Quasi-static evolutions in brittle fracture generated by gradient flows: sharp crack and phase-field approaches

Matteo Negri

Abstract In this paper we will describe how gradient flows, in a suitable norm, are natural and helpful to generate quasi-static evolutions in brittle fracture. First, we will consider the case of a brittle crack running along a straight line according to Griffith's law. Then, we will see how the same approach leads to quasi-static evolutions in the phase field setting, taking into account the alternate minimization scheme. In the latter, the norm associated to the gradient flow is not "user supplied", however the algorithm itself together with the separate quadratic structure of the energy define a family of norms which, in the limit, characterize the quasi-static evolution. Mathematically speaking, all of these evolutions are (parametrized) BV-evolutions.

1 Introduction

The idea of using monotone descent paths (among which the gradient flow) for quasi-static crack propagation goes back to the foundation of fracture mechanics: according to Griffith's principle [12] "the system can pass from the unbroken to the broken condition by a process involving a continuous decrease of potential energy". Choosing the gradient flow, in a suitable norm, as optimal and most common descent path, Griffith's criterion would be: "the system follows the gradient flow of the potential energy".

As a matter of fact, gradient flows usually refer to time dependent problems while Griffith's law does not make any reference to time, neither directly or indirectly (e.g. in terms of velocities). In our rate-independent setting, the gradient flows will provide in fact a *parametrization* of the path connecting "the unbroken" with "the broken condition". Such a parametrization will appear both in the construction of the

Matteo Negri
Department of Mathematics, Univerisity of Pavia, Via Ferrata 1, 27100 Pavia, Italy
e-mail: matteo.negri@unipv.it

solution, by time discretization, and in the quasi-static evolution itself, specifically in the instantaneous "catastrophic" propagations.

Our construction of quasi-static propagations follows closely a well known scheme in computational mechanics: we employ a uniform time discretization, say $t_k = k\Delta t$, together with an incremental law based on Griffith's principle: at each discrete time t_k the crack advances along a decreasing path of the energy, if any, until it reaches a stationary point of the energy; decreasing paths will then be written as suitable gradient flows.

In particular, for a straight crack this scheme provides in the limit, as $\Delta t \to 0$, a quasi-static evolution which can be described rigorously in several equivalent ways: by means of Karush-Kuhn-Tucker conditions (cf. Theorem 1 and [23]) by BV-solutions [18] (cf. Corollary 1 and [19]) or by parametrized BV-solutions [9] (cf. Corollary 2 and [22]). At this pont it is important to remark that in the limit, as $\Delta t \to 0$, the quasi-static evolution can be discontinuous in time. This is a common feature of BV-solutions for rate-independent systems and, most important, it is not a pure mathematical artefact. In the rate independent setting discontinuities represent catastrophic propagations of the crack, which can happen in real life: a numerical example (cf. §2) shows a clear jump discontinuity in a standard ASTM compact tension test. Mathematically, jump discontinuities are characterized by unstable regimes of propagation where instantaneously the crack advances following a gradient flow, which is indeed "a process involving a continuous decrease of potential energy".

For the general situation in which the crack path is unknown, both mathematical and numerical models involving geometrical and topological features of the crack become sensibly harder. Facing these problems is challenging but in practice it is more convenient to employ regularized models [26], [24] , [5], [17], [16], [1], [7], [13] which bypass the issues related to the morphology of the crack. One of the most successful choice is the phase field approach, which has been implemented in different ways and for several problems in fracture. Here we will focus our interest on the evolution obtained with a very efficient numerical method, known as *alternate minimization* [5]. In this scheme at each time t_k the evolution is obtained by a sequence of (quadratic) minimization problems, which produces a monotone decreasing path of the energy, in agreement with Griffith's criterion. Formally this scheme "defines" a discrete evolution law for the crack (represented by the phase-field variable). Our goal, in analogy with the straight crack problem, is to characterize the limit evolution obtained by letting $\Delta t \to 0$ and to show its main properties. First, we will see that in the limit we get a quasi-static BV-evolution, which in general does not coincide with the evolution obtained by global minimization problems [11]. Then, we recast the evolution by Karush-Kuhn-Tucker conditions where it appears an energy release rate; in this respect, note that the alternate minimization algorithm does not employ explicitly any kind of energy release. Finally, we show that the irreversibility constraint is thermodynamically consistent and that the evolution of the displacement field follows a sort of visco-elastic flow in the jumps.

Mathematically, in order to characterize the limit it is fundamental to recast alternate minimization as a gradient flow, with respect to a suitable family of norms,

induced by the separately quadratic structure of the phase-field energy; clearly this is a particular choice, which works extremely well, but other choices are also possible and worth studying, e.g. [22]. In this work, proofs and fine mathematical details are not included; the interested reader can make reference for instance to [23], [19], [22] and [14].

2 Sharp Crack

2.1 Setting: compact tension

In order to avoid technical issues as much as possible we will state our results only for a representative example, cf. Figure 1. Denote by Ω the open set in Figure 1 (obtained removing a couple of symmetric holes from a rectangle). Let $\partial\Omega = \partial_D\Omega \cup \partial\Omega_N$ where $\partial_N\Omega$ is the boundary of the rectangle while $\partial_D\Omega$ denotes (the union of) the boundaries of the circular holes. Assume that the initial crack K_0 is given by the line segment $(0,l_0]\times\{0\}$ for $l_0 > 0$. In our simple setting the crack will propagate horizontally, thus our family of admissible cracks will be given by the line segments of the form $K_l = (0,l]\times\{0\}$. Clearly such a family is simply parametrized by the scalar $l \in [l_0, L)$, which gives as well the position of the crack tip.

Consider on $\partial_D\Omega$ a proportional boundary condition of the form $u = \pm t\hat{e}$ where the sign $\pm$ is chosen as in Figure 1.

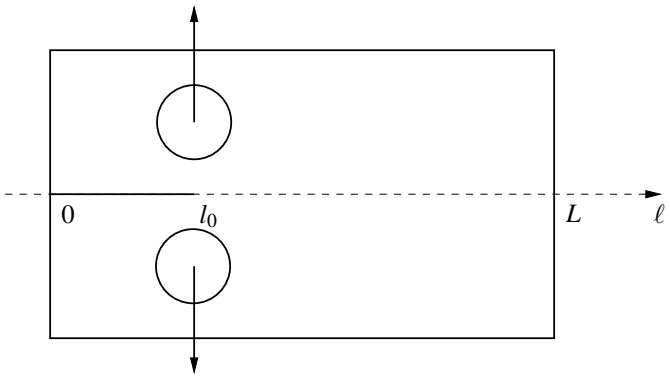

Fig. 1 An ASTM-Compact Tension geometry: the set $\Omega \setminus K_0$.

We consider in-plane elasticity with linearised energy density

$$W(Du) = \tfrac{1}{2}\, Du : \mathbf{C}[Du] = \tfrac{1}{2}\, \varepsilon(u) : \sigma(u)$$

where $\varepsilon(u) = (Du + Du^T)/2$ and $\mathbf{C}[Du] = \sigma(u) = 2\mu\varepsilon(u) + \lambda \mathrm{tr}(\varepsilon(u))I$, for $\lambda, \mu > 0$ the Lamé coefficients. For $t \in [0,T]$ and $l \in [l_0, L)$ the space of admissible configu-

rations is

$$\mathscr{U}_{t,l} = \{u \in H^1(\Omega \setminus K_l, \mathbf{R}^2) : u = \pm t\hat{e} \ \partial_D\Omega\}.$$

Hence, for $u \in \mathscr{U}_{t,l}$ the elastic energy will be

$$E(u) = \int_{\Omega\setminus K_l} W(Du)\,dx.$$

Before proceeding, it is convenient to introduce the *reduced elastic energy*: for $t \in [0,T]$ and $l \in [l_0, L)$ let

$$\mathscr{E}(t,l) = E(u_{t,l}),$$

where $u_{t,l} \in \operatorname{argmin}\{E(u) : u \in \mathscr{U}_{t,l}\}$. Note that in quasi-static evolutions it is not restrictive to employ $\mathscr{E}$ instead of E since it is assumed that the system is always in equilibrium and $u_{t,l}$ is indeed the only equilibrium point; in particular it solves the PDE

$$\begin{cases} \operatorname{div}(\sigma(u_{t,l})) = 0 & \Omega \setminus K_l \\ u_{t,l} = \pm t\hat{e} & \partial_D\Omega \\ \sigma(u_{t,l})\hat{n} = 0 & \partial_N\Omega \cup K_l^{\pm}. \end{cases} \tag{1}$$

Note that the Neumann homogeneous boundary condition holds on the boundary $\partial_N\Omega$ of the rectangle and on both the crack faces, above denoted by $K^{\pm}$.

Let us now turn to dissipation. Since we are interested in brittle fracture the energy dissipated by the crack will be provided by a potential $\mathscr{K} : [l_0, L) \to \mathbf{R}^+$ which is simply of the form $\mathscr{K}(l) = G_c(l - l_0)$, being $G_c > 0$ the material toughness.

In the sequel we will always work with the *reduced total energy* $\mathscr{F} : [0,T] \times [l_0, L) \to \mathbf{R}^+$ given by

$$\mathscr{F}(t,l) = \mathscr{E}(t,l) + \mathscr{K}(l).$$

Before proceeding it is fundamental to have at our disposal the partial derivatives of the energy $\mathscr{F}$.

Lemma 1. *The energy $\mathscr{F} : [0,T] \times [l_0, L) \to \mathbf{R}^+$ is differentiable with respect to both its variables with*

$$\partial_t\mathscr{F}(t,l) = \partial_t\mathscr{E}(t,l) = \int_{\partial_D\Omega} (\pm\hat{e}) \cdot \sigma(u_{t,l})\hat{n}\,ds = \mathscr{P}^{ext}(t,l),$$
$$-\partial_l\mathscr{F}(t,l) = G(t,l) + G_c\,,$$

where $\mathscr{P}^{ext}$ is the power of the external forces while G denotes as usual the energy release rate. Moreover, $G(t,\cdot)$ is non-negative and locally Lipschitz continuous in $[l_0, L)$.

A proof can be adapted e.g. from [19] or [23]. In this setting, by irreversibility, equilibrium reads

$$\partial_l\mathscr{F}(t,l) = -G(t,l) + G_c \geq 0 \quad \Leftrightarrow \quad G(t,l) \leq G_c. \tag{2}$$

2.2 Discrete in time evolution

Now, we will define the discrete in time evolution by a sequence of incremental problems. Denote by ℓ the evolution in time. Given $\Delta t > 0$ let $t_k = k\Delta t$ for $k = 1, ..., [T/\Delta t]$ and let $\ell(t_0) = l_0$. Knowing $\ell(t_k)$ we define $\ell(t_{k+1})$ as

$$\ell(t_{k+1}) = \min\{l \geq \ell(t_k) : G(t_{k+1}, l) \leq G_c\}. \tag{3}$$

In other terms, we advance the crack up to the closest equilibrium point. This is in some sense a *return mapping* algorithm on the set $\{G(t_{k+1}, l) \leq G_c\}$ of equilibrium points at time t_{k+1}, which is usually at the core of many crack tracking algorithms.

Now, let us see how to recast the incremental problem as a gradient flow. By irreversibility the crack cannot heal, for this reason it convenient to introduce the one sided "slope"

$$|\partial_l \mathscr{F}(t, l)|^-$$

where $|\cdot|^-$ denotes the negative part. Next, let us introduce an auxiliary parameter $s \in \mathbf{R}^+$ and an auxiliary function $l : \mathbf{R}^+ \to [l_0, L)$. We set $\ell(t_{k+1}) = \sup_s l(s)$ where l solves the gradient flow

$$\begin{cases} \dot{l}(s) = |\partial_l \mathscr{F}(t_{k+1}, l(s))|^- \\ l(0) = \ell(t_k). \end{cases}$$

In this simply setting the gradient flow boils down to an autonomous Cauchy problem for a non-linear, first order ODE. Intuitively, l grows when $\partial_l \mathscr{F}(t_{k+1}, l) < 0$, i.e. when $G(t_{k+1}, l) > G_c$ and thus when the crack is not in equilibrium. It is easy to see that there exists a unique solution and that the definition $\ell(t_{k+1}) = \sup_s l(s)$ coincides with (3) (for a proof, see [19])

At this point we have defined $\ell(t_k)$ for $t_k = k\Delta t$. Now consider a sequence of time steps $\Delta t_n \searrow 0$ and denote by ℓ_n the corresponding discrete evolutions, defined in the discrete points $t_{n,k} = k\Delta t_n$. Denote again by $\ell_n : [0, T] \to [l_0, L)$ the piecewise affine interpolate of $\ell_n(t_{n,k})$. By Helly's Theorem it follows that (up to subsequences) ℓ_n converge pointwise to a limit evolution ℓ. Our goal is now the characterization of ℓ: we will provide two characterizations, the first in terms of Karush-Kuhn-Tucker conditions, the second in terms of parametrized BV-evolutions. In order to better understand the meaning of these characterizations it is useful to show first an explicit example, which has been computed numerically.

2.3 Example

First, let us comment on Figure 2. Remember that the "loading" is monotone increasing. The set of *critical points* of the energy, i.e. $\{(t, l) : G(t, l) = G_c\}$ is represented with a dotted curve. This curve splits the (t, l)-plane into two regions: on the left is the set $\{(t, l) : G(t, l) < G_c\}$ of *stable points* while on the right is the

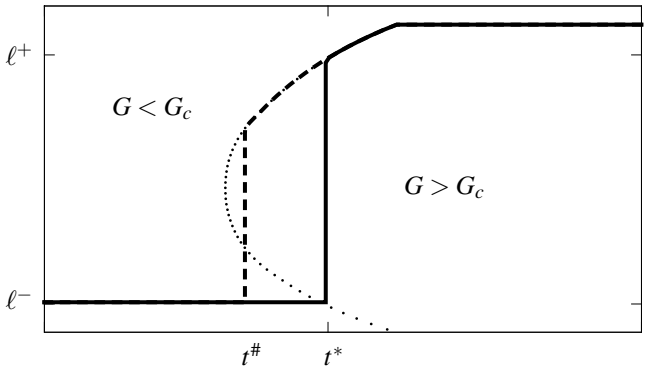

Fig. 2 Quasi-static evolutions: of energetic type (dashed) and of BV-type (solid).

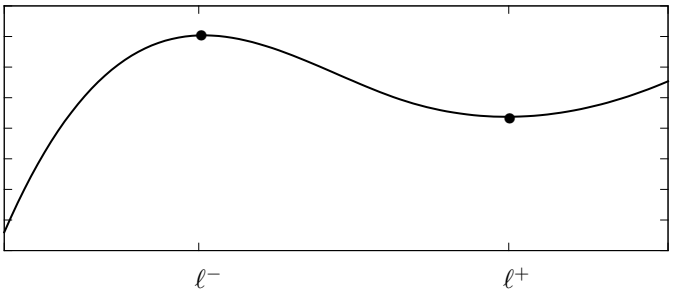

Fig. 3 A detail of the energy landscape $\mathscr{F}(t^*,l)$.

set $\{(t,l) : G(t,l) > G_c\}$ of *unstable points*. From the picture it is clear that up to time t^* the crack is not moving, since it is inside the *stable region*. At time t^* a"catastrophic evolution" occurs: in the istantaneous transition from ℓ^- to ℓ^+ the system crosses the unstable region, since $G(t^*,l) > G_c$ for every $l \in (\ell^-,\ell^+)$. The energy landscape at time t^* is reported in Figure 3: it is clear that $\mathscr{F}(t^*,\cdot)$ is not convex and that ℓ makes a transition from ℓ^- to ℓ^+ following a descent path, in particular $\mathscr{F}(t^*,\ell^-) > \mathscr{F}(t^*,\ell^+)$.

Figure 2 shows (in dashed bold line) also the energetic evolution, obtained by global energy minimization; this evolution presents a discontinuity as well, however the qualitative behaviour is quite different: the system crosses first the *stable* and then the *unstable* region, in particular a propagation occurs even if $G(t^\#,\ell^-) < G_c$.

2.4 Characterization by Karush-Kuhn-Tucker conditions

The next Theorem provides the first characterization of the limit evolution. For a proof, see [19].

Theorem 1. *The limit evolution ℓ, obtained letting $\Delta t_n \searrow 0$, is non-decreasing and belongs to $BV(0,T)$. Moreover*

$$G(t,\ell^-(t)) \le G_c \quad \text{for } t \in [0,T], \tag{4}$$

$$\big(G(t,\ell^-(t)) - G_c\big)\, d\ell(t) = 0 \quad \text{in the sense of measures in } [0,T]. \tag{5}$$

Furthermore, for $t \in J(\ell)$ (the set of jumps) we have

$$G(t,l) \ge G_c \qquad \text{for every } l \in [\ell^-(t), \ell^+(t)]. \tag{6}$$

Let us make some comments on the above Theorem. Clearly (4) and (5) play the role of the "classical" Karush-Kuhn-Tucker conditions, with some technical differences: the left limit ℓ^- is used instead of ℓ and (weak) the measure theoretic derivative $d\ell$ is used instead of the speed $\dot\ell$. Both these technical details are due to the fact that in general the evolution belongs to $BV(0,T)$ and thus it may have jump discontinuities. However the qualitative meaning of (4) and (5) is quite clear and consistent with standard KKT conditions. What is instead not common in the study of quasi-static evolutions is condition (6) which characterizes the behaviour in the jumps in terms of *unstable* branches of propagation (cf. also Figure 2).

Finally, in terms of derivatives of the energy, the above Theorem reads as follows.

Corollary 1. *The limit ℓ, obtained letting $\Delta t_n \searrow 0$, is non-decreasing and belongs to $BV(0,T)$. Moreover*

$$\partial_l \mathscr{F}(t,\ell^-(t)) \ge 0 \quad \text{for } t \in [0,T], \tag{7}$$

$$\partial_l \mathscr{F}(t,\ell^-(t))\, d\ell(t) = 0 \quad \text{in the sense of measures in } [0,T]. \tag{8}$$

Furthermore, for $t \in J(\ell)$ (the set of jumps) we have

$$\partial_l \mathscr{F}(t,\ell^-(t)) \le 0 \qquad \text{for every } l \in [\ell^-(t), \ell^+(t)]. \tag{9}$$

At this point it is necessary to introduce the concept of BV-solution. Here we will give the "simplest" possible definition, for a general treatise see [18].

Corollary 2. *The limit ℓ is non-decreasing and of class $BV(0,T)$. Moreover*

$$|\partial_l \mathscr{F}(t,\ell^-(t))|^- = 0 \quad \text{for } t \in [0,T] \tag{10}$$

and for every $t \in [0,T]$ the following energy identity holds:

$$\mathscr{F}(t,\ell^-(t)) = \mathscr{F}(0,l_0) + \int_0^t \partial_t \mathscr{F}(\tau,\ell(\tau))\, d\tau - \sum_{t \in J(\ell)} \mathrm{diss}(\mathscr{F}(t,\cdot)), \tag{11}$$

where

$$\mathrm{diss}(\mathscr{F}(t,\cdot)) = \int_{\ell^-(t)}^{\ell^+(t)} |\partial_l \mathscr{F}(t,l)|^-\, dl$$

denotes the "energy gap" in the discontinuity points.

An evolution ℓ which satisfies (10) *and* (11) *is called a BV-solution.*

It is important to note that equilibrium (10) and energy balance (11) provide a very concise and mathematically convenient way of characterizing the quasi-static evolution; they are indeed equivalent to the Karush-Kuhn-Tucker conditions of Theorem 1. Without entering too much into the technical details (for a complete proof the reader can follow [20] or [22]) let us see how (7)-(9) follow from (10)-(11). First,

$$|\partial_l \mathscr{F}(t,\ell^-(t))|^- = 0 \;\Leftrightarrow\; \partial_l \mathscr{F}(t,\ell^-(t)) \geq 0\,,$$

gives (7). Next, by the chain rule in $BV(0,T)$

$$\mathscr{F}(t,\ell^-(t)) = \mathscr{F}(0,l_0) + \int_0^t \partial_t \mathscr{F}(\tau,\ell(\tau))\,d\tau \;+\; \int_0^t \partial_l \mathscr{F}(\tau,\ell^-(\tau))\,d_{ac}\ell(\tau) \;+ \\ + \sum_{t\in J(\ell)} [\![\mathscr{F}(t,\cdot)]\!],$$

where $d_{ac}\ell$ denotes the (weak) measure theoretic derivative of ℓ in $[0,T]\setminus J(\ell)$. Comparing with (11) we get

$$\int_0^t \partial_l \mathscr{F}(\tau,\ell^-(\tau))\,d_{ac}\ell(\tau) + \sum_{t\in J(\ell)} [\![\mathscr{F}(t,\cdot)]\!] = -\sum_{t\in J(\ell)} \mathrm{diss}(\mathscr{F}(t,\cdot)).$$

Since the measures $d_{ac}\ell$ is supported in $[0,T]\setminus J(\ell)$ it follows that

$$\partial_l \mathscr{F}(t,\ell^-(t))\,d_{ac}\ell(t) = 0 \qquad \text{in the sense of measures in } [0,T]$$

and that

$$[\![\mathscr{F}(t,\cdot)]\!] = \int_{\ell^-(t)}^{\ell^+(t)} \partial_l \mathscr{F}(t,l)\,dl = -\int_{\ell^-(t)}^{\ell^+(t)} |\partial_l \mathscr{F}(t,l)|^-\,dl, \qquad \text{for every } t\in J(\ell).$$

The former leads to (8), thanks to the continuity of G, while the latter leads to

$$\partial_l \mathscr{F}(t,l) = -|\partial_l \mathscr{F}(t,l)|^- \qquad \text{for every } l\in[\ell^-(t),\ell^+(t)],$$

which is en turn equivalent to

$$\partial_l \mathscr{F}(t,l) \leq 0 \qquad \text{for every } l\in[\ell^-(t),\ell^+(t)],$$

that is (9).

2.5 Characterization as a graph parametrized BV-evolution

We have seen that the limit evolution ℓ can have jump discontinuities in time. For this reason it is convenient, both for theoretical and numerical purposes, to represent

ℓ by a (Lipschitz) parametrization of the form $s \mapsto (t(s), l(s))$ of the extended graph. Remember that the extended graph is just the set $\{(t,l) : \ell^-(t) \le l \le \ell^+(t)\}$ obtained "completing" the jumps with a vertical line segments (see Figure2). Therefore, using the map $s \mapsto (t(s), l(s))$ a jump of ℓ at time t^* will be characterized by $t(s) = t^*$ in $[s_1, s_2]$ together with $l(s_1) = \ell^-(t^*)$ and $l(s_2) = \ell^+(t^*)$. In essence, continuity points (in time) will correspond points with $t'(s) > 0$ while discontinuity points (in time) will correspond points with $t'(s) = 0$. The advantages of this representation, originally suggested in [9], will become more clear in §3 and in general are quite evident in the case of infinite dimensional systems [22].

Here for sake of simplicity we will skip any argument on the different ways which provide existence of a parametrized evolution. We will instead assume that $s \mapsto (t(s), l(s))$ is a parametrization of the extended graph of the solution ℓ, obtained as above by letting $\Delta t_n \searrow 0$. We will consider also $t' \ge 0$ (in order to avoid physically meaningless cases) and we will normalize the parametrization with $t'(s) + l'(s) = 1$ (for a.e. $s \in [0, S]$). The resulting parametrization satisfies the following properties.

Theorem 2. *The (normalized) parametrization $s \mapsto (t(s), l(s))$ satisfies $t'(s) \ge 0$, $l'(s) \ge 0$ and $t'(s) + l'(s) \le 1$. Moreover it satisfies the following conditions:*

$$|\partial_l \mathscr{F}(t(s), l(s))|^- = 0 \qquad \text{for every } s \text{ with } t'(s) > 0, \tag{12}$$

$$\begin{aligned} \mathscr{F}(t(s), l(s)) = \; & \mathscr{F}(0, l(0)) + \int_0^s \partial_t \mathscr{F}(t(r), l(r))\, t'(r)\, dr + \\ & - \int_0^s |\partial_\ell \mathscr{F}(t(r), l(r))|^-\, l'(r)\, dr. \qquad \text{for every } s. \end{aligned} \tag{13}$$

A (normalized) parametrization $s \mapsto (t(s), \ell(s))$ which satisfies (10) *and* (11) *is called a parametrized BV-solution.*

It is not difficult to see that, upon choosing the right parametrization, (12)-(13) follows from (10)-(11) and viceversa (for a proof see [22]).

3 Alternate minimization scheme in the phase-field approach

In this section we will deal with the phase-field approach for fracture, which in the last decade has been an effective and popular method for the simulation of crack propagation, see e.g. [5], [17], [16], [1], [7], [13] and many others. In particular we will consider evolutions defined by the alternate minimization scheme; we will see that in the limit they will provide indeed parametrized BV-evolution, with respect to a suitable family of norms.

3.1 Setting

For sake of simplicity let us consider the same geometry and the same boundary conditions of §2.1. In the phase-field framework it is however necessary to re-define the space of admissible displacements by

$$\mathscr{U}_t = \{u \in H^1(\Omega, \mathbf{R}^2) : u = \pm t\hat{e} \text{ on } \partial_D\Omega\}$$

and to introduce a set for the phase field variables

$$\mathscr{Z} = \{z \in H^1(\Omega) : 0 \le z \le 1\}.$$

By linearity we can always write $\mathscr{U}_t = t\mathscr{U}$ where

$$\mathscr{U} = \{u \in H^1(\Omega, \mathbf{R}^2) : u = \pm\hat{e} \text{ on } \partial_D\Omega\}.$$

In the sequel we will work indeed with the spaces $\mathscr{U}$ and $\mathscr{Z}$ which are independent of time. For $\varepsilon > 0$ and $\eta_\varepsilon > 0$, typically with $\eta_\varepsilon = o(\varepsilon)$, the phase field elastic and dissipated energy [2] will be respectively

$$\mathscr{E}_\varepsilon(t,u,z) = \tfrac{1}{2}\int_\Omega t^2(z^2+\eta_\varepsilon)W(Du)\,dx,$$
$$\mathscr{K}_\varepsilon(z) = \tfrac{1}{2}G_c\int_\Omega (z-1)^2/2\varepsilon + \varepsilon|\nabla z|^2\,dx.$$

The total energy $\mathscr{F}_\varepsilon : [0,T]\times\mathscr{U}\times\mathscr{Z} \to \mathbb{R}$ will be $\mathscr{F}_\varepsilon(t,u,z) = \mathscr{E}_\varepsilon(t,u,z) + \mathscr{K}_\varepsilon(z)$.

In the sequel it will be fundamental to have at hand the partial derivatives of the energy,

$$\partial_t\mathscr{F}_\varepsilon(t,u,z) = \int_\Omega t(z^2+\eta_\varepsilon)W(Du)\,dx, \tag{14}$$

$$\partial_u\mathscr{F}_\varepsilon(t,u,z)[\phi] = \int_\Omega t^2(z^2+\eta_\varepsilon)\sigma(u) : \varepsilon(\phi)\,dx, \tag{15}$$

$$\partial_z\mathscr{F}_\varepsilon(t,u,z)[\xi] = \int_\Omega t^2 z\xi\, W(Du)\,dx + G_c\int_\Omega (z-1)\xi/2\varepsilon + \varepsilon\nabla z\cdot\nabla\xi\,dx. \tag{16}$$

For our purposes the spaces of admissible variations for $\mathscr{Z}$ and $\mathscr{U}$ respectively will be provided by

$$\Xi = \{\xi \in H^1(\Omega) : \xi \le 0\}, \qquad \Phi = \{\phi \in H^1(\Omega, \mathbf{R}^2) : \phi = 0 \text{ on } \partial_D\Omega\}.$$

To conclude this section, let us see how to define a notion of energy release, with respect to a variation ξ, and how to get the power of external forces. To this end, denoting $u(t,z) \in \operatorname{argmin}\{\mathscr{E}_\varepsilon(t,u,z) : u \in \mathscr{U}\}$ we will call "energy release functional" (with respect to a variation ξ)

$$\mathscr{G}_\varepsilon(t,z)[\xi] = -\lim_{h\to 0^+}\frac{\mathscr{E}_\varepsilon(t,u(t,z+h\xi),z+h\xi) - \mathscr{E}_\varepsilon(t,u(t,z),z)}{h}$$

In this way the displacement field changes "simultaneously" with the variation of the phase field variable. Actually, by minimality of $u(t,z)$ the derivative can be represented explicitly as (see e.g. [15])

$$\mathscr{G}_\varepsilon(t,z)[\xi] = -\partial_z \mathscr{E}_\varepsilon(t,u,z)[\xi] = -\int_\Omega t^2 z \xi\, W(Du)\, dx. \tag{17}$$

Now, let us introduce the set of normalized variations

$$\hat{\Xi}_z = \Big\{ \xi \in \Xi : \int_\Omega (z-1)\xi/4\varepsilon + \varepsilon \nabla z \cdot \nabla \xi \, dx \le 1 \Big\}.$$

Note that these variations normalize the "varation of crack lenght" since

$$d\mathscr{K}_\varepsilon(z)[\hat{\xi}] = G_c \int_\Omega (z-1)\hat{\xi}/4\varepsilon + \varepsilon \nabla z \cdot \nabla \hat{\xi}\, dx \le G_c\,.$$

Then, we can define the energy release as

$$G_\varepsilon(t,z) = \sup\{\mathscr{G}_\varepsilon(t,z)[\hat{\xi}] : \hat{\xi} \in \hat{\Xi}_z\}. \tag{18}$$

Finally, Green's formula allows to rewrite (14) as

$$\partial_t \mathscr{F}_\varepsilon(t,u(t,z),z) = \int_{\partial_D \Omega} (\pm \hat{e}) \cdot \sigma_z(tu(t,z))\, \hat{n}\, ds = \mathscr{P}^{ext}_\varepsilon(t,u(t,z),z), \tag{19}$$

where $\sigma_z(w)$ denotes the phase field stress, that is

$$\sigma_z(w) = (z^2 + \eta_\varepsilon)\sigma(w).$$

3.2 Discrete in time evolution by alternate minimization

As we did in §2.2, given $\Delta t > 0$ let $t_k = k\Delta t$ and set the intial conditions $u(t_0) = u_0$ and $z(t_0) = z_0$. Known $u(t_{k-1})$ and $z(t_{k-1})$ we will introduce a couple of auxiliary sequences, u^m and z^m, with $u^0 = u(t_{k-1})$ and $z^0 = z(t_{k-1})$ defined recursively by the following *alternate minimization scheme* [5]

$$\begin{cases} u^m \in \operatorname{argmin}\big\{ \mathscr{F}_\varepsilon\big(t_k, \cdot, z^{m-1}\big) : u \in \mathscr{U} \big\}, \\ z^m \in \operatorname{argmin}\big\{ \mathscr{F}_\varepsilon(t_k, u^m, \cdot) : z \in \mathscr{Z} \text{ with } z \le z^{m-1} \big\}, \end{cases} \tag{20}$$

where the constraint $z \le z^{m-1}$ models the irreversibility of the crack. Then we define the updates $u(t_k) = \lim_{m\to+\infty} u^m$ and $z(t_k) = \lim_{m\to+\infty} z^m$. More precisely, we have the following result (for a proof see [14]).

Proposition 1. *The sequence u^m converge to $u(t_k)$ strongly in $H^1(\Omega, \mathbf{R}^2)$ while z^m converge to $z(t_k)$ strongly in $H^1(\Omega)$. Further,*

$$\partial_u \mathscr{F}_\varepsilon(t_k, u(t_k), z(t_k))[\phi] = 0 \quad \textit{for every } \phi \in \Phi,$$
$$\partial_z \mathscr{F}_\varepsilon(t_k, u(t_k), z(t_k))[\xi] \geq 0 \quad \textit{for every } \xi \in \Xi.$$

In other terms, $u(t_k)$ is an equilibrium point for $\mathscr{F}_\varepsilon(t_k, \cdot, z(t_k))$ while $z(t_k)$ is an equilibrium point for $\mathscr{F}_\varepsilon(t_k, u(t_k), \cdot)$ (for the latter remember the irreversibility constraint).

In this way, given $\Delta t > 0$ a time discrete evolution is provided in terms of the equilibrium configurations $(u(t_k), z(t_k))$ for $t_k = k\Delta t$. In order to understand the limit evolution, obtained by letting $\Delta t \to 0$, we have first to recast the alternate minimization scheme as a gradient flow. This is the goal of the next section.

3.3 Minimization as a gradient flow

3.3.1 An illustrative example

In order to better understand the gradient flow structure behind (20) let us start with an example: the minimization of a quadratic functional in a finite dimensional setting. Let $F(x) = \frac{1}{2}x^T Ax + b^T x + c$ for $x \in \mathbb{R}^n$ and $A^T = A > 0$. Let $\|x\|_A = \sqrt{x^T Ax}$ be the "natural" norm induced by the symmetric, positive define matrix A, with associated scalar product $\langle \cdot, \cdot \rangle_A$.

Our problem is the following: given x_0 find the increment x_* in such a way that $x_0 + x_*$ is the minimizer of F. Since F is quadratic we can write

$$F(x_0 + x_*) = F(x_0) + \nabla F(x_0)x_* + \tfrac{1}{2}x_*^T Ax_*$$

and we can characterize x_* by stationarity of the energy, i.e.,

$$\nabla^T F(x_0) + Ax_* = 0 \quad \Leftrightarrow \quad x_* = -A^{-1}\nabla^T F(x_0) = -\nabla_A^T F(x_0).$$

In the last term of the previous row we have introduced the notation $\nabla_A^T F(x_0)$ which denotes the gradient of F (computed in x_0) with respect to the norm $\|\cdot\|_A$, that is the (unique) vector such that $dF(x_0)[x'] = \langle \nabla_A F(x_0), x' \rangle_A$ for every $x' \in \mathbb{R}^n$. For our pourposes it is just important to remark that $\nabla_A F(x_0) = \nabla F(x_0)A^{-1}$ and that

$$-\widehat{\nabla}_A F(x_0) = -\nabla_A F(x_0)/\|\nabla_A F(x_0)\|_A = \operatorname{argmin}\big\{dF(x_0)[x'] : \|x'\|_A \leq 1\big\}.$$

In other terms, the normalized gradient $\nabla_A F(x_0)$ provides the steepest descent direction with respect to the norm $\|\cdot\|_A$. Now, let $k = \|\nabla_A F(x_0)\|_A$ and define $x(s) = x_0 - \lambda(s)\widehat{\nabla}_A F(x_0)$ where $\lambda(s) = 1 - e^{-ks}$ for $s \in [0, +\infty)$. Then, by homogeneity, $x(s)$ solves the gradient flow

$$\begin{cases} x'(s) = -\nabla_A F(x(s)) & \text{for } s \in [0, +\infty), \\ x(0) = x_0. \end{cases} \tag{21}$$

Note that the right hand side is evaluated in $x(s)$, and not in x_0, and that $\lim_{s\to+\infty} x(s) = x_0 + x_*$ is exactly the minimizer. In other terms, the linear interpolation of the points x_0 and $x_0 + x_*$ with parametrization $\lambda(s)$ is the solution of the gradient flow (21).

It is now time to turn back to the phase field setting.

3.3.2 A family of "intrinsic norms" for the phase field energy

As we have seen above in order to recast minimization as a gradient flow as a first step it is necessary to single out an "intrinsic norm" which is nothing but the quadratic part of the energy. In our setting, for the phase field variable we will employ

$$\|z\|_{t,u}^2 = \int_\Omega z^2\big(G_c/2\varepsilon + t^2 W(Du)\big) + \varepsilon G_c|\nabla z|^2\,dx,$$
$$\langle z,\xi\rangle_{t,u} = \int_\Omega z\xi\big(G_c/2\varepsilon + t^2 W(Du)\big) + \varepsilon\nabla z\cdot\nabla\xi\,dx,$$

while for the displacement field u we will employ

$$|u|_{t,z}^2 = \int_\Omega t^2(z^2+\eta_\varepsilon)W(Du)\,dx,$$
$$\langle u,\phi\rangle_{t,z} = \int_\Omega t^2(z^2+\eta_\varepsilon)\sigma(u):\varepsilon(\phi)\,dx.$$

With the above definitions the quadratic structure of the energy looks very clear, indeed we can write the energy as

$$\mathscr{F}_\varepsilon(t,u,z) = \tfrac{1}{2}|u|_{t,z}^2 + c_z \qquad \text{for } c_z = \tfrac{1}{2}G_c\int_\Omega (z-1)^2/2\varepsilon + \varepsilon|\nabla z|^2\,dx,$$
$$\mathscr{F}_\varepsilon(t,u,z) = \tfrac{1}{2}\|z\|_{t,u}^2 - b(z) + c_{t,u} \qquad \text{for } \begin{cases} b(z) = G_c\displaystyle\int_\Omega z/2\varepsilon\,dx, \\ c_{t,u} = \tfrac{1}{2}\displaystyle\int_\Omega \eta_\varepsilon t^2 W(Du) + G_c/2\varepsilon\,dx. \end{cases}$$

Note that c_z is independent of u and viceversa $c_{t,u}$ is independent of z, while $b(\cdot)$ is linear. As a consequence, the partial derivatives as well take a particularly simple form, being

$$\partial_u\mathscr{F}_\varepsilon(t,u,z)[\phi] = \langle u,\phi\rangle_{t,z}, \qquad \partial_z\mathscr{F}_\varepsilon(t,u,z)[\xi] = \langle z,\xi\rangle_{t,u} - b(\xi).$$

We will see §3.4.3 how to write more explicitly the gradient flows originating from alternate minimization.

Finally, it will be very convenient, if not necessary, to define a couple of slopes, with respect to the "norms" defined above, that is

$$|\partial_u \mathscr{F}_\varepsilon(t,u,z)|^-_{t,z} = |\min\{\partial_u \mathscr{F}_\varepsilon(t,u,z)[\phi] : \phi \in \Phi,\ |\phi|_{t,z} \le 1\}|^-,$$
$$|\partial_z \mathscr{F}_\varepsilon(t,u,z)|^-_{t,u} = |\min\{\partial_z \mathscr{F}_\varepsilon(t,u,z)[\xi] : \xi \in \Xi,\ \|\xi\|_{t,z} \le 1\}|^-,$$

where $|\cdot|^-$ is once again the negative part.

3.4 A parametrized "BV-evolution"

Consider a sequence $\Delta t_n \searrow 0$. For each Δt_n, let $u(t_{n,k})$ and $z(t_{n,k})$ (for $t_k = k\Delta t_n$) be given by the alternate minimization scheme (20). In order to define the limit as $\Delta t_n \searrow 0$ it would be natural to introduce first, at every time t_k, an arc length interpolation of the alternate minimizing path (u^m, z^m) which links $(u(t_{k-1}), z(t_{k-1})) = (u^0, z^0)$ with $(u(t_k), z(t_k)) = \lim_{m\to+\infty}(u^m, z^m)$. This point, even if intuitive, is technically quite delicate since at the current stage it is not known if the lenght of the alternate minimizing path has fine length. However, it is possible to bypass this technical issue defining a suitable normalized parametrization $s \mapsto (t_n(s), u_n(s), z_n(s))$ which, roughly speaking, interpolates the points $(t_k, u(t_k), z(t_k))$ (for the detail the reader should make reference to the forthcoming [14]). We can then apply an abstract results developed in [22] which yields the following Theorem.

Theorem 3. *Given $\Delta t_n \searrow 0$, let $s \mapsto (t_n(s), u_n(s), z_n(s))$ be the parametrizations of the discrete evolutions provided by the alternate minimization scheme* (20)*. Then, up to subsequences, there exists a limit normalized parametrization $s \mapsto (t(s), u(s), z(s))$ with $t'(s) \ge 0$, $z'(s) \le 0$ and $t'(s) + |u'(s)|_{t(s),z(s)} + \|z'(s)\|_{t(s),u(s)} \le 1$. Moreover, for every s with $t'(s) > 0$ the following equilibrium conditions holds*

$$|\partial_u \mathscr{F}_\varepsilon(t(s),u(s),z(s))|^-_{t(s),z(s)} = |\partial_z \mathscr{F}_\varepsilon(t(s),u(s),z(s))|^-_{t(s),z(s)} = 0. \tag{22}$$

Finally, for every s it holds the energy balance

$$\begin{aligned}\mathscr{F}(t(s),u(s),z(s)) = \mathscr{F}(0,u_0,z_0) &+ \int_0^s \partial_t \mathscr{F}(t(r),u(r),z(r))\,t'(r)\,dr + \\ &- \int_0^s |\partial_u \mathscr{F}(t(r),u(r),z(r))|^-_{t(r),z(r)}\ |u'(r)|_{t(r),z(r)}\,dr + \\ &- \int_0^s |\partial_v \mathscr{F}(t(r),u(r),z(r))|^-_{t(r),u(r)}\ \|z'(r)\|_{t(r),u(r)}\,dr. \end{aligned} \tag{23}$$

In the next section we will explain better the meaning of the previous theorem, which is by itself quite technical. However, the analogy with Theorem 2 should be quite evident.

3.4.1 Continuity points: equilibrium

Let us start considering the point where $t'(s) > 0$, which corresponds in the parametric setting to continuity points in time. Equation (22) gives equilibrium. Indeed by definition of the slopes

$$|\partial_u \mathscr{F}_\varepsilon(t(s),u(s),z(s))|^-_{t(s),z(s)} = 0 \;\Leftrightarrow\; \partial_u \mathscr{E}_\varepsilon(t(s),u(s),z(s))[\phi] = 0 \quad \text{for } \phi \in \Phi.$$
$$|\partial_z \mathscr{F}_\varepsilon(t(s),u(s),z(s))|^-_{t(s),u(s)} = 0 \;\Leftrightarrow\; \partial_z \mathscr{F}_\varepsilon(t(s),u(s),z(s))[\xi] \geq 0 \quad \text{for } \xi \in \Xi.$$

Let us write more explicitly the equilibrium conditions. Introducing the phase field stress $\sigma_z(u) = (z^2 + \eta_\varepsilon)\,\sigma(u)$ we get

$$\partial_u \mathscr{E}_\varepsilon(t(s),u(s),z(s))[\phi] = \int_\Omega \sigma_{z(s)}(u(s)) : \varepsilon(\phi)\,dx = 0 \qquad \text{for } \phi \in \Phi$$

and thus

$$\begin{cases} \operatorname{div}(\sigma_{z(s)}(u(s))) = 0 & \Omega \\ u(s) = \pm\hat{e} & \partial_D\Omega \\ \sigma_{z(s)}(u(s))\,\hat{n} = 0 & \partial_N\Omega, \end{cases} \tag{24}$$

which is the phase-field counterpart of (1).

Now, let us see discuss the physical meaning to the equilibrium condition with respect to the phase field variable z. By the definition (17) of energy release the equilibrium condition

$$\partial_z \mathscr{F}_\varepsilon(t(s),u(s),z(s))[\xi] \geq 0 \quad \text{for } \xi \in \Xi$$

reads

$$-\mathscr{G}_\varepsilon(t(s),z(s))[\xi] + \partial_z \mathscr{K}_\varepsilon(z(s))[\xi] \geq 0 \quad \text{for } \xi \in \Xi \tag{25}$$

where

$$\partial_z \mathscr{K}_\varepsilon(z(s))[\xi] = G_c \int_\Omega (z-1)\xi/4\varepsilon + \varepsilon \nabla z \cdot \nabla \xi\,dx.$$

Employing the normalized set $\hat{\Xi}_{z(s)}$ and taking the supremum with respect to $\hat{\xi} \in \hat{\Xi}_{z(s)}$ from (25) it follows

$$G_\varepsilon(t(s),z(s)) \leq G_c, \tag{26}$$

which plays the role of (4).

To conclude this section, we remark that by the separate quadratic structure of the energy we have the following *separate minimimality* property

$$u(s) \in \operatorname{argmin}\big\{\mathscr{E}_\varepsilon(t(s),u,z(s)) : u \in \mathscr{U}\big\}$$
$$z(s) \in \operatorname{argmin}\big\{\mathscr{F}_\varepsilon(t(s),u(s),z) : z \in \mathscr{Z}, z \leq z(s)\big\}.$$

Note that $z(s)$ is only a costrained minimizer and that in general it is not true that

$$(u(s),z(s)) \in \operatorname{argmin}\big\{\mathscr{F}_\varepsilon(t(s),u,z) : u \in \mathscr{U},\ z \in \mathscr{Z}, z \le z(s)\big\}$$

as it would be in an energetic evolution.

3.4.2 Continuity points: thermodynamic consistency

In this section we will discuss a couple of thermodynamic issues: the first is simply the energy balance (which will also lead to a Karush-Kuhn-Tucker condition) while the second originates from the relationship between irreversibility constraint and dissipated energy. In both the cases we will assume that $t'(s) > 0$ in a parametrization interval $[s_1, s_2]$ (so that the corresponding evolution is continuous in time). By (19) and (22) we can rewrite (23) as

$$\mathscr{F}_\varepsilon(t(s_2),u(s_2),z(s_2)) = \mathscr{F}_\varepsilon(t(s_1),u(s_1),z(s_1)) + \int_{s_1}^{s_2} \mathscr{P}^{ext}_\varepsilon(t(r),u(r),z(r))\,t'(r)\,dr,$$

which is the usual energy balance in parametrized integral form. Using the chain rule the above energy identity reads: for every $s \in (s_1, s_2)$

$$\begin{aligned}\mathscr{F}'_\varepsilon(t(s),u(s),z(s)) &= \partial_t\mathscr{F}_\varepsilon(t(s),u(s),z(s))\,t'(s) + \partial_u\mathscr{F}'_\varepsilon(t(s),u(s),z(s))[u'(s)] + \\ &\quad + \partial_z\mathscr{F}'_\varepsilon(t(s),u(s),z(s))[z'(s)] \\ &= \mathscr{P}^{ext}_\varepsilon(t(s),u(s),z(s))\,t'(s)\,.\end{aligned}$$

Again by (19) and (22) it follows that

$$\partial_z\mathscr{F}'_\varepsilon(t(s),u(s),z(s))[z'(s)] = 0\,.$$

Here, note that equilibrium gives $\partial_z\mathscr{F}'_\varepsilon(t(s),u(s),z(s))[\xi] \ge 0$ for every $\xi \in \Xi$. Using the notation of (25) the above identity in "Karush-Kuhn-Tucker" fashion becomes

$$\big(\mathscr{G}_\varepsilon(t(s),z(s)) - \partial_z\mathscr{H}_\varepsilon(z(s))\big)[z'(s)] = 0\,, \tag{27}$$

which plays the role of (5).

Now, let us study the relationship between the irreversibility constraint and the dissipated energy $\mathscr{H}_\varepsilon(z)$. In general if $z_1 \le z_2$ it is not true that $\mathscr{H}_\varepsilon(z_1) \le \mathscr{H}_\varepsilon(z_2)$! This is simply due to the fact that

$$\mathscr{H}_\varepsilon(z) = \tfrac{1}{2}G_c\int_\Omega (z-1)^2/2\varepsilon + \varepsilon|\nabla z|^2\,dx$$

includes a gradient term which is not monotone. For instance, consider $z_1 \le z_2$ with z_1 constant and with z_2 (highly) oscillating. Then

$$\int_\Omega (z_1-1)^2/4\varepsilon\,dx \ \ge\ \int_\Omega (z_2-1)^2/4\varepsilon\,dx$$

while

$$\int_\Omega \varepsilon|\nabla z_1|^2\,dx \;<\; \int_\Omega \varepsilon|\nabla z_2|^2\,dx.$$

If $z_1 \approx z_2$ but the energy of ∇z_2 is big enough it may be that $\mathscr{K}_\varepsilon(z_1) < \mathscr{K}_\varepsilon(z_2)$. Thus, the irreversibility constraint, given by the monotonicity of z, does not always match with the monotonicity of the dissipated energy. However, this is not what happens in the evolution, at least in those interval $[s_1,s_2]$ where $t'(s) > 0$. Indeed, $z'(s) \in \Xi$ and thus by (27)

$$\partial_z \mathscr{E}_\varepsilon(t(s),u(s),z(s))[z'(s)] + \partial_z \mathscr{K}_\varepsilon(z(s))[z'(s)] = 0.$$

Note that

$$\partial_z \mathscr{E}_\varepsilon(t(s),u(s),z(s))[z'(s)] = \int_\Omega t^2(s)\,z(s)\,z'(s)\,W(Du(s))\,dx \le 0,$$

because the only negative term is $z'(s)$. It follows that $\partial_z \mathscr{K}_\varepsilon(z(s))[z'(s)] \ge 0$ and thus the energy $s \mapsto \mathscr{K}_\varepsilon(z(s))$ is non-decreasing.

3.4.3 Discontinuity points: which gradient flow?

In this section we want to collect some properties of the evolution in the jumps, i.e., in the parametrization intervals $[s_1,s_2]$ where $t(s)$ is constant, and thus $t'(s) = 0$. It is fair to say that at the current stage the picture is not fully clear and detailed. In order to understand the main qualitative features it is not too restrictive to assume that for $s \in [s_1,s_2]$ it holds

$$|\partial_u \mathscr{F}_\varepsilon(t(s),u(s),z(s))|^-_{t(s),z(s)} = |\partial_v \mathscr{F}_\varepsilon(t(s),u(s),z(s))|^-_{t(s),u(s)} = 1\,.$$

Under these assumptions, (23) implies (by the chain rule and convexity arguments) that for a.e. $s \in [s_1,s_2]$ we have

$$u'(s) \in \operatorname{argmin}\{\partial_u \mathscr{F}_\varepsilon(t(s),u(s),z(s))[\phi] : \phi \in \Phi,\ |\phi|_{t(s),z(s)} = 1\}$$
$$z'(s) \in \operatorname{argmin}\{\partial_v \mathscr{F}_\varepsilon(t,u,v)[\xi] : \xi \in \Xi,\ \|\xi\|_{t(s),u(s)} = 1\}.$$

In other terms, u' and z' are the steepest descent direction for $\mathscr{F}_\varepsilon$ with respect to the "intrinsic norms". If the mathematical meaning is formally clear, the physical behaviour is understood only for the gradient flow for u. Indeed (cf. [14]) on the jumps the displacment field evolves like a "phase-field visco-elastic flow"

$$\begin{cases} \operatorname{div}(\sigma_{z(s)}(u(s)+u'(s))) = 0 & \Omega \\ u(s) = \pm\hat{e} & \partial_D\Omega \\ \sigma_{z(s)}(u(s)+u'(s))\,\hat{n} = 0 & \partial_N\Omega. \end{cases}$$

On the contrary, it seems not easy to provide a meaningful PDE for the evolution of the phase field variable z since Ξ is not a space, but just a convex set.

4 Open problems

In a broader perspective, the most interesting, and probably most difficult, open problem is the convergence of the parametrized quasi-static BV-evolutions, say $s \mapsto (t_\varepsilon(s), u_\varepsilon(s), z_\varepsilon(s))$, as $\varepsilon \to 0$. On the base on Γ-convergence [2], [6] it is expected a sharp crack evolution, possibly in the space *SBD* [4] or *GSBD* [8]. However, Γ-convergence has been crafted to study the convergence of energies and global minimizers but it is not enough to provide convergence of equilibrium points and slopes which are the main ingredients for gradient flows and BV-evolutions. In general to have have convergence of BV-evolutions it is necessary to have at least a sort of Γ-liminf inequality for the slopes [22], as it is for gradient flows [25]. However, it is not yet known any reasonable notion of slope in *SBD* spaces.

In this direction some partial results have been published: for instance [10] (in the one dimensional setting) [21] and [27] (with geometrical restriction on the crack) and [3] (with a regularized energy).

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