IMPLICIT PDES WITH A LINEAR CONSTRAINT

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Abstract. We study implicit differential systems with a linear constraint on the gradient variable and we prove the existence of infinitely many Lipschitz continuous solutions. The result is obtained by a density argument in a suitable complete metric space.

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1. Introduction

We deal with the existence of Lipschitz continuous solutions to a first order differential system of the following kind:

$$\begin{cases} F(x,u(x),Du(x)) = 0 \text{ a.e. } x \in \Omega \\ L(x,u(x),Du(x)) = 0 \text{ a.e. } x \in \Omega \\ u \in \phi + W_0^{1,\infty}(\Omega,\mathbb{R}^n) \end{cases} \tag{1.1}$$

where the function F satisfies a certain coercivity condition with respect to the gradient variable, while the function L is *quasi-affine* on the same variable. The boundary datum ϕ is affine on Ω.

Differential systems of this type, without the quasi-linear constraint $L = 0$, have been recently introduced and investigated by Dacorogna and Marcellini in the context of *implicit* partial differential equations and systems [1]. The book (and the related wide bibliography) gives conditions (in particular a compatibility condition on the boundary datum) in order to obtain a Lipschitz solution to the system. Several extensions are also considered. However, the constrained problem (1.1) did not enter in the results in [1], and in fact in Section 1.5.7 the authors posed the constrained problem as an open one.

Problems with quasi-affine constraints were first studied by Müller and Sverak [4] and by Dacorogna and Tanteri [2], with two different approaches. In such papers $L = L(Du) = \det(Du)$.

Here we assume that L is any linear function of Du, i.e. $L(Du) = \sum_{i,j=1}^n \alpha_{ij} u^i_{x_j}$ for some matrix (α_{ij}) and, by some algebraic manipulations of the involved matrices, in particular by some product decomposition (see the book [3] by G. H. Golub and C. F. Van Loan), we obtain existence of $W_0^{1,\infty}(\Omega,\mathbb{R}^n)$ solutions to (1.1).

As a particular case of our general results, we will prove for example that the differential problem

$$\begin{cases} F(Du(x)) = 0 & \text{a.e. } x \in \Omega \\ L(Du(x)) = \ell & \text{a.e. } x \in \Omega \\ u = \phi & \text{on } \partial\Omega \end{cases} \tag{1.2}$$

has a $W^{1,\infty}(\Omega,\mathbb{R}^n)$ solution under the assumption that F is a continuous coercive quasi-convex function and $L = L(\xi) = \displaystyle\sum_{i,j=1}^n \alpha_{ij}\xi_{ij}$ is any linear function. Moreover

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we emphasize that the method of proof in the two cases, respectively L linear function of Du and L the determinant of the matrix Du, is different and doesn't allow us to treat at the same time any quasi-affine function L of Du.

A wide bibliography on implicit partial differential equations can be found in [1].

2. The approximation lemma

In this section we give a technical lemma which is the main tool to prove the existence Theorem 3.2.

Lemma 2.1. *Let $\Omega \subset \mathbb{R}^n$ be an open bounded set. Let $\phi\colon \overline{\Omega} \to \mathbb{R}^n$ be an affine function such that $D\phi(x) = tA + (1-t)B$, where $A,\ B \in \mathbb{R}^{n\times n}$ are such that* $\operatorname{rank}(A-B) = 1$ *i.e. there exist $\mu,\ v \in \mathbb{R}^n$, $\|v\| = 1$ such that $A - B = \mu \otimes v$ and $t \in (0,1)$. Let $L\colon \mathbb{R}^{n\times n} \to \mathbb{R}$, $L\colon \xi = (\xi_{ij})_{i,j=1,\dots,n} \to \sum_{i,j=1}^{n} \alpha_{ij}\xi_{ij}$ be a linear operator, $\mathcal{L} = (\alpha_{ij})_{i,j=1,\dots,n}$ a given matrix. Let us assume $L(A) = L(B)$ and $\mathcal{L}v \neq 0$. Let $\varepsilon > 0$. Then there exists $u \in \phi + W_0^{1,\infty}(\Omega,\mathbb{R}^n)$ and an open set $\Omega_\varepsilon \Subset \Omega$ with the following properties:*

$$\begin{cases} |\Omega \setminus \Omega_\varepsilon| < \varepsilon \\ Du(x) \in \{A, B\} \qquad a.e.\ x \in \Omega_\varepsilon \\ L(Du(x)) = L(D\phi(x)) = L(A) = L(B) \qquad a.e.\ x \in \Omega \\ \operatorname{dist}(Du(x), Rco\{A,B\}) < \varepsilon \qquad a.e.\ x \in \Omega \\ \|u - \phi\|_\infty < \varepsilon \end{cases}$$

Remark 2.2. *Since L is a linear operator, $L(A) = L(B)$ is equivalent to $L(A-B) = 0$ and to $L(D\phi) = L(A)$. In Section 4 we will discuss the assumption $\mathcal{L}v \neq 0$ in Lemma 2.1.*

Proof. Step 1 Let us assume $A - B = \mu \otimes e_1$, where $e_1 = (1,0,\dots,0)$ is the first element of the canonical basis of $\mathbb{R}^n$ and $\mathcal{L}$ is an upper triangular matrix of the following kind i.e.

$$\mathcal{L} = (\alpha_{ij}) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1r} & \cdots & & \alpha_{1n} \\ 0 & \alpha_{22} & 0 & \cdots & & & 0 \\ 0 & 0 & \ddots & & & & \\ & & & \alpha_{rr} & \ddots & & \vdots \\ \vdots & & & \ddots & 0 & & \\ & & & & & \ddots & 0 \\ 0 & 0 & & \cdots & & 0 & 0 \end{pmatrix} \tag{2.1}$$

with $r = \operatorname{rank}\mathcal{L}$ (i.e. $\prod_{i=1}^{r} \alpha_{ii} \neq 0$). $\mathcal{L}e_1 \neq 0$ is equivalent to $\alpha_{11} \neq 0$, since $L(A-B) = \alpha_{11}\mu_1$ this implies $\mu_1 = 0$.

Without loss of generality we can assume $\Omega = (0,1)^n$. If this is not the case we know that Ω can be covered with a finite family $\{\Omega_k\}$ of subsets homotetics to $(0,1)^n$ and a set of small measure. We solve the problem in each set Ω_k and we are done.

We are given a linear operator

$$L(X) = \sum_{j=1}^{n} \sum_{i=1}^{j} \alpha_{ij} X_{ij}, \quad X \in \mathbb{R}^{n\times n}$$

We want to find a linear differential functional $\Phi\colon C^1(\mathbb{R}^n, \mathbb{R}^n) \to C^0(\mathbb{R}^n, \mathbb{R}^n)$ such that

$$L(D\Phi(v)) \equiv 0 \quad \forall v \in C^2(\Omega, \mathbb{R}^n)\,. \tag{2.2}$$

We will not define the differential operator Φ explicitly, but only prove that an operator Φ and a function $w = w(x_1)$ with the required properties exist. Let us call Φ_s, $s = 1, \ldots, n$ its components. Φ_s will be of the following kind

$$\Phi_s(v) = \sum_{l=1}^{n} \sum_{m=1}^{n} a^s_{lm} v^l_{x_m}$$

Condition (2.2) gives us for any $s = 1, \ldots, n$ the following linear algebraic system in the unknowns a^i_{sk} where $k, i = 1, \ldots, n$.

$$\begin{aligned}
&\alpha_{11} a^1_{s1} = 0 && && (2.3)\\
&\alpha_{1k} a^1_{sk} + \alpha_{kk} a^k_{sk} = 0 && \forall k = 2, \ldots, r && (2.4)\\
&\alpha_{11} a^1_{sk} = -\alpha_{1k} a^1_{s1} - \alpha_{kk} a^k_{sm} && \forall k = 2, \ldots, r && (2.5)\\
&\alpha_{1m} a^1_{sk} + \alpha_{mm} a^m_{sk} = -\alpha_{1k} a^1_{sm} - \alpha_{kk} a^k_{sm} && \forall k = 3, \ldots, r,\ \forall m = 2, \ldots, k-1 && (2.6)\\
&\alpha_{1k} a^1_{sk} = 0 && \forall k = r+1, \ldots, n && (2.7)\\
&\alpha_{11} a^1_{sk} = -\alpha_{1k} a^1_{s1} && \forall k = r+1, \ldots, n && (2.8)\\
&\alpha_{1m} a^1_{sk} + \alpha_{mm} a^m_{sk} = -\alpha_{1k} a^1_{sm} && \forall k = r+1, \ldots, n,\ \forall m = 2, \ldots, r && (2.9)\\
&\alpha_{1m} a^1_{sk} = -\alpha_{1k} a^1_{sm} && \forall k = r+2, \ldots, n,\ \forall m = r+1, \ldots, k-1 && (2.10)
\end{aligned}$$

Of course all these systems admit the trivial solution $a^i_{sk} = 0\,,\ \forall s, k, i = 1, \ldots, n$, but this is not what we want. For some reason we will see later on we want the matrix $(a^i_{s1})_{s,i=2,\ldots,n}$ to be non singular. Let us consider the equations where **only** the unknowns a^i_{s1} appear. We have only equation (2.3). Since we assume $\alpha_{11} \neq 0$ we have $a^1_{s1} = 0\ \forall s = 1, \ldots, n$ but this is not important since we are interested only in $(a^i_{s1})_{s,i=2,\ldots,n}$.

From now on we shall fix s and solve the system by induction on the index $k = 2, \ldots, n$. Let us assume $\alpha_{22} \neq 0$ and let us consider the equations where a^i_{s2} appear and the other a^i_{sk} appear only if $k = 1$. They are

$$\begin{aligned}
&(2.4)_{k=2} \quad \alpha_{12} a^1_{s2} + \alpha_{22} a^2_{s2} = 0\\
&(2.5)_{k=2} \quad \alpha_{11} a^1_{s2} = -\alpha_{22} a^1_{s2}
\end{aligned}$$

It is a diagonal system in the unknown a^1_{s2}, a^2_{s2} and the determinant of the incomplete matrix associated to the system is $\alpha_{11}\alpha_{22} \neq 0$ so we can solve it for any value of the right hand side. Let us assume we solved the subsystem of equations where $a^i_{s,k}$ appears only if $k \leq K - 1 < r$.

Let us consider the equations where a^i_{sK} appear and the other a^i_{sk} appear only if $k \le K-1$. They are

$$\begin{array}{lll}
(2.4)_{k=K} & \alpha_{1K}a^1_{sK} + \alpha_{KK}a^K_{sK} & = 0 \\
(2.6)_{k=K,m=K-1} & \alpha_{1,K-1}a^1_{sK} + \alpha_{K-1,K-1}a^K_{sK} - 1 & = -\alpha_{1K}a^1_{s,K-1} - \alpha_{KK}a^K_{s,K-1} \\
\dots & \dots & \dots \\
(2.5)_{k=K} & \alpha_{11}a^1_{sK} & = -\alpha_{1K}a^1_{s1} - \alpha_{KK}a^K_{s1}
\end{array}$$

It is a triangular system in the unknown a^i_{sK}, $i = 1, \dots, K$ and the determinant of the incomplete matrix associated to the system is $\prod_{i=1}^K \alpha_{ii} \neq 0$ so we can solve it for any value of the right hand side.

Note that at each step k the unknown a^i_{sk}, $i > k$ remain undetermined.

Now we want to show that we can solve the sub-system of equations where the unknown $a^i_{s,r+1}$ appear and the other a^i_{sk} appear only if $k \le r$. It is given by the following equations

$$\begin{array}{llll}
(2.7)_{k=r+1} & \alpha_{1,r+1}a^1_{s,r+1} & = 0 & \\
(2.9)_{k=r+1} & \alpha_{1m}a^1_{s,r+1} + \alpha_{r+1,r+1}a^m_{s,r+1} & = -\alpha_{1,r+1}a^1_{s,m} & \forall m = 2, \dots, r \\
(2.8)_{k=r+1} & \alpha_{11}a^1_{s,r+1} & = 0 &
\end{array}$$

which gives

$$\begin{cases} a^1_{s,r+1} = 0 \\ a^m_{s,r+1} = \dfrac{-\alpha_{1,r+1}a^1_{sm}}{\alpha_{mm}} \qquad \forall m = 2, \dots, r \end{cases}$$

while $a^{r+1}_{s,r+1}$ remains undetermined.

Now let us assume we solved the system up to step $k = K-1$ with $K-1 > r-1$ and that in doing so we determined the following coefficients:

$$\begin{array}{cccc}
a^1_{s1} = 0 & & & \\
a^1_{s2} & & & \\
\vdots & & & \\
a^1_{sr} & a^1_{sr} & \cdots & a^r_{sr} \\
a^1_{s,r+1} = 0 & a^2_{s,r+1} & \cdots & a^r_{s,r+1} \\
\vdots & \vdots & \vdots & \vdots \\
a^1_{s,K-1} = 0 & a^2_{s,K-1} & \cdots & a^r_{s,K-1}
\end{array}$$

Using this induction hypothesis the system at step K becomes:

$$\begin{array}{llll}
(2.7)_{k=K} & \alpha_{1K}a^1_{sK} & = 0 & \\
(2.8)_{k=K} & \alpha_{11}a^1_{sK} & = 0 & \\
(2.9)_{k=K} & \alpha_{1m}a^1_{s,K} + \alpha_{KK}a^m_{s,K} & = -\alpha_{1K}a^1_{s,m} & \forall m = 2, \dots, r \\
(2.10)_{k=K} & \alpha_{1m}a^1_{s,K} & = 0 & \forall m = r+1, \dots, K-1
\end{array}$$

which gives

$$\begin{cases} a^1_{sK} = 0 \\ a^m_{sK} = \dfrac{-\alpha_{1K}a^1_{sm}}{\alpha_{mm}} \qquad \forall m = 2, \dots, r \end{cases}$$

while $a^{r+1}_{s,K}, \ldots, a^K_{s,K}$ remain undetermined.

In this way we solve the systems for any prescribed a^i_{s1}, $s, i = 2, \ldots, n$ and we choose these in such a way that $(a^i_{s1})_{s,i=2,\ldots,n}$ is a non singular matrix.

Now, let $w = w(x_1)$, we want $D\Phi(w)(x) \in \{-t\mu \otimes e_1, (1-t)\mu \otimes e_1\}$ a.e. $x_1 \in (0,1)$.

But $w = w(x_1)$ implies $\Phi_s(w) = \displaystyle\sum_{l=1}^{n} a^s_{l1} w'_l(x_1)$ We choose $w_1(x_1) \equiv 0$. Since $a^1_{s1} = 0\ \forall s = 1, \ldots, n$, $\dfrac{\partial \Phi(w)}{\partial x_1} = 0$ is satisfied (we have $\mu_1 = 0$). So we need

$$\begin{aligned} &\sum_{l=2}^{n} a^s_{l1} w''_l(x_1) = -t\mu_s \quad \forall s = 2, \ldots, n \\ \text{or } &\sum_{l=2}^{n} a^s_{l1} w''_l(x_1) = (1-t)\mu_s \quad \forall s = 2, \ldots, n \end{aligned}$$

This can be done since by construction $\det(a^s_{l1})_{l,s=2,\ldots,n} \neq 0$.

Moreover, for any $\delta > 0$ we can choose w such that $\|w\|_\infty < \varepsilon$, $\|w\prime\|_\infty < \varepsilon$. So just call

$$\begin{aligned} I &= \left\{ x_1 \in (0,1) \colon \sum_{l=2}^{n} a^s_{l1} w''_l(x_1) = -t\mu_s \quad \forall s = 2, \ldots, n \right\} \\ J &= \left\{ x_1 \in (0,1) \colon \sum_{l=2}^{n} a^s_{l1} w''_l(x_1) = (1-t)\mu_s \quad \forall s = 2, \ldots, n \right\} \end{aligned}$$

We can assume I and J to be the union of pairwise disjoint intervals of $(0,1)$ such that $I \cap J = \emptyset$, $\overline{I} \cup \overline{J} = [0,1]$.

Let $\Omega_\varepsilon \Subset \Omega$ such that $|\Omega \setminus \Omega_\varepsilon| < \varepsilon$ and let $\eta \in C^\infty_0(\Omega, \mathbb{R})$ such that

$$\eta|_{\Omega_\varepsilon} \equiv 1 \quad |D\eta| < \frac{L}{\varepsilon}$$

Define

$$u(x) = \phi(x) + \Phi(\eta(x) w(x_1))$$

We want to show that u is the function we are looking for. Near $\partial\Omega$ $\eta \equiv 0$ hence $u = \phi$. In Ω_ε $\eta \equiv 1$ so $u = \phi + \Phi(w)$ and $Du = D\phi + D\Phi(w)$. We now want to evaluate $D\Phi(w)$. Define

$$\Omega_A = \{x \in \Omega_\varepsilon \colon x_1 \in J\} \qquad \Omega_B = \{x \in \Omega_\varepsilon \colon x_1 \in I\}$$

We have

$$\begin{aligned} Du &= B + t\mu \otimes e_1 - t\mu \otimes e_1 = B \quad \forall x \in \Omega_B \\ Du &= B + t\mu \otimes e_1 + (1-t)\mu \otimes e_1 = A \quad \forall x \in \Omega_A \end{aligned}$$

Moreover for any $x \in \Omega$ we have $L(Du) = L(D\phi) + L(D\Phi(\eta w)) = L(D\phi) = L(A)$ by definition of Φ.

We only have to evaluate $\mathrm{dist}(Du, \mathrm{Rco}A, B)$ and $\|u - \phi\|_\infty$. Let us consider $\Phi(\eta w) = \eta\Phi(w) + G(D\eta\, w)$ where G is a linear function. So

$$\|u - \phi\|_\infty \leq \|\Phi(w)\|_\infty + \|G(D\eta w)\|_\infty \leq C \max\{\delta, \delta\varepsilon^{-1}\} < \varepsilon$$

if we choose a small enough δ.

With further computation we get

$$Du = D\phi + \eta D\Phi(w) + H(D\eta\, w\prime, D^2\eta\, w)$$

where H is a linear function. Since $D\phi + \eta D\Phi(w) \in \mathrm{Rco}\{A, B\}$ we have

$$\mathrm{dist}(Du, \mathrm{Rco}\{A, B\}) \le C \max\{\delta\varepsilon^{-1}, \delta\varepsilon^{-2}\} < \varepsilon$$

if we choose δ small enough.

Step 2 Now let us assume we are in a slightly more general situation; we assume $A - B = \mu \otimes e_1$ and $\alpha_{i1} = 0\ \forall i = 2, \ldots, n$. Hence $\mathcal{L}e_1 \neq 0$ if and only if $\alpha_{11} \neq 0$. Let $\widehat{\mathcal{L}} = (\alpha_{ij})_{i,j=2,\ldots,n}$. We know (SVD decomposition, see [3], pages 16-20) that there exist $\widehat{V}$, $\widehat{U}$, both in $SO(n-1)$ such that

$$\widehat{U}^t \widehat{\mathcal{L}} \widehat{V} = \begin{pmatrix} \sigma_2 & 0 & & \cdots & & 0 \\ 0 & \ddots & & & & \\ & & \sigma_r & \ddots & & \vdots \\ \vdots & & \ddots & 0 & & \\ & & & & \ddots & 0 \\ 0 & & \cdots & & 0 & 0 \end{pmatrix}$$

Define

$$V = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & \widehat{V} & \\ 0 & & & \end{pmatrix} \quad U = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & \widehat{U} & \\ 0 & & & \end{pmatrix}$$

Then $U^t\mathcal{L}V$ is as in (2.1). Let $\psi\colon V^t\Omega \to \mathbb{R}^n$, $\psi\colon y \to U^t\phi(Vy)$, then

$$\phi(x) = U\psi(V^t x), \qquad \phi^i_{x_j} = \sum_{i,j=1}^{n} u_{ik}\psi^k_{y_l} v_{jl}$$

and

$$\begin{aligned} L(D\phi) &= \sum_{i,j=1}^{n} \sum_{k,l=1}^{n} \alpha_{ij} u_{ik} \psi^k_{y_l} v_{jl} = \sum_{k,l=1}^{n} \left(\sum_{i,j=1}^{n} u_{ik}\alpha_{ij}v_{jl} \right) \psi^k_{y_l} \\ &= \sum_{k,l=1}^{n} \left(U^t\mathcal{L}V\right)_{kl} \psi^k_{y_l} \end{aligned}$$

while

$$\begin{aligned} D\psi =& U^t D\phi V = U^t(B + t\mu \otimes e_1)V = U^t BV + t(U^t\mu) \otimes (V^t e_1) \\ =& U^t BV + t(U^t\mu) \otimes e_1 \end{aligned}$$

From step 1 we can solve the problem for ψ in $V^t\Omega$ with respect to $U^t\mathcal{L}V$. Let $u_\psi\colon V^t\Omega \to \mathbb{R}^n$ be a function with the required properties, with its associated sets Ω_{AV} and Ω_{BV}, then $u\colon \Omega \to \mathbb{R}^n$, $u\colon x \to U^t u_\psi(Vx)$ and $\Omega_A \equiv V^t\Omega_{AV}$ and $\Omega_B \equiv V^t\Omega_{BV}$ are a solution to our problem.

Step 3 Now let us assume we are in the general situation $A - B = \mu \otimes v$, $\mu \in \mathbb{R}^n$, $v \in S^{n-1}$ and $\mathcal{L} = (\alpha_{ij})_{ij}$ is such that $\mathcal{L}v \neq 0$. Since $v \in S^{n-1}$, there exists $R \in SO(n)$ such that $R^t v = e_1$. Moreover there exists a symmetric matrix $P \in SO(n)$ (actually P is an Householder matrix, see [3], page. 38) such that the first column of $P\mathcal{L}R$ is parallel to e_1. Let $\theta\colon R^t\Omega \to \mathbb{R}^n$, $\theta\colon y \to P\phi(Ry)$.

$$\phi(x) = P\theta(R^t x), \qquad \phi^i_{x_j} = \sum_{i,j=1}^{n} p_{ik}\theta^k_{y_l} r_{jl}$$

and

$$\begin{aligned} L(D\phi) &= \sum_{i,j=1}^{n}\sum_{k,l=1}^{n} \alpha_{ij} p_{ik}\theta^k_{y_l} r_{jl} = \sum_{k,l=1}^{n}\left(\sum_{i,j=1}^{n} p_{ki}\alpha_{ij} r_{jl}\right)\theta^k_{y_l} \\ &= \sum_{k,l=1}^{n} (P\mathcal{L}R)_{kl}\,\theta^k_{y_l} \end{aligned}$$

while

$$\begin{aligned} D\theta =& PD\phi RV = P(B + t\mu \otimes v)R = PBV + t(P\mu) \otimes (R^t v) \\ =& PBV + t(P\mu) \otimes e_1 \end{aligned}$$

Thanks to step 2 we can solve the problem for θ in $R^t\Omega$ with respect to the matrix $P\mathcal{L}R$. Let $u_\theta\colon R^t\Omega \to \mathbb{R}^n$ be a solution with the required properties and let Ω_{AR} and Ω_{BR} its associated sets, then $u\colon \Omega \to \mathbb{R}^n$, $u\colon x \to Pu_\theta(R^t x)$ and $\Omega_A \equiv R\Omega_{AR}$ and $\Omega_B \equiv R\Omega_{BR}$ are a solution to our problem. ∎

3. Existence

In this section we define the relaxation property and trough the approximation Lemma 2.1 we prove the existence theorem.

Definition 3.1 (Relaxation property). *Let $L\colon \mathbb{R}^{n\times n} \to \mathbb{R}$ be a linear operator, and for any $\ell \in \mathbb{R}$ let $\mathcal{L}_\ell = \{\eta\colon L(\eta) = \ell\}$. We say that a set $K \subset \mathcal{L}_\ell$ has the relaxation property with respect to a set $E \subset \mathcal{L}_\ell$ if $\forall\Omega \subset \mathbb{R}^n$ open bounded subset, $\forall u_\xi$ affine in Ω such that $\xi \equiv Du_\xi \in \operatorname{int}K$ (the interior is relative to the affine manifold $\mathcal{L}_\ell$) there exists a sequence $u_\nu \in W_\theta$, $\theta > 0$ such that*

$$\begin{aligned} & u_\nu = u_\xi \text{ on } \partial\Omega \\ & u_\nu \overset{*}{\rightharpoonup} u_\xi \text{ in } W^{1,\infty}(\Omega,\mathbb{R}^n) \\ & Du_\nu \in E \cup \operatorname{int}K \text{ a.e. in } \Omega \\ & \int_\Omega \operatorname{dist}(Du_\nu(x), E)\,dx \to 0 \text{ when } \nu \to +\infty \end{aligned}$$

where

$$W_\theta = \left\{ \begin{matrix} u \in C^1_{piec}(\overline{\Omega},\mathbb{R}^n)\colon \exists\Omega_\theta \subset \Omega \text{ open} \\ |\Omega \setminus \Omega_\theta| < \theta \text{ and } u \text{ is piecewise affine in } \Omega_\theta \end{matrix} \right\}$$

Theorem 3.2. *Let $\Omega \subset \mathbb{R}^n$ open. Let $F_i\colon \mathbb{R}^{n\times n} \to \mathbb{R}$ $i = 1,\dots,I$ be continuous, quasi-convex functions, $L\colon \mathbb{R}^{n\times n} \to \mathbb{R}$, $L\colon \xi = (\xi_{ij}) \to \sum_{i,j=1}^{n} \alpha_{ij}\xi_{ij}$ a linear operator and $\ell \in \mathbb{R}$ a given number. Let*

$$E = \left\{\xi \in \mathbb{R}^{n\times n}\colon F_i(\xi) = 0 \quad \forall i = 1,\dots,I\,, \quad L(\xi) = \ell\right\}$$

Let us assume that RcoE has the relaxation property with respect to E and is compact. Let ϕ be a piecewise affine function such that $D\phi(x) \in E \cup \mathit{int}\, RcoE$. Then there exists $u \in W^{1,\infty}(\Omega, \mathbb{R}^n)$ such that

$$\begin{cases} F_i(Du) = 0 & \forall i = 1, \dots, I \quad a.e.\ x \in \Omega \\ L(Du) = \ell & a.e.\ x \in \Omega \\ u = \phi & on\ \partial\Omega \end{cases}$$

Proof. Without any loss of generality we can assume that Ω is bounded. Define

$$V_0 = \{u \in W_\theta \colon u = \phi \text{ on } \partial\Omega\,, \quad Du(x) \in E \cup \operatorname{int} \mathrm{Rco}E\}$$

and let V be its closure in the $W^{1,\infty}(\Omega, \mathbb{R}^n)$ weak* topology.

The functions F_i are quasi-convex hence

$$V \subset \left\{ \begin{array}{c} u \in \phi + W_0^{1,\infty}(\Omega, \mathbb{R}^n) \\ F_i(Du(x)) \leq 0 \quad \forall i = 1, \dots, I \text{ a.e. } x \in \Omega \\ L(Du(x)) = \ell \end{array} \right\}$$

Define $\mathcal{F}(u) = \displaystyle\sum_{i=1}^{I} \int_\Omega F_i(Du(x))\, dx$. The functions F_i are quasi-convex therefore $\forall u \in V \quad \displaystyle\liminf_{\substack{u_s \overset{*}{\rightharpoonup} u \\ u_s \in V}} \mathcal{F}(u_s) \geq \mathcal{F}(u)$. If $u \in V$ then $\mathcal{F}(u) = 0$ if and only if $Du(x) \in E \quad$ a.e. $x \in \Omega$.

Define $V^k = \left\{ u \in V \colon \mathcal{F}(u) > \dfrac{-1}{k} \right\}$. V^k is open since the functions F_i are quasi-convex. It suffices to show that it is also dense in V.

Let $u \in V$, $\varepsilon > 0$, we have to show that there exists $u_\varepsilon \in V^k$ such that $\|u - u_\varepsilon\|_\infty < \varepsilon$. It suffices to show that this property holds for any $u \in V_0$. Since $u \in V_0$ we also have $u \in W_\theta$. Define $u_\varepsilon = u$ in $\Omega \setminus \Omega_\theta$. $\Omega_\theta = \cup \Omega_\theta^i$ and $Du(x) = \xi_i \in E \cup \operatorname{int} \mathrm{Rco}E$ in Ω_θ^i. If $\xi_i \in E$ we are done, if $\xi_i \in \operatorname{int} \mathrm{Rco}E$ we use the relaxation property. ∎

Theorem 3.3. *Let F be a continuous, coercive rank one convex function. Define*

$$E = \left\{ \xi \in \mathbb{R}^{n \times n} \colon F(\xi) = 0\,, \quad L(\xi) = \ell \right\}$$

Then

$$RcoE = \left\{ \xi \in \mathbb{R}^{n \times n} \colon F(\xi) \leq 0\,, \quad L(\xi) = \ell \right\}$$

and the relaxation property holds.

Proof. Define $X \equiv \{\xi \in \mathbb{R}^{n \times n} \colon F(\xi) \leq 0\,, \quad L(\xi) = \ell\}$. Since $E \subset X$, and X is rank one convex $\mathrm{Rco}E \subset X$.

We have to prove that $X \subset \mathrm{Rco}E$. Let $\xi \in X$: if $F(\xi) = 0$ there is nothing left to prove: $\xi \in E \subset \mathrm{Rco}E$.

If $F(\xi) < 0$: let $\eta = a \otimes b$ with $\mathcal{L}b \neq 0$ and $L(\eta) = 0$ (as previously remarked it suffices to choose $a \perp \mathcal{L}b$). F is coercive therefore there exist $t_1 < 0 < t_2$ such that

$$\begin{cases} F(\xi + t\eta) < 0 & \forall t \in (t_1, t_2) \\ F(\xi + t_i\eta) = 0 & \forall i = 1, 2 \\ L(\xi + t_i\eta) = L(\xi) = \ell & \forall i = 1, 2 \end{cases}$$

Since $\xi = \dfrac{t_2}{t_2 - t_1}(\xi + t_1\eta) + \dfrac{-t_1}{t_2 - t_1}(\xi + t_2\eta)$ and $\xi + t_i\eta \in E \quad \forall i = 1, 2$ we have shown that $\xi \in \mathrm{Rco}E$.

Let us show that the relaxation property holds. Let $\Omega \subset \mathbb{R}^n$ be an open bounded set, and let u_ξ be affine in Ω such that $Du_\xi = \xi \in \mathrm{Rco}E$.

We have to find a sequence $u_\nu \in W_\theta$, $\theta > 0$ such that

$$\begin{cases} u_\nu = u_\xi & \text{on } \partial\Omega \\ u_\nu \overset{*}{\rightharpoonup} u_\xi & \text{in } W^{1,\infty}(\Omega, \mathbb{R}^n) \\ Du_\nu \in \operatorname{int}\mathrm{Rco}E & \text{a.e. in } \Omega \\ \displaystyle\int_\Omega F(Du_\nu(x))\,dx \to 0 & \text{as } \nu \to +\infty \end{cases}$$

If $F(\xi) = 0$ we let $u_\nu = u_\xi$; if $F(\xi) < 0$, $L(\xi) = \ell$, consider a rank one matrix $\eta = a \otimes b$ such that $\mathcal{L}b \neq 0$ and $L(\eta) = 0$: there exist $t_1 < 0 < t_2$ such that

$$\begin{cases} F(\xi + t\eta) < 0 & \forall t \in (t_1, t_2) \\ F(\xi + t_i\eta) = 0 & \forall i = 1, 2 \\ L(\xi + t_i\eta) = L(\xi) = \ell & \forall i = 1, 2 \end{cases}$$

Now it suffices to apply the approximation lemma to $A = \xi + (t_1 + \varepsilon)\eta$, $B = \xi + (t_2 - \varepsilon)\eta$ and $\xi = \dfrac{t_2 - \varepsilon}{t_2 - t_1 - 2\varepsilon}A + \dfrac{-(t_1 + \varepsilon)}{t_2 - t_1 - 2\varepsilon}B$. □

We now prove that if we have only one equation $F(Du) = 0$, with suitable compatibility conditions then a Lipschitz solution to (1.2) exists (actually infinitely many solutions exist).

Corollary 3.4 (Existence theorem for one equation)**.** *Let $\Omega \subset \mathbb{R}^n$ be an open set. Let $F\colon \mathbb{R}^{n\times n} \to \mathbb{R}$ be a continuous, coercive, quasi-convex function, let $L\colon \mathbb{R}^{n\times n} \to \mathbb{R}$, $L\colon \xi = (\xi_{ij}) \to \displaystyle\sum_{i,j=1}^{n} \alpha_{ij}\xi_{ij}$ be a linear operator and $\ell \in \mathbb{R}$ a given real number. Let ϕ be a piecewise affine function such that $F(D\phi(x)) \leq 0$ and $L(D\phi(x)) = \ell$. Then there exists $u \in W^{1,\infty}(\Omega, \mathbb{R}^n)$ such that*

$$\begin{cases} F(Du(x)) = 0 & \text{a.e. } x \in \Omega \\ L(Du(x)) = \ell & \text{a.e. } x \in \Omega \\ u = \phi & \text{on } \partial\Omega \end{cases}$$

Remark 3.5. *The coercivity hypothesis on F can be weakened. Let us give the following definition:*

Definition 3.6. *We say that a function $F\colon \mathbb{R}^{n\times n} \to \mathbb{R}$ is coercive in a rank one direction $\eta = a \otimes b$ if for any bounded subset B of $\mathbb{R}^{n\times n}$ there exist m, $q \in \mathbb{R}$, with $m > 0$ such that*

$$F(\xi + t\eta) \geq m\,|t| - q \qquad \forall t \in \mathbb{R}, \quad \forall \xi \in B\,.$$

It suffices F to be coercive in a rank one direction $\eta = a\otimes b$ such that $\mathcal{L}b \neq 0$ and $L(\eta) = 0$. Our chances of finding such an η increase when the rank of $\mathcal{L} = (\alpha_{ij})$ increases.

4. A note on the approximation lemma

Here we want to prove that in Lemma 2.1 the hypothesis $\mathcal{L}v \neq 0$ cannot be removed. In fact we show that a certain Dirichlet problem does not admit a solution, and that, if the lemma held even if $\mathcal{L}v = 0$ we could show the existence of solutions.

Let $\Omega = (0,1)\times(0,1)$. The Dirichlet problem we consider is as follows. We want to find $u = (u_1, u_2) \in W^{1,\infty}(\Omega, \mathbb{R}^2)$ such that

$$\begin{cases} \dfrac{\partial u_1}{\partial x_2} = 0 & \text{a.e. in } \Omega \\ \left|\dfrac{\partial u_1}{\partial x_1}\right| - 1 = 0 & \text{a.e. in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

Let (u_1, u_2), be a solution to this problem, then $u_1 = u_1(x_1)$ and ${u_1}'(x_1) = \pm 1$ a.e. $x_1 \in (0,1)$. Hence the boundary condition $u_1 = 0$ cannot be fulfilled and the problem doesn't admit any Lipschitz solution.

Let us consider $F(\xi) = |\xi_{11}| - 1$. It is coercive in the rank-1 direction $\Lambda = (a_1, a_2) \otimes (b_1, b_2)$ if and only if $a_1 b_1 \neq 0$. In this problem $\mathcal{L} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ therefore $\mathcal{L}\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} b_2 \\ 0 \end{pmatrix}$ and $L(\Lambda) = a_1 b_2$. So the only rank-1 directions Λ such that F is coercive in the direction Λ and $L(\Lambda) = 0$ are those which satisfy $\begin{cases} a_1 b_1 \neq 0 \\ a_1 b_2 = 0 \end{cases}$ i.e. those $\Lambda = (a_1, a_2) \otimes (b_1, b_2)$ such that $\begin{cases} b_2 = 0 \\ a_1 b_1 \neq 0 \end{cases}$. For all such directions $\mathcal{L}\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. So we can always assume $\Lambda = (a_1, a_2) \otimes (1, 0)$ with $a_1 \neq 0$. Now let us assume Lemma 2.1 holds even if $\mathcal{L}v = 0$. We want to show that in such case we could prove this Dirichlet problem admits a solution.

Let

$$\begin{aligned} E &= \{\xi \in \mathbb{R}^{2\times 2} \colon L(\xi) = 0\,,\ F(\xi) = 0\} = \{\xi \in \mathbb{R}^{2\times 2} \colon \xi_{12} = 0\,,\ |\xi_{11}| = 1\}, \\ X &= \{\xi \in \mathbb{R}^{2\times 2} \colon L(\xi) = 0\,,\ F(\xi) \leq 0\} = \{\xi \in \mathbb{R}^{2\times 2} \colon \xi_{12} = 0\,,\ |\xi_{11}| \leq 1\}. \end{aligned}$$

Step 1 (RcoE = X): X is rank-1 convex, $E \subset X$ hence $\mathrm{Rco}E \subset X$. The reverse inequality holds: let $\xi \in X$: if $|\xi_{11}| = 1$, $\xi \in E \in \mathrm{Rco}E$ and there is nothing left to prove. If $|\xi_{11}| < 1$, let $\Lambda = (a_1, a_2) \otimes (b_1, b_2)$ be a rank-1 matrix such that $L(\Lambda) = 0$ and F is coercive in the direction Λ. We have shown that we must have $\Lambda = (a_1, a_2) \otimes (1, 0)$ with $a_1 \neq 0$.

Since F is coercive in the direction Λ, there exist $t_1 < 0 < t_2$ such that

$$\begin{cases} F(\xi + t\Lambda) = 0 & \forall t \in (t_1, t_2) \\ F(\xi + t_1\Lambda) = F(\xi + t_2\Lambda) = 0 & \\ L(\xi + t\Lambda) = L(\xi) + tL(\Lambda) = 0 & \forall t \in \mathbb{R} \end{cases}$$

We may write $\xi = \dfrac{t_2}{t_2 - t_1}(\xi + t_1\Lambda) + \dfrac{-t_1}{t_2 - t_1}(\xi + t_2\Lambda)$ hence $\xi \in \mathrm{Rco}E$.

Step 2 (the relaxation property holds): Let $\Omega \subset \mathbb{R}^2$ be an open bounded set, let u_ξ be an affine function on Ω such that $Du_\xi = \xi \in \mathrm{Rco}E$. We must find a sequence of positive real numbers, $\{\theta_n\}$, decreasing to 0 and a sequence of functions $\{u_n\}$, $u_n \in C^1_{\text{piec}}(\overline{\Omega}, \mathbb{R}^2)$ such that for any $n \in \mathbb{N}$

$$\begin{cases} \exists \Omega_n \subset \Omega, \text{ open subset such that } |\Omega \setminus \Omega_n| < \theta_n \\ u_n \text{ is piecewise affine in } \Omega_n \\ u_n = u_\xi \text{ on } \partial\Omega \\ u_n \overset{*}{\rightharpoonup} u_\xi \text{ in } W^{1,\infty}(\Omega, \mathbb{R}^2) \\ Du_n \in \int \mathrm{Rco}E \text{ a.e. in } \Omega \\ \displaystyle\int_\Omega F(Du_n(x))\,dx \to 0 \text{ as } n \to \infty \end{cases}$$

If $F(\xi) = 0$ there is nothing to be done, if $F(\xi) < 0$, with the same Λ, t_1 and t_2 of the previous step and $\varepsilon > 0$, let

$$A = \xi + (t_1 + \varepsilon)\Lambda\,, \quad B = \xi + (t_2 - \varepsilon)\Lambda\,.$$

We may write

$$\xi = \frac{t_2 - \varepsilon}{t_2 - t_1 - 2\varepsilon} A + \frac{-(t_1 + \varepsilon)}{t_2 - t_1 - 2\varepsilon} B\,.$$

If we could apply the approximation lemma to these matrixes we would have proved that the relaxation property holds, and therefore the problem would admit a solution, see Corollary 3.4 and Remark 3.5.

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