FURTHER EXISTENCE RESULTS FOR EVOLUTION EQUATIONS OF MIXED TYPE AND FOR A GENERALIZED TRICOMI EQUATION

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Abstract. In this note we give several existence results: for first order evolution equation of the type \( Ru' + Au = f \) where \( R \) may be a function depending also on time assuming positive, null and negative sign, then the equation may be elliptic-parabolic, both forward and backward; for second order evolution equation of the type \( Ru'' + A'u + Au = f \) where \( R \) may be a function depending also on time assuming positive, null and negative sign. The setting is very general, \( R \) may be an operator and some varying with time Banach spaces are considered.

We also generalize known existence results for the equations \((Ru')' + Au = f\) and \((Ru')' + Au + Au = f\) in the setting of varying spaces.

Finally we give existence results for the generalized Tricomi equations \( Ru'' + Au = f \) and \((Ru')' + Au = f\) with \( R \) as before.

We also give some time regularity result for the solutions and at the end we give many examples of different possible choices of \( R \).

Mathematics subject classification: 35M10, 35R20, 35K90, 35L90

Keywords: abstract equations; mixed type equations; elliptic-parabolic equations; forward-backward parabolic equations; Tricomi equation; elliptic-hyperbolic equations.

1. Introduction

In this paper we consider many abstract differential equations of mixed type in abstract form which we specify below. This equations are abstract equations of first and second order and may be of elliptic-parabolic type, parabolic both forward and backward, and of elliptic-hyperbolic type.

Equations of mixed type have been considered since at least one century ago, since, as far as many authors say, they are mentioned in [7]. Here we recall here some simple and more known examples:

\[ x \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0, \quad \text{sgn}(x) \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} + ku = f. \]

The first was considered in [1] in 1968, the second one in [16] in 1971. They are particular cases of a class of “parabolic type” equations like

\[ \sigma(x)u_t + Au = f \]

where \( A \) is an elliptic operator and \( \sigma \) a changing sign function. They seem to be interesting in many areas and have arisen in connection with many different problems: in the study of some stochastic differential equation, in the kinetic theory, in some physical models (like electron scattering, neutron transport). For these applications we confine to quote the recent paper [10] and for the many others we refer to the references contained in the already quoted paper [16] and in [2], [3]. Just in these papers Beals treated equations like (1), but always
with simple $\sigma$. For instance, in [2] the equation
\[ x \frac{\partial u}{\partial t} - \frac{\partial}{\partial x} \left( (1 - x^2) \frac{\partial u}{\partial x} \right) = 0 \]
is considered, but, as Gevrey seems to suggest and as Beals says, some coefficients like
\[ \sigma(x) = x^m, \quad m \text{ odd} \quad \text{or} \quad \sigma(x) = \text{sgn}(x) |x|^p \]
are of interest in some applications. Among papers taking into account equations with some more general $\sigma$ we recall [24] and [9]. In particular in [9] a coefficient depending also on time is considered and a condition of regularity in time is requested, i.e.
\[ \sigma = \sigma(x, t), \quad \sigma, \sigma_t \in L^\infty. \]
A recent paper where the author considers $\sigma = \sigma(x, t)$ (and $A$ linear) is [8]. Another paper we want to recall is [17] which is more general, even if incomplete.

As regards elliptic-parabolic equations, i.e. equations like (1) where $\sigma \geq 0$, the known results were more general than those regarding forward-backward parabolic equations. We recall the paper [26] and the book (see chapter 3) [27] for some general results. Finally we want to recall [4] for many examples and applications of equations with non-negative coefficients.

As regards equations of second order we recall the Tricomi equations
\[ x \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0 \]
which is of hyperbolic type where $x > 0$ and becomes elliptic where $x < 0$. About equations of this type, and their applications, we refer to the review papers [6] and [14] (and the references therein) for an overview about Tricomi equation, equations of this type and their applications. Even in this case one can consider more general equations, like for instance
\[ \sigma u_{tt} + Bu_t + Au = f \]
where (here we are intentionally vague) $A$ is an elliptic operator, $B$ is an operator and $\sigma$ is a function which may change sign. In the already quoted paper [8] also a second order equation is considered and also in this case, besides to the first order equation, the coefficient $\sigma$ is depending also on time, i.e.
\[ \sigma = \sigma(x, t). \]
Besides to this paper, about equations of this type we want to mention a very general result due to Showalter and contained in [25] where $\sigma \geq 0$, but in return it is quite general since at the place of $\sigma$ in (3) an operator $S$ is considered.

As regards first order equations all the results we cite above are generalized in the paper [20], where it is considered the abstract equation
\[ (Ru)' + Au = f \]
with suitable boundary data, where $A$ is a monotone operator and $R$ is a linear operator depending also on time which can be not invertible. When $R$ is a multiplication operator, i.e. $(Ru)(x, t) := r(x, t)u(x, t)$, also discontinuous and unbounded coefficients, i.e. $r \in L^1_{\text{loc}}$, are admitted. Just to show an example, an equation included in the result in [20] is
\[ (r(x, t)u)_t - \text{div}(|Du|^{p-2}Du) = f, \quad p \geq 2. \]
An analogous result for second order equations like
\[ (Ru')' + Au' + Bu = f, \]
where $\mathcal{R}$, $\mathcal{A}$, $\mathcal{B}$ are operators, $\mathcal{R}$ and $\mathcal{B}$ linear, which generalizes the result contained in [25] to $\mathcal{R}$ which depends also on time and which can be also negative, is given in [21].

A simple example which suits to (6) one can keep in mind is the following:

$$ (r(x,t)u_t)_t - \text{div}(a \cdot Du_t) - \text{div}(b \cdot Du) = f $$

where $a, b$ elliptic matrices. The function $r$ may be discontinuous and unbounded as in (5).

The aim of this paper is giving existence results for problems, with suitable boundary data, involving the following abstract equations

$$ \mathcal{R}u' + \mathcal{A}u = f, $$
$$ \mathcal{R}u'' + \mathcal{A}u' + \mathcal{B}u = f, $$
$$ (\mathcal{R}u')' + \mathcal{A}u' + \mathcal{B}u = f, $$
$$ (\mathcal{R}u')' + \mathcal{B}u = f. $$

(7)

Honestly we already treated the equation $(\mathcal{R}u')' + \mathcal{A}u' + \mathcal{B}u = f$, in [21], but in the present paper we consider a more general setting, so we extend the results contained in [21] to this setting (see (8) and Section 2).

As regards the equation $(\mathcal{R}u')' + \mathcal{A}u = f$, we already treated it in [20] and generalized it to the more general setting of the present paper in [22]. The motivation which drove us to study also the equations $\mathcal{R}u' + \mathcal{A}u = f$ ($\mathcal{R}$ is an operator possibly depending also on time) is that also situations like

$$ r(x,t)u_t + \mathcal{A}u = f $$

are of interest (see, for instance, [9] and [29]), so we give a quite general existence and uniqueness result. Moreover we give a time regularity result for the solutions to such equations (with suitable boundary data satisfying some suitable compatibility conditions). Clearly when $\mathcal{R}$ does not depend on time the present result and the one contained in [20] coincide. We want to mention that a very deep study of regularity both in space and time for solutions to this equation has been recently given in [19] (but see also [18]).

Besides to this equation we also study the second order equation

$$ \mathcal{R}u'' + \mathcal{A}u' + \mathcal{B}u = f, \quad (x,t) \in \Omega \times (0,T), $$

using the result given for $\mathcal{R}u' + \mathcal{A}u = f$ to prove existence and uniqueness result for this equation with suitable boundary data. Also in this case, when $\mathcal{R}$ does not depend on time, the present result and the one given in [21] coincide.

Roughly speaking, in the first order equation we give the initial datum about $u$ at time 0 where $r$ “is positive”, we prescribe a final datum for $u$ at time $T$ where $r$ “is negative”, no datum is prescribed where $r = 0$, both at time $t = 0$ and $t = T$, but below (and in the examples in the last section) you can find the detailed problems. The analogous thing is done for the second order equation, but with the data depending on the sign of $r$ prescribed for $u_t$. Also in this case for the details see below or the last section, where many examples are given.

Besides giving the analogous result given in [21] and generalize [25], a motivation to study this equation is also to prepare a result needed to prove a theorem of Section 5, regarding some generalized Tricomi equations.

Since we consider abstract equation we consider functions defined in $[0,T]$ and valued in a Banach space. But the setting is not standard, we consider a more general situation, we have a family of triplets

$$ V(t) \subset H(t) \subset V'(t), \quad t \in [0,T], $$

(8)

where $V(t)$ is a reflexive Banach space which continuously embeds in the Hilbert space $H(t)$, while $V'(t)$ denotes the dual space of $V(t)$. In this way our functions will be defined in $[0,T]$.
and, for each \( t \in [0, T] \), \( u(t) \) will denote an element in \( V(t) \), \( H(t) \) or \( V'(t) \). For this reason, some technical results are needed (we are referring to Lemma 3.17 and Lemma 3.19).

In the last section we show with some examples why this setting can be interesting. Finally the last goal of this paper is to consider the generalized Tricomi equations

\[
\mathcal{R}u'' + Bu = f, \quad (\mathcal{R}u')' + Bu = f
\]

and give an existence (and uniqueness) result for some problems with suitable boundary data. The idea is to use the result given for the two equations \( \mathcal{R}u'' + Au' + Bu = f \) and \( (\mathcal{R}u')' + Au' + Bu = f \), consider \( A = \epsilon B \) for \( \epsilon > 0 \) and take the limit as \( \epsilon \) goes to zero. Then, as before, both \( B \) and \( \mathcal{R} \) are linear operators. Clearly, if \( \mathcal{R} \) does not depend on time the two results coincide.

As we will show this procedure will not be easy because the natural estimates on the solutions \( u_\epsilon \) goes as \( \epsilon^{-1/2} \), therefore some more fine estimates are needed. The simplest cases one can keep in mind are the

\[
(r(x,t) u)_t - \text{div}(b \cdot Du) = f, \quad r(x,t) u_t - \text{div}(b \cdot Du) = f,
\]

where \( Bu := -\text{div}(b \cdot Du) \) with \( b \) elliptic matrix and the operator \( \mathcal{R} \) is a multiplication operator defined as \( (\mathcal{R}u)(x,t) := r(x,t)u(x,t) \). For the equation

\[
(\mathcal{R}u')' + Bu = f
\]

the assumptions we need about \( \mathcal{R} \) and \( B \) are the same needed to prove the result for the equation (6), while for the equation

\[
\mathcal{R}u'' + Bu = f
\]

we need to require about \( \mathcal{R} \) some more regularity (see subsection 5.1, and in particular point 5, for details). Also for the details about boundary data for these equations we refer to Section 5 and to the last section where we show some examples.

2. Notations, hypotheses and preliminary results

Consider the following family of evolution triplets

\[
V(t) \subset H(t) \subset V'(t) \quad t \in [0, T]
\]

where \( H(t) \) is a separable Hilbert space, \( V(t) \) a reflexive Banach space which continuously and densely embeds in \( H(t) \) and \( V'(t) \) the dual space of \( V(t) \), and we suppose there is a constant \( k \) which satisfies

\[
\|w\|_{V'(t)} \leq k \|w\|_{H(t)}, \quad \text{and} \quad \|v\|_{H(t)} \leq k \|v\|_{V'(t)}
\]

for every \( w \in H(t) \), \( v \in V(t) \) and every \( t \in [0, T] \).

We will suppose the existence of a Banach space \( U \) such that

\[
U \subset V(t) \quad \text{and} \quad U \text{ dense in } V(t) \quad \text{for a.e. } t \in [0, T]
\]

and define, for some \( p \geq 2 \), the set

\[
\mathcal{U} := L^p(0, T; U).
\]

Moreover we will suppose that the functions

\[
t \mapsto \|u(t)\|_{V(t)}, \quad t \mapsto \|u(t)\|_{H(t)}, \quad t \mapsto \|u(t)\|_{V'(t)}, \quad t \in [0, T],
\]

are measurable for every \( u \in \mathcal{U} \) and for the same \( p \in [2, +\infty) \) used to define \( \mathcal{U} \) we define the spaces

\[
\mathcal{V} \quad \text{and} \quad \mathcal{H}
\]
as the completion of $\mathcal{U}$ with respect to the natural norms

$$
\|v\|_{V} := \left( \int_{0}^{T} \|v(t)\|_{V(t)}^{p} \, dt \right)^{1/p}, \quad \|v\|_{\mathcal{H}} := \left( \int_{0}^{T} \|v(t)\|_{\mathcal{H}(t)}^{2} \, dt \right)^{1/2}.
$$

Finally by $\mathcal{V}'$ we denote the dual space of $\mathcal{V}$ endowed with the norm

$$
\|f\|_{\mathcal{V}'} := \left( \int_{0}^{T} \|f(t)\|_{\mathcal{V}'(t)}^{p'} \, dt \right)^{1/p'}.
$$

Notice that if there is a Banach space $Z$ such that

$$
(13) \quad V(t) \subset Z \quad \text{and} \quad V(t) \text{ dense in } Z \quad \text{for a.e. } t \in [0, T]
$$

then the space $\mathcal{V}$ (and similarly the space $\mathcal{H}$) turns out to be

$$
\{ v \in L^p(0, T; Z) \mid v(t) \in V(t) \quad \text{for a.e. } t \in [0, T] \}.
$$

For our purposes the existence of a space $Z$ is not needed, but for the examples we have in mind such a Banach space exists. We will show some examples in the last section.

**Definition 2.1.** Given a family of linear operators $R(t)$ such that

$$
(14) \quad R \text{ depends on a parameter } t \in [0, T] \quad \text{and} \quad R(t) \in \mathcal{L}(H(t)),
$$

being $\mathcal{L}(H(t))$ the set of linear and bounded operators from $H(t)$ in itself, instead of (14) we sometimes will write improperly

$$
(15) \quad R : [0, T] \rightarrow \mathcal{L}(H(t)), \quad t \in [0, T].
$$

Now consider an abstract function $R : [0, T] \rightarrow \mathcal{L}(H(t))$. We say that $R$ belongs to the class $\mathcal{E}(C_1, C_2)$, $C_1, C_2 > 0$, if it satisfies what follows for every $u, v \in U$:

- $R(t)$ is self-adjoint and $\|R(t)\|_{\mathcal{L}(H(t))} \leq C_1$ for every $t \in [0, T]$,
- $t \mapsto \langle R(t)u, v \rangle_{H(t)}$ is absolutely continuous on $[0, T]$,
- $\left| \frac{d}{dt} \langle R(t)u, v \rangle_{H(t)} \right| \leq C_2 \|u\|_{V(t)} \|v\|_{V(t)}$ for a.e. $t \in [0, T]$.

Now, given two positive constants $C_1$ and $C_2$, consider $R \in \mathcal{E}(C_1, C_2)$. For every $t \in [0, T]$ we consider the spectral decomposition of $R(t)$ (see, e.g., Section 8.4 in [15]) and define $R_+(t)$, and respectively $R_-(t)$, the operator connected to the positive, respectively negative, part of the spectrum, so that $R(t) = R_+(t) - R_-(t)$ and $R_+(t) \circ R_-(t) = R_-(t) \circ R_+(t) = 0$ (see, e.g., Theorem 10.37 in [15]) and $R_+(t)$ and $R_-(t)$ turn out to be invertible. By $|R(t)|$ we will mean $R_+(t) + R_-(t)$. By this decomposition we can also write $H(t) = H_+(t) \oplus H_0(t) \oplus H_-(t)$ where $H_+(t) = (\text{Ker } R_+(t))^\perp$ and $H_-(t) = (\text{Ker } R_-(t))^\perp$ and $H_0(t)$ is the kernel of $R(t)$. Finally we denote $\hat{H}_0(t) = H_0(t) = \text{Ker } R(t)$ and

$$
(16) \quad \hat{H}(t), \hat{H}_+(t), \hat{H}_-(t) = \text{the completion respectively of } H(t), H_+(t), H_-(t)
$$

with respect to the norm $\|w\|_{\hat{H}(t)} = \|R(t)^{1/2}w\|_{H(t)}$.

Clearly the operation $\hat{\cdot}$ depends on $R$.

Given an operator $R \in \mathcal{E}(C_1, C_2)$ it is possible to define two other linear operators. First we can define the derivative of $R$ which, unlike $R$, is valued in $\mathcal{L}(V(t), V'(t))$, i.e. the set
of linear and bounded operators from $V(t)$ to $V'(t)$: since $R \in \mathcal{E}(C_1, C_2)$ we can define a family of equibounded operators

$$R'(t), \quad t \in [0, T], \quad R'(t) : V(t) \to V'(t)$$

by

$$\langle R'(t)u, v \rangle_{V'(t) \times V(t)} := \frac{d}{dt} \langle R(t)u, v \rangle_{H(t)}, \quad u, v \in U.$$ 

By the density of $U$ in $V(t)$ we can extend $R'(t)$ to $V(t)$. Then we can also define

$$(17) \quad \mathcal{R} : \mathcal{H} \to \mathcal{H}, \quad (\mathcal{R}u)(t) := R(t)u(t)$$

which turns out to be linear and bounded by the constant $C_1$ and, by density of $U$ in $V$, an operator

$$(18) \quad \mathcal{R} : V \to V' \quad \text{by} \quad \langle \mathcal{R}u, v \rangle_{V' \times V} := \int_0^T \langle \mathcal{R}'(t)u(t), v(t) \rangle_{V'(t) \times V(t)} dt$$

which turns out to be linear, self-adjoint and bounded by $C_2$. As done before we can define, in a way analogous to that done for the spaces (16),

$$(19) \quad \mathcal{H}, \mathcal{H}_+, \mathcal{H}_- = \text{the completion respectively of } \mathcal{H}, \mathcal{H}_+, \mathcal{H}_-$$

with respect to the norm $||w||_{\mathcal{H}} = |||R|||^{1/2} w||_{\mathcal{H}}$. Analogously, we define $\mathcal{H}_+$ and $\mathcal{H}_-$ and $\mathcal{P}_+$ and $\mathcal{P}_-$ the orthogonal projections from $\mathcal{H}$ onto $\mathcal{H}_+$ and $\mathcal{H}_-$ respectively. $\mathcal{H}_0$ is the kernel of $\mathcal{R}$ and $\mathcal{P}_0$ the projection defined in $\mathcal{H}$ onto $\mathcal{H}_0$.

**Remark 2.2.** - Notice that since $R$ is self-adjoint and bounded we can define $|R|(t)^{1/2}$, $R_+(t)^{1/2}$, $R_-(t)^{1/2}$ (see, e.g., Chapter 3 in [15]).

### 3. The existence result for first order equations

In this section we will give one of the main results of the paper. We will consider a function $R$ such that, given two positive constants,

$$(20) \quad R \in \mathcal{E}(C_1, C_2)$$

and all the spaces we introduced in (9), (11), (12), (16).

Our goal is to give an existence result for an abstract equation like

$$\mathcal{R}u' + Au = f$$

for some suitable operator $\mathcal{A}$ we will specify below.

We want to stress that, despite of the fact that no derivative of $\mathcal{R}$ appears in the equation, we require $\mathcal{R}$ to be differentiable, i.e. $R \in \mathcal{E}(C_1, C_2)$. This fact will be needed to get the existence of a solution to the previous equation and we will also show (see example 6 in the last section) that without this assumption at least uniqueness of the solution fails. Anyway to require that $R$ is differentiable is not so restrictive (as shown in the examples in the last section) because if, for instance, $\mathcal{R}$ is a multiplication operator, i.e. $\mathcal{R}u = r(x, t)u(x, t)$ for some function $r$, $\mathcal{R}$ could be differentiable even if $r$ is discontinuous.

We will use this assumption about $\mathcal{R}$ to split properly the operator $u \mapsto \mathcal{R}u' + Au$ as indicated in (30) to give the existence of a solution.

We will use the operator $\mathcal{R}'$ to define $\mathcal{R}u'$ in an apparently involute way. First for a function $u \in V$, we consider the generalized derivative of $\mathcal{R}u$ and require that it belongs to $V'$, where the generalized derivative is defined as a function $w \in V'$ such that

$$\langle w(t), v \rangle_{V'(t) \times V(t)} = \frac{d}{dt} \langle \mathcal{R}u(t), v \rangle_{H(t)} \quad \text{for every } v \in U.$$
that

\[ U \subset V(t) \subset H(t) \subset V'(t) \subset U', \]

than we can define in a classical way the generalized derivative of \( Ru \) in \( L^{p'}(0, T; U') \), and then to require that \( Ru \in V' \) (here \( p' \) denotes \( p/(p-1) \)).

For a function \( u \in V \) for which \( Ru \) admits generalized derivative in \( V' \) we define

\[ (21) \quad Ru' := (Ru)' - Ru. \]

With this definition in mind we now define the space

\[ (22) \quad \mathcal{W}_R := \{ u \in V \mid Ru' \in V' \}, \quad \|u\|_{\mathcal{W}_R} = \|u\|_V + \|Ru'\|_{V'}. \]

**Remark 3.1.** - Notice that one could also define \( \mathcal{W}_R \) as the space \( \{ u \in V \mid (Ru)' \in V' \} \) endowed with the norm \( \| \cdot \|_{\mathcal{W}_R} \) defined in (22), in view of the definition of \( Ru' \) given in (21), since we require that both \( Ru' \) and \( (Ru)' \) belong to \( V' \).

**Proposition 3.2.** The space \( C^1([0, T]; U) \) is dense in \( \mathcal{W}_R \).

**Proof** - Clearly \( C^1([0, T]; U) \) is dense in the space \( \mathcal{U} \) with respect to the topology of \( L^{p'}(0, T; U) \). Fix a function \( v \in \mathcal{W}_R \). Since \( v \in V \) we can find a sequence \((u_j)_j \subset \mathcal{U}\) such that

\[
\lim_{j \to \infty} \|u_j - v\|_V = 0.
\]

Now we can choose some (suitable) kernels \((\rho_\epsilon)_{\epsilon > 0}\) and consider \( u_{\epsilon, j} := u_j * \rho_\epsilon \) and prove that \( Ru'_{\epsilon, j} \to Ru' \) in \( V' \) following the proof of Proposition 2.4 in [20]. \( \square \)

**Remark 3.3.** - Notice that if \( V(t) = V \) and \( H(t) = H \) for every \( t \in [0, T] \), assumptions (20) on \( R \) simply means

\[ R \in L^{\infty}(0, T; \mathcal{L}(H)) \cap W^{1, \infty}(0, T; \mathcal{L}(V, V')). \]

**Proposition 3.4.** Under assumption (20) we have that for every \( u, v \in \mathcal{W}_R \) the following holds:

\[
\frac{d}{dt} \langle Ru(t), v(t) \rangle_{H(t)} = \langle Ru'(t), v(t) \rangle_{V'(t) \times V(t)} + \langle Ru'(t), v(t) \rangle_{V' \times V(t)} + \langle Ru'(t), u(t) \rangle_{V' \times V(t)}.
\]

Moreover the function \( t \mapsto \langle R(t)u(t), v(t) \rangle_{H(t)} \) is continuous and there exists a constant \( c \), which depends only on \( T \), such that

\[
\max_{[0, T]} \|R(t)u(t), v(t)\|_{H(t)} \leq c \left[ \|Ru'\|_V \|v\|_V + \|Ru'\|_V \|u\|_V + \|Ru\|_V \|v\|_V + \|Ru\|_V \|u\|_H \|v\|_H \right].
\]

In particular if \( u = v \) we have

\[
\int_s^t \langle Ru'(\tau), u(\tau) \rangle_{V'(\tau) \times V(\tau)} d\tau = \frac{1}{2} \left[ \langle R(t)u(t), u(t) \rangle_{H(t)} - \langle R(s)u(s), u(s) \rangle_{H(s)} \right] - \frac{1}{2} \int_s^t \langle Ru'(\tau), u(\tau) \rangle_{V'(\tau) \times V(\tau)} d\tau
\]

and

\[ (23) \quad \max_{[0, T]} \|R(t)u(t), u(t)\|_{H(t)} \leq c \|u\|^2_{\mathcal{W}_R} \]

where \( c \) depends (only) on \( T, \|Ru\|_{\mathcal{L}(V, V')}, \|Ru\|_{\mathcal{L}(H)} \).
Proof - One can follow the proof in [30] (see ex. 23.10) or the one contained in [28] (see Lemma 40.2) to obtain the thesis for every \( u,v \in C^1([0,T];U) \) and then use the density result in Proposition 3.2. \( \square \)

Remark 3.5. - Since the function \( t \mapsto (R(t)u(t), u(t))_{H(t)} \) is \( L^\infty(0,T) \) we have that
\[
(|R(t)|u(t), u(t))_{H(t)} \quad \text{is finite for every} \quad t \in [0,T]
\]
but, in general,
\[
(\mathcal{R} u(t), u(t))_{H(t)} \leq c \|u\|_{\mathcal{W}_R}^2
\]
may not hold, as an example in [20] shows.

Theorem 3.6. Suppose \( R \neq 0 \). Then the space \( \mathcal{W}_R \) compactly embeds in \( \tilde{H} \).

Proof - We give a sketch of the proof, since one can follow the proof of Theorem 5.1 in [12]. Consider a sequence \( (u_h)_h \) such that \( \|u_h\|_{\mathcal{W}_R} \leq c \). In particular, up to a subsequence, \( u_h \to u \) weakly in \( \mathcal{V} \) and suppose \( u = 0 \). Then by Lemma 2.11 in [20]
\[
\|\mathcal{R}^{1/2} u_h\|_{\mathcal{H}} \leq \eta \|u_h\|_{\mathcal{V}} + c \eta \|\mathcal{R} u_h\|_{\tilde{\mathcal{V}}}
\]
where \( \tilde{\mathcal{V}} \) is the completion of \( \mathcal{U} \) with respect to the norm
\[
\|v\|_{\tilde{\mathcal{V}}} := \left( \int_0^T \|v(t)\|_{\mathcal{V}(t)}^p dt \right)^{1/p}.
\]
and defining \( v_h = u_h/\|u_h\|_{\mathcal{V}} \) we have
\[
\|\mathcal{R}^{1/2} v_h\|_{\mathcal{H}} \leq \eta + c \eta \|\mathcal{R} v_h\|_{\tilde{\mathcal{V}}}
\]
Since \( \mathcal{R} v_h \in \mathcal{H} \) and \( (\mathcal{R} v_h)' \in \mathcal{V}' \) (see Remark 3.1) by Proposition 3.4 we have that \( t \mapsto (\mathcal{R} v_h(t), v_h(t))_{H(t)} \) is a continuous function, so it is sufficient to prove that \( (\mathcal{R} v_h(t), v_h(t))_{H(t)} \to 0 \) for every \( t \in [0,T] \). To prove this fact it is sufficient to follow the proof in [12]. \( \square \)

Before to state the existence result we recall some definitions and a classical result, for which we refer to [30] (see Section 32.4).

Definition 3.7. Given a Banach space \( X \), we say that an operator \( M : X \to X' \) is coercive if
\[
\lim_{\|x\| \to +\infty} \frac{\langle Mx, x \rangle}{\|x\|} \to +\infty,
\]
is bounded if it maps a bounded set in a bounded set, is pseudomonotone if
\[
x_n \to x \text{ in } X\text{-weak} \quad \text{and} \quad \limsup_n \langle Mx_n, x_n - x \rangle \leq 0
\]
implies that
\[
\langle Mx, x - y \rangle \leq \liminf_n \langle Mx_n, x_n - y \rangle \quad \text{for every } y \in X.
\]

Theorem 3.8. Let \( M : X \to X' \) (\( X' \) the dual space of \( X \), \( X \) Banach space) be pseudomonotone, bounded and coercive. Suppose \( L : X \to 2^{X'} \) to be maximal monotone. Then for every \( f \in X' \) the following equation has a solution
\[
Lu + Mu \ni f
\]
and in particular if \( L,M \) are single-valued the equation \( Lu + Mu = f \) has a solution.
Now we pass to introduce our assumptions. Consider a family of operators

\[ A(t) : V(t) \to V'(t) \]

such that

\[ t \mapsto \langle A(t)u, v \rangle_{V'(t) \times V(t)} \text{ measurable on } [0, T] \text{ for every } u, v, \in U \]

such that if we define the abstract operator

\[ (25) \quad A : V \to V', \quad Au(t) = A(t)u(t) \quad 0 \leq t \leq T. \]

this turns out to be

\[ A \text{ pseudomonotone, coercive, bounded.} \]

We denote

\[ P : W_\mathbb{R} \to V', \quad (Pu)(t) = \mathcal{R}u'(t) + Au(t), \quad 0 \leq t \leq T \]

where \( \mathcal{R} \) is defined in (17) and satisfies (20).

**Remark 3.9.** Thanks to Remark 3.5 we have that

\[ (R_+(t)u(t), u(t))_{H(t)} \text{ and } (R_-(t)u(t), u(t))_{H(t)} \text{ are finite for every } t \in [0, T] \]

and then in particular it makes sense to consider for \( u \in W_\mathbb{R} \)

\[ R_+(t)u(t) \text{ and } R_-(t)u(t) \text{ for every } t \in [0, T] \]

and then \( u(t) \) belongs to \( \hat{H}(t) \). Then if we denote the orthogonal projections

\[ (26) \quad P_+(t) : \hat{H}(t) \to \hat{H}_+(t) \quad \text{and} \quad P_-(t) : \hat{H}(t) \to \hat{H}_-(t) \]

and since \( R_+(t) \) and \( R_-(t) \) are invertible we get that for every \( \varphi \in \hat{H}_+(t) \) and \( \psi \in \hat{H}_-(t) \) it makes sense to consider

\[ P_+(t)u(t) = \varphi \quad \text{in} \quad \hat{H}_+(t), \quad P_-(t)u(t) = \psi \quad \text{in} \quad \hat{H}_-(t). \]

We will use also the projections \( P_+ \) and \( P_- \) defined in \( \mathcal{H} \).

Thanks to the previous remark we can define

\[ (27) \quad W^0_\mathbb{R} = \{ u \in W_\mathbb{R} \mid P_+(0)u(0) = 0, P_-(T)u(T) = 0 \}. \]

Now we consider the two operators

\[ \mathcal{L}_1 u = \mathcal{R}u' + \frac{1}{2} \mathcal{R}'u, \quad \mathcal{L}_2 u = \mathcal{R}u', \quad D(\mathcal{L}_1) = D(\mathcal{L}_2) = W^0_\mathbb{R} \]

and state the following result, whose proof can be obtained by the analogous in [20].

**Proposition 3.10.** The operator \( \mathcal{L}_1 : W^0_\mathbb{R} \subset V \to V' \) is maximal monotone. The operator \( \mathcal{L}_2 : W^0_\mathbb{R} \subset V \to V' \) is maximal monotone if

\[ (28) \quad \langle \mathcal{R}'u, v \rangle_{V' \times V} \leq 0 \quad \text{for every } u \in V. \]

**Definition 3.11.** We say \( u \) is a solution of the problem

\[ \begin{cases}
\mathcal{R}u' + Au = f \\
P_+(0)u(0) = \varphi \\
P_-(T)u(T) = \psi,
\end{cases} \]

where \( f \in V', \varphi \in \hat{H}_+(0), \psi \in \hat{H}_-(T), \) if \( u \in W_\mathbb{R} \) and

\[ \mathcal{R}u'(t) + Au(t) = f(t) \quad \text{for a.e. } t \in [0, T] \]

and the two conditions \( P_+(0)u(0) = \varphi \) and \( P_-(T)u(T) = \psi \) are satisfied.
We start now giving an existence result for the following problem

\[
\begin{cases}
Ru' + Au = f \\
P_+(0)u(0) = 0 \\
P_-(T)u(T) = 0
\end{cases}
\]  

(29)

The idea now is to apply the previous proposition and Theorem 3.8 to the equation \(Ru' + Au = f\) adding and subtracting a term involving the derivative of \(R\) and, in this way, to get the sum of two operators satisfying assumption of Theorem 3.8. We will see with an example (see example 6) that, even if the derivative of \(R\) is not involved in the equation, the lack of regularity in time for \(R\) can cause problems, at least lack of uniqueness of solutions.

With this in mind we write

\[
Ru' + Au = \left( Ru' + \frac{1}{2} R' u \right) + \left( Au - \frac{1}{2} R' u \right).
\]

(30)

**Theorem 3.12.** Suppose \(R\) satisfies assumptions (20). Suppose true one of the following:

i) \(M = A - \frac{1}{2} R'\) is pseudomonotone, coercive, bounded.

ii) \(M = A\) is pseudomonotone, coercive, bounded and \(\langle R' u, u \rangle_{V' \times V} \leq 0\) for every \(u \in V\).

Then problems (29) admits a solution for every \(f \in V'\). If moreover \(M\) is strictly monotone the solution is unique.

**Proof -** Again for the proof we refer to the analogous in [20]. \(\Box\)

**Remark 3.13.** - In fact we obtain an existence result also for the Cauchy problem

\[
Ru' + Au \ni f, \quad u \in W_0^0.
\]

Moreover the solution is unique if the operator \(M\) of Theorem 3.12 is strictly monotone (see [27], chap. 4).

Now we want to consider the Cauchy-Dirichlet problem with non-zero “initial” data

\[
\begin{cases}
Ru' + Au = f \\
P_+(0)u(0) = \varphi \\
P_-(T)u(T) = \psi.
\end{cases}
\]

(31)

To do that we have to add some assumptions, both on \(A\) and \(R\). Assumptions on \(A\) are explicitly given in the theorem which follows. Assumptions on \(R\) are more implicit and are hidden in (32). Consider the following spaces:

\[
U_+(0) = \{ w \in U \mid [P_+(0) + P_0(0)]w \in U \} = U \cap (\tilde{H}_+(0) \oplus \tilde{H}_0(0)),
\]

\[
U_-(T) = \{ w \in U \mid [P_-(T) + P_0(T)]w \in U \} = U \cap (\tilde{H}_-(T) \oplus \tilde{H}_0(T)).
\]

(see (16) for the definition of \(\tilde{H}_-, \tilde{H}_0, \tilde{H}_+\)). Then we suppose

\[
U_+(0)\text{ dense in } \tilde{H}_+(0), \quad U_-(T)\text{ dense in } \tilde{H}_-(T).
\]

(32)

This assumptions indirectly involves the operator \(R\), as we show with an example at the end of the paper (see example 2 in the last section).

**Remark 3.14.** - We want to stress that we remove the assumption \(H_+(0) \cap H_-(T) = \{0\}\) which, on the contrary, was made in [20].

Then the following theorem holds.
Theorem 3.15. Suppose (32) holds. Define an operator $P : W_R \rightarrow V'$ by $Pu = R'u + Au$ where $R$ is defined in (17), $R \in E(C_1,C_2)$ and $A : V \rightarrow V'$ is continuous. Suppose that there exist two constants $\alpha, \beta > 0$ such that

$$
\langle Au - A\varphi - \frac{1}{2}(R'u - R'v), u - v \rangle_{V' \times V} \geq \alpha \|u - v\|^2_V,
$$

or for some $p \in (2, +\infty)$

$$
\|Au - \frac{1}{2}R'u\|_{V'} \leq \beta \|u\|_{V'}
$$

for every $u,v \in V$. Then there is a constant $c = c(\alpha, \beta, p)$ (depending only on $\alpha, \beta, p$ and proportional to $\alpha^{-\frac{1}{2}} + \alpha^{-\frac{p-2}{2}}$) such that for every $u \in W_R$

$$
\|u\|_{W_R} \leq c \left[ \|Pu\|_{V'} + \|Pu\|_{V'}^{1/(p-1)} + \|R_1^{1/2}(T)u(T)\|_{H_+(T)} + \|R_+^{1/2}(0)u(0)\|_{H_+(0)}^{2(p-1)/p} + \|R_-^{1/2}(T)u(T)\|_{H_-^2(T)} + \|R_-^{1/2}(0)u(0)\|_{H_-^2(0)}^{2/p} \right].
$$

Moreover for every $f \in V'$, $\varphi, \psi \in \tilde{H}_+(0)$, $\psi \in \tilde{H}_-(T)$ problem (31) has a unique solution.

Proof - The proof is similar to the analogous in [20], but we prefer to give it for two reasons: first because in Theorem 3.8 in [20] is required the assumption that $H_+(0) \cap H_-(T) = \{0\}$; but this in fact can be dropped, then also to show the precise estimate of $\|u\|_{W_R}$, which in [20] is different.

First we show the existence result: consider $\Phi, \Psi, \varphi, \psi \in U$ with $P_+(0)\Phi = \varphi$, $P_-(T)\Psi = \psi$ and define

$$
\eta(t) := \frac{T-t}{T} \varphi + \frac{t}{T} \psi = \varphi + \frac{t}{T} (\psi - \varphi), \quad t \in [0,T].
$$

Now solve the problem (see the proof of Theorem 3.8 in [20])

$$
\begin{cases}
Rv' + A(v + \eta) = f - R\eta' \\
P_+(0)v(0) = 0 \\
P_-(T)v(T) = 0
\end{cases}
$$

and consider $u = v + \eta$; then the function $u$ solves (31). Now to conclude one can consider $\varphi \in \tilde{H}_+(0)$, $\psi \in \tilde{H}_-(T)$, two sequences $(\varphi_n)_n \subset U_+(0)$, $(\psi_n)_n \subset U_-(T)$, $\varphi_n \rightarrow \varphi$ in $H_+(0)$, $\psi_n \rightarrow \psi$ in $H_-(T)$. This can be done thanks to assumption (32).

To come to the estimate, notice that, since $Pu = Ru' + Au$ and, for $p = 2$, $Pu = (Ru' +
\[ \frac{1}{2} \mathcal{R}'u + (Au - \frac{1}{2} \mathcal{R}'u) \] and by Proposition 3.4 we get for \( p > 2 \)
\[
\alpha \|u\|_{V'}^p \leq \left\langle \mathcal{A}u - \frac{1}{2} \mathcal{R}'u, u \right\rangle_{V' \times V} = \left\langle \mathcal{P}u, u \right\rangle_{V' \times V} - \left\langle \mathcal{R}'u, u \right\rangle_{V' \times V} = \left\langle \mathcal{P}u, u \right\rangle_{V' \times V} - \frac{1}{2} \left[ (R(T)u(T), u(T))_{H(T)} - (R(0)u(0), u(0))_{H(0)} \right] + \left\langle \frac{1}{2} \mathcal{R}'u, u \right\rangle_{V' \times V} = \left\langle \mathcal{P}u, u \right\rangle_{V' \times V} + \frac{1}{2} \left[ (R_-(T)u(T), u(T))_{H(T)} + (R_+(0)u(0), u(0))_{H(0)} \right] - \frac{1}{2} \left[ (R_-(T)u(T), u(T))_{H(T)} + (R_+(0)u(0), u(0))_{H(0)} \right] + \left\langle \frac{1}{2} \mathcal{R}'u, u \right\rangle_{V' \times V} \leq \left\langle \mathcal{P}u, u \right\rangle_{V' \times V} + \frac{1}{2} \left[ (R_-(T)u(T), u(T))_{H(T)} + (R_+(0)u(0), u(0))_{H(0)} \right].
\]
and for \( p = 2 \)
\[
\alpha \|u\|_{V'}^2 \leq \left\langle \mathcal{A}u - \frac{1}{2} \mathcal{R}'u, u \right\rangle_{V' \times V} = \left\langle \mathcal{P}u, u \right\rangle_{V' \times V} - \left\langle \mathcal{R}'u, u \right\rangle_{V' \times V} = \left\langle \mathcal{P}u, u \right\rangle_{V' \times V} - \frac{1}{2} \left[ (R(T)u(T), u(T))_{H(T)} - (R(0)u(0), u(0))_{H(0)} \right] \leq \left\langle \mathcal{P}u, u \right\rangle_{V' \times V} + \frac{1}{2} \left[ (R_-(T)u(T), u(T))_{H(T)} + (R_+(0)u(0), u(0))_{H(0)} \right].
\]
Now, if we denote by \( q \) the quantity \( p/(p - 1) \), for every \( \epsilon > 0 \) we can write
\[
(35) \quad \|\mathcal{P}u\|_{V'} \|u\|_{V} \leq \frac{\epsilon^p}{p} \|u\|^p + \frac{1}{q} \left( \frac{1}{\epsilon} \right)^{q/p} \|\mathcal{P}u\|^q
\]
by which we get the existence of a constant \( c = c(\alpha, p) \) which is proportional to \( \alpha^{-1/p} \) (e.g. choosing \( \epsilon \) such that \( \epsilon^p/p = \alpha/2 \)) such that
\[
\|u\|_{V} \leq c(\alpha, p) \left[ \|\mathcal{P}u\|_{V'}^{1/(p-1)} + (R_-(T)u(T), u(T))_{H(T)}^{1/p} + (R_+(0)u(0), u(0))_{H(0)}^{1/p} \right]
\]
for every \( p \in [2, +\infty) \). Since \( \mathcal{R}u = \mathcal{P}u - Au \) we immediately get
\[
\|\mathcal{R}u\|_{V'} \leq \|\mathcal{P}u\|_{V'} + \|Au\|_{V'} \leq \|\mathcal{P}u\|_{V'} + c \|u\|_{V'}^{p-1}
\]
where \( c \) depends on \( \beta \) and \( \|\mathcal{R}'\| \) when \( p = 2 \) and \( c \) depends only on \( \beta \) when \( p > 2 \), by which we get the thesis.

\[ \square \]

**Remark 3.16.** - If \( \mathcal{R} \geq 0 \) (\( \mathcal{R} \leq 0 \)) and \( \mathcal{A} \) is linear we also have the corresponding existence results for the problem
\[
\begin{cases}
\mathcal{R}u' + Au + \lambda \mathcal{R}u = f \\
P_+(0)u(0) = \varphi \quad (P_-(T)u(T) = \psi)
\end{cases}
\]
for every \( \lambda \in \mathbb{R} \). It is sufficient indeed to consider the change of variable
\[
v(t) = e^{\lambda t} u(t) \quad (v(t) = e^{\lambda(t-T)} u(t))
\]
to obtain
\[
\begin{cases}
\mathcal{R}v' + Av = \tilde{f} = f e^{\lambda t} \quad (f e^{\lambda(t-T)}) \\
P_+(0)u(0) = \varphi \quad (P_-(T)u(T) = \psi)
\end{cases}
\]
Moreover we will suppose that \( \mathcal{A}(37) \) \( (38) \). Now for a function \( v \) varying spaces, may be not obvious or not true. The goal of the next proposition is to show this and other results which, in the situation of \( v \) it is not obvious at all that \( Jv \equiv V \) for every \( t \), but less obvious in our case. To prove this result in particular one needs points \( Jv \equiv V \) for every \( t \) and \( C \) needs points \( \mathcal{A}(36) \).

By \( \mathcal{A}(36) \) one can define an operator \( \mathcal{A}'(t) \) \( (37) \) as done for \( R'' \) as done before. Moreover we will suppose that there is \( C_3 > 0 \) for which the operator \( \mathcal{A} \) satisfies

\[
\begin{align*}
\mathcal{A} & \quad \text{is given via a family } A : [0, T] \to \mathcal{L}(V(t), V'(t)), \\
\text{and } A(t) & \quad u(t) := A(t)u(t), \\
(36) & \quad t \to \langle A(t)u, v \rangle_{V'(t) \times V(t)} \quad \text{absolutely continuous in } [0, T], \\
& \quad \left| \frac{d}{dt} \langle A(t)u, v \rangle_{V'(t) \times V(t)} \right| \leq C_3 \| u(t) \| \| v(t) \|_{V(t)} \quad \text{for a.e. } t \in [0, T],
\end{align*}
\]

where the two last point has to hold for every \( u, v \in U \). Consistently with (15), we improperly use the notation \( A : [0, T] \to \mathcal{L}(V(t), V'(t)) \) to mean that

\[
A \quad \text{depends on } t \in [0, T] \quad \text{and } A(t) \in \mathcal{L}(V(t), V'(t)),
\]

where \( \mathcal{L}(V(t), V'(t)) \) is the set of linear and bounded operators from \( V(t) \) into \( V'(t) \).

By (36) one can define an operator \( \mathcal{A}' \) as done for \( R' \) as follows:

\[
(37) \quad \langle A'(t)u, v \rangle_{V'(t) \times V(t)} := \frac{d}{dt} \langle A(t)u, v \rangle_{V'(t) \times V(t)}; \quad u, v \in U,
\]

and for \( u \in V \) we define \( A'u(t) := A'(t)u(t) \).

Now for a function \( v \in \mathcal{V} \) we define the following functional:

\[
(38) \quad J_{t_o} v(t) := \int_{t_o}^t v(s)ds.
\]

In connection with this functional we now define the spaces

\[
\mathcal{W}_{t_o} := \{ v \in \mathcal{V} \mid Jv \in \mathcal{V} \}
\]

endowed with the norm \( \| v \|_{\mathcal{W}_{t_o}} := \| v \|_{\mathcal{V}} + \| J_{t_o} v \|_{\mathcal{V}} \). Notice that if we consider a function \( v \in \mathcal{V} \) it is not obvious at all that \( J_{t_o} v \) belongs to \( \mathcal{V} \), as the example in Remark 3.18 shows. The goal of the next proposition is to show this and other results which, in the situation of varying spaces, may be not obvious or not true.

**Lemma 3.17.** Consider a function \( v \in \mathcal{V} \). Then

a) for every \( t_o \in [0, T] \) \( J_{t_o} v \in \mathcal{V} \), i.e. \( \mathcal{W}_{t_o} = \mathcal{V} \);

b) \( \langle f, \int_0^t v(s)ds \rangle_{V'(t) \times V(t)} = \int_0^t \langle f, v(s) \rangle_{V'(s) \times V(s)}ds \quad \text{for every } f \in \bigcap_{t \in [0, T]} V'(t); \)
c) for each \([s, t] \subset [0, T]\) we have that \(\int_s^t v(\tau) d\tau \in V(\sigma)\) for every \(\sigma \in [s, t]\) and 
\[
\left\| \int_s^t v(\tau) d\tau \right\|_{V(\sigma)} \leq \int_s^t \|v(\tau)\|_{V(\tau)} d\tau \quad \text{for every } \sigma \in [s, t];
\]
d) if \(v, v' \in V\) the function \([0, T] \ni t \mapsto \|v(t)\|_{V(t)}\) belongs to \(L^\infty(0, T)\) and there is a constant \(c > 0\), depending on \(T\), such that 
\[
\|v(t)\|_{V(t)} \leq c \left[ \|v\|_V + \|v'\|_V \right] \quad \text{for every } t \in [0, T];
\]
e) if \(v, v' \in V\) and the function \([0, T] \ni t \mapsto \|v\|_{V(t)}\) is continuous for every \(u \in U\) then the function \([0, T] \ni t \mapsto \|v(t)\|_{V(t)}\) is continuous;
f) if \(v, v' \in V\) then for every \(t \in [0, T]\) \(v(t) \in \bigcap_{s \in [0, T]} V(s)\).

**Remark 3.18.** - Notice that without the existence of a space \(U\) dense in every space \(V(t)\) the previous proposition may be not true, in particular point a). Take for instance \(V(t) = H^1_0(\Omega(t))\) for some family of varying domains \(\Omega(t)\), \(t \in [0, T]\), \(\Omega(t) \subset A \subset \mathbb{R}^n\) and consider \(V\) to be the completion of \(C^1_c(\bigcup_{t \in [0, T]} \Omega(t))\) with respect to the norm 
\[
|\|u\|^2 := \int_0^T \int_{\Omega(t)} |Du(x, t)|^2 dx dt.
\]
Only if \(\Omega(t)\) is increasing in \(t\) then \(\int_0^t u(s) ds\) belongs to \(V(t)\), otherwise this is not guaranteed.

**Proof** - First we show some preliminary results for \(u \in U\), then prove the lemma. Consider the Banach spaces \(X = \bigcap_{t \in [0, T]} V(t)\) and \(Y = \bigcup_{t \in [0, T]} V(t)\) endowed with the norms 
\[
\|x\|_X := \sup_{t \in [0, T]} \|x\|_{V(t)} dt, \quad \|y\|_Y := \int_0^T \|y\|_{V(t)} dt.
\]
and the dual space \(Y' = \bigcap_{t \in [0, T]} V'(t)\) with the norm 
\[
\|f\|_{Y'} := \sup_{t \in [0, T]} \|f\|_{V'(t)}.
\]
Notice that also \(\bigcap_{t \in [0, T]} V'(t)\) is not empty, indeed \(U \subset V'(t)\) for every \(t \in [0, T]\). Consider \(f \in \bigcap_{t \in [0, T]} V'(t)\) and \(u \in U\). Clearly 
\[
u(t) \in U \subset X \subset Y.
\]
We can find a sequence \((u_n)_n\), of step functions, \(u_n : [0, T] \rightarrow U\), such that 
\[
u_n(\tau) \rightarrow u(\tau) \quad \text{in } V(\tau) \quad \text{for almost every } \tau \in [0, T],
\]
\[
\int_s^t u_n(\tau) d\tau \rightarrow \int_s^t u(\tau) d\tau \quad \text{for every } s, t \in [0, T].
\]
On this sequence, since \(\int_s^t u_n(\tau) d\tau\) is a finite sum and since \(\int_s^t u_n(\tau) d\tau\) in particular belongs to \(V(\sigma)\) for every \(\sigma \in [s, t]\), it holds 
\[
\left\langle f, \int_s^t u_n(\tau) d\tau \right\rangle_{V'(\sigma) \times V(\sigma)} = \int_s^t \left\langle f, u_n(\tau) \right\rangle_{V'(\tau) \times V(\tau)} d\tau.
\]
Notice that 
\[
\left\langle f, u_n(\tau) \right\rangle_{V'(\tau) \times V(\tau)} \rightarrow \left\langle f, u(\tau) \right\rangle_{V'(\tau) \times V(\tau)} \quad \text{for almost every } \tau \in [0, T].
\]
and moreover
\[
\int_0^T \| \langle f, u_n(\tau) \rangle_{V'(\tau) \times V(\tau)} - \langle f, u_m(\tau) \rangle_{V'(\tau) \times V(\tau)} \| d\tau \leq \int_0^T \| f \|_{V'(\tau)} \| u_m(\tau) - u_m(\tau) \|_{V(\tau)} d\tau < \varepsilon \int_0^T \| f \|_{V'(\tau)} d\tau.
\]

Then
\[
\lim_{n \to +\infty} \int_s^t \langle f, u_n(\tau) \rangle_{V'(\tau) \times V(\tau)} d\tau = \int_s^t \langle f, u(\tau) \rangle_{V'(\tau) \times V(\tau)} d\tau.
\]

This then yields the following equalities for every \( s, t \in [0, T] \)
\[
\int_s^t \langle f, u(\tau) \rangle_{V'(\tau) \times V(\tau)} d\tau = \lim_{n \to +\infty} \int_s^t \langle f, u_n(\tau) \rangle_{V'(\tau) \times V(\tau)} d\tau = \lim_{n \to +\infty} \left\langle f, \int_s^t u_n(\tau) d\tau \right\rangle_{V'(\sigma) \times V(\sigma)} = \left\langle f, \int_s^t u(\tau) d\tau \right\rangle_{V'(\sigma) \times V(\sigma)}.
\]
(39)

Notice that
\[
\int_s^t u(\tau) d\tau \in V(\sigma) \quad \text{for every } \sigma \in [s, t], \quad \text{in particular } \int_s^t u(\tau) d\tau \in Y_{s,t}
\]
where \( Y_{s,t} = \bigcup_{\sigma \in [s, t]} V(\sigma) \). Then there exists \( f \in \bigcap_{\sigma \in [s, t]} V'(\sigma) \) such that
\[
\sup_{\sigma \in [s, t]} \| f \|_{V'(\sigma)} = 1 \quad \text{and } \quad \left\| \int_s^t u(\tau) d\tau \right\|_{V(\sigma)} = \left\langle f, \int_s^t u(\tau) d\tau \right\rangle_{V'(\sigma) \times V(\sigma)}.
\]

By the previous equality we get
\[
\left\| \int_s^t u(\tau) d\tau \right\|_{V(\sigma)} = \left\langle f, \int_s^t u(\tau) d\tau \right\rangle_{V'(\sigma) \times V(\sigma)} = \int_s^t \langle f, u(\tau) \rangle_{V' \times V} d\tau = \int_s^t \| | \langle f, u(\tau) \rangle_{V' \times V(\tau)} \| d\tau \leq \int_s^t \| f \|_{V'(\tau)} \| u(\tau) \|_{V(\tau)} d\tau \leq \max_{\tau \in [s,t]} \| f \|_{V'(\tau)} \int_s^t \| u(\tau) \|_{V(\tau)} d\tau = \int_s^t \| u(\tau) \|_{V(\tau)} d\tau.
\]
(40)

a) and c) - Given \( v \in V \) we can find a sequence \((u_n)_n\) of step functions, \( u_n : [0, T] \to U \), such that
\[
u_n(\tau) \to v(\tau) \quad \text{in } V(s) \quad \text{for almost every } \tau \in [0, T],
\]
\[
\int_s^t u_n(\tau) d\tau \to \int_s^t v(\tau) d\tau \quad \text{for every } s, t \in [0, T].
\]
Moreover for each $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that for every $n, m \geq N$ such that
\[
\int_0^t \| u_n(s) - u_m(s) \|_{V(s)} \, ds < \varepsilon,
\]
that is
\[
\lim_{n \to +\infty} \int_s^t \| u_n(\tau) \|_{V(\tau)} \, d\tau = \int_s^t \| v(\tau) \|_{V(\tau)} \, d\tau.
\]
Then, applying (40) and by the semicontinuity of the norm, we get
\[
\left\| \int_s^t v(\tau) \, d\tau \right\|_{V(\sigma)} \leq \lim_{n \to +\infty} \left\| \int_s^t u_n(\tau) \, d\tau \right\|_{V(\sigma)} \leq
\leq \lim_{n \to +\infty} \int_s^t \| u_n(\tau) \|_{V(\tau)} \, d\tau = \int_s^t \| v(\tau) \|_{V(\tau)} \, d\tau.
\]
In particular $\int_s^t v(\tau) \, d\tau \in V(\sigma)$ for every $\sigma \in [s, t]$ and
\[
\| Jv \|_{V'}^p = \int_0^T \left( \int_0^t \| v(\tau) \, d\tau \right)^p dt \leq \int_0^T \left( \int_0^t \| v(\tau) \|_{V(\tau)} \, d\tau \right)^p dt \leq \int_0^T \left( T^{p/p'} \int_0^T \| v(\tau) \|_{V(\tau)} \, d\tau \right)^p dt = T^{1+\frac{p}{p'}} \| v \|_{V'}^p.
\]
\[b) \ - \text{To prove this point it is sufficient to consider (39), use it for the sequence } (u_n)_n \text{ introduced in point a) and take the limit.}
\]
\[d) \ - \text{To get point d) fix an arbitrary } \varepsilon > 0 \text{ and choose } u \in C^1([0, T]; U) \text{ such that } \| v - u \|_V + \| v' - u' \|_V < \varepsilon \text{ (such a function exists thanks to Proposition 3.2). For a generic function } w \in V \text{ such that } w' \in V' \text{ we define } \tilde{w} \text{ in } [-T, T] \text{ as } \tilde{w}(t) = w(t) \text{ for } t \in [0, T] \text{ and } \tilde{w}(t) = w(-t) \text{ for } t \in [-T, 0], \text{ consider } \phi : [-T, T] \to [0, 1] \text{ such that } \phi(t) = 1 + t/T \text{ for } t \in [-T, 0], \phi(t) = 1 \text{ for } t \in [0, T]. \text{ Then, integrating the function } (\phi \tilde{w})' \text{ between } s \text{ and } t, \text{ where } s, t \in [-T, T], \text{ we get}
\]
\[
(\phi \tilde{w})(t) - (\phi \tilde{w})(s) = \int_s^t (\phi \tilde{w})'(\tau) \, d\tau.
\]
\[\text{Notice that the integral on the right hand side makes sense in } U'. \text{ If we consider } s = -T \text{ and } t \geq 0 \text{ and since } w \in V \text{ we get that the result is in fact a function in } V(t). \text{ Then, taking } s = -T \text{ and } w = v, \text{ and using point c) we get}
\]
\[
\| v(t) \|_{V(t)} \leq c \left( \| v \|_V + \| v' \|_V \right) \quad \text{for every } t \in [0, T].
\]
\[e) \ - \text{Now to get this point fix } \varepsilon > 0 \text{ and consider } u \in C^1([0, T]; U) \text{ such that } \| v - u \|_V + \| v' - u' \|_V < \varepsilon, \text{ then take } s = -T \text{ and } w = v - u \text{ in (41); we get that}
\]
\[
\| v(t) - u(t) \|_{V(t)} \leq c \| v - u \|_V + \| v' - u' \|_V < c \varepsilon \quad \text{for every } t \in [0, T].
\]
By this estimate we first get that
\[
\| v(t) \|_{V(t)} - \| v(s) \|_{V(s)} \leq
\leq \| v(t) - u(t) \|_{V(t)} + \| u(t) \|_{V(t)} - \| u(s) \|_{V(s)} + \| u(s) - v(s) \|_{V(s)} \leq
\leq \| v(t) - u(t) \|_{V(t)} + \| u(t) \|_{V(t)} - \| u(t) \|_{V(t)} + \| u(t) - u(s) \|_U +
+ \| u(s) - v(s) \|_{V(s)} \leq
\leq 2c\varepsilon + \| u(t) \|_{V(t)} - \| u(t) \|_{V(t)} + \| u(t) - u(s) \|_U
\]
and by the continuity of \( u \) in \( U \) and the continuity of \( \| \cdot \|_{V(t)} \) we conclude.

f) - Since both \( v \) and \( v' \) belong to \( V \), by point a) and c) we get that for each \( s \in [0,T] \) the function

\[
J_s v' \in V.
\]

We derive that, thanks to point d), or every \( t \in [0,T] \)

\[
(J_s v')(t) \in V(t)
\]

and this holds for every \( s \in [0,T] \). Since

\[
(J_s v')(t) = v(t) - v(s) \quad \text{and} \quad v(t) \in V(t)
\]

we can conclude that

\[
v(s) \in V(t) \quad \text{for every } t, s \in [0,T]. \quad \square
\]

In a complete analogous way one can prove the following lemma.

**Lemma 3.19.** Suppose \( f \in V' \). Then

a) \( \int_0^t \langle f(s), v \rangle_{V'(s) \times V(s)} \, ds = \int_0^t \langle f(s), v \rangle_{V'(s) \times V(s)} \, ds \) for every \( t \in [0,T] \) and for every \( v \in \bigcap_{s \in [0,T]} V(s) \);

b) suppose \( w \in H \) and \( w' = f \); then \( w(t) \in \bigcap_{s \in [0,T]} H(t) \) and for every \( \sigma \in [s,t] \) it holds (\( k \) is defined in (10))

\[
\left\| \int_s^t f(\tau) d\tau \right\|_{H(\sigma)} \leq k \int_s^t |f(\tau)|_{V'(\tau)} \, d\tau.
\]

Lemma 3.17 (in particular points b) and c)), is immediately needed for the following result, even if the proof can be obtained following the analogous one in [23], to which we refer to.

**Theorem 3.20.** Under the assumption of Theorem 3.15 and given \( f \in V', \varphi \in \tilde{H}_+(0) \), \( \psi \in \tilde{H}_-(T) \) denote by \( u \) the solution of problem (31). Suppose moreover that \( A \) satisfies (36) and \( R, R' \in \mathcal{E}(C_1,C_2) \) and that there is a positive constant \( \gamma \) such that

\[
(A(t)w,w)_{V'(t) \times V(t)} \geq \gamma \|w\|^2_{V(t)}
\]

for every \( t \in [0,T] \) and every \( w \in U \). Assume that \( f' \), the generalized derivative of \( f \), belongs to \( V' \) and suppose there exist \( u_0 \in V(0), u_T \in V(T) \) such that \( P_+(0)u_0 = \varphi, P_-(T)u_T = \psi, f(0) - A(0)u_0 \in \text{Im } R(0) \) and \( f(T) - A(T)u_T \in \text{Im } R(T) \). Then

i) \( u' \in V; \)

ii) there is \( c > 0 \) depending (only) on \( \alpha_{1/2}, \beta, ||A||, ||A'||, ||A'|| \), such that

\[
\|u\|_{W_2} + \|u'\|_{W_2} + \sup_{t \in [0,T]} \|u(t)\|_{V(t)} \leq c \left[ \|f\|_V + \|R^{1/2}_+(0)u_0\|_{H_+(0)} + \|R^{1/2}_-(0)u_T\|_{H_-(T)} + \|f'\|_V + \|R^{1/2}_+(0)[P_+(0)(f(0) - A(0)u_0)]\|_{H_+(0)} + \|R^{1/2}_-(0)[P_-(T)(f(T) - A(T)u_T)]\|_{H_-(T)} \right]
\]

**Remark 3.21.** Following the proof one realizes that in fact it is sufficient to require

\( A \) coercive and bounded \quad and \quad

\[
(A(t)u,u)_{V'(t) \times V(t)} \geq \gamma \|u\|_{V(t)}^2 \quad \text{for every } u \in U \quad \text{only for } t = 0 \text{ and } t = T. \]
Remark 3.22. - If \( V(t) \equiv V \) for every \( t \), in which case one can consider \( U = V \), points \( i \) of Theorem 3.20 and points \( b \) and \( c \) of Lemma 3.17 reduce to say that \( u \in H^1(0,T;V) \subset C^0([0,T];V) \).

Remark 3.23. - If \( R \) is defined by a function \( r \), i.e. \( (Ru(t))(x) = r(x,t)u(x,t) \), to require that \( R, R' \in \mathcal{E} \) implies that \( r \) admits a weak derivative with respect to time, so \( (R'(t)u(t))(x) = \partial_t r(x,t)u(x,t) \). In particular assumptions of Theorem 3.20 are satisfied if \( A(t) \) is strictly monotone and \( \partial_t r \leq 0 \). See example 5 in the last section for a practical example.

4. The existence result for second order equations

In [21] an existence result for the abstract equation

\[
(Ru')' + Au' + Bu = f
\]

is presented, where \( \mathcal{R}, \mathcal{A}, \mathcal{B} \) are operators defined in

\[ \mathcal{V} \quad \text{or in one of its subspaces}, \]

where \( \mathcal{V} \) is the reflexive Banach space defined in (12), but in this section with \( p = 2 \).

The aim of this section is twofold: use the result of the previous section to give an analogous result for an abstract equation like

\[
\mathcal{R}u'' + \mathcal{A}u' + \mathcal{B}u = f
\]

with suitable boundary data, then to extend the result contained in [21] for the equation

\[
(Ru')' + Au' + Bu = f
\]

to the situation presented in Section 2, i.e. to a situation in which some moving spaces are admitted, like that considered in (9). Clearly when \( \mathcal{R} \) is an operator independent of time the two results coincide.

About \( \mathcal{R} \) we will require, as done in the previous section, one derivative (even if the equation is of second order) in the weak sense given in Definition 2.1.

All the Banach spaces we are going to consider are those defined in Section 2. Besides about the spaces \( V'(t) \)'s we will suppose that

(42) the function \([0,T] \ni t \mapsto \|u\|_{V'(t)}\) is continuous for every \( u \in U \).

We need to consider three operators, \( \mathcal{R}, \mathcal{A}, \mathcal{B} \): first fix two positive constant \( C_1 \) and \( C_2 \) and (the class \( \mathcal{E}(C_1,C_2) \) is defined in Definition 2.1)

(43) \( R \in \mathcal{E}(C_1,C_2) \).

Then we can define \( \mathcal{R} \) and \( \mathcal{R}' \) as done in (17) and (18). About the operator \( \mathcal{A} \) we do not require, for the moment, any assumption, only that

(44) \( \mathcal{A} : \mathcal{V} \longrightarrow \mathcal{V}' \).

About \( \mathcal{B} \) we make the following assumptions: that there is a family of operators (with the notations already used in the previous section), a positive constant \( C_3 \) and \( t_o \in [0,T] \) such
that
\[ B : [0, T] \rightarrow \mathcal{L}(V(t), V'(t)), \]
\[ B(t) \text{ linear, monotone and symmetric for a.e. } t \in [0, T], \]
\[ \max_{t \in [0, T]} \| B(t) \|_{\mathcal{L}(V(t), V'(t))} = C_3, \]
\[ t \mapsto \langle B(t)u, v \rangle_{V'(t) \times V(t)} \text{ is absolutely continuous in } [0, T] \text{ for every } u, v \in U, \]
\[ \frac{d}{dt} \langle B(t)u, v \rangle_{V'(t) \times V(t)} \leq 0 \text{ for every } u, v \in U \text{ and for } t \in [t_0, T]\]
\[ \frac{d}{dt} \langle B(t)u, v \rangle_{V'(t) \times V(t)} \geq 0 \text{ for every } u, v \in U \text{ and for } t \in [0, t_0]. \]

In this way we define an abstract operator \( B \) as follows
\[ B : V \rightarrow V', \quad Bu(t) = B(t)u(t) \quad 0 \leq t \leq T, \]
this turns out to be linear, monotone and symmetric. We moreover can define an operator \( B' \) as defined in (37). The assumption about the derivative of the operator \( B \) is needed because of the following result (used in the following with \( Q = B \), which can be proved adapting the analogous result contained in [21] and using Lemma 3.17, point a).

**Proposition 4.1.** Consider \( Q : [0, T] \rightarrow \mathcal{L}(V(t), V'(t)) \) satisfying (45) and consider the two operators
\[ Q : V \rightarrow V', \quad Qv(t) = Q(t)v(t), \]
\[ J_{t_o} : V \rightarrow V, \quad (J_{t_o} v)(t) := \int_{t_o}^{t} v(\sigma)d\sigma \text{ for some } t_o \in [0, T]. \]

If \( Q'(t) \) is monotone for a.e. \( t \in [0, t_o] \) and \(-Q'(t) \) is monotone for a.e. \( t \in [t_o, T] \) then the operator
\[ Q \circ J_{t_o} : V \rightarrow V', \quad (Q \circ J_{t_o}) v(t) = QJ_{t_o}v(t) = Q(t)\int_{t_o}^{t} v(\sigma)d\sigma \]
is monotone. If \( Q \) is bounded, \( QJ_{t_o} \) is bounded by \( T\|Q\|_{\mathcal{L}(V,V')} \).

We want to stress that the proof is based essentially in the following inequality which can be obtained following the proof of Proposition 2.3 in [21]:
\[ \int_{t_1}^{t_2} \langle QJ_{t_o}v(\sigma), v(\sigma) \rangle_{V'(\sigma) \times V(\sigma)} d\sigma \geq \frac{1}{2} \int_{t_1}^{t_2} \langle Q(\sigma) \int_{t_o}^{\sigma} v(s)ds, \int_{t_o}^{\sigma} v(s)ds \rangle_{V'(\sigma) \times V(\sigma)} d\sigma \]
for every \( t_1, t_2 \in [0, T], t_1 < t_2, v \in V. \)

### 4.1. The first result

Now we want to study the problem
\[ \begin{cases} \mathcal{R}u'' + Au' + Bu = f \\ P_+(0)u'(0) = \varphi \\ P_-(T)u'(T) = \psi \\ u(t_o) = \eta \end{cases} \]
(48)
with \( f \in V', \varphi \in \tilde{H}_+(0), \psi \in \tilde{H}_-(T), \eta \in V(t_o) \) and \( P_+(0) \) and \( P_-(T) \) are the orthogonal projections defined in (26), where \( u \) will be a solution of (48), according to Definition 3.11,
if (the space $\mathcal{W}_R$ is defined in (22))
\[
\begin{align*}
u \in \mathcal{Z}_R := \{u \in \mathcal{V} \mid u' \in \mathcal{W}_R\},
\mathcal{R}u''(t) + Au'(t) + Bu(t) = f(t) \quad \text{for a.e. } t \in [0, T],
P_+(0)u'(0) = \varphi \quad \text{in } \tilde{H}_+(0), \quad P_-(T)u'(T) = \psi \quad \text{in } \tilde{H}_-(T), \quad u(t_o) = \eta \in V(t_o).
\end{align*}
\]

The boundary conditions with respect to the variable $t$, i.e. the initial-final conditions, are given as follows: we give an initial condition for $u'$ at time zero where $\mathcal{R}$ is positive (i.e. the datum $\varphi$) while a final condition at time $T$ where $\mathcal{R}$ is negative (i.e. the datum $\psi$). Where $\mathcal{R}$ is null, no conditions for $u'$ are given.

About $u$ we impose a datum at time $t_o \in [0, T]$ (the datum $\eta$).

By Proposition 3.4 and, thanks to assumption (42), by Theorem 3.20, one has that
\[
\begin{align*}
u, v \in \mathcal{Z}_R \quad \implies \quad & t \mapsto (R(t)u(t), v(t))_{H(t)} \quad \text{is continuous} \\
& t \mapsto \|v(t)\|_{V(t)} \quad \text{is continuous}
\end{align*}
\]
so the data $\varphi$, $\psi$ and $\eta$ makes perfectly sense.

If $\mathcal{R} \equiv 0$ the initial-final conditions about $u'$ make no sense and the problem simply becomes
\[
\begin{align*}
\begin{cases}
Au' + Bu = f \\
u(t_o) = \eta
\end{cases}
\end{align*}
\]

The initial/final conditions we require about $u'$ and $u$ are easily understood by explaining how we prove the existence result: indeed the idea to solve problem (48) is to consider an operator $J_{t_o}$ defined in Proposition 4.1 for some arbitrary $t_o \in [0, T]$ and the change of variable $v = u'$ in (48) and then solve, once set $g = f - B\eta$, the first order problem
\[
\begin{align*}
\begin{cases}
\mathcal{R}u' + Av + BJ_{t_o}v = g \\
P_+(0)v(0) = \varphi \\
P_-(T)v(T) = \psi.
\end{cases}
\end{align*}
\]

Then it is natural to impose for the equation some data as done in the previous section for problem (31).

**Definition 4.2.** We say that $u \in \mathcal{Z}_R$ is a solution of problem (48) with $f \in \mathcal{V}'$, $\varphi \in \tilde{H}_+(0)$, $\psi \in \tilde{H}_-(T)$, $\eta \in V(t_o)$, if
\[
\begin{align*}
\mathcal{R}u''(t) + Au'(t) + Bu(t) = f(t) \quad \text{in } \mathcal{V}'(t) \quad \text{for a.e. } t \in [0, T],
P_+(0)u'(0) = \varphi, \quad P_-(T)u'(T) = \psi, \quad u(t_o) = \eta.
\end{align*}
\]

If $\mathcal{R} \equiv 0$ the solution of (49) will be a function in the space $\mathcal{Z} := \{u \in \mathcal{V} \mid u' \in \mathcal{V}\}$.

For the definition of solution of problem (50) we refer to Definition 3.11.

Now to solve problem (50) we write the right hand term in the equation (50) as
\[
\mathcal{R}u' + Av + BJ_{t_o}v = \left(\mathcal{R}u' + \frac{1}{2}\mathcal{R}'v\right) + \left(-\frac{1}{2}\mathcal{R}'v + Av + BJ_{t_o}v\right) = \mathcal{L}v + \mathcal{M}v
\]
and use Theorem 3.8. Since by Proposition 3.10 the operator ($\mathcal{W}_R^0$ is defined in (27))
\[
\mathcal{L} : \mathcal{W}_R^0 \rightarrow \mathcal{V}'
\]
is maximal monotone we can apply Theorem 3.8 if $-\frac{1}{2}\mathcal{R}' + A + BJ_{t_o}$ is pseudomonotone, coercive, bounded. Indeed the following result is an immediate consequence of Theorem 3.12 and Theorem 3.15.
Corollary 4.3. Fix $t_o \in [0, T]$ and suppose the existence of three positive constants $C_1, C_2, C_3$ such that $R \in \mathcal{E}(C_1, C_2)$ and $B$ satisfies (45). Then

i) if $-\frac{1}{2} R' + A + BJ_{t_o}$ is pseudomonotone, coercive, bounded then for $\varphi = 0$ and $\psi = 0$ problem (50) has a solution for every $g \in \mathcal{V}'$, if moreover $-\frac{1}{2} R' + A + BJ_{t_o}$ is strictly monotone the solution is unique.

If there are two positive constants $\alpha, \beta$ such that

$$\mathcal{A} \text{ is continuous and } \|Au\|_{\mathcal{V}'} \leq \beta \|u\|_{\mathcal{V}}$$

for every $u, v \in \mathcal{V}$ then

ii) there is a constant $c > 0$ depending only on $\alpha, \beta$ and $C_3$ (and proportional to $\alpha^{-1/2}$) such that for every $u \in W_R$

$$\|u\|_{W_R} \leq c \left[ \|\tilde{P}u\|_{\mathcal{V}'} + \|R^{1/2}(T)u(T)\|_{H_{-}(T)} + \|R_{+}^{1/2}(0)u(0)\|_{H_{+}(0)} \right]$$

where, for $v \in W_R$, $\tilde{P}v := Rv' + Av + BJ_{t_o}v$;

iii) finally, if moreover (32) holds, then for every $g \in \mathcal{V}'$, $\varphi \in \tilde{H}_{+}(0)$, $\psi \in \tilde{H}_{-}(T)$ problem (50) has a unique solution.

Proof - The proof of point i) is an immediate consequence of Theorem 3.12. Points ii) and iii) follow by Theorem 3.15, but we have to make a little remark regarding the proof of point ii). To apply Theorem 3.15 we need, as in (33),

$$-\frac{1}{2} R'v + A + BJ_{t_o} \text{ continuous, }$$

$$\left\| -\frac{1}{2} R' + A + BJ_{t_o} \right\|_{\mathcal{V}'} \leq \tilde{\beta} \|u\|_{\mathcal{V}},$$

$$\left\langle \left( -\frac{1}{2} R' + A + BJ_{t_o} \right)(u - v), u - v \right\rangle_{\mathcal{V}' \times \mathcal{V}} \geq \tilde{\alpha} \|u - v\|_{\mathcal{V}}^2$$

for some positive $\tilde{\alpha}$ and $\tilde{\beta}$. The first and the second conditions immediately follow since $R'$ and $B$ are linear and bounded. Finally, the third condition follows, in fact with $\tilde{\alpha} = \alpha$, by Proposition 4.1, since

$$BJ_{t_o} \text{ is linear and } \langle BJ_{t_o}v, v \rangle_{\mathcal{V}' \times \mathcal{V}} \geq 0 \text{ for every } v \in \mathcal{V}. \quad \Box$$

Now we want to solve problem (48) for some fixed $t_o \in [0, T]$ and for $f \in \mathcal{V}'$, $\varphi \in \tilde{H}_{+}(0)$, $\psi \in \tilde{H}_{-}(T)$, $\eta \in V(t_o)$. Compared with the result in [21] also here, as done in the previous section, we remove the assumption mentioned in Remark 3.14.

Theorem 4.4. Fix $t_o \in [0, T]$ and suppose (32) holds, suppose the existence of five positive constants $C_1, C_2, C_3, \alpha, \beta$ such that $S \in \mathcal{E}(C_1, C_2)$, $A$ satisfies (45) and

$$\mathcal{A} \text{ is continuous and } \|Au\|_{\mathcal{V}'} \leq \beta \|u\|_{\mathcal{V}},$$

$$\left\langle Au - Av - \frac{1}{2} R'(u - v), u - v \right\rangle_{\mathcal{V}' \times \mathcal{V}} \geq \alpha \|u - v\|_{\mathcal{V}}^2$$

for every $u, v \in \mathcal{V}$. Then for every $f \in \mathcal{V}'$, $\varphi \in \tilde{H}_{+}(0)$, $\psi \in \tilde{H}_{-}(T)$, $\eta \in V(t_o)$ problem (48) admits a unique solution $u \in Z_R$ and there is a positive constant $c$, depending only on
analogous to that given at the end of the previous section. Also here we confine ourselves to the case that $A$ is linear and satisfies (36) and moreover we suppose

\[ (51) \quad \text{the function } [0, T] \ni t \mapsto \|u(t)\|_{V(t)} \text{ is continuous for every } u \in U. \]

The following result is in fact a corollary of Theorem 4.20 and of Lemma 3.17.

**Theorem 4.5.** Suppose the assumptions of Theorem 4.4. Suppose moreover that (51) holds, $B$ and $A$ satisfy (36), $R, R' \in E(C_1, C_2)$ and that there are two positive constant $\gamma, \Gamma$ such that

\[ \gamma \|w\|^2_{V(t)} \leq \langle (A(t) + BJ_{\xi}(t))w, w \rangle_{V'(t) \times V(t)} \leq \Gamma \|w\|^2_{V(t)} \]

for every $t \in [0, T]$ and every $w \in U$. For each $f \in \mathcal{V}$, $\varphi \in \hat{H}_+(0)$, $\psi \in \hat{H}_-(T)$ and $\eta \in U$ denote by $u$ the solution of problem (48) and assume that $f'$ belongs to $\mathcal{Y}$ and suppose there exist $v_0 \in V(0), v_T \in V(T)$ such that $P_+(0)v_0 = \varphi$, $P_-(T)v_T = \psi$, $f(0) - A(0)v_0 - B(0)u(0) \in \text{Im } R(0)$ and $f(T) - A(T)v_T - B(T)u(T) \in \text{Im } R(T)$. Then

i) $u'' \in V$;

ii) the function $[0, T] \ni t \mapsto \|u'(t)\|_{V(t)}$ is continuous and there is a constant $c > 0$, depending on $T$, such that

\[ \|u'(t)\|_{V(t)} \leq c [\|u\|_V + \|u''\|_V] \quad \text{for every } t \in [0, T]; \]
iii) there is $c > 0$ depending (only) on $\alpha^{-1/2}$, $\beta$, $\|A + BJ_\alpha\|$, $\|R\|$, $\|(A + BJ_\alpha)\|$ such that

$$
\|R u'' + u''\| \leq c \left( \|f\| + \|R_{1/2}(0)\|_{H_{1}(0)} + \|R_{1/2}(T)\|_{H_{-}(T)} + \|f'\| + \|R_{-1/2}(0)\|_{H_{1}(0)} + \|R_{-1/2}(T)\|_{H_{-}(T)} \right).
$$

4.2. The second result. Here we confine ourselves to state the main result, since one can follow the proof in [21] and adapt it to the situation with moving spaces (9) as done before in the previous section and in the previous part of the present section.

Here we want to give an existence (and uniqueness) result for the following problem

$$
\begin{aligned}
\begin{cases}
(Ru')' + Au' + Bu = f \\
P_+(0)u'(0) = \varphi \\
P_-(T)u(T) = \psi \\
u(t_o) = \eta
\end{cases}
\end{aligned}
$$

(52)

with $f \in V'$, $\varphi \in \bar{H}_+, \psi \in \bar{H}_-$, $\eta \in V(t_o)$. This problem has been already treated in [21], the only difference is that here we want to state the main result contained in [21] in the more general situation of spaces possibly varying in time, as set in (9). The solution of this problem will live in the space $Z_R$ since, even if in this case the space $W_R$ should be defined as $\{u \in V \mid (Ru)' \in V'\}$, it coincides with the space $W_R$ defined in (22), as observed in Remark 3.1. The idea is, as done before for the first result, to solve a first order problem, precisely

$$
\begin{aligned}
\begin{cases}
(Rv)' + Av + BJ_\alpha v = g - B\eta \\
P_+(0)v(0) = \varphi \\
P_-(T)v(T) = \psi
\end{cases}
\end{aligned}
$$

(53)

take the solution $v$ and then consider $u(t) := \eta + \int_{t_o}^t v(s)ds$, which will be the solution of (52).

Definition 4.6. We say that $u \in Z_R$ is a solution of problem (52) with $f \in V'$, $\varphi \in \bar{H}_+, \psi \in \bar{H}_-$, $\eta \in V(t_o)$, if

$$
\begin{aligned}
&(Ru')'(t) + Au'(t) + Bu(t) = f(t) \quad \text{in } V'(t) \quad \text{for a.e. } t \in [0, T], \\
P_+(0)u'(0) = \varphi, \quad P_-(T)u(T) = \psi, \quad u(t_o) = \eta.
\end{aligned}
$$

If $R \equiv 0$ the solution of (49) will be a function in the space $Z := \{u \in V \mid u' \in V\}$.

Theorem 4.7. Fix $t_o \in [0, T]$ and suppose (32) holds, suppose the existence of five positive constants $C_1, C_2, C_3, \alpha, \beta$ such that $S \in E(C_1, C_2)$, $B$ satisfies (45) and $A$ is continuous and $\|Au\| \leq \beta \|u\|_V$

$$
\langle Au - Av + \frac{1}{2} R'(u - v), u - v \rangle_{V' \times V} \geq \alpha \|u - v\|^2_V,
$$

for every $u, v \in V$. Then for every $f \in V'$, $\varphi \in \bar{H}_+(0)$, $\psi \in \bar{H}_-(T)$, $\eta \in V(t_o)$ problem (48) admits a unique solution $u \in Z_R$ and there is a positive constant $c$, depending only on
Suppose the assumptions of Theorem 4.8. Suppose moreover that (51) holds, A and B satisfy (36), \( R, R' \in \mathcal{E}(C_1, C_2) \) and that there are two positive constant \( \gamma, \Gamma \) such that
\[
\gamma \| w \|_{V(t)}^2 \leq \langle (A(t) + BJ_{t_u}(t))w, w \rangle_{V'(t) \times V(t)} \leq \Gamma \| w \|_{V(t)}^2
\]
for every \( t \in [0, T] \) and every \( w \in U \). For each \( f \in V' \), \( \varphi \in \tilde{H}_+(0) \), \( \psi \in \tilde{H}_-(T) \) denote by \( u \) the solution of problem (52) and assume that \( f' \) belongs to \( V' \) and suppose there exist \( v_0 \in V(0), v_T \in V(T) \) such that \( P_+(0)v_0 = \varphi, P_-(T)v_T = \psi, f(0) - R'(0)v_0 - A(0)v_0 - B(0)u(0) \in \text{Im } R(0) \) and \( f(T) - R'(T)v_T - A(T)v_T - B(T)u(T) \in \text{Im } R(T) \). Then
\( i) \) \( u'' \in V; \)
\( ii) \) the function \( [0, T] \ni t \mapsto \| u'(t) \|_{V(t)} \) is continuous and there is a constant \( c > 0 \), depending on \( T \), such that
\[
\| u'(t) \|_{V(t)} \leq c \left[ \| u' \|_V + \| u'' \|_V \right] \quad \text{for every } t \in [0, T];
\]
\( iii) \) there is \( c > 0 \) depending (only) on \( \alpha^{-1/2}, \beta, \| A + BJ_{t_u} \|, \| R' \|, \| (A + BJ_{t_u})' \| \), such that
\[
\| Ru'' \|_V + \| u'' \|_V + \sup_{t \in [0, T]} \| u'(t) \|_{V(t)} + \sup_{t \in [0, T]} \| u(t) \|_{V(t)} \leq \frac{c}{\beta} \left[ \| f' \|_V + \| R_{+1/2}^{-1}(0)v_0 \|_{H_+(0)} + \| R_{-1/2}^{1/2}(T)v_T \|_{H_-(T)} + \| f' \|_V + \| R_{+1/2}^{-1}(0)(f(0) - R'(0)v_0 - A(0)v_0 - B(0)u(0)) \|_{H_+(0)} + \| R_{-1/2}^{1/2}(T)(f(T) - R'(T)v_T - A(T)v_T - B(T)u(T)) \|_{H_-(T)} \right].
\]

5. The existence result for two generalized Tricomi equations

In this section we want to give some existence results for some generalized Tricomi equation using the results of the previous section. We recall that Tricomi equation is
\[ xu_{tt} - u_{xx} = 0 \]
where \( u = u(x, t) \), and then the equation is of hyperbolic type in the half-plane \( x > 0 \) and is of elliptic type in the half-plane \( x < 0 \).

Our goal is to give existence results for equations like
\[ Ru'' + Bu = f \quad \text{and} \quad (Ru')' + Bu = f \]
with \( R \) and \( B \) suitable operators. Consider (the spaces are defined in Section 2)
\[
f, f' \in V', \varphi \in \tilde{H}_+(0), \psi \in \tilde{H}_-(T), \eta \in \bigcap_{t \in [0, T]} V(t)
\]
(54)
and the two problems

\begin{equation}
\begin{aligned}
\mathcal{R}u'' + Bu &= f \\
P_+(0)u'(0) &= \varphi \\
P_-(T)u'(T) &= \psi \\
\left( (P_+(0) + P_-(0))u(0) = (P_+(0) + P_-(0))\eta, \right.
\end{aligned}
\end{equation}

\begin{equation}
\begin{aligned}
(\mathcal{R}u')' + Bu &= f \\
P_+(0)u'(0) &= \varphi \\
P_-(T)u'(T) &= \psi \\
\left( (P_+(0) + P_-(0))u(0) = (P_+(0) + P_-(0))\eta. \right.
\end{aligned}
\end{equation}

Notice that we do not simply require \( f \in \mathcal{V}' \), but also that

\begin{equation}
\text{the datum } f \in \mathcal{V}' \text{ admits a derivative and } f' \in \mathcal{V}'.
\end{equation}

This is needed in the proof we present below. About the operators we will need that

\begin{itemize}
\item \( R \) satisfies (43),
\item \( B \) satisfies (45) and: there are \( \alpha, \beta, C > 0 \) such that
\end{itemize}

\begin{equation}
\begin{aligned}
\alpha \| u \|^2_{V(t)} &\leq (B(t)u, u)_{V'(t) \times V(t)}, \\
(B(t)u, v)_{V'(t) \times V(t)} &\leq \beta \| u \|_{V(t)} \| v \|_{V(t)}, \\
\frac{d}{dt}(B(t)u, v)_{V'(t) \times V(t)} &\leq C \| u \|_{V(t)} \| v \|_{V(t)}
\end{aligned}
\end{equation}

for a.e. \( t \in [0, T] \) and every \( u, v \in U \).

\begin{itemize}
\item in (45) we consider \( t_o = 0 \).
\end{itemize}

As usual \( \mathcal{R} \) and \( \mathcal{R}' \) are defined as done in (17) and (18), \( \mathcal{B} \) as done in (46) and \( \mathcal{B}' \) as done for \( \mathcal{A}' \) in (37).

\subsection*{5.1. The equation \( \mathcal{R}u'' + Bu = f \)}

For problem (55) we suppose

\begin{equation}
\mathcal{R}' \leq 0
\end{equation}

and, coherently with (45) and the fact that \( t_o = 0 \),

\begin{equation}
\mathcal{B}' \leq 0.
\end{equation}

Moreover, since we lean on the results of the previous sections, we will also need (32).

Finally we make the further assumption

\begin{equation}
R' \in \mathcal{E}(C_1, C_2),
\end{equation}

then, beyond to \( \mathcal{R} \) and \( \mathcal{R}' \) we can define and consider \( \mathcal{R}'' : \mathcal{V} \to \mathcal{V}' \) and, most of all,

\( \mathcal{R}' : \mathcal{H} \to \mathcal{H} \).

In this case the solution will belong to the space

\[ \mathcal{V}_R := \{ u \in \mathcal{V} \mid |R|^{1/2}u' \in \mathcal{H} \text{ and } \mathcal{R}u'' \in \mathcal{V}' \}. \]
5.2. The equation $(Ru')'+Bu = f$. The differences between this case and the previous are, first of all, that we will assume

\begin{equation}
R' \geq 0 \quad \text{and} \quad B' \leq 0
\end{equation}

instead of (59) and moreover that we will not need $R' : \mathcal{H} \to \mathcal{H}$ and therefore we drop assumption (61).

Also in this case we will also need (32). In this case the solution will belong to the space

$$\mathcal{X}_R := \{ u \in \mathcal{V} \mid |R|^1/2 u' \in \mathcal{H} \text{ and } (Ru')' \in \mathcal{V}' \}.$$  

5.3. Statements.

**Theorem 5.1.** For every $f, \varphi, \psi, \eta$ as in (54) and under assumptions (32), (58), (59), (60) and (61) problem (55) admits a unique solution in the space $\mathcal{Y}_R$.

**Theorem 5.2.** For every $f, \varphi, \psi, \eta$ as in (54) and under assumptions (32), (58) and (62) problem (56) admits a unique solution in the space $\mathcal{X}_R$.

5.4. Proofs. In this subsection we present the proofs of the two theorems just stated above. The computations are very similar, so we confine to prove Theorem 5.1, being the other proof very similar and actually equal in many parts. To prove the results we consider the family of second order problems $Ru'' + \epsilon Bu' + Bu = f$ with suitable boundary data and take the limit when $\epsilon$ goes to zero. The main difference between the two problems we are going to consider is due to some difficulty when taking the limit to get the Tricomi equations: the different and additional assumption needed to prove this passage for the first problem with respect to the second is well explained in point 5 below.

For the same reasons, i.e. the same problems, the spaces $\mathcal{Y}_R$ and $\mathcal{X}_R$ could be different, because, a priori, we do not know anything about $R'u'$.

1. A family of approximating problems - The idea is to consider a second order problem, like those considered in the previous section, and choose $B = \epsilon B$ where $\epsilon$ is a positive parameter which will be sent to zero. Then for $\epsilon > 0$ we consider the family of problems (remember that $t_o = 0$)

\begin{equation}
\begin{cases}
Ru'' + \epsilon Bu' + Bu = f \\
P_+(0)u'(0) = \varphi \\
P_-(T)u'(T) = \psi \\
u(0) = \eta
\end{cases}
\end{equation}

for $f \in \mathcal{V}'$, $\varphi \in \tilde{H}_+(0)$, $\psi \in \tilde{H}_-(T)$, $\eta \in \bigcap_{t \in [0,T]} \mathcal{V}(t)$ and denote by $u_\epsilon$ the solution. Then try to get a limit $u$, choosing if necessary a sequence from $(u_\epsilon)_{\epsilon > 0}$, of $u_\epsilon$ and see that $u$ satisfies (55).

Notice that the estimate in Theorem 4.4 does not help to have some boundedness of the solution, since the constant on the right hand side is proportional to $\epsilon^{-1/2}$ since $R' \leq 0$ and $B = \epsilon B$ (see Theorem 4.4), so we have

\begin{equation}
\|Ru''\| + \|u'\| + \sup_{t \in [0,T]} \|u_\epsilon(t)\| \leq \frac{c}{\sqrt{\epsilon}} (\|f\| + \|R^{1/2} \psi\|_{H_-(T)} + \|R^{1/2} \varphi\|_{H_+(0)})
\end{equation}

with $c$ depending on $T$ and $\max_{t \in [0,T]} \|B(t)\|_{C(V(t),V'(t))}$.

Another attempt can be done multiplying the equation by $2u'_\epsilon$, following what done in [13],
chap. 3, section 8.5, but, as we will see, also this will be not sufficient. Anyway this procedure gives one of the two steps needed to get an estimate on \( u_\epsilon \), (71) and (72).

2 - Modifying the initial datum \( \eta \) - Indeed we cannot choose every possible \( \eta \). Since passing from problems (63) to (55) we lost some boundary conditions, and precisely \( P_0(0)\eta \) does not appear in the limit problem (55), we can modify the information which will be lost without modifying the limit problem. This will allow to get uniqueness of the solution (see point 7). First of all we state the following lemma, needed only because we are concerned with moving spaces.

**Lemma 5.3.** Because of the assumption made about \( f \) and \( B \) we have the following facts:

a) since \( f, f' \in V' \) then for every \( t \in [0, T] \) \( f(t) \in \bigcap_{s \in [0, T]} V'(s) \);

b) since \( B \) satisfies (45) then for every \( t \in [0, T] \) we have that \( B(t) \in \mathcal{L}(V(s), V'(s)) \) for every \( s \in [0, T] \).

**Proof** - The proofs can be obtain adapting the analogous of point f) in Lemma 3.17, using the Banach spaces \( V'(t)'s \) in point a) and \( \mathcal{L}(V(t), V'(t))'s \) in point b) instead of \( V(t) \).

Now, thanks to the previous lemma, we can consider the following problems (a family of problems depending on \( t \)). Once defined the space \( V_0(t) := V(t) \cap \text{Ker} \ R(t) \), for every fixed \( t \in [0, T] \) we solve the problem

\[
(65) \quad \begin{cases} 
\langle B(0)w, \phi \rangle_{V'(t) \times V(t)} = \langle f(0) - B(0)\eta, \phi \rangle_{V'(t) \times V(t)} & \text{for every} \ \phi \in V_0(t), \\
\ w \in V_0(0). 
\end{cases}
\]

We will explain better the meaning of this problem in the last section (see a comment in the example 15). Notice that the function \( w \) depends on \( t \), so we denote it by \( w(t) \), and belongs to \( V_0(t) \). Since problem (65) has a unique solution and since (65) holds in particular for every \( \phi \in U \cap \text{Ker} \ R(t) \), which is dense in \( V_0(t) \), we infer that \( w(t) \) is the same for every \( t \in [0, T] \). So denote by \( w \) the solution of problem (65), which is the same for every \( t \); then we denote by \( \tilde{w} \) the function

\[
\tilde{w} := \begin{cases} 
0 & \text{in} \ H_+(0) \oplus H_-(0), \\
\ w & \text{in} \ \text{Ker} \ R(0). 
\end{cases}
\]

Finally consider the function \( \tilde{\eta} \) defined by

\[
(66) \quad \tilde{\eta} = \eta + \tilde{w} = \begin{cases} 
\eta & \text{in} \ H_+(0) \oplus H_-(0), \\
\eta + w & \text{in} \ \text{Ker} \ R(0). 
\end{cases}
\]

Notice that \( \tilde{\eta} \in \bigcap_{t \in [0, T]} V(t) \) since \( w \in \bigcap_{t \in [0, T]} V_0(t) \). Then we will consider, instead of (63), the following family of problems:

\[
(67) \quad \begin{cases} 
\mathcal{R}u'' + \epsilon \mathcal{B}u' + \mathcal{B}u = f \\
P_+(0)u'(0) = \varphi \\
P_-(T)u'(T) = \psi \\
\ u(0) = \tilde{\eta} 
\end{cases}
\]

3 - Boundedness for the solutions \( u_\epsilon \) - Now we multiply the equation in (67) by \( 2u'_\epsilon \) and integrate between 0 and \( t \); we will derive (71). Notice that if \( S \) were positive (and so invertible) this would be sufficient to conclude. On the contrary, in our situation this estimate is not sufficient. We will couple this estimate with (72) and, since \( f \) is differentiable, get (76).
Then we get (to lighten the notation we sometimes omit to write \( H(t) \) as subscript in the scalar product of \( H(t) \) and \( V(t) \) as subscript in the duality product between \( V'(t) \) and \( V(t) \))

\[
\langle Ru'_c(t), u'_c(t) \rangle + \langle Bu_c(t), u_c(t) \rangle - \int_0^t \left[ \langle R'u'_c(s), u'_c(s) \rangle + \langle B'u_c(s), u_c(s) \rangle \right] ds + 2\epsilon \int_0^t \langle Bu'_c(s), u'_c(s) \rangle ds = 2 \int_0^t \langle f(s), u'_c(s) \rangle ds + \langle Ru'_c(0), u'_c(0) \rangle + \langle Bu_c(0), u_c(0) \rangle.
\]

(68)

Since \( R' \leq 0 \) and \( B' \leq 0 \) we get

\[
\langle Ru'_c(t), u'_c(t) \rangle + \langle Bu_c(t), u_c(t) \rangle \leq 2 \int_0^t \langle f(s), u'_c(s) \rangle ds + \langle Ru'_c(0), u'_c(0) \rangle + \langle Bu_c(0), u_c(0) \rangle
\]

(69)

and in particular

\[
\langle Ru'_c(t), u'_c(t) \rangle + \langle Bu_c(t), u_c(t) \rangle \leq 2 \int_0^t \langle f(s), u'_c(s) \rangle ds + \langle Ru'_c(0), u'_c(0) \rangle + \langle Bu_c(0), u_c(0) \rangle
\]

(70)

It is clear that this procedure cannot lead to an estimate, since \( R \) is not necessarily positive and neither non-negative. Indeed if \( R \) were positive we would have a bound both on \( \|u_c\|^2 \) and on \( \|u'_c\|_H^2 \) besides to a bound on \( \epsilon \|u'_c\|_H^2 \). But since

\[
\langle Ru'_c(t), u'_c(t) \rangle = \langle R_+(t)u'_c(t), u'_c(t) \rangle - \langle R_-(t)u'_c(t), u'_c(t) \rangle
\]

this is not possible. Anyway, consider now \( t = T \) in (69) and divide by 2 to derive

\[
\frac{1}{2} \langle R_+(T)u'_c(T), u'_c(T) \rangle + \frac{1}{2} \langle R_-(0)u'_c(0), u'_c(0) \rangle + \frac{1}{2} \langle B(T)u_c(T), u_c(T) \rangle \leq \int_0^T \langle f(s), u'_c(s) \rangle ds + \frac{1}{2} \langle R_-(T)\psi, \psi \rangle + \frac{1}{2} \langle R_+(0)\varphi, \varphi \rangle + \frac{1}{2} \langle B(0)\eta, \eta \rangle.
\]

(71)

This last inequality is the first one of the two we need to get the desired estimate. Notice that in the right hand side there is a term \( \int_0^T \langle f(s), u'_c(s) \rangle ds \) which we cannot control.

So now we proceed and make a more detailed and refined estimate using (47) with \( t_1 = 0 \), \( t_2 = T \), \( t_\alpha = 0 \) and with \( Q = B \) and \( v = u'_c \) (we denote for simplicity by \( J \) the operator \( J_0 \)).
Then we have

\[ \epsilon \alpha \int_0^T \| u'_x(s) \|^2_{V(s)} ds + \frac{\alpha}{2} \int_0^T \| u_x(s) - u_x(0) \|^2_{V(s)} ds \leq \]

\[ \leq \int_0^T \langle \epsilon B(s)u'_x(s) + BJ(s)u'_x(s), u'_x(s) \rangle_{V'(s) \times V(s)} ds = \]

\[ = \int_0^T \langle B(s)\tilde{u}_x(s) + B(s)(u_x(s) - u_x(0)), u'_x(s) \rangle_{V'(s) \times V(s)} ds = \]

\[ = - \int_0^T \langle R(s)u''_x(s), u'_x(s) \rangle_{V'(s) \times V(s)} ds + \int_0^T \langle f(s), u'_x(s) \rangle_{V'(s) \times V(s)} ds + \]

\[ - \int_0^T \langle B(s)\tilde{u}, u'_x(s) \rangle_{V'(s) \times V(s)} ds = \]

\[ = - \frac{1}{2} \langle R(T)u'_x(T), u'_x(T) \rangle_{H(T)} + \frac{1}{2} \langle R(0)u'_x(0), u'_x(0) \rangle_{H(0)} + \]

\[ + \frac{1}{2} \int_0^T \langle R'(s)u'_x(s), u'_x(s) \rangle_{V'(s) \times V(s)} ds + \int_0^T \langle f(s), u'_x(s) \rangle_{V'(s) \times V(s)} ds + \]

\[ - \int_0^T \langle B(s)\tilde{u}, u'_x(s) \rangle_{V'(s) \times V(s)} ds \]

by which, using also the inequality \(2^{-1}||a|| \leq ||a - b||^2 + ||b||^2\) to estimate \(||u_x - u_x(0)||_{V(s)}^2\),

\[ \epsilon \alpha \| u'_x \|^2_{V'} + \frac{\alpha}{4} \| u_x \|^2_{V'} - \frac{1}{2} \int_0^T \langle R'(s)u'_x(s), u'_x(s) \rangle_{V'(s) \times V(s)} ds + \]

\[ + \frac{1}{2} \langle R(T)u'_x(T), u'_x(T) \rangle_{H(T)} + \frac{1}{2} \langle R(0)u'_x(0), u'_x(0) \rangle_{H(0)} \leq \]

\[ \leq \frac{1}{2} \langle R(T)\psi, \psi \rangle_{H(T)} + \frac{1}{2} \langle R(0)\varphi, \varphi \rangle_{H(0)} + \frac{\alpha}{2} \| \tilde{u} \|^2_{V'} + \]

\[ + \int_0^T \langle f(s), u'_x(s) \rangle_{V'(s) \times V(s)} ds - \int_0^T \langle B(s)\tilde{u}, u'_x(s) \rangle_{V'(s) \times V(s)} ds. \]

We estimate \(- \int_0^T \langle B(s)\tilde{u}, u'_x(s) \rangle_{V'(s) \times V(s)} ds\) as follows, using the assumptions on \(B'\):

\[ - \int_0^T \langle B(s)\tilde{u}, u'_x(s) \rangle_{V'(s) \times V(s)} ds = \int_0^T \langle B'(s)\tilde{u}, u_x(s) \rangle_{V'(s) \times V(s)} ds + \]

\[ + \langle B(0)\tilde{u}, u_x(0) \rangle_{V'(0) \times V(0)} - \langle B(T)\tilde{u}, u_x(T) \rangle_{V'(T) \times V(T)} \leq \]

\[ \leq C \left[ \beta \| u_x \|^2_{V'} + \frac{1}{\theta} \| \tilde{u} \|^2_{V'} + \frac{\beta}{\theta} \| \tilde{u} \|^2_{V'(0)} + \frac{1}{\beta} \| u_x \|^2_{V'(T)} + \frac{1}{\beta} \| \tilde{u} \|^2_{V'(T)} \right] \]

for any \(\theta, \tilde{\theta} > 0\). Then we will use (71) to treat the terms \(\| u_x(T) \|^2_{V(T)}\) and \(\| u_x \|^2_{V'}\). To estimate \(\int_0^T \langle f(s), u'_x(s) \rangle_{V'(s) \times V(s)} ds\) we can write

\[ 2 \int_0^T \langle f(s), u'_x(s) \rangle ds = -2 \int_0^T \langle f'(s), u_x(s) \rangle ds + 2 \langle f(T), u_x(T) \rangle - 2 \langle f(0), u_x(0) \rangle, \]
but in this way we need both (71) and (72) to treat the terms on the right hand side $\|u_e(T)\|_{V(T)}^2$ and $\|u_e\|_{V'}^2$. By the last equality, for every $\delta, \tilde{\delta} > 0$, we can estimate

$$2 \int_0^T \langle f(s), u'_e(s) \rangle \, ds = -2 \int_0^T \langle f'(s), u_e(s) \rangle \, ds + 2 \langle f(T), u_e(T) \rangle - 2 \langle f(0), u_e(0) \rangle \leq$$

$$\leq \frac{1}{\delta} \|f\|_{V'}^2 + \delta \|u_e\|_{V}^2 + \sup_{t \in [0,T]} \|f(t)\|_{V'(t)}^2 + \tilde{\delta} \|u_e(T)\|_{V(T)}^2 + \delta \|\tilde{\eta}\|^2_{V(0)} \leq$$

$$\leq \tilde{c} \frac{\delta + \tilde{\delta}}{\delta} \left[ \|f\|_{V'}^2 + \|f'\|_{V'}^2 \right] + \delta \|u_e\|_{V}^2 + \tilde{\delta} \|u_e(T)\|_{V(T)}^2 + \delta \|\tilde{\eta}\|^2_{V(0)}$$

where $\tilde{c}$ depends only on $T$. Now summing (71) and (72)

$$\langle R_+(T) u'_e(T), u'_e(T) \rangle + \langle R_-(0) u'_e(0), u'_e(0) \rangle +$$

$$+ \frac{\alpha}{2} \|u_e(T)\|_{V(T)}^2 + \epsilon \alpha \|u'_e\|_{V}^2 + \frac{\alpha}{4} \|u_e\|_{V}^2 - \frac{1}{2} \langle R' u'_e, u'_e \rangle_{V \times V} \leq$$

$$\leq \frac{1}{2} \langle R_-(T) \psi, \psi \rangle + \frac{1}{2} \langle R_+(0) \varphi, \varphi \rangle + \frac{\alpha}{2} \|\tilde{\eta}\|_{V}^2 +$$

$$+ \int_0^T \langle f(s), u'_e(s) \rangle_{V'(s) \times V(s)} \, ds - \int_0^T \langle B(s) \tilde{\eta}, u'_e(s) \rangle_{V'(s) \times V(s)} \, ds.$$  

Now using the estimates (73) and (74) with $C\delta/2 = \delta = \alpha/16$ and $\beta \tilde{\delta}/2 = \tilde{\delta} = \alpha/8$ by (75) we finally get

$$\langle R_+(T) u'_e(T), u'_e(T) \rangle + \langle R_-(0) u'_e(0), u'_e(0) \rangle + \epsilon \alpha \|u'_e\|_{V}^2 +$$

$$+ \frac{\alpha}{8} \|u_e\|_{V}^2 + \frac{\alpha}{4} \|u_e(T)\|_{V(T)}^2 - \frac{1}{2} \langle R' u'_e, u'_e \rangle_{V \times V} \leq$$

$$\leq \frac{1}{2} \langle R_-(T) \psi, \psi \rangle + \frac{1}{2} \langle R_+(0) \varphi, \varphi \rangle + \left( \beta + \frac{\alpha}{8} \right) \|\tilde{\eta}\|_{V(0)}^2 + \frac{2 \beta^2}{\alpha} \|\tilde{\eta}\|_{V(T)}^2 +$$

$$+ \left( \frac{\alpha}{2} + \frac{4 C^2}{\alpha} \right) \|\tilde{\eta}\|_{V}^2 + c \left( \|f\|_{V'}^2 + \|f'\|_{V'}^2 \right)$$

with $c$ depending on $T$ and $\alpha$.

Then we conclude, recalling that $R'$ is self-adjoint and $R' \leq 0$, that there is a positive constant $c$ and a sequence $(u_{e_j})_{j \in \mathbb{N}}$ such that (we for simplicity write $u_e$ instead of $u_{e_j}$)

$$\|u_e\|_{V'} \leq c, \quad u_e \rightharpoonup u \quad \text{in } V'-\text{weak}$$

$$\sqrt{\epsilon} \|u'_e\|_{V} \leq c, \quad \sqrt{\epsilon} u'_e \rightharpoonup v \quad \text{in } V-\text{weak}$$

$$\epsilon u'_e \rightarrow 0 \quad \text{in } V$$

$$\|(-R')^{1/2} u'_e\|_{V'} \leq c, \quad (-R')^{1/2} u'_e \rightharpoonup \tilde{w} \quad \text{in } V'-\text{weak}.$$

4 - Passage to the limit on the boundary conditions concerning $u'_e$ - Since

$$R u''_e = -\epsilon B u'_e - B u_e + f$$

by (77) we derive that also $R u''_e$ is bounded in $V'$ and then it, or one of its subsequences, is weakly converging to some $z \in V'$. Now since

$$(R u'_e)' = R' u'_e + R u''_e,$$

and

$$\frac{d}{dt} \langle R(t) u'_e(t), u'_e(t) \rangle_{H(t)} = 2 \langle R(t) u''_e(t), u'_e(t) \rangle_{V'(t) \times V(t)} + \langle R'(t) u'_e(t), u'_e(t) \rangle_{V'(t) \times V(t)}$$
and since, by (77), $R' u'_e$ is bounded in $V'$, integrating between $t_1$ and $t_2$, we get

$$R(t_2)u'_e(t_2) - R(t_1)u'_e(t_1) = \int_{t_1}^{t_2} R' u'_e(s)ds + \int_{t_1}^{t_2} Ru''(s)ds$$

and

$$(R(t_2)u'_e(t_2), u'_e(t_2))_{H(t_2)} - (R(t_1)u'_e(t_1), u'_e(t_1))_{H(t_1)} =$$

$$= \int_{t_1}^{t_2} \langle R'(s)u'_e(s), u'_e(s) \rangle_{V'(s) \times V(s)}ds + 2 \int_{t_1}^{t_2} \langle R(s)u''(s), u'_e(s) \rangle_{V'(s) \times V(s)}ds$$

Since by (78) we have $Ru'_e \in H$ and $(Ru'_e)' \in V'$ by Lemma 3.19, point b), we get that

$$Ru'_e(t) \in H(\sigma) \quad \text{for every } t \in [0, T],$$

and then

$$\int_{t_1}^{t_2} Ru'_e(s)ds + \int_{t_1}^{t_2} Ru''(s)ds \in H(\sigma) \quad \text{for every } \sigma \in [0, T].$$

Notice that

$$\langle (R(0)u'_e(0), u'_e(0))_{H(0)} \rangle \leq \langle (R_+(0)\varphi, \varphi)_{H(0)} + (R_-(0)u'_e(0), u'_e(0))_{H(0)}$$

and then, by (76), we know that $Ru'_e(0)$ is bounded. Similarly also $Ru'_e(T)$ turns out to be bounded. Then taking $t_1 = 0$ and $t_2 = t$ in (80) we first get that

$$Ru'_e(t) = Ru'_e(0) + \int_0^t R' u'_e(s)ds + \int_0^t Ru''(s)ds$$

and by Lemma 3.19, point b) and choosing $\sigma = t$ in (82) we first get

$$t \mapsto R(t)u'_e(t) \quad \text{belongs to } L^\infty(0, T)$$

and in particular $Ru'_e$ are equibounded in $H$. So there are $y \in H$ and a sequence $\epsilon_j \to 0$ such that

$$Ru'_e \to y \quad \text{weakly in } H.$$

Notice that, for every $\phi \in C^1([0, T]; U)$ with $\phi(0) = \phi(T) = 0$ we have that

$$\langle (y, \phi)_{H} = \lim_{j \to +\infty} (Ru'_e, \phi)_{H} = \lim_{j \to +\infty} (u'_e, Ru\phi)_{H} =$$

$$= - \lim_{j \to +\infty} \langle (R\phi)', u_e \rangle_{V' \times V} = - \langle (R\phi)', u \rangle_{V' \times V}.$$

We denote by $\tilde{R}$ the isomorphism

$$R|_{H_+ \oplus H_-} : H_+ \oplus H_- \to H_+ \oplus H_-$$

and for each $\phi$ we denote by $\psi$ the function $R\phi \in H_+ \oplus H_-$ which has a derivative in $V'$. Then we have

$$\langle (y, \tilde{R}^{-1}\psi)_{H} = \langle \tilde{R}^{-1}y, \psi \rangle_{H} = - \langle \psi', u \rangle_{V' \times V}.$$  

We conclude that

$$u' = \tilde{R}^{-1}y \quad \text{in } H_+ \oplus H_-,$$

i.e. $y = Ru'$ by which we get

$$Ru'_e \to Ru' \quad \text{weakly in } H.$$
Now from (83), thanks to Lemma 3.19, point b) and choosing \( \sigma = 0 \) in (82) we first get that there is a constant \( c > 0 \) such that

\[
\mathcal{R}u'_{\epsilon}(t) \in H(0) \quad \text{for every } t \in [0, T] \quad \text{and} \quad \|\mathcal{R}u'_{\epsilon}(t)\|_{H(0)} \leq c.
\]

Since also \( \mathcal{R}u'_0 \) and \( \mathcal{R}u'_{\sigma} \) are bounded in \( \mathcal{V}' \) we get by (80) and Lemma 3.19, point b), that

\[
\|R(t_2)u'_\epsilon(t_2) - R(t_1)u'_\epsilon(t_1)\|_{H(0)} = \left\| \int_{t_1}^{t_2} \mathcal{R}'u'_\epsilon(s)ds + \int_{t_1}^{t_2} \mathcal{R}u''(s)ds \right\|_{H(0)} \leq k \left[ \int_{t_1}^{t_2} \|\mathcal{R}'u'_\epsilon(s)\|_{\mathcal{V}'(s)}ds + \int_{t_1}^{t_2} \|\mathcal{R}u''(s)\|_{\mathcal{V}'(s)}ds \right] \leq k |t_2 - t_1|^{1/2} \left[ \left( \int_{t_1}^{t_2} \|\mathcal{R}'u'_\epsilon(s)\|^2_{\mathcal{V}'(s)}ds \right)^{1/2} + \left( \int_{t_1}^{t_2} \|\mathcal{R}u''(s)\|^2_{\mathcal{V}'(s)}ds \right)^{1/2} \right] \leq k |t_2 - t_1|^{1/2} (\|\mathcal{R}'u'_\epsilon\|_{\mathcal{V}'} + \|\mathcal{R}u''\|_{\mathcal{V}'})
\]

Then we derive that the family

\[
\left\{ R(t)u'_\epsilon(t) \right\}_{\epsilon > 0}
\]

is equibounded and equicontinuous in \( [0, T] \) with respect to the topology of \( H(0) \)

and then \( \left\{ R(t)u'_\epsilon(t) \right\}_{\epsilon > 0} \) is weakly relatively compact in \( H(0) \), i.e. there is a sequence \( (\epsilon_j) \) such that, by (84), for every \( \phi \in H(0) \)

\[
(\phi, R(t)u'_\epsilon(t))_{H(0)} \to (\phi, R(t)u'(t))_{H(0)} \quad \text{uniformly in } [0, T].
\]

Since we also have

\[
\mathcal{R}u'_\epsilon(t) = R\mathcal{R}u'_\epsilon(T) + \int_T^t \mathcal{R}'u'_\epsilon(s)ds + \int_T^t \mathcal{R}u''(s)ds
\]

the analogous argument can be used to get that for every \( \phi \in H(T) \)

\[
(\phi, R(t)u'_\epsilon(t))_{H(T)} \to (\phi, R(t)u'(t))_{H(T)} \quad \text{uniformly in } [0, T].
\]

Thanks to these uniform convergences and since \( R_+(0)u'_\epsilon(0) = R_+(0)\varphi \) and \( R_-(T)u'_\epsilon(T) = R_-(T)\psi \) for every \( \epsilon > 0 \) we in particular get that the conditions

\[
R_+(0)u'_\epsilon(0) = R_+(0)\varphi \quad \text{and} \quad R_-(T)u'_\epsilon(T) = R_-(T)\psi
\]

are maintained also at the limit. i.e.

\[
R_+(0)u'(0) = R_+(0)\varphi \quad \text{and} \quad R_-(T)u'(T) = R_-(T)\psi.
\]

Notice that applying \( \mathcal{R}_+^{-1/2} \) to \( \mathcal{R}u' \) we get

\[
\mathcal{R}_+^{-1/2}(\mathcal{R}u') = \mathcal{R}_+^{-1/2}(\mathcal{R}_+u' + \mathcal{R}_-u') = \mathcal{R}_+^{1/2}u'.
\]

We conclude that \( \mathcal{R}_+^{1/2}u' \in \mathcal{H} \) and similarly that \( \mathcal{R}_-^{1/2}u' \in \mathcal{H} \). In particular \( |\mathcal{R}|^{1/2}u' \in \mathcal{H} \).

5 - Passage to the limit in the equation  - Before going on we recall the following simple result, needed shortly.

**Lemma 5.4.** Consider \( X \) and \( Y \) Banach spaces and \( L : X \to Y \) linear and continuous. Then \( L \) is weakly continuous, i.e. if \( (x_n)_{n \in \mathbb{N}} \) is a weakly converging sequence in \( X \) to \( x \in X \) then \( (Lx_n)_{n \in \mathbb{N}} \) is a weakly converging sequence in \( Y \) to \( Lx \in X \).
By (77) and the previous point we have got that $\mathcal{R}u''_\epsilon$ and $\mathcal{R}'u'_\epsilon$ are bounded in $\mathcal{V}'$ and then there are $z, w \in \mathcal{V}'$ such that

$$
\begin{align*}
&u_\epsilon \to u \quad \text{in } \mathcal{V} - \text{weak}, \\
&\epsilon u'_\epsilon \to 0 \quad \text{in } \mathcal{V}, \\
&\mathcal{R}'u'_\epsilon \to w = (\mathcal{R}')^{1/2}\tilde{w} \quad \text{in } \mathcal{V}' - \text{weak}, \\
&\mathcal{R}u''_\epsilon \to z \quad \text{in } \mathcal{V}' - \text{weak}.
\end{align*}
$$

Notice that 

$$(\mathcal{R}u'_\epsilon)' \to (\mathcal{R}u')' \quad \text{in } \mathcal{V}' - \text{weak}.$$ 

Indeed by (84) we know that $\mathcal{R}u'_\epsilon \to \mathcal{R}u'$ weakly in $\mathcal{H}$ and then we have 

$$
\langle \mathcal{R}u', \phi' \rangle = \lim_{\epsilon \to 0} \langle \mathcal{R}u'_\epsilon, \phi' \rangle = -\lim_{\epsilon \to 0} \langle (\mathcal{R}u'_\epsilon)', \phi \rangle = -\lim_{\epsilon \to 0} \langle z + w, \phi \rangle
$$

for every $\phi \in C^1_0([0, T]; U)$, then 

$$z + w = (\mathcal{R}u')'.$$

Nevertheless without (61) we are not able to prove that (86)

$$z = \mathcal{R}u'' \quad \text{and} \quad w = \mathcal{R}'u'.$$

For this reason we require (61) and with this assumption we have that 

$\mathcal{R}' : \mathcal{H} \to \mathcal{H}.$

Since moreover obviously 

$$\text{Ker} \mathcal{R}' \supset \text{Ker} \mathcal{R}$$

by these informations and by (84) we easily get that 

$$w = \mathcal{R}'u'.$$

Then by (86) we finally get 

$$\mathcal{R}u''_\epsilon \to \mathcal{R}u'' \quad \text{in } \mathcal{V}' - \text{weak}.$$ 

Now taking the limit in the equation 

$$\mathcal{R}u''_\epsilon + \epsilon B u'_\epsilon + B u_{\epsilon} = f$$

and using Lemma 5.4 we finally get that $u$ satisfies 

$$\mathcal{R}u'' + B u = f.$$ 

6 - Passage to the limit on the boundary conditions concerning $u_\epsilon$ - The last thing to be verified to have existence is that the condition about $u(0)$ is maintained. 

We know that 

$$\|u_\epsilon\|_\mathcal{V} \leq c, \quad \|\mathcal{R}u'_\epsilon\|_\mathcal{H} \leq c.$$ 

This does not allow to conclude that $u_\epsilon(0)$ converges to $u(0)$ (in some sense) because we have not enough information in $\text{Ker} \mathcal{R}$. Anyway we have enough information in $(\text{Ker} \mathcal{R})^\perp$. Indeed we have that there is a positive constant $c'$ such that 

$$\|\mathcal{R}u_\epsilon\|_\mathcal{H} \leq c' \quad \text{and} \quad \|\mathcal{R}u'_\epsilon\|_\mathcal{V} \leq c'.$$

This because 

$$\|\mathcal{R}u_\epsilon\|_\mathcal{H} \leq \|\mathcal{R}\| \|u_\epsilon\|_\mathcal{H} \leq \|\mathcal{R}\| \|u_\epsilon\|_\mathcal{V} \leq c \|\mathcal{R}\|$$

and 

$$\|\mathcal{R}u'_\epsilon\|_\mathcal{V} \leq \|\mathcal{R}'u'_\epsilon\|_\mathcal{V} + \|\mathcal{R}u'_\epsilon\|_\mathcal{H} \leq \|\mathcal{R}'\| \|u'_\epsilon\|_\mathcal{V} + \|\mathcal{R}u'_\epsilon\|_\mathcal{H} \leq c \|\mathcal{R}'\| + c.$$
As done in the previous point we can get that $\{(R(t)u_\epsilon(t))_{\epsilon>0}\}$ is weakly relatively compact in $H(0)$, i.e. there is a sequence $(\epsilon_j)_j$ such that for every $\phi \in H(0)$

$$(R(t)u_\epsilon(t), \phi)_{H(0)} \rightarrow (R(t)u(t), \phi)_{H(0)}$$

uniformly in $[0, T]$. This in particular holds for $t = 0$ and then we conclude that

$$R(0)u(0) = R(0)\tilde{\eta} \quad \text{in } H(0).$$

7 - Uniqueness - Point 2 is devoted to explain how to modify the initial datum $\eta$ with a function $\tilde{\eta}$ defined in (66). This modification does not affect the limit problem (55) as got in (87), but, on the other side, forces this problem to have only one solution. Indeed suppose now problem (55) has two solutions $u_1$ and $u_2$ and suppose $u_j$ ($j = 1, 2$) is obtained as limit of some sequences selected from the solutions $(u^{(1)}_\epsilon)_{\epsilon>0}$ and $(u^{(2)}_\epsilon)_{\epsilon>0}$ of the following problems

$$
\begin{align*}
\left\{ \begin{array}{ll}
R\phi'' + \epsilon Bu' + Bu &= f \\
P_+(0)u'(0) &= \varphi \\
P_-(T)u'(T) &= \psi \\
u(0) &= \eta_1 
\end{array} \right. \\
\left\{ \begin{array}{ll}
R\phi'' + \epsilon Bu' + Bu &= f \\
P_+(0)u'(0) &= \varphi \\
P_-(T)u'(T) &= \psi \\
u(0) &= \eta_2 
\end{array} \right.
\end{align*}
$$

with $\eta_1$ and $\eta_2$, for the moment and a priori, different, but will be chosen to satisfy (90). Now call $u$ the function $u_2 - u_1$; then $u$ satisfies

$$
\begin{align*}
\left\{ \begin{array}{ll}
R\phi'' + Bu &= 0 \\
P_+(0)u'(0) &= 0 \\
P_-(T)u'(T) &= 0 \\
(P_+(0) + P_-(-0))u(0) &= (P_+(0) + P_-(-0))(\eta_2 - \eta_1).
\end{array} \right.
\end{align*}
$$

Anyway $u^{(2)}_\epsilon - u^{(1)}_\epsilon$ satisfies (63) with $f = 0$, $\psi = 0$, $\varphi = 0$; then in particular, by (64), we get

$$
\| R(u^{(2)}_\epsilon - u^{(1)}_\epsilon)'' \|_\infty + \|(u^{(2)}_\epsilon - u^{(1)}_\epsilon)''\|_\infty + \sup_{t \in [0, T]} \| u^{(2)}_\epsilon(t) - u^{(1)}_\epsilon(t) \|_{V(t)} \leq \| \eta_2 - \eta_1 \|_{V(0)}
$$

and this inequality has to be satisfied by each sequence selected from $(u^{(1)}_\epsilon)_{\epsilon>0}$ and from $(u^{(2)}_\epsilon)_{\epsilon>0}$. Now we choose first $\eta_1$ and $\eta_2$ in such a way that

$$
(P_+(0) + P_-(-0))\eta_1 = (P_+(0) + P_-(-0))\eta_2,
$$

but it is clear that this choice is not sufficient, since $P_0(0)\eta_1$ could differ from $P_0(0)\eta_2$. But if in (88) we replace $\eta_1$ with $\tilde{\eta}_1$ and $\eta_2$ with $\tilde{\eta}_2$, the suitable modifications of $\eta_1$ and $\eta_2$ defined in point 2, we get that also $P_0(0)\tilde{\eta}_1$ and $P_0(0)\tilde{\eta}_2$ are the same. Indeed denote by $w_1$ and $w_2$ the solutions to (65) respectively with $\eta_1$ and $\eta_2$. Then it is easy to check that

$$
\eta_1 + w_1 = \eta_2 + w_2
$$

that is, $\tilde{\eta}_1 = \tilde{\eta}_2$ (remember that $f \equiv 0$ and in particular $f(0) = 0$) by which we conclude.

6. Examples

In this section we present some simple examples of possible choices of $\mathcal{R}$ and $\mathcal{A}$ for the equation considered in Section 3 and examples of possible choices of $\mathcal{R}$, $\mathcal{A}$, $\mathcal{A}$ for the equations considered in Section 4 and the generalized Tricomi equations considered in Section 5. These examples should help to understand which kind of operators $\mathcal{R}$ are admissible and combining
these examples one could imagine some more general situations which satisfy the conditions assumed in the theorems given in the previous sections.

1 - The equation \( Ru' + Au = f \)

In the first examples which follow we consider \( T > 0, \Omega \subset \mathbb{R}^n \) open set with Lipschitz boundary and

\[
U \equiv V(t) = H^1_0(\Omega) \quad \text{and} \quad H(t) = L^2(\Omega) \quad \text{for every} \ t \in [0, T],
\]

\[
A(t) : H^1_0(\Omega) \to H^{-1}(\Omega)
\]

\[
(A(t)u)(x) := - \text{div} a(x, t, Du(x)) + b(x, t, u),
\]

with \( a : \Omega \times (0, T) \times \mathbb{R}^n \to \mathbb{R}^n, \ b : \Omega \times (0, T) \times \mathbb{R} \to \mathbb{R}, \)

verifying \( \lambda_0 |\xi|^2 \leq a(x, t, \xi) \cdot \xi \leq \Lambda_0 |\xi|^2 \) and \( |b(x, t, u)| \leq M |u(x)| \)

for every \( \xi \in \mathbb{R}^n \) and for some positive \( \lambda_0, \Lambda_0 \) and some \( M \geq 0 \). Then \( A \) will be defined as in (25). We fix now our attention on the operator \( \mathcal{R} \). Consider a function

\[
r : \Omega \times [0, T] \to \mathbb{R}, \quad r \in L^\infty(\Omega \times (0, T))
\]

and

\[
R(t) : L^2(\Omega) \to L^2(\Omega), \quad (R(t)u)(x) := r(x, t)u(x).
\]

Finally for every \( t \in [0, T] \) we denote

\[
\Omega_+(t) := \{ x \in \Omega \mid r(x, t) > 0 \},
\]

\[
\Omega_-(t) := \{ x \in \Omega \mid r(x, t) < 0 \},
\]

\[
\Omega_0(t) := \Omega \setminus (\Omega_+ \cap \Omega_-)
\]

and (see also (16))

\[
r_+ \quad \text{the positive part of} \ r, \quad r_- \quad \text{the negative part of} \ r,
\]

\[
\hat{H}_+(0) = L^2(\Omega_+(0), r_+(\cdot, 0)) \quad \text{the completion of} \ C_c(\Omega_+(0))
\]

\[
\begin{align*}
&\text{w.r.t. the norm} \ ||w||^2 = \int_{\Omega_+(0)} w^2(x) r_+(x, 0) dx, \\
&\hat{H}_-(T) = L^2(\Omega_-(T), r_-(\cdot, T)) \quad \text{the completion of} \ C_c(\Omega_+(0))
\end{align*}
\]

\[
\begin{align*}
&\text{w.r.t. the norm} \ ||w||^2 = \int_{\Omega_-(T)} w^2(x) r_-(x, T) dx.
\end{align*}
\]

Then consider the problem, for some \( f \in L^2(0, T; H^{-1}(\Omega)), \varphi \in \hat{H}_+(0), \psi \in \hat{H}_-(T) \)

\[
\begin{cases}
  r(x, t) u_t + A u = f(x, t) & \text{in} \ \Omega \times (0, T) \\
  u(x, t) = 0 & \text{for} \ (x, t) \in \partial \Omega \times (0, T) \\
  u(x, 0) = \varphi(x) & \text{for} \ x \in \Omega_+(0) \\
  u(x, T) = \psi(x) & \text{for} \ x \in \Omega_-(T).
\end{cases}
\]
1 - Clearly Theorem 3.15 includes the “standard” equations. If \( r \equiv 1 \) we have the forward parabolic equation

\[
\begin{aligned}
  \begin{cases}
    u_t + A u = f(x,t) & \text{in } \Omega \times (0,T) \\
    u(x,t) = 0 & \text{for } (x,t) \in \partial\Omega \times (0,T) \\
    u(x,0) = \phi(x) & \text{for } x \in \Omega,
  \end{cases}
\end{aligned}
\]

if \( r \equiv -1 \) we have the backward parabolic equation

\[
\begin{aligned}
  \begin{cases}
    -u_t + A u = f(x,t) & \text{in } \Omega \times (0,T) \\
    u(x,t) = 0 & \text{for } (x,t) \in \partial\Omega \times (0,T) \\
    u(x,T) = \psi(x) & \text{for } x \in \Omega,
  \end{cases}
\end{aligned}
\]

if \( r \equiv 0 \) we have a family of elliptic equations

\[
\begin{aligned}
  \begin{cases}
    A(t) u_t = f(t) & \text{in } \Omega \text{ for a.e. } t \in (0,T) \\
    u(\cdot, t) = 0 & \text{for } (x,t) \in \partial\Omega \times (0,T).
  \end{cases}
\end{aligned}
\]

2 - Suppose \( r = r(x) \), \( r \in L^\infty(\Omega) \). As long as \( r \geq 0 \) every function is admitted, even, for example,

\[
\begin{aligned}
  r(x) = 1 & \text{ in } \Omega_+ , \quad r(x) = 0 & \text{ in } \Omega_0 ,
\end{aligned}
\]

\( \Omega_+ \) and \( \Omega_0 \) Cantor-type sets of positive measure.

This because clearly \( R \) belongs to the class \( \mathcal{E} \) defined in Definition 2.1 and because assumption (32) is satisfied. This last assumption might not be satisfied if one consider a generic \( r \in L^\infty(\Omega) \), if for instance

\[
\begin{aligned}
  r(x) = 1 & \text{ in } \Omega_+ , \quad r(x) = 0 & \text{ in } \Omega_0 , \quad r(x) = -1 & \text{ in } \Omega_- ,
\end{aligned}
\]

\( \Omega_+ , \Omega_0 , \Omega_- \) Cantor-type sets of positive measure.

The request (32) is surely satisfied if there are two open sets \( A_1, A_2 \) with

\[
\begin{aligned}
  A_1 \cap A_2 &= \emptyset , \quad \Omega_+ \subset A_1 , \quad \Omega_- \subset A_2 .
\end{aligned}
\]

3 - Suppose \( r = r(t) \). Assumptions (20) are satisfied if \( r \in W^{1,\infty}(0,T) \), therefore every \( r \in W^{1,\infty}(0,T) \) is admitted. Two interesting situations are the following: the first when \( r(0) \geq 0 \) and \( r(T) \leq 0 \) leads to the problem

\[
\begin{aligned}
  \begin{cases}
    r(t)u_t + r'(t)u + Au = f & \text{in } \Omega \times (0,T) \\
    u(x,t) = 0 & \text{for } (x,t) \in \partial\Omega \times (0,T) \\
    u(x,0) = \phi(x) & \text{for } x \in \Omega \\
    u(x,T) = \psi(x) & \text{for } x \in \Omega
  \end{cases}
\end{aligned}
\]

where a datum is given in the whole \( \Omega \) both at time 0 and at time \( T \); the second where \( r(0) \leq 0 \) and \( r(T) \geq 0 \), which leads to the problem

\[
\begin{aligned}
  \begin{cases}
    r(t)u_t + r'(t)u + Au = f & \text{in } \Omega \times (0,T) \\
    u(x,t) = 0 & \text{for } (x,t) \in \partial\Omega \times (0,T)
  \end{cases}
\end{aligned}
\]

where no information is given in the whole \( \Omega \) both at time 0 and at time \( T \).

4 - More interesting is the case when \( r = r(x,t) \). As long as

\[
\begin{aligned}
  r & \text{ and } \frac{\partial r}{\partial t} \in L^\infty(\Omega \times (0,T))
\end{aligned}
\]
the situation is very similar to that analyzed in example 3, so every \( r \) such that \( r, r_t \in L^\infty(\Omega \times (0, T)) \) is admitted, provided that assumption (32) is satisfied. Suppose now

\[ r \] does not admit a partial derivative with respect to time.

Well, assumption (20) could be satisfied anyway. To show it we consider a very simple example: suppose \( n = 1, \Omega = (a, b), T > 0, \) consider a function

\[ \gamma : [0, T] \to (a, b), \quad \gamma \in W^{1,\infty}(0, T) \]

and define the sets

\[(95) \quad \omega_+ := \{(x, t) \in \Omega \times (0, T) \mid x < \gamma(t)\}, \quad \omega_0 := (\Omega \times (0, T)) \setminus \omega_+ \]

and the function \( r \)

\[(96) \quad r(x, t) = \chi_{\omega_+}(x, t) := \begin{cases} 1 & \text{in } \omega_+ \\ 0 & \text{in } \omega_0 \end{cases}. \]

To verify that \( R \in \mathcal{E} \) we consider \( w_1, w_2 \in H^1_0(a, b) \) and evaluate

\[
\frac{d}{dt}(R(t)w_1, w_2)_{L^2(a, b)} = \frac{d}{dt} \int_a^b w_1(x)w_2(x)r(x, t) \, dx = \frac{d}{dt} \int_a^{\gamma(t)} w_1(x)w_2(x) \, dx = w_1(\gamma(t))w_2(\gamma(t))\gamma'(t),
\]

then for \( u \in \mathcal{V} = L^2(0, T; H^1_0(\omega)) \) we have

\[ \langle R'u, u \rangle_{\mathcal{V}' \times \mathcal{V}} = \int_0^T (u(\gamma(t), t))^2 \gamma'(t) \, dt. \]

Moreover notice that, as long as \( \gamma \) is decreasing so that \( \gamma' \leq 0 \), assumption (33) in Theorem 3.15 is easily satisfied, since

\[ A - \frac{1}{2} R' : \mathcal{V} \to \mathcal{V}' \]

turns out to be bounded thanks to the fact that \( \gamma' \) is bounded, and

\[ \langle Au - Av - \frac{1}{2}(R'u - R'v), u - v \rangle_{\mathcal{V}' \times \mathcal{V}} \geq \lambda_o \|u - v\|^2_{\mathcal{V}}. \]

In Figure 1.a below two possible admissible configurations are shown, the first one referring to a situation when \( \omega_+ \) and \( \omega_0 \) are like those defined in (95), the second one refers to a possible configuration with

\[ \omega_+ := \{(x, t) \in \Omega \times (0, T) \mid x > \gamma(t)\}, \quad \omega_0 := (\Omega \times (0, T)) \setminus \omega_+. \]
Now suppose that
\[ \text{ess sup}_{[0,T]} \gamma'(t) > 0. \]

Notice that there is a constant \( C \) such that for every \( w \in H^1_0(a,b) \)
\[ |w(x)| \leq C \|w\|_{H^1_0(a,b)} \quad \text{for every } x \in [a,b], \]
then, finally, for \( u \in \mathcal{V} = L^2(0,T;H^1_0(\Omega)) \) we have
\[
-\frac{1}{2} \langle \mathcal{R}'u, u \rangle_{\mathcal{V}' \times \mathcal{V}} = -\frac{1}{2} \int_0^T (u(\gamma(t),t))^2 \gamma'(t) dt \geq
\[
\geq \frac{1}{2} \int_0^T (u(\gamma(t),t))^2 \text{ess inf}_{[0,T]} (-\gamma'(t)) dt \geq
\[
\geq -\frac{1}{2} \int_0^T (u(\gamma(t),t))^2 \text{ess sup}_{[0,T]} \gamma'(t) dt \geq
\]
\[
\geq -\frac{C^2}{2} \text{ess sup}_{[0,T]} \gamma'(t) \|u\|_{\mathcal{V}}^2.
\]

and then
\[
\langle A u - A v - \frac{1}{2} (\mathcal{R}'u - \mathcal{R}'v), u - v \rangle_{\mathcal{V}' \times \mathcal{V}} \geq \left( \lambda_o - \frac{C^2}{2} \text{ess sup}_{[0,T]} \gamma'(t) \right) \|u - v\|_{\mathcal{V}}^2.
\]

Then the first assumption required in (33) in Theorem 3.15 is satisfied if
\[
\lambda_o - \frac{C^2}{2} \text{ess sup}_{[0,T]} \gamma'(t) > 0
\]
and then \( \gamma \) can be also increasing, provided that
\[
\text{ess sup}_{[0,T]} \gamma'(t) < \frac{2 \lambda_o}{C^2}.
\]

A possible configuration is shown in Figure 2, where \( \gamma' \) is not necessarily negative, but it has to satisfy (97).
Analogous considerations can be made if \( n \geq 2 \): taking the simplest example

\[
r(x, t) = \chi_{\omega^+}(x, t)
\]

where for each \( t \in [0, T] \) we have \( \Omega = \Omega^+(t) \cup \Omega_0(t) \) and

\[
\omega^+ := \bigcup_t \Omega^+(t), \quad \omega_0 := \bigcup_t \Omega_0(t).
\]

In this case we need that for \( w_1, w_2 \in H^1_0(\Omega) \) the following hold:

\[
t \mapsto \int_{\Omega^+(t)} w_1(x)w_2(x) \, dx \quad \text{is differentiable},
\]

\[
\left| \frac{d}{dt} \int_{\Omega^+(t)} w_1(x)w_2(x) \, dx \right| \leq \tilde{C} \| w_1 \|_{H^1_0(\Omega)} \| w_2 \|_{H^1_0(\Omega)}
\]

for some positive \( \tilde{C} \), since

\[
(R(t)w_1, w_2)_{L^2(\Omega)} = \int_{\Omega} w_1(x)w_2(x)r(x, t) \, dx = \int_{\Omega^+(t)} w_1(x)w_2(x) \, dx.
\]

These hold if \( \Omega^+(t) \) is open and the interface separating \( \Omega^+(t) \) and \( \Omega_0(t) \) is Lipschitz continuous (see, e. g., Proposition 3, section 3.4.4, in [5]). Moreover, since \( u, v \in H^1_0(\Omega) \), it makes sense to consider the trace on this interface (see, e. g., Theorem 1, section 4.3, in [5]).

5 - We want to show a little example where the regularity result stated in Theorem 3.20 holds. First suppose that (32) and (33) are satisfied, so to have a solution.

About \( r \) consider

\[
r \quad \text{and} \quad \frac{\partial r}{\partial t} \in L^\infty(\Omega \times (0, T)).
\]

As regards the operator \( \mathcal{A} \), consider, for instance, it is like in (91). Then, since

\[
\langle A(t)u, v \rangle_{H^{-1}(\Omega) \times H^1_0(\Omega)} = \int_{\Omega} \left( a(x, Du(x), Dv(x)) + b(x, t)u(x)v(x) \right) \, dx,
\]

in order that (36) is satisfied we need \( a \) satisfying

\[
a \quad \text{differentiable with respect to} \quad t \quad \text{and}
\]

\[
\left| \int_{\Omega} \left( \frac{\partial a}{\partial t} (x, t, Du(x)), Dv(x) \right) \, dx \right| \leq C_3 \left( \int_{\Omega} |Du(x)|^2 \, dx \int_{\Omega} |Dv(x)|^2 \, dx \right)^{1/2}
\]
for every \( u, v \in H^1_0(\Omega) \) and for some positive constant \( C_3 \), while for \( b \) we need
\[
 t \mapsto \int_{\Omega} b(x, t)u(x)v(x) \, dx \quad \text{is to be absolutely continuous and satisfying}
\]
\[
 \left| \frac{d}{dt} \int_{\Omega} b(x, t, u(x))v(x) \, dx \right| \leq C_3 \left( \int_{\Omega} |Dv(x)|^2 \, dx \int_{\Omega} |Du(x)|^2 \, dx \right)^{1/2}.
\]

To consider a simpler example suppose \( b(x, t, u) = b(x, t)u \) with \( b \in L^\infty(\Omega \times (0, T)) \). Then the assumptions about \( b \) are similar to those regarding \( r \) in the previous example. Now consider \( f \in H^1(0, T; H^{-1}(\Omega)) \) and the two functions \( u_0 \) and \( u_T \) solutions respectively of the two problems
\[
\begin{cases}
-\text{div} \, a(x, 0, Du) + b(x, 0)u = f(0) & \text{in } \Omega \\
u = 0 & \text{in } \partial \Omega
\end{cases}
\]
\[
\begin{cases}
-\text{div} \, a(x, T, Du) + b(x, T)u = f(T) & \text{in } \Omega \\
u = 0 & \text{in } \partial \Omega
\end{cases}
\]
and consider \( \varphi \) the restriction to \( \Omega_+(0) \) (see (92)) of \( u_0 \) and \( \psi \) the restriction to \( \Omega_-(T) \) of \( u_T \).
Then the solution \( u \) of (94) belongs to \( H^1(0, T; H^1_0(\Omega)) \).

6 - The equation we considered in Section (3) is \( Ru' + Au = f \). Nevertheless we required some regularity assumption about \( R \), precisely that \( R \in \mathcal{E} \), the class defined in Definition 2.1. With this example we want to show that at least uniqueness is lost if \( R \notin \mathcal{E} \).
Consider
\[
r = r(t) = \begin{cases} 
-1 & \text{for } t < T/2 \\
1 & \text{for } t \geq T/2
\end{cases}
\]
and the problem (94) with this \( r \). Clearly
\[
\Omega_+(0) = \Omega_-(T) = \emptyset.
\]
Then we can fix \( \eta \in H^1_0(\Omega) \) and solve separately the two problems
\[
\begin{cases}
-u_t + Au = f & \text{in } \Omega \times (0, T/2) \\
u(x, t) = 0 & \text{in } \partial \Omega \times (0, T/2) \\
u(x, T/2) = \eta(x) & \text{in } x \in \Omega
\end{cases}
\]
\[
\begin{cases}
-u_t + Au = f & \text{in } \Omega \times (T/2, 0) \\
u(x, t) = 0 & \text{in } \partial \Omega \times (0, T) \\
u(x, T/2) = \eta(x) & \text{in } x \in \Omega
\end{cases}
\]
and call \( u_1 \) the solution of the first problem, \( u_2 \) the solution of the second problem. Notice that the function \( u^n(t) = u_1(t) \) for \( t \in [0, T/2] \), \( u(t) = u_2(t) \) for \( t \in [T/2, T] \) solves problem
\[
\begin{cases}
r u_t + Au = f & \text{in } \Omega \times (0, T/2) \\
u(x, t) = 0 & \text{in } \partial \Omega \times (0, T)
\end{cases}
\]
and this is true for every \( \eta \in H^1_0(\Omega) \), and so we have infinite different solutions.
Notice that if \( r \) depends only on \( t \), \( r(0) < 0 \), \( r(T) > 0 \), \( r \) increasing and continuous the problem above has a unique solution, even if there are no initial and final data.

In the following example we modify (91) and consider for \( A \) a monotone operator whose growth is more than linear. We consider the simple example where \( p > 2 \) (one could also consider \( p > 2n/(2 + n) \) is such a way that \( W^{1,p} \subset L^2 \), but for simplicity we confined to
\( p \geq 2 \)

\[
U \equiv V(t) = W_0^{1,p}(\Omega) \quad \text{and} \quad H(t) = L^2(\Omega) \quad \text{for every } t \in [0, T],
\]

\[
A(t) : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)
\]

\[
(A(t)u)(x) := -\text{div} a(x, t, Du(x)),
\]

with \( a : \Omega \times (0, T) \times \mathbb{R}^n \to \mathbb{R}^n \),

verifying

\[
\lambda_o |\xi|^p \leq a(x, t, \xi) \cdot \xi \leq \Lambda_o |\xi|^p
\]

for every \( \xi \in \mathbb{R}^n \) and for some positive \( \lambda_o, \Lambda_o \).

7 - With \( A \) like in (98) all things we said in examples 1, 2, 3, 4 hold, except one. Since \( R' \)

is linear, \( R' \) is not comparable with \( A \), then in this case to have assumption (34) in

Theorem 3.15 satisfied we have to confine to some \( R \) such that

\[-\frac{1}{2} R' \]

is a positive operator.

Then we can consider the functions considered in example 4, but we have to confine to non-increasing \( \gamma \) in the first example and to some \( \Omega_+ \) such that

\[
\frac{d}{dt} \int_{\Omega_+ (t)} w_1(x) w_2(x) \, dx \leq 0 \quad \text{and clearly bounded}
\]

in the more general case. So in this case examples like those shown in Figure 1 are admissible, but that in Figure 2 is not.

8 - Now consider the following \( R : [0, T] \to \mathcal{L}(L^2(\Omega)) \). For a fixed function \( r \in L^\infty (\Omega \times \Omega \times [0, T]) \) we define

\[
(R(t)u)(x) := \int_{\Omega} r(x, y, t) u(y) \, dy \quad u \in L^2(\Omega).
\]

Clearly \( r \) could be a convolution kernel, i.e. \( r(x, y, t) = r(x - y, t) \) (suitable extended to zero outside of \( \Omega \times (0, T) \)). If assumptions (32) and (33) are satisfied if \( p = 2 \), e.g. if the situation is like that in (91), or if (32) and (34) are satisfied if \( p > 2 \), e.g. if the situation is like that in (98), we have the existence and uniqueness of the solution of the following problem

\[
\begin{cases}
\int_{\Omega} r(x, y, t) u_t(y) \, dy + Au = f & \text{in } \Omega \times (0, T), \\
u = 0 & \text{in } \partial \Omega \times (0, T), \\
u(\cdot, 0) = \varphi & \text{in } \tilde{H}_+(0), \\
u(\cdot, T) = \psi & \text{in } \tilde{H}_-(T).
\end{cases}
\]

Notice that \( Ru' \) in (31) belongs, a priori, to \( V' \), but this is well defined since we recall that \( Ru' = (Ru)' - R'u \).

In this case we have to give the initial and final data respectively in the space \( \tilde{H}_+(0) \) and \( \tilde{H}_-(T) \) (defined in (16)) which in the previous cases are those defined in (93).

Now we want to show some examples of varying spaces in which the Banach spaces \( V(t) \) are varying with time.

9 - **Unbounded coefficients.** Another admissible situation is the following. Consider two functions

\[
\mu, \lambda \in L^1(\Omega \times (0, T)).
\]
Suppose \( \lambda > 0 \) a.e. while \( \mu \) can change sign and also be zero. Denote by \( |\tilde{\mu}| \) a suitable function (see \[22\] or \[19\] for this detail) such that \( |\tilde{\mu}| > 0 \) a.e. (we choose \( |\tilde{\mu}| = \lambda \) where \( \mu \equiv 0 \)) and

\[
|\tilde{\mu}| = \begin{cases} 
\mu & \text{in } \{(x,t) \in \Omega \times (0,T) \mid \mu(x,t) > 0\} \\
-\mu & \text{in } \{(x,t) \in \Omega \times (0,T) \mid \mu(x,t) < 0\}
\end{cases}
\]

and the weighted Sobolev spaces for \( p \geq 2 \) (also for these details about these spaces we refer to \[22\] or to \[19\])

\[
H(t) := L^2(\Omega, |\tilde{\mu}|(\cdot, t)) , \quad V(t) := W^1_0(\Omega, |\mu|((\cdot, t), \lambda(\cdot, t)) .
\]

In this case (see again \[22\]) one has that there is \( q > p \) such that \( W^1_0(\Omega) \) is dense in \( V(t) \) for every \( t \in [0,T] \). Then we consider

\[
U = W^1_0(\Omega) , \quad V(t) \text{ and } H(t) \text{ as above ,}
\]

\[
A(t) : V(t) \to V'(t)
\]

(99)

\[
(A(t)u)(x) := - \text{div } a(x, t, Du(x)) ,
\]

with \( a : \Omega \times (0, T) \times \mathbb{R}^n \to \mathbb{R}^n \), verifying

\[
\lambda(x, t) |\xi|^p \leq a(x, t, \xi) \cdot \xi \leq L \lambda(x, t) |\xi|^p
\]

for every \( \xi \in \mathbb{R}^n \) and for some \( L \geq 1 \).

Consider the spaces and the operator just introduced and once defined

\[
\Omega_+(t) := \{ x \in \Omega \mid \mu(\cdot, t) > 0 \} ,
\]

\[
\Omega_-(t) := \{ x \in \Omega \mid \mu(\cdot, t) < 0 \} ,
\]

define the operators

\[
R(t) : L^2(\Omega, |\mu|((\cdot, t)) \to L^2(\Omega, |\mu|((\cdot, t)) , \quad R(t) := P_+(t) - P_-(t) ,
\]

\[
P_+(t) : L^2(\Omega, |\mu|((\cdot, t)) \to L^2(\Omega_+(t), |\mu|((\cdot, t)) \quad \text{the orthogonal projection ,}
\]

\[
P_-(t) : L^2(\Omega, |\mu|((\cdot, t)) \to L^2(\Omega_-(t), |\mu|((\cdot, t)) \quad \text{the orthogonal projection .}
\]

In this way \( R(t) \) turns out to be bounded for every \( t \) even if \( \mu \) is unbounded and we will need (see Definition (15)) that the following function is absolutely continuous (and differentiable) for every \( u, v \in W^1_0(\Omega) \):

\[
t \mapsto (R(t)u, v)_{H(t)} = \int_{\Omega_+(t)} u(x) v(x) |\tilde{\mu}|(x, t) \, dx - \int_{\Omega_-(t)} u(x) v(x) |\tilde{\mu}|(x, t) \, dx =
\]

\[
= \int_{\Omega} u(x) v(x) \mu(x, t) \, dx .
\]

Then for every \( \varphi \in L^2(\Omega_+(0), \mu_+((\cdot), 0)) \), \( \psi \in L^2(\Omega_-(T), \mu_-((\cdot), T)) \) and \( f \in \mathcal{V} \) the problem

\[
\begin{cases}
\mu(x, t) u_t + \mathcal{A} u = f(x, t) \quad \text{in } \Omega \times (0, T) \\
u(x, t) = 0 \quad \text{in } \partial \Omega \times (0, T) \\
u(x, 0) = \varphi(x) \quad \text{in } \Omega_+(0) \\
u(x, T) = \psi(x) \quad \text{in } \Omega_-(T)
\end{cases}
\]

has a unique solution.
10 - The analogous of example 8 with unbounded coefficient can be considered, then, adapting examples 8 and 9, one can consider

\[
\begin{cases}
\int_\Omega \mu(x,y,t)u_t(y,t)dy + Au(x,t) = f(x,t) & \text{in } \Omega \times (0,T), \\
u = 0 & \text{in } \partial\Omega \times (0,T), \\
u(\cdot,0) = \varphi & \text{in } \hat{H}_+(0), \\
u(\cdot,T) = \psi & \text{in } \hat{H}_-(T).
\end{cases}
\]

where

\[\mu \in L^1(\Omega \times \Omega \times (0,T))\]

and \(A\), for instance, as in (99).

11 - Another example of varying spaces is the following: consider first a function \(q : \Omega \to [1, +\infty)\)

\[
L^{q(\cdot)}(\Omega) := \left\{ u \in L^{1,1}_\text{loc}(\Omega) \mid \int_\Omega |u(x)|^{q(x)} dx < +\infty \right\}
\]

endowed with the norm (see, for instance, [11] for definitions and properties of these spaces)

\[\|u\|_{L^{q(\cdot)}(\Omega)} := \inf \left\{ \lambda > 0 \mid \int_\Omega \left| \frac{u(x)}{\lambda} \right|^{q(x)} dx \leq 1 \right\}.\]

Clearly \(W_0^{1,q(\cdot)}(\Omega)\) is defined as the space

\[W_0^{1,q(\cdot)}(\Omega) := \left\{ u \in W^{1,1}_\text{loc}(\Omega) \mid u \in L^{q(\cdot)}(\Omega) \text{ and } Du \in L^{q(\cdot)}(\Omega) \right\}\]

endowed with the norm \(\|u\|_{W^{1,q(\cdot)}(\Omega)} + \|Du\|_{L^{q(\cdot)}(\Omega)}\).

If now we have a function

\[p : \Omega \times [0,T] \to [2,p_0]\]

for some \(p_0 \geq 2\) we can consider

\[
U = W_0^{1,p(\cdot)}(\Omega), \quad V(t) = W_0^{1,p(\cdot,t)}(\Omega), \quad H(t) = L^2(\Omega),
\]

\[A(t) : V(t) \to V'(t)\]

\[(A(t)u)(x) := -\text{div} a(x,t,Du(x)),\]

with \(a : \Omega \times (0,T) \times \mathbb{R}^n \to \mathbb{R}^n\),

verifying

\[\lambda_o |\xi|^{p(x,t)} \leq a(x,t,\xi) \cdot \xi \leq \Lambda_o |\xi|^{p(x,t)}\]

for every \(\xi \in \mathbb{R}^n\) and for some positive \(\lambda_o, \Lambda_o\). If

\[p : \Omega \times [0,T] \to [2, +\infty)\]

one can simply consider, if \(U\) does not need to be a Banach space,

\[U = C^1_0(\Omega)\]

Then problem (94) has a unique solution for \(r\) like in the examples 1-8.

II - The equation \(Ru'' + Au' + Bu = f\)
First we consider the following situation:

\[ U \equiv V(t) = H^1_0(\Omega) \quad \text{and} \quad H(t) = L^2(\Omega) \quad \text{for every} \ t \in [0,T], \]

\[ A(t) : H^1_0(\Omega) \to H^{-1}(\Omega), \quad B(t) : H^1_0(\Omega) \to H^{-1}(\Omega) \]

\[ (A(t)u)(x) := -\text{div}(a(x,t,Du)) + b(x,t,u), \]

\[ (B(t)u)(x) := -\text{div}(\tilde{a}(x,t) \cdot Du) + \tilde{b}(x,t)u, \]

(100) with \( \tilde{a} : \Omega \times (0,T) \times \mathbb{R}^n \to \mathbb{R}^n, \quad a_{ij}, b \in L^\infty(\Omega \times (0,T)). \)

verifying

\[ \lambda_o \| \xi \|^2 \leq a(x,t,\xi) \cdot \xi \leq \Lambda_o \| \xi \|^2 \]

and

\[ \tilde{\lambda}_o \| \xi \|^2 \leq (\tilde{a}(x,t) \cdot \xi, \xi) \leq \tilde{\Lambda}_o \| \xi \|^2 \]

for every \( \xi \in \mathbb{R}^n \) and for some positive \( \lambda_o, \Lambda_o, \tilde{\lambda}_o, \tilde{\Lambda}_o. \) So the operators \( A \) and \( B \) are (have to be) of the same type and they can be the same, i.e. \( A = B, \) the only difference is that \( A \) can be non-linear, while \( B \) has to be linear. Via \( A \) and \( B \) we define two operators \( A \) and \( B \) as done in (46). Once we have fixed \( t_o \in [0,T], \) we will need (see (45)),

\[ \partial_t (\tilde{a}_{ij}) \in L^\infty(\Omega \times (0,T)) \]

and denoting by \( \frac{\partial a}{\partial t} \) the matrix whose entries are \( \partial_t (\tilde{a}_{ij}) \) we also will need

\[
\begin{align*}
\int_\Omega \frac{\partial \tilde{a}}{\partial t}(x,t) \cdot Du(x), Du(x) \, dx &\geq 0 \quad \text{for} \ t \in [0,t_o], \\
\frac{d}{dt} \int_\Omega \tilde{b}(x,t)(u(x))^2 \, dx &\geq 0 \quad \text{for} \ t \in [0,t_o], \\
\int_\Omega \frac{\partial \tilde{a}}{\partial t}(x,t) \cdot Du(x), Du(x) \, dx &\leq 0 \quad \text{for} \ t \in [t_o,T], \\
\frac{d}{dt} \int_\Omega \tilde{b}(x,t)(u(x))^2 \, dx &\leq 0 \quad \text{for} \ t \in [t_o,T]
\end{align*}
\]

(101) for every \( u \in H^1_0(\Omega) \), so \( b \) can be discontinuous (as can be \( r \) in example 4). In particular this is satisfied if

\[ \tilde{a} = \tilde{a}(x) \quad \text{and} \quad \tilde{b} = \tilde{b}(x). \]

12 - As regards the operator \( R \) we can consider all the examples from 1 to 10 considered before, all except 7 where \( A \) is non-linear. For each \( t_o \in [0,T] \) and every

\[ f \in H^1(0,T; H^{-1}(\Omega)), \ \varphi \in L^2(\Omega^+(0), r_+, 0)), \ \psi \in L^2(\Omega^-(T), r_-, T)), \ \eta \in H^1_0(\Omega) \]

we have that for every

\[ r \quad \text{as in the examples 1-6} \]

the following problem has a unique solution

\[
\begin{cases}
    r(x,t) u_{tt} + A u_t + B u = f(x,t) & \text{in} \ \Omega \times (0,T) \\
    u(x,t) = 0 & \text{in} \ \partial \Omega \times (0,T) \\
    u_t(x,0) = \varphi(x) & \text{in} \ \Omega^+(0) \\
    u_t(x,T) = \psi(x) & \text{in} \ \Omega^-(T) \\
    u(x,t_o) = \eta(x) & \text{in} \ \Omega,
\end{cases}
\]

\[ 0 \leq a_{ij} \leq \Lambda_o, \quad 0 \leq b \leq \tilde{\Lambda}_o. \]
we have that with \( r \) as in the example 8 the following problem has a unique solution
\[
\begin{aligned}
\int_\Omega r(x,t) u_{tt}(x,t) \, dx + A u_t + B u & = f(x,t) \quad \text{in } \Omega \times (0,T) \\
u(x,t) & = 0 \quad \text{in } \partial \Omega \times (0,T) \\
u_t(x,0) = \varphi(x) & \quad \text{in } \Omega_+(0) \\
u_t(x,T) = \psi(x) & \quad \text{in } \Omega_-(T) \\
u(x,t_0) = \eta(x) & \quad \text{in } \Omega.
\end{aligned}
\]

13 - Another admissible situation is the analogous of example 9. Consider \( \mu, \lambda, V(t), H(t), U \) and \( A \) as in the example 9, while \( B \) as in (100) but satisfying
\[
\lambda(x,t) |\xi|^p \leq (\tilde{a}(x,t) \cdot \xi, \xi) \leq L \lambda(x,t) |\xi|^p
\]
and, for simplicity, \( \tilde{b} \equiv 0 \) and moreover (101).

14 - Also example 10 can be adapted in the obvious way to the second order equation.

III - The Tricomi type equations \( R u'' + A u = f \) and \( (R u')' + A u = f \)

First we consider the following situation
\[
U \equiv V(t) = H^1_0(\Omega) \quad \text{and} \quad H(t) = L^2(\Omega) \quad \text{for every } t \in [0,T],
\]
\[
B(t) : H^1_0(\Omega) \to H^{-1}(\Omega)
\]
\[
(B(t)u)(x) := - \text{div} (a(x,t) \cdot Du) + b(x,t)u,
\]
with \( a : \Omega \times (0,T) \times \mathbb{R}^n \to \mathbb{R}^n \), \( a_{ij} \in L^\infty(\Omega \times (0,T)) \),
\[
\text{verifying} \quad \lambda_0 |\xi|^2 \leq (a(x,t) \cdot \xi, \xi) \leq \Lambda_0 |\xi|^2
\]
for every \( \xi \in \mathbb{R}^n \) and for some positive \( \lambda_0, \Lambda_0 \). The function \( b \) can be discontinuous while \( a_{ij} \) has to satisfy (see (60) and (62))
\[
\partial_t a_{ij} \in L^\infty(\Omega \times (0,T))
\]
and, denoting by \( \partial_t \tilde{a} \) the matrix whose entries are \( \partial_t (\tilde{a}_{ij}) \),
\[
\int_\Omega \left( \partial_t \tilde{a}(x,t) \cdot Du(x), Du(x) \right) dx \leq 0 \quad \text{and} \quad \frac{d}{dt} \int_\Omega b(x,t)(u(x))^2 dx \leq 0
\]
for every \( t \in [0,T] \). Clearly a simpler example is to consider
\[
a_{ij} = a_{ij}(x) \quad \text{and} \quad b = b(x).
\]

15 - Consider the data, as supposed in Section 5,
\[
f \in H^1(0,T; H^{-1}(\Omega)), \quad \varphi \in L^2(\Omega), \quad \psi \in L^2(\Omega), \quad \eta \in H^1_0(\Omega).
\]

First of all notice that if \( r \equiv 1 \) we have the wave type equation
\[
\begin{aligned}
u_{tt} + B u & = f(x,t) \quad \text{in } \Omega \times (0,T) \\
u(x,t) & = 0 \quad \text{in } \partial \Omega \times (0,T) \\
u_t(x,0) = \varphi(x) & \quad \text{in } \Omega \\
u(x,0) = \eta(x) & \quad \text{in } \Omega.
\end{aligned}
\]
If \( r \equiv -1 \) we have the elliptic equation (in dimension \( n+1 \)) with Dirichlet type condition in \( \Omega \times \{0\} \) and Neumann type condition \( \Omega \times \{T\} \)

\[
\begin{cases}
-u_{tt} + B u = f(x,t) & \text{in } \Omega \times (0,T) \\
u(x,t) = 0 & \text{in } \partial \Omega \times (0,T) \\
\left|u(x,T) = \psi(x)\right| & \text{in } \Omega \\
u(x,0) = \eta(x) & \text{in } \Omega 
\end{cases}
\]

while we have a family of elliptic equations (in dimension \( n \)) if \( r \equiv 0 \) (notice that \( f \) is continuous with respect to time)

\[
\begin{cases}
B(t) u(t) = f(\cdot, t) & \text{in } \Omega \times (0,T) \\
u(x,t) = 0 & \text{in } \partial \Omega \times (0,T) 
\end{cases}
\]

In the following examples consider, for \( r = r(x,t) \in L^\infty(\Omega \times (0,T)) \), the data, as supposed in Section 5,

\[ f \in H^1(0,T;H^{-1}(\Omega)), \quad \varphi \in L^2(\Omega_+(0),\nu_+(\cdot,0)), \quad \psi \in L^2(\Omega_-(T),\nu_-(\cdot,T)), \quad \eta \in H^1_0(\Omega) \]

(see (92) and below for the definition of these spaces).

16 - If we consider \( r = r(x) \in L^\infty(\Omega) \) then

\[
\begin{cases}
\left|\begin{array}{c}
 r(x)u_{tt} + B u = f(x,t) \\
u(x,t) = 0 \\
u_t(x,0) = \varphi(x) \\
u_t(x,T) = \psi(x) \\
u(x,0) = \eta(x)
\end{array}\right| & \text{in } \Omega \times (0,T) \\
\text{in } \partial \Omega \times (0,T) \\
\text{in } \Omega_+ \\
\text{in } \Omega_- \\
\text{in } \Omega_+ \cup \Omega_-
\end{cases}
\]

(here \( \Omega_+ \) and \( \Omega_- \) do not depend on time) has a unique solution provided that (32) holds (see example 2 for more details). About \( \eta \) we want to stress the we can choose every \( \eta \in H^1_0(\Omega) \). The technical point 2 in Section 5 is needed only to prove uniqueness of the solution, but since after that modification we consider the information only in \( \Omega_+ \cap \Omega_- \), modifying \( \eta \) is not needed to solve the above problem, but is needed if we approximate this problem with the approximating problems (63). To modify \( \eta \) we need to consider the solution \( w \) of the following problem

\[
\begin{cases}
- \text{div} \left( a(x,0) \cdot Dw \right) + b(x,0)w = f(x,0) + \text{div} \left( a(x,0) \cdot D\eta \right) - b(x,0)\eta, \\
w \in H^1_0(\Omega_0).
\end{cases}
\]

and then consider \( \tilde{\eta} = \eta + w \). Clearly \( \tilde{\eta} \equiv \eta \) in \( \Omega_+ \cap \Omega_- \).

Also unbounded \( r \) can be considered: if \( r \) is as \( \mu = \mu(x) \) in example 9 with the analogous modifications for \( B \) we get that (103) has a unique solution.

17 - If \( r = r(t) \) the assumptions are more or less the same, in the sense that if

\[ r \in W^{1,\infty}(0,T) \]
(remember that also unbounded coefficients are admitted) and we consider
\[
\begin{cases}
   r(t)u_{tt} + \mathcal{B} u = f & \text{in } \Omega \times (0, T) \\
   u(x, t) = 0 & \text{in } \partial \Omega \times (0, T) \\
   u_t(x, 0) = \varphi(x) & \text{in } \Omega_+(0) \\
   u_t(x, T) = \psi(x) & \text{in } \Omega_-(T) \\
   u(x, 0) = \eta(x) & \text{in } \Omega_+(0) \cup \Omega_-(T)
\end{cases}
\]

\[
\begin{cases}
   (r(t)u'_t) + \mathcal{B} u = f & \text{in } \Omega \times (0, T) \\
   u(x, t) = 0 & \text{in } \partial \Omega \times (0, T) \\
   u_t(x, 0) = \varphi(x) & \text{in } \Omega_+(0) \\
   u_t(x, T) = \psi(x) & \text{in } \Omega_-(T) \\
   u(x, 0) = \eta(x) & \text{in } \Omega_+(0) \cup \Omega_-(T)
\end{cases}
\]

we need
\[
r'(t) \leq 0 \quad \text{in the first case,} \\
r'(t) \geq 0 \quad \text{in the second case.}
\]

Notice that in the first case, if \( r(0) > 0 \) and \( r(T) < 0 \) we have both initial and final datum for \( u_t \) in the whole \( \Omega \); in the second, if \( r(0) < 0 \) and \( r(T) > 0 \) we have no information for \( u_t \) both at time \( t = 0 \) and time \( t = T \) in the whole \( \Omega \).

18 - The main differences between the two problems
\[
\begin{cases}
   r(t)u_{tt} + \mathcal{B} u = f & \text{in } \Omega \times (0, T) \\
   u(x, t) = 0 & \text{in } \partial \Omega \times (0, T) \\
   u_t(x, 0) = \varphi(x) & \text{in } \Omega_+(0) \\
   u_t(x, T) = \psi(x) & \text{in } \Omega_-(T) \\
   u(x, 0) = \eta(x) & \text{in } \Omega_+(0) \cup \Omega_-(T)
\end{cases}
\]

are seen when \( r = r(x, t) \) because for the first problem we need an additional assumption about \( r \). If for the second problem we only need
\[
r \in L^\infty(\Omega \times (0, T)),
\]

for the first one we need
\[
r, \frac{\partial r}{\partial t} \in L^\infty(\Omega \times (0, T)).
\]

Moreover, due to (59) and (62), in the first case we need
\[
\frac{\partial r}{\partial t} \leq 0 \quad \text{a.e. in } \Omega \times (0, T)
\]

while in the second case we need
\[
\frac{d}{dt} \int_{\Omega} (u(x))^2 r(x, t) \, dx \quad \text{absolutely continuous and}
\]

\[
\int_{\Omega} (u(x))^2 r(x, t) \, dx \geq 0 \quad \text{for every } u \in H_0^1(\Omega).
\]

For more details for possible choice of \( r \) we refer to example 4.

The same remarks can be made about the problems
\[
\begin{cases}
   \mathcal{R} u'' + \mathcal{B} u = f & \text{in } \Omega \times (0, T) \\
   u = 0 & \text{in } \partial \Omega \times (0, T) \\
   u'(-, 0) = \varphi & \text{in } \tilde{H}_+(0) \\
   u'(-, T) = \psi & \text{in } \tilde{H}_-(T) \\
   u(-, 0) = \eta & \text{in } \tilde{H}_+(0) \oplus \tilde{H}_-(T)
\end{cases}
\]

\[
\begin{cases}
   (\mathcal{R} u')' + \mathcal{B} u = f & \text{in } \Omega \times (0, T) \\
   u = 0 & \text{in } \partial \Omega \times (0, T) \\
   u'(-, 0) = \varphi & \text{in } \tilde{H}_+(0) \\
   u'(-, T) = \psi & \text{in } \tilde{H}_-(T) \\
   u(-, 0) = \eta & \text{in } \tilde{H}_+(0) \oplus \tilde{H}_-(T)
\end{cases}
\]
where
\[ \mathcal{R}u(x, t) := \int_{\Omega} r(x, y, t) u(y, t) \, dy, \]
i.e. we will need (104) and (105) in the first case, (106) in the second case.

19 - Finally we want to note that also unbounded coefficient and moving spaces can be considered, adapting examples 9, 10, 11.

References


