

STABILITY FOR THE BRUNN-MINKOWSKI AND RIESZ REARRANGEMENT INEQUALITIES, WITH APPLICATIONS TO GAUSSIAN CONCENTRATION AND FINITE RANGE NON-LOCAL ISOPERIMETRY

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ABSTRACT. We provide a simple, general argument to obtain improvements of concentration-type inequalities starting from improvements of their corresponding isoperimetric-type inequalities. We apply this argument to obtain robust improvements of the Brunn-Minkowski inequality (for Minkowski sums between generic sets and convex sets) and of the Gaussian concentration inequality. The former inequality is then used to obtain a robust improvement of the Riesz rearrangement inequality under certain natural conditions. These conditions are compatible with the applications to a finite-range nonlocal isoperimetric problem arising in statistical mechanics.

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1. INTRODUCTION

In this paper we present a general argument to deduce robust improvements of the Brunn-Minkowski inequality and of the Gaussian concentration inequality starting from the corresponding quantitative isoperimetric inequalities. We then exploit the former result to obtain a robust improvement of the Riesz rearrangement inequality in the case of a strictly decreasing interaction kernel that acts on nested sets. Finally, we discuss how this last result may be applied to provide a quantitative geometric

¹Work partially supported by U.S. National Science Foundation grant DMS-1201354.

²Work partially supported by U.S. National Science Foundation grants DMS 1265910 and DMS 1361122.

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description of near-minimizing droplets for the Gates-Lebowitz-Penrose free energy, a problem arising in statistical mechanics that motivated this research.

1.1. Stability for the Brunn-Minkowski inequality. If E and F are Lebesgue measurable sets in \mathbb{R}^d , $E + F = \{x + y : x \in E, y \in F\}$ is their Minkowski sum, and $|G|$ denotes the (outer) Lebesgue measure of a set $G \subset \mathbb{R}^d$, then the Brunn-Minkowski inequality ensures that

$$|E + F|^{1/d} \geq |E|^{1/d} + |F|^{1/d}. \quad (1.1)$$

Henstock and Macbeath [HM53] proved that if $0 < |E||F| < \infty$, then equality holds in (1.1) if and only if E and F are equivalent to their convex hulls, which in turn, up to translations, are homothetic to each other. A natural question is then how to relate the size of the gap between the left-hand side and the right-hand side of (1.1) to the distance of (suitably scaled and translated copies of) E and F from a suitably chosen convex set. This problem has been solved in the case that both E and F are convex sets in [FMP09]. In this case, it was shown that

$$|E + F|^{1/d} \geq (|E|^{1/d} + |F|^{1/d}) \left\{ 1 + \frac{\alpha(E; F)^2}{C(d) \sigma(E; F)^{1/d}} \right\}, \quad (1.2)$$

where $\sigma(E; F) = \max\{|E|/|F|, |F|/|E|\}$, and where $\alpha(E; F)$ is defined as

$$\alpha(E; F) = \frac{1}{2} \inf \left\{ \frac{|E \Delta (x_0 + r F)|}{|E|} : x_0 \in \mathbb{R}^d, r^d = \frac{|E|}{|F|} \right\}. \quad (1.3)$$

(The factor $1/2$ is included so to have $\alpha(E; F) \in [0, 1)$). We shall find it convenient to restate (1.2) as

$$C(d) \delta(E; F) \geq \alpha(E; F)^2, \quad (1.4)$$

where we have set

$$\delta(E; F) = \sigma(E; F)^{1/d} \left\{ \frac{|E + F|^{1/d}}{|E|^{1/d} + |F|^{1/d}} - 1 \right\}. \quad (1.5)$$

The advantage of formulation (1.4) of (1.2) is that $\delta(E; F)$ and $\alpha(E; F)$ are both scale invariant quantities, meaning that

$$\delta(\lambda E; \lambda F) = \delta(E; F), \quad \alpha(\lambda E; \mu F) = \alpha(E; F), \quad \forall \lambda, \mu > 0.$$

(Note that, in general, if $\lambda \neq \mu$ then $\delta(\lambda E; \mu F)$ may differ from $\delta(E; F)$.) Our first main result is a quantitative improvement of (1.1) in the spirit of (1.2) in the case that one of two sets E and F is a convex set with positive measure. In the following we shall thus fix K to be an open, bounded, convex set in \mathbb{R}^d containing the origin.

Theorem 1.1. *For every $d \geq 1$ there exists a positive constant $C(d)$ with the following property. If $E \subset \mathbb{R}^d$ is a Lebesgue measurable set with $0 < |E| < \infty$, then*

$$\alpha(E; K) \leq C(1) \delta(E; K), \quad \text{if } d = 1; \quad (1.6)$$

$$\alpha(E; K)^4 \leq C(d) \max \left\{ 1, \frac{|K|}{|E|} \right\}^m \delta(E; K), \quad \text{if } d \geq 2. \quad (1.7)$$

Here we can take $C(1) = 2$, $m = (4d + 2)/d$, and

$$C(d) = \frac{d}{\log(2)} \left(\frac{\sqrt{2^{5d}} C_0(d)}{d} + d 2^{d-1} + 2^d \right)^4, \quad \text{if } d \geq 2,$$

where $C_0(d)$ is defined in (1.10) below.

Remark 1.1. The estimate (1.6) for the one-dimensional case $d = 1$ is sharp in the decay rate of $\alpha(E; K)$ as $\delta(E; K) \rightarrow 0$. On the other hand, if E is a convex set with $0 < |E| < \infty$, then by (1.2) we have $\alpha(E; K)^2 \leq C(d)\delta(E; K)$ for every $d \geq 2$, and this inequality is sharp in the decay rate of $\alpha(E; K)$ as $\delta(E; K) \rightarrow 0$, see [FMP10, Section 4]. It is thus natural to conjecture that the power 4 on the left-hand side of (1.7) should be replaced by the power 2. Moreover, it should be possible to remove the corrective factor $\max\{1, |K|/|E|\}^m$ on the right-hand side of (1.7). *In any case, in the application of this result to be discussed here, we shall have $|K| < |E|$, in which case the factor is 1.*

Remark 1.2. Improvements of the Brunn-Minkowski inequality (1.1) in the general case when none of the two sets is assumed to be convex have been recently obtained by Figalli and Jerison in [FJ13a, FJ13b]. For example, in [FJ13a] it is shown that if $E \subset \mathbb{R}^d$ and $|E + E| \leq |2E|(1 + \delta(d))$, then, with $\text{co}(E)$ denoting the convex hull of E ,

$$c(d) \left(\frac{|\text{co}(E) \setminus E|}{|E|} \right)^{8 \cdot 16^{d-1} \cdot d! \cdot (d-1)!} \leq \frac{|E + E|}{|2E|} - 1,$$

where $\delta(d)$ and $c(d)$ are positive computable constants.

To prove Theorem 1.1, we exploit known quantitative improvements of the (Wulff) isoperimetric inequality (associated to the convex set K) through the use of the coarea formula. Precisely, given an open bounded convex set K containing the origin, one sets

$$\|\nu\|_* = \sup \left\{ x \cdot \nu : x \in K \right\}, \quad \nu \in S^{n-1},$$

and correspondingly introduces a notion of *anisotropic perimeter* by setting

$$P_K(E) = \int_{\partial E} \|\nu_E\|_* \, d\mathcal{H}^{d-1} = \limsup_{r \rightarrow 0^+} \frac{|E + rK| - |E|}{r},$$

in case E is an open set with Lipschitz boundary in \mathbb{R}^d . The most important case is that in which $K = B = \{x \in \mathbb{R}^d : |x| < 1\}$, in which case $\|\cdot\|_*$ is simply the Euclidian norm and we obtain the (usual) *perimeter*

$$P(E) := P_B(E) = \mathcal{H}^{d-1}(\partial E) = \limsup_{r \rightarrow 0^+} \frac{|E + rB| - |E|}{r}.$$

(Here \mathcal{H}^s stands for the s -dimensional Hausdorff measure on \mathbb{R}^d , and $rF = \{rx : x \in F\}$. We shall also set $sB = B_s$ and $B_{x,s} = x + B_s$ for every $x \in \mathbb{R}^d$ and $s > 0$.)

It is well known that the Brunn-Minkowski inequality implies the Wulff inequality

$$P_K(E) \geq d|K|^{1/d}|E|^{(d-1)/d}, \quad 0 < |E| < \infty, \quad (1.8)$$

where equality holds if and only if $|E\Delta(x + rK)| = 0$ for some $x \in \mathbb{R}^d$ and $r > 0$. In [FMP10] a quantitative improvement of (1.8) was proved, in the form

$$P_K(E) \geq d|K|^{1/d}|E|^{(d-1)/d} \left(1 + \frac{\alpha(E; K)^2}{C_0(d)} \right), \quad \text{if } 0 < |E| < \infty, \quad (1.9)$$

where

$$C_0(d) = \frac{181 d^7}{4(2 - 2^{1-(1/d)})^{3/2}}. \quad (1.10)$$

(See [FMP08] for the case $K = B$ of (1.9).) Our starting point in the proof of Theorem 1.1 is then the remark that, if $|E| = |K|$ and $r > 0$, then by the coarea formula (and provided E is closed, see Lemma 2.1)

$$\begin{aligned} |E + rK| - (|E|^{1/d} + |rK|^{1/d})^d &= |E + rK| - |K + rK| \\ &= \int_0^r P_K(E + sK) - P_K(K + sK) ds. \end{aligned} \quad (1.11)$$

The integrand here is positive for every $s \in (0, r)$: indeed, $|E + sK| \geq |K + sK|$ by the Brunn-Minkowski inequality, and thus $P_K(E + sK) \geq P_K(K + sK)$ by the Wulff inequality (1.8). If, instead of the Wulff inequality (1.8) one applies its improved form (1.9), then one gets

$$\begin{aligned} |E + rK| - (|E|^{1/d} + |rK|^{1/d})^d & \\ \geq d|K|^{1/d} \int_0^r |E + sK|^{(d-1)/d} \frac{\alpha(E + sK; K)^2}{C_0(d)} ds. & \end{aligned} \quad (1.12)$$

The main difficulty in proving Theorem 1.1 is that $\alpha(E + sK; K)$ may decrease to zero very rapidly as s increases: For example, suppose $K = B$, and E is the ball of radius 2 that has been “perforated” by removing a large number of small disjoint balls of radius at most ϵ from the interior – think of a Swiss cheese with many tiny holes. We can arrange this construction so that $|E| = |B|$. Then, for $s > \epsilon$, one has $\alpha(E + sB; B) = 0$, while

$$\delta(E, sB) = \frac{1}{s^d} \left(\frac{2+s}{1+s} - 1 \right).$$

Thus, while “Swiss cheese” sets E are such that $\alpha(E + sB; B)$ can go to zero rapidly as s increases away from zero, such sets have a large Brunn-Minkowski deficit. The proof of Theorem 1.1 that we give turns on showing that if $\alpha(E + sB; B)$ is much smaller than $\alpha(E; B)$ for small s , then $\delta(E, sB)$ is sizable for small s .

The main idea may be obscured by the details in the proof given in Section 2, and so we provide here a sketch of a proof for the special case $K = B$, $|E| = |B|$ and $d \geq 2$. In this special case, we easily deduce from the definition of $\delta(E; B)$, (1.12), $|E + rB| \geq |B|$, and Hölder inequality, that

$$C(d)\sqrt{\delta(E; B)} \geq \int_0^1 \alpha(E + rB; B) dr. \quad (1.13)$$

Next, by (the elementary) Lemma 2.2 below, one finds that

$$|\alpha(E; B) - \alpha(F; B)| \leq \frac{2|E\Delta F|}{\max\{|E|, |F|\}}, \quad (1.14)$$

for every $E, F \subset \mathbb{R}^d$ with positive and finite Lebesgue measure. Before applying this with $F = E + rB$, we first pick $\epsilon \in (0, 1)$ and restrict the domain of integration from $r \in (0, 1)$ to $r \in (0, \epsilon)$, and then absorb a factor of $|E + rB|^{-1} \leq |E|^{-1} = |B|^{-1}$ into $C(d)$, to obtain that

$$C(d) \left\{ \sqrt{\delta(E; B)} + \int_0^\epsilon |E\Delta(E + rB)| dr \right\} \geq \int_0^\epsilon \alpha(E; B) dr \geq \epsilon \alpha(E; B). \quad (1.15)$$

The key step is to bound $\int_0^\epsilon |E\Delta(E+rB)|dr$ in terms of $\delta(E; B)$ and ϵ . Note that

$$\begin{aligned}\int_0^\epsilon |E\Delta(E+rB)|dr &= \int_0^\epsilon (|E+rB| - |E|)dr \\ &= \int_0^\epsilon dr \int_0^r P(E+tB) dt,\end{aligned}$$

where, again by the integration formula (1.11) and by $P(E+tB) \geq P(B+tB)$,

$$\begin{aligned}\int_0^r P(E+tB) dt &= \int_0^r (P(E+tB) - P(B+tB)) dt + \int_0^r P(B+tB) dt \\ &\leq |E+B| - |B+B| + C(d)r \leq C(d)(\delta(E) + r).\end{aligned}$$

Integrating over $r \in (0, \epsilon)$ we eventually prove

$$\epsilon \alpha(E; B) \leq C(d) \left\{ \sqrt{\delta(E; B)} + \epsilon \delta(E; B) + \epsilon^2 \right\},$$

and then optimize the choice of ϵ by setting $\epsilon = \delta(E; B)^{1/4}$.

1.2. Improvements of the Gaussian concentration inequality. The strategy for proving Theorem 1.1 that we have just described is applicable in other situations. We illustrate this by considering the Gaussian concentration inequality. Let us denote by γ_d the Gaussian measure on \mathbb{R}^d , so that

$$\gamma_d(E) = \frac{1}{(2\pi)^{d/2}} \int_E e^{-|x|^2/2} dx, \quad E \subset \mathbb{R}^d.$$

Given $\nu \in S^{d-1}$ and $s \in \mathbb{R}$ we set $H_\nu(s) = \{x \in \mathbb{R}^d : x \cdot \nu < s\}$, $H(s) = H_{e_1}(s)$,

$$\phi(s) = \gamma_d(H_\nu(s)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^s e^{-z^2/2} dz, \quad (1.16)$$

and for every $E \subset \mathbb{R}^d$ we let $s_E \in \mathbb{R}$ be such that

$$\gamma_d(E) = \phi(s_E).$$

With this notation, the Gaussian concentration inequality says that

$$\gamma_d(E+rB) \geq \gamma_d(H(s_E)+rB), \quad \forall r > 0, \quad (1.17)$$

with equality if and only if $E = H_\nu(s_E)$ for some $\nu \in S^{d-1}$. We now want to improve this inequality into a quantitative statement, and we shall do this by exploiting Gaussian isoperimetry. Let us recall that given an open set E with Lipschitz boundary, the quantity

$$P_\gamma(E) = \frac{1}{(2\pi)^{(d-1)/2}} \int_{\partial E} e^{-|x|^2/2} d\mathcal{H}^{d-1} = \limsup_{r \rightarrow 0^+} \frac{\gamma_d(E+rB) - \gamma_d(E)}{r},$$

is the Gaussian perimeter of E , and we have the Gaussian isoperimetric inequality,

$$P_\gamma(E) \geq P_\gamma(H(s_E)), \quad (1.18)$$

with equality if and only if $E = H_\nu(s_E)$ for some $\nu \in S^{d-1}$.

An important point of contrast with the Wulff inequality (1.8) is that while $P_K(rK) = d|K|r^{(d-1)/d}$ is monotone increasing in r , $P_\gamma(H(s))$ is not monotone in s . In fact, $\lim_{s \rightarrow \pm\infty} P_\gamma(H(s)) = 0$.

The quantitative analysis of (1.18) was initiated in [CFMP11, MN12] using a natural Gaussian analog of $\alpha(E; F)$ defined by

$$\alpha_\gamma(E) = \inf_{\nu \in S^{d-1}} \gamma_d(E \Delta H_\nu(s_E)).$$

The best result to date is that

$$P_\gamma(E) - P_\gamma(H(s_E)) \geq \frac{e^{s_E^2/2}}{c(1+s_E^2)} \alpha_\gamma(E)^2, \quad c = 80 \pi^2 \sqrt{2\pi}, \quad (1.19)$$

proved in [BBJ14].

A key property of (1.19) is that it is dimension independent, and we definitely desire that this strong property be reflected in a quantitative version of (1.17). To this end, given $E \subset \mathbb{R}^d$ and $r > 0$ we set

$$\delta_\gamma^r(E) = \max \left\{ 1, \frac{1}{r} \right\} \sup_{0 < t < r} \gamma_d(E + B_t) - \gamma_d(H(s_E) + B_t), \quad r > 0. \quad (1.20)$$

Notice that the factor $1/r$ is needed if r is very small, because in that regime one needs to consider an isoperimetric type deficit. The same feature appears in the Euclidean case, see (2.16). The supremum over $t \in (0, r)$ in the definition of the deficit is necessary because of the non-monotonicity of $P_\gamma(H(s))$ as a function of s , as noted above.

Next, given $\lambda \in (\gamma_d(E), 1)$, we define

$$r_E(\lambda) = \sup \left\{ r > 0 : \gamma_d(E + rB) < \lambda \right\}. \quad (1.21)$$

With this notation in force, we have the following theorem.

Theorem 1.2. *Given $E \subset \mathbb{R}^d$ with $\gamma_d(E) < 1$ and $\lambda \in (\gamma_d(E), 1)$, one has*

$$\alpha_\gamma(E)^4 \leq C_*(\lambda) \delta_\gamma^{r_E(\lambda)}(E), \quad (1.22)$$

where, by definition,

$$C_*(v) = (5 + 1280 \pi^3)^2 (1 + \phi^{-1}(v))^2, \quad \forall v \in (0, 1).$$

Remark 1.3. Notice that (1.22) degenerates as we allow $\lambda \rightarrow 1^-$.

The relation between (1.17) and (1.18) is similar – but not entirely analogous – to that existing between the Brunn-Minkowski inequality (1.1) (with $F = K$ convex) and the Wulff inequality (1.8). Indeed, we can still write the deficit in (1.17) as an integral of Gaussian isoperimetric deficits, so that (1.11) now takes the form

$$\gamma_d(E + rB) - \gamma_d(H(s_E) + rB) = \frac{1}{\sqrt{2\pi}} \int_0^r P_\gamma(E + tB) - P_\gamma(H(s_E) + tB) dt. \quad (1.23)$$

However, now we cannot infer the non-negativity of the integrand by isoperimetry (compare with the argument below (1.11)), because of the non-monotonicity of $P_\gamma(H(s))$ in s . Indeed, if we consider the decomposition

$$\begin{aligned} P_\gamma(E + B_t) - P_\gamma(H(s_E) + B_t) &= P_\gamma(E + B_t) - P_\gamma(H(s_{E+B_t})) \\ &\quad + P_\gamma(H(s_{E+B_t})) - P_\gamma(H(s_E) + B_t), \end{aligned} \quad (1.24)$$

then the first term in the sum on the right-hand side is non-negative by (1.18), while the sign of second term depends on the values of $\gamma_d(E)$ and $\gamma_d(E + B_t)$. In particular, it is not clear if the left-hand side of (1.23) is increasing in r , in contrast to the Euclidean case. The possible lack of this monotonicity property is the ultimate reason for including the supremum over $t \in (0, r)$ in the definition (1.20) of $\delta_\gamma^r(E)$.

1.3. A finite range non-local perimeter functional. We shall apply Theorem 1.1 to a finite range non-local perimeter functional that arises in statistical mechanics. In mathematical terms, our main result is Theorem 1.5, a quantitative version of the Riesz rearrangement inequality in a case that is relevant to statistical mechanics. We now briefly discuss the variational problem that motivates Theorem 1.5.

Let Λ denote the d dimensional torus with period L , and hence volume L^d . For smooth functions m on Λ , the van der Waals free energy functional is

$$\mathcal{F}(m) = \int_{\Lambda} W(m(x))dx + \frac{\theta}{2} \int_{\Lambda} |\nabla m(x)|^2 dx$$

where $W(m) = \frac{1}{4}m^2(1 - m)^2$. The function $m(x)$ specifies the mixture of two “phases” (think liquid and vapor, for example) at x , so that where $m(x) = 1$, the system is in one phase, and where $m(x) = 0$ it is in the other (and thus $m(x) \in (0, 1)$ corresponds to some mixture of the phases).

Let $n \in (0, 1)$, and consider the problem of determining

$$\inf \left\{ \mathcal{F}(m) : \int_{\Lambda} m(x)dx = nL^d \right\} \quad (1.25)$$

For $\theta = 0$, the problem is trivial. Let D be any measurable subset of Λ with $|D| = nL^d$, and define

$$m(x) = \begin{cases} 1 & x \in D \\ 0 & x \notin D \end{cases} .$$

Any such function is a minimizer. We may think of D as a “droplet” of the $m = 1$ phase in a sea of the $m = 0$ phase. For $\theta = 0$, the shape of the droplet is irrelevant.

For $\theta > 0$, surface tension plays a role and tries to minimize the perimeter of the droplet. A classic argument of Modica and Mortola, that we now briefly sketch, shows how isoperimetry comes into play. Use the co-area formula, and then the arithmetic-geometric mean inequality to write

$$\begin{aligned} \mathcal{F}(m) &= \int_{\mathbb{R}} \int_{\{m=h\}} \left(\frac{1}{4} \frac{(1 - h^2)^2}{|\nabla m(x)|} + \frac{\theta^2}{2} |\nabla m(x)| \right) d\mathcal{H}^{d-1} dh \\ &\geq \int_{\mathbb{R}} \frac{\theta}{\sqrt{2}} |1 - h^2| \mathcal{H}^{d-1}(\{m = h\}) dh \end{aligned}$$

where \mathcal{H}^{d-1} is $d - 1$ dimensional Hausdorff measure. It is possible to nearly saturate the arithmetic-geometric mean inequality by choosing m to cross the boundary between the phases with a certain profile, and then, to nearly minimize \mathcal{F} , the quantitative isoperimetric inequality forces the phase boundary to be nearly spherical – at least when n is small enough that the droplet cannot wrap around the torus. Thus, near minimizers of the van der Waals free energy functional, which by the rules of statistical mechanics are what one is likely to observe in equilibrium, are “round droplets”. There is a cost to any departure from this optimal shape that is determined through the quantitative isoperimetric inequality.

The van der Waals free energy function is purely phenomenological; it cannot be derived from any underlying particle system. However, other free energy functionals, such as the Gates-Penrose-Lebowitz free energy functional [LP66, GP69], do arise from particle systems, and are therefore more physically significant. While they have

a similar structure, the gradient term in \mathcal{F} is replaced by a finite range non-local interaction functional that we now describe.

Let $J : [0, \infty) \rightarrow [0, \infty)$ be a decreasing Lipschitz function supported in $[0, 1]$ such that

$$\int_{\mathbb{R}^d} J(|x|) dx = 1.$$

On square integrable functions $m(x)$ on \mathbb{R}^d , we define the functional \mathcal{P}_J by setting

$$\mathcal{P}_J(m) = \int_{\Lambda \times \Lambda} J(|x - y|) |m(x) - m(y)|^2 dx dy . \quad (1.26)$$

If one replaces the gradient term in $\mathcal{F}(m)$ by $\mathcal{P}_J(m)$, one obtains a variant of the Gates-Penrose-Lebowitz free energy functional [LP66, GP69]:

$$\mathcal{G}(m) = \int_{\Lambda} W(m(x)) dx + \mathcal{P}_J(m) .$$

(The actual GPL functional has a different “double well” potential function W in it, but this does not matter here.) We would like to solve the minimization problem (1.25) with \mathcal{G} in place of \mathcal{F} .

The functional $\mathcal{P}_J(m)$ can be thought of as a finite range non-local perimeter functional in the following sense: Let m be the characteristic function of a set D with $|D| = nL^d$, $n \in (0, 1)$. Should the boundary of D be smooth enough (with a graphicality scale much larger than the interaction range of J), we would then have

$$\mathcal{P}_J(m) \asymp \mathcal{H}^{d-1}(\partial D) . \quad (1.27)$$

This motivates the intuition that \mathcal{P}_J is a non-local perimeter functional, and suggests that as for the van der Waals free energy functionals, near minimizers for \mathcal{G} will necessarily be “droplets” D that are almost spherical, at least when n is small enough that the droplets cannot wrap around the torus.

However, the Modica-Mortola strategy cannot be directly applied to the functional \mathcal{G} since the absence of gradients prevents one from making the same argument with the co-area formula. What we do instead is to investigate the behavior of \mathcal{P}_J under spherically symmetric decreasing rearrangements. The first thing we do is to specialize to the case in which m is supported in a set whose diameter is less than L , in which case we may extend m , and the integration in (1.26) to all of \mathbb{R}^d . (See [CCE⁺09] for the reduction to this case in the statistical mechanics problem.) We then have the functional

$$\mathcal{P}_J(m) = \int_{\mathbb{R}^d \times \mathbb{R}^d} J(|x - y|) |m(x) - m(y)|^2 dx dy , \quad (1.28)$$

and are in a position to apply rearrangement inequalities.

1.4. Riesz rearrangement and Lieb’s Theorem. If E is a measurable subset of \mathbb{R}^d with finite measure, then we let E^* denote the ball in \mathbb{R}^d centered at 0 with $|E^*| = |E|$. If f is a non-negative function on \mathbb{R}^d such that for each $\lambda \geq 0$, $|\{f > \lambda\}| < \infty$, the symmetric decreasing rearrangement of f is the function f^* given by

$$f^*(x) = \int_0^\infty 1_{\{f > \lambda\}^*}(x) d\lambda . \quad (1.29)$$

By construction, f^* is measurable, and for all $\lambda > 0$, $|\{f^* > \lambda\}| = |\{f > \lambda\}|$, and so for any non-negative function G on \mathbb{R}_+ ,

$$\int_{\mathbb{R}^d} G(f^*(x))dx = \int_{\mathbb{R}^d} G(f(x))dx .$$

In particular, the double well potential energy term in the GPL free energy \mathcal{G} is conserved in passing from m to m^* , as it takes the form $\int_{\mathbb{R}^d} W(m(x))dx$. The interaction energy $\mathcal{P}_J(m)$ is instead decreased, as a consequence of the following deep theorem about symmetric decreasing rearrangements (for a proof, and for more discussion of rearrangements, see [HLP34, LL01]):

Theorem 1.3 (Riesz rearrangement inequality). *Let f , g and h be non-negative integrable functions on \mathbb{R}^d . Then*

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x)g(x-y)h(y)dxdy \leq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f^*(x)g^*(x-y)h^*(y)dxdy .$$

To apply this to the functional \mathcal{P}_J , note that since $\int_{\mathbb{R}^d} J(|x|)dx = 1$,

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} J(|x-y|)m(x)^2dxdy = \int_{\mathbb{R}^d} m^2 = \int_{\mathbb{R}^d} (m^*)^2 ,$$

and thus

$$\mathcal{P}_J(m) - \mathcal{P}_J(m^*) = 2 \left(\mathcal{I}_J(m^*) - \mathcal{I}_J(m) \right) , \quad (1.30)$$

where

$$\mathcal{I}_J(m) = \int_{\mathbb{R}^d \times \mathbb{R}^d} m(x)J(|x-y|)m(y)dxdy . \quad (1.31)$$

Thus Theorem 1.3 implies that $\mathcal{P}_J(m) - \mathcal{P}_J(m^*) \geq 0$. In particular, if m is a minimizer, then equality holds in the Riesz inequality with $f = h = m$ and $g = J$. Should this necessary condition for optimality imply that m is radially decreasing, then one could try to perturb it in order to infer that near minimizers are nearly spherical.

Generally speaking, the cases of equality in the Riesz rearrangement inequality have been fully determined by Burchard [Bur96]. The matter is quite complex, as there are many ways that equality can hold without f , g and h being translates of their rearrangements. For example, suppose that $f = 1_F$, $g = 1_G$ and $h = 1_H$ where F , G and H are Borel sets of finite measure. Define $A = \{y \in \mathbb{R}^d : y + G \subset F\}$. Then, with \star denoting convolution, $1_G \star 1_F(y) = |G|$ everywhere on A . Then if $H \subset A$, $H^* \subset A^*$, and there will be equality in Riesz's inequality regardless of the "shapes" of F , G and H .

Things are different, however, when one of the functions involved, say g , is symmetric decreasing and, in addition, *every ball* (centered at the origin) is a super-level set of g . The following theorem is due to Lieb [Lie77]:

Theorem 1.4 (Lieb's theorem on cases of equality in the Riesz rearrangement inequality). *Let f , g and h be non-negative integrable functions on \mathbb{R}^d . Suppose that $g = g^*$, and that for every $r > 0$, there is a $\lambda_r > 0$ so that*

$$\{g > \lambda_r\} = rB .$$

Then whenever

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x)g(x-y)h(y)dxdy = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f^*(x)g^*(x-y)h^*(y)dxdy , \quad (1.32)$$

there is an $a \in \mathbb{R}^d$ so that $f(x) = f^*(x - a)$ and $h(y) = h^*(y - a)$ almost everywhere in x and y .

Lieb's proof of this theorem was by induction on the dimension. A different proof, based on the Brunn-Minkowski inequality, will allow us to make two extensions: The first is relatively simple and could be done within the framework of Lieb's proof: We relax the requirement that *every* centered ball is a super level set of g . The more significant extension is a quantitative version asserting, roughly speaking, that when (1.32) holds with near equality instead of equality, then f and h are still *nearly* translates of their rearrangements. The quantitative version of the Brunn-Minkowski inequality proved in this paper is basis of this. To explain the connection between this inequality and Lieb's Theorem, we now sketch a proof of Theorem 1.4 based on the Brunn-Minkowski inequality.

The starting point is the layer-cake representation, see [LL01],

$$\int_{\mathbb{R}^d} f \star g(x)h(x)dx = \int_0^\infty dr \int_0^\infty ds \int_0^\infty dt \int_{\mathbb{R}^d} 1_{F_r} \star 1_{G_s}(x)1_{H_t}(x)dx, \quad (1.33)$$

where for f, g and h as above and $r, s, t > 0$ we have set

$$F_r = \{f > r\}, \quad G_s = \{g > s\} \quad \text{and} \quad H_t = \{h > t\}.$$

For each fixed r, s , the continuous function $1_{F_r} \star 1_{G_s}(x)$ is supported on the closure of the Minkowski sum $F_r + G_s$. (One must be careful about sets of measure zero as explained in section 4.) One way to prove Lieb's theorem is to prove that for fixed $r, t > 0$, there is a set $A \subset \mathbb{R}_+$ of strictly positive measure such that when $t \in A$,

$$\int_{\mathbb{R}^d} 1_{F_r} \star 1_{G_s}(x)1_{H_t}(x)dx < \int_{\mathbb{R}^d} 1_{F_r^*} \star 1_{G_s^*}(x)1_{H_t}(x)dx. \quad (1.34)$$

(Here, since $g = g^*$, we have $G_t = G_t^*$.)

Without loss of generality, we may suppose that $|F_r| < |H_t|$. Note that F_r^* is a ball of radius $(|F_r|/|B|)^{1/d}$, G_s^* is a ball of radius $(|G_s|/|B|)^{1/d}$ and H_t^* is a ball of radius $(|H_t|/|B|)^{1/d}$. By hypothesis, there exists an s such that

$$\left(\frac{|F_r|}{|B|}\right)^{1/d} + \left(\frac{|G_s|}{|B|}\right)^{1/d} = \left(\frac{|H_t|}{|B|}\right)^{1/d}$$

Then $1_{F_r^*} \star 1_{G_s^*}$ is supported in a ball of radius $(|F_r|/|B|)^{1/d} + (|G_s|/|B|)^{1/d}$, which is the radius of H_t^* . Hence

$$1_{F_r^*} \star 1_{G_s^*}(x)1_{H_t^*}(x) = 1_{F_r^*} \star 1_{G_s^*}(x),$$

and consequently,

$$\int_{\mathbb{R}^d} 1_{F_r^*} \star 1_{G_s^*}(x)1_{H_t^*}(x)dx = \int_{\mathbb{R}^d} 1_{F_r^*} \star 1_{G_s^*}(x)dx = |F_r||G_s|. \quad (1.35)$$

However, if F_r is not a ball, the Brunn-Minkowski inequality says that the support of $1_{F_r} \star 1_{G_s}$ has a measure that is strictly larger than that of H_t . Hence the set

$$\{x \notin H_t \quad \text{and} \quad 1_{F_r} \star 1_{G_s}(x) > 0\}$$

has positive measure. Therefore

$$\int_{\mathbb{R}^d} 1_{F_r} \star 1_{G_s}(x)1_{H_t}(x)dx = |F_r||G_s| - \int_{\mathbb{R}^d \setminus H_t} 1_{F_r} \star 1_{G_s}(x)dx. \quad (1.36)$$

Comparing (1.35) and (1.36), we see that

$$\int_{\mathbb{R}^d} 1_{F_r} \star 1_{G_s}(x) 1_{H_t}(x) dx < \int_{\mathbb{R}^d} 1_{F_r^*} \star 1_{G_s^*}(x) 1_{H_t^*}(x) dx$$

when F_r is not (equivalent to) a ball. By a dominated convergence argument this strict inequality remains valid when s is replaced by $s' \in [s, s+a]$ for some $a > 0$. From here it is easy to prove Theorem 1.4.

The two features of the this proof that are relevant to us are the following: (1) it is “localizable” in t , r and s , in the sense that if we consider r and s lying in some interval, then we only need values of t such that $|G_t|$ matches $(|E_r|^{1/d} + |F_s|^{1/d})^d$, and not any arbitrary positive number; (2) it is based on the Brunn-Minkowski inequality, for which we have a quantitative improvement.

The main difficulty to be overcome in proving a quantitative version is that while the quantitative Brunn-Minkowski inequality gives us an estimate on the measure of the set $(F_r + G_s) \cap H_t^c$, it is evident from (1.36) that what we really need is a lower bound on

$$\int_{\mathbb{R}^d \setminus H_t} 1_{F_r} \star 1_{G_s}(x) dx . \quad (1.37)$$

Even if $(F_r + G_s) \cap H_t^c$ has a large measure, $1_{F_r} \star 1_{G_s}$ may well be small on this set, and then the integral may be small.

Indeed, if G_s is a ball of radius ρ , then $1_{F_r} \star 1_{G_s}(x) = 1_{F_r \cap B_\rho(x)}$, and so if F_r is the union of many small and well separated components – think of a cloud of dust – then $\|1_{F_r} \star 1_{G_s}\|_\infty$ will be very small. However, in this case, we will be far from having equality in the Riesz rearrangement inequality.

This suggests making a decomposition of any set E (here, F_r), as follows: Given $\lambda, \tau > 0$ we set

$$E^{\lambda, \tau} = E \setminus D^{\lambda, \tau} , \quad (1.38)$$

$$D^{\lambda, \tau} = \left\{ x \in E : \frac{|E \cap B_{x, \tau}|}{|B_{x, \tau}|} < \lambda \right\} . \quad (1.39)$$

For small λ , and any τ , $D^{\lambda, \tau}$ is the “dusty” component of E . The key to obtaining a lower bound on the integral in (1.37) is to show that for appropriately chosen λ and τ , the dusty component of F_r must be very small whenever

$$\int_{\mathbb{R}^d} 1_{F_r} \star 1_{G_s}(x) 1_{H_t}(x) dx \approx \int_{\mathbb{R}^d} 1_{F_r^*} \star 1_{G_s^*}(x) 1_{H_t^*}(x) dx .$$

This proof of Lieb’s Theorem via the Brunn-Minkowski inequality also shows that the heart of the matter is a geometric inequality for super level sets. Indeed, consider a function m with values in $[0, 1]$, and notice that if $E_t = \{m > t\}$, then $m(x) = \int_0^1 1_{E_t}(x) dt$ and

$$\mathcal{I}_J(m) = \int_0^1 \int_0^1 \mathcal{E}_J(E_t, E_s) ds dt , \quad (1.40)$$

where we have set

$$\mathcal{E}_J(E, F) = \int_E \int_F J(|x - y|) dx dy , \quad E, F \subset \mathbb{R}^d . \quad (1.41)$$

Then defining $\delta_J(E, F)$ by

$$\delta_J(E, F) = \mathcal{E}_J(E^*, F^*) - \mathcal{E}_J(E, F) , \quad (1.42)$$

we obtain

$$\mathcal{I}_J(m^*) - \mathcal{I}_J(m) = \int_0^1 dt \int_0^1 \delta_J(E_t, E_s) ds.$$

Notice also that for $s < t$, $E_t \subset E_s$, so we are interested in $\mathcal{E}_J(E, F)$ when $E \subset F$. Under mild conditions of the distribution function of m , for each t one can bound from below the length of the interval of those values of s , with $s < t$, such that

$$\frac{1}{4} \leq \left(\frac{|E_s|}{|B|} \right)^{1/d} - \left(\frac{|E_t|}{|B|} \right)^{1/d} \leq \frac{3}{4},$$

and in the statistical mechanical application, we are chiefly interested in “droplets” that are large compared with the unit ball. The following theorem is our quantitative version of Lieb’s Theorem. The application to statistical mechanics that motivates it will be made elsewhere.

Theorem 1.5. *Let us consider a decreasing Lipschitz function $J : [0, \infty) \rightarrow [0, \infty)$ with $\text{spt}(J) \subset [0, 1]$ such that*

$$\int_{\mathbb{R}^d} J(|x|) dx = 1, \quad -J' \geq \frac{r}{k} \quad \text{on } [0, 3/4], \quad \|J\|_{C^0(\mathbb{R}^d)} \leq k, \quad (1.43)$$

for some $k > 0$. For subsets E, F of \mathbb{R}^d , let $\delta_J(E; F)$ be defined by (1.42).

If $E \subset F \subset \mathbb{R}^d$ are such that

$$\frac{1}{4} \leq \left(\frac{|F|}{|B|} \right)^{1/d} - \left(\frac{|E|}{|B|} \right)^{1/d} \leq \frac{3}{4}, \quad |E| \geq 2|B|, \quad (1.44)$$

then one has

$$|E|^{1-1/d} \alpha(E; B)^{8(d+2)} \leq C(d, k) \delta_J(E; F). \quad (1.45)$$

Remark 1.4. Note the factor of $|E|^{1-1/d}$ on the left-hand side of (1.45), which is proportional to $\mathcal{H}^{d-1}(\partial E^*)$. That is, the size of the “remainder term” is a multiple, depending on the asymmetry $\alpha(E; B)$, of the perimeter of E^* . This is the size we would expect since we are measuring the deficit under rearrangement of a finite range non-local perimeter functional.

2. IMPROVEMENT IN THE BRUNN-MINKOWSKI INEQUALITY

In this section we prove Theorem 1.1. Recall that K is a bounded open convex set containing the origin, so that

$$K = \{x \in \mathbb{R}^d : \|x\| < 1\},$$

where $\|\cdot\| : \mathbb{R}^d \rightarrow [0, \infty)$ is the convex one-homogenous function on \mathbb{R}^d defined as

$$\|x\| = \inf \left\{ t > 0 : \frac{x}{t} \in K \right\}, \quad x \in \mathbb{R}^d.$$

Let us consider the convex one-homogenous function $\|\cdot\|_* : \mathbb{R}^d \rightarrow [0, \infty)$ defined by setting

$$\|y\|_* = \sup \left\{ x \cdot y : \|x\| < 1 \right\}, \quad y \in \mathbb{R}^d.$$

Given a set of locally finite perimeter E in \mathbb{R}^d , and a bounded open set $A \subset \mathbb{R}^d$, we set

$$P_K(E; A) = \int_{A \cap \partial^* E} \|\nu_E\|_* d\mathcal{H}^{d-1},$$

where $\partial^* E$ denotes the reduced boundary of E and where ν_E is the measure theoretic outer unit normal to E (see [Mag12, Chapter 16] for these definitions). When E is an open set with Lipschitz boundary, one can replace $\partial^* E$ with the topological boundary in this definition. Notice that because we do not assume that $K = -K$, it may well be that $\|y\|_* \neq \|-y\|_*$, and thus that $P_K(E; A) \neq P_K(\mathbb{R}^d \setminus E; A)$.

Given this notion of anisotropic perimeter we can consider the isoperimetric problem of determining

$$\inf \left\{ P_K(E) : |E| = m \right\}, \quad m > 0. \quad (2.1)$$

It turns out that if $r > 0$ is such that $|rK| = m$, then $\{x + rK\}_{x \in \mathbb{R}^d}$ is the family of minimizers in (2.1). By taking into account that

$$P_K(K) = d|K|, \quad (2.2)$$

this assertion can in fact be reformulated as the so-called Wulff inequality (1.8). By repeating *verbatim* the classical proof of the coarea formula (see, for example, [Mag12, Theorem 13.1]) we find that

$$\int_A \|\nabla u(x)\|_* dx = \int_{\mathbb{R}} P_K(\{u > t\}; A) dt,$$

(as elements of $[0, \infty]$) whenever $u : \mathbb{R}^d \rightarrow \mathbb{R}$ is a Lipschitz function and A is an open set. If we use sub-level sets instead of super-level sets of u we find of course

$$\int_A \|\nabla u(x)\|_* dx = \int_{\mathbb{R}} P_K(\{u < t\}; A) dt. \quad (2.3)$$

Setting $K_s = sK = \{sx : x \in K\}$, $s > 0$, we now prove the following lemma.

Lemma 2.1. *If E is a closed set in \mathbb{R}^d , then*

$$|E + K_r| = |E| + \int_0^r P_K(E + K_s) ds. \quad (2.4)$$

Proof. If we set

$$g_E(x) = \inf \left\{ \|x - y\| : y \in E \right\}, \quad x \in \mathbb{R}^d,$$

then g_E is a Lipschitz function with

$$|g_E(x) - g_E(y)| \leq \|x - y\|, \quad \forall x, y \in \mathbb{R}^d, \quad (2.5)$$

and $\{g_E = 0\} = E$, $\{g_E < s\} = E + K_s$ for every $s > 0$. Let x be a point of differentiability for g_E . By (2.5), we certainly have

$$|\nabla g_E(x) \cdot e| \leq \|e\|, \quad \forall e \neq 0, \quad (2.6)$$

If now $g_E(x) > 0$ then there exists $z \in E$ such that $g_E(x) = \|x - z\|$ and for $0 < h < \|x - z\|$ and $e_0 = -(x - z)/\|x - z\|$ we easily find

$$g_E(x + h e_0) \leq \|x + h e_0 - z\| = \|x - z\| \left(1 - \frac{h}{\|x - z\|} \right) = g_E(x) - h,$$

that gives $\nabla g_E(x) \cdot e_0 \leq -1$, or, in other terms

$$\nabla g_E(x) \cdot (-e_0) \geq 1, \quad \text{for some } e_0 \text{ with } \|-e_0\| = 1. \quad (2.7)$$

Combining (2.6) and (2.7) with Rademacher's theorem we thus find that $\|\nabla g_E\|_* = 1$ a.e. on $\{g_E > 0\}$ so that, by the coarea formula (2.3) (applied to the open set $A = \{0 < g_E < r\}$)

$$|\{0 < g_E < r\}| = \int_{\mathbb{R}} P_K(\{g_E < s\}; \{0 < g_E < r\}) ds.$$

Since we have

$$\begin{aligned} |\{0 < g_E < r\}| &= |E + K_r| - |E|, \\ \int_{\mathbb{R}} P_K(\{g_E < s\}; \{0 < g_E < r\}) ds &= \int_0^r P_K(\{g_E < s\}) ds, \end{aligned}$$

the proof is complete. \square

We shall also need the following elementary lemma.

Lemma 2.2. *If $E, F \subset \mathbb{R}^d$ are Lebesgue measurable sets, with $0 < |E| |F| < \infty$, then*

$$\left| |E| \alpha(E; K) - |F| \alpha(F; K) \right| \leq |E \Delta F|. \quad (2.8)$$

Proof. Let $x \in \mathbb{R}^d$ be such that $2|E| \alpha(E; K) = |E \Delta(x + r_E K)|$, where $r_E = (|E|/|K|)^{1/d}$. If $r_F = (|F|/|K|)^{1/d}$, then we have,

$$\begin{aligned} 2|F| \alpha(F; K) &\leq |F \Delta(x + r_F K)| \\ &\leq |F \Delta E| + |(x + r_E K) \Delta(x + r_F K)| + 2|E| \alpha(E; F). \end{aligned}$$

Since K is star-shaped with respect to the origin we have

$$|(x + r_E K) \Delta(x + r_F K)| = ||r_E K| - |r_F K|| = ||F| - |E|| \leq |E \Delta F|,$$

and thus we conclude

$$|F| \alpha(F; K) - |E| \alpha(E; K) \leq |E \Delta F|.$$

By symmetry, we find (2.8). \square

Proof of Theorem 1.1. Step one: We start showing that, in proving Theorem 1.1, we can directly assume that E is a compact set. Indeed, let E be a Lebesgue measurable set, and consider a sequence of compact sets $\{E_h\}_{h \in \mathbb{N}}$ with $E_h \subset E$, $|E_h| > 0$, and $|E \setminus E_h| \rightarrow 0$ as $h \rightarrow \infty$. By Lemma 2.2 we have $\alpha(E_h; K) \rightarrow \alpha(E; K)$ as $h \rightarrow \infty$, while the inclusion $E_h + K \subset E + K$ implies

$$\delta(E_h; K) \leq \sigma(E_h; K)^{1/d} \left\{ \frac{|E + K|^{1/d}}{|E_h|^{1/d} + |K|^{1/d}} - 1 \right\},$$

so that, in particular, $\limsup_{h \rightarrow \infty} \delta(E_h; K) \leq \delta(E; K)$. Therefore, if Theorem 1.1 holds true on compact sets, then it holds true on Lebesgue measurable sets.

Step two: We address the one-dimensional case $d = 1$. We want to prove that

$$2 \delta(E; K) \geq \alpha(E; K),$$

where $K = (a, b)$ for some $a < 0 < b$ and where $E \subset \mathbb{R}$ is compact. Exploiting the scale invariance properties of δ and α we can equivalently prove that

$$2 \max \left\{ r, \frac{1}{r} \right\} \left(\frac{|E + K_r|}{|K + K_r|} - 1 \right) \geq \alpha(E; K), \quad \forall r > 0, \quad (2.9)$$

where E is a compact set with $|E| = |K|$. Let us now set

$$\alpha = \|-1\|_*, \quad \beta = \|1\|_*,$$

so that $\alpha, \beta > 0$ and, if $\{(a_i, b_i)\}_{i=1}^m$ is a family of bounded open intervals in \mathbb{R} lying at mutually positive distances, then

$$P_K\left(\bigcup_{i=1}^m (a_i, b_i)\right) = m(\alpha + \beta).$$

Since $E + K_s$ is a bounded open set in \mathbb{R} for every $s > 0$, with $E + K_s \subset E + K_r$ if $s < r$, and since, by Lemma 2.1,

$$\infty > |E + K_r| = |E| + \int_0^r P_K(E + K_s) ds,$$

we deduce that $E + K_s$ is a finite union of intervals for every $s > 0$. In particular, if we set

$$N(r) = \frac{P_K(E + K_r)}{\alpha + \beta}, \quad r > 0,$$

then $N(r) \in \mathbb{N}$ for every $r > 0$, $N(r)$ is decreasing on $r > 0$, and $N(r) \geq 1$ for every $r > 0$. Since $P(K + K_s) = 1$ for every $s > 0$, by (2.4) we find that

$$|E + K_r| - |K + K_r| = (\alpha + \beta) \int_0^r (N(s) - 1) ds. \quad (2.10)$$

Let us now set

$$r_0 = \inf\{r > 0 : N(r) = 1\}.$$

(Notice that, trivially, $r_0 < \infty$.) If $r_0 = 0$, then $E + K_r$ is an interval for every $r > 0$, thus $\alpha(E; K) = 0$ and (2.9) follows immediately. If $r_0 > r$, then by (2.10), and since $\alpha(E; K) < 1$, we find

$$|E + K_r| - |K + K_r| \geq (\alpha + \beta) r (N(r) - 1) \geq (\alpha + \beta) r \geq (\alpha + \beta) r \alpha(E; K). \quad (2.11)$$

Since $\alpha + \beta = P_K(K) = |K|$ (by (2.2)) and $|K + K_r| = (1 + r)|K|$, we conclude from (2.11)

$$\frac{|E + K_r|}{|K + K_r|} - 1 \geq \frac{r}{1 + r} \alpha(E; K), \quad (r \leq r_0),$$

which is easily seen to imply (2.9). We are thus left to consider (2.9) in the case that $r > r_0$. In this case from (2.10), the definition of r_0 and, again, by $\alpha + \beta = |K|$, we find

$$|E + K_r| - |K + K_r| = (\alpha + \beta) r_0 = |K| r_0, \quad (2.12)$$

as well as that

$$|E + K_r| - |K + K_r| = |E + K_{r_0}| - |K + K_{r_0}|. \quad (2.13)$$

Up to a translation, $E + K_{r_0} = (-Ra, Rb) = K_R$ for some $R > 0$. Therefore, by (2.13),

$$|E + K_r| - |K + K_r| = |K| (R - (1 + r_0)). \quad (2.14)$$

Adding up (2.12) and (2.14), and since $E \subset E + K_{r_0} = K_R$, we find

$$\begin{aligned} 2(|E + K_r| - |K + K_r|) &= |K| (R - 1) = |K_R \setminus K| \geq |E \setminus K| = \frac{|E \Delta K|}{2} \\ &\geq |K| \alpha(E; K). \end{aligned}$$

that in turn gives

$$\frac{|E + K_r|}{|K + K_r|} - 1 \geq \frac{\alpha(E; K)}{2(1 + r)}, \quad (r > r_0).$$

Since this last inequality implies (2.9), we have completed the proof of step two.

Step three: We now prove the theorem in dimension $d \geq 2$. By step one and by exploiting the scale invariance of δ and α , we need to prove that if E is a compact set in \mathbb{R}^d with $|E| = |K|$, then

$$\alpha(E; K)^4 \leq C(d) \max\{1, r^{4d+2}\} \delta(E; K_r), \quad \forall r > 0, \quad (2.15)$$

where

$$\delta(E; K_r) = \max\left\{r, \frac{1}{r}\right\} \left(\frac{|E + K_r|^{1/d}}{|K + K_r|^{1/d}} - 1\right), \quad r > 0. \quad (2.16)$$

Let us thus fix a value of $r > 0$, and set for the sake of brevity

$$\eta = \frac{|E + K_r|}{|K + K_r|}.$$

By (1.1), $\eta \geq 1$. We claim that we may directly assume

$$\eta \leq 1 + \kappa(r), \quad (2.17)$$

where

$$\kappa(r) = \min\left\{r, \frac{1}{r}\right\}. \quad (2.18)$$

Indeed $\kappa(r) \in (0, 1]$ for every $r > 0$ and

$$(1 + \kappa)^{1/d} - 1 \geq (2^{1/d} - 1) \kappa, \quad \forall \kappa \in [0, 1]. \quad (2.19)$$

Therefore, if (2.17) does not hold true, then, as $\alpha(E; K) < 1$,

$$\begin{aligned} \delta(E; K_r) &= \max\left\{r, \frac{1}{r}\right\} (\eta^{1/d} - 1) \geq \max\left\{r, \frac{1}{r}\right\} \left((1 + \kappa(r))^{1/d} - 1\right) \\ &\geq (2^{1/d} - 1) \max\left\{r, \frac{1}{r}\right\} \kappa(r) = (2^{1/d} - 1) \geq (2^{1/d} - 1) \alpha(E; K)^4, \end{aligned}$$

and (2.15) follows provided

$$C(d) \geq \frac{1}{2^{1/d} - 1}. \quad (2.20)$$

We have thus reduced to consider the case that (2.17) holds true. In this case, by (2.19) we find that

$$\delta(E; K_r) \geq \max\left\{r, \frac{1}{r}\right\} (2^{1/d} - 1) (\eta - 1). \quad (2.21)$$

Having this lower bound for $\delta(E; K_r)$ in mind, we now apply Lemma 2.1 to find

$$|E + K_r| - |K + K_r| = \int_0^r \left(P_K(E + K_s) - P_K(K + K_s)\right) ds. \quad (2.22)$$

From now, for the sake of brevity, we directly set $\alpha(G; K) = \alpha(G)$ for every $G \subset \mathbb{R}^d$. By applying the quantitative Wulff inequality (1.9) to $E + K_s$ we deduce that

$$\begin{aligned} |E + K_r| - |K + K_r| &\geq n|K|^{1/d} \int_0^r |E + K_s|^{1/d'} \frac{\alpha(E + K_s)^2}{C_0(d)} ds \\ &\quad + n|K|^{1/d} \int_0^r \left(|E + K_s|^{1/d'} - |K + K_s|^{1/d'} \right) ds, \end{aligned} \quad (2.23)$$

where the second integral on the right-hand side of (2.23) is non-negative by the Brunn-Minkowski inequality. By Hölder inequality, we thus find

$$\begin{aligned} &\frac{C_0(d)}{d|K|^{1/d}} \left(|E + K_r| - |K + K_r| \right) \int_0^r |E + K_s|^{1/d'} ds \\ &\geq \left(\int_0^r |E + K_s|^{1/d'} \alpha(E + K_s) ds \right)^2 \\ &\geq |E + K_r|^{-2/d} \left(\int_0^r |E + K_s| \alpha(E + K_s) ds \right)^2. \end{aligned} \quad (2.24)$$

Now, by Wulff's inequality (1.8), by Lemma 2.1, and by (2.17)

$$\begin{aligned} n|K|^{1/d} \int_0^r |E + K_s|^{1/d'} ds &\leq \int_0^r P(E + K_s) ds = |E + K_r| - |E| \\ &\leq \eta |K + K_r| - |K| \\ &\leq |K| \left((1 + \kappa(r))(1 + r)^d - 1 \right) \\ &\leq 2^{d+1} |K| \max\{r, r^d\}; \end{aligned} \quad (2.25)$$

in particular, having shown that $|E + K_r| - |E| \leq 2^{d+1} |K| \max\{r, r^d\}$, we certainly have

$$|E + K_r| \leq 2^{d+2} |K| \max\{1, r^d\}. \quad (2.26)$$

Thus, by (2.24), (2.25) and (2.26), we find that

$$\begin{aligned} &\left(\int_0^r |E + K_s| \alpha(E + K_s) ds \right)^2 \\ &\leq \frac{C_0(d)}{d|K|^{1/d}} \left(|E + K_r| - |K + K_r| \right) \frac{2^{d+1} |K| \max\{r, r^d\}}{d|K|^{1/d}} \times \\ &\quad \times \left(2^{d+2} |K| \max\{1, r^d\} \right)^{2/d} \\ &= \frac{2^{d+3+4/d} C_0(d)}{d^2} \left(|E + K_r| - |K + K_r| \right) |K| \max\{r, r^d\} \max\{1, r^2\} \\ &= \frac{2^{d+3+4/d} C_0(d)}{d^2} (\eta - 1) |K|^2 (1 + r)^d \max\{r, r^d\} \max\{1, r^2\} \\ &\leq \frac{2^{5d} C_0(d)}{d^2} |K|^2 \max\{r, r^{2(d+1)}\} (\eta - 1), \end{aligned} \quad (2.27)$$

where in the last inequality we have used $(1+r)^d \leq 2^d \max\{1, r^d\}$ and $2d+3+(4/d) \leq 5d$. Let us now consider $\varepsilon \in (0, \min\{r, 1\})$, and apply Lemma 2.2 to compare E and $E + K_s$ for $s \in (0, \varepsilon)$. In this way we find that

$$\int_0^\varepsilon |E + K_s| \alpha(E + K_s) ds \geq \varepsilon |K| \alpha(E) - \int_0^\varepsilon |E \Delta (E + K_s)| ds, \quad (2.28)$$

where

$$\begin{aligned}
& \int_0^\varepsilon |E\Delta(E + K_s)| ds = \int_0^\varepsilon (|E + K_s| - |E|) ds = \int_0^\varepsilon ds \int_0^s P_K(E + K_t) dt \\
& = \int_0^\varepsilon ds \int_0^s (P_K(E + K_t) - P_K(K + K_t)) dt + d|K| \int_0^\varepsilon ds \int_0^s (1+t)^{d-1} dt \\
& \leq \varepsilon (|E + K_r| - |K + K_r|) + |K| \left(\frac{(1+\varepsilon)^{d+1}}{d+1} - \frac{1}{d+1} - \varepsilon \right) \\
& \leq \varepsilon (|E + K_r| - |K + K_r|) + d2^{d-1} |K| \varepsilon^2 \\
& = \varepsilon (1+r)^d |K| (\eta - 1) + d2^{d-1} |K| \varepsilon^2 \\
& \leq \varepsilon 2^d \max\{1, r^d\} |K| (\eta - 1) + d2^{d-1} |K| \varepsilon^2 \tag{2.29}
\end{aligned}$$

where we have also used the elementary inequality

$$\frac{(1+x)^{d+1}}{d+1} - \frac{1}{d+1} - x \leq d2^{d-1} x^2, \quad \forall x \in [0, 1].$$

We now combine (2.27), (2.28), and (2.29) to prove that

$$\alpha(E) \leq a \max\{r^{1/2}, r^{d+1}\} \frac{\sqrt{\eta-1}}{\varepsilon} + 2^d \max\{1, r^d\} (\eta - 1) + b\varepsilon, \tag{2.30}$$

for every $\varepsilon \in (0, \min\{1, r\})$, where we have set

$$a = \frac{\sqrt{2^{5d} C_0(d)}}{d}, \quad b = d2^{d-1}.$$

In the case $r < 1$, by (2.18), we have $\eta - 1 \leq r$, and thus

$$\varepsilon = \left(\frac{\eta - 1}{r} \right)^{1/4} r,$$

is an admissible choice in (2.30); correspondingly we find

$$\begin{aligned}
\alpha(E) & \leq a r^{1/2} \frac{(\eta - 1)^{1/4}}{r^{3/4}} + 2^d (\eta - 1) + b \left(\frac{\eta - 1}{r} \right)^{1/4} r \\
& \leq (a + b + 2^d r) \left(\frac{\eta - 1}{r} \right)^{1/4}.
\end{aligned}$$

Since, by (2.21), $\delta(E; K_r) \geq (\log(2)/d) ((\eta - 1)/r)$ when $r < 1$, we conclude that

$$\alpha(E)^4 \leq (a + b + 2^d)^4 \frac{d}{\log(2)} \delta(E; K_r), \quad \text{if } r \leq 1.$$

This implies (2.15) for every $r \leq 1$, provided we set

$$C(d) = \frac{d}{\log(2)} \left(\frac{\sqrt{2^{5d} C_0(d)}}{d} + d2^{d-1} + 2^d \right)^4. \tag{2.31}$$

(Notice that this value of $C(d)$ satisfies (2.20).) If, instead, $r > 1$, then by (2.18) we have $r(\eta - 1) \leq 1$, and

$$\varepsilon = (r(\eta - 1))^{1/4},$$

is admissible in (2.30), that gives

$$\begin{aligned}
\alpha(E) & \leq a r^{d+(3/4)} (\eta - 1)^{1/4} + 2^d r^d (\eta - 1) + b r^{1/4} (\eta - 1)^{1/4} \\
& \leq \left(a r^{d+(3/4)} + 2^d r^{d-(3/4)} + b r^{1/4} \right) (\eta - 1)^{1/4}.
\end{aligned}$$

At the same time, by (2.21) we have $\delta(E; K_r) \geq (\log(2)/d) r (\eta - 1)$, so that

$$\begin{aligned} \alpha(E)^4 &\leq \left(a r^{d+(3/4)} + 2^d r^{d-(3/4)} + b r^{1/4} \right)^4 (\eta - 1) \\ &\leq \frac{d}{\log(2)} \left(a + b + 2^d \right)^4 r^{4d+3} \frac{\delta(E; K_r)}{r}. \end{aligned}$$

This concludes the proof of (2.15). \square

3. IMPROVEMENT IN THE GAUSSIAN CONCENTRATION INEQUALITY

This section is devoted to the proof of Theorem 1.2. As in the case of the proof of Theorem 1.1, we shall need two preliminary facts: first, if $E \subset \mathbb{R}^d$ is closed, then

$$\gamma_d(E + rB) - \gamma_d(E) = \frac{1}{\sqrt{2\pi}} \int_0^r P_\gamma(E + B_t) dt; \quad (3.1)$$

second, if $E, F \subset \mathbb{R}^d$, then

$$|\alpha_\gamma(E) - \alpha_\gamma(F)| \leq 2 \gamma_d(E \Delta F). \quad (3.2)$$

Since the proofs are entirely analogous to the arguments of Lemma 2.1 and Lemma 2.2 we omit them. We notice that, since $\alpha_\gamma(E) \leq 1$ for every $E \subset \mathbb{R}^d$, then (3.2) immediately implies

$$|\alpha_\gamma(E)^2 - \alpha_\gamma(F)^2| \leq 2 |\alpha_\gamma(E) - \alpha_\gamma(F)| \leq 4 \gamma_d(E \Delta F). \quad (3.3)$$

It will be convenient to set $\sigma_E : [0, \infty) \rightarrow [s_E, \infty)$,

$$\gamma_d(H(\sigma_E(t))) = \gamma_d(E + B_t) = \gamma_d(H(s_{E+B_t})), \quad t \geq 0, \quad (3.4)$$

i.e. $\sigma_E(t) = s_{E+B_t}$, and, in particular, $\sigma_E(0) = s_E$. It is useful to keep in mind that since $\phi(s)$ is increasing, see (1.16), and since $\gamma_d(E + B_t) \geq \gamma_d(H(s_E) + B_t)$ with $H(s_E) + B_t = H(s_E + t)$, we clearly have that

$$\sigma_E(t) \geq s_E + t, \quad \forall t > 0.$$

However, taking into account that

$$P_\gamma(H(s)) = e^{-s^2/2}, \quad \forall s \in \mathbb{R}, \quad (3.5)$$

we easily see that $P_\gamma(H(s_{E+B_t})) - P_\gamma(H(s_E) + B_t)$ has no definite sign, and that

$$P_\gamma(H(s_{E+B_t})) - P_\gamma(H(s_E) + B_t) \geq 0 \quad \text{if and only if} \quad \sigma_E(t) \leq |s_E + t|. \quad (3.6)$$

Proof of Theorem 1.2. We fix $\lambda \in (\gamma_d(E), 1)$ and $r < r_E(\lambda)$. By an approximation argument we may directly assume that E is closed, and since $\alpha_\gamma(E) \leq 1$ and $C_* \geq 1$, we can definitely assume that

$$\delta_\gamma^r(E) \leq 1.$$

Next we exploit (3.1) to deduce (1.23), which combined with (1.24) and (3.5) gives, for every $r > 0$,

$$\sqrt{2\pi} \delta_\gamma^r(E) + \int_0^r e^{-(s_E+t)^2/2} - e^{-\sigma_E(t)^2/2} dt \geq \int_0^r P_\gamma(E + B_t) - P_\gamma(H(\sigma_E(t))) dt. \quad (3.7)$$

As noticed in (3.6), the integral on the left-hand side could be positive depending on the value of $\gamma_d(E)$ and t . To estimate its size, we shall use the fact that

$$|e^{-b^2/2} - e^{-a^2/2}| \leq \sqrt{2\pi} \max\{a, b\} |\phi(a) - \phi(b)|, \quad \forall a, b > 0. \quad (3.8)$$

where ϕ is defined as in (1.16). The proof of (3.8) is immediate: if we set

$$\alpha = e^{-a^2/2} \in (0, 1), \quad a = \sqrt{\log\left(\frac{1}{\alpha^2}\right)} \in (0, \infty), \quad \psi(\alpha) = \phi\left(\sqrt{\log\left(\frac{1}{\alpha^2}\right)}\right)$$

and, similarly, $\beta = e^{-b^2/2}$, then by a simple computation $\psi'(\alpha) = \frac{-1}{\sqrt{2\pi \log(\alpha^{-2})}}$ and thus

$$|\psi(\beta) - \psi(\alpha)| \geq \frac{|\beta - \alpha|}{\sqrt{2\pi \log(\min\{\alpha, \beta\}^{-2})}}, \quad \forall \alpha, \beta \in (0, 1),$$

which immediately gives us (3.8). If t is such that $\sigma_E(t) \leq |s_E + t|$, then $e^{-(s_E+t)^2/2} - e^{-\sigma_E(t)^2/2} \leq 0$. Otherwise, by (3.8) we find

$$e^{-(s_E+t)^2/2} - e^{-\sigma_E(t)^2/2} \leq \sqrt{2\pi} \sigma_E(t) \delta_\gamma^t(E),$$

and since $\phi(\sigma_E(t)) = \gamma_d(E + B_t) \leq \gamma_d(E + rB) < \lambda$ thanks to $r < r_E(\lambda)$, we conclude that

$$e^{-(s_E+t)^2/2} - e^{-\sigma_E(t)^2/2} \leq \sqrt{2\pi} \phi^{-1}(\lambda) \delta_\gamma^t(E).$$

By (3.7) we thus infer

$$\begin{aligned} \sqrt{2\pi} \left(1 + \phi^{-1}(\lambda)\right) \delta_\gamma^r(E) &\geq \int_0^r P_\gamma(E + B_t) - P_\gamma(H(\sigma_E(t))) dt \\ &\geq \int_0^r \frac{e^{\sigma_E(t)^2/2}}{c(1 + \sigma_E(t)^2)} \alpha_\gamma(E + B_t)^2 dt, \end{aligned}$$

where in the last inequality we have used (1.19). By exploiting the trivial estimate

$$\frac{e^{s^2/2}}{1 + s^2} \geq \frac{e^{s^2/4}}{4}, \quad s > 0,$$

together with (3.3), we find that for every $\rho \leq r$

$$\sqrt{2\pi} \left(1 + \phi^{-1}(\lambda)\right) \delta_\gamma^r(E) \geq \frac{e^{s_E^2/4}}{4c} \int_0^\rho \left(\alpha_\gamma(E)^2 - 4\gamma_d(E\Delta(E + B_t))\right) dt.$$

Now, since $\gamma_d(H(s_E) + B_t) - \gamma_d(E) = \phi(s_E + t) - \phi(s_E) \leq t/\sqrt{2\pi}$, one gets

$$\begin{aligned} \int_0^\rho \gamma_d(E\Delta(E + B_t)) dt &\leq \frac{\rho}{\max\{1, 1/r\}} \delta_\gamma^r(E) + \int_0^\rho \gamma_d(H(s_E) + B_t) - \gamma_d(E) dt \\ &\leq \frac{\rho}{\max\{1, 1/r\}} \delta_\gamma^r(E) + \frac{\rho^2}{2\sqrt{2\pi}}, \end{aligned}$$

so that, in conclusion, for $\rho \leq r < r_E(\lambda)$,

$$\alpha_\gamma(E)^2 \leq 4\sqrt{2\pi} c e^{-s_E^2/2} (1 + \phi^{-1}(\lambda)) \frac{\delta_\gamma^r(E)}{\rho \max\{1, 1/r\}} + 4\delta_\gamma^r(E) + \frac{\rho}{2\sqrt{2\pi}}. \quad (3.9)$$

If $r > \delta_\gamma^r(E)^{1/2}$, then we choose $\rho = \delta_\gamma^r(E)^{1/2}$ and thus obtain from (3.9) and $\delta_\gamma^r(E) \leq 1$

$$\alpha_\gamma(E)^2 \leq \left(4\sqrt{2\pi} c e^{-s_E^2/2} (1 + \phi^{-1}(\lambda)) + 4 + \frac{1}{2\sqrt{2\pi}}\right) \sqrt{\delta_\gamma^r(E)};$$

in, instead, $r \leq \delta_\gamma^r(E)^{1/2}$, then $r \leq 1$ and setting $\rho = r$ we obtain

$$\alpha_\gamma(E)^2 \leq \left(4\sqrt{2\pi} c e^{-s_E^2/2} (1 + \phi^{-1}(\lambda)) + 4\right) \delta_\gamma^r(E) + \frac{\sqrt{\delta_\gamma^r(E)}}{2\sqrt{2\pi}}.$$

By taking into account that $c = 80\pi^2\sqrt{2\pi}$, one finds

$$4\sqrt{2\pi} c e^{-s_E^2/2} (1 + \phi^{-1}(\lambda)) + 4 + \frac{1}{2\sqrt{2\pi}} \leq (5 + 1280\pi^3) (1 + \phi^{-1}(\lambda)),$$

and the proof of (1.22) is complete. \square

4. STABILITY IN THE RIESZ REARRANGEMENT INEQUALITY

The goal of this section is proving Theorem 1.5. Let us recall that we are considering a decreasing Lipschitz function $J : [0, \infty) \rightarrow [0, \infty)$ with $\text{spt}(J) \subset [0, 1]$ such that

$$\int_{\mathbb{R}^d} J(|x|) dx = 1, \quad -J' \geq \frac{r}{k} \quad \text{on } [0, 3/4], \quad \|J\|_{C^0(\mathbb{R}^d)} \leq k, \quad (4.1)$$

for some $k > 0$, and that given $E, F \subset \mathbb{R}^d$, we set

$$\begin{aligned} \mathcal{E}_J(E, F) &= \int_F \int_E J(|x - y|) dx dy, \\ \delta_J(E, F) &= \mathcal{E}_J(E^*, F^*) - \mathcal{E}_J(E, F). \end{aligned}$$

We shall actually assume that $E \subset F \subset \mathbb{R}^d$, and denote by

$$r_{E,F} = \frac{|F|^{1/d}}{|B|^{1/d}} - \frac{|E|^{1/d}}{|B|^{1/d}},$$

the radius such that $|E^* + B_{r_{E,F}}| = |F^*|$. We assume that

$$\frac{1}{4} \leq r_{E,F} \leq \frac{3}{4}, \quad |E| \geq 2|B|, \quad (4.2)$$

and aim to prove

$$|E|^{1-1/d} \alpha(E; B)^{8(d+2)} \leq C(d, k) \delta_J(E; F).$$

Proof of Theorem 1.5. Step one: Given $\lambda, \tau > 0$ we set

$$E^{\lambda, \tau} = E \setminus D^{\lambda, \tau}, \quad (4.3)$$

$$D^{\lambda, \tau} = \left\{ x \in E : \frac{|E \cap B_{x, \tau}|}{|B_{x, \tau}|} < \lambda \right\}. \quad (4.4)$$

We claim that for every $\lambda > 0$ and $\tau \in (0, r_{E,F})$ one has

$$k \delta_J(E, F) \geq \lambda \tau^{d+1} \int_\tau^{r_{E,F}} |(E^{\lambda, \tau} + (r - \tau)B) \cap F^c| dr. \quad (4.5)$$

(Later on we shall specify the size of λ and τ , and they both will be small in terms of $\alpha(E; B)$.) To prove (4.5), since $r_{E,F}$ is the difference of the radii of F^* and E^* , one has, for every $r < r_{E,F}$,

$$\begin{aligned} \int_{\mathbb{R}^d} (1_{rB} \star 1_{E^*}) 1_{F^*} &= \int_{F^*} |E^* \cap B_{x,r}| dx = \int_{\mathbb{R}^d} |E^* \cap B_{x,r}| dx \\ &= |E^*| |rB| = |E| |rB| = \int_{\mathbb{R}^d} 1_E \star 1_{rB}. \end{aligned} \quad (4.6)$$

By the layer-cake representation $J(|x|) = \int_{|x|}^{\infty} -J'(r)dr = \int_0^1 -J'(r)1_{rB}(x)dr$ and by (4.6) we find

$$\begin{aligned}
\delta_J(E, F) &= \int_0^1 -J'(r) \left(\int_{\mathbb{R}^d} (1_{rB} \star 1_{E^*}) 1_{F^*} - (1_{rB} \star 1_E) 1_F \right) dr \\
&\geq \int_0^{r_{E,F}} -J'(r) \left(\int_{\mathbb{R}^d} (1_{rB} \star 1_{E^*}) 1_{F^*} - (1_{rB} \star 1_E) 1_F \right) dr \\
&= \int_0^{r_{E,F}} -J'(r) \left(\int_{\mathbb{R}^d} 1_E \star 1_{rB} - \int_{\mathbb{R}^d} 1_{rB} \star 1_E(x) 1_F(x) dx \right) dr \\
&\geq \frac{\tau}{k} \int_{\tau}^{r_{E,F}} \left(\int_{F^c} |E \cap B_{x,r}| dx \right) dr, \tag{4.7}
\end{aligned}$$

where in the last inequality we have used (4.1). We now notice that

$$|E \cap B_{x,r}| \geq \lambda \tau^d |B|, \quad \forall x \in E^{\lambda, \tau} + B_{r-\tau}, \forall \tau < r. \tag{4.8}$$

Indeed, by assumption on x , there exists $y \in B_{x, r-\tau} \cap E^{\lambda, \tau}$, so that, in particular, $B_{y, \tau} \subset B_{x, r}$, and thus $y \in E^{\lambda, \tau}$ implies $|B_{x, r} \cap E| \geq |B_{y, \tau} \cap E| \geq \lambda |B| \tau^d$. By combining (4.7) with (4.8) we thus find the lower bound (4.5).

Step two: We notice that the volumes of $|E|$ and $|F|$ differ by a ‘‘surface term’’,

$$|F| - |E| \leq C(d) |E|^{1-1/d}. \tag{4.9}$$

Indeed, by definition of $r_{E,F}$ and by (4.2) we have

$$\begin{aligned}
|F| - |E| &\leq \left(|E|^{1/d} + \frac{3}{4} |B|^{1/d} \right)^d - |E| = d \int_0^{3|B|^{1/d}/4} (|E|^{1/d} + t)^{d-1} dt \\
&\leq C(d) (|E|^{1/d} + |B|^{1/d})^{d-1} \leq C(d) |E|^{1-1/d}.
\end{aligned}$$

Step three: Given $\tau \in (0, r_{E,F})$, let us set $\ell = \int_{\tau/4 \leq |x| \leq \tau/2} J(|x|) dx$, (We shall pick τ so that ℓ will be small in terms of $\alpha(E; B)$.) We claim that if

$$\delta_J(E, F) \leq \ell^2 |B|^{1/d} |E|^{1-1/d}, \tag{4.10}$$

and λ is small enough in terms of d , then

$$|D^{\lambda, \tau}| \leq C(d) (\lambda + \ell) |E|^{1-1/d}. \tag{4.11}$$

To this end, let us consider the truncated kernel

$$J_1(r) = \begin{cases} J(r)/\ell & r \in (\tau/4, \tau/2), \\ 0 & r \notin (\tau/4, \tau/2), \end{cases}$$

and notice that

$$\tau^d \int_0^{\infty} (-J_1'(r)) dr \leq C(d). \tag{4.12}$$

Indeed,

$$\tau^d \int_0^{\infty} (-J_1'(r)) dr \leq C(d) \int_{\tau/4}^{\tau/2} (-J_1'(r)) |rB| dr \leq C(d) \int_{\mathbb{R}^d} J_1(|y|) dy = C(d).$$

By a similar argument we find that

$$J_1 \star 1_{F^*}(x) \geq c(d), \quad \forall x \in F^*. \quad (4.13)$$

To see this, notice that since $|F^*| \geq |E^*| = |E| \geq 2|B|$, one has

$$|F^* \cap B_{x,r}| \geq c(d) |rB|, \quad \forall x \in F^*, r < \frac{3}{4},$$

and thus

$$\begin{aligned} J_1 \star 1_{F^*}(x) &= \int_{\tau/4}^{\tau/2} (-J_1'(r)) |F^* \cap B_{x,r}| dr \geq c(d) \int_{\tau/4}^{\tau/2} (-J_1'(r)) |rB| dr \\ &\geq c(d) \int_{\mathbb{R}^d} J_1(|x|) dx = c(d), \end{aligned}$$

as claimed. By (4.13) we have

$$|D^{\lambda,\tau}| = |E^* \setminus (E^{\lambda,\tau})^*| \leq C(d) \left(\mathcal{E}_{J_1}(E^*, F^*) - \mathcal{E}_{J_1}((E^{\lambda,\tau})^*, F^*) \right). \quad (4.14)$$

We now notice that thanks to (4.10)

$$\begin{aligned} \mathcal{E}_{J_1}(E^*, F^*) - \mathcal{E}_{J_1}(E, F) &= \int_0^\infty (-J_1'(r)) dr \int_{\mathbb{R}^d} (1_{E^*} \star 1_{rB}) 1_{F^*} - (1_E \star 1_{rB}) 1_F \\ &\leq \frac{1}{\ell} \int_0^\infty (-J'(r)) dr \int_{\mathbb{R}^d} (1_{E^*} \star 1_{rB}) 1_{F^*} - (1_E \star 1_{rB}) 1_F \\ &\leq \frac{\delta_J(E, F)}{\ell} \leq \ell |B|^{1/d} |E|^{1-1/d}, \end{aligned}$$

while $\mathcal{E}_{J_1}(E^{\lambda,\tau}, F) \leq \mathcal{E}_{J_1}((E^{\lambda,\tau})^*, F^*)$ by Riesz inequality, so that (4.14) implies

$$\begin{aligned} |D^{\lambda,\tau}| &\leq C(d) \left(\ell |E|^{1-1/d} + \mathcal{E}_{J_1}(E, F) - \mathcal{E}_J(E^{\lambda,\tau}, F) \right) \\ &= C(d) \left(\ell |E|^{1-1/d} + \mathcal{E}_{J_1}(D^{\lambda,\tau}, F) \right). \end{aligned} \quad (4.15)$$

Having in mind the decomposition $\mathcal{E}_{J_1}(D^{\lambda,\tau}, F) = \mathcal{E}_{J_1}(D^{\lambda,\tau}, E) + \mathcal{E}_{J_1}(D^{\lambda,\tau}, F \setminus E)$, we first notice that

$$\begin{aligned} \mathcal{E}_{J_1}(D^{\lambda,\tau}, E) &= \int_{D^{\lambda,\tau}} dx \int_{\tau/4}^{\tau/2} (-J_1'(r)) |E \cap B_{x,r}| dr \\ &\leq \int_{D^{\lambda,\tau}} |E \cap B_{x,\tau}| dx \int_{\tau/4}^{\tau/2} (-J_1'(r)) dr \\ &\leq \lambda |D^{\lambda,\tau}| \int_{\tau/4}^{\tau/2} (-J_1'(r)) |B_\tau| dr \leq C(d) \lambda |D^{\lambda,\tau}|, \end{aligned} \quad (4.16)$$

where in the last inequality we have used (4.12). At the same time

$$\mathcal{E}_{J_1}(D^{\lambda,\tau}, F \setminus E) = \int_{F \setminus E} dx \int_{\tau/4}^{\tau/2} (-J_1'(r)) |D^{\lambda,\tau} \cap B_{x,r}| dr,$$

where, given $x \in F \setminus E$, either we have $D^{\lambda,\tau} \cap B_{x,r} = \emptyset$, or there exists $y \in D^{\lambda,\tau} \cap B_{x,r}$, in which case, by $r < \tau/2$, $B_{x,r} \subset B_{y,2r} \subset B_{y,\tau}$, and $y \in D^{\lambda,\tau} \subset E$, we obtain

$$|D^{\lambda,\tau} \cap B_{x,r}| \leq |E \cap B_{y,\tau}| \leq \lambda |B_\tau|;$$

we thus find, thanks to (4.12) and (4.9)

$$\mathcal{E}_{J_1}(D^{\lambda,\tau}, F \setminus E) \leq C(d) \lambda |F \setminus E| \leq C(d) \lambda |E|^{1-1/d}. \quad (4.17)$$

By combining (4.15), (4.16) and (4.17) we thus find

$$|D^{\lambda,\tau}| \leq C(d) \left((\ell + \lambda) |E|^{1-1/d} + \lambda |D^{\lambda,\tau}| \right).$$

In particular, if λ is small enough depending on d , obtain (4.11).

Step four: We complete the proof of the theorem. We start by choosing the values of τ and λ . For a small value of $a > 0$ to be fixed in the argument, and for some $p \geq 4$, let us set

$$\lambda = \tau = a \alpha(E; B)^p \leq a. \quad (4.18)$$

(Recall that $\alpha(E; B) \leq 1$ by definition.) Since $r_{E,F} \geq 1/4$, we can definitely entail $\tau < r_{E,F}$, and thus infer from (4.5) that

$$k \delta_J(E, F) \geq \lambda \tau^{d+1} \int_0^{r_{E,F}-\tau} |(E^{\lambda,\tau} + B_s) \cap F^c| ds \quad (4.19)$$

holds. Now, since $(\tau/4, \tau/2) \subset (0, 3/4)$, by (4.1) we find

$$\ell = \int_{\tau/4}^{\tau/2} (-J'(r)) \omega_d r^d dr \geq \frac{\omega_d}{k} \int_{\tau/4}^{\tau/2} r^{d+1} dr \geq \frac{\tau^{d+2}}{C(d, k)} = \frac{\alpha(E; B)^{p(d+2)}}{C(d, k, a)}.$$

Hence, by step three, either

$$\delta_J(E, F) \geq \ell^2 |B|^{1/d} |E|^{1-1/d} \geq \frac{|E|^{1-1/d} \alpha(E; B)^{2p(d+2)}}{C(d, k, a)}, \quad (4.20)$$

or (4.10) holds, and thus

$$|D^{\lambda,\tau}| \leq C(d) (\lambda + \ell) |E|^{1-1/d}. \quad (4.21)$$

Let us now notice that, provided a is small enough in terms of d and k ,

$$\begin{aligned} \ell &= \int_{\tau/4 \leq |x| \leq \tau/2} J(|x|) dx \leq C(d) \|J\|_{C^0(\mathbb{R}^d)} \tau^d \\ &\leq C(d, k) a^d \alpha(E, B)^{pd} \leq a \alpha(E, B)^p, \end{aligned}$$

so that (4.18) and (4.21) give us

$$|D^{\lambda,\tau}| \leq C(d) a |E|^{1-1/d} \alpha(E; B)^p. \quad (4.22)$$

Summarizing, either (4.20) holds, and then we are done, or the bad set $D^{\lambda,\tau}$ is actually small in terms of $\alpha(E; B)$. In this latter case we effectively exploit the lower bound (4.19) together with the quantitative Brunn-Minkowski inequality of Theorem 1.1 in order to infer an estimate similar to (4.20).

The argument goes as follows. By applying Theorem 1.1 to $E^{\lambda,\tau}$ and B_s with $s \in (0, r_{E,F} - \tau)$, we find that

$$\frac{\alpha(E^{\lambda,\tau}; B)^4}{C(d)} \leq \max \left\{ \frac{|E^{\lambda,\tau}|}{|B_s|}, \frac{|B_s|}{|E^{\lambda,\tau}|} \right\}^{1/d} \left\{ \left(\frac{|E^{\lambda,\tau} + B_s|}{|(E^{\lambda,\tau})^* + B_s|} \right)^{1/d} - 1 \right\}. \quad (4.23)$$

By (4.22), $|E| \geq 2|B|$, and provided a is small enough in terms of d ,

$$|E^{\lambda,\tau}| \geq |E| \left(1 - \frac{C(d) a}{|E|^{1/d}} \right) \geq 2|B| \left(1 - \frac{C(d) a}{|B|^{1/d}} \right) \geq |B|,$$

so that $|E^{\lambda,\tau}| \geq |B_s|$ for $s \in (0, r_{E,F} - \tau)$ and (4.23) gives us

$$\alpha(E^{\lambda,\tau}; B)^4 \leq C(d) \frac{|E^{\lambda,\tau}|^{1/d}}{s} \left(\frac{|E^{\lambda,\tau} + B_s|}{|(E^{\lambda,\tau})^* + B_s|} - 1 \right), \quad (4.24)$$

where we have also used the concavity of $\eta \mapsto \eta^{1/d}$. We notice that by (4.21) and by $|E| \geq 2|B|$, if a is small enough depending on d , then

$$\begin{aligned} r_{E^{\lambda,\tau},F} - r_{E,F} &= \frac{|E|^{1/d}}{|B|^{1/d}} \left(1 - \left(1 - \frac{|D^{\lambda,\tau}|}{|E|} \right)^{1/d} \right) \\ &\leq \frac{|E|^{1/d}}{|B|^{1/d}} \left(1 - \left(1 - \frac{C(d) a \alpha(E; B)^p}{|E|^{1/d}} \right)^{1/d} \right) \\ &\leq C(d) a \alpha(E; B)^p. \end{aligned} \quad (4.25)$$

In particular,

$$r_{E,F} - \tau = r_{E,F} - a \alpha(E; B)^p > r_{E^{\lambda,\tau},F} - C_*(d) a \alpha(E; B)^p, \quad (4.26)$$

for some specific constant $C_*(d)$. In particular, if we set

$$I = \left[r_{E^{\lambda,\tau},F} - 2C_*(d) a \alpha(E; B)^p, r_{E^{\lambda,\tau},F} - C_*(d) a \alpha(E; B)^p \right],$$

then for a small enough

$$I \subset (0, r_{E,F} - \tau), \quad \text{with } \mathcal{H}^1(I) = C_*(d) a \alpha(E; B)^p; \quad (4.27)$$

moreover, if $s \in I$, then $|(E^{\lambda,\tau})^* + B_{r_{E^{\lambda,\tau},F}}| = |F|$ gives

$$|(E^{\lambda,\tau})^* + B_s|^{1/d} = |F|^{1/d} - (r_{E^{\lambda,\tau},F} - s) |B|^{1/d} \geq |F|^{1/d} - C(d) a \alpha(E; B)^p,$$

that is (thanks to $|F| \geq |E| \geq 2|B|$)

$$|(E^{\lambda,\tau})^* + B_s| \geq |F| \left(1 - C(d) a \alpha(E; B)^p \right),$$

and thus

$$|E^{\lambda,\tau} + B_s| - |(E^{\lambda,\tau})^* + B_s| \leq |(E^{\lambda,\tau} + B_s) \setminus F| + C(d) a |F| \alpha(E; B)^p.$$

By combining this inequality with (4.24) (and with $|(E^{\lambda,\tau})^* + B_s| \geq |F|/C(d)$)

$$\begin{aligned} \alpha(E^{\lambda,\tau}; B)^4 &\leq C(d) \frac{|E^{\lambda,\tau}|^{1/d}}{s |F|} \left(|(E^{\lambda,\tau} + B_s) \setminus F| + 2|F| a^{1/4} \alpha(E; B)^p \right) \\ &\leq \frac{C(d)}{s} \frac{|E^{\lambda,\tau}|^{1/d}}{|F|} \left(|(E^{\lambda,\tau} + B_s) \setminus F| + \frac{C(d)}{s} a \alpha(E; B)^p \right). \end{aligned}$$

Of course $r_{E^{\lambda,\tau},F} \geq r_{E,F} \geq 1/4$ so that if $s \in I$, then $s \geq 1/8$, and thus we conclude by $p \geq 4$ and for a small enough in terms of d that

$$\alpha(E^{\lambda,\tau}; B)^p \leq C(d) \frac{|E^{\lambda,\tau}|^{1/d}}{|F|} |(E^{\lambda,\tau} + B_s) \setminus F|.$$

On the one hand by (1.14) and by $|E| \geq 2|B|$

$$|\alpha(E^{\lambda,\tau}; B) - \alpha(E; B)| \leq \frac{2|D^{\lambda,\tau}|}{|E|} \leq C(d) a \alpha(E; B)^p,$$

so that

$$\alpha(E^{\lambda,\tau}; B)^p \geq \left(\alpha(E; B) - C(d) a \alpha(E; B)^p \right)^p \geq \frac{\alpha(E; B)^p}{2},$$

while on the other hand $|E^{\lambda,\tau}|^{1/d}|F|^{-1} \leq |E|^{(1/d)-1}$ and thus

$$|E|^{1-1/d} \alpha(E; B)^p \leq C(d) |(E^{\lambda,\tau} + B_s) \setminus F|, \quad \forall s \in I.$$

By (4.19), (4.27), and the choices of λ and τ we thus find

$$k \delta_J(E, F) \geq \frac{|E|^{1-1/d}}{C(d, a)} \lambda \tau^{d+1} \alpha(E; B)^{2p} = \frac{|E|^{1-1/d}}{C(d, a)} \alpha(E; B)^{(d+4)p}.$$

By (4.20), setting $p = 4$ and recalling that $a = a(d, k)$ we deduce that

$$|E|^{1-1/d} \min\{\alpha(E; B)^{4(d+4)}, \alpha(E; B)^{8(d+2)}\} \leq C(d, k) \delta_J(E; F),$$

where the left-side is actually equal to $|E|^{1-1/d} \alpha(E; B)^{8(d+2)}$ as $\alpha(E; B) \leq 1$. \square

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