

# $L^p$ -ESTIMATES FOR PARABOLIC SYSTEMS WITH UNBOUNDED COEFFICIENTS COUPLED AT ZERO AND FIRST ORDER

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ABSTRACT. We consider a class of nonautonomous parabolic first-order coupled systems in the Lebesgue space  $L^p(\mathbb{R}^d; \mathbb{R}^m)$ , ( $d, m \geq 1$ ) with  $p \in [1, +\infty)$ . Sufficient conditions for the associated evolution operator  $\mathbf{G}(t, s)$  in  $C_b(\mathbb{R}^d; \mathbb{R}^m)$  to extend to a strongly continuous operator in  $L^p(\mathbb{R}^d; \mathbb{R}^m)$  are given. Some  $L^p$ - $L^q$  estimates are also established together with  $L^p$  gradient estimates.

## 1. INTRODUCTION

Second order elliptic and parabolic operators with unbounded coefficients have received a great deal of attention because of their analytical interest as well as their applications to stochastic analysis, both in the autonomous and, more recently, in the nonautonomous case. Due to the applications in Stochastics, much of the work has been done in spaces of continuous and bounded functions and in the  $L^p$ -spaces with respect to *the invariant measure*, in the autonomous, and *evolution systems of measures*, in the nonautonomous case. The existence of a unique classical solution for homogeneous parabolic Cauchy problems associated with operators with unbounded coefficients in spaces of continuous and bounded functions, or equivalently the existence of a *semigroup*  $T(t)$  or an *evolution operator*  $G(t, s)$ , respectively, can be shown under mild assumptions on the growth of the coefficients. Let us refer the reader to [22, 10, 19] and their bibliographies for more information.

On the other hand, the analysis in the  $L^p$  setting with respect to the Lebesgue measure has an independent analytical interest and it turns out to be much more difficult than the analysis in the space of continuous and bounded functions or in  $L^p$ -spaces with respect to the invariant measure (resp. evolution system of measures). Even in the autonomous case, the Cauchy problem may be not well posed in  $L^p(\mathbb{R}^d, dx)$  if the coefficients are unbounded, unless they satisfy very restrictive assumptions. For instance, in the 1-dimensional case very simple operators, such as  $D^2 - |x|^\varepsilon xD$ , with  $\varepsilon > 0$ , do not generate any semigroup in  $L^p(\mathbb{R}, dx)$  and in this situation, the lack of the potential term plays a crucial role, see also [3] for further examples and comments.

Since nowadays many of the results obtained concern the single equations, the aim of this paper is the study of parabolic systems with unbounded coefficients, coupled in the zero and first order terms, in the Lebesgue space  $L^p(\mathbb{R}^d, \mathbb{R}^m)$ . We consider the Cauchy problem

$$\begin{cases} D_t \mathbf{u}(t, x) = (\mathcal{A}(t)\mathbf{u})(t, x), & t > s \in I, \quad x \in \mathbb{R}^d, \\ \mathbf{u}(s, x) = \mathbf{f}(x), & x \in \mathbb{R}^d \end{cases} \quad (1.1)$$

where  $I$  is an open right-halfline or the whole  $\mathbb{R}$  and the elliptic operators

$$\mathcal{A}\mathbf{v} = \sum_{i,j=1}^d D_i(q_{ij}D_j\mathbf{v}) + \sum_{i=1}^d B_i D_i\mathbf{v} + C\mathbf{v} \quad (1.2)$$

have unbounded coefficients  $q_{ij} : I \times \mathbb{R}^d \rightarrow \mathbb{R}$  and  $B_i, C : I \times \mathbb{R}^d \rightarrow \mathbb{R}^{m^2}$  ( $m \geq 1$ ).

Second order elliptic and parabolic systems have been already studied in the simplest case of *zero order coupling*, i.e., when  $B_i = b_i I_m$  (see [15, 13]). The more general frame of *first order coupling*, i.e., uncoupled diffusion and coupled drift and potential, has been very recently studied in the space of continuous and bounded functions in [2], where the existence of an *evolution operator*  $\mathbf{G}(t, s)$  associated with  $\mathcal{A}(t)$  in  $C_b(\mathbb{R}^d; \mathbb{R}^m)$  has been shown.

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Here, we take advantage of such construction and of a pointwise estimate shown in [2] to start our investigation on the properties of  $\mathbf{G}(t, s)$  in the  $L^p$  context. We refer to [20, 11] for the abstract theory of evolution operators.

We assume that the coefficients are regular enough, namely locally  $C^{\alpha/2, \alpha}$ , for some  $\alpha \in (0, 1)$ , together with the first order spatial derivatives of  $q_{ij}$  and of the entries of  $B_i$ , for any  $i, j = 1, \dots, d$ , and that the matrix  $Q(t, x) = [q_{ij}(t, x)]_{i,j=1,\dots,d}$  is uniformly positive definite, see Hypotheses 2.1.

The  $L^p$  analysis is carried out under two different sets of assumptions, Hypotheses 2.2 and 2.3, which we compare in Remark 2.5. The two approaches give slightly different results. Indeed, under Hypotheses 2.2 we deal directly with the vectorial problem. Using the pointwise estimate proved in [2] (and recalled in the Appendix), an interpolation argument and requiring a balance between the growth of the potential matrix  $C$  and the derivative of the drift matrices  $B_i$  ( $i = 1, \dots, d$ ), we prove that the evolution operator  $\mathbf{G}(t, s)$  extends to a bounded and strongly continuous operator in  $L^p(\mathbb{R}^d; \mathbb{R}^m)$  for any  $p \in [1, +\infty)$ .

On the other hand, when Hypotheses 2.3 are satisfied, we estimate  $|\mathbf{G}(t, s)\mathbf{f}|^p$  in terms of  $G(t, s)|\mathbf{f}|^p$  for any  $t > s \in I$ ,  $p \in [p_0, +\infty)$  and some  $p_0 > 1$ . Here,  $G(t, s)$  is the evolution operator which governs an auxiliary scalar problem. As a consequence of this comparison result, the boundedness of  $\mathbf{G}(t, s)$  in  $\mathcal{L}(L^p(\mathbb{R}^d; \mathbb{R}^m))$  for  $p \in [p_0, +\infty)$  can be obtained as a byproduct of the boundedness of  $G(t, s)$  in  $\mathcal{L}(L^1(\mathbb{R}^d))$ . Sufficient conditions in order that  $G(t, s)$  is bounded in  $L^p$  for any  $p \in [1, +\infty)$  can be found in [7]. Notice however that slightly strengthening Hypothesis 2.3(ii) we can deal with the whole scale of  $1 < p < \infty$  rather than  $p \geq p_0$ , see Remark 2.7.

Going further, we find conditions for the hypercontractivity of  $\mathbf{G}(t, s)$ . More precisely, under suitable assumptions, we prove that

$$\|\mathbf{G}(t, s)\mathbf{f}\|_{L^q(\mathbb{R}^d; \mathbb{R}^m)} \leq c\|\mathbf{f}\|_{L^p(\mathbb{R}^d; \mathbb{R}^m)}, \quad (1.3)$$

for any  $t \in (s, T]$ ,  $T > s \in I$ ,  $\mathbf{f} \in L^p(\mathbb{R}^d; \mathbb{R}^m)$ ,  $q \geq p$  and some positive constant  $c$  depending on  $p, q, s$  and  $T$ . Actually, whenever Hypotheses 2.2 are satisfied, under the same assumptions which guarantee that  $L^p(\mathbb{R}^d; \mathbb{R}^m)$  is preserved by the action of  $\mathbf{G}(t, s)$ , we prove (1.3) for any  $2 \leq p \leq q$ . Then, arguing by duality we establish (1.3) also when  $1 \leq p \leq q \leq 2$ . Applying this hypercontractivity result to the scalar evolution operator  $G(t, s)$  and using the pointwise estimate of  $|\mathbf{G}(t, s)\mathbf{f}|^p$  in terms of  $G(t, s)|\mathbf{f}|^p$ , we provide conditions for (1.3) to hold for  $p_0 \leq p \leq q$ , when Hypotheses 2.3 are satisfied.

The hypercontractivity estimate (1.3), in this generality, seems to be new also in the autonomous scalar case. Some  $L^p$ - $L^q$  estimates have been recently proved in [16] for a special class of homogeneous operators with unbounded diffusion.

Next, we prove some pointwise estimates for the spatial derivatives of  $\mathbf{G}(t, s)\mathbf{f}$ . Under additional assumptions, which are essentially growth conditions on the coefficients of the operator  $\mathcal{A}(t)$  and their derivatives, we show that there exist positive constants  $c_1, c_2$  such that

$$|D_x \mathbf{G}(t, s)\mathbf{f}|^p \leq c_1 G(t, s)(|\mathbf{f}|^p + |D\mathbf{f}|^p) \quad (1.4)$$

and, under more restrictive conditions, that

$$|D_x \mathbf{G}(t, s)\mathbf{f}|^p \leq c_2 (t - s)^{-\frac{p}{2}} G(t, s)|\mathbf{f}|^p, \quad (1.5)$$

for any  $t \in (s, T]$ ,  $T > s \in I$ ,  $\mathbf{f} \in C_c^1(\mathbb{R}^d; \mathbb{R}^m)$  and  $p \in [p_1, +\infty)$  for some  $p_1 > 1$ .

Now, if the scalar evolution operator  $G(t, s)$  preserves  $L^1(\mathbb{R}^d)$ , estimates (1.4) and (1.5) yield that the evolution operator  $\mathbf{G}(t, s)$  belongs to  $\mathcal{L}(W^{1,p}(\mathbb{R}^d; \mathbb{R}^m))$  and to  $\mathcal{L}(L^p(\mathbb{R}^d; \mathbb{R}^m), W^{1,p}(\mathbb{R}^d; \mathbb{R}^m))$ , respectively. As a consequence of this fact, we show that  $\mathbf{G}(t, s)$  is bounded from  $W^{\theta_1, p}(\mathbb{R}^d; \mathbb{R}^m)$  into  $W^{\theta_2, p}(\mathbb{R}^d; \mathbb{R}^m)$  for any  $0 \leq \theta_1 \leq \theta_2 \leq 1$  and any  $p \geq p_1$ .

We believe that estimates (1.4) and (1.5) could represent a helpful tool to study the evolution operator  $\mathbf{G}(t, s)$  in  $L^p$ -spaces with respect to a natural extension to the vector case of evolution systems of measures, whose definition and analysis is deferred to a future paper. Indeed, already in the scalar case, (see [4, 5]), pointwise gradient estimates have been a key tool to study the asymptotic behaviour of the evolution operator associated with the problem and in establishing some summability improving results for such operator in the  $L^p$  spaces with respect the tight time dependent family of invariant measures.

The last section of the paper is devoted to exhibit some classes of operators which satisfy our assumptions.

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**Notations.** Functions with values in  $\mathbb{R}^m$  are displayed in bold style. Given a function  $\mathbf{f}$  (resp. a sequence  $(\mathbf{f}_n)$ ) as above, we denote by  $f_i$  (resp.  $f_{n,i}$ ) its  $i$ -th component (resp. the  $i$ -th component of the function  $\mathbf{f}_n$ ). By  $B_b(\mathbb{R}^d; \mathbb{R}^m)$  we denote the set of all the bounded Borel measurable functions  $\mathbf{f} : \mathbb{R}^d \rightarrow \mathbb{R}^m$ . For any  $k \geq 0$ ,  $C_b^k(\mathbb{R}^d; \mathbb{R}^m)$  is the space of all the functions whose components belong to  $C_b^k(\mathbb{R}^d)$ , where the notation  $C^k(\mathbb{R}^d)$  ( $k \geq 0$ ) is standard and we use the subscript “ $c$ ” and “ $b$ ” for spaces of functions with compact support and bounded, respectively. Similarly, when  $k \in (0, 1)$ , we use the subscript “loc” to denote the space of all  $f \in C(\mathbb{R}^d)$  which are Hölder continuous in any compact set of  $\mathbb{R}^d$ . We assume that the reader is familiar also with the parabolic spaces  $C^{\alpha/2, \alpha}(I \times \mathbb{R}^d)$  ( $\alpha \in (0, 1)$ ) and  $C^{1,2}(I \times \mathbb{R}^d)$ , and we use the subscript “loc” with the same meaning as above.

The Euclidean inner product of the vectors  $x, y \in \mathbb{R}^d$  is denoted by  $\langle x, y \rangle$ . For any square matrix  $M$ , we denote by  $M_{ij}$ ,  $\text{Tr}(M)$  and  $M^*$  the  $ij$ -th element of the matrix  $M$ , the trace of  $M$  and the matrix transposed to  $M$ , respectively. Finally,  $\lambda_M$  and  $\Lambda_M$  denote the minimum and the maximum eigenvalue of the (symmetric) matrix  $M$ . For any  $k \in \mathbb{N}$ , by  $I_k$  we denote the identity matrix of size  $k$ . Square matrices of size  $m$  are thought as elements of  $\mathbb{R}^{m^2}$ .

By  $\chi_A$ ,  $\mathbb{1}$  and  $\mathbf{e}_j$  we denote the characteristic function of the set  $A \subset \mathbb{R}^d$ , the function which is identically equal to 1 in  $\mathbb{R}^d$  and the  $j$ -th vector of the Euclidean basis of  $\mathbb{R}^m$ . Finally, the Euclidean open ball with centre  $x_0$  and radius  $R > 0$  and its closure are denoted by  $B_R(x_0)$  and  $\overline{B}_R(x_0)$ ; when  $x_0 = 0$  we simply write  $B_R$  and  $\overline{B}_R$ .

For any interval  $J \subset \mathbb{R}$  we denote by  $\Sigma_J$  the set  $\{(t, s) \in J \times J : t > s\}$ .

## 2. PRELIMINARY RESULTS

Let  $I$  be an open right-halfline (possibly  $I = \mathbb{R}$ ) and  $\{\mathcal{A}(t)\}_{t \in I}$  be the family of second order uniformly elliptic operators defined in (1.2). In this paper we study the Cauchy problem (1.1) when  $\mathbf{f} \in L^p(\mathbb{R}^d; \mathbb{R}^m)$  and  $s \in I$ , under the following standing assumptions.

**Hypotheses 2.1.** (i) *The matrices  $Q = [q_{ij}]_{i,j=1,\dots,d}$ ,  $B_i$  ( $i = 1, \dots, d$ ) and  $C$  are symmetric. Further,  $q_{ij}, (B_i)_{lk} \in C_{\text{loc}}^{\alpha/2, 1+\alpha}(I \times \mathbb{R}^d)$  and  $C_{lk} \in C_{\text{loc}}^{\alpha/2, \alpha}(I \times \mathbb{R}^d)$  for any  $i, j = 1, \dots, d$  and  $l, k = 1, \dots, m$ ;*  
(ii) *the matrix  $Q$  is uniformly elliptic, i.e.,  $\nu_0 := \inf_{I \times \mathbb{R}^d} \lambda_Q(t, x) > 0$  where*

$$\lambda_Q(t, x) := \min\{\langle Q(t, x)\xi, \xi \rangle : \xi \in \mathbb{R}^d, |\xi| = 1\}, \quad t \in I, x \in \mathbb{R}^d$$

*is the minimum eigenvalue of  $Q(t, x)$ .*

Besides Hypotheses 2.1 we consider one of the following two sets of assumptions.

**Hypotheses 2.2.** (i) *The function  $\mathcal{K}_\eta : I \times \mathbb{R}^d \rightarrow \mathbb{R}$ , defined by*

$$\mathcal{K}_\eta = \sum_{i,j=1}^d (Q^{-1})_{ij} [\langle B_i \eta, \eta \rangle \langle B_j \eta, \eta \rangle - \langle B_i \eta, B_j \eta \rangle] - 4 \langle C \eta, \eta \rangle, \quad (2.1)$$

*is nonnegative in  $I \times \mathbb{R}^d$ , for any  $\eta \in \partial B_1$ ;*

(ii) *for any bounded interval  $J \subset I$  there exist a constant  $\lambda_J$  and a positive (Lyapunov) function  $\varphi_J \in C^2(\mathbb{R}^d)$ , blowing up as  $|x| \rightarrow +\infty$ , such that*

$$\sup_{\eta \in \partial B_1} \sup_{(t,x) \in J \times \mathbb{R}^d} (\mathcal{A}_\eta(t)\varphi_J)(x) - \lambda_J \varphi_J(x) < +\infty,$$

*where*

$$\mathcal{A}_\eta = \text{div}(QD_x) + \langle b_\eta, D_x \rangle, \quad (b_\eta)_i = \langle B_i \eta, \eta \rangle. \quad (2.2)$$

Condition 2.2(i) is already used by [21] in the case of bounded coefficients.

**Hypotheses 2.3.** (i) *There exist functions  $b_i : I \times \mathbb{R}^d \rightarrow \mathbb{R}$  and  $\tilde{B}_i : I \times \mathbb{R}^d \rightarrow \mathbb{R}^{m^2}$  such that  $B_i := b_i I_m + \tilde{B}_i$ , for any  $i = 1, \dots, d$ ,  $\sigma > 0$ , and a function  $\xi : I \rightarrow (0, +\infty)$  such that*

$$|(\tilde{B}_i)_{jk}(t, x)| \leq \xi(t) \lambda_Q^\sigma(t, x), \quad (t, x) \in I \times \mathbb{R}^d,$$

*for any  $j, k = 1, \dots, m$  and  $i = 1, \dots, d$ ;*

(ii) *for any bounded interval  $J \subset I$  there exists  $\beta \geq 1/4$  such that*

$$H_{\beta, J} := \sup_{J \times \mathbb{R}^d} (\Lambda_C + \beta d m^2 \xi^2 \lambda_Q^{2\sigma-1}) < +\infty; \quad (2.3)$$

(iii) *for any bounded interval  $J \subset I$  there exist  $\lambda_J > 0$  and a positive function  $\varphi_J \in C^2(\mathbb{R}^d)$  blowing up as  $|x| \rightarrow +\infty$  such that  $\sup_{J \times \mathbb{R}^d} (\mathcal{A}\varphi_J - \lambda_J \varphi_J) < +\infty$ , where*

$$\mathcal{A} = \operatorname{div}(QD_x) + \langle b, D_x \rangle, \quad b = (b_1, \dots, b_m). \quad (2.4)$$

**Remark 2.4.** Hypothesis 2.2(i) can be replaced with the weaker condition

$$\inf_{\eta \in \partial B_1} \inf_{J \times \mathbb{R}^d} \mathcal{K}_\eta > -\infty \quad (2.5)$$

for any bounded interval  $J \subset I$ . Indeed, in this latter case, for any bounded interval  $J \subset I$  there exists a positive constant  $c_J$  such that  $\mathcal{K}_\eta \geq -c_J$  in  $J \times \mathbb{R}^d$  for any  $\eta \in \partial B_1$ . Let us notice that  $\mathbf{u}$  is a classical solution of the Cauchy problem (1.1) if and only if the function  $\mathbf{v}$ , defined by  $\mathbf{v}(t, x) := e^{-c_J(t-s)/4} \mathbf{u}(t, x)$  for any  $(t, x) \in (s, +\infty) \times \mathbb{R}^d$ , is a classical solution of the problem

$$\begin{cases} D_t \mathbf{v}(t, x) = \left( \mathcal{A}(t) - \frac{c_J}{4} \right) \mathbf{v}(t, x), & (t, x) \in (s, +\infty) \times \mathbb{R}^d \\ \mathbf{v}(s, x) = \mathbf{f}(x), & x \in \mathbb{R}^d. \end{cases} \quad (2.6)$$

The elliptic operator in problem (2.6) satisfies Hypothesis 2.2(i) and, clearly, the uniqueness of  $\mathbf{v}$  is equivalent to the uniqueness of  $\mathbf{u}$ .

**Remark 2.5.** A comparison between Hypotheses 2.2 and 2.3 is in order. First of all, notice that writing the matrices  $B_i$  as in 2.3(i) the function  $\mathcal{K}_\eta$  depends only upon  $\tilde{B}_i$ , because the diagonal part cancels. The two sets of hypotheses are independent in general: 2.3(i) and (ii) imply 2.2(i), whereas 2.2(ii) is stronger than 2.3(iii). Indeed, assuming 2.3(i) it is easily seen that

$$\sum_{i,j=1}^d (Q^{-1})_{ij} [\langle B_i \eta, \eta \rangle \langle B_j \eta, \eta \rangle - \langle B_i \eta, B_j \eta \rangle]$$

is negative and of order  $\lambda_Q^{2\sigma-1}$ . This fact together with 2.3(ii) implies 2.2(i) (taking Remark 2.4 into account). On the other hand, assuming 2.3(i), the function  $\mathcal{K}_\eta$  can be of order less than  $\lambda_Q^{1-2\sigma}$ . For instance, assume  $d = m = 2$ ,  $Q = \operatorname{diag}(\lambda_Q, \Lambda_Q)$ ,  $B_1 = b_1 I_2$  diagonal and  $\tilde{B}_2 \neq 0$ . Then, we have

$$\mathcal{K}_\eta = \Lambda_Q^{-1} (\langle \tilde{B}_2 \eta, \eta \rangle^2 - |\tilde{B}_2 \eta|^2) - 4 \langle C \eta, \eta \rangle \geq 0 \quad \text{if} \quad \Lambda_C + 2 \xi^2 \lambda_Q^{2\sigma} \Lambda_Q^{-1} < +\infty,$$

which is weaker than (2.3) if  $\lambda_Q = o(\Lambda_Q)$ .

Concerning 2.2(ii) and 2.3(iii), the latter requires the existence of a Lyapunov function for *one* decomposition of each drift matrix, while the former requires the existence of a Lyapunov function for *any* decomposition  $B_i = b_\eta I_m + \tilde{B}_{\eta, i}$ ,  $\eta \in \partial B_1$ .

We start by recalling some known results used in the sequel and proved in [2]. The evolution operator on  $C_b(\mathbb{R}^d; \mathbb{R}^m)$  which gives a solution of problem (1.1) is obtained as the limit of the sequence of the evolution operators related to the following Cauchy-Dirichlet problem in  $I \times B_n$ :

$$\begin{cases} D_t \mathbf{u}_n(t, x) = (\mathcal{A}(t) \mathbf{u}_n)(t, x), & t > s, x \in B_n, \\ \mathbf{u}_n(t, x) = \mathbf{0}, & t > s, x \in \partial B_n, \\ \mathbf{u}_n(s, x) = \mathbf{f}(x), & x \in \overline{B_n}. \end{cases} \quad (2.7)$$

We shall also be concerned with the Cauchy-Neumann problem in  $I \times B_n$ :

$$\begin{cases} D_t \mathbf{u}_n(t, x) = (\mathcal{A}(t) \mathbf{u}_n)(t, x), & t > s, x \in B_n, \\ \frac{\partial \mathbf{u}_n}{\partial \nu}(t, x) = \mathbf{0}, & t > s, x \in \partial B_n, \\ \mathbf{u}_n(s, x) = \mathbf{f}(x), & x \in \overline{B_n}, \end{cases} \quad (2.8)$$

where  $\nu$  denotes the unit exterior normal vector to  $\partial B_n$ . Throughout the paper, we denote by  $\mathbf{G}_n^{\mathcal{D}}(t, s)$  and  $\mathbf{G}_n^{\mathcal{N}}(t, s)$  the Dirichlet and Neumann evolution operators associated with problems (2.7), (2.8) in  $C_b(B_n; \mathbb{R}^m)$ .

**Proposition 2.6.** *Under Hypotheses 2.1 and 2.2 (resp. 2.3), for any  $\mathbf{f} \in C_b(\mathbb{R}^d; \mathbb{R}^m)$ , problem (1.1) admits a unique classical solution  $\mathbf{u}$  which is bounded in the strip  $[s, T] \times \mathbb{R}^d$  for any  $T > s \in I$ . Setting  $\mathbf{G}(t, s)\mathbf{f} := \mathbf{u}(t, \cdot)$  for any  $t > s$  and  $\mathbf{f} \in C_b(\mathbb{R}^d; \mathbb{R}^m)$ ,  $\mathbf{G}(t, s)$  is a bounded linear operator in  $C_b(\mathbb{R}^d; \mathbb{R}^m)$  and*

$$\|\mathbf{G}(t, s)\mathbf{f}\|_{\infty} \leq \gamma(t-s)\|\mathbf{f}\|_{\infty}, \quad t \in (s, T), \quad (2.9)$$

where  $\gamma(r) = 1$  (resp.<sup>1</sup>  $\gamma(r) = e^{H_{1/4, [s, T]} r}$ ) for any  $r > 0$ . Moreover, for any  $s \in I$  and  $\mathbf{f} \in C_b(\mathbb{R}^d; \mathbb{R}^m)$ , both  $\mathbf{G}_n^{\mathcal{N}}(\cdot, s)\mathbf{f}$  and  $\mathbf{G}_n^{\mathcal{D}}(\cdot, s)\mathbf{f}$  converge to  $\mathbf{G}(\cdot, s)\mathbf{f}$  in  $C_{loc}^{1,2}((s, +\infty) \times \mathbb{R}^d; \mathbb{R}^m)$ .

The uniqueness of the solution of the problem (1.1) shows that the family  $\{\mathbf{G}(t, s)\}_{t \geq s \in I}$  is an evolution operator in  $C_b(\mathbb{R}^d; \mathbb{R}^m)$ .

**Remark 2.7.** Notice that working in  $L^p$  is allowed provided that Hypothesis 2.3(ii) holds for some  $\beta \geq [4(p-1)]^{-1}$ , as we shall see in the proof of Proposition 2.8 below. We are supposing  $\beta \geq 1/4$  in order to encompass the case  $p = 2$ : indeed, estimate (2.9) has been obtained as consequence of a pointwise estimate for  $|\mathbf{u}|^2$  in terms of the solution of a suitable scalar problem.

Moreover, we point out that if (2.3) holds with  $\lambda_Q^\alpha$  in place of  $\lambda_Q^{2\sigma-1}$  for some  $\alpha < 2\sigma - 1$ , then every  $\beta > 0$  is allowed and we can extend our results to the whole scale of  $p > 1$ . We shall not mention this extension anymore.

Since in this paper we are interested in studying the evolution operator  $\mathbf{G}(t, s)$  in the  $L^p(\mathbb{R}^d; \mathbb{R}^m)$  setting under Hypotheses 2.3, we extend the just mentioned pointwise estimate to  $|\mathbf{u}|^p$  for any  $p \in [1 + \frac{1}{4\beta}, +\infty)$ .

**Proposition 2.8.** *Assume that Hypotheses 2.3 hold true; then, for every bounded interval  $J \subset I$  and  $p \geq 1 + \frac{1}{4\beta}$ , there exists a positive constant  $K_J$  such that*

$$|(\mathbf{G}(t, s)\mathbf{f})(x)|^p \leq e^{pK_J(t-s)}(G(t, s)|\mathbf{f}|^p)(x), \quad (2.10)$$

for any  $(t, s) \in \Sigma_J$ ,  $x \in \mathbb{R}^d$  and  $\mathbf{f} \in C_b(\mathbb{R}^d; \mathbb{R}^m)$ , where  $G(t, s)$  denotes the evolution operator in  $C_b(\mathbb{R}^d)$  associated with the operator  $\mathcal{A}$  defined in (2.4). Here,  $K_J = H_{1/4, J}$  if  $p \geq 2$  whereas  $K_J = H_{\beta, J}$  if  $p \in [1 + \frac{1}{4\beta}, 2)$ .

*Proof.* Estimate (2.10) has been already proved when  $p = 2$  in [2, Prop. 2.8] with  $K_J = H_{1/4, J}$ ; for a general  $p$ , its proof is similar, so that we limit ourselves to sketch it. Moreover, it suffices to prove (2.10) only for  $p \in [1 + \frac{1}{4\beta}, 2)$ . Indeed, if  $p > 2$ , the integral representation formula of  $G(t, s)|\mathbf{f}|^2$  in terms of the transition kernels associated with  $\mathcal{A}$  in  $C_b(\mathbb{R}^d)$  (see [17, Prop. 2.4]) and the Jensen inequality yield

$$|\mathbf{G}(t, s)\mathbf{f}|^p \leq (e^{2H_{1/4, J}(t-s)}G(t, s)|\mathbf{f}|^2)^{p/2} \leq e^{pH_{1/4, J}(t-s)}G(t, s)|\mathbf{f}|^p$$

for any  $(t, s) \in \Sigma_J$ . Hence, (2.10) follows.

Now, let  $J \subset I$  be a bounded interval. Fix  $p \in [1 + \frac{1}{4\beta}, 2]$ ,  $\varepsilon > 0$ , and, for brevity, let  $H = H_{\beta, J}$  be as in Hypotheses 2.3(ii) and  $\mathbf{u} = \mathbf{G}(\cdot, s)\mathbf{f}$ . We set  $w_\varepsilon = (|\mathbf{u}|^2 + \varepsilon)^{p/2}$  and

$$u_\varepsilon(t, \cdot) = e^{-pH(t-s)}w_\varepsilon(t, \cdot) - G(t, s)(|\mathbf{f}|^2 + \varepsilon)^{p/2}, \quad t > s \in I.$$

The function  $u_\varepsilon$  belongs to  $C^{1,2}((s, +\infty) \times \mathbb{R}^d) \cap C_b([s, +\infty) \times \mathbb{R}^d)$  and verifies

$$D_t u_\varepsilon - \mathcal{A}u_\varepsilon = pe^{-pH(t-s)}w_\varepsilon^{1-2/p} \left[ \sum_{i=1}^d \langle \mathbf{u}, \tilde{B}_i D_i \mathbf{u} \rangle + \langle \mathbf{u}, C\mathbf{u} \rangle - \sum_{i,j=1}^d q_{ij} \langle D_i \mathbf{u}, D_j \mathbf{u} \rangle \right]$$

<sup>1</sup>Here  $H_{1/4, [s, T]}$  is the constant in (2.3).

$$+ (2-p)(|\mathbf{u}|^2 + \varepsilon)^{-1} \sum_{i,j=1}^d q_{ij} \langle \mathbf{u}, D_i \mathbf{u} \rangle \langle \mathbf{u}, D_j \mathbf{u} \rangle - H(|\mathbf{u}|^2 + \varepsilon) \Big]$$

in  $(s, \infty) \times \mathbb{R}^d$ . Since

$$\begin{aligned} \sum_{i,j=1}^d q_{ij} \langle \mathbf{u}, D_i \mathbf{u} \rangle \langle \mathbf{u}, D_j \mathbf{u} \rangle &\leq \sum_{h,k=1}^m |u_h| |u_k| | \langle Q D_x u_h, D_x u_k \rangle | \leq \sum_{h,k=1}^m |u_h| |u_k| |Q^{1/2} D_x u_h| |Q^{1/2} D_x u_k| \\ &= \left( \sum_{h=1}^m |u_h| |Q^{1/2} D_x u_h| \right)^2 \\ &\leq \left( \sum_{h=1}^m |u_h|^2 \right) \left( \sum_{h=1}^m |Q^{1/2} D_x u_h|^2 \right) = |\mathbf{u}|^2 \sum_{i,j=1}^d q_{ij} \langle D_i \mathbf{u}, D_j \mathbf{u} \rangle, \end{aligned} \quad (2.11)$$

by the assumptions it follows that

$$D_t u_\varepsilon - \mathcal{A}(t) u_\varepsilon \leq p e^{-pH(t-s)} w_\varepsilon^{1-\frac{2}{p}} \left[ \sum_{i=1}^d \langle \mathbf{u}, \tilde{B}_i D_i \mathbf{u} \rangle + (1-p) \lambda_Q |D_x \mathbf{u}|^2 + (\Lambda_C - H) |\mathbf{u}|^2 \right] \quad (2.12)$$

in  $(s, \infty) \times \mathbb{R}^d$ . The Young and the Cauchy-Schwarz inequalities and Hypotheses 2.3(i) show that

$$\begin{aligned} \sum_{i=1}^d \langle \mathbf{u}, \tilde{B}_i D_i \mathbf{u} \rangle + (1-p) \lambda_Q |D_x \mathbf{u}|^2 &\leq m \xi \lambda_Q^\sigma |\mathbf{u}| \sum_{i=1}^d |D_i \mathbf{u}| + (1-p) \lambda_Q |D_x \mathbf{u}|^2 \\ &\leq (adm^2 \xi^2 + 1-p) \lambda_Q |D_x \mathbf{u}|^2 + \frac{\lambda_Q^{2\sigma-1}}{4a} |\mathbf{u}|^2 \end{aligned} \quad (2.13)$$

in  $J \times \mathbb{R}^d$  where and  $a = a(t)$  is an arbitrary positive function. Putting together (2.12), (2.13) and choosing  $a = (p-1)(dm^2 \xi^2)^{-1}$  yield that

$$D_t u_\varepsilon - \mathcal{A} u_\varepsilon \leq p e^{-pH(t-s)} w_\varepsilon^{1-2/p} \left[ \frac{dm^2 \xi^2}{4(p-1)} \lambda_Q^{2\sigma-1} + \Lambda_C - H \right] |\mathbf{u}|^2 \leq 0$$

in  $((s, \infty) \cap J) \times \mathbb{R}^d$ . The maximum principle in [17, Prop. 2.1] yields that  $u_\varepsilon \leq 0$  in  $((s, \infty) \cap J) \times \mathbb{R}^d$ , i.e.,

$$(|\mathbf{u}(t, \cdot)|^2 + \varepsilon)^{p/2} \leq e^{pH(t-s)} G(t, s) (|\mathbf{f}|^2 + \varepsilon)^{p/2}, \quad (t, s) \in \Sigma_J.$$

Letting  $\varepsilon \rightarrow 0^+$  we get (2.10) with  $K_J = H_{\beta, J}$ .  $\square$

### 3. THE EVOLUTION OPERATOR $\mathbf{G}(t, s)$ IN $L^p(\mathbb{R}^d; \mathbb{R}^m)$

As it has been already stressed in the introduction, even in the autonomous scalar case, the Cauchy problem (1.1) is not well posed in  $L^p(\mathbb{R}^d, dx)$  if the coefficients of  $\mathcal{A}$  are unbounded, unless they satisfy suitable assumptions.

Actually, in some cases the Lebesgue space  $L^p(\mathbb{R}^d, dx)$  is not preserved by the action of the evolution operator associated with  $\mathcal{A}$ . For example, the compactness in  $C_b(\mathbb{R}^d)$  implies that  $L^p(\mathbb{R}^d, dx)$  is not preserved (see e.g. [23, 7]) by the action of the evolution operator. Here, we are interested in studying properties of the evolution operators  $\mathbf{G}(t, s)$  in  $L^p(\mathbb{R}^d; \mathbb{R}^m)$  when this space is preserved by its action and when an estimate like

$$\|\mathbf{G}(t, s) \mathbf{f}\|_{L^p(\mathbb{R}^d; \mathbb{R}^m)} \leq c_p(t-s) \|\mathbf{f}\|_{L^p(\mathbb{R}^d; \mathbb{R}^m)} \quad (3.1)$$

holds true for some function  $c_p : [0, +\infty) \rightarrow (0, +\infty)$ .

In what follows we consider alternatively Hypotheses 2.2 and 2.3, under additional assumptions. See also Remark 2.7 in connection to Theorem 3.4 and Proposition 3.6.

We begin by considering the case when Hypotheses 2.2 are satisfied. Here, in order to use a duality argument we introduce the following conditions.

**Hypotheses 3.1.** *There exists a function  $\kappa : I \times \mathbb{R}^d \rightarrow \mathbb{R}$ , bounded from above by a constant  $\kappa_0$ , such that*

(i) the function  $\tilde{\mathcal{K}}_\eta : I \times \mathbb{R}^d \rightarrow \mathbb{R}$  defined by

$$\tilde{\mathcal{K}}_\eta = \mathcal{K}_\eta + 4 \sum_{k=1}^d \langle D_k B_k \eta, \eta \rangle + 4\kappa,$$

where  $\mathcal{K}_\eta$  is defined in (2.1), is nonnegative in  $I \times \mathbb{R}^d$  for any  $\eta \in \partial B_1$ ;

(ii) for any bounded interval  $J \subset I$  there exist a constant  $\lambda_J$  and a positive function  $\varphi_J \in C^2(\mathbb{R}^d)$ , blowing up as  $|x| \rightarrow +\infty$ , such that

$$\sup_{\eta \in \partial B_1} \sup_{(t,x) \in J \times \mathbb{R}^d} \left( (\tilde{\mathcal{A}}_\eta(t) \varphi_J)(x) - \lambda_J \varphi_J(x) \right) < +\infty,$$

where

$$\tilde{\mathcal{A}}_\eta = \operatorname{div}(Q D_x) - \langle b_\eta, D_x \rangle + 2\kappa$$

and  $b_\eta$  is defined in (2.2).

**Remark 3.2.** The same arguments as in Remark 2.4 show that the condition  $\tilde{\mathcal{K}}_\eta \geq 0$  in  $J \times \mathbb{R}^d$  can be replaced with the weaker condition  $\inf_{\eta \in \partial B_1} \inf_{J \times \mathbb{R}^d} \tilde{\mathcal{K}}_\eta > -\infty$  for any bounded interval  $J \subset I$ .

**Theorem 3.3.** Assume that Hypotheses 2.2 hold true. If for some interval  $J \subset I$  there exists a positive constant  $L_J$  such that

$$\Lambda_{2C - \sum_{i=1}^d D_i B_i}(t, x) \leq L_J, \quad (t, x) \in J \times \mathbb{R}^d, \quad (3.2)$$

then estimate (3.1) is satisfied for any  $(t, s) \in \Sigma_J$ ,  $\mathbf{f} \in C_c(\mathbb{R}^d; \mathbb{R}^m)$  and  $p \in [2, +\infty)$  with  $c_p(r) = e^{rL_J/p}$ . In addition, if Hypotheses 3.1 are satisfied, then estimate (3.1) holds also for  $p \in [1, 2)$  with  $c_p(r) = e^{r(L_J + \kappa_0(p' - 2))/p'}$ ,  $r \geq 0$  and  $p' = p/(p - 1)$ .

*Proof.* Let us fix  $s \in J$ ,  $\mathbf{f} \in C_c(\mathbb{R}^d; \mathbb{R}^m)$  and for any  $n \in \mathbb{N}$  consider the classical solution  $\mathbf{u}_n := \mathbf{G}_n(\cdot, s)\mathbf{f} = \mathbf{G}_n^{\mathcal{D}}(\cdot, s)\mathbf{f}$  of the Cauchy-Dirichlet problem (2.7). From Proposition 2.6,  $\mathbf{G}_n(\cdot, s)\mathbf{f}$  converges pointwise to  $\mathbf{G}(\cdot, s)\mathbf{f}$  as  $n \rightarrow +\infty$  and

$$\|\mathbf{G}_n(t, s)\mathbf{f}\|_\infty \leq \|\mathbf{f}\|_\infty, \quad t \in (s, +\infty). \quad (3.3)$$

Let us prove that estimate (3.1) holds true for  $p = 2$  with  $\mathbf{G}(t, s)$  replaced by  $\mathbf{G}_n(t, s)$  and some positive function  $c$  independent of  $n$ . To ease the notation, we use  $\|\cdot\|_p$  (resp.  $\|\cdot\|_{p,n}$ ) in place of  $\|\cdot\|_{L^p(\mathbb{R}^d; \mathbb{R}^m)}$  (resp.  $\|\cdot\|_{L^p(B_n; \mathbb{R}^m)}$ ). To this aim, first observe that from the symmetry of  $B_i$  it follows that  $2\langle \mathbf{v}, B_i D_i \mathbf{v} \rangle = \operatorname{Tr}(B_i D_i(\mathbf{v} \otimes \mathbf{v}))$  for any smooth function  $\mathbf{v} : \mathbb{R}^d \rightarrow \mathbb{R}^m$  and  $i = 1, \dots, d$ . Then, multiplying the differential equation  $D_t \mathbf{u}_n = \mathcal{A}(t)\mathbf{u}_n$  by  $\mathbf{u}_n$  and integrating by parts in  $B_n$ , we get

$$\begin{aligned} D_t \|\mathbf{u}_n(t, \cdot)\|_{2,n}^2 &= 2 \int_{B_n} \langle \mathbf{u}_n(t, \cdot), (\mathcal{A}(t)\mathbf{u}_n)(t, \cdot) \rangle dx \\ &= -2 \int_{B_n} \langle Q(t, \cdot) D_x \mathbf{u}_n(t, \cdot), D_x \mathbf{u}_n(t, \cdot) \rangle dx - \sum_{i=1}^d \int_{B_n} \langle (D_i B_i)(t, \cdot) \mathbf{u}_n(t, \cdot), \mathbf{u}_n(t, \cdot) \rangle dx \\ &\quad + 2 \int_{B_n} \langle C(t, \cdot) \mathbf{u}_n(t, \cdot), \mathbf{u}_n(t, \cdot) \rangle dx. \end{aligned}$$

Thus, from Hypotheses 2.1(ii) and (3.2) we deduce that

$$D_t \|\mathbf{u}_n(t, \cdot)\|_{2,n}^2 \leq L_J \|\mathbf{u}_n(t, \cdot)\|_{2,n}^2,$$

whence  $\|\mathbf{u}_n(t, \cdot)\|_{2,n}^2 = \|\mathbf{G}_n(t, s)\mathbf{f}\|_{2,n}^2 \leq e^{L_J(t-s)} \|\mathbf{f}\|_2^2$ , for any  $(t, s) \in \Sigma_J$  and any  $n \in \mathbb{N}$ . This latter inequality together with estimate (3.3) and the Riesz-Thorin interpolation theorem yields

$$\|\mathbf{G}_n(t, s)\mathbf{f}\|_{p,k} \leq e^{p^{-1}L_J(t-s)} \|\mathbf{f}\|_p$$

for any  $(t, s) \in \Sigma_J$ ,  $p \in [2, +\infty)$  and  $k, n \in \mathbb{N}$  with  $k \leq n$ .

Since  $\mathbf{G}_n(t, s)\mathbf{f}$  converges pointwise to  $\mathbf{G}(t, s)\mathbf{f}$  in  $\mathbb{R}^d$  as  $n \rightarrow +\infty$ , Fatou's lemma yields that  $\|\mathbf{G}(t, s)\mathbf{f}\|_{p,k} \leq e^{p^{-1}L_J(t-s)}\|\mathbf{f}\|_p$ , for any  $k \in \mathbb{N}$ . Letting  $k \rightarrow +\infty$  in the previous inequality and using Fatou's lemma again we get the first part of the claim.

Now, let us suppose that Hypotheses 3.1 are satisfied, too. Multiplying the differential equation  $(D_r - \mathcal{A}(r))\mathbf{G}_n(r, s)\mathbf{f} = \mathbf{0}$  by  $\mathbf{g} \in C_c^2([s, t] \times B_n; \mathbb{R}^m)$  and integrating by parts with respect to  $r$  and  $x$  in  $[s, t] \times B_n$ , we easily deduce that, for any  $\mathbf{f} \in C_c^\infty(B_n; \mathbb{R}^m)$ , the function  $\mathbf{v}_n(s, \cdot) = \mathbf{G}_n^*(t, s)\mathbf{f}$  is a weak solution of the backward Dirichlet Cauchy problem

$$\begin{cases} D_s \mathbf{v}_n(s, x) = -(\mathcal{A}^*(s)\mathbf{v}_n)(s, x), & t > s, x \in B_n, \\ \mathbf{v}_n(s, x) = \mathbf{0}, & t > s, x \in \partial B_n, \\ \mathbf{v}_n(t, x) = \mathbf{f}(x), & x \in \overline{B_n}, \end{cases} \quad (3.4)$$

where

$$\mathcal{A}^* \mathbf{v} = \sum_{i,j=1}^d D_i(q_{ij}D_j \mathbf{v}) - \sum_{i=1}^d B_i D_i \mathbf{v} + \left( C - \sum_{k=1}^d D_k B_k \right) \mathbf{v}$$

for any smooth function  $\mathbf{v} : \mathbb{R}^d \rightarrow \mathbb{R}^m$ . Actually, by the duality theory developed in [14] (see, in particular, Theorem 9.5.5),  $\mathbf{v}_n$  is the unique classical solution of problem (3.4) and from Hypotheses 3.1 it follows that  $\|\mathbf{G}_n^*(t, s)\mathbf{f}\|_\infty \leq e^{\kappa_0(t-s)}\|\mathbf{f}\|_\infty$ , for any  $t > s$  and  $\mathbf{f}$  as above (see [2] and the Appendix). We can then apply the arguments above to  $\mathbf{G}_n^*(t, s)$ , showing that (3.1) holds true with  $\mathbf{G}(t, s)\mathbf{f}$  replaced by  $\mathbf{G}^*(t, s)\mathbf{f}$  for any  $p \geq 2$ . Indeed, multiplying the differential equation in (3.4) by  $\mathbf{v}_n$  and integrating by parts in  $B_n$ , we get

$$\begin{aligned} D_s \|\mathbf{v}_n(s, \cdot)\|_{2,n}^2 &= -2 \int_{B_n} \langle \mathbf{v}_n(s, \cdot), (\mathcal{A}^*(s)\mathbf{v}_n)(s, \cdot) \rangle dx \\ &= \int_{B_n} \langle Q(s, \cdot) D_x \mathbf{v}_n(s, \cdot), D_x \mathbf{v}_n(s, \cdot) \rangle dx + \sum_{i=1}^d \int_{B_n} \langle (D_i B_i)(s, \cdot) \mathbf{v}_n(s, \cdot), \mathbf{v}_n(s, \cdot) \rangle dx \\ &\quad - 2 \int_{B_n} \langle C(s, \cdot) \mathbf{v}_n(s, \cdot), \mathbf{v}_n(s, \cdot) \rangle dx \\ &\geq \int_{B_n} \lambda_{\sum_{i=1}^d D_i B_i - 2C}(s, \cdot) |\mathbf{v}_n(s, \cdot)|^2 dx. \end{aligned}$$

Since  $-\lambda_A = \Lambda_{-A}$  for any symmetric matrix  $A$ , from (3.2) it follows that

$$D_r \|\mathbf{v}_n(r, \cdot)\|_{2,n}^2 \geq -L_J \|\mathbf{v}_n(r, \cdot)\|_{2,n}^2 \quad (3.5)$$

for any  $r \in (s, t)$  and  $n \in \mathbb{N}$ . Integrating (3.5) with respect to  $r$  from  $s$  to  $t$  and taking the final condition in (3.4) into account, we get

$$\|\mathbf{G}_n^*(t, s)\mathbf{f}\|_{2,n}^2 \leq e^{L_J(t-s)}\|\mathbf{f}\|_2^2.$$

Again, by the Riesz-Thorin theorem and the uniform estimate  $\|\mathbf{G}_n^*(t, s)\mathbf{f}\|_\infty \leq e^{\kappa_0(t-s)}\|\mathbf{f}\|_\infty$ , we obtain

$$\|\mathbf{G}_n^*(t, s)\mathbf{f}\|_{p,n} \leq e^{\frac{1}{p}(L_J + \kappa_0(p-2))(t-s)}\|\mathbf{f}\|_p,$$

for any  $(t, s) \in \Sigma_J$  and  $p \in [2, +\infty)$ . Arguing as above and letting  $n \rightarrow +\infty$  in the previous inequality we get

$$\|\mathbf{G}^*(t, s)\mathbf{f}\|_p \leq e^{\frac{1}{p}(L_J + \kappa_0(p-2))(t-s)}\|\mathbf{f}\|_p \quad (3.6)$$

for the same values of  $t, s$  and  $p$ .

Now, fix  $p \in [1, 2)$  and  $\mathbf{f} \in C_c(\mathbb{R}^d; \mathbb{R}^m)$ . Then, from (3.6)

$$\begin{aligned} \|\mathbf{G}(t, s)\mathbf{f}\|_p &= \sup \left\{ \int_{\mathbb{R}^d} \langle \mathbf{G}(t, s)\mathbf{f}, \mathbf{g} \rangle dx : \mathbf{g} \in C_c^\infty(\mathbb{R}^d; \mathbb{R}^m), \|\mathbf{g}\|_{p'} \leq 1 \right\} \\ &\leq \|\mathbf{f}\|_p \sup \{ \|\mathbf{G}^*(t, s)\mathbf{g}\|_{p'} : \mathbf{g} \in C_c^\infty(\mathbb{R}^d; \mathbb{R}^m), \|\mathbf{g}\|_{p'} \leq 1 \} \\ &\leq e^{\frac{1}{p'}(L_J + \kappa_0(p'-2))(t-s)}\|\mathbf{f}\|_p \end{aligned}$$



for any  $(t, s) \in \Sigma_J$ , which completes the proof.  $\square$

The case when the pointwise estimate (2.10) holds is much simpler. Indeed, estimate (3.1) can be obtained just requiring conditions on the scalar evolution operator  $G(t, s)$ . As an immediate consequence of estimate (2.10) we get the following

**Theorem 3.4.** *Assume that Hypotheses 2.3 hold true and fix  $p \in [1 + \frac{1}{4\beta}, +\infty)$ . If  $G(t, s)$  preserves  $L^1(\mathbb{R}^d)$  and satisfies (3.1) with  $p = m = 1$  and  $c_1 = \tilde{c}_1$ , then estimate (3.1) holds true for any  $(t, s) \in \Sigma_J$  and  $\mathbf{f} \in C_c(\mathbb{R}^d; \mathbb{R}^m)$  with  $c_p(r) = e^{K_J r} \tilde{c}_1(r)$ .*

**Remark 3.5.** Sufficient conditions for the scalar evolution operator  $G(t, s)$  to satisfy (3.1) with  $p \in [1, +\infty)$  can be found in [7, Thms. 5.3 & 5.4] when  $\mathcal{A}$  is not in divergence form. Adapting the cited theorems to our case, one can show that estimate (3.1) is satisfied with  $p = 1$  if there exists an interval  $J \subset I$  and a positive constant  $\Gamma_J$  such that either  $\operatorname{div}_x b \geq -\Gamma_J$  or  $|b|^2 \leq \Gamma_J \lambda_Q$  in  $J \times \mathbb{R}^d$ .

**Proposition 3.6.** *Let the assumptions of Theorem 3.3 (resp. Theorem 3.4) be satisfied. Then, the evolution operator  $\mathbf{G}(t, s)$  associated with  $\mathcal{A}(t)$  in  $C_c(\mathbb{R}^d; \mathbb{R}^m)$  admits a continuous extension to  $L^p(\mathbb{R}^d; \mathbb{R}^m)$  for any  $p \in [1, +\infty)$  (resp.  $p \in [1 + \frac{1}{4\beta}, +\infty)$ ). Moreover,  $\mathbf{G}(t, s)\mathbf{f}$  tends to  $\mathbf{f}$  in  $L^p(\mathbb{R}^d; \mathbb{R}^m)$  as  $t \rightarrow s^+$ , for any  $s \in I$ ,  $\mathbf{f} \in L^p(\mathbb{R}^d; \mathbb{R}^m)$  and  $p \in [1, +\infty)$  (resp.  $p \in [1 + \frac{1}{4\beta}, +\infty)$ ).*

*Proof.* The first part of the claim is an easy consequence of estimate (3.1). Indeed, fix  $(t, s) \in \Sigma_J$ ,  $\mathbf{f} \in L^p(\mathbb{R}^d; \mathbb{R}^m)$  and let  $(\mathbf{f}_n)$  be a sequence in  $C_c(\mathbb{R}^d; \mathbb{R}^m)$  converging to  $\mathbf{f}$  in  $L^p(\mathbb{R}^d; \mathbb{R}^m)$ , as  $n \rightarrow +\infty$ . Then, from (3.1) it follows that

$$\|\mathbf{G}(t, s)(\mathbf{f}_n - \mathbf{f}_k)\|_{L^p(\mathbb{R}^d; \mathbb{R}^m)} \leq c_p(t - s) \|\mathbf{f}_n - \mathbf{f}_k\|_{L^p(\mathbb{R}^d; \mathbb{R}^m)} \quad (3.7)$$

for any  $n, k \in \mathbb{N}$  and, consequently,  $(\mathbf{G}(t, s)\mathbf{f}_n)$  is a Cauchy sequence in  $L^p(\mathbb{R}^d; \mathbb{R}^m)$ . We can then define  $\mathbf{G}(t, s)\mathbf{f}$  as the  $L^p(\mathbb{R}^d; \mathbb{R}^m)$ -limit of  $\mathbf{G}(t, s)\mathbf{f}_n$  as  $n \rightarrow +\infty$ . Moreover, from (3.7) it follows that  $\|\mathbf{G}(t, s)\mathbf{f}\|_{L^p(\mathbb{R}^d; \mathbb{R}^m)} \leq c\|\mathbf{f}\|_{L^p(\mathbb{R}^d; \mathbb{R}^m)}$  for any  $\mathbf{f} \in L^p(\mathbb{R}^d; \mathbb{R}^m)$ .

To prove the remaining part of the claim it suffices to show that, for any  $t > s \in I$ , any  $x \in \mathbb{R}^d$  and any  $\mathbf{f} \in C_c^2(\mathbb{R}^d; \mathbb{R}^m)$ ,

$$(\mathbf{G}(t, s)\mathbf{f})(x) - \mathbf{f}(x) = - \int_s^t (\mathbf{G}(t, r)\mathcal{A}(r)\mathbf{f})(x) dr. \quad (3.8)$$

Indeed, fix  $[a, b] \subset I$ ; from estimates (3.8) and (3.1) we deduce that

$$\|\mathbf{G}(t, s)\mathbf{f} - \mathbf{f}\|_{L^p(\mathbb{R}^d; \mathbb{R}^m)} \leq \sup_{r \in [a, b]} \|\mathcal{A}(r)\mathbf{f}\|_{L^p(\mathbb{R}^d; \mathbb{R}^m)} \int_s^t c_p(r - s) dr$$

for any  $s \in [a, b]$  and  $t \geq s$ . Since, in our assumptions, the last integral vanishes as  $t \rightarrow s^+$ ,  $\mathbf{G}(t, s)\mathbf{f}$  tends to  $\mathbf{f}$  in  $L^p(\mathbb{R}^d; \mathbb{R}^m)$  as  $t \rightarrow s^+$  and  $s \in [a, b]$ . A standard density argument and the arbitrariness of  $[a, b]$  allow us to get the same result for  $\mathbf{f} \in L^p(\mathbb{R}^d; \mathbb{R}^m)$  and any  $s \in I$ .

Let us show formula (3.8). From [1, Thm 2.3 (ix)] (see also [6, Thm. A.1]), we know that, for any  $n$  such that  $\operatorname{supp}(f) \subset B_n$ ,

$$(\mathbf{G}_n^{\mathcal{D}}(t, s_1)\mathbf{f})(x) - (\mathbf{G}_n^{\mathcal{D}}(t, s_0)\mathbf{f})(x) = \int_{s_0}^{s_1} (\mathbf{G}_n^{\mathcal{D}}(t, r)\mathcal{A}(r)\mathbf{f})(x) dr \quad (3.9)$$

for any  $s_0 \leq s_1 \leq t$ ,  $x \in \mathbb{R}^d$ . Since the function  $\mathcal{A}(r)\mathbf{f}$  belongs to  $C_b(\mathbb{R}^d; \mathbb{R}^m)$ , by Proposition 2.6  $\mathbf{G}_n^{\mathcal{D}}(\cdot, r)\mathcal{A}(r)\mathbf{f}$  converges to  $\mathbf{G}(\cdot, r)\mathcal{A}(r)\mathbf{f}$  in  $C_{\text{loc}}^{1,2}((r, +\infty) \times \mathbb{R}^d; \mathbb{R}^m)$ . Thus, letting  $n \rightarrow +\infty$  in (3.9) and choosing  $s_1 = t$  we get (3.8).  $\square$

## 4. HYPERCONTRACTIVITY ESTIMATES

The aim of this section consists in proving that, under suitable assumptions, the evolution operator  $\mathbf{G}(t, s)$  maps  $L^p(\mathbb{R}^d; \mathbb{R}^m)$  into  $L^q(\mathbb{R}^d; \mathbb{R}^m)$  for any  $t > s$  and  $1 \leq p \leq q \leq +\infty$  and that

$$\|\mathbf{G}(t, s)\mathbf{f}\|_{L^q(\mathbb{R}^d; \mathbb{R}^m)} \leq c_{p,q}(t-s)\|\mathbf{f}\|_{L^p(\mathbb{R}^d; \mathbb{R}^m)}, \quad t > s, \mathbf{f} \in L^p(\mathbb{R}^d; \mathbb{R}^m), \quad (4.1)$$

for suitable functions  $c_{p,q} : (0, +\infty) \rightarrow (0, +\infty)$ .

**Theorem 4.1.** *Assume that Hypotheses 2.2 hold true and that, for some interval  $J \subset I$ , estimate (3.2) is satisfied for any  $(t, s) \in \Sigma_J$ . Then, the following properties are satisfied.*

- (i) *Estimate (4.1) holds true for any  $2 \leq p \leq q \leq +\infty$ ,  $(t, s) \in \Sigma_J$  and  $\mathbf{f} \in L^p(\mathbb{R}^d; \mathbb{R}^m)$ . Moreover,  $c_{2,\infty}(r) \leq k_1 e^{k_2 r}$  for some positive  $k_1, k_2$  depending on  $m, d, \inf_{J \times \mathbb{R}^d} \lambda_Q, L_J$ , and  $c_{p,q}(r) = (c_p(r))^{p/q} (c_{2,\infty}(r))^{2(q-p)/pq}$ , for any  $r > 0$  and  $(p, q) \neq (2, \infty)$ .*
- (ii) *If, in addition, Hypotheses 3.1 are satisfied, then estimate (4.1) holds true for any  $1 \leq p \leq q \leq +\infty$ ,  $t, s$  and  $\mathbf{f}$  as in (i). Moreover,  $c_{1,2}(r) \leq k_1 e^{k_2 r}$  for some positive  $k_1, k_2$  as in (i) and*

$$c_{p,q}(r) = (c_p(r))^{\frac{p(2-q)}{q(2-p)}} (c_{1,2}(r))^{\frac{2(q-p)}{pq}} c_2^{\frac{4(q-p)(p-1)}{pq(2-p)}}$$

for any  $r > 0$ , if  $q \leq 2$ , and  $c_{p,q}(r) = c_{p,2}(r/2)c_{2,q}(r/2)$  for any  $r > 0$ , if  $p < 2 < q$ .

*Proof.* Taking the result of the Proposition 3.6 into account, we confine ourselves to proving (4.1) for functions belonging to  $C_c(\mathbb{R}^d; \mathbb{R}^m)$ .

- (i) Fix  $\mathbf{f} \in C_c(\mathbb{R}^d; \mathbb{R}^m)$  and let  $J$  be as in the assumptions. Note that it suffices to prove that

$$\|\mathbf{G}(t, s)\mathbf{f}\|_\infty \leq c_{2,\infty}(t-s)\|\mathbf{f}\|_{L^2(\mathbb{R}^d; \mathbb{R}^m)}, \quad (t, s) \in \Sigma_J \quad (4.2)$$

for some positive function  $c_{2,\infty} : (0, +\infty) \rightarrow (0, +\infty)$ . Indeed, once (4.2) is proved, using the estimate  $\|\mathbf{G}(t, s)\mathbf{f}\|_\infty \leq \|\mathbf{f}\|_\infty$ , which holds for any  $t > s \in I$ , and the Riesz-Thorin theorem, we deduce that  $\|\mathbf{G}(t, s)\mathbf{f}\|_\infty \leq c_{p,\infty}(t-s)\|\mathbf{f}\|_{L^p(\mathbb{R}^d; \mathbb{R}^m)}$  for any  $p \in [2, +\infty]$ ,  $(t, s) \in \Sigma_J$  where  $c_{p,\infty}(t-s) = [c_{2,\infty}(t-s)]^{\frac{2}{p}}$  for any  $p > 2$ . On the other hand, Theorem 3.3 shows that  $\|\mathbf{G}(t, s)\mathbf{f}\|_{L^p(\mathbb{R}^d; \mathbb{R}^m)} \leq c_p(t-s)\|\mathbf{f}\|_{L^p(\mathbb{R}^d; \mathbb{R}^m)}$ , for any  $(t, s) \in \Sigma_J$  and  $p \geq 2$ . Hence, again by interpolation we deduce that

$$\|\mathbf{G}(t, s)\mathbf{f}\|_{L^q(\mathbb{R}^d; \mathbb{R}^m)} \leq c_{p,q}(t-s)\|\mathbf{f}\|_{L^p(\mathbb{R}^d; \mathbb{R}^m)}, \quad (t, s) \in \Sigma_J$$

for any  $2 \leq p \leq q < +\infty$ , where  $c_{p,q}(t-s) = [c_p(t-s)]^{\frac{p}{q}} [c_{p,\infty}(t-s)]^{1-\frac{p}{q}}$ .

So, let us prove (4.2). First, observe that for any  $n \in \mathbb{N}$ , any  $\mathbf{h} \in C^2(\overline{B_n}; \mathbb{R}^m)$ , which vanishes on  $\partial B_n$ , and  $\lambda > 0$ , it holds that

$$\begin{aligned} \int_{B_n} \langle \lambda \mathbf{h} - \mathcal{A}^*(s)\mathbf{h}, \mathbf{h} \rangle dx &= \sum_{i=1}^d \int_{B_n} \langle Q D_x h_i, D_x h_i \rangle dx + \lambda \|\mathbf{h}\|_2^2 + 2^{-1} \sum_{i=1}^d \int_{B_n} \text{Tr}(B_i D_i(\mathbf{h} \otimes \mathbf{h})) dx \\ &\quad - \int_{B_n} \left\langle \left( C - \sum_{i=1}^d D_i B_i \right) \mathbf{h}, \mathbf{h} \right\rangle dx \\ &\geq \nu_0 \|D_x \mathbf{h}\|_{L^2(B_n; \mathbb{R}^m)}^2 + \lambda \|\mathbf{h}\|_{L^2(B_n; \mathbb{R}^m)}^2 - \int_{B_n} \left\langle \left( C - \frac{1}{2} \sum_{i=1}^d D_i B_i \right) \mathbf{h}, \mathbf{h} \right\rangle dx \\ &\geq \nu_0 \|D_x \mathbf{h}\|_{L^2(B_n; \mathbb{R}^m)}^2 + (\lambda - L_J/2) \|\mathbf{h}\|_{L^2(B_n; \mathbb{R}^m)}^2 \end{aligned}$$

for any  $s \in J$ , with  $L_J$  as in (3.2), where  $\nu_0$  is the ellipticity bound in Hypotheses 2.1(ii). Nash's inequality (see [12, Thm. 2.4.6]) together with the latter estimate yield

$$\int_{\mathbb{R}^d} \langle (\lambda - \mathcal{A}^*(s))\mathbf{h}, \mathbf{h} \rangle dx \geq c_1 \|\mathbf{h}\|_{W^{1,2}(B_n; \mathbb{R}^m)}^2 \geq c_2 \|\mathbf{h}\|_{L^2(B_n; \mathbb{R}^m)}^{2+4/d} \|\mathbf{h}\|_{L^1(B_n; \mathbb{R}^m)}^{-4/d} \quad (4.3)$$

<sup>2</sup>Here and below  $c_p$ ,  $1 < p < \infty$ , is the constant in Theorem 3.4.

for any  $\lambda > L_J/2$ ,  $s \in J$  and some positive constants  $c_1, c_2$  depending on  $\nu_0, L_J$  and  $m$ . Now, fix  $\mathbf{g} \in C_c^\infty(\mathbb{R}^d; \mathbb{R}^m)$  and  $\lambda > L_J/2$ . For any  $n \in \mathbb{N}$ , such that  $\text{supp}(f) \subset B_n$ , we set

$$v_n(s) = \|e^{-\lambda(t-s)} \mathbf{G}_n^*(t, s) \mathbf{g}\|_{L^2(B_n; \mathbb{R}^m)}^2, \quad (t, s) \in \Sigma_J,$$

where, as in the proof of Theorem 3.3,  $\mathbf{G}_n^*(t, s) \mathbf{g}$  denotes the unique classical solution of (3.4). Estimate (4.3) implies

$$\begin{aligned} v_n'(s) &= 2e^{-2\lambda(t-s)} \int_{\mathbb{R}^d} \langle (\lambda - \mathcal{A}^*(s)) \mathbf{G}_n^*(t, s) \mathbf{g}, \mathbf{G}_n^*(t, s) \mathbf{g} \rangle dx \\ &\geq 2c_2 \|e^{-\lambda(t-s)} \mathbf{G}_n^*(t, s) \mathbf{g}\|_{L^2(B_n; \mathbb{R}^m)}^{2+4/d} \|e^{-\lambda(t-s)} \mathbf{G}_n^*(t, s) \mathbf{g}\|_{L^1(B_n; \mathbb{R}^m)}^{-4/d} \\ &\geq 2c_2 e^{\frac{4}{d}\lambda(t-s)} \|e^{-\lambda(t-s)} \mathbf{G}_n^*(t, s) \mathbf{g}\|_{L^2(B_n; \mathbb{R}^m)}^{2+4/d} \|\mathbf{g}\|_{L^1(B_n; \mathbb{R}^m)}^{-4/d}, \end{aligned} \quad (4.4)$$

where in the last inequality we have used the estimate  $\|\mathbf{G}_n^*(t, s) \mathbf{g}\|_{L^1(B_n; \mathbb{R}^m)} \leq \|\mathbf{g}\|_{L^1(B_n; \mathbb{R}^m)}$  which holds true for any  $\mathbf{g} \in C_c^\infty(\mathbb{R}^d; \mathbb{R}^m)$ . Indeed, the function  $\mathbf{G}_n^*(t, s) \mathbf{g}$  belongs to  $L^1(B_n; \mathbb{R}^m)$  and

$$\begin{aligned} \left| \int_{B_n} \langle \mathbf{G}_n^*(t, s) \mathbf{g}, \mathbf{f} \rangle dx \right| &= \left| \int_{B_n} \langle \mathbf{g}, \mathbf{G}_n(t, s) \mathbf{f} \rangle dx \right| \\ &\leq \|\mathbf{g}\|_{L^1(B_n; \mathbb{R}^m)} \|\mathbf{G}_n(t, s) \mathbf{f}\|_{L^\infty(B_n; \mathbb{R}^m)} \\ &\leq \|\mathbf{g}\|_{L^1(B_n; \mathbb{R}^m)} \|\mathbf{f}\|_{L^\infty(B_n; \mathbb{R}^m)} \end{aligned}$$

for any  $\mathbf{f} \in C_b(B_n; \mathbb{R}^m)$ , since the proof of Proposition A.1 shows that  $\|\mathbf{G}_n(t, s) \mathbf{f}\|_{L^\infty(B_n; \mathbb{R}^m)} \leq \|\mathbf{f}\|_{L^\infty(B_n; \mathbb{R}^m)}$  for any  $t \geq s$ . By approximating any  $\mathbf{f} \in L^\infty(B_n; \mathbb{R}^m)$  by a bounded sequence  $(\mathbf{f}_n) \subset C_b(B_n; \mathbb{R}^m)$  converging to  $\mathbf{f}$  in a dominated way, we conclude that

$$\left| \int_{B_n} \langle \mathbf{G}_n^*(t, s) \mathbf{g}, \mathbf{f} \rangle dx \right| \leq \|\mathbf{g}\|_{L^1(B_n; \mathbb{R}^m)} \|\mathbf{f}\|_{L^\infty(B_n; \mathbb{R}^m)}$$

for any such  $\mathbf{f}$ . This estimate shows that  $\|\mathbf{G}_n^*(t, s) \mathbf{g}\|_{L^1(B_n; \mathbb{R}^m)} \leq \|\mathbf{g}\|_{L^1(B_n; \mathbb{R}^m)}$ , as claimed.

From (4.4) it thus follows that

$$\frac{d}{ds} [(v_n(s))^{-2/d}] \leq -\frac{4c_2}{d} e^{\frac{4}{d}\lambda(t-s)} \|\mathbf{g}\|_{L^1(B_n; \mathbb{R}^m)}^{-4/d}, \quad (t, s) \in \Sigma_J,$$

whence, integrating from  $s$  to  $t$  and estimating  $\int_s^t e^{\frac{4}{d}\lambda(t-r)} dr$  from below by 1, we get

$$(v_n(t))^{-2/d} - (v_n(s))^{-2/d} \leq -\frac{4c_2}{d} \|\mathbf{g}\|_{L^1(B_n; \mathbb{R}^m)}^{-4/d}.$$

Consequently,  $v_n(s) = \|e^{-\lambda(t-s)} \mathbf{G}_n^*(t, s) \mathbf{g}\|_{L^2(B_n; \mathbb{R}^m)}^2 \leq d^{d/2} (4c_2)^{-d/2} \|\mathbf{g}\|_{L^1(B_n; \mathbb{R}^m)}^2$ , for any  $(t, s) \in \Sigma_J$ . Thus, we have established that

$$\|\mathbf{G}_n^*(t, s) \mathbf{g}\|_{L^2(B_n; \mathbb{R}^m)} \leq c_0 e^{\lambda(t-s)} \|\mathbf{g}\|_{L^1(B_n; \mathbb{R}^m)},$$

for any  $\mathbf{g} \in C_c(\mathbb{R}^d; \mathbb{R}^m)$ ,  $(t, s) \in \Sigma_J$ ,  $\lambda \geq L_J/2$  and  $c_0 := d^{d/4} (4c_2)^{-d/4}$ . By duality, the latter inequality leads to

$$\begin{aligned} \|\mathbf{G}_n(t, s) \mathbf{f}\|_\infty &= \sup \left\{ \int_{\mathbb{R}^d} \langle \mathbf{f}, \mathbf{G}_n^*(t, s) \mathbf{g} \rangle dx : \mathbf{g} \in C_c^\infty(B_n; \mathbb{R}^m), \|\mathbf{g}\|_{L^1(B_n; \mathbb{R}^m)} \leq 1 \right\} \\ &\leq c_0 e^{\lambda(t-s)} \|\mathbf{f}\|_{L^2(B_n; \mathbb{R}^m)} \end{aligned} \quad (4.5)$$

for any  $(t, s) \in \Sigma_J$ . Letting  $n \rightarrow +\infty$  in (4.5) yields estimate (4.2) with  $c_{2, \infty}(t-s) = c_0 e^{\lambda(t-s)}$ .

(ii) The second part of the statement can be easily obtained arguing again by interpolation as in (i). In this case, since  $\|\mathbf{G}(t, s) \mathbf{f}\|_{L^p(\mathbb{R}^d; \mathbb{R}^m)} \leq c_p(t-s) \|\mathbf{f}\|_{L^p(\mathbb{R}^d; \mathbb{R}^m)}$ , for any  $(t, s) \in \Sigma_J$  and  $p \in [1, 2]$ , it is enough to prove that

$$\|\mathbf{G}(t, s) \mathbf{f}\|_{L^2(\mathbb{R}^d; \mathbb{R}^m)} \leq c_{1,2}(t-s) \|\mathbf{f}\|_{L^1(\mathbb{R}^d; \mathbb{R}^m)}, \quad (t, s) \in \Sigma_J, \quad (4.6)$$

Once (4.6) is proved, using Riesz-Thorin theorem and interpolating between (3.1), with  $p = 2$ , and (4.6), we get (4.1) with  $q = 2$ . Next, interpolating between this latter estimate and, again, (3.1), we get (4.1) for any  $1 \leq p < q \leq 2$ , with  $c_{p,q}(r) = (c_p(r))^{\frac{2-q}{q(2-p)}} (c_{1,2}(r))^{\frac{2(q-p)}{pq}}$ . Finally, splitting  $\mathbf{G}(t, s) = \mathbf{G}(t, (t+s)/2) \mathbf{G}((t+s)/2, s)$ , we get (4.1) with  $p < 2 < q$  and  $c_{p,q}(r) = c_{p,2}(r/2) c_{2,q}(r/2)$ .

The proof of (4.6) can be obtained arguing as in (i) replacing the function  $v_n$  defined there by the function  $u_n(t) = \|e^{-\lambda(t-s)}\mathbf{G}(t,s)\mathbf{g}\|_{L^2(\mathbb{R}^d;\mathbb{R}^m)}^2$  for any  $(t,s) \in \Sigma_J$ .  $\square$

Theorem 4.1 can now be used to prove that the hypercontractivity estimate (4.1) holds true also when Hypotheses 2.3 are satisfied, see also Remark 2.7.

**Theorem 4.2.** *Let us assume that Hypotheses 2.3 hold true and that for some interval  $J \subset I$  there exist a positive constant  $\lambda_J$  and two functions  $\kappa_J : J \times \mathbb{R}^d \rightarrow \mathbb{R}$ , bounded from above, and  $\varphi_J \in C^2(\mathbb{R}^d)$ , blowing up as  $|x| \rightarrow +\infty$ , such that  $\operatorname{div}_x b + \kappa_J \geq 0$ , in  $J \times \mathbb{R}^d$  and  $\sup_{J \times \mathbb{R}^d} (\tilde{A}\varphi_J - \lambda\varphi_J) < +\infty$ , where  $\tilde{A} = \operatorname{div}(QD_x) - \langle b, D_x \rangle + 2\kappa_J$ . Then,  $\mathbf{G}(t,s)$  maps  $L^p(\mathbb{R}^d;\mathbb{R}^m)$  into  $L^q(\mathbb{R}^d;\mathbb{R}^m)$  for any  $1 + \frac{1}{4\beta} \leq p \leq q \leq +\infty$ . Moreover,  $\|\mathbf{G}(t,s)\mathbf{f}\|_{L^q(\mathbb{R}^d;\mathbb{R}^m)} \leq \tilde{c}_{p,q}(t-s)\|\mathbf{f}\|_{L^p(\mathbb{R}^d;\mathbb{R}^m)}$  for any  $(t,s) \in \Sigma_J$ ,  $1 + \frac{1}{4\beta} \leq p \leq q \leq +\infty$  and some function  $\tilde{c}_{p,q} : (0, +\infty) \rightarrow (0, +\infty)$ .*

*Proof.* Note that all the assumptions of Theorem 4.1(ii) are satisfied by the scalar operator  $\mathcal{A}$  in (2.4). As a consequence, the evolution operator  $G(t,s)$  associated with  $\mathcal{A}$  satisfies (4.1) for any  $p, q$  as in the statement. In particular  $G(t,s)$  maps  $L^1(\mathbb{R}^d)$  into  $L^{q/p}(\mathbb{R}^d)$  and

$$\|G(t,s)\psi\|_{L^{q/p}(\mathbb{R}^d)} \leq c_{1,q/p}(t-s)\|\psi\|_{L^1(\mathbb{R}^d)}, \quad (t,s) \in \Sigma_J, \psi \in L^1(\mathbb{R}^d). \quad (4.7)$$

Therefore, from (2.10) and (4.7) it follows that

$$\begin{aligned} \|\mathbf{G}(t,s)\mathbf{f}\|_{L^q(\mathbb{R}^d;\mathbb{R}^m)}^q &= \int_{\mathbb{R}^d} |\mathbf{G}(t,s)\mathbf{f}|^q dx \leq e^{qK_p(t-s)/p} \int_{\mathbb{R}^d} (G(t,s)|\mathbf{f}|^p)^{q/p} dx \\ &\leq e^{qK_p(t-s)/p} [c_{1,q/p}(t-s)]^{q/p} \|\mathbf{f}\|_{L^1(\mathbb{R}^d;\mathbb{R}^m)}^{q/p} \\ &= e^{qK_p(t-s)/p} [c_{1,q/p}(t-s)]^{q/p} \|\mathbf{f}\|_{L^p(\mathbb{R}^d;\mathbb{R}^m)}^q \end{aligned}$$

for any  $\mathbf{f} \in C_c(\mathbb{R}^d;\mathbb{R}^m)$  and  $(t,s) \in \Sigma_J$ . The density of  $C_c(\mathbb{R}^d;\mathbb{R}^m)$  in  $L^p(\mathbb{R}^d;\mathbb{R}^m)$  allows us to obtain the claim with  $\tilde{c}_{p,q}(r) = e^{K_p r/p} [c_{1,q/p}(r)]^{1/p}$ ,  $r \geq 0$ .  $\square$

## 5. POINTWISE GRADIENT ESTIMATES

In this section we prove some gradient estimates satisfied by the evolution operator  $\mathbf{G}(t,s)\mathbf{f}$  when  $\mathbf{f} \in C_c^\infty(\mathbb{R}^d;\mathbb{R}^m)$  when Hypotheses 2.3 are satisfied. Notice that  $p > 1$  could be allowed in all the results if  $\beta$  is arbitrary in (2.3), according to Remark 2.7. We also add the following assumptions.

**Hypotheses 5.1.** *There exist  $\gamma \geq 1/4$  and a function  $k$  such that  $|D_x q_{ij}| \leq k\lambda_Q$  in  $I \times \mathbb{R}^d$  for any  $i, j = 1, \dots, d$  and*

$$\sup_{J \times \mathbb{R}^d} \left[ \sqrt{d}m\xi\lambda_Q + \left( \sum_{i=1}^d |D_i C|^2 \right)^{\frac{1}{2}} + 2\Lambda_C \right] < +\infty \quad (5.1)$$

$$\sup_{J \times \mathbb{R}^d} \left[ \sqrt{d} \left( \sum_{i,j,l=1}^d |D_{il} q_{ij}|^2 \right)^{\frac{1}{2}} + \left( \sum_{i,j=1}^d |D_j \tilde{B}_i|^2 \right)^{\frac{1}{2}} + \Lambda_{D_x b} + \Lambda_C + M_\gamma \lambda_Q + \frac{1}{2} \left( \sum_{i=1}^d |D_i C|^2 \right)^{\frac{1}{2}} \right] < +\infty \quad (5.2)$$

where  $M_\gamma := \gamma(\sqrt{d}m\xi + dk)^2 + \frac{1}{2}\sqrt{d}m\xi + \frac{1}{4\gamma}$  (see Hypotheses 2.3).

**Theorem 5.2.** *Assume that Hypotheses 2.3 (with  $\sigma = 1$ ) and Hypotheses 5.1 are satisfied. Then, for any  $p \geq 1 + \frac{1}{4(\beta \wedge \gamma)}$ ,*

$$|D_x \mathbf{G}(t,s)\mathbf{f}|^p \leq c_p e^{C_{p,J}(t-s)} G(t,s)(|\mathbf{f}|^p + |D\mathbf{f}|^p) \quad (5.3)$$

for any  $(t,s) \in \Sigma_J$ ,  $\mathbf{f} \in C_c^\infty(\mathbb{R}^d;\mathbb{R}^m)$  and some positive constants  $c_p$  and  $C_{p,J}$ , where  $G(t,s)$  is the evolution operator associated with  $\mathcal{A}(t)$  in  $C_b(\mathbb{R}^d)$ .

*Proof.* From [17, Prop. 2.4] it follows that  $|G(t, s)\psi|^p \leq G(t, s)|\psi|^p$ , for any  $\psi \in C_b(\mathbb{R}^d)$ ,  $t \geq s \in I$  and  $p \in [1, +\infty)$ . Thus, it suffices to prove the claim only for  $p \in [1 + \frac{1}{4(\beta \wedge \gamma)}, 2]$ . Let  $J$  be as in Hypotheses 5.1,  $\mathbf{f} \in C_c^\infty(\mathbb{R}^d; \mathbb{R}^m)$  and for large  $n \in \mathbb{N}$ , we consider the classical solution  $\mathbf{u}_n = \mathbf{G}_n^{\mathcal{N}}(\cdot, s)\mathbf{f}$  of the Cauchy-Neumann problem (2.8). The core of the proof consists in proving that

$$|D_x \mathbf{u}_n(t, \cdot)|^p \leq e^{C_{p,J}(t-s)} G_n^{\mathcal{N}}(t, s)(|\mathbf{f}|^2 + |D\mathbf{f}|^2)^{\frac{p}{2}} \quad (5.4)$$

for any  $(t, s) \in \Sigma_J$ ,  $\mathbf{f} \in C_c^\infty(\mathbb{R}^d; \mathbb{R}^m)$ ,  $p \in [1 + \frac{1}{4(\beta \wedge \gamma)}, 2]$  and some positive constant  $C_{p,J}$ . Here,  $G_n^{\mathcal{N}}(t, s)$  denotes the evolution operator associated with the restriction of  $\mathcal{A}(t)$  (see (2.4)) to  $B_n$ , with homogeneous Neumann boundary conditions. Indeed, once (5.4) is proved, estimate (5.1) follows, from Proposition 2.6, with  $c_p = 2^{(p/2-1)\vee 0}$ .

So, let us prove (5.4). For any  $\varepsilon > 0$ , let us consider the function  $v_n = (|\mathbf{u}_n|^2 + |D_x \mathbf{u}_n|^2 + \varepsilon)^{\frac{p}{2}}$ . From [18, Thm. IV.5.5] it follows that  $v_n \in C^{1,2}([s, +\infty) \times \mathbb{R}^d) \cap C_b([s, T] \times \mathbb{R}^d)$  for any  $T > s$ . Moreover,  $v_n$  solves the problem

$$\begin{cases} D_t v_n - \mathcal{A}(t)v_n = p v_n^{1-2/p} \left( \sum_{i=1}^5 \psi_i + (2-p)v_n^{-2/p} \psi_6 \right), & (s, +\infty) \times B_n, \\ \frac{\partial v_n}{\partial \nu} \leq 0 & (s, +\infty) \times \partial B_n, \\ v_n(s) = (|\mathbf{f}|^2 + |D_x \mathbf{f}|^2 + \varepsilon)^{p/2} & B_n, \end{cases} \quad (5.5)$$

where

$$\begin{aligned} \psi_1 &= \sum_{i,j,l=1}^d \sum_{k=1}^m D_{li} q_{ij} D_l u_{n,k} D_j u_{n,k} + \sum_{i,l=1}^d \sum_{k,j=1}^m D_l (\tilde{B}_i)_{kj} D_l u_{n,k} D_i u_{n,j} \\ &\quad + \sum_{j=1}^m \langle D_x b D_x u_{n,j}, D_x u_{n,j} \rangle + \sum_{i=1}^d \langle C D_i \mathbf{u}_n, D_i \mathbf{u}_n \rangle, \\ \psi_2 &= \sum_{i,j,l=1}^d \sum_{k=1}^m D_l q_{ij} D_{ij} u_{n,k} D_l u_{n,k} + \sum_{i,l=1}^d \sum_{k,j=1}^m (\tilde{B}_i)_{kj} D_l u_{n,j} D_l u_{n,k}, \\ \psi_3 &= \sum_{i=1}^d \langle \mathbf{u}_n, \tilde{B}_i D_i \mathbf{u}_n \rangle + \sum_{l=1}^d \sum_{k,j=1}^m D_l C_{kj} u_{n,j} D_l u_{n,k}, \\ \psi_4 &= \langle C \mathbf{u}_n, \mathbf{u}_n \rangle, \\ \psi_5 &= - \sum_{k=1}^m \langle Q D_x u_{n,k}, D_x u_{n,k} \rangle - \sum_{i=1}^d \sum_{k=1}^m \langle Q D_x D_i u_{n,k}, D_x D_i u_{n,k} \rangle, \\ \psi_6 &= - \sum_{i,j=1}^d q_{ij} \left( \langle \mathbf{u}, D_i \mathbf{u} \rangle + \sum_{l=1}^d \langle D_{il} \mathbf{u}, D_l \mathbf{u} \rangle \right) \left( \langle \mathbf{u}, D_j \mathbf{u} \rangle + \sum_{m=1}^d \langle D_{jm} \mathbf{u}, D_m \mathbf{u} \rangle \right) \end{aligned}$$

and the boundary condition in (5.5) follows since the normal derivative of  $|D_x u_{n,k}|^2$  is nonpositive in  $(s, +\infty) \times \partial B_n$  for any  $k = 1, \dots, m$  (see e.g., [8, 9]).

Using Hypotheses 2.3(i)-(ii) and the inequality  $|D_x q_{ij}| \leq k \lambda_Q$ , we get the following estimates for the functions  $\psi_i$ , for  $i = 1, 2, 3$ :

$$\begin{aligned} \psi_1 &\leq \left[ \sqrt{d} \left( \sum_{i,j,l=1}^d |D_{li} q_{ij}|^2 \right)^{1/2} + \left( \sum_{i,l=1}^d |D_l \tilde{B}_i|^2 \right)^{1/2} + \Lambda_{D_x b} + \Lambda_C \right] |D_x \mathbf{u}_n|^2 \\ \psi_2 &\leq \left[ \left( \sum_{i=1}^d |D_i Q|^2 \right)^{1/2} + \left[ \left( \sum_{i=1}^d |\tilde{B}_i|^2 \right)^{1/2} \right] |D_x \mathbf{u}_n| |D_x^2 \mathbf{u}_n| \leq a(dk + \sqrt{dm} \xi)^2 \lambda_Q |D_x^2 \mathbf{u}_n|^2 + \frac{1}{4a} \lambda_Q |D_x \mathbf{u}_n|^2, \end{aligned}$$

$$\psi_3 \leq \frac{1}{2} \left[ \sqrt{dm} \xi \lambda_Q + \left( \sum_{i=1}^d |D_i C|^2 \right)^{1/2} \right] (|\mathbf{u}_n|^2 + |D_x \mathbf{u}_n|^2)$$

in  $J \times \mathbb{R}^d$ . To estimate  $\psi_6$ , we observe that

$$\begin{aligned} \psi_6 &= \sum_{h,k=1}^m \sum_{i,j=1}^d q_{ij} \left( u_{n,h} D_i u_{n,h} + \sum_{l=1}^d D_{il} u_{n,h} D_l u_{n,h} \right) \left( u_{n,k} D_j u_{n,k} + \sum_{m=1}^d D_{jm} u_{n,k} D_m u_{n,k} \right) \\ &= \sum_{h,k=1}^m u_{n,h} u_{n,k} \langle Q D_x u_{n,h}, D_x u_{n,k} \rangle + 2 \sum_{h,k=1}^m u_{n,h} \sum_{l=1}^d D_l u_{n,k} \langle Q D_x u_{n,h}, D_x D_l u_{n,k} \rangle \\ &\quad + \sum_{h,k=1}^m \sum_{l,m=1}^d D_l u_{n,h} D_m u_{n,k} \langle Q D_x D_l u_{n,h}, D_x D_m u_{n,k} \rangle. \end{aligned}$$

It thus follows that

$$\begin{aligned} \psi_6 &\leq \left( \sum_{h=1}^m |u_{n,h}| |Q^{1/2} D_x u_{n,h}| \right)^2 + 2 \sum_{h,k=1}^m |u_{n,h}| |Q^{1/2} D_x u_{n,h}| \sum_{l=1}^d |D_l u_{n,k}| |Q^{1/2} D_x D_l u_{n,k}| \\ &\quad + \sum_{h,k=1}^m \sum_{l,m=1}^d |D_l u_{n,h}| |D_m u_{n,k}| |Q^{1/2} D_x D_l u_{n,h}| |Q^{1/2} D_x D_m u_{n,k}| \\ &\leq |\mathbf{u}_n|^2 \sum_{k=1}^d \langle Q D_x u_{n,k}, D_x u_{n,k} \rangle \\ &\quad + 2 |\mathbf{u}_n| |D_x \mathbf{u}_n| \left( \sum_{k=1}^d \langle Q D_x u_{n,k}, D_x u_{n,k} \rangle \right)^{\frac{1}{2}} \left( \sum_{i=1}^d \sum_{k=1}^m \langle Q D_x D_i u_{n,k}, D_x D_i u_{n,k} \rangle \right)^{\frac{1}{2}} \\ &\quad + |D_x \mathbf{u}_n|^2 \sum_{i=1}^d \sum_{k=1}^m \langle Q D_x D_i u_{n,k}, D_x D_i u_{n,k} \rangle \\ &= \left[ |\mathbf{u}_n| \left( \sum_{k=1}^d \langle Q D_x u_{n,k}, D_x u_{n,k} \rangle \right)^{\frac{1}{2}} + |D_x \mathbf{u}_n| \left( \sum_{i=1}^d \sum_{k=1}^m \langle Q D_x D_i u_{n,k}, D_x D_i u_{n,k} \rangle \right)^{\frac{1}{2}} \right]^2 \\ &\leq (|\mathbf{u}_n|^2 + |D_x \mathbf{u}_n|^2) \left( \sum_{k=1}^d \langle Q D_x u_{n,k}, D_x u_{n,k} \rangle + \sum_{i=1}^d \sum_{k=1}^m \langle Q D_x D_i u_{n,k}, D_x D_i u_{n,k} \rangle \right) \\ &\leq v_n^{\frac{2}{p}} \left( \sum_{k=1}^d \langle Q D_x u_{n,k}, D_x u_{n,k} \rangle + \sum_{i=1}^d \sum_{k=1}^m \langle Q D_x D_i u_{n,k}, D_x D_i u_{n,k} \rangle \right). \end{aligned}$$

Putting everything together, we get

$$\begin{aligned} \sum_{i=1}^5 \psi_i + (2-p) \psi_6 v_n^{-2/p} &\leq \left[ \sqrt{d} \left( \sum_{i,j,l=1}^d |D_{il} q_{ij}|^2 \right)^{1/2} + \left( \sum_{i,j=1}^d |D_j \tilde{B}_i|^2 \right)^{1/2} + \Lambda_{D_x b} + \Lambda_C \right. \\ &\quad \left. + \left( \frac{1}{4a} + p - 1 + \frac{1}{2} \sqrt{dm} \xi \right) \lambda_Q + \frac{1}{2} \left( \sum_{i=1}^d |D_i C|^2 \right)^{1/2} \right] |D_x \mathbf{u}_n|^2 \\ &\quad + [a(dk + \sqrt{dm} \xi)^2 - (1-p)] \lambda_Q |D_x^2 \mathbf{u}_n|^2 \\ &\quad + \left\{ \Lambda_C + \frac{1}{2} \left[ \sqrt{dm} \xi \lambda_Q + \left( \sum_{i=1}^d |D_i C|^2 \right)^{1/2} \right] \right\} |\mathbf{u}_n|^2 \end{aligned}$$

for any  $a = a(t)$  and, choosing  $a = (p-1)(dk + \sqrt{dm}\xi)^{-2}$ , we conclude that

$$\begin{aligned} \sum_{i=1}^5 \psi_i + (2-p)\psi_6 v_n^{-2/p} &\leq \left[ \sqrt{d} \left( \sum_{i,j,l=1}^d |D_{il}q_{ij}|^2 \right)^{1/2} + \left( \sum_{i,j=1}^d |D_j \tilde{B}_i|^2 \right)^{1/2} + \Lambda_{D_x b} + \Lambda_C + M_\gamma \lambda_Q \right. \\ &\quad \left. + \frac{1}{2} \left( \sum_{i=1}^d |D_i C|^2 \right)^{1/2} \right] |D_x \mathbf{u}_n|^2 \\ &\quad + \left[ \frac{1}{2} \sqrt{dm}\xi \lambda_Q + \frac{1}{2} \left( \sum_{i=1}^d |D_i C|^2 \right)^{1/2} + \Lambda_C \right] |\mathbf{u}_n|^2 \end{aligned}$$

in  $J \times \mathbb{R}^d$ . Using estimates (5.1) and (5.2) we conclude that  $D_t v_n - \mathcal{A}(t)v_n \leq C_{p,J} v_n$  in  $J \times \mathbb{R}^d$  for some positive constant  $C_{p,J}$ . Hence, the function  $w_n(t, \cdot) = v_n(t, \cdot) - e^{C_{p,J}(t-s)} G_n^N(t, s)(|\mathbf{f}|^2 + |\mathbf{D}\mathbf{f}|^2 + \varepsilon)^{p/2}$  solves the problem

$$\begin{cases} D_t w_n - (\mathcal{A}(t) + C_{p,J})w_n \leq 0, & (s, T] \times B_n, \\ \frac{\partial w_n}{\partial \nu} \leq 0, & (s, T] \times \partial B_n, \\ w_n(s) = 0, & B_n. \end{cases}$$

The classical maximum principle yields that  $w_n \leq 0$  in  $(s, T) \times B_n$ , whence, letting  $\varepsilon \rightarrow 0^+$ , estimate (5.4) follows at once.  $\square$

**Theorem 5.3.** *Assume that Hypotheses 2.3 (with  $\sigma = 1$ ) and Hypotheses 5.1 are satisfied with  $J = I$ . If  $\Lambda_C \leq -2\gamma dm^2 \xi^2 \lambda_Q$  in  $I \times \mathbb{R}^d$ , where  $\gamma$  is as in Hypotheses 5.1, then the estimate*

$$|D_x \mathbf{G}(t, s) \mathbf{f}|^p \leq k_p e^{h_p(t-s)} (t-s)^{-\frac{p}{2}} G(t, s) |\mathbf{f}|^p, \quad (5.6)$$

holds in  $\Sigma_I \times \mathbb{R}^d$ , for any  $p \in [1 + \frac{1}{4(\beta \wedge \gamma)}, +\infty)$ ,  $\mathbf{f} \in C_c^\infty(\mathbb{R}^d, \mathbb{R}^m)$  and some positive constants  $k_p$  and  $h_p$ .

*Proof.* Using the same arguments as in the proof of Theorem 5.2 we can limit ourselves to proving (5.6) when  $p \in [1 + \frac{1}{4(\beta \wedge \gamma)}, 2]$ . Note that, under our assumptions, the estimates (2.10) and (5.3) hold true for any  $p \in [1 + \frac{1}{4(\beta \wedge \gamma)}, 2]$ ,  $\mathbf{f} \in C_c^\infty(\mathbb{R}^d, \mathbb{R}^m)$  and  $t > s \in I$ , with positive constants  $K_J$  in (2.10) and  $C_p$  in (5.3), independent of  $J$ . Moreover, after a rescaling argument we can assume that  $K_J < 0$ . Thus, for any fixed  $p \in [1 + \frac{1}{4(\beta \wedge \gamma)}, 2]$ ,  $\mathbf{f} \in C_c^\infty(\mathbb{R}^d, \mathbb{R}^m)$ , from (5.3) and the evolution law it follows that

$$\begin{aligned} |D_x \mathbf{G}(t, s) \mathbf{f}|^p &= |D_x \mathbf{G}(t, \sigma) \mathbf{G}(\sigma, s) \mathbf{f}|^p \\ &\leq c_p e^{C_p(t-\sigma)} G(t, \sigma) [|\mathbf{G}(\sigma, s) \mathbf{f}|^p + |D_x \mathbf{G}(\sigma, s) \mathbf{f}|^p] \\ &\leq c_p e^{C_p(t-\sigma)} [G(t, s) |\mathbf{f}|^p + G(t, \sigma) |D_x \mathbf{G}(\sigma, s) \mathbf{f}|^p] \end{aligned}$$

for any  $\sigma \in (s, t)$ . Since the transition kernel  $p_{t,s}(x, y)$  associated with the evolution operator  $G(t, s)$  is a positive  $L^1$ -function with respect to the variable  $y$  with  $L^1$ -norm equal to one (see [17, Prop. 2.4]), using the Hölder inequality we can estimate

$$\begin{aligned} G(t, \sigma) |D_x \mathbf{G}(\sigma, s) \mathbf{f}|^p &= G(t, \sigma) \left[ |D_x \mathbf{G}(\sigma, s) \mathbf{f}|^p (|\mathbf{G}(\sigma, s) \mathbf{f}|^2 + \delta)^{\frac{p(p-2)}{4}} (|\mathbf{G}(\sigma, s) \mathbf{f}|^2 + \delta)^{\frac{p(2-p)}{4}} \right] \\ &\leq \left( G(t, \sigma) (|D_x \mathbf{G}(\sigma, s) \mathbf{f}|^2 (|\mathbf{G}(\sigma, s) \mathbf{f}|^2 + \delta)^{\frac{p-2}{2}}) \right)^{\frac{p}{2}} \left( G(t, \sigma) (|\mathbf{G}(\sigma, s) \mathbf{f}|^2 + \delta)^{\frac{p}{2}} \right)^{\frac{2-p}{2}} \\ &\leq \varepsilon^{\frac{2}{p}} \frac{p}{2} G(t, \sigma) \left( |D_x \mathbf{G}(\sigma, s) \mathbf{f}|^2 (|\mathbf{G}(\sigma, s) \mathbf{f}|^2 + \delta)^{\frac{p-2}{2}} \right) \\ &\quad + \left( 1 - \frac{p}{2} \right) \varepsilon^{\frac{2}{p-2}} G(t, \sigma) (|\mathbf{G}(\sigma, s) \mathbf{f}|^2 + \delta)^{\frac{p}{2}} \end{aligned}$$

for any  $\varepsilon, \delta > 0$ , whence

$$e^{-C_p(t-\sigma)} |D_x \mathbf{G}(t, s) \mathbf{f}|^p \leq c_p G(t, s) |\mathbf{f}|^p + c_p \left( 1 - \frac{p}{2} \right) \varepsilon^{\frac{2}{p-2}} G(t, \sigma) (|\mathbf{G}(\sigma, s) \mathbf{f}|^2 + \delta)^{\frac{p}{2}}$$

$$+ \frac{p}{2} c_p \varepsilon^{\frac{2}{p}} G(t, \sigma) \left( |D_x \mathbf{G}(\sigma, s) \mathbf{f}|^2 (|\mathbf{G}(\sigma, s) \mathbf{f}|^2 + \delta)^{\frac{p-2}{2}} \right).$$

Integrating the previous estimate with respect to  $\sigma \in (s, t)$ , we deduce

$$|D_x \mathbf{G}(t, s) \mathbf{f}|^p \leq \frac{C_p c_p}{1 - e^{-C_p(t-s)}} \left\{ (t-s) G(t, s) |\mathbf{f}|^p + \left(1 - \frac{p}{2}\right) \varepsilon^{\frac{2}{p-2}} \int_s^t G(t, \sigma) (|\mathbf{G}(\sigma, s) \mathbf{f}|^2 + \delta)^{\frac{p}{2}} d\sigma \right. \\ \left. + \frac{p}{2} \varepsilon^{\frac{2}{p}} \int_s^t G(t, \sigma) \left( |D_x \mathbf{G}(\sigma, s) \mathbf{f}|^2 (|\mathbf{G}(\sigma, s) \mathbf{f}|^2 + \delta)^{\frac{p-2}{2}} \right) d\sigma \right\}. \quad (5.7)$$

The claim reduces to proving that there exists a positive constant  $k_p$  such that

$$\int_s^t G(t, \sigma) \left( |D_x \mathbf{G}(\sigma, s) \mathbf{f}|^2 (|\mathbf{G}(\sigma, s) \mathbf{f}|^2 + \delta)^{\frac{p-2}{2}} \right) d\sigma \leq k_p G(t, s) (|\mathbf{f}|^2 + \delta)^{\frac{p}{2}} \quad (5.8)$$

for any  $(t, s) \in \Sigma_I$ . Indeed, once (5.8) is proved, we replace (5.8) into (5.7) and, using [17, Prop. 3.1], we let  $\delta \rightarrow 0^+$ . Finally, using again (2.10) to estimate  $G(t, \sigma) |\mathbf{G}(\sigma, s) \mathbf{f}|^p \leq G(t, \sigma) G(\sigma, s) |\mathbf{f}|^p = G(t, s) |\mathbf{f}|^p$ , we get

$$|D_x \mathbf{G}(t, s) \mathbf{f}|^p \leq \frac{C_p c_p}{1 - e^{-C_p(t-s)}} \left\{ \left[ 1 + \left(1 - \frac{p}{2}\right) \varepsilon^{\frac{2}{p-2}} \right] (t-s) + \frac{p}{2} \varepsilon^{\frac{2}{p}} k_p \right\} G(t, s) |\mathbf{f}|^p$$

and, minimising on  $\varepsilon$ ,

$$|D_x \mathbf{G}(t, s) \mathbf{f}|^p \leq \frac{C_p c_p}{1 - e^{-C_p(t-s)}} \left[ (t-s) + k_p^{\frac{2}{p}} (t-s)^{1-\frac{p}{2}} \right] G(t, s) |\mathbf{f}|^p$$

whence the claim follows. Therefore, to conclude we prove (5.8). To this aim, we set

$$\psi_n(\sigma) = G_n^N(t, \sigma) (|\mathbf{G}_n^N(\sigma, s) \mathbf{f}|^2 + \delta)^{\frac{p}{2}} = G_n^N(t, \sigma) (|\mathbf{u}_n(\sigma, \cdot)|^2 + \delta)^{\frac{p}{2}} = G_n^N(t, \sigma) (v_n(\sigma, \cdot))$$

for any  $\sigma \in [s, t]$  and  $n \in \mathbb{N}$ , where  $G_n^N(t, \sigma)$  and  $\mathbf{G}_n^N(t, \sigma)$  are the same evolution operator considered in the proof of Theorem 5.2. Since the normal derivative of the function  $v_n(\sigma, \cdot)$  vanishes on  $\partial B_n$  for any  $\sigma \in (s, t)$ , classical results on evolution operators show that the function  $\psi_n$  is differentiable in  $(s, t)$  and a straightforward computation yields

$$\psi_n'(\sigma) = G_n^N(t, \sigma) [D_\sigma v_n(\sigma, \cdot) - \mathcal{A}(\sigma) v_n(\sigma, \cdot)] \\ = p G_n^N(t, \sigma) \left[ (v_n(\sigma))^{1-\frac{2}{p}} \left( \sum_{i=1}^d \langle \mathbf{u}_n, \tilde{B}_i D_i \mathbf{u}_n \rangle + \langle \mathbf{u}_n, C \mathbf{u}_n \rangle - \sum_{i,j=1}^d q_{ij} \langle D_i \mathbf{u}, D_j \mathbf{u} \rangle \right) \right. \\ \left. + (2-p) (v_n(\sigma))^{1-\frac{4}{p}} \sum_{i,j=1}^d q_{ij} \langle \mathbf{u}, D_i \mathbf{u} \rangle \langle \mathbf{u}, D_j \mathbf{u} \rangle \right].$$

Using (2.11), we get

$$\psi_n'(\sigma) \leq p G_n^N(t, \sigma) \left[ (v_n(\sigma))^{1-\frac{2}{p}} \left( \sum_{i=1}^d \langle \mathbf{u}_n, \tilde{B}_i D_i \mathbf{u}_n \rangle + \langle \mathbf{u}_n, C \mathbf{u}_n \rangle + (1-p) \lambda_Q |D_x \mathbf{u}_n|^2 \right) \right].$$

Thus, taking Hypotheses 2.3(i) into account, we deduce

$$\sum_{i=1}^d \langle \mathbf{u}_n, \tilde{B}_i D_i \mathbf{u}_n \rangle + \langle \mathbf{u}_n, C \mathbf{u}_n \rangle \leq m \xi \lambda_Q |\mathbf{u}_n| \sum_{i=1}^d |D_i \mathbf{u}_n| + \Lambda_C |\mathbf{u}_n|^2 \\ \leq (\varepsilon d m^2 \xi^2) \lambda_Q |D_x \mathbf{u}_n|^2 + \left( \frac{\lambda_Q}{4\varepsilon} + \Lambda_C \right) |\mathbf{u}_n|^2$$

for any  $\varepsilon = \varepsilon(t) > 0$ . Consequently,

$$\psi_n'(\sigma) \leq p G_n^N(t, \sigma) \left[ (v_n(\sigma))^{1-\frac{2}{p}} \left( (\varepsilon d m^2 \xi^2 + 1 - p) \lambda_Q |D_x \mathbf{u}_n|^2 + \left( \frac{\lambda_Q}{4\varepsilon} + \Lambda_C \right) |\mathbf{u}_n|^2 \right) \right].$$



Choosing  $\varepsilon = (p-1)(2dm^2\xi^2)^{-1}$  implies

$$\psi'_n(\sigma) \leq 2^{-1}p(1-p)\nu_0 G_n^N(t, \sigma) \left[ (v_n(\sigma))^{1-\frac{2}{p}} |D_x \mathbf{u}_n|^2 \right] \quad (5.9)$$

Integrating both sides of (5.9) with respect to  $\sigma$  in  $[s+h, t-h]$  and then letting  $n$  to  $+\infty$  and  $h$  to 0 we get (5.8) with  $k_p = 2[p(p-1)\nu_0]^{-1}$ . The proof is so completed.  $\square$

**Corollary 5.4.** *Under the same Hypotheses as in Theorem 5.3 and assuming that  $G(t, s)$  satisfies estimate (3.1) with  $p = 1$ , the evolution operator  $\mathbf{G}(t, s)$  is bounded from  $W^{\theta_1, p}(\mathbb{R}^d; \mathbb{R}^m)$  in  $W^{\theta_2, p}(\mathbb{R}^d; \mathbb{R}^m)$ , for any  $p \in [1 + \frac{1}{4(\beta\wedge\gamma)}, +\infty)$ ,  $0 \leq \theta_1 \leq \theta_2 \leq 1$  and  $(t, s) \in \Sigma_I$ .*

*Proof.* From Theorem 3.4 it follows that  $\|\mathbf{G}(t, s)\mathbf{f}\|_p \leq c_p(t-s)\|\mathbf{f}\|_p$  for any  $t > s \in I$ ,  $\mathbf{f} \in C_c^\infty(\mathbb{R}^d; \mathbb{R}^m)$  and some positive function  $c_p : (0, +\infty) \rightarrow (0, +\infty)$ . Moreover, integrating the estimates (5.3) and (5.6) in  $\mathbb{R}^d$ , writing (3.1) with  $p = 1$  and  $G(t, s)$  instead of  $\mathbf{G}(t, s)$  and using the above estimate for  $\|\mathbf{G}(t, s)\mathbf{f}\|_p$ , it follows that

$$\|\mathbf{G}(t, s)\mathbf{f}\|_{W^{1,p}(\mathbb{R}^d; \mathbb{R}^m)} \leq c_p^1(t-s)\|\mathbf{f}\|_{W^{1,p}(\mathbb{R}^d; \mathbb{R}^m)}, \quad \|\mathbf{G}(t, s)\mathbf{f}\|_{W^{1,p}(\mathbb{R}^d; \mathbb{R}^m)} \leq c_p^2(t-s)\|\mathbf{f}\|_{L^p(\mathbb{R}^d; \mathbb{R}^m)}, \quad (5.10)$$

for any  $t > s \in I$ ,  $p \in [1 + \frac{1}{4(\beta\wedge\gamma)}, +\infty)$ ,  $\mathbf{f} \in C_c^\infty(\mathbb{R}^d; \mathbb{R}^m)$  and some positive functions  $c_p^i : (0, +\infty) \rightarrow (0, +\infty)$ ,  $i = 1, 2$ . By density, the first estimate in (5.10) can be extended to any  $\mathbf{f} \in W^{1,p}(\mathbb{R}^d; \mathbb{R}^m)$  and the second to  $\mathbf{f} \in L^p(\mathbb{R}^d; \mathbb{R}^m)$ . Thus, the claim is proved for  $\theta_2 = 1$  and  $\theta_1 = 0, 1$ . The remaining cases follows by interpolation, taking into account that for any  $\theta \in (0, 1)$  and  $p \in [1, +\infty)$ ,  $W^{\theta, p}(\mathbb{R}^d; \mathbb{R}^m)$  equals the real interpolation space  $(L^p(\mathbb{R}^d; \mathbb{R}^m); W^{1,p}(\mathbb{R}^d; \mathbb{R}^m))_{\theta, p}$  with equivalence of the respective norms (see [24, Thm. 2.4.1(a)]).  $\square$

## 6. EXAMPLES

Here we exhibit some classes of elliptic operators to which Theorem 3.3 can be applied. Indeed examples of operators which satisfy the hypotheses of Theorem 3.4 can be found in [7].

**Example 6.1.** Let  $\mathcal{A}$  be as in (1.2) with  $Q = I_m$ ,  $B_i(x) = -x_i(1+|x|^2)^a \hat{B}_i$  and  $C(x) = -|x|^2(1+|x|^2)^b \hat{C}$  for any  $x \in \mathbb{R}^d$ ,  $i = 1, \dots, d$ . Here,  $\hat{B}_i$  ( $i = 1, \dots, d$ ) and  $\hat{C}$  are constant, symmetric and positive definite matrices and  $b > 2a \geq 0$ . It is easy to check that

$$\mathcal{K}_\eta(x) \geq -(1+|x|^2)^{2a} \sum_{i=1}^d x_i^2 |\hat{B}_i|^2 + 4|x|^2(1+|x|^2)^b \lambda_{\hat{C}}$$

for any  $x \in \mathbb{R}^d$ . Moreover, choosing  $\kappa(x) = -|x|^c$  with  $c \in (2+2a, 2+2b)$ , we get

$$\begin{aligned} \tilde{\mathcal{K}}_\eta(x) &\geq -(1+|x|^2)^{2a} \sum_{i=1}^d x_i^2 |\hat{B}_i|^2 + 4|x|^2(1+|x|^2)^b \lambda_{\hat{C}} - 4(1+|x|^2)^a \sum_{i=1}^d \Lambda_{\hat{B}_i} \\ &\quad - 8a(1+|x|^2)^{a-1} \sum_{i=1}^d \Lambda_{\hat{B}_i} x_i^2 - 4|x|^c \end{aligned}$$

for any  $x \in \mathbb{R}^d$ . Since  $b > 2a$  and  $c < 2+2b$ , the functions  $\mathcal{K}_\eta$  and  $\tilde{\mathcal{K}}_\eta$  blow up at infinity as  $|x| \rightarrow \infty$ , uniformly with respect to  $\eta \in \partial B_1$ . Therefore, assumption (2.5) is satisfied both by  $\mathcal{K}_\eta$  and  $\tilde{\mathcal{K}}_\eta$ . On the other hand, taking into account that  $c > 2+2a$ , the function  $\varphi(x) = 1+|x|^2$ ,  $x \in \mathbb{R}^d$ , satisfies Hypotheses 2.2(ii) and 3.1(ii) for any  $\lambda > 0$ . Finally, a straightforward computation shows that

$$\Lambda_{2C - \sum_{i=1}^d D_i B_i}(x) \leq -2|x|^2(1+|x|^2)^b \lambda_{\hat{C}} + (1+|x|^2)^a \sum_{i=1}^d \Lambda_{\hat{B}_i} + 2a(1+|x|^2)^{a-1} \sum_{i=1}^d x_i^2 \Lambda_{\hat{B}_i}$$

for any  $x \in \mathbb{R}^d$ . The choice of  $a$  and  $b$  yields that estimate (3.2) is satisfied, too. Since, all the assumptions in Theorem 3.3 are satisfied, the evolution operator  $\mathbf{G}(t, s)$  associated with  $\mathcal{A}$  is well-defined in  $L^p(\mathbb{R}^d; \mathbb{R}^m)$  for any  $p \geq 1$ . Moreover, estimate (3.1) holds true, where  $c_p(t-s)$  is defined in Theorem 3.3.

In the following example we consider the operator  $\mathcal{A}$  with  $B_i, C$  as above, but allow the diffusion coefficients  $q_{ij}$  to be unbounded as well.

**Example 6.2.** Let  $\mathcal{A}$  be as in (1.2) with  $Q(x) = (1 + |x|^2)^\delta I_m$ ,  $B_i(x) = -x_i(1 + |x|^2)^a I_m + (1 + |x|^2)^b \hat{B}_i$  ( $i = 1, \dots, d$ ) and  $C(x) = -(1 + |x|^2)^c \hat{C}$  for any  $x \in \mathbb{R}^d$ . We assume that  $\hat{B}_i$  ( $i = 1, \dots, d$ ) and  $\hat{C}$  are constant, symmetric and positive definite matrices. Finally,  $\delta, a, b \in [0, +\infty)$  satisfy  $2b \leq \delta < a + 1$  and  $c > 2a \vee (a + 1)$ . We have that

$$\mathcal{K}_\eta(x) = (1 + |x|^2)^{-\delta+2b} \sum_{i=1}^d \left[ \langle \hat{B}_i \eta, \eta \rangle^2 - |\hat{B}_i \eta|^2 \right] + 4(1 + |x|^2)^c \langle \hat{C} \eta, \eta \rangle,$$

for any  $x \in \mathbb{R}^d$  and  $\eta \in \partial B_1$ . Since  $\delta \geq 2b$ , the first term in the previous formula is bounded in  $\mathbb{R}^d$ , therefore (2.5) is clearly satisfied by  $\mathcal{K}_\eta$  and also by  $\tilde{\mathcal{K}}_\eta$ , where  $\kappa(x) = -|x|^s$  and  $s \in (2 + 2a, 2c)$ . Indeed,

$$\tilde{\mathcal{K}}_\eta(x) \geq \mathcal{K}_\eta(x) - 4(1 + |x|^2)^a - 8b|x|^2(1 + |x|^2)^{a-1} + 8b(1 + |x|^2)^{b-1} \sum_{i=1}^d x_i \langle \hat{B}_i \eta, \eta \rangle - |x|^s$$

for any  $x \in \mathbb{R}^d$ . The choice of  $\delta, a, b$  and  $s$  yields that the function  $\varphi$ , defined in (i) is a Lyapunov function in  $\mathbb{R}^d$  for both  $\mathcal{A}$  and  $\tilde{\mathcal{A}}$ . Moreover,

$$\Lambda_{2C - \sum_{i=1}^d D_i B_i}(x) \leq -2(1 + |x|^2)^c \lambda_{\hat{C}} + (1 + |x|^2)^b + 2b(1 + |x|^2)^{b-1} |x|^2 + 2c(1 + |x|^2)^{c-1} \sum_{i=1}^d |x_i| \Lambda_{\hat{B}_i},$$

and, since the leading term in the previous estimate is the first term in the right-hand side, estimate (3.2) is clearly satisfied. Thus, Theorem 3.3 can be applied. Moreover, since  $c > \delta$ ,  $2c > 2b - 1$  and  $b \leq \delta$ , the assumptions of Theorems 5.2 and 5.3 are satisfied and estimates (5.3) and (5.6) hold true in  $\mathbb{R}^d$  for any  $(t, s) \in \Sigma_I$ .

**Remark 6.3.** In the previous examples we can replace the constant matrices  $I_m, \hat{B}_j$  ( $j = 1, \dots, d$ ) and  $\hat{C}$  by matrices of the same type, i.e., by  $\text{diag}(q_i(t)), \hat{B}_j(t)$  ( $i = 1, \dots, m, j = 1, \dots, d$ ) and  $\hat{C}(t)$  respectively, whose entries are functions which belong to  $C_{\text{loc}}^{\alpha/2}(I) \cap C_b(I)$  and such that  $q_i, \lambda_{\hat{B}_i}$  ( $i = 1, \dots, m, j = 1, \dots, d$ ) and  $\lambda_{\hat{C}}$ , have positive infima on  $I$ .

#### APPENDIX A. UNIFORM ESTIMATES

Now, we prove that the  $L^\infty$ -norm of the classical solutions of the Cauchy problems (1.1) and (3.4) can be estimated in terms of the  $L^\infty$ -norm of the initial datum. The proof of this result can be found in [2] in the case when  $\mathcal{A}$  is not in divergence form.

**Proposition A.1.** *Let us assume that Hypotheses 2.1 hold true. If there exists a function  $h : I \times \mathbb{R}^d \rightarrow \mathbb{R}$  bounded from above, such that Hypotheses 2.2 are satisfied with  $\mathcal{K}_\eta$  replaced by  $\mathcal{K}_\eta + 4h$  and  $\mathcal{A}_\eta$  replaced by  $\mathcal{A}_\eta + 2h$  then the evolution operator associated with  $\mathcal{A}$  in  $C_b(\mathbb{R}^d; \mathbb{R}^m)$  satisfies the estimate*

$$\|\mathbf{G}(t, s)\mathbf{f}\|_\infty \leq e^{h_0(t-s)} \|\mathbf{f}\|_\infty,$$

for any  $t > s \in I$ ,  $\mathbf{f} \in C_b(\mathbb{R}^d; \mathbb{R}^m)$ , where  $h_0 = \sup_{I \times \mathbb{R}^d} h$ .

*Proof.* Let  $T > s$  and  $J := [s, T]$ . Up to replacing  $\lambda := \lambda_J$  with a larger constant if needed, we can assume that there exists a function  $\varphi := \varphi_J$  as in Hypothesis 2.2(ii) satisfying  $\sup_{\eta \in \partial B_1} \sup_{J \times \mathbb{R}^d} (\mathcal{A}_\eta \varphi - \lambda \varphi) < 0$  with  $\lambda > 2h_0$ . Now, for any  $t \in J$ ,  $x \in \mathbb{R}^d$  and  $n \in \mathbb{N}$ , we set

$$v_n(t, x) := e^{-\lambda(t-s)} |\mathbf{u}(t, x)|^2 - e^{-(\lambda-2h_0)(t-s)} \|\mathbf{f}\|_\infty^2 - \frac{\varphi(x)}{n}.$$

where  $\mathbf{u} = \mathbf{G}(\cdot, s)\mathbf{f}$ . Our aim consists in proving that  $v_n \leq 0$  in  $[s, T] \times \mathbb{R}^d$  for any  $n \in \mathbb{N}$ . Indeed in this case letting  $n \rightarrow +\infty$  and recalling that  $T$  has been arbitrarily fixed, we obtain  $|\mathbf{u}(t, \cdot)|^2 \leq e^{2h_0(t-s)} \|\mathbf{f}\|_\infty^2$  in  $\mathbb{R}^d$ , for any  $t \in [s, T]$  and the claim follows from the arbitrariness of  $T > s$ .

A straightforward computation shows that

$$D_t v_n(t, x) = e^{-\lambda(t-s)} \left[ (\mathcal{A}_0(t) + 2h - \lambda) |\mathbf{u}(t, \cdot)|^2 - 2V(D_1 \mathbf{u}(t, \cdot), \dots, D_d \mathbf{u}(t, \cdot), \mathbf{u}(t, \cdot)) \right]$$

$$+(\lambda - 2h_0)e^{2h_0(t-s)}\|\mathbf{f}\|_\infty^2],$$

in  $(s, T] \times \mathbb{R}^d$ , where  $\mathcal{A}_0(t) = \operatorname{div}(Q(t, \cdot)D_x)$  and

$$V(\cdot, \cdot, \xi^1, \dots, \xi^d, \zeta) := \sum_{i,j=1}^d q_{ij} \langle \xi^i, \xi^j \rangle - \sum_{j=1}^d \langle B_j \xi^j, \zeta \rangle - \langle (C - h)\zeta, \zeta \rangle$$

for any  $\xi^1, \dots, \xi^d, \zeta \in \mathbb{R}^m$ . Since  $\lambda > 2h_0$ , we can estimate

$$\begin{aligned} & D_t v_n(t, \cdot) - (\mathcal{A}_0(t) + 2h - \lambda)v_n(t, \cdot) - 2(h - h_0)e^{-(\lambda - 2h_0)(t-s)}\|\mathbf{f}\|_\infty^2 \\ & < \frac{1}{n}(\mathcal{A}_0(t) + 2h - \lambda)\varphi - 2e^{-\lambda(t-s)}V(D_1 \mathbf{u}(t, \cdot), \dots, D_d \mathbf{u}(t, \cdot), \mathbf{u}(t, \cdot)), \end{aligned} \quad (\text{A.1})$$

in  $\mathbb{R}^d$  for any  $t \in (s, T]$ . Since  $\lim_{|x| \rightarrow +\infty} v_n(t, x) = -\infty$ , uniformly with respect to  $t \in [s, T]$ ,  $v_n$  attains its maximum at some point  $(t_0, x_0) \in [s, T] \times \mathbb{R}^d$ . If  $t_0 = s$  the proof is complete since  $v_n(s, \cdot) < 0$ . If  $t_0 > s$ , assume by contradiction that  $v_n(t_0, x_0) > 0$ . In this case, since  $\lambda - 2h \geq 0$  in  $I \times \mathbb{R}^d$ , the left-hand side of (A.1) is strictly positive at  $(t_0, x_0)$ .

Thus, it suffices to prove that the right-hand side of (A.1) is nonpositive at  $(t_0, x_0)$  to get a contradiction and to conclude that  $v_n \leq 0$  in  $[s, T] \times \mathbb{R}^d$ .

Since  $D_x v_n(t_0, x_0) = 0$ , it holds that  $\langle D_j \mathbf{u}(t_0, x_0), \mathbf{u}(t_0, x_0) \rangle = D_j \tilde{\varphi}(x_0)/(2n)$  for any  $j = 1, \dots, d$ , where  $\tilde{\varphi} = e^{\lambda(t_0-s)}\varphi$ . Thus it is enough to show that the maximum of the function

$$F_{n,\zeta}(\xi^1, \dots, \xi^d) := \frac{1}{n}(\mathcal{A}_0(t_0) + 2h(t_0, \cdot) - \lambda)\tilde{\varphi}(x_0) - 2V(t_0, x_0, \xi^1, \dots, \xi^d, \zeta),$$

in the set  $\Sigma = \{(\xi^1, \dots, \xi^d) \in \mathbb{R}^{md} : \langle \xi^j, \zeta \rangle = (2n)^{-1}D_j \tilde{\varphi}(x_0), j = 1, \dots, d\}$  is nonpositive. Note that the function  $(\xi^1, \dots, \xi^d) \mapsto V(t_0, x_0, \xi^1, \dots, \xi^d, \zeta)$  tends to  $+\infty$  as  $\|(\xi^1, \dots, \xi^d)\| \rightarrow +\infty$ , for any  $\zeta \in \mathbb{R}^m$ . Hence,  $F_{n,\zeta}$  has a maximum in  $\Sigma$  attained at some point  $(\xi_0^1, \dots, \xi_0^d)$ . Applying the Lagrange multipliers theorem, it can be proved that

$$\xi_0^j = \frac{1}{2n}|\zeta|^{-2}\zeta D_j \tilde{\varphi}(x_0) + \frac{1}{2} \sum_{k=1}^d (Q^{-1})_{jk}(t_0, x_0) [B_k(t_0, x_0)\zeta - |\zeta|^{-2}\langle B_k(t_0, x_0)\zeta, \zeta \rangle \zeta],$$

for  $j = 1, \dots, d$  and, consequently, that

$$\begin{aligned} V(t_0, x_0, \xi_0^1, \dots, \xi_0^d) &= \frac{1}{4n^2|\zeta|^2}|Q^{1/2}(t_0, x_0)D\tilde{\varphi}(x_0)|^2 - \langle (C(t_0, x_0) - h(t_0, x_0))\zeta, \zeta \rangle \\ &\quad - \frac{1}{4} \sum_{i,k=1}^d (Q^{-1})_{ik} \langle B_i(t_0, x_0)\zeta, B_k(t_0, x_0)\zeta \rangle \\ &\quad + \frac{1}{4|\zeta|^2} \sum_{i,k=1}^d (Q^{-1})_{ik} \langle B_i(t_0, x_0)\zeta, \zeta \rangle \langle B_k(t_0, x_0)\zeta, \zeta \rangle \\ &\quad - \frac{1}{2n|\zeta|^2} \sum_{j=1}^d D_j \tilde{\varphi}(x_0) \langle B_j(t_0, x_0)\zeta, \zeta \rangle. \end{aligned}$$

It thus follows that

$$\begin{aligned} \max_{\Sigma} F_{n,\zeta} &= \frac{1}{n}(\mathcal{A}_{\zeta/|\zeta|}(t_0)\tilde{\varphi}(x_0) - \lambda\tilde{\varphi}(x_0)) \\ &\quad - \frac{1}{2n^2|\zeta|^2}|Q^{1/2}(t_0, x_0)D\tilde{\varphi}(x_0)|^2 - \frac{1}{2}|\zeta|^2\mathcal{K}(t_0, x_0, |\zeta|^{-1}\zeta) \leq 0, \end{aligned}$$

and the proof is complete.  $\square$

**Corollary A.2.** *Let assume that Hypotheses 2.1 hold true. Then,*

- (i) if Hypotheses 2.3 are satisfied, then the classical solution  $\mathbf{u}$  of the problem (1.1) satisfies the estimate  $\|\mathbf{u}(t, \cdot)\|_\infty \leq \|\mathbf{f}\|_\infty$ , for any  $t > s \in I$  and  $\mathbf{f} \in C_b(\mathbb{R}^d; \mathbb{R}^m)$ ;
- (ii) if Hypotheses 3.1 are satisfied, then the classical solution of the problem (3.4) satisfies the estimate  $\|\mathbf{v}(t, \cdot)\|_\infty \leq e^{\kappa_0(t-s)} \|\mathbf{f}\|_\infty$ , for any  $t > s \in I$  and  $\mathbf{f} \in C_b(\mathbb{R}^d; \mathbb{R}^m)$ .

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