_k, \widehat{p}^h_k) \in A(0) \cap C^\infty(\overline{\Omega}; \mathbb{R}^n \times \mathbb{M}^{n\times n}_{sym} \times \mathbb{M}^{n\times n}_D)$$

such that

$$\widehat{u}_k^h \to \widehat{u}_k \quad \text{in } L^1(\Omega;\mathbb{R}^n), \quad \widehat{e}_k^h \to \widehat{e}_k \quad \text{in } L^2(\Omega;\mathbb{M}^{n\times n}_{sym}), \quad \widehat{p}_k^h \xrightarrow{s} \widehat{p}_k \quad \text{in } M_b(\overline{\Omega};\mathbb{M}^{n\times n}_D)$$

as $h \to \infty$. Indeed, by (6.3) it is enough to show that $\widehat{p}_k^h \xrightarrow{s} \widehat{p}_k$ in $M_b(\Omega;\mathbb{M}^{n\times n}_D)$, and this holds again by (6.3) since the regularization by convolution of a measure entails strict convergence on open subsets whose boundaries are not charged by the measure itself (see [3, Theorem 2.2]).

By a diagonal argument we obtain (u_k, e_k, p_k) as $(\widehat{u}_k^{h_k}, \widehat{e}_k^{h_k}, \widehat{p}_k^{h_k})$ with $h = h_k$ sufficiently large. ☐

We now show the relaxation property for perfect plasticity triples in a bidimensional domain. The construction of the approximants is similar to the one made in [13, Theorem 6.2, Step 1].

Lemma 6.4 (Approximation, $n = 2$)**.** *Let $\Omega \subset \mathbb{R}^2$ be open, bounded, and Lipschitz. Then for every $(u, e, p) \in A_{\rm pp}(0)$ there exists a sequence of triples $(u_k, e_k, p_k) \in A(0)$ such that*

$$u_k \to u \quad \text{in } L^1(\Omega;\mathbb{R}^n), \qquad e_k \to e \quad \text{in } L^2(\Omega;\mathbb{M}^{n\times n}_{sym}), \qquad p_k \xrightarrow{s} p \quad \text{in } M_b(\overline{\Omega};\mathbb{M}^{n\times n}_D).$$

Proof. Let us define

$$\widehat{u} := \begin{cases} u & \text{in } \Omega \\ 0 & \text{in } \mathbb{R}^2\setminus\Omega \end{cases}, \qquad \widehat{e} := \begin{cases} e & \text{in } \Omega \\ 0 & \text{in } \mathbb{R}^2\setminus\Omega \end{cases}, \qquad \widehat{p} := \begin{cases} p & \text{in } \overline{\Omega} \\ 0 & \text{in } \mathbb{R}^2\setminus\overline{\Omega} \end{cases}.$$

Since $(u, e, p) \in A_{\rm pp}(0)$, we get that

$$E\widehat{u} = \widehat{e} + \widehat{p} \quad \text{in } \mathbb{R}^2.$$

Let $\{Q_{\nu_k}(x_k, r_k)\}_{k\in I}$ be a finite covering of $\partial\Omega$ made of open cubes with centers $x_k \in \partial\Omega$, side $2r_k$, with $r_k > 0$, and a face orthogonal to $\nu_k \in \mathbb{R}^2$ such that $\Omega \cap Q_{\nu_k}(x_k, r_k)$ is a Lipschitz subgraph in the direction ν_k. Let $\{\phi_k\}_{k\in I}$ be an associated partition of unity of $\partial\Omega$. Then

$$\widehat{u} = \sum_{k\in I} \phi_k \widehat{u} + \left(1 - \sum_{k\in I}\phi_k\right)\widehat{u},$$

and the last term has a support compactly contained in Ω. Set

$$\widehat{e}_k := \phi_k \widehat{e} + \nabla\phi_k \odot \widehat{u} \quad \text{and} \quad \widehat{p}_k := \phi_k \widehat{p}, \tag{6.4}$$

so that $\widehat{e}_k \in L^2(\mathbb{R}^2;\mathbb{M}^{2\times 2}_{sym})$ (indeed $\widehat{u} \in BD(\mathbb{R}^2) \subset L^2(\mathbb{R}^2;\mathbb{R}^2)$) and $\widehat{p}_k \in M_b(\mathbb{R}^2;\mathbb{M}^{2\times 2}_D)$ with

$$E(\phi_k\widehat{u}) = \widehat{e}_k + \widehat{p}_k \quad \text{in } \mathbb{R}^2.$$

For $h \in \mathbb{N}$ so large that the support of the functions $\widehat{\phi}_k(x) := \phi_k\left(x + \frac{\nu_k}{h}\right)$ is compactly contained in $Q_{\nu_k}(x_k, r_k)$ for every $k \in I$, let us define

$$u_{k,h}(x) := \phi_k\left(x + \frac{\nu_k}{h}\right)\widehat{u}\left(x + \frac{\nu_k}{h}\right);$$

we also define $e_{k,h}$, $p_{h,k}$ following (6.4). Set

$$u_h := \sum_{k\in I} u_{k,h} + \left(1 - \sum_{k\in I}\phi_k\right)\widehat{u}, \quad e_h := \sum_{k\in I} e_{k,h} + \left(1 - \sum_{k\in I}\phi_k\right)\widehat{e} - \sum_{k\in I}\nabla\phi_k \odot \widehat{u}$$

$$p_h := \sum_{k\in I} p_{k,h} + \left(1 - \sum_{k\in I}\phi_k\right)\widehat{p}$$

Notice that

$$(u_h, e_h, p_h) \in BD(\mathbb{R}^2) \times L^2(\mathbb{R}^2;\mathbb{M}^{2\times 2}_{sym}) \times M_b(\mathbb{R}^2;\mathbb{M}^{2\times 2}_D)$$

with

$$Eu_h = e_h + p_h \quad \text{in } \mathbb{R}^2,$$

and that u_h, e_h, p_h vanish outside a compact subset of Ω. This last condition and fact that we have only used local translations imply that restricting to $\overline{\Omega}$

$$u_h \to u \quad \text{in } L^2(\Omega;\mathbb{R}^2), \quad e_h \to e \quad \text{in } L^2(\Omega;\mathbb{M}^{2\times 2}_{sym}), \quad p_h \xrightarrow{s} p \quad \text{in } M_b(\overline{\Omega};\mathbb{M}^{2\times 2}_D).$$

Moreover, if we regularize (u_h, e_h, p_h) by convolution with a sequence of mollifiers $(\varrho_{\frac{1}{m}})_m$, we get for m sufficiently large that

$$(u_h^m, e_h^m, p_h^m) \in C_c^\infty(\Omega; \mathbb{R}^2 \times \mathbb{M}^{2\times 2}_{sym} \times \mathbb{M}^{2\times 2}_D) \cap A(0)\,,$$

using again that u_h, e_h, p_h have compact support in Ω. Recalling that the regularization by convolution of a measure entails strict convergence on open subsets whose boundaries are not charged by the measure itself, and that $p_h = 0$ on $\partial\Omega$, we have

$$u_h^m \to u_h \quad \text{in } L^2(\Omega;\mathbb{R}^2)\,, \quad e_h^m \to e_h \quad \text{in } L^2(\Omega;\mathbb{M}^{2\times 2}_{sym})\,, \quad p_h^m \xrightarrow{s} p_h \quad \text{in } M_b(\overline{\Omega};\mathbb{M}^{2\times 2}_D)\,,$$

and then we conclude by a diagonal argument. □

We are now ready to prove Theorem 6.1.

Proof of Theorem 6.1. The proof is divided into two steps.

Step 1: Compactness and global stability. By definition of $\mathcal{H}_k$ we have that for every $\beta \in W^{1,\gamma}(\Omega)$, $q \in BV(\Omega;\mathbb{M}^{n\times n}_D)$, and $k \in \mathbb{N}$

$$\mathcal{H}_k(\beta, q) \geq S_1(0)\|q\|_1\,,$$

and then

$$\mathcal{V}_{\mathcal{H}_k}(\alpha_k, p_k; 0, t) \geq S_1(0)\mathcal{V}_1(p_k; 0, t)\,,$$

with $\mathcal{V}_1(p_k; 0, t)$ the variation of p_k with respect to $L^1(\Omega;\mathbb{M}^{n\times n}_D)$ in $(0,t)$. Then, by (6.1), the fact that $\mathcal{Q}_1$ is quadratic, and Korn's inequality, we get that there exists a constant C independent of k and t such that

$$\|\alpha_k(t)\|_{1,\gamma} + \|u_k(t)\|_{BD} + \|e_k(t)\|_2 + \mathcal{V}_1(p_k; 0, t) \leq C\,. \tag{6.5}$$

Let $\widetilde{\Omega}$ be a smooth open set such that $\overline{\Omega} \subset \widetilde{\Omega}$, and let us define for every k and t the functions $\widehat{u}_k(t) \in W^{1,\frac{n}{n-1}}(\widetilde{\Omega};\mathbb{R}^n)$, $\widehat{e}_k(t) \in L^2(\widetilde{\Omega};\mathbb{M}^{n\times n}_{sym})$, and $\widehat{p}_k(t) \in BV(\widetilde{\Omega};\mathbb{M}^{n\times n}_D)$ as

$$\widehat{u}_k(t) := \begin{cases} u_k(t) & \text{in } \Omega \\ w(t) & \text{in } \widetilde{\Omega}\setminus\Omega \end{cases}, \quad \widehat{e}_k(t) := \begin{cases} e_k(t) & \text{in } \Omega \\ \mathrm{E}w(t) & \text{in } \widetilde{\Omega}\setminus\Omega \end{cases}, \quad \widehat{p}_k(t) := \begin{cases} p_k(t) & \text{in } \Omega \\ 0 & \text{in } \widetilde{\Omega}\setminus\Omega \end{cases}.$$

The α_k are nonincreasing in time and $\alpha_k(t,x) \in [0,1]$ with $\|\alpha_k(t)\|_{1,\gamma} \leq C$ and the functions p_k from $[0,T]$ to $L^1(\widetilde{\Omega};\mathbb{M}^{n\times n}_D)$ have uniformly bounded variations; therefore, taking into account (6.1) we get the existence of two functions $\alpha\colon [0,T] \to W^{1,\gamma}(\Omega;[0,1])$ nonincreasing in time and $\widehat{p}\colon [0,T] \to M_b(\widetilde{\Omega};\mathbb{M}^{n\times n}_D)$ with bounded variation such that up to a subsequence (not relabeled)

$$\alpha_k(t) \rightharpoonup \alpha(t) \quad \text{in } W^{1,\gamma}(\Omega)\,, \qquad \widehat{p}_k(t) \xrightarrow{*} \widehat{p}(t) \quad \text{in } M_b(\widetilde{\Omega};\mathbb{M}^{n\times n}_D)$$

for every $t \in [0,T]$. Notice that we have applied [10, Theorem 7.2] considering $M_b(\widetilde{\Omega};\mathbb{M}^{n\times n}_D)$ as a subspace of $L^1(\widetilde{\Omega};\mathbb{M}^{n\times n}_D)$.

Let us fix $t \in [0,T]$. By the a priori estimate (6.5) we deduce that there exist an increasing sequence $(k_j)_j$ (that could depend on t) and two functions $\widehat{u} \in BD(\widetilde{\Omega})$ and $\widehat{e} \in L^2(\widetilde{\Omega};\mathbb{M}^{n\times n}_{sym})$ such that

$$\widehat{u}_{k_j} \xrightarrow{*} \widehat{u} \quad \text{in } BD(\widetilde{\Omega})\,, \qquad \widehat{e}_{k_j} \rightharpoonup \widehat{e} \quad \text{in } L^2(\widetilde{\Omega};\mathbb{M}^{n\times n}_{sym})\,.$$

As in [16, Lemma 9.1] (that holds in our assumptions on Ω), we obtain that

$$u_{k_j}(t) \xrightarrow{*} \widehat{u} \quad \text{in } BD(\Omega), \quad e_{k_j}(t) \rightharpoonup \widehat{e} \quad \text{in } L^2(\Omega;\mathbb{M}^{n\times n}_{sym}), \quad p_k(t) \xrightarrow{*} p(t) \quad \text{in } M_b(\overline{\Omega};\mathbb{M}^{n\times n}_D)\,, \tag{6.6}$$

and

$$(\widehat{u}, \widehat{e}, p(t)) \in A_{\mathrm{pp}}(w(t))\,,$$

where $p(t)$ denotes the restriction of $\widehat{p}(t)$ to $\overline{\Omega}$ and we have not relabeled the restrictions of $\widehat{u}$, $\widehat{e}$ to Ω. We claim that the quadruple $(\alpha(t), \widehat{u}, \widehat{e}, p(t))$ satisfies the stability condition $(\mathrm{qs1})_{\mathrm{pp}}$, namely

$$\mathcal{E}_{\mathrm{pp}}(\alpha(t), \widehat{e}) \leq \mathcal{E}_{\mathrm{pp}}(\beta, \eta) + \mathcal{H}_{\mathrm{pp}}(\beta, q - p(t)) \tag{6.7}$$

for every $(\beta,(v,\eta,q))\in\mathcal{A}_{\rm pp}(\alpha(t),w(t))$. Then, since $(\widehat{u},\widehat{e})$ minimizes the functional $(v,\eta)\mapsto\mathcal{E}_{\rm pp}(\beta,\eta)$ on the convex set $\{(v,e)\colon(v,e,p(t))\in A_{\rm pp}(w(t))\}$, we have that $(\widehat{u},\widehat{e})=(u(t),e(t))$ and

$$u_k(t)\xrightharpoonup{*}u(t)\quad\text{in } BD(\Omega)\,,\qquad e_k(t)\rightharpoonup e(t)\quad\text{in } L^2(\Omega;\mathbb{M}^{n\times n}_{sym})\,,\tag{6.8a}$$

for the whole subsequence. We have already shown that

$$\alpha_k(t)\rightharpoonup\alpha(t)\quad\text{in } W^{1,\gamma}(\Omega)\,,\qquad p_k(t)\xrightharpoonup{*}p(t)\quad\text{in } M_b(\overline{\Omega};\mathbb{M}^{n\times n}_D)\,.\tag{6.8b}$$

Let us now prove the claim (6.7); since we work with a given t, we can neglect the dependence on j in (6.6). By assumption, for every k we have the stability condition:

$$\mathcal{E}_k(\alpha_k(t),e_k(t),\mathrm{curl}\;p_k(t))\le\mathcal{E}_k(\beta,\eta,\mathrm{curl}\;q)+\mathcal{H}_k(\beta,q-p_k(t))\tag{6.9}$$

for every $(\beta,v,\eta,q)\in\mathcal{A}(\alpha_k(t),w_k(t))$.

Let us fix $(\beta,v_0,\eta_0,q_0)\in\mathcal{A}(\alpha(t),0)$, and test (6.9) by

$$(\widehat{\alpha}_k,\widehat{v}_k,\widehat{\eta}_k,\widehat{q}_k):=(\beta\wedge\alpha_k(t),u_k(t)+v_0,e_k(t)+\eta_0,p_k(t)+q_0)\in\mathcal{A}(\alpha_k(t),w_k(t))\,.$$

Arguing as in Theorem 3.6 we deduce that

$$\begin{aligned}\gamma_k&:=\mathcal{Q}_1(\alpha_k(t),e_k(t))-\mathcal{Q}_1(\widehat{\alpha}_k,e_k(t))+\mathcal{D}(\alpha_k(t))+\|\nabla(\beta\vee\alpha_k(t))\|_\gamma^\gamma-\|\nabla\beta\|_\gamma^\gamma\\&\le\tfrac12\langle\mathbb{C}(\widehat{\alpha}_k)(\eta_0+2e_k(t)),\eta_0\rangle+\tfrac{L_k^2}{2}\langle\mu(\widehat{\alpha}_k)\,\mathrm{curl}\,(q_0+2p_k(t)),\mathrm{curl}\,q_0\rangle+\mathcal{D}(\widehat{\alpha}_k)\\&\quad+\mathcal{H}_{\rm pp}(\widehat{\alpha}_k,q_0)+l_k\int_\Omega S_2(\widehat{\alpha}_k)\,\mathrm{d}|\mathrm{D}q_0|=:\delta_k\,.\end{aligned}\tag{6.10}$$

To get the above inequality we have also used that

$$\frac{L_k^2}{2}\int_\Omega(\mu(\alpha_k(t))-\mu(\widehat{\alpha}_k))|\,\mathrm{curl}\;p_k(t)|^2\,\mathrm{d}x\ge0$$

and that for every $\alpha\in W^{1,\gamma}(\Omega)$ and $p\in BV(\Omega;\mathbb{M}^{n\times n}_D)$

$$\mathcal{H}_k(\alpha,p)\le\mathcal{H}_{\rm pp}(\alpha,p)+l_k\int_\Omega S_2(\alpha)\,\mathrm{d}|\mathrm{D}p|\,.$$

By (6.1) and the energy balance for (α_k,u_k,e_k,p_k) we get

$$\frac{L_k^2}{2}\int_\Omega\mu(\alpha_k(t))|\,\mathrm{curl}\;p_k(t)|^2\,\mathrm{d}x\le C\,,$$

for C independent of k; by the Hölder inequality and the monotonicity of μ it follows that

$$L_k^2\langle\mu(\widehat{\alpha}_k)\;\mathrm{curl}\;p_k(t),\mathrm{curl}\,q_0\rangle\le L_k\left(\int_\Omega L_k^2\,\mu(\alpha_k(t))|\,\mathrm{curl}\;p_k(t)|^2\,\mathrm{d}x\right)^{\frac12}\left(\int_\Omega\mu(\widehat{\alpha}_k)|\,\mathrm{curl}\;q_0|^2\,\mathrm{d}x\right)^{\frac12}.$$

Thus, letting $k\to0$ in (6.10) we obtain as in Theorem 3.6 the inequality

$$\mathcal{E}_{\rm pp}(\alpha(t),\widehat{e})-\mathcal{Q}_1(\beta,\widehat{e})-\|\nabla\beta\|_\gamma^\gamma\le\tfrac12\langle\mathbb{C}(\beta)(\eta_0+2\widehat{e}),\eta_0\rangle+\mathcal{D}(\beta)+\mathcal{H}_{\rm pp}(\beta,q_0)\,.\tag{6.11}$$

Let us consider a triple $(v,\eta,q)\in A_{\rm pp}(w(t))$; then $(v-\widehat{u},\eta-\widehat{e},q-p(t))\in A_{\rm pp}(0)$. By Lemmas 6.3 and 6.4 there exist triples $(v_k,\eta_k,q_k)\in A(0)$ such that

$$\begin{gathered}v_k\to v-\widehat{u}\quad\text{in } L^1(\Omega;\mathbb{R}^n),\qquad\eta_k\to\eta-\widehat{e}\quad\text{in } L^2(\Omega;\mathbb{M}^{n\times n}_{sym}),\\ q_k\xrightarrow{s}q-p(t)\quad\text{in } M_b(\overline{\Omega};\mathbb{M}^{n\times n}_D)\,.\end{gathered}$$

In particular Reshetnyak's Continuity Theorem (cf. [3, Theorem 2.39]) implies that

$$\mathcal{H}_{\rm pp}(\beta,q_k)\to\mathcal{H}_{\rm pp}(\beta,q-p(t))\,.$$

Therefore, considering (v_k,η_k,q_k) in place of (v_0,η_0,q_0) in (6.11) and taking the limit of the right-hand side as $k\to\infty$ we deduce (6.7).

Step 2: Energy balance. From (6.8b) it follows that

$$\mathcal{V}_{\mathcal{H}_{\rm pp}}(\alpha, p; 0, T) \leq \liminf_{k\to\infty} \mathcal{V}_{\mathcal{H}_k}(\alpha_k, p_k; 0, T)\,. \tag{6.12}$$

Indeed, for every $\beta_k \rightharpoonup \beta$ in $W^{1,\gamma}(\Omega)$ and $(q_k)_k \subset BV(\Omega;\mathbb{M}^{n\times n}_D)$ with $q_k \xrightharpoonup{*} q$ in $M_b(\overline{\Omega};\mathbb{M}^{n\times n}_D)$, it holds

$$\mathcal{H}_{\rm pp}(\beta, q) \leq \liminf_{k\to\infty} \int_{\overline{\Omega}} S_1(\beta_k)\,\mathrm{d}|q_k| = \liminf_{k\to\infty} \int_\Omega S_1(\beta_k(x))|q_k(x)|\,\mathrm{d}x \leq \liminf_{k\to\infty} \mathcal{H}_k(\beta_k, q_k)\,,$$

and then we get (6.12) by the definition of $\mathcal{V}_{\mathcal{H}_{\rm pp}}$ and $\mathcal{V}_{\mathcal{H}_k}$. By lower semicontinuity and the fact that $\mathcal{Q}_2(\alpha_k(t), \mathrm{curl}\ p_k(t))$ is nonnegative it follows that

$$\mathcal{E}_{\rm pp}(\alpha(t), e(t)) \leq \liminf_{k\to\infty} \mathcal{E}_k(\alpha_k(t), e_k(t), \mathrm{curl}\ p_k(t))\,. \tag{6.13}$$

Collecting (6.1), (6.12), and (6.13) we deduce that

$$\mathcal{E}_{\rm pp}(\alpha(T), e(T)) + \mathcal{V}_{\mathcal{H}_{\rm pp}}(\alpha, p; 0, T) \leq \mathcal{E}_{\rm pp}(\alpha(0), e(0)) + \int_0^T \langle \sigma(s), E\dot{w}(s)\rangle\,\mathrm{d}s\,.$$

From the stability condition $(\mathrm{qs1})_{\rm pp}$, with arguments similar to those in Lemma 4.3 (cf. [10, Theorem 4.7]), we can prove that the opposite energy imbalance holds and then (α, u, e, p) is a quasistatic evolution for the coupled perfect plasticity-damage model. By (6.1), (6.12), (6.13), and the energy balance (evaluated in $[0,t]$) it follows that for every $t \in [0,T]$

$$\mathcal{E}_{\rm pp}(\alpha(t), e(t)) = \lim_{k\to\infty} \mathcal{E}_k(\alpha_k(t), e_k(t), \mathrm{curl}\ p_k(t))\,,$$

which implies

$$\mathcal{Q}_1(\alpha_k(t), e_k(t)) \to \mathcal{Q}_1(\alpha(t), e(t)), \quad \|\nabla\alpha_k(t)\|_\gamma \to \|\nabla\alpha(t)\|_\gamma, \quad \mathcal{Q}_2^k(\alpha_k(t), \mathrm{curl}\ p_k(t)) \to 0\,,$$

and then (6.2a) and (6.2c). This concludes the proof. □

Acknowledgments. The author wishes to thank Gianni Dal Maso and Giuliano Lazzaroni for many helpful suggestions and discussions. He also gratefully acknowledges a discussion with Stefano Almi, who brought to his attention A. Giacomini's paper on Gurtin-Anand model. This work is part of the Project "Calculus of Variations" 2010-11, supported by the Italian Ministry of Education, University, and Research. The author is member of the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM).

References

[1] A. Acharya and J. L. Bassani, *Lattice incompatibility and a gradient theory of crystal plasticity*, J. Mech. Phys. Solids, 48 (2000), pp. 1565–1595.

[2] R. Alessi, J.-J. Marigo, and S. Vidoli, *Gradient damage models coupled with plasticity and nucleation of cohesive cracks*, Arch. Ration. Mech. Anal., 214 (2014), pp. 575–615.

[3] L. Ambrosio, N. Fusco, and D. Pallara, *Functions of bounded variation and free discontinuity problems*, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 2000.

[4] G. Buttazzo, *Semicontinuity, relaxation and integral representation in the calculus of variations*, vol. 207 of Pitman Research Notes in Mathematics Series, Longman Scientific & Technical, Harlow; copublished in the United States with John Wiley & Sons, Inc., New York, 1989.

[5] J. Casado-Diaz and G. Dal Maso, *A weak notion of convergence in capacity with applications to thin obstacle problems*, in Calculus of variations and differential equations (Haifa, 1998), vol. 410 of Chapman & Hall/CRC Res. Notes Math., Chapman & Hall/CRC, Boca Raton, FL, 2000, pp. 56–64.

[6] P. Cermelli and M. Gurtin, *On the characterization of geometrically necessary dislocations in finite plasticity*, J. Mech. Phys. Solids, 49 (2000), pp. 1539–1568.

[7] A. H. Cottrell, *Theory of brittle fracture in steel and similar metals*, Transactions of the metallurgical society of AIME, 212 (1958), pp. 192–203.

[8] V. Crismale, *Globally stable quasistatic evolution for a coupled elastoplastic-damage model.* To appear on ESAIM Control Optim. Calc. Var. DOI 10.1051/cocv/2015037, 2015.

[9] V. Crismale and G. Lazzaroni, *Viscous approximation of evolutions for a coupled elastoplastic-damage model.* Preprint SISSA 05/2015/MATE, 2015.

[10] G. Dal Maso, A. DeSimone, and M. G. Mora, *Quasistatic evolution problems for linearly elastic-perfectly plastic materials*, Arch. Ration. Mech. Anal., 180 (2006), pp. 237–291.

[11] M. Duchoň and P. Maličký, *A Helly theorem for functions with values in metric spaces*, Tatra Mt. Math. Publ., 44 (2009), pp. 159–168.
[12] G. B. Folland, *Real analysis*, Pure and Applied Mathematics (New York), John Wiley & Sons, Inc., New York, second ed., 1999. Modern techniques and their applications, A Wiley-Interscience Publication.
[13] G. A. Francfort and A. Giacomini, *Small-strain heterogeneous elastoplasticity revisited*, Comm. Pure Appl. Math., 65 (2012), pp. 1185–1241.
[14] H. Gao, Y. Huang, W. D. Nix, and J. W. Hutchinson, *Mechanism-based strain gradient plasticity. I. Theory*, J. Mech. Phys. Solids, 47 (1999), pp. 1239–1263.
[15] A. Giacomini, *On the energetic formulation of the Gurtin and Anand model in strain gradient plasticity*, Discrete Contin. Dyn. Syst. Ser. B, 17 (2012), pp. 527–552.
[16] A. Giacomini and L. Lussardi, *Quasi-static evolution for a model in strain gradient plasticity*, SIAM J. Math. Anal., 40 (2008), pp. 1201–1245.
[17] C. Goffman and J. Serrin, *Sublinear functions of measures and variational integrals*, Duke Math. J., 31 (1964), pp. 159–178.
[18] M. E. Gurtin, *A gradient theory of single-crystal viscoplasticity that accounts for geometrically necessary dislocations*, J. Mech. Phys. Solids, 50 (2002), pp. 5–32.
[19] M. E. Gurtin and L. Anand, *A theory of strain-gradient plasticity for isotropic, plastically irrotational materials. I. Small deformations*, J. Mech. Phys. Solids, 53 (2005), pp. 1624–1649.
[20] J. Heinonen, T. Kilpeläinen, and O. Martio, *Nonlinear potential theory of degenerate elliptic equations*, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 1993. Oxford Science Publications.
[21] Y. Huang, H. Gao, W. D. Nix, and J. W. Hutchinson, *Mechanism-based strain gradient plasticity. II. Analysis*, J. Mech. Phys. Solids, 48 (2000), pp. 99–128.
[22] D. Hull and D. Bacon, *Introduction to Dislocations*, Butterworth Heinemann, Butterworth-Heinemann, 2011.
[23] D. Knees, R. Rossi, and C. Zanini, *A vanishing viscosity approach to a rate-independent damage model*, Math. Models Methods Appl. Sci., 23 (2013), pp. 565–616.
[24] J. Lemaitre and J. Chabouche, *Mechanics of solid materials*, Cambridge University Press, Avon, 1990.
[25] E. Lorentz and A. Benallal, *Gradient constitutive relations: numerical aspects and application to gradient damage*, Computer methods in applied mechanics and engineering, 194 (2005), pp. 5191–5220.
[26] A. Mielke, *Evolution of rate-independent systems*, in Evolutionary equations. Vol. II, Handb. Differ. Equ., Elsevier/North-Holland, Amsterdam, 2005, pp. 461–559.
[27] A. Mielke and T. Roubíček, *Rate-independent damage processes in nonlinear elasticity*, Math. Models Methods Appl. Sci., 16 (2006), pp. 177–209.
[28] M. G. Mora, *Relaxation of the Hencky model in perfect plasticity*. Preprint, 2015.
[29] K. Pham and J.-J. Marigo, *Approche variationnelle de l'endommagement: II. Les modèles à gradient*, Comptes Rendus Mécanique, 338(4) (2010), pp. 199–206.
[30] B. D. Reddy, F. Ebobisse, and A. McBride, *Well-posedness of a model of strain gradient plasticity for plastically irrotational materials*, Int. J. Plasticity, 24 (2008), pp. 55–73.
[31] W. Rudin, *Real and Complex Analysis*, McGraw-Hill, New York, 1966.
[32] E. Smith and J. T. Barnby, *Crack nucleation in crystalline solids*, Metal Science, 1 (1967), pp. 56–64.
[33] F. Solombrino, *Quasistatic evolution problems for nonhomogeneous elastic plastic materials*, J. Convex Anal., 16 (2009), pp. 89–119.
[34] A. Stroh, *A theory of the fracture of metals*, Advances in Physics, 6 (1957), pp. 418–465.
[35] R. Temam, *Mathematical problems in plasticity*, Gauthier-Villars, Paris, 1985. Translation of Problèmes mathématiques en plasticité. Gauthier-Villars, Paris, 1983.
[36] M. Thomas and A. Mielke, *Damage of nonlinearly elastic materials at small strain—existence and regularity results*, ZAMM Z. Angew. Math. Mech., 90 (2010), pp. 88–112.
[37] A. I. Vol′pert, *Spaces* BV *and quasilinear equations*, Mat. Sb. (N.S.), 73 (115) (1967), pp. 255–302.
[38] C. Zener, *The macro-mechanism of fracture*, in Fracturing of Metals, American Society of Metals, Cleveland, 1948, pp. 3–31.

SISSA, Via Bonomea 265, 34136 Trieste, Italy
E-mail address: vito.crismale@sissa.it