Anisotropic mean curvature on facets and relations with capillarity

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May 18, 2015

Abstract

Given an anisotropy ϕ on $\mathbb{R}^3$, we discuss the relations between the ϕ-calibrability of a facet $F \subset \partial E$ of a solid crystal E, and the capillary problem on a capillary tube with base F. When F is parallel to a facet $\widetilde{B}_\phi^F$ of the unit ball of ϕ, ϕ-calibrability is equivalent to show the existence of a ϕ-subunitary vector field in F, with suitable normal trace on ∂F, and with constant divergence equal to the ϕ-mean curvature of F. Assuming E convex at F, $\widetilde{B}_\phi^F$ a disk, and F (strictly) ϕ-calibrable, such a vector field is obtained by solving the capillary problem on F in absence of gravity and with zero contact angle. We show some examples of facets for which it is possible, even without the strict ϕ-calibrability assumption, to build one of these vector fields. The construction provides, at least for convex facets of class $\mathcal{C}^{1,1}$, the solution of the total variation flow starting at 1_F.

1 Introduction

The aim of this paper is to point out some connections between crystalline mean curvature of facets of a solid set $E \subset \mathbb{R}^3$, and the capillary problem in absence of gravity. In particular, we are interested in examples of facets $F \subset \partial E$ which admit a subunitary vector field allowing to define an anisotropic mean curvature not easily expressible in terms of a scalar function. The study of anisotropic mean curvature of facets is related to crystalline mean curvature flow [68], [70], [71], [2], [48], [49]: for instance, the constancy of the crystalline mean curvature makes a facet to translate parallely to itself in normal direction, at least for a short time, thus preventing the facet-breaking and bending phenomena [21].

Let us start with a brief overview of the action principle for a capillary, referring the reader for instance to [44], [59], [33] and references therein, for a more complete discussion on this topic.

In the absence of gravity, the *capillary problem* on a bounded connected Lipschitz open set $\Omega \subset \mathbb{R}^m$ ($m = 2$ being the physical case) can be stated as follows: given $b, \mu \in \mathbb{R}$, solve

$$\inf \left\{ \mathscr{G}_\mu(u) \ : \ u \in BV(\Omega), \ \int_\Omega u \, dx = b \right\}, \tag{1.1}$$

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where $BV(\Omega)$ is the space of functions with bounded variation in Ω, and $\mathscr{G}_\mu$ is the strictly convex functional

$$\mathscr{G}_\mu(u) := \int_\Omega \sqrt{1+|Du|^2} - \int_{\partial\Omega} \mu u \, d\mathcal{H}^{m-1}. \tag{1.2}$$

Here, $\int_\Omega \sqrt{1+|Du|^2}$ is the area of the (generalized) graph of u [59], [53], $\mathcal{H}^{m-1}$ is the $(m-1)$-dimensional Hausdorff measure in $\mathbb{R}^m$ [43], u can be thought of as the height of the liquid, and the last term in (1.2) involves the trace of u on $\partial\Omega$. Let $\mu \geq 0$.(1) Then, one can show [59] that, when $\mu > 1$, the functional $\mathscr{G}_\mu$ is unbounded from below. In what follows, we shall confine ourselves to the case(2)

$$\mu \in (0, 1].$$

We then set $\mu = \cos\gamma$, where γ represents, for $m = 2$, the (assigned) contact angle between the liquid and the bounding walls of the capillary tube $\Omega\times\mathbb{R}$. From the first variation computation of $\mathscr{G}_\mu$, supposing for simplicity that $\partial\Omega$ is of class $\mathcal{C}^1$, it turns out that if $\mu \in (0, 1)$, then solving (1.1) is equivalent to find

$$u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^1(\overline{\Omega}) \tag{1.3}$$

such that

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = h \qquad \text{in } \Omega \tag{1.4}$$

for a suitable constant $h \in \mathbb{R}$ independent of b. The *prescribed mean curvature equation* (1.4) is coupled with the Neumann-type boundary condition

$$\frac{\nabla u}{\sqrt{1+|\nabla u|^2}} \cdot \nu^\Omega = \mu \qquad \text{on } \partial\Omega, \tag{1.5}$$

where ν^Ω is the unit normal vector field to $\partial\Omega$ pointing outside of Ω. The constant h is identified integrating by parts, since

$$h = \frac{1}{|\Omega|}\int_\Omega \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) dx = \frac{1}{|\Omega|}\int_{\partial\Omega} \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} \cdot \nu^\Omega \, d\mathcal{H}^{m-1} = \frac{\mu P(\Omega)}{|\Omega|}, \tag{1.6}$$

where $P(\Omega)$ denotes the perimeter of Ω in $\mathbb{R}^m$ and $|\Omega|$ is the Lebesgue measure of Ω. From (1.5), it follows that solutions of (1.1) can be expected only when $\mu < 1$. Once μ has been chosen, the problem becomes to find necessary and sufficient conditions on the set Ω ensuring existence of solutions of (1.3), (1.4) and (1.5). In this respect, it is convenient to introduce the *prescribed mean curvature functionals* defined, for $\lambda \in \mathbb{R}$, and $\mu \in [-1, 1]$, as

$$\mathscr{F}_{\lambda,\mu}(B) := P(B, \Omega) + \mu\mathcal{H}^{m-1}(\partial^* B \cap \partial\Omega) - \lambda|B|, \qquad B \subseteq \Omega,$$

where $\partial^* B$ denotes the reduced boundary [6] of the finite perimeter set B, and $P(\cdot, \Omega)$ is the perimeter in Ω (if $\mu = 1$ we have $\mathscr{F}_{\lambda,1}(B) = P(B) - \lambda|B|$ for any $B \subseteq \Omega$). The problem

$$\inf\{\mathscr{F}_{\lambda,\mu}(B) : B \text{ of finite perimeter}, B \subseteq \Omega\} \tag{1.7}$$

(1)Up to a change of sign of b, this is not restrictive, since $\mathscr{G}_\mu(u) = \mathscr{G}_{-\mu}(-u)$.

(2)If $\mu = 0$, then problem (1.1) is trivially solved by a suitable constant.

has been studied by several authors, see for instance [67], [44], [25], [40], (see also [12], [27], [28], [29], [30], [31], [32]) and references therein. By direct methods, it turns out that there exists a solution of (1.7) and, again, if such a solution is sufficiently regular, its boundary inside Ω has mean curvature$^{(3)}$ equal to λ, and contact angle with $\partial\Omega$ equal to $\arccos\mu$.

Now, let $\mu \in (0,1)$ and h be as in (1.6).$^{(4)}$ Then [44, Chapter 7] there exists a solution of (1.3), (1.4) and (1.5) if and only if

$$0 = \mathscr{F}_{h,\mu}(\emptyset) = \mathscr{F}_{h,\mu}(\Omega) < \mathscr{F}_{h,\mu}(B), \qquad B \subset \Omega,\ B \neq \emptyset; \tag{1.8}$$

moreover, the solution is unique up to an additive constant, and it is bounded from below in Ω. On the other hand [45], if (1.8) is violated, still (1.4) admits a solution in some nonempty set $B^* \subset \Omega$, and such a solution becomes unbounded on $\Omega \cap \partial B^*$. In this situation, the expected physical phenomenon is that the height of the fluid increases unboundedly on $\Omega \setminus B^*$, until part of the base in B^* remains uncovered.
In connection with the case $\mu = 1$, and for taking into account unbounded functions u, we mention that problem (1.1) can be generalized into a minimization over subsets which are not necessarily subgraphs of a function. This formulation is originally due to M. Miranda [60], [61], and has led to the notion of *generalized solution*.
In [52], Giusti proved that (1.8) is a necessary and sufficient condition also in the case $\mu = 1$, thus identifying a "maximal" set Ω where the elliptic equation (1.4) has a solution.$^{(5)}$

Theorem 1.1 ([52]). *Let $\Omega \subset \mathbb{R}^m$ be a bounded connected open set with Lipschitz boundary, and let $h := \frac{P(\Omega)}{|\Omega|}$. Then there exists a solution $u \in \mathcal{C}^2(\Omega)$ of (1.4) if and only if*

$$h < \frac{P(B)}{|B|}, \qquad B \subset \Omega,\ B \neq \emptyset. \tag{1.9}$$

Moreover, if Ω is of class $\mathcal{C}^2$, the solution is unique up to an additive constant, bounded from below in Ω, and its graph is vertical at the boundary of Ω, in the sense that

$$\frac{\nabla u}{\sqrt{1+|\nabla u|^2}} \to \nu^\Omega \quad \text{uniformly on } \partial\Omega. \tag{1.10}$$

Finally, if $m = 2$ and Ω is convex, (1.9) is in turn equivalent to assume that the curvature of $\partial\Omega$, at all points of $\partial\Omega$ where it is defined, is less than or equal to h.

Similarly to the case $\mu \in (0,1)$, if Ω does not satisfy (1.9), the fluid height is expected to become unbounded in correspondence of the complement of some nonempty regular set $B^\star \subset \Omega$ (see Remark 3.6), such that $\Omega \cap \partial B^\star$ has mean curvature equal to h. Moreover, it is proven in [52, Theorem 3.2] that u is unbounded from above around a relatively open region (if any) of $\partial\Omega$ where the maximum of the mean curvature of $\partial\Omega$ equals $P(\Omega)/|\Omega|$.

$^{(3)}$Here and throughout the paper, if $B \subset \mathbb{R}^n$ is of class $\mathcal{C}^2$, we shall call mean curvature of ∂B the sum of the $(n-1)$-principal curvatures of ∂B.

$^{(4)}$Note that, for any $B \subseteq \Omega$, there holds $\mathscr{F}_{h,\mu}(\Omega \backslash B) = \tilde{\mathscr{F}}(B)$, where we set $\tilde{\mathscr{F}}(B) := P(B,\Omega) - \mu\mathcal{H}^{m-1}(\partial^* B \cap \partial\Omega) + h|B|$.

$^{(5)}$The result in [52] is given, more generally, for a right hand side of (1.4) belonging to $\mathrm{Lip}(\Omega) \cap L^\infty(\Omega)$.

Theorem 1.1 provides a subunitary vector field with constant divergence on Ω and satisfying (1.10). Interestingly enough [64], an equation of the form

$$\mathrm{div}\widetilde{N} = \mathrm{const} \qquad \text{in } \Omega, \tag{1.11}$$

hence very similar to (1.4), appears, supposing for simplicity $m = 2$, in connection with the motion of a solid set $E \subset \mathbb{R}^3 = \mathbb{R}^2 \times \mathbb{R}$ by its anisotropic mean curvature. Equation (1.11) is coupled with (1.10) (where the subunitary vector field $\widetilde{N}$ replaces $\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}$), and it gives the ϕ_c-mean curvature of a facet $F \subset \partial E$ which is parallel to the horizontal plane $\mathbb{R}^2$; here, ϕ_c is the norm of $\mathbb{R}^3$ induced by the (portion of) "Euclidean" cylinder

$$B_{\phi_c} := \left\{ (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 \ : \max\left(\sqrt{\xi_1^2 + \xi_2^2}, |\xi_3| \right) \leq 1 \right\}. \tag{1.12}$$

If E is evolving under ϕ_c-mean curvature flow, and ∂E has constant ϕ_c-mean curvature at F, then F is called ϕ_c-calibrable, and it is expected to move parallely to itself for short times.

The notion of calibrability can be given for any convex anisotropy ϕ [20], [21], and in any dimension $m \geq 1$ [5], [37].(6) Noncalibrable facets allow to construct explicit examples of facet breaking-bending phenomena, see again [20], [21]: indeed, it seems reasonable that, at least at time $t = 0$, the facet breaks in correspondance of the jump set of its curvature and bends if the curvature is continuous and not constant.
For simplicity, let us state the problem when $m = 2$, and for a facet $F \subset \partial E$ of a solid set E which is convex at F (Definition 4.3). Let $\phi \colon \mathbb{R}^3 \to [0, +\infty)$ be a convex nonregular anisotropy, and let B_ϕ be the unit ball of ϕ (the Wulff shape). Assume that F is parallel to a facet $\widetilde{B}_\phi^F$ of the Wulff shape, and let $\Pi_F \cong \mathbb{R}^2$ be the affine plane spanned by F. Then, F is said to be ϕ-calibrable if there exists a vector field $\widetilde{N} \in L^\infty(F; \mathbb{R}^2)$ satisfying

$$\begin{cases} \widetilde{N}(x) \in \widetilde{B}_\phi^F & \text{a.e. } x \in F, \\ \mathrm{div}\widetilde{N} = h & \text{a.e. in } F, \\ \langle \widetilde{\nu}^F, \widetilde{N} \rangle = 1 & \mathcal{H}^1 - \text{a.e. on } \partial F, \end{cases} \tag{1.13}$$

where $\widetilde{\nu}^F \in \Pi_F$ is the unit normal vector field to ∂F pointing outside of F, $\langle \cdot, \cdot \rangle$ is a suitable notion of normal trace, and the constant $h > 0$ is again determined by an integration by parts (Section 5). What is more informative, is that (1.13) is obtained as a by-product of a minimization of a functional defined on divergences of vector fields (Section 4), which remains interesting also for facets which are *not* ϕ-calibrable. This corresponds to the case when the right hand side of the second line in (1.13) is not constant anymore. We shall call any vector field solution of the above mentioned minimization problem an *optimal selection* in the (possibly non ϕ-calibrable) facet F. Remarkably, it is possible to prove [21] that a facet is ϕ-calibrable if and only if its "mean velocity" is less than or equal to the mean velocity of

(6) Actually, the case $m = 1$ is trivial, since all edges contained in the boundary of a planar domain have constant anisotropic mean curvature. See for instance [71], and references therein.

any subset of the facet.[7] We say that F is strictly ϕ-calibrable if it is ϕ-calibrable and there is no $B \subset F$, $B \neq \emptyset$, having mean velocity equal to that of F.

The analogy with (1.4), (1.10) is now apparent: a strictly ϕ_c-calibrable facet F such that E is convex at F is nothing but a set Ω where the problem addressed in Theorem 1.1 has a solution. As a consequence (Proposition 6.2), in such a facet there exists an optimal selection which is induced by any scalar function u solving the capillary problem in the relative interior of F with zero contact angle; moreover, this optimal selection is (disregarding the sign) the horizontal component of the outer unit normal vector to the graph of u. Incidentally, we recall [50] that it is not possible for an optimal selection in F to coincide with the gradient of a scalar function, unless the facet is the unit disk. Actually, constructing an optimal selection in non ϕ_c-calibrable facets is the main scope of the present paper (Section 6). Indeed, even if the facet is ϕ_c-calibrable but not strictly ϕ_c-calibrable, it is possible in some case to extend the selection out of the maximal subset of F where the capillary problem is solvable. This extension, in general, may not be induced by a scalar function, nevertheless it still provides information on the regularity of the anisotropic mean curvature at F. Our construction is based on the characterization of sublevel sets of the anisotropic mean curvature (Section 4).

It is worth to notice that, by virtue of [5, Theorem 17], and for a convex facet F of class $\mathcal{C}^{1,1}$, our construction provides also the solution of the total variation flow in $\mathbb{R}^2$ with initial datum the characteristic function of F (see (7.1)). Heuristically, if u is a solution of (7.1), and $p(t) = (x, u(t,x))$ is a point of graph$(u(t)) \subset \mathbb{R}^3$ around which $u(t)$ is sufficiently smooth with nonzero gradient, then the vertical velocity of $p(t)$ equals the mean curvature of the level set of $u(t)$ passing through x; strictly ϕ_c-calibrable flat regions F of graph$(u(t))$ evolve in vertical direction[8] with velocity equal to $P(F)/|F|$; vertical walls (provided $u(t)$ is discontinuous) of graph$(u(t))$ do not move; finally, isolated points where the gradient of $u(t)$ vanishes, such as local minima or local maxima, may develop instantaneously flat horizontal regions. See also [15], [16], [5], [37], and Section 7. Therefore, there are analogies between the total variation flow in $\mathbb{R}^2$ and the anisotropic mean curvature flow of ϕ_c-calibrable facets; however the two motions differ immediately after the initial time. Indeed, even for ϕ_c-calibrable facets, the graph of $v = 1_F$ decreases its height without distortion of the boundary, while the shape of F is expected in general to change for $t > 0$.

The plan of this paper is the following. In Section 2, we recall the definitions of anisotropic perimeter, duality maps, and anisotropic mean curvature, and we fix the family of regular sets we shall deal with. In Section 3, we briefly collect some results on the anisotropic and Euclidean Cheeger problem. In Section 4, we focus on the three dimensional case: we recall the definition of normal trace at a facet, needed to localize the anisotropic mean curvature at a facet. Theorems 4.12-4.16 enlight the relation between sublevel sets of the restriction of κ_ϕ on facets and Cheeger-like problems. In this respect, Theorem 4.15 plays a role also for its application to the construction of optimal selections. This is done in Section 6, which, together with Section 5, contains the main results of this paper. In Section 5, we consider the problem of ϕ-calibrable facets. Let $\widetilde{\phi}$ be the bidimensional metric induced by $\widetilde{B}^F_\phi$, let $P_{\widetilde{\phi}}(F)$

[7] See Theorem 5.2. See also [37] for an extension to convex facets in dimension $n \geq 2$.

[8] Upwards or downwards depending on whether F consists of points of local minimum or local maximum of $u(t)$.

be the $\widetilde{\phi}$-perimeter of F, and denote by $\kappa^F_{\widetilde{\phi}}$ the $\widetilde{\phi}$-mean curvature of ∂F. Then, we show in Theorem 5.8 that

$$\kappa^F_{\widetilde{\phi}} \leq \frac{P_{\widetilde{\phi}}(F)}{|F|} \tag{1.14}$$

is a necessary condition for calibrability when F is $\widetilde{\phi}$-convex. This result was already known for convex facets [21], and in that context the two conditions are actually equivalent. For general $\widetilde{\phi}$-convex facets (Definition 5.6), condition (1.14) is not sufficient for ϕ-calibrability (Example 5.7). In Section 5.1, we prove some facts on the calibrability of "annular" facets. Theorems 5.9-5.11 could be considered as a first step towards an extension to the crystalline setting of the study of "oscillating towers" given in [16]. Finally, in Section 7, we provide a very brief overview on the relations between the total variation flow in $\mathbb{R}^2$ and the arguments considered in the paper.

2 Anisotropic mean curvature

General references for this section are for instance [26], [24], [22], [23].

Let $n := m+1 \geq 2$. A *convex anisotropy* (an anisotropy, for short) on $\mathbb{R}^n$ is any even convex function $\phi : \mathbb{R}^n \to [0, +\infty)$ such that $\phi(\xi) \geq \Lambda|\xi|$, for some $\Lambda > 0$, and $\phi(t\xi) = |t|\phi(\xi)$, for $\xi \in \mathbb{R}^n$ and $t \in \mathbb{R}$.

The *dual* of an anisotropy ϕ on $\mathbb{R}^n$ is the function $\phi^o : \mathbb{R}^{n\star} \to [0, +\infty)$ defined as $\phi^o(\xi^\star) := \sup\{\xi^\star \cdot \xi \,:\, \phi(\xi) \leq 1\}$, which is an anisotropy on the dual $\mathbb{R}^{n\star}$ of $\mathbb{R}^n$. We will usually denote by $B_\phi \subset \mathbb{R}^n$ and $B_{\phi^o} \subset \mathbb{R}^{n\star}$ the closed unit ϕ-ball and ϕ^o-ball respectively,(9) *i.e.*

$$B_\phi := \{\xi \in \mathbb{R}^n \,:\, \phi(\xi) \leq 1\} \qquad B_{\phi^o} := \{\xi^\star \in \mathbb{R}^{n\star} \,:\, \phi^o(\xi^\star) \leq 1\}.$$

By $\mathcal{M}(\mathbb{R}^n)$ we denote the class of all anisotropies on $\mathbb{R}^n$. We say that $\phi \in \mathcal{M}(\mathbb{R}^n)$ is *regular*, and we write $\phi \in \mathcal{M}_{\rm reg}(\mathbb{R}^n)$, if B_ϕ and B_{ϕ^o} have uniformly convex boundary of class $\mathcal{C}^2$. However, the relevant cases for this paper are when $\phi \in \mathcal{M}(\mathbb{R}^n) \setminus \mathcal{M}_{\rm reg}(\mathbb{R}^n)$, namely when ϕ is nonregular, and more precisely:

- when B_ϕ (and B_{ϕ^o}) is an n-dimensional polyhedron. In this case we say that ϕ is a *crystalline anisotropy.*

- when $B_\phi = C \times [-1, 1]$, C being an $(n-1)$-dimensional centrally symmetric convex body. In this case, we say that ϕ a *cylindrical anisotropy.* We say that ϕ is the Euclidean cylindrical norm if $n = 3$ and C is the Euclidean unit disk (see (1.12)).

Definition 2.1 (Duality maps). *Let $\phi \in \mathcal{M}(\mathbb{R}^n)$. We define the (maximal monotone possibly multivalued) one-homogeneous maps $T_\phi : \mathbb{R}^n \to 2^{\mathbb{R}^n}$, $T_{\phi^o} : \mathbb{R}^{n\star} \to 2^{\mathbb{R}^{n\star}}$, as*

$$T_\phi(\xi) := \frac{1}{2}\partial(\phi^2)(\xi), \quad T_{\phi^o}(\xi^\star) := \frac{1}{2}\partial((\phi^o)^2)(\xi^\star), \qquad \xi \in \mathbb{R}^n,\ \xi^\star \in \mathbb{R}^{n\star},$$

where ∂ denotes the subdifferential.

(9) In the literature, B_{ϕ^o} is sometimes called Frank diagram.

The ϕ-*anisotropic perimeter* of a finite perimeter set $E \subset \mathbb{R}^n$ in the open set $\Omega \subseteq \mathbb{R}^n$ is defined as

$$P_\phi(E,\Omega) := c_n \int_{\Omega\cap\partial^* E} \phi^o(\nu^E)\ d\mathcal{H}^{n-1}, \tag{2.1}$$

where ν^E is the generalized unit normal to $\partial^* E$ [6], $c_n := \frac{\omega_n}{|B_\phi|}$ and ω_n is the Lebesgue measure of the Euclidean unit ball of $\mathbb{R}^n$.(10) It turns out that B_ϕ satisfies the following isoperimetric property:(11) for every set $E \subset \mathbb{R}^n$ of finite perimeter and finite Lebesgue measure, we have

$$P_\phi(E) \geq \left(\frac{|E|}{|B_\phi|}\right)^{\frac{n-1}{n}} P_\phi(B_\phi), \tag{2.2}$$

with equality if and only if E coincides (up to a translation) with B_ϕ.
For simplicity, we shall always assume ϕ to be such that the constant c_n in (2.1) is 1.

Let $E \subseteq \mathbb{R}^n$ be a Lipschitz set, and $\nu_{\phi^o} := \frac{\nu^E}{\phi^o(\nu^E)}$. For $x, y \in \mathbb{R}^n$, we set $\mathrm{dist}_\phi(x,y) := \phi(y-x)$, $\mathrm{dist}_\phi(x,E) := \inf_{y\in E} \mathrm{dist}_\phi(x,y)$, and we define the ϕ-signed distance function d^E_ϕ from ∂E as $d^E_\phi(x) := \mathrm{dist}_\phi(x,E) - \mathrm{dist}_\phi(x,\mathbb{R}^n\setminus E)$. It turns out that d^E_ϕ is Lipschitz in a neighbourhood U of ∂E, and it satisfies the eikonal-type equation

$$\phi^o(\nabla d^E_\phi) = 1 \quad \text{a.e. in } U. \tag{2.3}$$

2.1 Regular case

Suppose that $\phi \in \mathcal{M}_{\rm reg}(\mathbb{R}^n)$, and let $E \subset \mathbb{R}^n$ be Lipschitz. The *Cahn-Hoffman vector field* n_ϕ on ∂E is defined as $n_\phi := T_{\phi^o}(\nu_{\phi^o})$, $\mathcal{H}^{n-1}$ almost everywhere on ∂E. When ∂E is compact and of class $\mathcal{C}^2$, there exists a tubular neighbourhood U of ∂E where d^E_ϕ is of class $\mathcal{C}^2$; hence, by (2.3), $\nabla d^E_\phi = \nu_{\phi^o}$ on ∂E. We extend the Cahn-Hoffman vector field n_ϕ on the whole of U as $N_\phi := T_{\phi^o}(\nabla d^E_\phi)$ in U, and we define the $\phi-$*anisotropic mean curvature* κ^E_ϕ of ∂E as $\kappa^E_\phi := \mathrm{div} N_\phi$ on ∂E.
Anisotropic mean curvature appears in the first variation of the anisotropic perimeter functional. More precisely, let $(\Psi_\lambda)_{\lambda\in\mathbb{R}} \subset \mathcal{C}^{1,1}(\mathbb{R}^n;\mathbb{R}^n)$ be a family of diffeomorphisms of the form $\Psi_\lambda(x) := x + \lambda\psi(x)N_\phi(x) + o(\lambda)$ for any $x \in \mathbb{R}^n$, where the scalar function ψ is Lipschitz with compact support in $\mathbb{R}^n$. Then(12)

$$\inf_{\substack{\psi\in \mathrm{Lip}(\partial E),\\ \int_{\partial E}\psi^2\ \phi^o(\nu^E)\ d\mathcal{H}^{n-1}\leq 1}} \frac{d}{d\lambda} P_\phi(\Psi_\lambda(E))_{|\lambda=0} = -\left(\int_{\partial E} (\kappa^E_\phi)^2\ \phi^o(\nu^E)\ d\mathcal{H}^{n-1}\right)^{\frac{1}{2}}, \tag{2.4}$$

and the infimum is realized by a suitable scalar multiple of κ^E_ϕ.

(10) The constant c_n plays a role in the definition of the ϕ-anisotropic volume $|\cdot|_\phi$, see for instance [26], [24]. We recall that $|\cdot|_\phi = c_n|\cdot|$, so that $|B_\phi|_\phi = \omega_n$ for any $\phi \in \mathcal{M}(\mathbb{R}^n)$.
(11) See [68], [69], [70], [46], [47], and [42] for a quantitative version.
(12) See [25], [17], and [13] for an extension to nonconvex smooth anisotropies.

2.2 Nonregular case

When $\phi \in \mathcal{M}(\mathbb{R}^n) \setminus \mathcal{M}_{\rm reg}(\mathbb{R}^n)$, there can be (for instance for ϕ crystalline or cylindrical) several possible choices of vector fields $N : \partial E \to \mathbb{R}^n$ satisfying $N(x) \in T_{\phi^o}(\nu_{\phi^o}(x))$ for $\mathcal{H}^{n-1}$-almost every $x \in \partial E$.

Definition 2.2 (Selection). *A* selection *on ∂E is an element of*

$$\mathrm{Nor}_\phi(\partial E) := \{N\colon \partial E \to \mathbb{R}^n \ : \ N(x) \in T_{\phi^o}(\nu_{\phi^o}(x)) \text{ for } \mathcal{H}^{n-1}\text{–a.e. } x \in \partial E\}.$$

Definition 2.3 (Neighbourhood ϕ-regular sets). *We say that E is* neighbourhood-Lip ϕ-regular *if there exists a tubular neighbourhood U of ∂E and a bounded vector field $\eta \in \mathrm{Lip}(U;\mathbb{R}^n)$ such that $\eta(z) \in T_{\phi^o}(\nabla d^E_\phi(z))$ for almost every $z \in U$.*

Remark 2.4. In the Euclidean case, E is neighbourhood-Lip ϕ-regular if and only if ∂E is of class $\mathcal{C}^{1,1}$. Neighbourhood regularity of boundaries has some connection with the notion of inner-outer tangent ball: given $r > 0$, we say that E satisfies the rB_ϕ-condition if, for any $x \in \partial E$, there exists $y \in \mathbb{R}^n$ such that $rB_\phi + y \subseteq E$, and $x \in \partial(rB_\phi + y)$. It turns out [14] that, if E is neighbourhood-Lip ϕ-regular, then there exists $r > 0$ such that E and $\overline{\mathbb{R}^n \setminus E}$ satisfy the rB_ϕ-condition. Moreover, if E is convex, then E is neighbourhood-L^∞ ϕ-regular if and only if E and $\overline{\mathbb{R}^n \setminus E}$ satisfy the rB_ϕ-condition for some $r > 0$.

Neighbourhood Lipschitz regularity has been used in [37] to give a characterization of convex subsets of $\mathbb{R}^{n-1}$ which are ϕ-calibrable, see Section 5. In this paper, we shall adopt a second notion of regular sets.

Definition 2.5 (Lip ϕ-regular sets). *We say that E is* Lip ϕ-regular *if there exists a vector field $N \in \mathrm{Nor}_\phi(\partial E) \cap \mathrm{Lip}(\partial E;\mathbb{R}^n)$.*

It turns out that a Lip ϕ-regular set is also neighbourhood Lip ϕ-regular, in the sense of Definition 2.3. Indeed, for any $N \in \mathrm{Nor}_\phi(\partial E)$ it is possible to exhibit a Lipschitz extension of N inside a tubular neighbourhood U of ∂E, see [22].(13)
Definitions 2.3 and 2.5 make sense also when $\phi \in \mathcal{M}_{\rm reg}(\mathbb{R}^n)$; in this case, if U is the tubular neighbourhood of ∂E where (2.3) holds, then the unique vector field $\eta \in \mathrm{Lip}(U;\mathbb{R}^n)$ satisfying Definition 2.3 is $\eta := N_\phi$.
Anisotropic mean curvature is defined, as in formula (2.4), by computing the first variation of the perimeter functional. For $\lambda \in \mathbb{R}$ and $z \in U$, define $\Psi_\lambda(z) := z + \lambda\psi^e(z)N^e(z)$, where $\psi \in \mathrm{Lip}(U)$ and $N^e \in \mathrm{Lip}(U;\mathbb{R}^n)$ is a Lipschitz extension of N on U. It is convenient to introduce the family

$$\mathscr{H}^2_{\rm div}(\partial E) := \big\{N \in \mathrm{Nor}_\phi(\partial E) \ : \ \mathrm{div}_\tau N \in L^2(\partial E)\big\}, \tag{2.5}$$

(13) When ϕ is crystalline, a polyhedron E is Lip ϕ-regular if and only if, at every vertex $v \in \partial E$,

$$\bigcap_{\substack{F \text{ facet of } \partial E,\\ v\in F}} T_{\phi^o}(\nu^F) \neq \emptyset,$$

where the definition of facet is given in Section 4 below.

where the *tangential divergence* of a vector field $N \in \mathrm{Nor}_\phi(\partial E)$ is defined as in [22]. Set

$$\mathcal{K}(N) := \int_{\partial E} (\mathrm{div}_\tau N)^2 \ \phi^o(\nu^E) \ d\mathcal{H}^{n-1}, \qquad N \in \mathscr{H}^2_{\mathrm{div}}(\partial E). \tag{2.6}$$

The following result is proven in [22].

Theorem 2.6 (**First variation in the nonregular case**). *Suppose that E is Lip ϕ-regular. Then*

$$\inf_{\substack{\psi \in \mathrm{Lip}(\partial E), \\ \int_{\partial E} \psi^2 \ \phi^o(\nu^E) \ d\mathcal{H}^{n-1} \leq 1}} \ \liminf_{\lambda \to 0^+} \frac{P_\phi(\Psi_\lambda(E)) - P_\phi(E)}{\lambda} = - \inf_{N \in \mathscr{H}^2_{\mathrm{div}}(\partial E)} (\mathcal{K}(N))^{\frac{1}{2}} \,. \tag{2.7}$$

The minimization problem in (2.7) may admit, in general, more than one solution. Nevertheless, by the strict convexity of $\mathcal{K}$ in the divergence, two minimizers have the same divergence. In the following, we denote by

$$N_{\min} \in \mathscr{H}^2_{\mathrm{div}}(\partial E)$$

any minimizer of (2.6).

Definition 2.7 (**Anisotropic mean curvature**). *The* ϕ-mean curvature κ_ϕ^E *of ∂E is defined as*

$$\kappa_\phi^E := \mathrm{div}_\tau N_{\min}.$$

Actually, Lip ϕ-regular sets have anisotropic mean curvature which is more than just square integrable on ∂E: indeed, the following result holds [23].

Theorem 2.8 (**Boundedness of κ_ϕ^E**). *We have $\kappa_\phi^E \in L^\infty(\partial E)$.*

Some further regularity properties of κ_ϕ^E are expected for those $(n-1)$-dimensional portions of ∂E which correspond (via the map T_{ϕ^o}) to $(n-1)$-dimensional portions of ∂B_ϕ. We shall collect some of these results in Section 4.
When ϕ is the euclidean norm, we omit the dependence on ϕ of the various symbols, thus letting ν in place of ν_ϕ, P in place of P_ϕ, κ in place of κ_ϕ, and so on.

3 Prescribed mean curvature problem

Let $m \geq 2$, $\psi \in \mathcal{M}(\mathbb{R}^m)$, $\Omega \subset \mathbb{R}^m$ be a bounded open set with Lipschitz boundary, and $\beta > 0$. In the following, we shall consider solutions C_β to the *prescribed mean curvature problem*, namely solutions to

$$\inf \left\{ P_\psi(B) - \beta|B| : B \subseteq \Omega, B \neq \emptyset \right\}. \tag{3.1}$$

Existence of solutions of (3.1) can be proved by direct methods. The following regularity result holds.

Theorem 3.1. *Let ψ be the Euclidean norm. Then $\Omega \cap \partial^* C_\beta$ is an analytic hypersurface with constant mean curvature equal to β, and the set $\Omega \cap (\partial C_\beta \setminus \partial^* C_\beta)$ is a closed set with Hausdorff dimension at most $(m-8)$. Moreover, $\partial^* C_\beta$ can meet $\partial^* \Omega$ only tangentially, that is, $\nu^\Omega = \nu^{C_\beta}$ on $\partial^* C_\beta \cap \partial^* \Omega$.*

Proof. The analyticity of $\Omega \cap \partial^* C_\beta$, the closedness and the estimate on the dimension of $\Omega \cap (\partial C_\beta \setminus \partial^* C_\beta)$ follow from classical regularity results, see for instance [67] or [58]. We refer the reader to the latter reference for a proof of the tangentiality condition on $\partial^* C_\beta \cap \partial^* \Omega$. $\square$

For $\psi \in \mathcal{M}_{\mathrm{reg}}(\mathbb{R}^m)$ of class $\mathcal{C}^{3,\alpha}$ on $\mathbb{R}^m \setminus \{0\}$, and $\alpha \in (0,1)$, solutions of (3.1) are hypersurfaces of class $\mathcal{C}^{1,\alpha}$, out of a closed singular set of zero $\mathcal{H}^{m-1}$-measure, see [3].(14) For $m = 2$, in [7] the authors study the problem for a more general notion of perimeter, and prove that the inner boundary of a solution of (3.1) is a Lipschitz curve out of a closed singular set of zero $\mathcal{H}^1$-measure. The result has been improved in [63, Theorem 4.5], with the following theorem.

Theorem 3.2. *Let $\psi \in \mathcal{M}(\mathbb{R}^2)$, $\beta > 0$, and let C_β be a solution of* (3.1)*. Then, every connected component of $\Omega \cap \partial C_\beta$ is contained in a translated of $\beta^{-1} \partial B_\psi$.*

Remark 3.3. In dimension $m > 2$, even with the Euclidean metric, we cannot deduce from Theorem 3.1 that any connected component of $\Omega \cap \partial C_\beta$ is contained in the boundary of a ball of radius β^{-1}, see for instance [54] for an explicit example.

The ψ-Cheeger problem(15) for Ω consists in solving

$$\inf \left\{ \frac{P_\psi(B)}{|B|} : B \subseteq \Omega,\ B \neq \emptyset \right\} =: h_\psi(\Omega), \tag{3.2}$$

see [37], [40]. A minimizer of (3.2) is sometimes called a ψ-Cheeger subset of Ω, while $h_\psi(\Omega)$ is called the ψ-Cheeger constant of Ω. Notice that, when $\beta := h_\psi(\Omega)$, a nonempty set $B \subseteq \Omega$ solves (3.1) if and only if B is a minimizer of (3.2).

Definition 3.4 (**Cheeger and strict Cheeger sets**). *If Ω is a solution of* (3.2)*, we say that Ω is a ψ-*Cheeger set*. If*

$$\frac{P_\psi(\Omega)}{|\Omega|} < \frac{P_\psi(B)}{|B|}, \quad B \subset \Omega,\ B \neq \emptyset, \tag{3.3}$$

we say that Ω is a strict *ψ-Cheeger set.*

If $B \subseteq \Omega$ is a ψ-Cheeger subset of Ω, then B is a ψ-Cheeger set (namely, $h_\psi(B) = h_\psi(\Omega) = \frac{P_\psi(B)}{|B|}$). We say that B is a *strict* ψ-Cheeger subset of Ω provided that B is a ψ-Cheeger subset of Ω, and $\frac{P_\psi(B)}{|B|} < \frac{P_\psi(B')}{|B'|}$, for every $B' \subset B$, $B' \neq \emptyset$.
It can be proved [58] that the union of ψ-Cheeger subsets of Ω is still a ψ-Cheeger subset of Ω.

Definition 3.5 (**Maximal/minimal Cheeger subsets**). *We denote by*

$$\mathrm{Ch}_\psi(\Omega)$$

the maximal *ψ-Cheeger subset of Ω, which is defined as the union of all ψ-Cheeger subsets of Ω.*
Moreover, we say that a ψ-Cheeger subset C of Ω is minimal *if, for any ψ-Cheeger subset $C' \subseteq \Omega$, either $C \subseteq C'$ or $C \cap C' = \emptyset$.*

(14) See also [1], [66] for the case $\psi \in \mathcal{M}_{\mathrm{reg}}(\mathbb{R}^m)$ of class $\mathcal{C}^{2,1}$.
(15) Problem (3.2) has been introduced in [41], in the effort to give an estimate from below for the spectrum of the Laplacian operator.

We observe that any minimal ψ-Cheeger subset of Ω is connected. Existence of $\mathrm{Ch}_\psi(\Omega)$ and of a finite number of minimal ψ-Cheeger subsets is proven for example in [40], [38].
When ψ is the euclidean norm, we omit the dependence on ψ of the various symbols, thus letting $h(\Omega)$ in place of $h_\psi(\Omega)$, $\mathrm{Ch}(\Omega)$ in place of $\mathrm{Ch}_\psi(\Omega)$, and so on.

Remark 3.6. Let $B^\star \subseteq \Omega$ be a minimal Cheeger subset of Ω. Then $B^\star$ satisfies (1.9) (with $B^\star$ replacing Ω, and $h := h(\Omega) = h(B^\star)$). Hence, by Theorem 1.1, the capillary problem in $B^\star$ admits a solution of class $\mathcal{C}^2(\mathrm{int}(B^\star))$.

Concerning uniqueness, examples of planar sets Ω admitting more then one (Euclidean) Cheeger subset, and also an uncountable family of Cheeger subsets, can be found in [55], [58].(16) Further results hold for a convex $\Omega \subset \mathbb{R}^m$, see [4].

Theorem 3.7. *Let $\Omega \subset \mathbb{R}^m$ be convex. Then* $\mathrm{Ch}(\Omega)$ *is the unique Cheeger subset of* Ω*, and it is convex.*

In the anisotropic case(17) $\psi \in \mathcal{M}(\mathbb{R}^m) \setminus \mathcal{M}_{\mathrm{reg}}(\mathbb{R}^m)$, instead, the uniqueness of the Cheeger subset of a convex set $\Omega \subset \mathbb{R}^m$ is proven, at our best knowledge, only in dimension $m = 2$ (see Theorem 3.9); anyway, when Ω is convex, $\mathrm{Ch}_\psi(\Omega)$ is also convex [37, Theorem 6.3]. Both in the Euclidean and in the anisotropic case, there is also a necessary and sufficient condition for a smooth enough convex body to be a ψ-Cheeger set. It appeared at first in [51] for $m = 2$ and ψ Euclidean; in [21] for $m = 2$, $\psi \in \mathcal{M}(\mathbb{R}^2)$; in [4] for $m \geq 2$ and ψ the Euclidean norm; finally in [37] in the whole generality. This latter result is recalled in Theorem 3.8 below.

Theorem 3.8. *Let $\Omega \subset \mathbb{R}^m$ be a convex body satisfying the rB_ϕ-condition(18) for some $r > 0$. Then Ω is a ψ-Cheeger set if and only if*

$$\operatorname*{ess\,sup}_{\partial\Omega} \kappa_\psi^\Omega \leq \frac{P_\psi(\Omega)}{|\Omega|}.$$

Finally we have a complete characterization of the (unique) Cheeger subset of a planar convex domain, proven in [55] for the Euclidean norm and in [56] for a general anisotropy.(19)

Theorem 3.9 (**Cheeger subset of a planar convex domain**). *If $\Omega \subset \mathbb{R}^2$ is a bounded, open and convex set, then* $\mathrm{Ch}_\psi(\Omega)$ *is the union of all ψ-balls of radius $r = h_\psi(\Omega)^{-1}$ that are contained in Ω. Moreover, setting $\Omega_r^- := \{x \in \Omega : \mathrm{dist}_\psi(x, \partial\Omega) > r\}$, we have*

$$\mathrm{Ch}_\psi(\Omega) = \Omega_r^- + rB_\psi,$$

and $|\Omega_r^-| = r^2|B_\phi|$.

(16) Anyway, even when uniqueness fails, it is possible to prove [38] that any connected open set $\Omega \subset \mathbb{R}^m$ with finite volume *generically admits a unique Cheeger subset*, namely it has a unique Cheeger subset up to small perturbations in volume. More precisely, for any compact $K \subset \Omega$, there exists an open set $\Omega_K \subseteq \Omega$ such that $K \subset \Omega_K$, and Ω_k admits a unique Cheeger subset.
(17) In [36, Remark 3.6] the authors extend the uniqueness result in Theorem 3.7 to the case of an anisotropy $\psi \in \mathcal{M}_{\mathrm{reg}}(\mathbb{R}^m)$ and a uniformly convex set $\Omega \subset \mathbb{R}^m$ of class $\mathcal{C}^2$.
(18) Recall Remark 2.4.
(19) In [58] the authors prove that in the Euclidean case most of the peculiarities of the planar convex case can be proven also for bi-dimensional (non necessarily convex) strips.

4 Anisotropic mean curvature on facets

From now on, we shall focus on the case $n = 3$, and

$$\phi \in \mathcal{M}(\mathbb{R}^3) \setminus \mathcal{M}_{\mathrm{reg}}(\mathbb{R}^3).$$

Let E be a Lip ϕ-regular set. We say that $F \subset \partial E$ is a (two-dimensional) *facet* of ∂E if F is the closure of a connected component of the relative interior of $\partial E \cap T_x \partial E$, for some $x \in \partial E$ such that the tangent space $T_x \partial E$ of ∂E at x exists. Given a facet $F \subset \partial E$, by $\Pi_F \subset \mathbb{R}^3$ we denote the affine plane spanned by F. Whenever necessary, we identify Π_F with the plane parallel to Π_F and passing through the origin, and F with its orthogonal projection on this latter plane.

Definition 4.1 (**Facets of ∂E corresponding to facets of the Wulff shape**). *We write $F \in \mathrm{Facets}_\phi(\partial E)$ if F is parallel to a facet $\widetilde{B}^F_\phi$ of ∂B_ϕ, and $\nu_{\phi^o}(F) = \nu_{\phi^o}(\widetilde{B}^F_\phi)$.*

If $F \in \mathrm{Facets}_\phi(\partial E)$, then $\widetilde{B}^F_\phi = T_{\phi^o}(\nu_{\phi^o}(F))$. With a slight abuse of notation, we can see $\widetilde{B}^F_\phi$ as a subset of Π_F. We shall assume, unless otherwise specified, that $\widetilde{B}^F_\phi$ is a convex body which is symmetric with respect to the origin of Π_F. Let $\widetilde{\phi}_F : \Pi_F \to [0, +\infty)$ be the (convex) anisotropy on Π_F such that $\{\widetilde{\phi}_F \leq 1\} = \widetilde{B}^F_\phi$. We denote by $\widetilde{\phi}^o_F$ the dual of $\widetilde{\phi}_F$. we denote by $\kappa^B_{\widetilde{\phi}}$ the $\widetilde{\phi}$-curvature of the boundary of a Lip $\widetilde{\phi}$-regular set $B \subset \Pi_F$. If no confusion is possible, we shall omit the dependence on F of $\widetilde{\phi}_F$, thus writing $\widetilde{\phi}$ in place of $\widetilde{\phi}_F$.

The following regularity result is proven in [23].

Theorem 4.2 (**Bounded variation of anisotropic mean curvature**). *Let $F \in \mathrm{Facets}_\phi(\partial E)$. Then $\kappa^E_\phi \in BV(\mathrm{int}(F))$.*

Another result related to $\mathrm{Facets}_\phi(\partial E)$ allows to detect the anisotropic mean curvature of ∂E at a facet F from a minimization problem on F (Proposition 4.9). We need the following definition.

Definition 4.3 (**Convexity at a facet**). *We say that E is convex (resp. concave) at F if E lies, locally around F, in the half-space obtained as that side of Π_F opposite to (resp. same as) the exterior normal to E at F.*

We recall from [23] a regularity result for the boundary of F, which will be used to give a meaning to the normal trace of a selection (Definition 4.5).

Theorem 4.4. *Let $F \in \mathrm{Facets}_\phi(\partial E)$. Then there exists a finite set $Z_F \subset \partial F$ such that, for any $x \in \partial F \setminus Z_F$, ∂F is a Lipschitz graph locally around x. Moreover, if E is convex (or concave) at F, then F is Lipschitz.*

Now, let $N \in \mathrm{Nor}_\phi(\partial E) \cap \mathrm{Lip}(\partial E; \mathbb{R}^3)$. Notice that the orthogonal component of N with respect to the plane Π_F is constant. Hence,

$$\mathrm{div}_\tau N = \mathrm{div}(\mathrm{proj}_F(N)), \tag{4.1}$$

where $\mathrm{proj}_F(N) \colon F \to \Pi_F$ is the projection of N on F, and its divergence is computed in Π_F. Let $\widetilde{\nu}^F$ be the outer Euclidean unit normal to ∂F (when it exists).

It turns out that

$$\widetilde{\nu}^F \cdot \mathrm{proj}_F(N) = \begin{cases} \widetilde{\phi}^o(\widetilde{\nu}^F(x)) & \text{if } \widetilde{\nu}^F(x) \text{ points outside } E, \\ -\widetilde{\phi}^o(\widetilde{\nu}^F(x)) & \text{if } \widetilde{\nu}^F(x) \text{ points inside } E, \end{cases} \tag{4.2}$$

for any $x \in \partial^* F$ (see [22], [23]).

Definition 4.5 (**Maximal/minimal normal trace c_F^ϕ**). *Let E be a Lip ϕ-regular set, and $F \in \mathrm{Facets}_\phi(\partial E)$. The ϕ*-normal trace *at ∂F,*

$$c_F^\phi \in L^\infty(\partial F),$$

is defined as the right hand side of (4.2)*.*

When E is convex (resp. concave) at F, we have $c_F^\phi = \widetilde{\phi}^o(\widetilde{\nu}^F)$ (resp. $c_F^\phi = -\widetilde{\phi}^o(\widetilde{\nu}^F)$).
We recall [10] that any $N \in \mathscr{H}^2_{\mathrm{div}}(\partial E)$ admits a *normal trace*[20] $\langle \widetilde{\nu}^F, \mathrm{proj}_F(N)\rangle \in L^\infty(\partial F)$. However, we cannot say in general that $\langle \widetilde{\nu}^F, \mathrm{proj}_F(N)\rangle = c_F^\phi$, for any $N \in \mathscr{H}^2_{\mathrm{div}}(\partial E)$. The result is true under stronger regularity assumptions on the behaviour of ∂E around F. We refer the reader to [21] for a related discussion. To our purposes, we can confine ourselves to the case described by Proposition 4.6 below.

Proposition 4.6. *Suppose*

$$F \text{ Lipschitz}, \qquad \partial E \setminus F \text{ intersects transversally } F. \tag{4.3}$$

Then $\langle \widetilde{\nu}^F, \mathrm{proj}_F(N)\rangle = c_F^\phi$, for any $N \in \mathscr{H}^2_{\mathrm{div}}(\partial E)$.

It is now natural to look at the family

$$\mathscr{H}^2_{\mathrm{div}}(F) := \big\{\widetilde{N} \in \mathrm{Nor}_\phi(F) : \ \mathrm{div}\widetilde{N} \in L^2(F), \ \langle \widetilde{\nu}^F, \widetilde{N}\rangle = c_F^\phi \ \ \mathcal{H}^1\text{-a.e. on } \partial F\big\},$$

where $\mathrm{Nor}_\phi(F) := \big\{\widetilde{N} \in L^\infty(F;\Pi_F) \ : \ \widetilde{N}(x) \in \widetilde{B}^F_\phi \text{ for } \mathcal{H}^2\text{-a.e. } x \in F\big\}$.[21] Set also[22]

$$\mathcal{K}_F(\widetilde{N}) := \int_F (\mathrm{div}\widetilde{N})^2 \, dx, \qquad \widetilde{N} \in \mathscr{H}^2_{\mathrm{div}}(F). \tag{4.4}$$

The minimum problem

$$\inf\Big\{\mathcal{K}_F(\widetilde{N}) \ : \ \widetilde{N} \in \mathscr{H}^2_{\mathrm{div}}(F)\Big\} \tag{4.5}$$

admits a solution, and two minimizers have the same divergence.[23]

Definition 4.7 (**Optimal selection**). *Given $F \in \mathrm{Facets}_\phi(\partial E)$, we call* optimal selection *in F, and we denote by $\widetilde{N}_{\min} \in \mathscr{H}^2_{\mathrm{div}}(F)$ any solution of* (4.5)*.*

---

[20] See (5.3) below, with $X := \mathrm{proj}_F(N)$.
[21] Notice that $\mathscr{H}^2_{\mathrm{div}}(F) \neq \emptyset$, by the Lip ϕ-regularity of E.
[22] For notational simplicity, hereafter we shall identify the $\mathcal{H}^2$-measure on F with the two-dimensional Lebesgue measure on Π_F.
[23] Notice that the minimum problem (4.5) is *nonlocal*, in the sense that it depends on the shape of ∂E around F.

Remark 4.8 (Minimality criterion). Let $\widetilde{N}_0 \in \mathscr{H}^2_{\rm div}(F)$ be such that

$$\int_F \mathrm{div}(\widetilde{N}_0)\,\mathrm{div}(\widetilde{N}_0 - \widetilde{N})\ dx \le 0, \qquad \widetilde{N} \in \mathscr{H}^2_{\rm div}(F). \tag{4.6}$$

Then $\widetilde{N}_0$ is an optimal selection in F. In particular, if there exists $\widetilde{N}_0 \in \mathscr{H}^2_{\rm div}(F)$ such that $\mathrm{div}\widetilde{N}_0$ is constant on F, then $\widetilde{N}_0$ is optimal ((4.6) is satisfied with equality instead of the inequality), and necessarily

$$\mathrm{div}\widetilde{N}_0 = \frac{1}{|F|}\int_F \mathrm{div}\widetilde{N}_0\,dx = \frac{1}{|F|}\int_{\partial F} c_F^{\phi}\,d\mathcal{H}^1.$$

Let $\widetilde{N}_{\min} \in \mathscr{H}^2_{\rm div}(F)$ be an optimal selection in F, and set

$$\kappa_{\phi,F} := \mathrm{div}(\widetilde{N}_{\min}).$$

Proposition 4.9 (Restriction and localization on facets). *Assume* (4.3)*. Let $N_{\min} \in \mathscr{H}^2_{\rm div}(\partial E)$ be so that $\kappa^E_\phi = \mathrm{div}_\tau N_{\min}$. Then $\mathrm{proj}_F(N_{\min})$ is an optimal selection in F. In particular,*

$$\kappa^E_\phi = \kappa_{\phi,F} \qquad \mathcal{H}^2\text{-}a.e. \quad in\ F. \tag{4.7}$$

Proof. We follow [21, Remark 4.4 and Proposition 4.6]. Let $N_{\min} \in \mathscr{H}^2_{\rm div}(\partial E)$ (resp. $\widetilde{N}_{\min} \in \mathscr{H}^2_{\rm div}(F)$) be a minimizer of $\mathcal{K}$ (resp. of $\mathcal{K}_F$). Let $N \in L^\infty(\partial E;\mathbb{R}^3)$ be such that $N = N_{\min}$ on $\partial E \setminus F$, and such that $\mathrm{proj}_F(N) = \widetilde{N}_{\min}$. By Proposition 4.6, $N \in \mathscr{H}^2_{\rm div}(\partial E)$. Thus

$$\begin{aligned}
\mathcal{K}(N_{\min}) \le \mathcal{K}(N) &= \int_F (\mathrm{div}\widetilde{N}_{\min})^2\ d\mathcal{H}^2 + \int_{\partial E\setminus F} (\mathrm{div}_\tau N_{\min})^2\ d\mathcal{H}^2 \\
&\le \int_F (\mathrm{div}_\tau N_{\min})^2\ d\mathcal{H}^2 + \int_{\partial E\setminus F} (\mathrm{div}_\tau N_{\min})^2\ d\mathcal{H}^2 \\
&= \int_{\partial E} (\mathrm{div}_\tau N_{\min})^2\ d\mathcal{H}^2 = \mathcal{K}(N_{\min}),
\end{aligned}$$

which gives the statement. □

Despite its obviousness, the following observation will be used repeatedly in Section 6.

Remark 4.10. If there exists $\widetilde{N}_0 \in \mathscr{H}^2_{\rm div}(F)$ such that $\mathrm{div}\widetilde{N}_0 = \kappa_{\phi,F}$ in $\mathrm{int}(F)$, then $\widetilde{N}_0$ is an optimal selection in F, since

$$\int_F \left(\mathrm{div}\widetilde{N}_0\right)^2\ d\mathcal{H}^2 = \int_F (\kappa_{\phi,F})^2\ d\mathcal{H}^2 = \int_F \left(\mathrm{div}\widetilde{N}_{\min}\right)^2\ d\mathcal{H}^2 \le \int_F \left(\mathrm{div}\widetilde{N}\right)^2\ d\mathcal{H}^2,$$

for any $\widetilde{N} \in \mathscr{H}^2_{\rm div}(F)$.

For notational simplicity, and when no confusion is possible, we set

$$\kappa_{\min} := \operatorname{ess\,inf} \kappa_{\phi,F}, \qquad \kappa_{\max} := \operatorname{ess\,sup} \kappa_{\phi,F}. \tag{4.8}$$

Now, we recall from [21] and [22] some results on regularity of facets and on the function $\kappa_{\phi,F}$.

Theorem 4.11 (**Regularity of facets**). *Let $F \in \mathrm{Facets}_\phi(\partial E)$, and let E be convex (or concave) at F. Then F is Lip $\widetilde{\phi}$-regular.*

For $\beta \in [\kappa_{\min}, \kappa_{\max}]$, define

$$\Omega_\beta^F := \{x \in \mathrm{int}(F) \ : \ \kappa_{\phi,F}(x) < \beta\}, \qquad \Theta_\beta^F := \{x \in \mathrm{int}(F) \ : \ \kappa_{\phi,F}(x) \leq \beta\}. \tag{4.9}$$

Theorem 4.12 (**Sublevels of the anisotropic mean curvature**). *Let $F \in \mathrm{Facets}_\phi(\partial E)$, and suppose that E is convex at F. Then $\kappa_{\min} > 0$. Moreover, for any $\beta \in [\kappa_{\min}, \kappa_{\max}]$,*

$$\int_{\Omega_\beta^F} \kappa_{\phi,F} \ dx = P_{\widetilde{\phi}}(\Omega_\beta^F), \qquad \int_{\Theta_\beta^F} \kappa_{\phi,F} \ dx = P_{\widetilde{\phi}}(\Theta_\beta^F), \tag{4.10}$$

and Ω_β^F and Θ_β^F are solutions of the variational problem

$$\inf \left\{ P_{\widetilde{\phi}}(B) - \beta|B| \ : \ B \subseteq F \right\}. \tag{4.11}$$

Remark 4.13. In the setting of Theorem 4.11, assume further $\widetilde{\phi} \in \mathcal{M}_{\mathrm{reg}}(\Pi_F)$. Let $\beta \in [\kappa_{\min}, \kappa_{\max}]$. Since Θ_β^F solves (4.11), the $\widetilde{\phi}$-mean curvature of $\partial\Theta_\beta^F$ is less than or equal to β, and equality holds in $\mathrm{int}(F) \cap \partial\Theta_\beta^F$. A similar result holds for Ω_β^F.

Theorem 4.14 ([63]). *Let $F \in \mathrm{Facets}_\phi(\partial E)$, and suppose that E is convex at F. Then, for any $\beta \in [\kappa_{\min}, \kappa_{\max}]$, $\mathrm{int}(F) \cap \partial\Omega_\beta^F$ and $\mathrm{int}(F) \cap \partial\Theta_\beta^F$ are contained in a translated copy of $\beta^{-1}\partial\widetilde{B}_\phi^F$.*

Now, we want to show that the minimal level set of the curvature corresponds to the maximal $\widetilde{\phi}$-Cheeger subset of F (recall Section 3).

Theorem 4.15. *Let $F \in \mathrm{Facets}_\phi(\partial E)$, and assume that E is convex in F. Then*

$$\Theta_{\kappa_{\min}}^F = \mathrm{Ch}_{\widetilde{\phi}}(F). \tag{4.12}$$

Proof. We start with two preliminary steps.

Step 1. $|\Theta_{\kappa_{\min}}^F| > 0$. Essentially, this fact has been observed in [21, Remark 5.3]. We repeat the proof, for the sake of completeness. Let $\beta > \kappa_{\min}$, so that in particular $|\Theta_\beta^F| > 0$. From Theorem 4.12, using (2.2) (with $\widetilde{\phi}$ replacing ϕ), we get

$$0 = P_{\widetilde{\phi}}(\emptyset) - \beta|\emptyset| \geq P_{\widetilde{\phi}}(\Theta_\beta^F) - \beta|\Theta_\beta^F| \geq \gamma_{\widetilde{\phi}}\sqrt{|\Theta_\beta^F|} - \beta|\Theta_\beta^F|,$$

where $\gamma_{\widetilde{\phi}} := P_{\widetilde{\phi}}(\widetilde{B}_\phi^F)|\widetilde{B}_\phi^F|^{1/2}$. Thus, we deduce the estimate

$$|\Theta_\beta^F| \geq \beta^{-2}\,\gamma_{\widetilde{\phi}}^2 \geq \kappa_{\max}^{-2}\,\gamma_{\widetilde{\phi}}^2, \qquad \beta > \kappa_{\min}. \tag{4.13}$$

By (4.13), and since $\Theta_{\kappa_{\min}}^F = \bigcap_{\beta > \kappa_{\min}} \Theta_\beta^F$, we get Step 1.

Step 2. The $\widetilde{\phi}$-Cheeger constant of F equals $\kappa_{\min}$. By definition of $h_{\widetilde{\phi}}(F)$, using Step 1 and (4.10), we get

$$h_{\widetilde{\phi}}(F) \leq \frac{P_{\widetilde{\phi}}(\Theta_{\kappa_{\min}}^F)}{|\Theta_{\kappa_{\min}}^F|} = \frac{\int_{\Theta_{\kappa_{\min}}^F} \kappa_{\phi,F}\, dx}{|\Theta_{\kappa_{\min}}^F|} = \kappa_{\min}. \tag{4.14}$$

On the other hand, let C be a $\widetilde{\phi}$-Cheeger subset of F. Then, thanks to Theorem 4.12, we get

$$0 = P_{\widetilde{\phi}}(\Theta^F_{\kappa_{\min}}) - \kappa_{\min}|\Theta^F_{\kappa_{\min}}| \le P_{\widetilde{\phi}}(C) - \kappa_{\min}|C| = (h_{\widetilde{\phi}}(F) - \kappa_{\min})|C|. \tag{4.15}$$

Coupling (4.14) with (4.15), we get $h_{\widetilde{\phi}}(F) = \kappa_{\min}$. In particular, $\Theta^F_{\kappa_{\min}}$ is a $\widetilde{\phi}$-Cheeger subset of F and $\Theta^F_{\kappa_{\min}} \subseteq \mathrm{Ch}(F)$.

Now, we prove (4.12).
Suppose, by contradiction, that there exists a $\widetilde{\phi}$-Cheeger subset $C \subseteq F$ such that $|C \setminus \Theta^F_{\kappa_{\min}}| > 0$. We observe that $\kappa_{\phi,F} > \kappa_{\min}$ on $C \setminus \Theta^F_{\kappa_{\min}}$, hence

$$\kappa_{\min}|C| < \int_C \kappa_{\phi,F}\, dx = \int_C \mathrm{div}\widetilde{N}_{\min}\, dx = \int_{\partial^* C} \langle \widetilde{\nu}^F, \widetilde{N}_{\min}\rangle \, d\mathcal{H}^1 \le P_{\widetilde{\phi}}(C), \tag{4.16}$$

where $\widetilde{N}_{\min}$ is any optimal selection on F. At the same time, since C is a $\widetilde{\phi}$-Cheeger subset of F, using Step 2 we have $P_{\widetilde{\phi}}(C) = \kappa_{\min}|C|$, which, coupled with (4.16), leads to a contradiction. $\square$

In the same paper [21], the authors give a stronger regularity result for $\kappa_{\phi,F}$ in the case E is convex at F, and F itself is convex in the Euclidean sense.

Theorem 4.16. *Let $F \in \mathrm{Facets}_\phi(\partial E)$, and assume that E is convex at F. Assume further that F is convex. Then $\kappa_{\phi,F}$* is convex. *Moreover,*

$$\Omega^F_\beta = \bigcup \left\{ B \subseteq \mathrm{int}(F) \ : \ B \text{ is a translated copy of } \beta^{-1}\widetilde{B}^F_\phi \right\}, \qquad \beta > \kappa_{\min},$$

$$\Theta^F_\beta = \bigcup \left\{ B \subseteq F \ : \ B \text{ is a translated copy of } \beta^{-1}\widetilde{B}^F_\phi \right\}, \qquad \beta \ge \kappa_{\min}.$$

Finally, we recall $\kappa_{\phi_c,F} \in \mathrm{Lip}_{\mathrm{loc}}(\mathrm{int}(F))$, see [39, Theorem 2].

5 Calibrability of facets

Let $\phi \in \mathcal{M}(\mathbb{R}^3) \setminus \mathcal{M}_{\mathrm{reg}}(\mathbb{R}^3)$, and let E be a Lip ϕ-regular set. We shall focus on those $F \in \mathrm{Facets}_\phi(\partial E)$ such that $\kappa_{\phi,F}$ is constant. From now on in this section, we shall assume (4.3), and so $\kappa_{\phi,F}$ is the restriction of κ^E_ϕ to F (see (4.7)).

Recalling also Remark 4.8, it follows that $\kappa_{\phi,F}$ is constant in $F \in \mathrm{Facets}_\phi(\partial E)$ if and only if there exists $\widetilde{N} \in L^\infty(F; \Pi_F)$ such that

$$\begin{cases} \widetilde{N}(x) \in \widetilde{B}^F_\phi & \mathcal{H}^2\text{-a.e. } x \in F, \\ \mathrm{div}\widetilde{N} = \frac{1}{|F|}\int_{\partial F} c_F \, d\mathcal{H}^1 & \text{in } F, \\ \langle \widetilde{\nu}^F, \widetilde{N}\rangle = c^\phi_F & \mathcal{H}^1\text{-a.e. on } \partial F. \end{cases} \tag{5.1}$$

The following definition has been proposed in [21].

Definition 5.1 (Calibrability). *We say that* $F \in \mathrm{Facets}_\phi(\partial E)$ *is* ϕ-calibrable *if there exists a solution of* (5.1).

From the view point of crystalline mean curvature flow, the right hand side of the PDE in (5.1), namely

$$v_F := \frac{1}{|F|} \int_{\partial F} c_F^\phi \, d\mathcal{H}^1,$$

can be interpreted as the "mean velocity" of F (in direction normal to $\mathrm{int}(F)$), at time zero. We want to define a similar quantity also for subsets of the facet since, heuristically, subsets of F are expected to move not slower than F, consistently with the comparison principle for crystalline mean curvature flow [18], see Theorem 5.2 below.
Let $B \subseteq F$ be a nonempty set of finite perimeter. We define $c_B^\phi : \partial B \to \mathbb{R}$ as

$$c_B^\phi := \begin{cases} \widetilde{\phi}(\widetilde{\nu}^B) & \text{on } \partial^* B \setminus \partial F, \\ c_F^\phi & \text{otherwise}, \end{cases} \tag{5.2}$$

and we set

$$v_B := \frac{1}{|B|} \int_{\partial^* B} c_B^\phi \, d\mathcal{H}^1.$$

Let us recall [11], [10] that, given a function $u \in BV(\mathrm{int}(F))$ and a vector field $X \in L^\infty(F; \Pi_F)$ with $L^2(F)$-summable divergence, it is possible to define a Radon measure (X, Du) on F by setting

$$(X, Du) \colon \varphi \mapsto -\int_{\mathrm{int}(F)} u \, \varphi \, \mathrm{div} X \, dx - \int_{\mathrm{int}(F)} u \, X \cdot \nabla \varphi \, dx, \quad \varphi \in \mathcal{C}_c^\infty(\mathrm{int}(F));$$

moreover, there exists a function $\langle \widetilde{\nu}^F, X \rangle \in L^\infty(\partial F)$ such that the following generalized Gauss-Green formula holds:

$$\int_{\mathrm{int}(F)} u \, \mathrm{div} X \; dx + \int_{\mathrm{int}(F)} \theta(X, Du) \, d|Du| = \int_{\partial F} \langle \widetilde{\nu}^F, X \rangle \; u \; d\mathcal{H}^1; \tag{5.3}$$

here, $\theta(X, Du) \in L^\infty_{|Du|}(F)$ denotes the density [6] of the measure (X, Du) with respect to $|Du|$. We recall that in [23, Proposition 7.7] it has been shown that

$$-\theta(N_{\min}, D1_{\Omega_\beta^F}) = \widetilde{\phi}^o(\widetilde{\nu}^{\Omega_\beta^F}) = c_{\Omega_\beta^F}^\phi, \qquad \text{a.e. } \beta \in \mathbb{R}, \tag{5.4}$$

where Ω_β^F is the β-sublevel set of $\kappa_{\phi,F}$ (see (4.9)), and where 1_A denotes the characteristic function of a subset $A \subseteq F$.

Theorem 5.2 ([21], **Characterization of** ϕ**-calibrable facets**)**.** *Let* $F \in \mathrm{Facets}_\phi(\partial E)$*. Then,* F *is* ϕ*-calibrable if and only if*

$$v_F \leq v_B, \qquad B \subseteq F, \; B \neq \emptyset. \tag{5.5}$$

Proof. Assume $\widetilde{N}$ to be a solution of (5.1). In particular, $\mathrm{div}\widetilde{N} = v_F$ in F. Let $B \subseteq F$ be a nonempty set of finite perimeter. Integrating $\mathrm{div}\widetilde{N}$ on B and using (5.3) we get

$$v_F|B| = \int_B \mathrm{div}\widetilde{N}\, dx = \int_{\partial^* B} \langle \widetilde{\nu}^B, \widetilde{N}\rangle\, d\mathcal{H}^1 \le \int_{\partial^* B} c_B^\phi\, d\mathcal{H}^1,$$

where we used (5.2) and (5.1). This gives (5.5).
The converse implication can be proved as follows. Assume that F is not ϕ-calibrable. Let $\widetilde{N}_{\min} \in \mathscr{H}^2_{\mathrm{div}}(F)$ be an optimal selection on F. Recalling that almost every sublevel set of a BV function has finite perimeter, there exists $\beta < v_F$ such that $\Omega^F_\beta \neq \emptyset$, and Ω^F_β has finite perimeter. Applying (5.3) with the choice $u := 1_{\Omega^F_\beta}$ and $X := \widetilde{N}_{\min}$, we have

$$\begin{aligned}\int_{\Omega^F_\beta} \mathrm{div}\widetilde{N}_{\min}\, dx &= -\int_{\mathrm{int}(F)\cap\partial^*\Omega^F_\beta} \theta(\widetilde{N}_{\min}, D1_{\Omega^F_\beta})\, d\mathcal{H}^1 + \int_{\partial F} \langle \widetilde{\nu}^F, \widetilde{N}_{\min}\rangle\, 1_{\Omega^F_\beta}\, d\mathcal{H}^1 \\ &= -\int_{\mathrm{int}(F)\cap\partial^*\Omega^F_\beta} \theta(\widetilde{N}_{\min}, D1_{\Omega^F_\beta})\, d\mathcal{H}^1 + \int_{\partial F\cap\partial^*\Omega^F_\beta} \langle \widetilde{\nu}^F, \widetilde{N}_{\min}\rangle\, d\mathcal{H}^1.\end{aligned}$$

Observe that, by definition, $\langle \widetilde{\nu}^F, \widetilde{N}_{\min}\rangle = c^\phi_F = c^\phi_{\Omega^F_\beta}$ on $\partial F \cap \partial^*\Omega^F_\beta$. Therefore, recalling also (5.4), we get

$$\int_{\Omega^F_\beta} \mathrm{div}\widetilde{N}_{\min}\, dx = \int_{\partial^*\Omega^F_\beta} c^\phi_{\Omega^F_\beta}\, d\mathcal{H}^1.$$

Hence,

$$v_F > \beta > \frac{1}{|\Omega^F_\beta|}\int_{\Omega^F_\beta} \mathrm{div}\widetilde{N}_{\min}\, dx = \frac{1}{|\Omega^F_\beta|}\int_{\partial^*\Omega^F_\beta} c^\phi_{\Omega^F_\beta}\, d\mathcal{H}^1 = v_{\Omega^F_\beta},$$

which contradicts (5.5). □

In view of Theorem 5.2, we give the following definition.

Definition 5.3 (Strict ϕ-calibrability). *We say that F is* strictly ϕ-calibrable *if*

$$v_F < v_B \qquad \text{for every nonempty } B \subset F.$$

In the same paper [21], the authors characterize convex ϕ-calibrable facets $F \in \mathrm{Facets}_\phi(\partial E)$ such that E is convex at F.

Theorem 5.4 (ϕ-calibrability for convex E at F and convex F). *Suppose that E is convex at $F \in \mathrm{Facets}_\phi(\partial E)$, and that F is convex. Then, F is ϕ-calibrable if and only if*

$$\operatorname*{ess\,sup}_{\partial F} \kappa^F_{\widetilde{\phi}} \le \frac{P_{\widetilde{\phi}}(F)}{|F|}. \tag{5.6}$$

Hence, under the assumptions of Theorem 5.4, problem (5.1) is solvable if and only if the $\widetilde{\phi}$-curvature of ∂F is bounded above by the mean velocity of F; this means, roughly speaking, that the edges of ∂F cannot be too "short". When $\widetilde{\phi}$ is the Euclidean norm of Π_F, (5.6) has been given by Giusti in [52], compare Theorem 1.1.
The following observation clarifies in which sense calibrability extends Definition 3.4.

Remark 5.5 (**Calibrability versus Cheeger sets**)**.** Suppose that E *is convex* at F. In this case, the mean velocity of any nonempty finite perimeter set $B \subseteq F$ is

$$v_B = \frac{1}{|B|}\int_{\partial^* B} c_B^{\phi}\, d\mathcal{H}^1 = \frac{1}{|B|}\int_{\partial^* B} \widetilde{\phi}^o(\widetilde{\nu}^B)\, d\mathcal{H}^1 = \frac{P_{\widetilde{\phi}}(B)}{|B|}. \tag{5.7}$$

Then, using Theorem 5.2, and recalling also Section 3, ϕ-calibrability (resp. strict ϕ-calibrability) of F is equivalent to the property that F is a $\widetilde{\phi}$-Cheeger (resp. strict $\widetilde{\phi}$-Cheeger) set.

Definition 5.6 ($\widetilde{\phi}$**-convexity**)**.** *We say that* $F \in \mathrm{Facets}_\phi(\partial E)$ *is* $\widetilde{\phi}$-convex *if* $\kappa^F_{\widetilde{\phi}} \geq 0$.

One can ask whether the convexity assumption in Theorem 5.4 can be relaxed to just $\widetilde{\phi}$-convexity of F; the next example shows that this can not be expected in general.

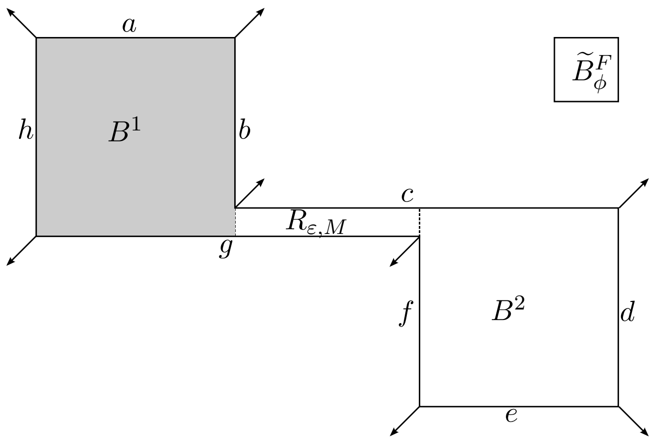

Figure 1: An example of $\widetilde{\phi}$-convex facet F satisfying (5.6), and not ϕ-calibrable ($\varepsilon > 0$ is sufficiently small and M sufficiently large). Here, $\widetilde{B}^F_\phi$ is the square of length ℓ represented on the top right. In grey, a subset of the facet with mean velocity smaller than the mean velocity of F.

Example 5.7. Let $\widetilde{\phi}$ be the two-dimensional crystalline metric having as unit ball the square with side $\ell > 0$, centered at the origin. Let F be as in Figure 1, where B^1 and B^2 are two copies of $\{\widetilde{\phi} \leq 1\}$, rescaled by a factor L/ℓ, and $R_{\varepsilon,M}$ is a rectangle of height ε and base M. We recall [71] that, for planar crystalline sets, $\kappa^F_{\widetilde{\phi}}$ is the derivative of the vector field obtained as the linear interpolation of the vectors at the vertices represented in the figure. Thus, $\kappa^F_{\widetilde{\phi}}$ equals ℓ/L on the sides a, d, e and h, while $\kappa^F_{\widetilde{\phi}}$ vanishes on the sides b, c, f, and g; hence, F is $\widetilde{\phi}$-convex.
Now, let ϕ be the cylindrical norm defined as $\phi(\xi) := \phi(\xi_1, \xi_2, \xi_3) := \max\{\widetilde{\phi}(\xi_1, \xi_2), |\xi_3|\}$, and let $E \subset \mathbb{R}^3$ be any prism with base F, for instance $E = F \times [0, 1]$; in particular, $F \in \mathrm{Facets}_\phi(\partial E)$, and E is convex at F.[24]

[24] Other choices of $\phi \in \mathcal{M}(\mathbb{R}^3) \setminus \mathcal{M}_{\mathrm{reg}}(\mathbb{R}^3)$ are possible, for which there exists $E \subset \mathbb{R}^3$, E Lip ϕ-regular, such that $F \in \mathrm{Facets}_\phi(\partial E)$.

Recalling (5.7), we can compute explicitely the mean velocity of F:

$$v_F = \frac{P_{\widetilde{\phi}}(F)}{|F|} = \frac{2\,(4L - \varepsilon + M)}{2L^2 + \varepsilon M}.$$

Hence $\kappa^F_{\widetilde{\phi}} \le v_F$ when

$$\varepsilon \le \frac{2L(-L\ell + 4L + M)}{\ell M + 2L},$$

the right hand side being positive for M large enough. Now, the mean velocity of B^1 is

$$v_{B^1} = \frac{P_{\widetilde{\phi}}(B^1)}{|B^1|} = \frac{4}{L}.$$

Therefore

$$v_{B^1} < v_F \Longleftrightarrow \varepsilon < \frac{ML}{2M + L}.$$

Hence, for $\varepsilon > 0$ small enough and M large enough, F is not ϕ-calibrable (Theorem 5.4).

However, it is still possible, for $\widetilde{\phi}$-convex facets, to recover one implication from Theorem 5.4.

Theorem 5.8. *Suppose that $\widetilde{\phi}$ is crystalline. Assume that E is convex at $F \in \mathrm{Facets}_\phi(\partial E)$, that F is $\widetilde{\phi}$-convex and ϕ-calibrable. Then* (5.6) *holds.*

Proof. We closely follow the argument in [21, Theorem 8.1]. By contradiction, let $x \in \partial F$ be a point where $\kappa^F_{\widetilde{\phi}}(x) > \frac{P_{\widetilde{\phi}}(F)}{|F|}$. Then, x belongs to the relative interior of an edge L that is parallel to an edge of $\widetilde{B}^F_\phi$, and such that F is convex at L (indeed, $\kappa^F_{\widetilde{\phi}}$ vanishes in all portions of ∂F that do not satisfy the previous requirements, see [71]); with a small abuse of language, we denote by L also the length of this edge, while ℓ is the length of the corresponding edge of $\widetilde{B}^F_\phi$. Since F is $\widetilde{\phi}$-regular, we can deduce that $B_{L/\ell} \cap U \subset F$, where U is a neighbourhood of the side L, while $B_{L/\ell}$ denote the rescaled copy of $\widetilde{B}^F_\phi$ having an edge in L, and lying on the same half-plane of F around L. Applying [21, Lemma 8.3], we get

$$\beta < \kappa^F_{\widetilde{\phi}}(x) < \frac{\ell}{L}, \tag{5.8}$$

where $\beta \in \left(\frac{P_{\widetilde{\phi}}(F)}{|F|}, \kappa^F_{\widetilde{\phi}}(x)\right)$ is such that Ω^F_β solves (4.11).

Following [21, Theorem 8.1, Step 3], let us define, for $\varepsilon > 0$ sufficiently small, the set F_ε of all points of F having Euclidean distance from the line through L greater than or equal to ε. Set $\widehat{F}_\varepsilon := F_\varepsilon \cup B_{L/\ell}$, see Figure 2.

It is possible to prove that, for ε sufficiently small,[(25)]

$$|F| = |\widehat{F}_\varepsilon| + o(\varepsilon^2), \qquad P_{\widetilde{\phi}}(F) = P_{\widetilde{\phi}}(\widehat{F}_\varepsilon). \tag{5.9}$$

[(25)] Clearly, we just need to justify the second equality in (5.9). Let Γ be a connected component of $\partial F \setminus \partial \widehat{F}_\varepsilon$, and let $\varepsilon > 0$ be so small that $\widetilde{\nu}^F_{|\Gamma}$ lies between two consecutive vertices ν_1, ν_2 of the unit ball of $\widetilde{\phi}^o$. Then, $\int_\Gamma \widetilde{\phi}^o(\widetilde{\nu}^F)\, d\mathcal{H}^1 = \widetilde{\phi}^o\Big(\int_\Gamma \widetilde{\nu}^F\, d\mathcal{H}^1\Big)$, where we used Jensen's inequality (which holds with equality, since the restriction of $\widetilde{\phi}^o$ on the segment between ν_1, ν_2 is a linear function). Now, a direct computation shows that the right hand side in the previous equality only depends on the ending points of Γ.

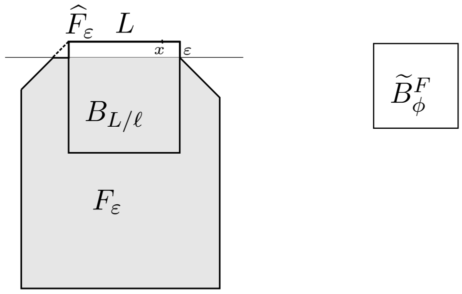

Figure 2: The construction used to prove Theorem 5.8: $\widehat{F}_\varepsilon$ is obtained by slightly modifying F near the edge L (the original boundary is drawn with a dotted line); $B_{L/\ell}$ is the rescaled copy of $\widetilde{B}_\phi^F$ (represented on the top right) having L as an edge; F_ε is the competitor subset.

Moreover, we notice that

$$|\widehat{F}_\varepsilon| - |F_\varepsilon| = \varepsilon L + o(\varepsilon), \tag{5.10}$$

and, using [21, Lemma 8.5],

$$P_{\widetilde{\phi}}(\widehat{F}_\varepsilon) - P_{\widetilde{\phi}}(F_\varepsilon) = \varepsilon l + o(\varepsilon). \tag{5.11}$$

Coupling (5.9), (5.10), and (5.11), also recalling (5.8), we get

$$\begin{aligned} P_{\widetilde{\phi}}(F_\varepsilon) - \beta|F_\varepsilon| =& P_{\widetilde{\phi}}(F) - \varepsilon l + \beta(\varepsilon L - |F|) + o(\varepsilon) \\ =& P_{\widetilde{\phi}}(F) - \beta|F| + \varepsilon(\beta L - l) + o(\varepsilon) < P_{\widetilde{\phi}}(F) - \beta|F|, \end{aligned} \tag{5.12}$$

for $\varepsilon > 0$ sufficiently small. But then, since F is ϕ-calibrable, $F = \Omega_\beta^F$, so that (5.12) violates Theorem 4.12, a contradiction. □

5.1 Annular facets

In this section we prove some facts about the ϕ-calibrability of "annular facets" $F \in \mathrm{Facets}_\phi(\partial E)$. A more general case with B_ϕ the Euclidean cylinder is covered in Theorem 7.3.
For $x \in \Pi_F$, and $\rho > 0$, we denote by $B(x;\rho)$ be the copy of $\rho\widetilde{B}_\phi^F$ centered at x.

Theorem 5.9. *Let $F \in \mathrm{Facets}_\phi(\partial E)$. Assume that there exist x_1, $x_2 \in \mathrm{int}(F)$, and $R > r > 0$ such that*

$$F = B(x_1;R) \setminus B(x_2;r), \qquad B(x_2;r) \subset\subset B(x_1;R),$$

and that $\widetilde{\nu}_F$ points outside of E along $\partial B(x_1;R)$, and inside of E along $\partial B(x_2;r)$.[26] *Then, F is ϕ-calibrable.*

Proof. We start by computing the mean normal velocity of F:

$$v_F = \frac{P_{\widetilde{\phi}}(\widetilde{B}_\phi^F)}{|\widetilde{B}_\phi^F|}\frac{R-r}{R^2-r^2} = \frac{P_{\widetilde{\phi}}(\widetilde{B}_\phi^F)}{|\widetilde{B}_\phi^F|}\frac{1}{R+r}. \tag{5.13}$$

[26] In particular, E is not convex at F.

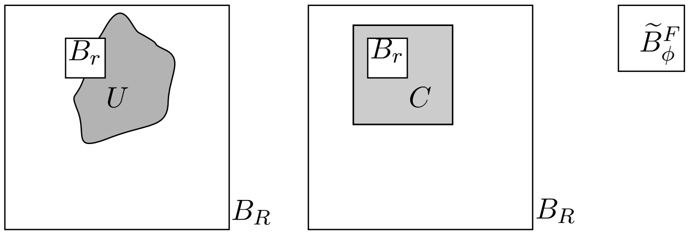

Figure 3: On the top right, as an example we take the square as the unit ball $\widetilde{B}_\phi^F$. We shorthand $B(x_1;R)$ with B_R and $B(x_2;r)$ with B_r. On the left, the facet F, which can be seen as an "annulus". We assume that $\widetilde{\nu}^F$ points outside (resp. inside) of E on ∂B_R (resp. on ∂B_r). In grey, the sets U and C used in Theorem 5.9 to prove the ϕ-calibrability of F.

Let C be any subset of F containing $B(x_2;r)$ and obtained as the difference of two rescaled $\widetilde{\phi}$-balls, namely $C = B(x;t) \setminus B(y;s)$ for suitable $x, y \in \operatorname{int}(F)$, such that $r \le s < t \le R$ and $B(x_2;r) \subset B(y;s) \subset\subset B(x;t)$. Then, recalling (5.2),

$$v_C = \begin{cases} \frac{P_{\widetilde{\phi}}(\widetilde{B}_\phi^F)}{|\widetilde{B}_\phi^F|} \frac{t+s}{t^2-s^2} = \frac{P_{\widetilde{\phi}}(\widetilde{B}_\phi^F)}{|\widetilde{B}_\phi^F|} \frac{1}{t-s}, & \text{if } s > r, \\ \frac{P_{\widetilde{\phi}}(\widetilde{B}_\phi^F)}{|\widetilde{B}_\phi^F|} \frac{t-s}{t^2-s^2} = \frac{P_{\widetilde{\phi}}(\widetilde{B}_\phi^F)}{|\widetilde{B}_\phi^F|} \frac{1}{t+s}, & \text{if } s = r; \end{cases} \tag{5.14}$$

in any case,

$$v_F \le v_C. \tag{5.15}$$

Now, let $U \subset F$ be a nonempty open finite perimeter set; we have to show that $v_U \ge v_F$. Write

$$\partial^- U := \partial U \cap \partial B(x_2;r), \qquad \partial^+ U := \partial U \setminus \partial^- U, \qquad \widehat{U} := U \cup B(x_2;r).$$

Let $t \in (r, R]$ be such that $|\widehat{U}| = |B(x;t)|$, where $x \in \operatorname{int}(F)$ is such that $B(x_2;r) \subset\subset B(x;t) \subset B(x_1;R)$. By the anisotropic isoperimetric inequality (2.2) (with ϕ replaced by $\widetilde{\phi}$), we get

$$P_{\widetilde{\phi}}(\widehat{U}) \ge P_{\widetilde{\phi}}(B(x;t)),$$

that is

$$\int_{\partial^+ U} \widetilde{\phi}^o(\widetilde{\nu}^U)\, d\mathcal{H}^1 + \int_{\partial B(x_2;r)} \widetilde{\phi}^o(\widetilde{\nu}^F)\, d\mathcal{H}^1 - \int_{\partial^- U} \widetilde{\phi}^o(\widetilde{\nu}^F)\, d\mathcal{H}^1 \ge \int_{\partial B(x;t)} \widetilde{\phi}^o(\widetilde{\nu}^{B(x;t)})\, d\mathcal{H}^1. \tag{5.16}$$

Let $C := B(x;t) \setminus B(x_2;r)$. Notice that $|C| = |U|$. Then, using also (5.16) and (5.15), we get

$$v_U = \frac{\int_{\partial^+ U} \widetilde{\phi}^o(\widetilde{\nu}^U)\, d\mathcal{H}^1 - \int_{\partial^- U} \widetilde{\phi}^o(\widetilde{\nu}^F)\, d\mathcal{H}^1}{|U|} \ge v_C \ge v_F.$$

$\square$

Remark 5.10. We cannot expect in general to prove ϕ-calibrability of a facet F such that E is convex at F, and which is obtained by removing from a ball a smaller ball. This is a difference compared to what happens when E is not convex at F (Theorem 5.9). To show this fact, let us consider the bidimensional anisotropy having a square of side ℓ as unit ball, and let us consider the facet F in Figure 4, obtained by removing from a rescaled ball $\frac{S}{\ell}\widetilde{B}^F_\phi(x_1)$ the ball $\frac{s}{\ell}\widetilde{B}^F_\phi(x_2)$, where S and s are the Euclidean lengths of the sides of the two squares. We assume also that the center of the smaller ball lies on the diagonal of the bigger one, and we denote by a the Euclidean distance between the boundaries of the two balls. The mean velocity of the facet is $v_F = \frac{4}{S-s}$. If we consider the set B in Figure 4 we get

$$v_B = \frac{4S}{S^2 - (a+s)^2},$$

and the inequality $v_B < v_F$ is verified when $a < -s + \sqrt{Ss}$.

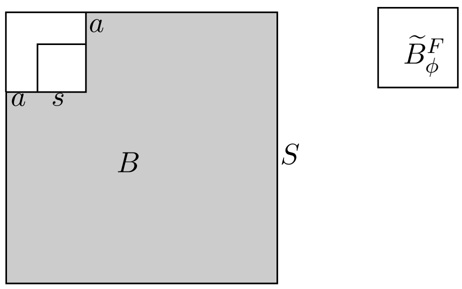

Figure 4: If F is a nonconcentric annulus and E is convex at F, then F is non ϕ-calibrable if the distance a between the two connected components of ∂F is small enough.

5.2 Closed strips

The case of strips has been investigated in [57] in the Euclidean setting. Our aim is to generalize it to the anisotropic setting.
Assume the facet F to have the following shape. Let $\Gamma := \partial\Omega$ be a closed planar simple curve, where Ω is a $\widetilde{\phi}$-regular and $\widetilde{\phi}$-convex set. For some positive integers $0 < l \le k$, we denote by Γ_i, $i = 1, .., l$, the relatively open edges of Γ parallel to some edges on the ball $\widetilde{B}^F_\phi$, and by Γ_j, $j = l+1, ..., k$, each relatively open connected component of Γ with zero $\widetilde{\phi}$-mean curvature (if $k = l$, we mean that there is no such a connected component); κ_i denotes the value of the $\widetilde{\phi}$-curvature of Γ_i. On Γ we take the optimal selection N_Γ, defined as the linear interpolation of the (uniquely determined) vectors on the vertices of Γ; while, on each Γ_j, N_Γ is a constant vector, which we denote by N_{Γ_j}.
For $a > 0$ such that $a \le \inf_{i=1,...,l} \kappa_i^{-1}$, set

$$F := \{x \in \mathbb{R}^2 \ : \ x = q + tN_\Gamma(q), \ q \in \Gamma, \ |t| \le a\}.$$

Due to the $\widetilde{\phi}$-convexity of Γ and to the bound on a, for any $x \in F$ the $\widetilde{\phi}$-projection $q(x)$ is uniquely determined, and it satisfies $x = q(x) + t(x)N_\Gamma(q(x))$ with $t(x) := d^\Omega_{\widetilde{\phi}}(x)$.

Theorem 5.11. *Assume that E is convex at F. Then F is ϕ-calibrable, and $\kappa_{\phi,F} = \frac{1}{a}$.*

Proof. In order to prove the statement, recalling also Remark 4.8, we want to construct a selection with divergence constantly equal to $\frac{1}{a}$. Following [57],(27) we define the vector field $\widetilde{N}$ on F as

$$\widetilde{N}(x) := \begin{cases} \left(1 - \frac{(\kappa_i^{-1}-a)(\kappa_i^{-1}+a)}{(\widetilde{\phi}(x-x_i))^2}\right) \frac{x-x_i}{2a}, & q(x) \in \Gamma_i,\ i = 1, ..., l \\ \\ \frac{d^{\Omega}_{\widetilde{\phi}}(x)}{a} N_{\Gamma_j}, & q(x) \in \Gamma_j,\ j = l+1, ..., k, \end{cases}$$

where, for $i = 1, \ldots, l$, x_i is the center of the copy of $\kappa_i^{-1} B_{\widetilde{\phi}}$ having Γ_i as an edge, lying in the side of Γ_i opposite to the direction of N_Γ. An immediate computation shows that $\widetilde{\phi}(\widetilde{N}(x)) \leq 1$, and $\langle \widetilde{\nu}^F, \widetilde{N} \rangle = 1 = c_F$, so that $\widetilde{N}$ is a selection on F.
Moreover, we notice that

$$\widetilde{N} \in \mathscr{H}^2_{\mathrm{div}}(F). \tag{5.17}$$

Indeed, for every $x \in F$, $\widetilde{N}(x)$ is parallel to $N_\Gamma(q(x))$,(28) which implies that $\mathrm{div}\widetilde{N} \in L^2(F)$, and hence (5.17).
Let us explicitely compute the divergence of $\widetilde{N}$. For any $i = 1, ..., l$ and for any $x \in F$ such that $q(x) \in \Gamma_i$, there holds:

$$\mathrm{div}\widetilde{N}(x) = \frac{1}{a}\left(\frac{(\widetilde{\phi}(x-x_i))^2 - (\kappa_i^{-1})^2 + a^2}{(\widetilde{\phi}(x-x_i))^2}\right) + \frac{((\kappa_i^{-1})^2 - a^2)\big(T_{\widetilde{\phi}}(x-x_i) \cdot (x-x_i)\big)}{a\,(\widetilde{\phi}(x-x_i))^4} = \frac{1}{a},$$

where in the last equality we noticed that $T_{\widetilde{\phi}}(x-x_i) \cdot (x-x_i) = (\widetilde{\phi}(x-x_i))^2$. When $x \in F$ is such that $q(x) \in \Gamma_j$, $j = l+1, .., k$ we get:

$$\mathrm{div}\widetilde{N}(x) = \frac{\nabla d^{\Omega}_{\widetilde{\phi}}(x) \cdot N_{\Gamma_j}}{a} = \frac{1}{a}.$$

Hence, $\widetilde{N}$ has constant divergence in F, and the proof is completed. □

Remembering Remark 5.10, we observe that in Theorem 5.11 we cannot easily drop the symmetry with respect to the curve Γ.

6 Optimal selections in facets for the ϕ_c-norm

In this section we shall restrict our attention to the case in which $\phi = \phi_c$ is the Euclidean cylindrical norm in $\mathbb{R}^3 = \mathbb{R}^2 \times \mathbb{R}$, *i.e.* the norm of $\mathbb{R}^3$ whose unit ball B_{ϕ_c} is given by the right hand side of (1.12). We shall assume that E is a Lip ϕ-regular set, $F \in \mathrm{Facets}_\phi(\partial E)$, and E is convex at F. Hence, by Theorems 4.12 and 2.8, we have $\kappa_{\min} > 0$ and $\kappa_{\max} < +\infty$. Notice that $\widetilde{\phi}_F = \widetilde{\phi}$ is the Euclidean norm in the plane Π_F (identified with the horizontal plane $\mathbb{R}^2$), so that F is of class $\mathcal{C}^{1,1}$ (Theorem 4.11). To avoid possible ambiguity in the notation,

(27) See also [15] for a similar computation.

(28) In general, $\widetilde{N}$ is not continuous in F, since it may jump on $\{x \in F \ :\ q(x) \text{ is a vertex of } \Gamma\}$.

in this section we shall restore symbol $\kappa^F_{\widetilde{\phi}}$ in order to denote the (Euclidean) curvature of ∂F.

From now on, by $h(F)$ we mean $h(\mathrm{int}(F))$, and by $\mathrm{Ch}(F)$ we mean $\mathrm{Ch}(\mathrm{int}(F))$. It is useful to remember that, by Theorem 4.15, we have $h(F) = \kappa_{\min}$.

Remark 6.1. Let u be a solution of (1.4), with $\Omega := \mathrm{int}(F)$ and $h := h(F)$. Repeating the proof in [52, Section 2], which is still valid assuming Ω of class $\mathcal{C}^{1,1}$, one proves that u is bounded from below in $\mathrm{int}(F)$ and satisfies (1.10).

We recall that, by Remark 5.5, F is strictly ϕ_c-calibrable if and only if F is a strict Cheeger set, which in turn is equivalent, when F is convex, to require that $\mathrm{ess\ sup}_{x\in\partial F}\kappa^F_{\widetilde{\phi}}(x) \le h(F)$.

Proposition 6.2. *Suppose that F is strictly ϕ_c-calibrable. Then there exists u solving* (1.4) *in* $\Omega := \mathrm{int}(F)$. *Moreover, the vector field*

$$\widetilde{N} := \begin{cases} \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} & \text{in } \mathrm{int}(F), \\ \\ \widetilde{\nu}^F & \text{on } \partial F, \end{cases} \tag{6.1}$$

is an optimal selection in F, continuous in F and analytic in $\mathrm{int}(F)$.

Proof. The first assertion follows recalling Remark 5.5 and Theorem 1.1. By construction, using also Remark 6.1, $\widetilde{N}$ belongs to $\mathscr{H}^2_{\mathrm{div}}(F)$ and satisfies (5.1). Analytic regularity of $\widetilde{N}$ follows from elliptic regularity. □

Clearly, the vector field $\widetilde{N}$ in (6.1) is, up to the sign, the "horizontal" component of the Euclidean outer normal to the subgraph of u.

Remark 6.3 (Lipschitz regularity). From [52, p. 125] it follows that if $\mathrm{ess\ sup}_{x\in\partial F}\kappa^F_{\widetilde{\phi}}(x) < \frac{P(F)}{|F|} = h(F)$, then

$$\widetilde{N} \in \mathrm{Lip}(F; \Pi_F).$$

6.1 Examples of optimal selections in non ϕ_c-calibrabile facets

In this section we give some examples of non ϕ_c-calibrable facets F for which we can exhibit an optimal *continuous* selection.

Example 6.4 (Non ϕ_c-calibrable convex facets). Let F be convex and not ϕ_c-calibrable (see Theorem 5.4). By virtue of Theorem 3.7, the maximal Cheeger subset $\mathrm{Ch}(F)$ of F is strictly Cheeger, and (Theorem 3.1) it is of class $\mathcal{C}^{1,1}$. Moreover (Theorem 4.15) $\mathrm{Ch}(F) = \Theta^F_{\kappa_{\min}}$. Applying Proposition 6.2 with $\mathrm{Ch}(F)$ in place of F, we get a function $u \in \mathcal{C}^2(\mathrm{int}(\mathrm{Ch}(F))$ solving (1.4) in $\mathrm{int}(\mathrm{Ch}(F))$ with $h := h(F)$. Set

$$\widetilde{N} := \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} \qquad \text{in } \ \mathrm{int}(\mathrm{Ch}(F)).$$

By Theorem 4.16, $\kappa_{\phi,F}$ is convex in F, so that there cannot be subsets of F with positive Lebesgue measure where $\kappa_{\phi,F}$ is constant, except for $\Theta^F_{\kappa_{\min}}$. Hence, for every $\beta \in [\kappa_{\min}, \kappa_{\max})$,

$\mathrm{int}(F) \cap \partial\Theta^F_\beta = \{x \in \mathrm{int}(F) : \kappa_{\phi,F}(x) = \beta\}$. From Theorems 4.12 and 4.16, each connected component of $\mathrm{int}(F) \cap \partial\Theta^F_\beta$ is contained in a circumference of radius β^{-1}. Thus, we extend $\widetilde{N}$ in $\mathrm{int}(F) \setminus \Theta^F_{\kappa_{\min}}$ as the outward normal unit vector to the level curves of $\kappa_{\phi,F}$ — namely, $\widetilde{N} := \widetilde{\nu}^{\partial\Theta^F_\beta}$ on $\{\kappa_{\phi,F} = \beta\}$. By construction, recalling also Remark 6.1, $\mathrm{div}\widetilde{N} = \kappa_{\phi,F}$ in $\mathrm{int}(F)$, and $\widetilde{N}$ verifies the third equation in (5.1). Hence, $\widetilde{N} \in \mathscr{H}^2_{\mathrm{div}}(F)$, and $\widetilde{N}$ is an optimal selection in F (Remark 4.10). Moreover, $\widetilde{N}$ is continuous in F, analytic in $\mathrm{int}(\Theta^F_{\kappa_{\min}})$, and $\widetilde{N}(x) \in \partial\widetilde{B}^F_\phi$ for any $x \in \mathrm{int}(F) \setminus \Theta^F_{\kappa_{\min}}$.

The following examples have been inspired by [44], [58]. For $r > 0$ and $(\bar{x}_1, \bar{x}_2) \in \mathbb{R}^2$, we set $B_r(\bar{x}_1, \bar{x}_2) := \{x = (x_1, x_2) \in \mathbb{R}^2 : (x_1 - \bar{x}_1)^2 + (x_2 - \bar{x}_2)^2 \leq r^2\}$.

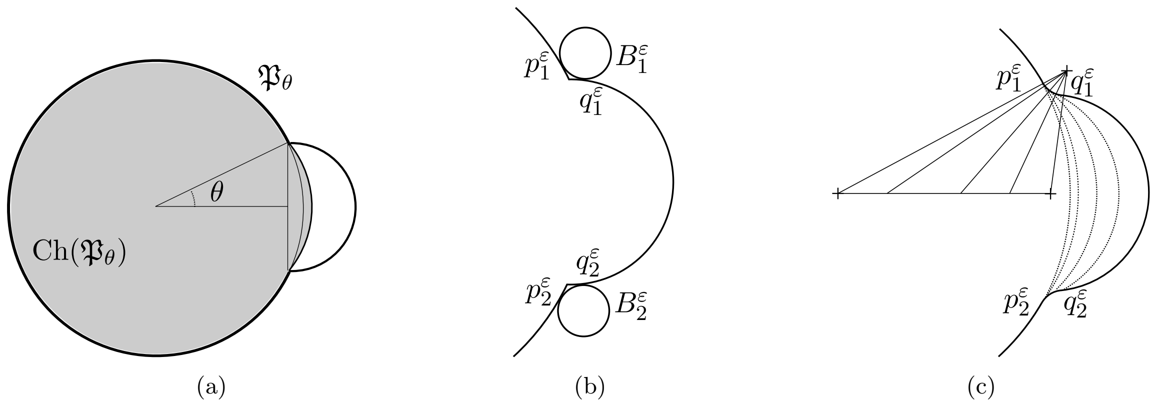

Figure 5: In (a), the set $\mathfrak{P}_\theta$ and its maximal Cheeger subset $\mathrm{Ch}(\mathfrak{P}_\theta)$ (in grey). In (b), the construction of the facet $F = F_\varepsilon$ in Example 6.5. In (c), some sublevel sets Θ^F_β of $\kappa_{\phi_c,F}$ are represented. For every $\beta \in (\kappa_{\min}, \kappa_{\max})$, $\mathrm{int}(F) \cap \partial\Theta^F_\beta$ is an arc of circumference with radius β^{-1}, and tangent to ∂F. For any $\beta \in (1, \frac{1}{\sin\theta})$, such an arc is unique, and its terminal points belong to the arcs bounded by p_j^ε and q_j^ε, for $j = 1, 2$.

Example 6.5 (Rounded two circle facets). Let $\theta \in (0, \frac{\pi}{2})$, and

$$\mathfrak{P}_\theta := B_1(0,0) \cup B_{\sin\theta}(\cos\theta, 0).$$

One can prove [58] the following facts: $\mathfrak{P}_\theta$ admits a unique (hence maximal) Cheeger subset $\mathrm{Ch}(\mathfrak{P}_\theta)$ (as in Figure 5(a)); moreover, there exists a unique $\theta_0 \in (0, \frac{\pi}{2})$ such that $\mathfrak{P}_{\theta_0}$ is Cheeger. Our idea is to construct an optimal selection, solving (1.4) in $\mathrm{Ch}(\mathfrak{P}_\theta)$ (for $\theta \neq \theta_0$), and then foliate the remaining part of $\mathfrak{P}_\theta$ with arcs of circles, taking as vector field the outward unit normal to the arcs. Fix $\theta \neq \theta_0$, so that

$$h(\mathfrak{P}_\theta) = \frac{P(\mathrm{Ch}(\mathfrak{P}_\theta))}{|\mathrm{Ch}(\mathfrak{P}_\theta)|} < \frac{P(\mathfrak{P}_\theta)}{|\mathfrak{P}_\theta|}. \tag{6.2}$$

Notice that

$$h(\mathfrak{P}_\theta) < \frac{1}{\sin\theta}, \tag{6.3}$$

since $h(\mathfrak{P}_\theta)$ equals the curvature of $\mathrm{int}(\mathfrak{P}_\theta) \cap \partial\mathrm{Ch}(\mathfrak{P}_\theta)$, which is strictly less than $\frac{1}{\sin\theta}$ by the geometry of $\mathfrak{P}_\theta$.

Even if $\mathfrak{P}_\theta$ is regarded as a facet of a three-dimensional set E convex at $\mathfrak{P}_\theta$, the set E cannot be Lip ϕ_c-regular, since $\mathfrak{P}_\theta$ is not of class $\mathcal{C}^{1,1}$.[29] Thus we perform the following smoothing construction near the non-differentiability points of $\partial\mathfrak{P}_\theta$. For $\varepsilon > 0$, let B_1^ε, B_2^ε be the two closed disks satisfying the following properties: for $j = 1, 2$, B_j^ε is externally tangent to $\mathfrak{P}_\theta$, and $\mathfrak{P}_\theta \cap B_j^\varepsilon = \{p_j^\varepsilon, q_j^\varepsilon\}$, for some $p_j^\varepsilon \in \partial B_1(0,0)$, and $q_j^\varepsilon \in \partial B_{\sin\theta}(\cos\theta, 0)$. According to Figure 5(b), we define $F = F_\varepsilon$ as the union of $\mathfrak{P}_\theta$ with the curved triangles having vertices p_j^ε, q_j^ε, and $(\cos\theta, (-1)^j \sin\theta)$, for $j = 1, 2$.
By construction F is of class $\mathcal{C}^{1,1}$ (and it is not convex). Recalling also (6.2), we choose $\varepsilon > 0$ so small that

$$\left| \frac{P(\mathfrak{P}_\theta)}{|\mathfrak{P}_\theta|} - \frac{P(F)}{|F|} \right| = O(\varepsilon) < \frac{P(\mathfrak{P}_\theta)}{|\mathfrak{P}_\theta|} - \frac{P(\mathrm{Ch}(\mathfrak{P}_\theta))}{|\mathrm{Ch}(\mathfrak{P}_\theta)|}. \tag{6.4}$$

In particular,

$$\frac{P(\mathrm{Ch}(\mathfrak{P}_\theta))}{|\mathrm{Ch}(\mathfrak{P}_\theta)|} < \frac{P(F)}{|F|},$$

which implies that F is not Cheeger, or equivalently (Remark 5.5) that F is not ϕ_c-calibrable.[30]
We observe that, for any $\beta \in (1, \frac{1}{\sin\theta})$, there is a *unique* circumference $\widehat{\Gamma}_\beta \subset F$, with curvature β, and tangent to ∂F at two points, lying on the arcs of ∂F bounded by p_j^ε, q_j^ε, for $j = 1, 2$: see Figure 5(c). We denote by Γ_β the shortest connected component of $\mathrm{int}(F) \cap \widehat{\Gamma}_\beta$. Then $\mathrm{Ch}(F)$ is determined as the subset of F containing $B_1(0,0)$,[31] and such that $\mathrm{int}(F) \cap \partial\mathrm{Ch}(F) = \Gamma_{h(F)}$. In particular, $\mathrm{Ch}(F)$ is strictly Cheeger and of class $\mathcal{C}^{1,1}$. Furthermore, recalling Remark 4.13, and taking into account the geometry of F, we have

$$\mathrm{int}(F) \cap \partial\Theta_\beta^F = \Gamma_\beta = \mathrm{int}(F) \cap \partial\Omega_\beta^F, \qquad \beta \in (\kappa_{\min}, \kappa_{\max}). \tag{6.5}$$

Now, we exclude the presence of regions in $\mathrm{int}(F) \setminus \mathrm{Ch}(F)$ where $\kappa_{\phi_c,F}$ is constant. Suppose by contradiction that there exists $\bar\beta \in (\kappa_{\min}, \kappa_{\max}]$ such that $\{\kappa_{\phi_c,F} = \bar\beta\}$ has positive Lebesgue measure. If $\bar\beta < \kappa_{\max}$, then

$$\mathrm{int}(F) \cap \partial\Theta_{\bar\beta}^F \neq \mathrm{int}(F) \cap \partial\Omega_{\bar\beta}^F, \tag{6.6}$$

which contradicts (6.5). If $\bar\beta = \kappa_{\max}$, then $\partial\Theta_{\kappa_{\max}}^F = \partial F$, and so (Remark 4.13) $\operatorname{ess\,sup} \kappa_{\widetilde{\phi}}^F = \frac{1}{\sin\theta} \leq \kappa_{\max}$. On the other hand, since we are assuming $\mathrm{int}(F) \cap \partial\Omega_{\kappa_{\max}}^F \neq \emptyset$, $\mathrm{int}(F) \cap \partial\Omega_{\kappa_{\max}}^F$ should be an arc of circumference with curvature $\kappa_{\max}$, and tangent to ∂F. In particular, by the geometry of F, $\kappa_{\max} < \frac{1}{\sin\theta}$, a contradiction.
As a consequence, we have

$$\kappa_{\max} = \frac{1}{\sin\theta},$$

otherwise $\kappa_{\phi_c,F}$ would be constantly equal to $\kappa_{\max}$ in the full-measure subset of F bounded by $\Gamma_{\kappa_{\max}}$, and not containing $B_1(0,0)$ — again a contradiction.
We define $\widetilde{N}$ in F as follows: $\widetilde{N} := \frac{\nabla u}{\sqrt{1+|\nabla u|^2}}$ in $\mathrm{int}(\mathrm{Ch}(F))$, where u is given by Theorem 1.1, with $\Omega = \mathrm{int}(\mathrm{Ch}(F))$ and $h = h(F)$; while, for $\beta \in (h(F), \frac{1}{\sin\theta})$ and $x \in \Gamma_\beta$, $\widetilde{N}(x)$ is the

[29] Therefore, strictly speaking, we cannot apply Theorem 2.6 in order to define κ_ϕ^E on $\mathfrak{P}_\theta$. In the present paper we do not want to insist on the minimal regularity assumptions on ∂F needed the study problem (4.5).
[30] Our argument neither provides nor excludes the ϕ_c-calibrability of $F := \mathfrak{P}_{\theta_0}^\varepsilon$.
[31] Actually, we have $B_1(0,0) \subset \mathrm{Ch}(F)$, since it can be proven that $B_1(0,0) \subset \mathrm{Ch}(\mathfrak{P}_\theta)$.

outward unit normal to Θ^F_β at x. Notice that $\widetilde{N} \in \mathscr{H}^2_{\mathrm{div}}(F)$ (Remark 6.1), and it is an optimal selection in F (Remark 4.10). Concerning the regularity of $\widetilde{N}$, we notice that $\widetilde{N}$ is continuous in F, and analytic in $\mathrm{int}(\mathrm{Ch}(F))$. Moreover, $\widetilde{N}(x) \in \partial \widetilde{B}^F_\phi$ for any $x \in \mathrm{int}(F) \setminus \mathrm{Ch}(F)$.

By modifying Example 6.5, we now build an optimal selection for a facet F admitting an open region outside of $\Theta^F_{\kappa_{\min}}$ where $\kappa_{\phi_c,F}$ is constant (and equal to $\kappa_{\max}$).

Example 6.6 (Rounded proboscis). Let $M > 0$, and let θ, θ_0 and $\mathfrak{P}_\theta$ be as in Example 6.5. Set

$$\mathfrak{P}_{\theta,M} := B_1(0,0) \cup \{x \in \mathbb{R}^2 : \ \ x = y + (c,0), \ \ y \in B_{\sin\theta}(\cos\theta, 0), \ \ c \in [0,M]\},$$

see Figure 6(a).

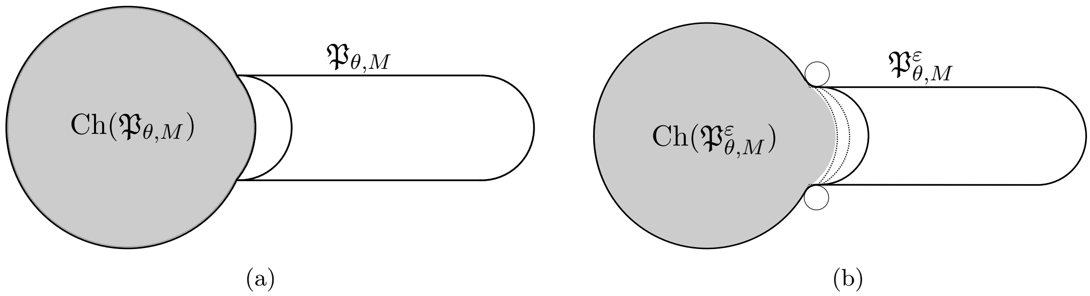

Figure 6: In (a), the set $\mathfrak{P}_{\theta,M}$ and its Cheeger subset $\mathrm{Ch}(\mathfrak{P}_{\theta,M})$. In (b), the facet $F = F_\varepsilon$ described in Example 6.6. In this case, there are two full-measure subsets where $\kappa_{\phi_c,F}$ is constant.

We claim that, for any $M > 0$ and any $\theta < \theta_0$,

$$\frac{P(\mathfrak{P}_\theta)}{|\mathfrak{P}_\theta|} < \frac{P(\mathfrak{P}_{\theta,M})}{|\mathfrak{P}_{\theta,M}|}. \tag{6.7}$$

Indeed, since $P(\mathfrak{P}_\theta) = 2(\pi - \theta) + \pi \sin\theta$, $|\mathfrak{P}_\theta| = \pi + \frac{\pi}{2}\sin^2\theta - (\theta - \sin\theta\,\cos\theta)$, $P(\mathfrak{P}_{\theta,M}) = P(\mathfrak{P}_\theta) + 2M$, and $|\mathfrak{P}_{\theta,M}| = |\mathfrak{P}_\theta| + 2M\sin\theta$, (6.7) is equivalent to $P(\mathfrak{P}_\theta)\sin\theta < |\mathfrak{P}_\theta|$, i.e.

$$(\pi - \theta)\,(2\sin\theta - 1) - \sin\theta\,\cos\theta + \frac{\pi}{2}\,\sin^2\theta < 0; \tag{6.8}$$

direct computations[(32)] show that the left hand side of (6.8) is strictly increasing in $[0, \frac{\pi}{2}]$, and it is zero just at one value of $\theta \in (0, \frac{\pi}{2})$, which must coincide with θ_0.

[(32)] Computing the first derivative (w.r.t. θ) of the left hand side of (6.8), we get $2(\pi-\theta)\cos\theta + \pi\cos\theta\sin\theta + 2\sin\theta(\sin\theta - 1)$. Notice that, since $\theta \in (0, \frac{\pi}{2})$, the first term in the previous line is greater than $\pi\cos\theta$. Now, using for instance the identities $\sin\theta = \frac{2t}{1+t^2}$, $\cos\theta = \frac{1-t^2}{1+t^2}$, since $t \in (0,1)$, it is easy to show that for every $\theta \in (0, \frac{\pi}{2})$

$$\begin{aligned}\pi\cos\theta + \pi\cos\theta\sin\theta + 2\sin\theta(\sin\theta - 1) &= (1-t)(1+t^2)^{-2}\left[\pi(1+t^2)(1+t) + 2\pi t(1+t) - 2t(1-t)\right] \\ &\geq (1-t)(1+t^2)^{-2}\left[\pi(1+t^2)(1+t) + 2\pi t(1+t) - 1/2\right] > 0.\end{aligned}$$

Fix $\theta \in (0, \theta_0)$. For $\varepsilon > 0$, let $\mathfrak{P}^{\varepsilon}_{\theta,M}$ be the set of class $\mathcal{C}^{1,1}$ obtained by taking the union of $\mathfrak{P}_{\theta,M}$ with the curved triangles, bounded by $\mathfrak{P}_{\theta,M}$ and a disk with radius ε and externally tangent to $\mathfrak{P}_{\theta,M}$: see Figure 6(b). Similarly to Example 6.5, we choose $\varepsilon > 0$ so small (depending on the difference between the two terms in (6.7)) that $F = F_\varepsilon := \mathfrak{P}^{\varepsilon}_{\theta,M}$ is not Cheeger.
By construction, F is neither convex nor ϕ_c-calibrable. Moreover, for any $\beta \in (1, \frac{1}{\sin\theta})$, there still exists a *unique* circumference $\widehat{\Gamma}_\beta \subset F$ with curvature β, and tangent to ∂F at two points; again, referring to Figure 6(b), these points must lie on the arcs bounded by p^{ε}_j, q^{ε}_j, where $j = 1, 2$. We denote by Γ_β the shortest connected component of $\mathrm{int}(F) \cap \widehat{\Gamma}_\beta$.
Similarly to Example 6.5, we can still determine $\mathrm{Ch}(F)$ as the unique subset of F (strictly) containing $B_1(0,0)$, and such that $\mathrm{int}(F) \cap \partial\mathrm{Ch}(F) = \Gamma_{h(F)}$. In particular, $\mathrm{Ch}(F)$ is strictly Cheeger. Moreover, reasoning as in Example 6.5,[33] there is no $\bar\beta \in (h(F), \frac{1}{\sin\theta})$ such that $\kappa_{\phi_c,F} = \bar\beta$ in some subset of F with positive Lebesgue measure.
Therefore:

- for any $\beta \in (h(F), \frac{1}{\sin\theta})$, Θ^F_β is the closed subset of F containing $B_1(0,0)$, and such that $\mathrm{int}(F) \cap \partial\Theta^F_\beta = \Gamma_\beta$;
- $\kappa_{\max} = \frac{1}{\sin\theta}$, and $\kappa_{\phi_c,F} = \kappa_{\max}$ in $\mathrm{int}(F) \setminus \bigcup_{\beta < \frac{1}{\sin\theta}} \Theta^F_\beta$.

Also in this case, we can exhibit an optimal selection $\widetilde{N}$ (Remarks 4.10-6.1) which is continuous in F, and analytic in $\mathrm{int}(\mathrm{Ch}(F))$. More precisely, $\widetilde{N}$ is defined as follows: $\widetilde{N} := \frac{\nabla u}{\sqrt{1+|\nabla u|^2}}$ in $\mathrm{int}(\mathrm{Ch}(F))$, where u is given in Theorem 1.1 (with the choice $\Omega := \mathrm{int}(\mathrm{Ch}(F))$, and $h := h(F)$); for $\beta \in (h(F), \frac{1}{\sin\theta})$ and $x \in \Gamma_\beta$, $\widetilde{N}(x)$ is the outer normal to Θ^F_β at x; finally, if $\kappa_{\phi_c,F}(x) = \kappa_{\max}$, we set $\widetilde{N}(x) := \widetilde{N}(\tilde{x})$, where $\tilde{x} \in \mathrm{int}(F) \cap \partial\Omega^F_{\frac{1}{\sin\theta}}$ is such that $\tilde{x}_2 = x_2$.
We notice the presence of a full-measure subset of F, unrelated to the maximal Cheeger subset of F, and where it is possible to construct an optimal selection without making use of Theorem 1.1.

We conclude this section with an example in which we are not able to provide an explicit optimal selection, even if we determine the ϕ_c-mean curvature of F.

Example 6.7 ("Dumbbell-like" facet). Let θ and θ_0 be as in Example 6.5, and suppose $\theta \in (0, \theta_0)$. Let $M > 2\sin\theta + 1$, and let $\mathfrak{D}_{\theta,M}$ be the set obtained as the union of $\mathfrak{P}_\theta \cup \mathfrak{P}'_\theta$, and the strip $[\cos\theta, \cos\theta + M] \times [-\sin\theta, \sin\theta]$, where $\mathfrak{P}_\theta$ is the set in Example 6.5, and $\mathfrak{P}'_\theta$ is its symmetric with respect to the straight line $\{(x_1, x_2) \in \mathbb{R}^2 : x_2 = \cos\theta + \frac{M}{2}\}$.
We observe that

$$\frac{P(\mathfrak{D}_{\theta,M})}{|\mathfrak{D}_{\theta,M}|} = \frac{M + 2(\pi - \theta)}{\cos\theta\,\sin\theta + \pi - \theta + M\sin\theta}, \tag{6.9}$$

which, as $M \to +\infty$, tends to $\frac{1}{\sin\theta}$. In particular, since $\theta < \theta_0$, recalling also (6.2) and (6.6),

$$\frac{P(\mathrm{Ch}(\mathfrak{P}_\theta))}{|\mathrm{Ch}(\mathfrak{P}_\theta)|} < \frac{P(\mathfrak{D}_{\theta,M})}{|\mathfrak{D}_{\theta,M}|}, \tag{6.10}$$

for $M > 0$ sufficiently large.

[33] Recall in particular the proof of (6.6).

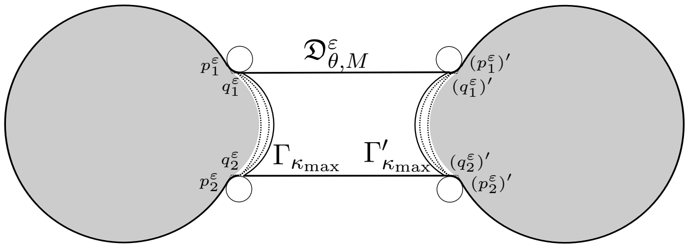

Figure 7: The dumbbell facet $\mathfrak{D}^{\varepsilon}_{\theta,M}$ in Example 6.7. In grey, its maximal Cheeger subset, some level sets of $\kappa_{\phi_c,F}$, and the set $\{\kappa_{\phi_c,F} = \kappa_{\max}\}$ bounded by the arcs $\Gamma_{\kappa_{\max}}$ and $\Gamma'_{\kappa_{\max}}$. Notice that in this case $\kappa_{\max} < \frac{1}{\sin\theta}$.

For $\varepsilon > 0$, let B_j^{ε} and $(B_j^{\varepsilon})'$ for $j = 1, 2$, be the four balls of radius ε externally tangent to $\mathfrak{D}_{\theta,M}$ in p_j^{ε} and q_j^{ε}, and in $(p_j^{\varepsilon})'$ and $(q_j^{\varepsilon})'$ respectively. For M such that (6.10) holds, let $F = F_{\varepsilon} := \mathfrak{D}^{\varepsilon}_{\theta,M}$ be the set of class $\mathcal{C}^{1,1}$ obtained by taking the union of $\mathfrak{D}_{\theta,M}$ with the four curved triangles, bounded by p_j^{ε}, q_j^{ε} and $(\cos\theta, (-1)^j \sin(\theta))$ and $(p_j^{\varepsilon})'$, $(q_j^{\varepsilon})'$ and $(\cos\theta + M, (-1)^j \sin(\theta))$ respectively, see Figure 6.7. Then, we choose $\varepsilon > 0$ so small that (6.10) holds with F replacing $\mathfrak{D}_{\theta,M}$; hence, F is not ϕ_c-calibrable.
For any $\beta \in (1, \frac{1}{\sin\theta})$ let Γ_β (resp. Γ'_β) be the arc of minimal length of the circumference of radius $\frac{1}{\beta}$, which is internally tangent to ∂F in two points, belonging to the arcs bounded by p_j^{ε} and q_j^{ε} (resp. $(p_j^{\varepsilon})'$ and $(q_j^{\varepsilon})'$), for $j = 1, 2$. Let $C_\beta \subset F$ be the disconnected set bounded by $\Gamma_\beta \cup \Gamma'_\beta$, let $C_{\frac{1}{\sin\theta}} := \cup_{\beta \in (1, \frac{1}{\sin\theta})} C_\beta$, and let $\Gamma_{\frac{1}{\sin\theta}}$ and $\Gamma'_{\frac{1}{\sin\theta}}$ be the two connected components of $\mathrm{int}(F) \cap \partial C_{\frac{1}{\sin\theta}}$.
Reasoning as in Example 6.5, $\mathrm{Ch}(F)$ is the disconnected subset of F bounded by $\Gamma_{h(F)}$ and $\Gamma'_{h(F)}$ (see again Figure 6.7). Moreover,[34] for all $\bar\beta \in (\kappa_{\min}, \kappa_{\max})$, we can still exclude the presence of regions of the form $\{\kappa_{\phi_c,F} = \bar\beta\}$ with positive Lebesgue measure. As a consequence,

$$\mathrm{int}(F) \cap \partial\Theta_\beta^F = \Gamma_\beta \cup \Gamma'_\beta = \mathrm{int}(F) \cap \partial\Omega_\beta^F, \quad \beta \in (\kappa_{\min}, \kappa_{\max}).$$

By the geometry of F, $\kappa_{\max} \le \frac{1}{\sin\theta}$. Therefore, we have $|F \setminus \Omega^F_{\kappa_{\max}}| > 0$: indeed, if $Q \subset F$ is the connected (full-measure) set bounded by $\Gamma_{\frac{1}{\sin\theta}} \cup \Gamma'_{\frac{1}{\sin\theta}}$, then $Q \subseteq F \setminus \Omega^F_{\kappa_{\max}}$.

[34] Recall once again the proof of (6.6).

It is interesting to show now that, differently from Example 6.6, the maximal value $\kappa_{\max}$ of $\kappa_{\phi_c,F}$ depends on M, and

$$\kappa_{\max} < \frac{1}{\sin\theta}. \tag{6.11}$$

Indeed, recalling (4.10) and the equality $F = \Theta^F_{\kappa_{\max}}$, the value $\kappa_{\max}$ must verify

$$P(F) - P(\Omega^F_{\kappa_{\max}}) = \kappa_{\max}|F \setminus \Omega^F_{\kappa_{\max}}|. \tag{6.12}$$

Notice that

$$P(F) - P(\Omega^F_{\kappa_{\max}}) = 2M - 2\mathcal{H}^1(\Gamma_{\kappa_{\max}}) + O(\varepsilon). \tag{6.13}$$

We estimate $\mathcal{H}^1(\Gamma_{\kappa_{\max}})$ with the length of the arc of circumference of curvature $\kappa_{\max}$ contained in $B_{\sin\theta}(\cos\theta, 0)$, and passing through the points $(\cos\theta, \pm\sin\theta)$. We denote by $\omega := \omega(\kappa_{\max})$ the angle such that

$$\sin\omega = \kappa_{\max}\sin\theta. \tag{6.14}$$

Notice that proving (6.11) is in turn equivalent to show that $\omega \neq \frac{\pi}{2}$. From (6.13), we get

$$P(F) - P(\Omega^F_{\kappa_{\max}}) = 2\left[M - 2\omega\frac{\sin\theta}{\sin\omega}\right] + O(\varepsilon). \tag{6.15}$$

Similarly, we estimate $|F\setminus\Omega^F_{\kappa_{\max}}|$ with the area of the connected subset of the strip $[\cos\theta, \cos\theta + M] \times [-\sin\theta, \sin\theta]$ bounded by two arcs of circumference of curvature $\kappa_{\max}$, and passing through the vertices of the strip. Thus

$$|F \setminus \Omega^F_{\kappa_{\max}}| = 2\left[M\sin\theta - \omega\,\frac{\sin^2\theta}{\sin^2\omega} + \sin^2\theta\,\frac{\cos\omega}{\sin\omega}\right] + O(\varepsilon). \tag{6.16}$$

Combining (6.12), (6.15), and (6.16) we get

$$M\,(1 - \sin\omega) = \omega\,\frac{\sin\theta}{\sin\omega} + \sin\theta\,\cos\omega + O(\varepsilon), \tag{6.17}$$

which does not admit $\omega = \frac{\pi}{2}$ as a solution, for $\varepsilon > 0$ sufficiently small. This proves (6.11).

Remark 6.8. Referring to Example 6.7, we notice that we can still apply Theorem 1.1 separately in each connected component of $\mathrm{Ch}(F)$, thus obtaining a subunitary vector field X satisfying $\mathrm{div}X = h(F)$ in $\mathrm{Ch}(F)$, and the third equation in (1.4) on $\partial F \cap \partial\mathrm{Ch}(F)$.
If we extend X following the normal direction of the curvature level lines in $\Omega^F_{\kappa_{\max}} \setminus \mathrm{Ch}(F)$, and then transporting the field parallelly to itself in $\Theta^F_{\kappa_{\max}} \setminus \Omega^F_{\kappa_{\max}}$, we end up with a field not belonging to $\mathscr{H}^2_{\mathrm{div}}(F)$. Indeed we cannot avoid the field to jump in the normal direction of some vertical discontinuity segment.
We observe that the difficulty for building an optimal selection seems to be related to the presence of two minimal Cheeger subsets of F. We are not aware whether there exists an optimal selection equal to X in $\mathrm{Ch}(F)$.

As we have already said, we are not able to find an optimal selection $\widetilde{N}_{\min}$ in F: we notice that [52, Theorem 1.1] cannot be applied with the choice of $h = \kappa_{\phi_c,F}$, since any Ω^F_β violates [52, formula (1.3)].

7 Calibrability and total variation flow

Here we want to recall some general facts concerning the total variation flow in $\mathbb{R}^2$,[35] namely the nonlinear degenerate parabolic equation

$$\frac{\partial u}{\partial t} = \mathrm{div}\left(\frac{Du}{|Du|}\right) \qquad \text{in } (0,T)\times\mathbb{R}^2, \tag{7.1}$$

coupled with an initial condition

$$u(0,\cdot) = 1_\Omega, \tag{7.2}$$

where $T > 0$, $\Omega \subset \mathbb{R}^2$ is the interior of a ϕ_c-calibrable facet F. We recall that equation (7.1) is the gradient flow in $L^2(\mathbb{R}^2)$ of the convex functional given by the total variation $\int_{\mathbb{R}^2} |Du|$, for $u \in L^2(\mathbb{R}^2) \cap BV(\mathbb{R}^2)$.

Remark 7.1. Let $v \in \mathcal{C}^1(\Omega)$, and let E be the subgraph of v. Then the ϕ_c-perimeter[36] of E in $\Omega \times \mathbb{R}$ is

$$\int_\Omega \phi_c^o\left(-\frac{\nabla v}{\sqrt{1+|\nabla v|^2}}, \frac{1}{\sqrt{1+|\nabla v|^2}}\right)\sqrt{1+|\nabla v|^2}\, dx = \int_\Omega (1+|\nabla v|)\, dx,$$

which, up to the constant $|\Omega|$, coincides with the total variation of v.

Well-posedness of (7.1)–(7.2) follows using the theory in [34], see also [15]. In particular, it is possible to prove that there exists a unique $u \in \mathcal{C}([0,T];L^2(\mathbb{R}^2)) \cap W^{1,2}(0,T;L^2(\mathbb{R}^2))$ and there exists $\eta \in L^\infty((0,T)\times\mathbb{R}^2;\mathbb{R}^2)$, with $||\eta||_\infty \le 1$, such that $u_t = \mathrm{div}\eta$ in the sense of distributions, and (recalling also (5.3) and setting $u(t) = u(t,\cdot)$)

$$\int_{\mathbb{R}^2} (u(t)-w)\partial_t u(t)\ dx = \int_{\mathbb{R}^2} (\eta(t), Dw) - \int_{\mathbb{R}^2} |Du(t)|, \quad \text{a.e. } t \in (0,T),$$

for all $w \in L^2(\mathbb{R}^2) \cap BV(\mathbb{R}^2)$; the initial datum is taken in the $L^2(\mathbb{R}^2)$ sense. We also mention the existence – uniqueness result for the entropic solution, well-suited for more general initial data [15].

It is interesting to consider also the time-step discretization of (7.1), which reads as the denoising problem

$$\min_{u\in BV(\mathbb{R}^2)} \left\{\int_{\mathbb{R}^2} |Du| + \frac{1}{2\tau}\int_{\mathbb{R}^2} |u-f|^2\ dx\right\}, \qquad \tau > 0, \tag{7.3}$$

originally proposed in [65].[37]

(35) Total variation flow in a bounded open set of $\mathbb{R}^2$, for $m \ge 2$, has been treated for instance in [9]. See also [62], [37], [40] for the anisotropic formulation of the flow.

(36) Notice that $\phi_c^o(\xi^\star) = \sqrt{(\xi_1^\star)^2 + (\xi_2^\star)^2} + |\xi_3^\star|$ for any $\xi^\star \in \mathbb{R}^{3\star}$.

(37) In dimension one, the denoising problem (7.3) is solved by the function $u(\tau)$, solution of the total variation flow with initial condition $u(0) = f$, see [35, Proposition 4.2].

In [8] (see also [19] for eventual regularity of solutions), solutions of (7.1)–(7.2) are shown to vanish in a finite time $T(u_0)$, and the rescaled function $\frac{u(t,\cdot)}{T(u_0)-t}$ converges along subsequences, as $t \to (T(u_0))^-$, to a solution[(38)] of the eigenvalue problem

$$\mathrm{div}\left(\frac{Dv}{|Dv|}\right) = -v. \tag{7.4}$$

The problem of detecting explicit solutions of (7.4) has an independent interest: first of all, given v solution of (7.4), the function $u(t,\cdot) := (1-t)^+ v(\cdot)$ is the solution of (7.1) starting at v. Secondly, as shown in [15], any solution of (7.4) allows to construct an explicit solution of problem (7.3).[(39)]

It is natural to look for special solutions of (7.4) of the form $v := 1_\Omega$, for some bounded open set $\Omega \subset \mathbb{R}^2$. This corresponds to characterizing all flat graphs in $\mathbb{R}^3$ which, under the total variation flow, decrease their height without distorsion of the boundary. The case of a connected Ω has been studied in [15]: as one can expect, this characterization leads to the same necessary and sufficient conditions obtained by Giusti within the framework of capillary problem (hence, in turn, within the study of ϕ_c-calibrable facets of solids).

The following result enlightens the relation between solutions of (7.4) and calibrability. We recall that by κ^B we denote the (Euclidean) curvature of ∂B, for $B \subset \mathbb{R}^2$ of class $\mathcal{C}^{1,1}$.

Theorem 7.2 ([15])**.** *Let $\Omega \subset \mathbb{R}^2$ be a bounded connected open set, and let $h := \frac{P(\Omega)}{|\Omega|}$. Then, $v := h1_\Omega$ is a solution of (7.4) if and only if Ω is convex, of class $\mathcal{C}^{1,1}$, and $\kappa^\Omega \le h$ on $\partial\Omega$. In particular Ω is a convex calibrable set, see Theorem 5.4.*

Concerning a non connected set Ω, the following result holds.

Theorem 7.3 ([15])**.** *Let $\Omega \subset \mathbb{R}^2$ be a bounded open set, and let $h := \frac{P(\Omega)}{|\Omega|}$. Then, $v := h1_\Omega$ is a solution of (7.4) if and only if Ω has a finite number of connected components $C_1, \ldots, C_q$, and*

(i) C_j is convex of class $\mathcal{C}^{1,1}$, for every $j = 1, \ldots, q$;

(ii) $\frac{P(C_j)}{|C_j|} = h$, and $\kappa^{C_j} \le h$, for every $j = 1, \ldots, q$;

(iii) for any $k = 1, \ldots, q$, and for any $J := \{j_1, \ldots, j_k\} \subseteq \{1, \ldots, q\}$, we have

$$P(B) \ge \sum_{l=1}^{k} P(C_{j_l}), \qquad \bigcup_{l=1}^{k} C_{j_l} \subseteq B \subseteq \big(\mathbb{R}^2 \setminus \bigcup_{j \notin J} \overline{C_j}\big).$$

Condition *(iii)* is a requirement on the mutual distance between the sets $C_1, \ldots, C_q$: roughly speaking, the sets cannot be too close. Theorems 7.2-7.3 have been extended to general dimension $n \ge 2$ in [5], and then to the anisotropic setting in [37], under convexity assumptions on the sets.

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(38) A function $v \in BV(\mathbb{R}^2)$ solves (7.4) if there exists a subunitary vector field $\eta \in L^\infty(\mathbb{R}^2;\mathbb{R}^2)$ such that $-\mathrm{div}\eta = v$ in the sense of distributions. See [16] and also [8] for more.

(39) In the most simple case, let $v \in BV(\mathbb{R}^2)$ be a solution of (7.4), let $b \in \mathbb{R}$, and let $f := bv$. Then, setting $a := \mathrm{sign}(b)(|b| - \tau)^+$, the function $u := av$ is the solution of (7.3).

The study of piecewise constant solutions of (7.4) has been extended in [16] to the case of an open bounded set $\Omega \subset \mathbb{R}^2$ of the form

$$\Omega := C_0 \setminus \bigcup_{j=1}^{q} \overline{C_j}, \tag{7.5}$$

where $C_0, \ldots, C_q$ are bounded open sets of $\mathbb{R}^2$ of class $\mathcal{C}^{1,1}$, such that $\overline{C_j} \subset C_0$, and $\overline{C_j} \cap \overline{C_l} = \emptyset$, for all j, $l = 1, \ldots, q$, $j \neq l$. Fix $k \in \{0, \ldots, q\}$, and set

$$h := \frac{1}{|\Omega|} \left(\sum_{j=0}^{k} P(C_j) - \sum_{j=k+1}^{q} P(C_j) \right). \tag{7.6}$$

In this setting, the existence of a solution v of (7.4) of the form $v := h 1_\Omega$ leads to the problem of the existence of a vector field $\eta \in L^\infty(\Omega; \mathbb{R}^2)$, with $||\eta||_{L^\infty(\Omega)} \leq 1$, and with $L^2(\Omega)$-summable divergence, which solves the following system

$$\begin{cases} \mathrm{div}\eta = h & \text{in } \Omega, \\ \langle \nu^\Omega, \eta \rangle = 1, & \text{on } \partial C_j, \ j = 0, \ldots, k, \\ \langle \nu^\Omega, \eta \rangle = -1, & \text{on } \partial C_j, \ j = k+1, \ldots, q. \end{cases} \tag{7.7}$$

In view of the identification $v = h 1_\Omega$, one can check that (7.6) corresponds to the mean velocity of $F := \{(x, 1) \subset \mathbb{R}^3 \ : \ x \in \Omega\}$, seen as a facet in $\mathrm{Facets}_\phi(\partial E)$ of a Lip ϕ_c-regular set $E \subset \mathbb{R}^3$ which is locally convex (resp., locally concave) at F around $\partial C_0, \ldots, \partial C_k$ (resp., around $\partial C_{k+1}, \ldots, \partial C_q$). The main result is summarized in Theorem 7.4 below. Let us set, for the sake of brevity, $J_i := \{0, \ldots, k\}$ and $J_e := \{k+1, \ldots, q\}$.

Theorem 7.4. *Let Ω, h be as in (7.5)-(7.6). The following assertions hold:*

(i) if system (7.7) has a solution, then

$$\kappa^{C_0} \leq h, \qquad \kappa^{C_j} \geq -h, \quad j \in J_i, \qquad \kappa^{C_j} \geq h, \quad j \in J_e; \tag{7.8}$$

(ii) if the third inequality in (7.8) holds true, the set $\Omega \cup (\cup_{j \in J_e} C_j)$ satisfies an interior $\frac{1}{h}$-ball condition, and

$$\mathrm{dist}(C_j, C_l) > \frac{2}{h}, \quad (j, l) \in (J_i \times J_i) \cup (J_e \times J_e), \ j \neq l,$$

then system (7.7) has a solution.

In order to obtain a solution $v := h 1_\Omega$ of (7.4), one has to couple the solution provided by Theorem 7.4 with the solution of $\mathrm{div}\eta = 0$ in $\mathbb{R}^2 \setminus \overline{\Omega}$, with proper boundary conditions. See again [16].

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