Special functions with bounded variation and with weakly differentiable traces on the jump set

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1 Introduction

"Free-discontinuity problems" are those problems in the Calculus of Variations where the unknown is a pair (u, K), with K a closed set and u a function (sufficiently) smooth on $\Omega \setminus K$ (Ω a fixed open set). As an example of problems of this type we can consider some models in fracture mechanics for brittle materials. A hyperelastic medium subject to fracture, following Griffith's theory, can be modeled by the introduction, besides the elastic volume energy, of a surface term which accounts for crack propagation. In its simplest formulation, the energy of a deformation u will be of the form

$$E(u, K) = \int_{\Omega \setminus K} f(\nabla u)\, dx + \lambda \mathcal{H}^{n-1}(K), \tag{1.1}$$

where ∇u is the deformation gradient, Ω the reference configuration, and K is the crack surface. The bulk energy density f accounts for elastic deformations outside the crack, while λ is a constant given by Griffith's criterion for fracture initiation (see [23], [25], [33], [32], [11]). The existence of equilibria, under appropriate boundary conditions, can be deduced from the study of minimum pairs (u, K) for the energy (1.1). Other models leading to similar minimization problems can be found in the theory of computer vision (see [30], [27], [14], [5]). Note that if $E(u, K) < +\infty$, then the Lebesgue measure of K is zero, u can be regarded as a measurable function defined on Ω, and the set K can be thought of as (a set containing) the set of discontinuity points for u, which explains the name "free discontinuity" problems. Note moreover that in general K will not be the boundary of a set (in this special case we talk of *free boundary* problems).

The application of the direct methods of the Calculus of Variations to minimum problems involving such functionals is forbidden by the impossibility of finding suitable topologies on closed sets that guarantee the compactness for minimizing sequences. The presence of two unknowns, the surface K and the deformation u, can be overcome by a weak formulation of the problem in spaces of discontinuous functions. The space of "special functions of bounded variation" $SBV(\Omega; \mathbf{R}^m)$ has been introduced by De Giorgi and Ambrosio ([16]) as the subset of $\mathbf{R}^m$-valued functions of bounded variation on the open set $\Omega \subset \mathbf{R}^n$, whose measure first derivative can be written in the form

$$Du = \nabla u\, \mathcal{L}^n \llcorner \Omega + (u^+ - u^-) \otimes \nu_u\, \mathcal{H}^{n-1} \llcorner S(u),$$

where ∇u is now the approximate gradient of u, $S(u)$ is the complement of the set of Lebesgue points of u, ν_u is the unit normal to $S(u)$, and $u^\pm$ the approximate trace values of u on both sides of $S(u)$. The measures $\mathcal{L}^n$ and $\mathcal{H}^{n-1}$ are the n-dimensional Lebesgue measure and the $(n-1)$-dimensional Hausdorff measure, respectively, and $\mu \llcorner A$ denotes the restriction of the measure μ to the set A.

The energy functional in (1.1) can be rewritten as

$$\mathcal{E}(u) = \int_\Omega f(\nabla u)\,dx + \lambda \mathcal{H}^{n-1}(S(u)),$$

which makes sense on $SBV(\Omega;\mathbf{R}^m)$. If f is quasiconvex and satisfies some standard growth conditions, then we can apply Ambrosio's lower semicontinuity and compactness theorems to obtain minimum points for some problems involving functionals like $\mathcal{E}$ (see [3], [4], [5], [6]). A complete regularity theory for minimum points u for $\mathcal{E}$ has not been developed yet, but in some cases it is possible to prove that the jump set $S(u)$ is $\mathcal{H}^{n-1}$-equivalent to its closure or even more regular, and that u is smooth on $\Omega\setminus\overline{S(u)}$, thus obtaining minimizing pairs $(u,K)=(u,\overline{S(u)})$ for the functional E (see [17], [8], [9]).

The viewpoint described above privileges the reference configuration, neglecting the effects of crack deformation. The aim of this paper is to define a sub-class of SBV functions which allows the statement (and solution) of problems which involve also the deformation of $S(u)$, *i.e.*, the shape of the crack surface in the deformed configuration. To fix ideas, we can think of an elastic body in two dimensions subject to fracture, so that a "hole" is formed, bounded by two curves Γ^+ and Γ^- which are the images of $S(u)$ by u^+ and u^-, respectively. If the traces are sufficiently smooth then the length of (the boundary of the hole) $\Gamma^+\cup\Gamma^-$ is given by

$$E_1(u) = \int_{S(u)} \Big(\big|\frac{\partial u^+}{\partial\tau}\big| + \big|\frac{\partial u^-}{\partial\tau}\big|\Big)\,d\mathcal{H}^1,$$

where τ is the tangent to $S(u)$. Similarly, if u is bounded and we have an "opening hole" (that is, $\Gamma^+\cup\Gamma^-$ is compactly contained in $u(\Omega)$) we can also consider the "area of the hole", given by

$$\begin{aligned} E_2(u) &= \int_{\text{hole}} dy_1 dy_2 = -\int_{\Gamma^+\cup\Gamma^-} y_1 dy_2 \\ &= -\int_{S(u)} \Big(u_1^+\frac{\partial u_2^+}{\partial\tau} - u_1^-\frac{\partial u_2^-}{\partial\tau}\Big)\,d\mathcal{H}^1, \end{aligned}$$

which again makes sense if the tangential derivatives of $u^\pm$ exist. An analogous formulation for three dimensional elasticity is possible, taking into account the orientation of the surface $\Gamma^+\cup\Gamma^-$.

It is clear that the crucial point in order to extend the definition of functionals as E_1 and E_2 to a wide-enough class to apply the direct methods of the Calculus of Variations is a weak definition of the tangential derivatives of u^+ and u^- on $S(u)$. Our method is close in spirit to the one adopted in [10] (see also

[31]), where a general theory of BV, SBV and Sobolev functions on rectifiable sets is derived. The idea is to look for suitable integrations by parts formulas characterizing the weak derivatives.

Let us consider the simplest case of $u : \Omega \subset \mathbf{R}^2 \to \mathbf{R}$, with $S(u)$ and $u_{|\Omega\setminus S(u)}$ smooth enough, and any $\phi \in C^1_0(\Omega)$, $\psi \in C^1_b(\mathbf{R})$. A simple application of the Gauss-Green formula gives

$$-\int_{S(u)} \Big(\psi(u^+)\frac{\partial u^+}{\partial \tau} - \psi(u^-)\frac{\partial u^-}{\partial \tau}\Big)\phi(x)\, d\mathcal{H}^1 = \int_\Omega \psi(u)\Big(\frac{\partial \phi}{\partial x_1}\frac{\partial u}{\partial x_2} - \frac{\partial \phi}{\partial x_2}\frac{\partial u}{\partial x_1}\Big)\, dx$$

so that an integration on Ω provides information on the integrability of tangential derivatives of $u^\pm$. On the other hand, a different integration by parts yields

$$\int_\Omega \psi(u)\Big(\frac{\partial \phi}{\partial x_1}\frac{\partial u}{\partial x_2} - \frac{\partial \phi}{\partial x_2}\frac{\partial u}{\partial x_1}\Big)\, dx = \int_{S(u)} \Big(\int_{u^-(x)}^{u^+(x)} \psi(y)\, dy\Big)\frac{\partial \phi}{\partial \tau}(x)\, d\mathcal{H}^1(x)$$

which provides a weak definition of tangential derivatives. We say that a bounded function u belongs to $SBV_0(\Omega, \mathbf{R})$ if the linear functional

$$L(\phi\psi) = \int_{S(u)} \Big(\int_{u^-(x)}^{u^+(x)} \psi(y)\, dy\Big)\frac{\partial \phi}{\partial \tau}\, d\mathcal{H}^1 \tag{1.4}$$

defines a measure (*i.e.*, it can be extended to a bounded linear functional on $C^0_0(\Omega \times \mathbf{R})$). If $\Omega \subset \mathbf{R}^n$ it is possible to define the space $SBV_0(\Omega, \mathbf{R})$, asking that there exist measures μ_α (α multi-index of order $n-2$) such that

$$\int_{\Omega\times\mathbf{R}} \varphi(x)\psi(y)\, d\mu_\alpha = \int_{S(u)} \Big(\int_{u^-(x)}^{u^+(x)} \psi(y)\, dy\Big)\Big(\frac{\partial \phi}{\partial x_{i_1}}\nu_{i_2} - \frac{\partial \phi}{\partial x_{i_2}}\nu_{i_1}\Big) d\mathcal{H}^{n-1},$$

where i_1, i_2 are indices such that

$$dx_{i_1} \wedge dx_{i_2} \wedge dx_\alpha = dx_1 \wedge \ldots \wedge dx_n.$$

This definition is coherent with the characterization by means of an integration by parts formula of the space SBV due to Ambrosio (see [7]): a bounded function u belongs to $SBV(\Omega)$ and $\mathcal{H}^{n-1}(S(u)) < +\infty$ if and only if the functionals

$$L_i(\phi\psi) = \int_\Omega \Big(\frac{\partial \phi}{\partial x_i}\psi(u) + \phi\psi'(u)\frac{\partial u}{\partial x_i}\Big)\, dx \tag{1.5}$$

define n measures for $i = 1, \ldots, n$. This characterization is a consequence of the chain rule formula for function in BV.

We can interpret formulas (1.4) and (1.5) above as a properties of the graph of u, which is given for BV functions by

$$\Gamma = \{(x, u(x)) : \ x \in \Omega,\ \exists \nabla u(x)\},$$

and is oriented by the unit n-covector

$$\eta(x,u(x)) = \frac{1}{\sqrt{1+|\nabla u|^2(x)}}(e_1, \frac{\partial u}{\partial x_1}(x)) \wedge \ldots \wedge (e_n, \frac{\partial u}{\partial x_n}(x)) \,,$$

where $\{e_1,\ldots,e_m\}$ is the standard orthonormal basis of $\mathbf{R}^n$. We can define the linear functional on n-forms (n-current) T_u "integration on the graph", by

$$\omega \mapsto \langle T_u, \omega\rangle = \int_\Gamma \langle \omega, \eta\rangle d\mathcal{H}^n \,,$$

and the *boundary of* T_u as the $(n-1)$-current ∂T_u given by

$$\langle \partial T_u, \omega\rangle = \langle T_u, d\omega\rangle \,.$$

We can read formula (1.5) as a property of ∂T_u. In fact, using the area formula, we have $L_i(\phi\psi) = \partial T_u(\phi\psi d\widehat{x}_i)$, where

$$d\widehat{x}_i = (-1)^{i+1} dx_1 \wedge \ldots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \ldots \wedge dx_n,$$

so that (1.5) states precisely that the boundary of T_u is a measure when computed on "horizontal forms" (*i.e.*, forms with no dy). Formula (1.4), which defines SBV_0, requires that ∂T_u be a measure also when computed on $(n-1)$-forms *with* a vertical part. Roughly speaking, this is equivalent to requiring that the traces $u^\pm$ be functions of bounded variation on $S(u)$ (this is not precisely true, since $S(u)$ may present a very complex structure). Moreover, we prove in Proposition 3.11 that the *approximate tangential derivatives* $\nabla u^\pm$ exist $\mathcal{H}^{n-1}$-a.e. on $S(u)$, and

$$\int_{S(u)} |\nabla u^\pm| \, d\mathcal{H}^{n-1} < +\infty.$$

We denote by $\partial_v T_u$ the vector of the measures μ_α; *i.e.*, the components of ∂T_u corresponding to differential forms $\varphi dx_\alpha \wedge dy$. The letter v refers to the fact that we have in mind "vertical components". The class $SBV_0(\Omega)$ has the following compactness property, proved in Theorem 3.10.

Compactness Theorem *Let (u_h) be a sequence in $SBV(\Omega,\mathbf{R}) \cap L^\infty(\Omega)$, $p > 1$ and assume that*

$$\sup_{h\in\mathbf{N}} \Big\{ \int_\Omega |\nabla u_h|^p \, dx + \mathcal{H}^{n-1}\big(S(u_h)\big) + \|u_h\|_\infty \Big\} < +\infty.$$

and that the sequence $\|\partial_v T_{u_h}\|(\Omega\times\mathbf{R})$ be bounded; then there exists a subsequence $(u_{h(k)})$ converging in $L^1_{\rm loc}(\Omega)$ to $u \in SBV_0(\Omega,\mathbf{R})$ and $\partial T_{u_{h(k)}}$ weakly converges to ∂T_u in $\Omega \times \mathbf{R}$.

As a subclass of $SBV_0(\Omega, \mathbf{R})$ (that is, "SBV-functions with BV-traces on $S(u)$") we can consider the family of "SBV-functions with Sobolev traces on $S(u)$", that is, those SBV_0 functions such that

$$\int_{S(u)} |\nabla u^{\pm}|^p \, dx < +\infty$$

for some $p \geq 1$, and such that the measure $\partial_v T_u$ is determined by $\nabla u^{\pm}$; *e.g.*, in the case $n = 2$

$$\langle \partial_v T_u, \phi\psi \rangle = -\int_{S(u)} \Big(\psi(u^+)\nabla u^+ - \psi(u^-)\nabla u^- \Big) \phi(x) d\mathcal{H}^1 \, .$$

Unfortunately, this subclass is not compact: it is possible to find (see Examples 4.6 and 4.7) an example such that all hypotheses of the compactness theorem are satisfied and in addition $\nabla u_h^{\pm}$ are equi-bounded, but the limit u does not possess Sobolev traces on $S(u)$. This phenomenon is due to the fact that $S(u_h)$ may converge only in a weak sense to $S(u)$, losing area in the limit; the phenomenon does not occur if we have no area loss, *i.e.*, $\mathcal{H}^{n-1}(S(u_h)) \to \mathcal{H}^{n-1}(S(u))$.

In the vector-valued case the definition of $SBV_0(\Omega, \mathbf{R}^m)$ is the same, requiring that ∂T_u be a vector measure. Notice however that now we must take into account all differential forms

$$\varphi \, dx_\alpha \wedge dy_\beta,$$

where α and β are multi-indices with $|\alpha| + |\beta| = n - 1$. This means that we will have to take into account also non-linear quantities involving minors of the matrix ∇u. As in the scalar case, we prove a compactness theorem for $SBV_0(\Omega, \mathbf{R}^m)$, under the additional hypothesis (analogous to the one in the classical compactness theorems by Ball [12]) that all approximate determinants are summable enough.

It is interesting to notice (see (5.6), (3.2) and Proposition 3.1) that the class $SBV_0^p(\Omega, \mathbf{R}^m)$ consists of functions u such that the (distributional) determinant of minors of ∇u can be split into an absolutely continuous measure with respect to the Lebesgue measure and a measure absolutely continuous with respect to $\mathcal{H}^{n-1} \llcorner S(u)$, computable in terms of $\nabla u^{\pm}$.

The plan of our paper is the following. In §2 we introduce the basic notions of Geometric Measure Theory (rectifiable sets, approximate continuity and approximate differentiability, BV functions, currents) that we will use in the sequel. In particular, following the ideas of [22], [21] and using essentially the area formula, we explain in Theorem 2.11 and Theorem 2.14 the structure of currents associated to graphs on rectifiable sets. In §3 we introduce the space $SBV_0(\Omega, \mathbf{R})$ and we prove some of its properties. Moreover, we notice that $SBV_0(\Omega, \mathbf{R})$ is a proper subset of $SBV(\Omega, \mathbf{R})$ and we give several examples showing to what extent the traces u^+, u^- are "functions with bounded variation on $S(u)$". In

§4 we define the class $SBV_0^p(\Omega,\mathbf{R})$ of functions $u \in SBV(\Omega,\mathbf{R})$ with "Sobolev traces on $S(u)$", we show that the weak derivatives of $\nabla u^{\pm}$ are precisely their approximate differentials (thus extending to this context the Calderon-Zygmund theorem) and we prove in Theorem 4.5 a closure property of this class. Finally, in §5 we extend the definitions of §3 and §4 to vector-valued functions.

2 Rectifiable sets, SBV functions, currents

2.1 Rectifiable sets

In this paper Ω denotes an open set in $\mathbf{R}^n$, $\mathcal{H}^k$ denotes the k-dimensional Hausdorff measure in $\mathbf{R}^n$, $\mathcal{H}^k \llcorner E$ the measure $\mathcal{H}^k$ restricted to the set E.

Definition 2.1 (rectifiable sets) (see [19, 3.2.14]) We say that $E \subset \mathbf{R}^n$ is *countably $\mathcal{H}^k$-rectifiable* if $\mathcal{H}^k$-almost all of E can be covered with a sequence of C^1 hypersurfaces Γ_i, *i.e.*

$$\mathcal{H}^k\Big(E \setminus \bigcup_{i=1}^{\infty} \Gamma_i\Big) = 0.$$

We say that E is *$\mathcal{H}^k$-rectifiable* if E is countably $\mathcal{H}^k$-rectifiable and $\mathcal{H}^k\big(E\big) < +\infty$.

The *approximate tangent space* $\mathrm{Tan}^k\big(E,x\big)$ of a $\mathcal{H}^k$-rectifiable set E at x is the k-plane $S \subset \mathbf{R}^n$ such that, denoting by $E_\rho = \rho^{-1}(E-x)$ the dilations of E around x, we have

$$\lim_{\rho\to 0} \int_{E_\rho} \phi(y)\, d\mathcal{H}^k(y) = \int_S \phi(y)\, d\mathcal{H}^k(y) \qquad \forall\, \phi \in C_0^1(\mathbf{R}^n).$$

The map $x \mapsto \mathrm{Tan}^k\big(E,x\big)$ is defined $\mathcal{H}^k$-a.e. on E and is $\mathcal{H}^k$-measurable (see [19, 3.2.25]). An additional, useful property of this map is the *locality* (see [34, Remark 11.5]):

$$\mathrm{Tan}^k\big(E,x\big) = \mathrm{Tan}^k\big(E',x\big) \qquad \text{for } \mathcal{H}^k\text{-a.e. } x \in E \cap E' \tag{2.1}$$

for any pair of $\mathcal{H}^k$-rectifiable sets E, E'. Using this property, the approximate tangent space can be defined even for countably $\mathcal{H}^k$-rectifiable sets and this extension still satisfies (2.1).

2.2 BV and SBV functions

Let $u \in BV(\Omega,\mathbf{R}^m)$, the space of functions whose components have bounded variation in Ω. We denote by $S(u)$ the jump set of u, defined as the complement of the Lebesgue set of u:

$$x \notin S(u) \quad \Longleftrightarrow \quad \exists z \in \mathbf{R} \text{ such that } \lim_{\rho\to 0} \rho^{-n} \int_{B_\rho(x)} |u(y)-z|\, dy = 0.$$

It is well-known (see for instance [19, 4.5.9(16)]) that $S(u)$ is countably $\mathcal{H}^{n-1}$-rectifiable. The matrix of measures Du representing the distributional derivative of u can be decomposed into its absolutely continuous part $\nabla u \mathcal{L}^n$ and its singular part $D^s u$ with respect to the Lebesgue measure $\mathcal{L}^n$. It is clear that u belongs to the Sobolev space $W^{1,1}(\Omega, \mathbf{R}^m)$ if and only if $Du = \nabla u \mathcal{L}^n$, or, equivalently, $D^s u = 0$.

Definition 2.2 **(*SBV* functions)** We say that u is a *special function with bounded variation* in Ω, and we write $u \in SBV(\Omega, \mathbf{R}^m)$, if $D^s u$ is supported in $S(u)$; *i.e.*, $|D^s u|\big(\Omega \setminus S(u)\big) = 0$.

It is well-known (see [35], [36]) that for any BV function u the restriction of Du to the jump set can be represented by the following formula

$$\int_{S(u)} \phi(x) Du = \int_{S(u)} \phi(x)(u^+(x) - u^-(x)) \otimes \nu(x)\, d\mathcal{H}^{n-1}(x) \tag{2.2}$$

where $\nu(x)$ is a unit normal to $\mathrm{Tan}^{n-1}\big(S(u), x\big)$ and $u^\pm$ are the traces on both sides of $S(u)$, defined for $\mathcal{H}^{n-1}$-almost every $x \in S(u)$:

$$\lim_{\rho \to 0^+} \rho^{-n} \int_{B^\pm_\rho(x)} |u(y) - u^\pm(x)|\, dy = 0 \tag{2.3}$$

where

$$B^\pm_\rho(x) = \big\{y : \pm\langle y - x, \nu(x)\rangle \geq 0\big\}. \tag{2.4}$$

Note that a change of sign of ν induces a permutation of u^+, u^-, and only for scalar functions there is a canonical choice of the sign of ν which ensures that $u^+(x) > u^-(x)$. To simplify our notation, in what follows we omit the dependence of ν on u and the dependence of u^+ and u^- on ν (our formulas will be invariant under changes of sign of ν).

By (2.2) we infer that $u \in SBV(\Omega, \mathbf{R}^m)$ if and only if the following integration by parts formula holds

$$\int_\Omega u \frac{\partial \phi}{\partial x_i}\, dx = -\int_\Omega \phi \frac{\partial u}{\partial x_i}\, dx - \int_{S(u)} \phi\big(u^+ - u^-\big)\nu_i\, d\mathcal{H}^{n-1} \tag{2.5}$$

for any $\phi \in C^1_0(\Omega)$ and $i \in \{1, \ldots, n\}$. Hence, for a SBV function u, Du is the sum of an absolutely continuous measure and of a measure absolutely continuous with respect to $\mathcal{H}^{n-1} \llcorner S(u)$.

The space SBV has been introduced in [16] to give a rigorous mathematical formulation to several variational problems involving both a "volume" energy and a "surface" energy. The utility of this space of functions in the Calculus of Variations is shown by the following theorem, proved in [3], [7].

Theorem 2.3 (compactness theorem) *Let* $(u_h) \subset SBV(\Omega) \cap L^\infty(\Omega)$*,* $p > 1$ *and assume that*

$$\sup_{h \in \mathbf{N}} \left\{ \int_\Omega |\nabla u_h|^p \, dx + \mathcal{H}^{n-1}\big(S(u_h)\big) + \|u_h\|_\infty \right\} < +\infty.$$

Then, there exists a subsequence $(u_{h(k)})$ *converging in* $L^1_{\mathrm{loc}}(\Omega)$ *to* $u \in SBV(\Omega)$*. Moreover,* $\nabla u_{h(k)}$ *weakly converges to* ∇u *in* $L^p(\Omega, \mathbf{R}^n)$ *and* $\mathcal{H}^{n-1} \llcorner S(u_{h(k)})$ *weakly converges in* Ω *to a measure* μ *greater than* $\mathcal{H}^{n-1} \llcorner S(u)$*.*

A similar compactness result is true for vector valued SBV functions. Recently, the partial regularity of solutions of some free discontinuity problems has been established (see [8], [9]).

2.3 Approximate continuity and differentiability

In this section we assume that $M \subset \mathbf{R}^n$ is $\mathcal{H}^k$-rectifiable for some integer $k \in [1, n]$, $u : M \to \mathbf{R}^m$ is a Borel function and $x_0 \in M$ is a point where $\mathrm{Tan}^k\big(M, x_0\big)$ is defined. In particular, the k-dimensional density of M at x_0 is 1.

Definition 2.4 We say that u is $\mathcal{H}^k$*-approximately continuous* at x_0 if there exists $y_0 \in \mathbf{R}^m$ such that all the sets

$$E_\epsilon := \big\{ x \in M \setminus \{x_0\} : |u(x) - y_0| > \epsilon \big\} \tag{2.6}$$

have k-dimensional density 0 at x_0. The vector y_0, if it exists, is unique and we write

$$y_0 = \text{ap-}\lim_{x \to x_0} u(x).$$

Remark 2.5 In the definition above, the convergence of $u(x)$ to y_0 is understood in a measure theoretic sense, that is, neglecting not only sets of zero Lebesgue measure, but also sets with zero density at x_0. Using Lusin's theorem, it can be proved that u is $\mathcal{H}^k$-approximately continuous at $\mathcal{H}^k$-almost every point $x_0 \in M$.

It is easy to see that any Lebesgue point for u, *i.e.*, a point $x_0 \in M$ such that

$$\lim_{\rho \to 0^+} \rho^{-k} \int_{M \cap B_\rho(x_0)} |u(x) - y_0| \, d\mathcal{H}^k(x) = 0$$

is a point of approximate continuity for u, and the opposite implication holds if u is bounded. The advantage of the notion of approximate continuity is that it makes sense even for maps which are not locally summable. For instance, one can notice that for a BV functions u the maps $u^+(x)$ and $u^-(x)$ need not be summable on $S(u)$.

In the "differential calculus on rectifiable sets" the role of the classical differential is played by the approximate one, defined as follows.

Definition 2.6 Let $m = 1$. We say that u is $\mathcal{H}^k$*-approximately differentiable* at x_0 if there exists a vector $v \in \mathrm{Tan}^k\big(M, x_0\big)$ such that

$$\text{ap-}\lim_{x \to x_0} \frac{|u(x) - u(x_0) - \langle v, x - x_0 \rangle|}{|x - x_0|} = 0.$$

The vector v, called *approximate differential*, is uniquely determined and denoted by $\nabla^M u(x_0)$ (we will drop the superscript M when there is no ambiguity on the domain of u, in particular when M is open). We say that u is $\mathcal{H}^k$-approximately differentiable on M if $\nabla^M u(x)$ is defined for $\mathcal{H}^k$-almost every $x \in M$. The notion of approximate differentiability can also be extended to vector functions, arguing componentwise, and the usual rules of calculus can be extended to approximate differentials.

A very useful property of approximate differentials is the *locality*, that we can state as follows.

Proposition 2.7 *Let M, M' be $\mathcal{H}^k$-rectifiable sets and let $u : M \to \mathbf{R}^m$, $v : M' \to \mathbf{R}^m$ be Borel functions. If u is $\mathcal{H}^k$-approximately differentiable on M and $u \equiv v$ on $M \cap M'$ then the function v is $\mathcal{H}^k$-approximately differentiable at x and $\nabla^M u(x) = \nabla^{M'} v(x)$ for $\mathcal{H}^k$-a.e. $x \in M \cap M'$.*

PROOF. For $\mathcal{H}^k$-a.e. $x_0 \in M \cap M'$ the two approximate tangent spaces $\mathrm{Tan}^k\big(M, x_0\big)$ and $\mathrm{Tan}^k\big(M', x_0\big)$ are both defined and coincide. Moreover, since M' and $M \cap M'$ are $\mathcal{H}^k$-rectifiable, $\mathcal{H}^k$-a.e. $x_0 \in M' \cap M$ is a point of density 1 for M' and $M \cap M'$ (see [18]). In particular, we have

$$\mathrm{Tan}^k\big(M, x_0\big) = \mathrm{Tan}^k\big(M', x_0\big) \quad \text{and} \quad \lim_{\rho \to 0^+} \frac{\mathcal{H}^k\big(B_\rho(x_0) \cap M' \setminus M\big)}{\rho^k} = 0$$

and there exists $\nabla^M u(x_0)$ for $\mathcal{H}^k$-a.e. $x_0 \in M \cap M'$. It is easy to check, using the definition of approximate differential, that $\nabla^{M'} v(x_0) = \nabla^M u(x_0)$ for any $x_0 \in M \cap M'$ with these properties. $\square$

We recall now a classical theorem, due to Calderon-Zygmund (see for instance [18]):

Theorem 2.8 *Let $u \in BV(\Omega, \mathbf{R}^m)$. Then,*

$$\lim_{\rho \to 0^+} \rho^{-n} \int_{B_\rho(x_0)} \frac{|u(x) - u(x_0) - \langle \nabla u(x_0), x - x_0 \rangle|}{|x - x_0|}\, dx = 0 \qquad \textit{for a.e. } x_0 \in \Omega.$$

In particular, u is $\mathcal{H}^n$-approximately differentiable on Ω and the approximate differential coincides almost everywhere with ∇u.

2.4 Currents

Let $A \subset \mathbf{R}^s$ be an open set. A k*-dimensional current* T in A is a linear functional on the space of k-differential forms in A with coefficients in $C_0^\infty(A)$ such that

$$\varphi \mapsto \langle T, \varphi dx_\alpha \rangle$$

is a distribution in A for any multi-index α of order k. Hence, a k-dimensional current can be viewed as a vector distribution whose components are indexed by multi-indices α of order k in $\mathbf{R}^s$.

A k-current T is *representable by integration* in A if all its components are measures of finite total variation in A. By Riesz's theorem, this is equivalent to require the existence of constants C_α such that

$$\bigl|\langle T, \varphi dx_\alpha \rangle\bigr| \le C_\alpha \|\varphi\|_\infty \qquad \forall \varphi \in C_0^1(A)$$

for any multi-index α of order k. Any current representable by integration can be extended in a natural way to forms with bounded Borel coefficients in A. Note that 0-dimensional and s-dimensional currents representable by integration can be identified with scalar measures (in this case there is only one component).

Among currents representable by integration, the integral ones are those associated with integration on rectifiable sets, endowed with a multiplicity and an orientation of the tangent space. Namely, we say that a k-current T is *integral* if there exist a $\mathcal{H}^k$-rectifiable set E and Borel functions $\theta : E \to \mathbf{N}$ and $\eta_1, \ldots, \eta_k : E \to \mathbf{R}^s$ such that the simple k-vector

$$\eta(x) := \eta_1(x) \wedge \ldots \wedge \eta_k(x)$$

has length equal to 1, provides an orientation of $\mathrm{Tan}^k\bigl(E, x\bigr)$ (*i.e.*, the vectors $\eta_i(x)$ span the space) for $\mathcal{H}^k$-a.e. $x \in E$ and

$$\langle T, \varphi dx_\alpha \rangle = \int_E \theta(x)\varphi(x)\langle \eta(x), dx_\alpha \rangle \, d\mathcal{H}^k(x) \qquad \forall \varphi \in C_0^\infty(A) \tag{2.7}$$

where $\langle \cdot, \cdot \rangle$ is the duality between k-vectors and k-covectors.

In order to be closer to the classical theory of integration of differential forms, we will use the shorter notation

$$\int_E \theta \varphi dx_\alpha$$

for the right hand side in (2.7). With this notation, the dependence on the orientation will not be emphasized, but of course we will always specify the orientation of the sets we are dealing with. In the particular case of integrations on codimension-one sets E (as the jump set of a BV function) we will choose a unit normal $\nu(x)$ to $\mathrm{Tan}^{s-1}\bigl(E, x\bigr)$ and we will orient the approximate tangent space to E by the simple, unit $(s-1)$ vector

$$\eta = \eta_1 \wedge \ldots \wedge \eta_{s-1}$$

uniquely determined by the condition

$$\nu \wedge \eta = dx_1 \wedge \ldots \wedge dx_s.$$

The vector η defined in this way is usually denoted by $*\nu$.

If $k \geq 1$ the *boundary* of a k-dimensional current T is the $(k-1)$-dimensional current ∂T defined by

$$\langle \partial T, \omega \rangle = \langle T, d\omega \rangle$$

where d denotes the usual exterior differential operator mapping $(k-1)$-forms in k-forms. When dealing with integral currents associated to the integration on smooth oriented manifolds, this definition of boundary is consistent with Stokes theorem (up to a change of sign).

A fundamental result in the theory of currents has been established by Federer-Fleming (see [19], [26]):

Theorem 2.9 (boundary rectifiability) *Let T be an integral k-current in A, and assume that ∂T is representable by integration. Then ∂T is integral.*

The Federer-Fleming theorem can be viewed as the generalization to the theory of currents of De Giorgi's theorem on the structure of the distributional derivative of sets of finite perimeter (that is, the boundary of the s-current associated to the integration on the set).

2.5 Currents associated to graphs

We are interested in the connection between the approximate differentiability properties of a function and the rectifiability properties of its graph (see also [22], [10]). We will use this connection on one hand to associate to a function $u \in BV(\Omega)$ an integral n-current T_u in $\Omega \times \mathbf{R}$ whose boundary gives much information on u, and on the other hand to obtain differentiability properties of the traces $u^+(x)$, $u^-(x)$ on the jump set $S(u)$ by the rectifiability of their graph.

Definition 2.10 For any $j \times l$ matrix A and any integer $k \in [1, \min\{j, l\}]$ we define

$$\mathbf{M}_k(A) = \sqrt{\sum_{B \subset A} (\det B)^2}$$

where B varies among the $k \times k$ minors of A.

If A is the Jacobian of a parametrization map of a k-dimensional manifold Γ contained in $\mathbf{R}^l$ then the quantity $\mathbf{M}_k(A)$ is the k-dimensional area element. It reduces to $|A|$ and to $|\det(A)|$ in the extreme cases $k = 1$ and $k = l$.

In the following theorem we assume that $M \subset \mathbf{R}^n$ is contained in countably many C^1 hypersurfaces of dimension k and $\mathcal{H}^k\big(M\big) < +\infty$ (this assumption is slightly stronger than $\mathcal{H}^k$-rectifiability).

Theorem 2.11 *Let $M \subset \mathbf{R}^n$ be contained in countably many C^1 hypersurfaces of dimension k with $\mathcal{H}^k\big(M\big) < +\infty$, let $u : M \to \mathbf{R}^m$ be $\mathcal{H}^k$-differentiable at every point in M, and let*

$$\Gamma := \big\{\big(x, u(x)\big) : x \in M\big\} \subset \mathbf{R}^n \times \mathbf{R}^m$$

be the graph of u on M. Then
(i) Γ *is countably $\mathcal{H}^k$-rectifiable and*

$$\mathcal{H}^k\big(\Gamma\big) = \int_M \mathbf{M}_k(J\Phi)\, d\mathcal{H}^k, \tag{2.8}$$

where $\Phi(x) = (x, u(x))$ and $J\Phi$ is the linear map

$$v \in \operatorname{Tan}^k\big(M, x\big) \mapsto \big(v, \nabla^M u(x)v\big).$$

(ii) *For $\mathcal{H}^k$ almost any $(x, y) \in \Gamma$ the unit k vector*

$$\eta = \frac{\sigma_1 \wedge \ldots \wedge \sigma_k}{\mathbf{M}_k(J\Phi)} \qquad \textit{with} \qquad \sigma_i(x, y) = \big(v_i, \nabla^M u(x)v_i\big)$$

(where $v_1, \ldots, v_k$ is an orthonormal basis of $\operatorname{Tan}^k\big(M, x\big)$) induces an orientation of Γ.
(iii) *If $\mathcal{H}^k\big(\Gamma\big) < +\infty$ and the orientation of Γ is induced by the orientation of M as in* (ii)*, then we have*

$$\int_\Gamma \varphi(x, y) dx_\alpha \wedge dy_\beta = \int_M \varphi(x, u(x)) dx_\alpha \wedge du_\beta \tag{2.9}$$

for any k-differential form $\varphi dx_\alpha \wedge dy_\beta$ in $\mathbf{R}^{n+m}$ with φ bounded Borel function.

Proof. All the statements follow by the classical area formula for Lipschitz functions. In fact, M can be covered by a family of sets M_h such that the restriction of u to M_h is a Lipschitz function (see [19], Theorem 3.1.8). The locality properties of the approximate differentials allow a restriction to M_h, where the area formula can be applied (see [22] for details). □

Definition 2.12 Let $u \in BV\big(\Omega, \mathbf{R}^m\big)$. By Theorem 2.8 the function u is $\mathcal{H}^n$-differentiable in Ω. Assuming $\mathbf{M}_n(J\Phi) \in L^1(\Omega)$ (with $\Phi(x) = (x, u(x))$), we will denote by T_u the n-current given by Theorem 2.11; *i.e.*, the integral current associated to the graph of u with multiplicity 1 and orientation induced by the canonical one in Ω. By (2.9), T_u acts as follows

$$\langle T_u, \varphi dx_\alpha \wedge dy_\beta\rangle = \int_\Omega \varphi(x, u(x)) dx_\alpha \wedge du_\beta \tag{2.10}$$

for any bounded Borel function φ defined in $\Omega \times \mathbf{R}^m$. Notice that in the integration on the right hand side is involved a minor of ∇u of the same order of β.

The compatibility condition between the orientation of the graph and the orientation of the basis used in Theorem 2.11(ii) can be rephrased as follows.

Definition 2.13 Assume that Γ is a $\mathcal{H}^k$-rectifiable set in $\mathbf{R}^{n+m}$ and let $\pi : \mathbf{R}^{n+m} \to \mathbf{R}^n$ be the orthogonal projection on the first n variables. Then, denoting by M the projection of Γ on $\mathbf{R}^n$ and denoting by $\eta(x)$, $\sigma(x,y)$ simple k-vectors orienting M and Γ respectively, we say that the orientations of Γ and M are *compatible* if

$$\pi(\sigma_1(x,y)) \wedge \ldots \wedge \pi(\sigma_k(x,y)) = \lambda(x,y)\,\eta_1(x) \wedge \ldots \wedge \eta_k(x)$$

with $\lambda(x,y) \geq 0$ for $\mathcal{H}^k$-almost every $(x,y) \in R$.

Theorem 2.14 *Let $M \subset \mathbf{R}^n$ be a k-rectifiable set, $u : M \to \mathbf{R}^m$, $\Gamma \subset M \times \mathbf{R}^m$ be the graph of u and assume that Γ is $\mathcal{H}^k$-rectifiable. Then*
(i) *the function u is $\mathcal{H}^k$-approximately differentiable on M and*

$$\mathcal{H}^k\big(\Gamma\big) \geq \int_M \mathbf{M}_k(J\Phi)\, d\mathcal{H}^k \tag{2.11}$$

where $\Phi(x) = (x, u(x))$ and $J\Phi$ is the linear map

$$v \in \mathrm{Tan}^k\big(M, x\big) \mapsto \big(v, \nabla^M u(x) v\big).$$

(ii) *If $\mathcal{H}^k\big(\Gamma\big) < +\infty$ and the orientations of M and Γ are compatible as in Definition* 2.13 *then we have*

$$\int_\Gamma \varphi(x,y) dx_\alpha = \int_M \varphi(x, u(x)) dx_\alpha \tag{2.12}$$

for any multi index α in $\mathbf{R}^n$ of order k and any bounded Borel function φ.

Proof. Using the locality properties of approximate tangent spaces and of approximate differentials it is not restrictive to assume that Γ is a compact subset of a k-dimensional C^1 surface Γ_0. Let $S(x,y)$ be the (classical) tangent space to Γ_0 at (x,y) and let $\pi : \mathbf{R}^{n+m} \to \mathbf{R}^n$ be the projection on the first n coordinates.

We denote by $\Gamma_1 \subset \Gamma$ the non vertical part of Γ; *i.e.*, the set of points $(x,y) \in \Gamma$ such that $\pi\big(S(x,y)\big)$ is a k-dimensional space. Since the rank of the restriction of π to $S(x,y)$ is strictly less than k for any $(x,y) \in \Gamma \setminus \Gamma_1$, the area formula (see [18]) implies that the set $\pi(\Gamma \setminus \Gamma_1)$ is $\mathcal{H}^k$-negligible. We will prove that u is approximately differentiable at any point $x_0 \in \pi(\Gamma_1)$ such that $\mathrm{Tan}^k\big(M, x_0\big)$ exists. Indeed, if $x_0 \in \pi(\Gamma_1)$ is such a point then, by the implicit function theorem the surface Γ_0 coincides for r sufficiently small with the graph of a C^1 function v defined in $\pi\big(\Gamma_0 \cap B_r(x_0, u(x_0))\big)$. Denoting by M' this set, the functions u and v coincide in $\pi\big(\Gamma \cap B_r(x_0, u(x_0))\big)$, which is contained in M'

and contains $M \cap B_s(x_0)$ for s sufficiently small. The definition of approximate differential shows that, in this situation, the classical differentiability of v at x_0 implies the approximate differentiability of u at x_0. Hence, the approximate differentiability of u is proved.

Finally, since u is $\mathcal{H}^k$-approximately differentiable in M, (i) and (ii) follow by the area formula arguing as in Theorem 2.11. □

Remark 2.15 The main difference between Theorem 2.11 and Theorem 2.14 is that in the latter the function u need not be approximately differentiable on *all* of M, hence Γ may contain "vertical parts" which are responsible for the strict inequality in (2.11). For this reason, in (2.12) we are only able to carry from Γ to M the integration of "horizontal" differential forms; *i.e.*, those containing no dy_j.

Let us consider, for instance, the graph of the Cantor-Vitali function $u(t) : [0,1] \to [0,1]$. In this case

$$\int_0^1 \sqrt{1+(u')^2}\,dt = 1$$

because $u' = 0$ almost everywhere in $[0,1]$. On the other hand, since Γ is a connected set containing $(0,0)$ and $(1,1)$, we have $\mathcal{H}^1(\Gamma) \geq \sqrt{2} > 1$ (a more accurate projection argument shows that $\mathcal{H}^1(\Gamma) = 2$). Similarly, we see that

$$\int_0^1 \varphi(t,u(t))u'(t)\,dt = 0$$

while an approximation argument shows that

$$\int_\Gamma \varphi(t,y)\,dy = \int_0^1 \varphi(t,u(t))Du(t)$$

and this integral is not equal to 0 if, for instance, $\varphi(t,u(t))$ is strictly positive on the Cantor set (the support of Du).

We conclude this section by introducing functionals on graphs corresponding to the integration of a power of the gradient on the basis.

Let $\Gamma \subset \mathbf{R}^n_x \times \mathbf{R}^m_y$ be a $\mathcal{H}^k$-rectifiable set, let $\{\xi_i(x,y)\}$ be an orthonormal basis of $\mathrm{Tan}^k\big(M,(x,y)\big)$. The vector $\xi = \xi_1 \wedge \ldots \wedge \xi_k$ provides an orientation of Γ. Similarly, we denote by $v_1 \wedge \ldots \wedge v_k$ an orientation of $\pi(\Gamma)$, where π is the projection on the x variables. We define

$$\xi^0 := \langle \xi, dv_1 \wedge \ldots \wedge dv_k \rangle$$

$$\xi^1_{i,j} := \langle \xi, dv_1 \wedge \ldots dv_{i-1} \wedge dy_j \wedge dv_{i+1} \wedge \ldots dv_k \rangle.$$

We assume that the two orientations are compatible; *i.e.*, $\xi^0(x,y) \geq 0$ for any $(x,y) \in \Gamma$. For $p > 1$ we set

$$\phi_p(\xi) := \begin{cases} \dfrac{|\xi^1|^p}{(\xi^0)^{p-1}} & \text{if } \xi_0 > 0; \\ +\infty & \text{otherwise.} \end{cases}$$

Using the inequality $|a+b|^p \leq 2^{p-1}(|a|^p + |b|^p)$ it is not hard to see that ϕ_p is a convex, lower semicontinuous, positively 1-homogeneous function in the space of k-vectors in $\mathbf{R}^n \times \mathbf{R}^m$. Finally, we set

$$\mathcal{F}_p(\Gamma) = \int_\Gamma \phi_p(\xi)\, d\mathcal{H}^k.$$

Theorem 2.16 *Under the assumptions of Theorem* 2.11*, we have*

$$\int_M |\nabla^M u|^p \, d\mathcal{H}^k = \mathcal{F}_p(\Gamma) \qquad\qquad \forall p > 1.$$

Conversely, assume that Γ *is the graph of a function* u *defined on a* $\mathcal{H}^k$*-rectifiable set* $M \subset \mathbf{R}^n$*, and let* $M' \subset M$ *be a set such that* $\mathcal{H}^k\big(M \setminus M'\big) = 0$ *and* (u, M') *satisfy the assumptions of Theorem* 2.11*. Then, denoting by* Γ' *the graph of* u *on* M'*,* $\mathcal{F}_p(\Gamma) < +\infty$ *for some* $p > 1$ *implies* $\mathcal{H}^k\big(\Gamma \setminus \Gamma'\big) = 0$.

Proof. We first assume that u, M, Γ are as in Theorem 2.11. Choosing the orientation as in statement (ii) of the theorem we find

$$\frac{\nabla^M u^j(v_i)}{\mathbf{M}_k(J\Phi)} = \xi^1_{i,j}, \qquad\qquad \frac{1}{\mathbf{M}_k(J\Phi)} = \xi^0$$

hence

$$\begin{aligned} \int_M |\nabla^M u(x)|^p \, d\mathcal{H}^k(x) &= \int_M \frac{|\nabla^M u(x)|^p}{\mathbf{M}_k(J\Phi(x))} \mathbf{M}_k(J\Phi(x))\, d\mathcal{H}^k(x) \\ &= \int_\Gamma \frac{|\xi^1|^p}{|\xi^0|^{p-1}}\, d\mathcal{H}^k = \mathcal{F}_p(\Gamma). \end{aligned}$$

The second part of the statement follows by the same argument of Theorem 2.14, noticing that in this case $\mathcal{H}^k\big(\Gamma \setminus \Gamma_1\big) = 0$. Indeed, $\mathcal{F}_p(\Gamma) < +\infty$ for some $p > 1$ implies $\xi^0 > 0$ for $\mathcal{H}^k$-a.e. $(x,y) \in \Gamma$, hence the rank of π restricted to the tangent space of Γ at (x,y) is k for $\mathcal{H}^k$-a.e. $(x,y) \in \Gamma$. $\square$

Remark 2.17 The functional $\mathcal{F}_p$ defined above is not lower semicontinuous with respect to the weak convergence of currents. We can consider $n = 2$,

$m = 1$, and the sequence (T_h) of 1-dimensional currents with support on the sets

$$\Gamma_h = \{(x, y, z) : \ x = \frac{1}{h}\cos(ht), \ y = \frac{1}{h}\sin(ht), \ z = t, \ 0 < t < 1\}.$$

and orientation given by

$$\xi_h(x, y, z) = \frac{1}{\sqrt{2}}(-hy, hx, 1).$$

We provide $\gamma_h = \pi(\Gamma_h)$ with the orientation given by $v_h(x, y) = (-hy, hx)$, so that $\xi^0_h = \xi^1_h = 1/\sqrt{2}$. The weak limit of T_h is the current with support on $\Gamma = \{(0, 0, t) : \ 0 < t < 1\}$, and constant orientation e_3, so that $\xi^0 = 0$, and $\mathcal{F}_p(\Gamma) = +\infty$, whatever orientation we provide $\gamma = \pi(\Gamma) = \{0\}$ with. Nevertheless we have

$$\mathcal{F}_p(\Gamma_h) = \int_{\Gamma_h} \frac{1}{\sqrt{2}} \, d\mathcal{H}^1 = 1\,.$$

Note that the vector measures $(\xi^0_h, \xi^1_h)\mathcal{H}^1 \llcorner \Gamma_h$ weakly converge to the measure $(1, 1)\mathcal{H}^1 \llcorner \Gamma$.

3 Functions of class $SBV_0(\Omega, \mathbf{R})$

Let $u \in SBV\big(\Omega\big) = SBV\big(\Omega, \mathbf{R}\big)$ and let us compute the boundary of the integral current T_u of Definition 2.12 on $(n-1)$-differential forms $\varphi d\hat{x}_i$, where $d\hat{x}_i$ by definition satisfies

$$dx_i \wedge d\hat{x}_i = (-1)^{i+1} dx_1 \wedge \ldots \wedge dx_n.$$

In what follows we will denote by $\mathcal{T}(\Omega \times \mathbf{R})$ the class of C^1-functions $\varphi(x, y)$ such that $|\varphi| + |\nabla\varphi|$ is bounded and the support of φ is contained in $K \times \mathbf{R}$ for some compact set $K \subset \Omega$. This class of functions, larger than $C^1_0(\Omega \times \mathbf{R})$, will be technically more convenient since we are going to integrate over graphs.

Let $\varphi \in \mathcal{T}(\Omega \times \mathbf{R})$ and $U(x) = \varphi(x, u(x))$; applying (2.5) to the SBV function U (see [[7]]) and using the identities

$$\frac{\partial U}{\partial x_i} = \frac{\partial \varphi}{\partial x_i} + \frac{\partial \varphi}{\partial y}\frac{\partial u}{\partial x_i}, \qquad U^+(x) = \varphi(x, u^+(x)), \ \ U^-(x) = \varphi(x, u^-(x))$$

we get (see also (2.10))

$$\begin{aligned} \langle T_u, d\big(\varphi d\hat{x}_i\big)\rangle &= \langle T_u, \frac{\partial \varphi}{\partial x_i} dx_i \wedge d\hat{x}_i + \frac{\partial \varphi}{\partial y} dy \wedge d\hat{x}_i\rangle \\ &= (-1)^{i+1}\int_\Omega \left(\frac{\partial \varphi}{\partial x_i}(x, u(x)) + \frac{\partial \varphi}{\partial y}(x, u(x))\frac{\partial u}{\partial x_i}(x)\right) dx \end{aligned}$$

$$= \quad (-1)^{i+1}\int_\Omega \frac{\partial U}{\partial x_i}\,dx = (-1)^i\int_{S(u)}\big(U^+(x)-U^-(x)\big)\nu_i\,d\mathcal{H}^{n-1}$$

$$= \quad (-1)^i\int_{S(u)}\Big(\varphi(x,u^+(x))-\varphi(x,u^-(x))\Big)\nu_i(x)\,d\mathcal{H}^{n-1}. \tag{3.1}$$

In particular, if $\varphi\in C_0^\infty(\Omega\times\mathbf{R})$ then this identity shows that for a function $u\in SBV\big(\Omega\big)$ with $\mathcal{H}^{n-1}$-rectifiable jump set the components $d\hat{x}_i$ of ∂T_u are representable by integration. Actually, it turns out that this is a characterization of SBV functions with this property. This characterization has been used in [7] to give a new proof of the compactness properties of SBV, stated in Theorem 2.3:

Theorem 3.1 (geometric characterization of SBV) *Let $u\in BV\big(\Omega\big)$ and let T_u be the graph of u, defined in* (2.12)*. Then, the following properties are equivalent:*
(i) *$u\in SBV\big(\Omega\big)$ and $\mathcal{H}^{n-1}\big(S(u)\big)<+\infty$;*
(ii) *For any $i\in\{1,\dots,n\}$ there exists a constant C_i such that*

$$\big|\langle T_u, d\big(\varphi d\hat{x}_i\big)\rangle\big|\le C_i\|\varphi\|_\infty \qquad\qquad \forall\varphi\in\mathcal{T}\big(\Omega\times\mathbf{R}\big).$$

In this case (3.1) *holds.*

Remark 3.2 The geometric characterization of SBV functions with a $\mathcal{H}^{n-1}$-rectifiable jump set can be restated without using currents by requiring the existence of Radon measures μ_i in $\Omega\times\mathbf{R}$ such that the following integration by parts formula holds

$$\int_\Omega\Big(\frac{\partial\varphi}{\partial x_i}(x,u(x))+\frac{\partial\varphi}{\partial y}(x,u(x))\frac{\partial u}{\partial x_i}(x)\Big)\,dx=\int_{\Omega\times\mathbf{R}}\varphi\,d\mu_i \tag{3.2}$$

for any $\varphi\in\mathcal{T}(\Omega\times\mathbf{R})$. Indeed, we need only to invoke Riesz representation theorem and to remark that the left hand side in (3.2) is $(-1)^{i+1}\langle T_u, d(\varphi d\hat{x}_i)\rangle$. Moreover, (3.1) shows that the measures μ_i are given by

$$\mu_i=-\Big(\Phi^+_\#\big(\nu_i\mathcal{H}^{n-1}\llcorner S(u)\big)-\Phi^-_\#\big(\nu_i\mathcal{H}^{n-1}\llcorner S(u)\big)\Big) \tag{3.3}$$

where $\Phi^\pm(x)=\big(x,u^\pm(x)\big)$.

Theorem 3.1 shows that membership to SBV is related to the properties of the boundary of T_u; *i.e.*, to the possibility of representing by integration some components (the "horizontal" ones) of ∂T_u. Assuming $n\ge 2$, this suggests the following definition, in which we require *all* components of ∂T_u to be representable by integration.

Definition 3.3 (the class SBV_0) Let $u \in SBV\big(\Omega\big)$ with a $\mathcal{H}^{n-1}$-rectifiable jump set. We say that u belongs to the class $SBV_0\big(\Omega\big)$ if for all multi-index α in $\mathbf{R}^n$ of order $(n-2)$ there exists a constant C_α such that

$$\big|\langle T_u, d\big(\varphi dx_\alpha \wedge dy\big)\rangle\big| \le C_\alpha \|\varphi\|_\infty \qquad \forall \varphi \in \mathcal{T}\big(\Omega \times \mathbf{R}\big). \tag{3.4}$$

In particular, ∂T_u is representable by integration in $\Omega \times \mathbf{R}$.

As in (3.2), we can formulate the condition (3.4) without using explicitly T_u. Let $C^1_b(\mathbf{R})$ be the space of bounded, Lipschitz, C^1 functions $\psi(y)$; for any function $\phi \in C^1_0(\Omega)$ the product $\varphi(x,y) := \phi(x)\psi(y)$ belongs to $\mathcal{T}(\Omega \times \mathbf{R})$. We use these products in the following

Theorem 3.4 *Let $u \in SBV\big(\Omega\big)$ with $\mathcal{H}^{n-1}(S(u)) < +\infty$. Then, $u \in SBV_0\big(\Omega\big)$ if and only if, for any multi index α in $\mathbf{R}^n$ of order $(n-2)$ there exists a measure μ_α with finite total variation in $\Omega \times \mathbf{R}$ such that*

$$\int_{\Omega\times\mathbf{R}} \phi(x)\psi(y)\, d\mu_\alpha(x,y) = \int_{S(u)} \bigg(\int_{u^-(x)}^{u^+(x)} \psi(y)\, dy\bigg) dx_\alpha \wedge d\phi \tag{3.5}$$

for any $\phi \in C^1_0(\Omega)$ and $\psi \in C^1_b(\mathbf{R})$. If (3.5) holds, then μ_α is the α-th component of ∂T_u.

Proof. Let i_1, i_2 be indices such that

$$dx_{i_1} \wedge dx_{i_2} \wedge dx_\alpha = dx_1 \wedge \ldots \wedge dx_n.$$

Taking into account (2.10), for any $\phi \in C^2_0(\Omega)$ and any $\psi \in C^1_b(\mathbf{R})$ we have

$$\begin{aligned}
& \langle T_u, d\big(\phi\psi dx_\alpha \wedge dy\big)\rangle = \sum_{j=1}^{2} \langle T_u, \psi \frac{\partial \phi}{\partial x_{i_j}} dx_{i_j} \wedge dx_\alpha \wedge dy\rangle \\
= \;& \sum_{j=1}^{2} \int_\Omega \psi(u) \frac{\partial \phi}{\partial x_{i_j}} dx_{i_j} \wedge dx_\alpha \wedge du \\
= \;& \sum_{j=1}^{2} \int_\Omega \frac{\partial \phi}{\partial x_{i_j}} dx_{i_j} \wedge dx_\alpha \wedge d\Psi(u) \\
= \;& (-1)^{n-2} \int_\Omega \Big(\frac{\partial \phi}{\partial x_{i_1}} \frac{\partial \Psi(u)}{\partial x_{i_2}} - \frac{\partial \phi}{\partial x_{i_2}} \frac{\partial \Psi(u)}{\partial x_{i_1}}\Big) dx_1 \wedge \ldots \wedge dx_n \\
= \;& (-1)^{n-1} \int_\Omega \Psi(u) \bigg(\frac{\partial^2 \phi}{\partial x_{i_2} \partial x_{i_1}} - \frac{\partial^2 \phi}{\partial x_{i_1} \partial x_{i_2}}\bigg) dx \\
& \quad + (-1)^{n-1} \int_{S(u)} \big(\Psi(u^+) - \Psi(u^-)\big) \Big(\frac{\partial \phi}{\partial x_{i_1}} \nu_{i_2} - \frac{\partial \phi}{\partial x_{i_2}} \nu_{i_1}\Big)\, d\mathcal{H}^{n-1} \\
= \;& (-1)^n \int_{S(u)} \big(\Psi(u^+) - \Psi(u^-)\big) d\phi \wedge dx_\alpha
\end{aligned}$$

where Ψ is a primitive of ψ. The last equality follows easily by the fact that $S(u)$ is oriented by $*\nu$ and

$$v_{i_1}\nu_{i_2} - v_{i_2}\nu_{i_1} = -\langle v \wedge dx_\alpha, *\nu\rangle$$

for any co-vector $v \in \mathbf{R}^n$. By approximation, the same formula holds in the case $\phi \in C^1_0(\Omega)$. Hence, if $u \in SBV_0\big(\Omega\big)$ then (3.5) holds for a suitable measure μ_α. Conversely, if (3.5) holds, then (3.4) is fulfilled by functions $\varphi(x,y)$ of the form $\phi(x)\psi(y)$, with $C_\alpha = |\mu_\alpha|(\Omega \times \mathbf{R})$. A standard approximation argument shows that the inequality is true in the general case. $\square$

Remark 3.5 In the case $n = 2$, (3.5) simply becomes

$$\int_{\Omega\times\mathbf{R}} \phi(x)\psi(y)\,d\mu(x,y) = \int_{S(u)} \left(\int_{u^-(x)}^{u^+(x)} \psi(y)\,dy\right) \frac{\partial\phi}{\partial\tau}\,d\mathcal{H}^1 \tag{3.6}$$

for any $\phi \in C^1_0\big(\Omega\big)$, $\psi \in C^1_b\big(\mathbf{R}\big)$, where $\tau = (\tau_1, \tau_2) = *\nu = (-\nu_2, \nu_1)$.

As ∂T_u is related to the integration of the graphs of $u^\pm$ (see (3.1)), we may heuristically think of $SBV_0\big(\Omega\big)$ as the class of functions $u \in SBV\big(\Omega\big)$ such that $S(u)$ is $\mathcal{H}^{n-1}$-rectifiable and u^+, u^- are "functions of bounded variation" on $S(u)$. The following examples illustrate this idea and show that there are functions in $SBV\big(\Omega\big)$ with $\mathcal{H}^{n-1}$-rectifiable jump set which do not belong to $SBV_0\big(\Omega\big)$.

Example 3.6 Let $n = 2$ and let f^+, $f^- \in L^1(\mathbf{R})$. Denoting by $H^+ = \{y > 0\}$, $H^- = \{y < 0\}$ the upper and lower half planes of $\mathbf{R}^2$, respectively, it is well-known (see Gagliardo [20], Alberti [2]) that we can find functions $u^\pm \in W^{1,1}(H^\pm)$ such that $u^\pm(x,y) = 0$ for $|y| \geq 1$, whose trace on $\partial H^\pm$ is $f^\pm$. Let

$$u(x,y) = \begin{cases} u^+(x,y) & \text{if } y > 0; \\ \\ u^-(x,y) & \text{if } y \leq 0. \end{cases}$$

Then, $u \in SBV\big(\mathbf{R}^2\big)$ and its jump set is contained (up to $\mathcal{H}^1$-negligible sets) in the horizontal line. Moreover, choosing as normal the vector e_2, we have

$$u^+(x,0) = f^+(x), \quad u^-(x,0) = f^-(x)$$

for a.e. $x \in \mathbf{R}$ such that $f^+(x) \neq f^-(x)$.

Let us first choose $f^- = 0$. Then, using (3.6) with $\psi \equiv 1$, it is not hard to see that $u \in SBV_0\big(\mathbf{R}^2\big)$ if and only if $f^+ \in BV\big(\mathbf{R}\big)$ and

$$\int_{\mathbf{R}^2} \phi(x_1,x_2)\,d\mu = \int_{\mathbf{R}} \phi(t,0)\,Df^+(t) \qquad\qquad \forall \phi \in C^1_0\big(\mathbf{R}\big).$$

In particular, choosing $f^+ \notin BV(\mathbf{R})$, we have an example of a function in $SBV(\mathbf{R}^2)$, with $\mathcal{H}^{n-1}$-rectifiable jump set, which does not belong to $SBV_0(\mathbf{R}^2)$.

Let $u_1(x,y) = y/|y|$, and let $u_2 \in W^{1,1}(\mathbf{R}^2)$ be a function with trace on the x-axis $f(x) = u_2(x,0)$ with $0 \le f \le 1$. Then, $u_i \in SBV_0(\Omega)$ for all bounded open subsets Ω of $\mathbf{R}^2$ but if $f \notin BV(I)$, where $\Omega \cap \{y=0\} = I \times \{0\}$, we deduce immediately from (3.6) that $u_1 + u_2 \notin SBV_0(\Omega)$. This shows that $SBV_0(\Omega)$ is not a vector space.

Example 3.7 Given a set $E \subset (0,1)$ with infinite perimeter, we choose $f^+ = \chi_E$ and $f^- = \chi_{(0,1)\setminus E}$. In this case, using (3.6) as in the previous example and assuming by contradiction that $u \in SBV_0\big((0,1)\times\mathbf{R}\big)$, we would find the identity

$$\int_{(0,1)\times\mathbf{R}^2} \phi(x_1,x_2)\,d\mu = -\int_0^1 (f^+ - f^-)\phi_{x_1}(x_1,x_2)\,dx_1$$

which would easily imply that $f^+ - f^-$ is a BV function. However, since $f^+ - f^- = \chi_E - \chi_{(0,1)\setminus E} = 2\chi_E - 1$, this is false.

Note that in this example $\{u^+(x), u^-(x)\} = \{0,1\}$ for $\mathcal{H}^1$-almost every $x \in S(u)$, hence there is an orientation of $S(u)$ for which $u^+ = 1$ and $u^- = 0$. This example shows that membership to SBV_0 should be understood as a "regularity" property of the triplet (u^+, u^-, ν) and not of the single components.

Example 3.8 We can easily build a $SBV(\Omega)$ function u with $\nabla u = 0$ in Ω, $\nabla u^+ = 0$, $\nabla u^- = 0$ $\mathcal{H}^{n-1}$ a.e. on $S(u)$, but such that $u \notin SBV_0$, as follows. Let $g \in W^{1,1}((0,1)) \cap C^1((0,1))$ be such that $0 \le g \le 1$ and $g(x) = 0 \iff 1/x \in \mathbf{N}$. We take $n = 2$ and $\Omega = (0,1)\times(-1,1) \subset \mathbf{R}^2$. We define $u : (0,1)\times(-1,1) \to \mathbf{R}$ by

$$u(x,y) = \begin{cases} 3 & \text{if } 0 < y < g(x),\ [1/x] \text{ even,} \\ 2 & \text{if } 0 < y < g(x),\ [1/x] \text{ odd,} \\ 1 & \text{if } y < 0, \\ 0 & \text{otherwise} \end{cases}$$

($[1/x]$ denotes the integer part of $1/x$). This example shows that we cannot expect the regularity of ∇u to increase the regularity of the traces $u^\pm$. At the same time, taking

$$u_h(x,y) = \begin{cases} u(x,y) & \text{if } x > 1/h, \\ 1 & \text{if } x \le 1/h, \\ 0 & \text{otherwise,} \end{cases}$$

since $u_h \in SBV_0$, and $u_h \to u$ in L^1, this example also shows that the limit of a sequence in SBV_0 may fail to belong to this space, even if $\nabla u_h = 0$ and $\mathcal{H}^{n-1}(S(u_h)) \to \mathcal{H}^{n-1}(S(u))$ (for closure properties of SBV_0 see Theorems 3.10 and Theorem 4.5 below).

Example 3.9 Let $K \subset [0,1]$ be the Cantor set and let $v : [0,1] \to \mathbf{R}$ be a continuous function with infinite total variation in $(0,1)$, locally constant in $(0,1) \setminus K$. Denoting by $J_i = (a_i, b_i)$ the connected components of $(0,1) \setminus K$, we can define f^+ and f^- as follows:

$$f^{\pm}(t) = \begin{cases} v(t) & \text{if } t \in K; \\ \\ v(t) \pm c_i(t-a_i)(b_i-t) & \text{if } t \in J_i \text{ for some } i. \end{cases}$$

Choosing properly the constants $c_i > 0$ we have

$$\int_{(0,1)\setminus K} |f^{\pm\prime}|(t)\,dt = \sum_{i=1}^{\infty} c_i \int_{a_i}^{b_i} |b_i - a_i - 2t|\,dt < +\infty.$$

Let $\psi \in C_b^1(\mathbf{R})$ and let Ψ be a primitive of ψ. Since K is negligible, $f^+ = f^-$ in K and v is constant in J_i, the distributional derivative of $u_\psi := \Psi(f^+) - \Psi(f^-)$ is given by

$$\sum_{i=1}^{\infty} \big[\psi(f^+) f^{+\prime} - \psi(f^-) f^{-\prime}\big] \mathcal{L}^1 \llcorner J_i,$$

and this is a finite measure by our choice of c_i.

In this situation, (3.6) becomes

$$\int_{(0,1)\times\mathbf{R}^2} \phi(x_1,x_2)\psi(y)\,d\mu = \int_{(0,1)} \phi(t,0)\,Du_\psi.$$

Hence, the formula is fulfilled by a finite measure μ, given by $S^+ - S^-$, where $S^{\pm}$ are the 1-currents in $\mathbf{R}^2 \times \mathbf{R}$ associated to the integration on the graphs of $f^{\pm}$ on $(0,1) \setminus K$ (viewed as a subset of $\mathbf{R}^2$).

This shows that the function u constructed as in Example 3.6 belongs to $SBV_0\big((0,1)\times\mathbf{R}\big)$. However, since

$$Df^{\pm} = Dv \pm \sum_{i=1}^{\infty} c_i\big(b_i - a_i - 2t\big)\chi_{(a_i,b_i)}\mathcal{L}^1$$

in the sense of distributions, neither f^+ nor f^- belong to $BV\big((0,1)\big)$.

In this case $S(u) = ((0,1)\setminus K) \times \{0\}$ and $f^{\pm} \in BV((0,1)\setminus K)$. However, modifying the construction above following Example 3.8, we can also give an example where $S(u) \supset (0,1)\times\{0\}$, and $u^{\pm}$ have unbounded variation on $(0,1)\times\{0\}$.

Similar examples can be repeated in any dimension (in this case it is necessary to use (3.5)). The last example shows that, even though $S(u)$ coincides (up to $\mathcal{H}^{n-1}$-negligible sets) with a smooth hypersurface, the functions u^+ and u^- need not be BV along $S(u)$.

Regarding ∂T_u as a vector measure in $\Omega \times \mathbf{R}$, we will denote by $\partial_v T_u$ the vector of the measures μ_α in (3.5); *i.e.*, the components of ∂T_u corresponding to differential forms $\varphi dx_\alpha \wedge dy$. The letter v refers to the fact that we have in mind "vertical components" (remember that for a function $u \in SBV\big(\Omega\big)$ the "horizontal components" of ∂T_u can be already controlled with the area of the jump set of u).

The class $SBV_0\big(\Omega\big)$ has the following compactness property.

Theorem 3.10 (compactness of $SBV_0\big(\Omega\big)$) *Let $(u_h) \subset SBV(\Omega)$ be as in Theorem 2.3 and assume in addition that the sequence $\|\partial_v T_{u_h}\|(\Omega \times \mathbf{R})$ be bounded. Then, the limit u of any subsequence $(u_{h(k)})$ given by Theorem 2.3 belongs to $SBV_0\big(\Omega\big)$ and $\partial T_{u_{h(k)}}$ weakly converges to ∂T_u in $\Omega \times \mathbf{R}$.*

Proof. By Theorem 2.3 we can assume that the whole sequence (u_h) converges in $L^1_{\rm loc}(\Omega)$ to $u \in SBV\big(\Omega\big)$ with $\mathcal{H}^{n-1}\big(S(u)\big) < +\infty$. Let $\varphi \in \mathcal{T}(\Omega \times \mathbf{R})$, and let ω be the $(n-1)$ form given by $\varphi dx_\alpha \wedge dy$. Then, the weak convergence of the gradients and the identity

$$\langle T_{u_h}, d\omega\rangle = \sum_{i=1}^{n} \int_\Omega \frac{\partial \varphi}{\partial x_i}(x, u_h) dx_i \wedge dx_\alpha \wedge du_h$$

(see (2.10)) imply that $\langle T_{u_h}, d\omega\rangle$ converges to $\langle T_u, d\omega\rangle$ as $h \to +\infty$, because

$$\lim_{h\to+\infty} \int_\Omega \left| \frac{\partial \varphi}{\partial x_i}(x, u_h) - \frac{\partial \varphi}{\partial x_i}(x, u)\right| dx = 0.$$

For any $\varphi \in C^1_0(\Omega \times \mathbf{R})$ we have

$$\big|\langle T_{u_h}, d\big(\varphi dx_\alpha \wedge dy\big)\rangle\big| \le \|\varphi\|_\infty \sup_{h\in\mathbf{N}} \|\partial_v T_{u_h}\|(\Omega \times \mathbf{R}).$$

By approximation, the same inequality holds with $\varphi \in \mathcal{T}(\Omega \times \mathbf{R})$, therefore passing to the limit as $h \to +\infty$ we obtain that $u \in SBV_0\big(\Omega\big)$. □

As in the case of SBV functions, it is desirable to have a characterization of $\partial_v T_u$ in terms of pointwise quantities, such as traces and approximate differentials. A first step in this direction is the following:

Proposition 3.11 *Let $u \in SBV_0\big(\Omega\big)$. Then, the traces $u^+(x)$, $u^-(x) : S(u) \to \mathbf{R}$ are $\mathcal{H}^{n-1}$-approximately differentiable in $S(u)$. Moreover,*

$$\int_{S(u)} \big(|\nabla u^+| + |\nabla u^-|\big)\, d\mathcal{H}^{n-1} < +\infty.$$

Proof. Since ∂T_u is representable by integration, by Theorem 2.9 we can find a $\mathcal{H}^{n-1}$-rectifiable set $R \subset \Omega \times \mathbf{R}$ with a suitable orientation ξ and a multiplicity function $\theta : R \to \mathbf{N}$ such that ∂T_u is representable by (R, θ). The idea of the proof is to compare this representation with (3.1).

Let us denote by $\pi : R \to \Omega$ the projection on the first n variables and by R_1 the nonvertical part of R as in the proof of Theorem 2.14. Denoting by R_2 the set $\pi(R_1)$, endowed with a suitable orientation η, π induces a map between the spaces $\mathrm{Tan}^{n-1}\big(R_1, (x,y)\big)$ and $\mathrm{Tan}^{n-1}\big(R_2, x\big)$. We set $\sigma(x,y) = 1$ if π is orientation-preserving and $\sigma(x,y) = -1$ if π is not.

For any choice of $\phi \in C_0^1(\Omega)$ and $\psi \in C_0^1(\mathbf{R})$ we have

$$\begin{aligned} & \int_{S(u)} \phi(x)\big(\psi(u^+(x)) - \psi(u^-(x))\big)\nu_i \, d\mathcal{H}^{n-1} \\ = \; & (-1)^i \int_R \theta(x,y)\phi(x)\psi(y)d\hat{x}_i = (-1)^i \int_{R_1} \theta(x,y)\phi(x)\psi(y)d\hat{x}_i \\ = \; & (-1)^i \int_{R_2} \phi(x) \sum_{(x,y)\in\pi^{-1}(x)} \sigma(x,y)\,\theta(x,y)\psi(y)d\hat{x}_i. \end{aligned}$$

In order to pass from the integration on R to the integration on R_2 we used the co-area formula (see [[18]]). Since ϕ and ψ are arbitrary, we obtain that $\mathcal{H}^{n-1}\big(S(u)\Delta R_2\big) = 0$. Possibly changing the sign of ν, we can assume that $S(u)$ and R_2 have the same orientation in their intersection. In this case we get

$$\begin{aligned} & \int_{S(u)} \phi(x)\Big(\psi(u^+(x)) - \psi(u^-(x))\Big)\nu_i \, d\mathcal{H}^{n-1} \\ & = -\int_{R_2} \phi(x) \sum_{(x,y)\in\pi^{-1}(x)} \sigma(x,y)\,\theta(x,y)\psi(y)\nu_i d\mathcal{H}^{n-1}. \end{aligned}$$

Denoting by D a countable dense subset of $C_0^1(\Omega)$, we infer the existence of a $\mathcal{H}^{n-1}$-negligible set $S \subset S(u) \cap R_2$ such that

$$\sum_{(x,y)\in\pi^{-1}(x)} \sigma(x,y)\,\theta(x,y)\psi(y) = \psi(u^-(x)) - \psi(u^+(x)) \qquad \forall \psi \in D$$

for any $x \in S(u) \cap R_2 \setminus S$. Denoting by M this set, we obtain

$$\pi^{-1}(x) = \big\{(x, u^+(x)), (x, u^-(x))\big\} \qquad \forall x \in M$$

so that, the graphs of $u^\pm$ on M are contained in R. Applying Theorem 2.14 to $u^\pm$ (with $k = (n-1)$) we obtain that the traces are $\mathcal{H}^{n-1}$-differentiable in M (hence in $S(u)$) and

$$\int_{S(u)} \big(|\nabla u^+| + |\nabla u^-|\big)\, d\mathcal{H}^{n-1} \leq \mathcal{H}^{n-1}\big(R\big) < +\infty$$

as desired. $\square$

Definition 3.12 By Proposition 3.11, we can find a set $M \subset S(u)$ contained in countably many hypersurfaces such that $\mathcal{H}^{n-1}\big(S(u) \setminus M\big) = 0$ and u^+ and u^- are $\mathcal{H}^{n-1}$-differentiable at every point of M. Under these assumptions, we denote by T_{u^+} and T_{u^-} the $(n-1)$-currents in $\Omega \times \mathbf{R}$ associated to the integration on the graphs of u^+ and u^- on M. By (2.9), $T_{u^\pm}$ acts as follows

$$\langle T_{u^\pm}, \varphi dx_\alpha \wedge dy\rangle = \int_{S(u)} \varphi(x, u(x)) dx_\alpha \wedge du^\pm \tag{3.7}$$

for any bounded Borel function φ defined in $\Omega \times \mathbf{R}$.

4 Functions with absolutely continuous traces on S(u)

We deal now with the special case where $u^\pm$ have neither jump part nor Cantor part of derivative on $S(u)$. Heuristically, we want the traces $u^\pm$ to belong to a "Sobolev space". This motivates the following definition, in which we require the "distributional derivative" $\partial_v T_u$ to be completely determined by $u^\pm$ and by their approximate differentials.

Definition 4.1 (the class SBV_0^p) Let $p \in [1, \infty)$. We denote by $SBV_0^p\big(\Omega\big)$ the class of all functions $u \in SBV\big(\Omega\big)$ such that $S(u)$ is $\mathcal{H}^{n-1}$-rectifiable and there exist two Borel functions w^+, w^- : $S(u) \to \mathbf{R}^n$ such that $w^\pm(x)$ is orthogonal to $\nu(x)$ for $\mathcal{H}^{n-1}$-almost every $x \in S(u)$,

$$\int_{S(u)} \big(|w^+|^p + |w^-|^p\big)\, d\mathcal{H}^{n-1} < +\infty$$

and

$$\begin{aligned} &\langle T_u, d\big(\varphi dx_\alpha \wedge dy\big)\rangle \\ = \;& (-1)^{n-1} \int_{S(u)} \Big(\varphi\big(x, u^+(x)\big) w^+(x) \wedge dx_\alpha - \varphi\big(x, u^-(x)\big) w^-(x) \wedge dx_\alpha\Big) \end{aligned} \tag{4.1}$$

for any multi-index α of order $(n-2)$ in $\mathbf{R}^n$ and any function $\varphi \in \mathcal{T}(\Omega \times \mathbf{R})$. In the case $p = \infty$ the definition is similar, requiring the functions $w^\pm$ to be essentially bounded on $S(u)$.

Remark 4.2 We will see that the functions (w^+, w^-) are uniquely determined and of course they depend on the orientation of $S(u)$; *i.e.*, a change of sign of ν induces a permutation of (u^+, u^-) and a permutation of (w^+, w^-) as well.

The integration by parts formula (4.1) implies the estimate

$$\big|\langle \partial T_u, \varphi dx_\alpha \wedge dt\rangle\big| \leq \|\varphi\|_\infty \int_{S(u)} \big(|w^+| + |w^-|\big)\, d\mathcal{H}^{n-1} \tag{4.2}$$

for any multi-index α in $\mathbf{R}^n$ of order $(n-2)$. Hence, $SBV_0^p\big(\Omega\big)$ is a subset of $SBV_0\big(\Omega\big)$. The inclusion is strict even with $p=1$. Indeed, using the same notations of Example 3.6 with $f^-=0$, and using again (3.6) with $\psi\equiv 1$, it can be seen that $u\in SBV_0^p\big((0,1)\times\mathbf{R}\big)$ if and only if $f^+\in W^{1,p}\big((0,1)\big)$ and in this case $w^+=(f^+)'e_1$.

Proposition 4.3 *Let $u\in SBV_0^p\big(\Omega\big)$. Then, the functions $w^\pm$ coincide with the approximate differentials of $u^\pm$ given by Proposition* 3.11 *and $\partial T_u=T_{u^-}-T_{u^+}$, with $T_{u^\pm}$ given by Definition* 3.12.

Proof. Using the locality properties of approximate differentials, it is not restrictive to prove the theorem with a particular choice of the orientation ν, the one for which $u^-<u^+$. The proof is based on a blow-up argument, starting from the identity (see (4.1), (3.5))

$$\int_{S(u)}\phi(x)\psi(u^+(x))w^+(x)\wedge dx_\alpha-\int_{S(u)}\phi(x)\psi(u^-(x))w^-(x)\wedge dx_\alpha=$$
$$=\ -\int_{S(u)}\Big(\int_{u^-(x)}^{u^+(x)}\psi(y)\,dy\Big)d\phi\wedge dx_\alpha$$

with $\phi\in C^1_0(\Omega)$ and $\psi\in C^1_0(\mathbf{R})$. We will prove that $\nabla u^+(x_0)=w^+(x_0)$ for any $x_0\in S(u)$ with the following properties (the argument for w^- is similar):
(i) $S(u)$ has an approximate tangent plane at x_0;
(ii) x_0 is an $\mathcal{H}^k$-approximate continuity point for the maps $u^\pm(x)$ and for the maps $w^\pm(x)$ on $S(u)$;
(iii) $u^+(x)$ is $\mathcal{H}^{n-1}$-approximately differentiable at x_0.

Let $S_\rho=\rho^{-1}(S(u)-x_0)$, $S=\mathrm{Tan}^{n-1}\big(S(u),x_0\big)$, $\varphi(z)\in C^1_0(\mathbf{R}^n)$ and $\gamma(s)\in C^1_0(\mathbf{R})$. Inserting in the identity $\phi(x)=\varphi\big((x-x_0)/\rho\big)$ and $\psi(y)=\gamma\big((y-u^+(x_0))/\rho\big)$ and changing variables we get

$$\begin{aligned}&-\rho^{-1}\int_{S_\rho}\Big(\int_{u^-(x_0+\rho z)}^{u^+(x_0+\rho z)}\gamma\big(\frac{y-u^+(x_0)}{\rho}\big)\,dy\Big)d\varphi\wedge dz_\alpha\\ =&\int_{S_\rho}\varphi(z)\,\gamma\Big(\frac{u^+(x_0+\rho z)-u^+(x_0)}{\rho}\Big)w^+(x_0+\rho z)\wedge dz_\alpha\\ -&\int_{S_\rho}\varphi(z)\,\gamma\Big(\frac{u^-(x_0+\rho z)-u^+(x_0)}{\rho}\Big)w^-(x_0+\rho z)\wedge dz_\alpha.\end{aligned}$$

Changing variables also in the inner integral and passing to the limit as $\rho\downarrow 0$, by (i), (ii), (iii) we infer

$$-\int_S\Big(\int_{-\infty}^{\langle\nabla u^+(x_0),z\rangle}\gamma(s)\,ds\Big)d\varphi\wedge dz_\alpha=\int_S\varphi(z)\gamma\big(\langle\nabla u^+(x_0),z\rangle\big)w^+(x_0)\wedge dz_\alpha.$$

Using Stokes theorem on the plane S we get

$$\int_S \varphi(z)\,\gamma\big(\langle \nabla u^+(x_0), z\rangle\big)\big(\nabla u^+(x_0) - w^+(x_0)\big) \wedge dz_\alpha = 0.$$

Since α, φ and γ are arbitrary we obtain that $\nabla u^+(x_0)$ must be equal to $w^+(x_0)$.

Finally, the equality $\partial T_u = T_{u^-} - T_{u^+}$ follows directly from the definition of $SBV_0^p\big(\Omega\big)$ and from (3.7). $\square$

The proposition above suggests an alternative, equivalent definition of the space $SBV_0^p\big(\Omega\big)$: a function $u \in SBV\big(\Omega\big)$ belongs to $SBV_0^p\big(\Omega\big)$ if $S(u)$ is $\mathcal{H}^{n-1}$-rectifiable, the traces $u^\pm(x)$ are $\mathcal{H}^{n-1}$-approximately differentiable on $S(u)$, their gradients belong to $L^p\big(\mathcal{H}^{n-1} \llcorner S(u)\big)$ and

$$\begin{aligned} &\langle T_u, d\big(\varphi dx_\alpha \wedge dy\big)\rangle \\ = \quad &(-1)^{n-1} \int_{S(u)} \Big(\varphi\big(x, u^+(x)\big)\nabla u^+(x) \wedge dx_\alpha - \varphi\big(x, u^-(x)\big)\nabla u^-(x) \wedge dx_\alpha\Big) \end{aligned} \tag{4.3}$$

for any $\varphi \in \mathcal{T}(\Omega \times \mathbf{R})$. However, Definition 4.1, based on "weak derivatives", is closer to the classical definition of the Sobolev space $W^{1,p}(\Omega)$. Notice also that (3.5) shows the possibility of defining the class $SBV_0^p\big(\Omega\big)$ without reference to currents, using an integration by parts formula involving test function $\phi(x) \in C_0^1(\Omega)$, $\psi(y) \in C_b^1(\mathbf{R})$.

Proposition 4.4 *If $u \in SBV_0\big(\Omega\big)$ then the following three conditions are equivalent:*
(i) (4.3) *holds;*
(ii) $\pi_\#(\|\partial_v T_u\|) = (|\nabla u^+| + |\nabla u^-|)\mathcal{H}^{n-1} \llcorner S(u)$;
(iii) $\pi_\#(\|\partial_v T_u\|)$ *is absolutely continuous with respect to* $\mathcal{H}^{n-1} \llcorner S(u)$.

Proof. Following the notation of Proposition 3.11 let R_1 be the non-vertical part of the support R of ∂T_u. From the proof of Proposition 3.11 it follows immediately that

$$\begin{aligned} &\int_{R_1} \theta(x,y)\varphi(x,y)\,dx_\alpha \wedge dy \\ = \quad &(-1)^{n-1} \int_{S(u)} \Big(\varphi\big(x, u^+(x)\big)\nabla u^+(x) \wedge dx_\alpha - \varphi\big(x, u^-(x)\big)\nabla u^-(x) \wedge dx_\alpha\Big) \end{aligned}$$

for all $\varphi \in \mathcal{T}(\Omega \times \mathbf{R})$, where θ is the multiplicity of ∂T_u. Thus, we get

$$\begin{aligned} &\langle T_u, d\big(\varphi dx_\alpha \wedge dy\big)\rangle \\ = \quad &(-1)^{n-1} \int_{S(u)} \Big(\varphi\big(x, u^+(x)\big)\nabla u^+(x) \wedge dx_\alpha - \varphi\big(x, u^-(x)\big)\nabla u^-(x) \wedge dx_\alpha\Big) \\ &+ \int_{R\setminus R_1} \theta(x,y)\varphi(x,y)\,dx_\alpha \wedge dy \end{aligned} \tag{4.4}$$

for any $\varphi \in \mathcal{T}(\Omega \times \mathbf{R})$. Let now $v_1 \wedge \ldots \wedge v_{n-1}$ be an orientation of $S(u)$ compatible with the orientation ξ of ∂T_u, and let $\xi^1 = (\xi_i^1)_i$, where

$$\xi_i^1 = \langle \xi, dv_1 \wedge \ldots dv_{i-1} \wedge dy \wedge dv_{i+1} \wedge \ldots dv_k \rangle$$

is defined as in Section 1. Recall that, up to $\mathcal{H}^{n-1}$-negligible sets, $R_1 = \{(x, u^+(x)), (x, u^-(x)) : x \in S(u)\}$ and

$$|\xi^1(x, u^\pm(x))| = \frac{|\nabla u^\pm(x)|}{\mathbf{M}_{n-1}(J\Phi)}$$

on R_1. From (4.4) it follows that

$$\begin{aligned} \pi_\#(\|\partial_v T_u\|)(A) &= \int_{S(u)\cap A} \Big(|\nabla u^+| + |\nabla u^-|\Big) d\mathcal{H}^{n-1} \\ &\quad + \int_{(R\setminus R_1)\cap(A\times\mathbf{R})} \theta |\xi^1| d\mathcal{H}^{n-1} \end{aligned} \tag{4.5}$$

for every open subset A of Ω.

The equivalence between (i) and (ii) now follows easily: if (ii) holds then by (4.5) $\mathcal{H}^{n-1}(R \setminus R_1) = 0$, since by definition $|\xi^1| > 0$ on $R \setminus R_1$. Hence, by (4.4) we obtain (4.3). Vice versa, by (4.3) we have

$$\pi_\#(\|\partial_v T_u\|) \le (|\nabla u^+| + |\nabla u^-|)\mathcal{H}^{n-1} \llcorner S(u),$$

while the opposite inequality is assured by (4.5).

Similarly we get the equivalence between (ii) and (iii). $\square$

If $p > 1$, we have the following closure theorem.

Theorem 4.5 (closure of SBV_0^p) *Let $(u_h) \subset SBV_0^p(\Omega)$ be a sequence converging in $L^1_{\mathrm{loc}}(\Omega)$ to $u \in SBV(\Omega)$ such that $\mathcal{H}^{n-1}(S(u_h)) \to \mathcal{H}^{n-1}(S(u))$ and, for constants p, $q > 1$ and $C \ge 0$*

$$\|u_h\|_\infty + \int_\Omega |\nabla u_h|^q \, dx + \int_{S(u_h)} \big(1 + |\nabla u_h^+|^p + |\nabla u_h^-|^p\big) \, d\mathcal{H}^{n-1} \le C \tag{4.6}$$

Then $u \in SBV_0^p(\Omega)$.

Proof. The inequality (4.2) and the Hölder inequality imply that the sequence $\|\partial_v T_{u_h}\|(\Omega \times \mathbf{R})$ is bounded. By the compactness properties of $SBV_0(\Omega)$ we have $u \in SBV_0(\Omega)$, ∇u_h weakly converges to ∇u in $L^q(\Omega, \mathbf{R}^n)$ and ∂T_{u_h} weakly converges to ∂T_u in $\Omega \times \mathbf{R}$. We have to prove that $u \in SBV_0^p(\Omega)$.

Let $\varphi \in C_0^1(\Omega)$ with $\varphi \ge 0$. Since $u_h \in SBV_0^p(\Omega)$ by Proposition 4.4, using Hölder's inequality, we get

$$\begin{aligned} \int_\Omega \varphi \, d\pi_\#(\|\partial_v T_{u_h}\|) &= \int_{S(u_h)} \varphi\big(|\nabla u_h^+| + |\nabla u_h^-|\big) \, d\mathcal{H}^{n-1} \\ \le \ & \Big(\int_{S(u_h)} \big(|\nabla u_h^+| + |\nabla u_h^-|\big)^p d\mathcal{H}^{n-1}\Big)^{1/p} \Big(\int_{S(u_h)} |\varphi|^{p'} d\mathcal{H}^{n-1}\Big)^{1/p'}. \end{aligned}$$

Then since $\mathcal{H}^{n-1} \llcorner S(u_h)$ weakly converges to $\mathcal{H}^{n-1} \llcorner S(u)$, taking the limit as $h \to +\infty$ we get

$$\int_\Omega \varphi d\pi_\#(\|\partial_v T_u\|) \le I\left(\int_{S(u)} |\varphi|^{p'} d\mathcal{H}^{n-1}\right)^{1/p'}, \tag{4.7}$$

where

$$I = \liminf_{h\to+\infty}\left(\int_{S(u_h)} \left(|\nabla u_h^+| + |\nabla u_h^-|\right)^p d\mathcal{H}^{n-1}\right)^{1/p},$$

which is finite by hypothesis. This implies that $\pi_\#(\|\partial_v T_u\|)$ is absolutely continuous with respect to $\mathcal{H}^{n-1} \llcorner S(u)$, and hence there exists $f \in L^1(S(u), \mathcal{H}^{n-1})$ such that $\pi_\#(\|\partial_v T_u\|) = f\, \mathcal{H}^{n-1} \llcorner S(u)$. Therefore, from (4.7) we obtain

$$\left|\int_{S(u)} \phi f\, d\mathcal{H}^{n-1}\right| \le I\left(\int_{S(u)} |\phi|^{p'} d\mathcal{H}^{n-1}\right)^{1/p'} \tag{4.8}$$

for every $\phi \in L^{p'}(S(u), \mathcal{H}^{n-1})$. By Riesz's Theorem $f \in L^p(S(u), \mathcal{H}^{n-1})$. The proof is concluded by applying Proposition 4.4. $\square$

Example 4.6 The thesis of Theorem 4.5 is in general false if only the boundedness assumption (4.6) holds, without assuming $\mathcal{H}^{n-1}(S(u_h)) \to \mathcal{H}^{n-1}(S(u))$. We can give a counterexample in the spirit of Remark 2.17. Let $n = 2$, and let the function $u : B(0,1) \to \mathbf{R}$ be given by

$$u(x,y) = \begin{cases} -\pi & \text{if } x < 0, \\ \arctan\left(\dfrac{x}{y}\right) & \text{if } x > 0, \\ \pi/2i & \text{if } x = 0,\ y \ge 0, \\ -\pi/2i & \text{if } x = 0,\ y < 0. \end{cases}$$

Note that u belongs to $SBV_0(B(0,1))$, but it does not belong to $SBV_0^p(B(0,1))$ for any $p \ge 1$. Moreover $\nabla u \in L^q(B(0,1))$ if $q < 2$. For every $h \in \mathbf{N}$ we can choose a non self-intersecting piecewise C^1 curve γ_h with the following properties: $\gamma_h \subset B(0,1/h)$, $\sup_{h\in\mathbf{N}} \mathcal{H}^1(\gamma_h) < +\infty$, the endpoints of γ_h are $(0,-1/h)$ and $(0,1/h)$, the restriction of u to γ_h is continuous, and its tangential derivative $\partial u/\partial\tau$ is equi-bounded (uniformly with respect to h). We can give an explicit example of such γ_h simply by choosing the piecewise linear curve connecting the points P_k $(k = 0, \ldots, 2h)$, where

$$P_k = \left(\frac{1}{h}\cos\left(\frac{\pi}{2}\left(\frac{k}{h} - 1\right)\right), \frac{1}{h}\sin\left(\frac{\pi}{2}\left(\frac{k}{h} - 1\right)\right)\right), \quad k \text{ even},$$

$$P_k = \left(\frac{1}{2h}\cos\left(\frac{\pi}{2}\left(\frac{k}{h} - 1\right)\right), \frac{1}{2h}\sin\left(\frac{\pi}{2}\left(\frac{k}{h} - 1\right)\right)\right), \quad k \text{ odd}.$$

We can define u_h equal to $-\pi$ on the region bounded by γ_h and the axis $\{x=0\}$, and equal to u otherwise. Clearly $u_h \in SBV_0^\infty(B(0,1))$. Moreover, $\|u_h\|_\infty = \pi$, $\int_{B(0,1)} |\nabla u_h|^q\,dx \le \int_{B(0,1)} |\nabla u|^q\,dx$, and

$$\sup_{h\in\mathbf{N}} \int_{S(u_h)} (1+|\nabla u_h^+|^p + |\nabla u_h^-|^p)\,d\mathcal{H}^1$$

$$\le (1+\sup_{h\in\mathbf{N}} \|\frac{\partial u}{\partial\tau}\|^p_{L^\infty(\gamma_h)}) \sup_{h\in\mathbf{N}} \mathcal{H}^1(\gamma_h) < +\infty$$

for all $p \ge 1$. Nevertheless, $u_h \to u$ in $L^1(B(0,1))$, so that the thesis of the theorem is violated.

Note that in the same way we can give an example where the traces $u^\pm$ of the limit function on $\{x=0\}$ are any pair of piecewise constant functions. Since the quantities involved in all the estimates depend only on the BV norm of $u^\pm$ we obtain by a diagonal argument that the example can be constructed with every pair of BV functions as traces.

Example 4.7 In the previous example low integrability on the gradient ∇u was required to obtain discontinuous traces on $S(u)$. Now we show that for all $q \ge 1$ a similar example can be constructed with $\nabla u \in L^q$.

First, let $K > 2$ and let F be the Cantor set-like fractal of unitary $\log 2/\log K$-dimensional Hausdorff measure generated by the two similitudes S_0 and S_1 of ratio $1/K$ which carry $(0,1)$ in $(0,1/K)$ and in $(1-1/K,1)$, respectively (see [24]). Note that the Cantor-Vitali-like function $f:(0,1)\to\mathbf{R}$ given by

$$f(t) = \mathcal{H}^{\log 2/\log K}((0,t)\cap F)$$

is $(\log 2/\log K)$-Hölder continuous, and hence it belongs to $W^{1-1/q,q}((0,1))$ for all $q < 2\log K/(\log K - \log 2)$ (see [1]).

Now, given $q \ge 1$, we choose K sufficiently close to 2 in such a way that $q < 2\log K/(\log K - \log 2)$. The function f can be approximated by the sequence (f_j) of piecewise affine functions given by

$$f_j(t) = \Big(\frac{K}{2}\Big)^j \mathcal{L}^1(A_j \cap (0,t)),$$

where $A_j = \bigcup\{(S_{i_1}\circ\cdots\circ S_{i_j})(0,1) : \ i_l \in \{0,1\}\}$ (this is the usual construction of the Cantor-Vitali function if $K=3$). The functions f_k belong to $W^{1-1/q,q}((0,1))$ if $q < 2\log K/(\log K - \log 2)$. If we define $u_k \in W^{1,q}((0,1)^2)$ by

$$u_k(x,y) = \begin{cases} \dfrac{1}{x}\displaystyle\int_y^{y+x} f_k(t)\,dt & \text{if } y < 1-x \\ 1 & \text{otherwise} \end{cases}$$

then $u_k(0,y) = f_k(y)$ in the sense of traces. The sequence (u_k) converges in $W^{1,q}((0,1)^2)$ to the function u constructed in the same way from f.

Let us define $\widetilde{u} : (-1,1) \times (0,1) \to \mathbf{R}$ by

$$\widetilde{u}(x,y) = \begin{cases} u(x,y) & \text{if } x > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, the function $\widetilde{u}$ does not belong to any $SBV_0^p((-1,1) \times (0,1))$.

We can construct a sequence $(\widetilde{u}_h)$ in $SBV_0^\infty((-1,1) \times (0,1))$ converging to $\widetilde{u}$ and satisfying (4.6) by modifying the sequence (u_k) in the spirit of Example 4.6. In fact, we can find a sequence of piecewise C^1 curves (γ_k) in $[0,1) \times (0,1)$ with equibounded length, such that each γ_k consists of the union of $\{0\} \times ((0,1) \setminus A_k)$ and 2^k piecewise C^1 arcs with endpoints $(S_{i_1} \circ \cdots \circ S_{i_k})(0)$ and $(S_{i_1} \circ \cdots \circ S_{i_k})(1)$ $((i_1, \ldots, i_k) \in \{0,1\}^k)$ on which the tangential derivative of u_k is bounded, uniformly with k. An explicit construction of such arcs is not difficult, as in Example 4.6. The functions

$$\widetilde{u}_k(x,y) = \begin{cases} 0 & \text{if } x \le 0 \\ 0 & \text{in the region bounded by } \{x = 0\} \text{ and } \gamma_k \\ u_k(x,y) & \text{otherwise} \end{cases}$$

provide the desired example.

5 Vector-valued SBV_0 functions

In this section we generalize the notions introduced in the previous chapters to the vector-valued case. The definition of SBV_0 remains unchanged as follows.

Definition 5.1 Let u be a function in $SBV\big(\Omega, \mathbf{R}^m\big)$ with a $\mathcal{H}^{n-1}$-rectifiable jump set, with all determinants of minors of ∇u in $L^1(\Omega)$. We say that u belongs to the class $SBV_0\big(\Omega, \mathbf{R}^m\big)$ if there exists a constant C such that

$$\big|\langle T_u, d\big(\varphi dx_\alpha \wedge dy_\beta\big)\rangle\big| \le C\|\varphi\|_\infty \qquad \forall \varphi \in \mathcal{T}\big(\Omega \times \mathbf{R}^m\big) \tag{5.1}$$

for every pair of multi-indices α, β with $|\alpha| + |\beta| = n - 1$. In particular, ∂T_u is representable by integration in $\Omega \times \mathbf{R}^m$.

Remark 5.2 If $u \in SBV_0\big(\Omega, \mathbf{R}^m\big)$, then there exist bounded measures $\mu_{\alpha\beta}$ such that

$$\langle \partial T_u, \varphi dx_\alpha \wedge dy_\beta\rangle = \int_{\Omega \times \mathbf{R}^m} \varphi \, d\mu_{\alpha\beta} \tag{5.2}$$

for every $\varphi \in \mathcal{T}\big(\Omega \times \mathbf{R}^m\big)$. In particular there exist n bounded measures μ_i on $\Omega \times \mathbf{R}^m$ (corresponding to $\mu_{\hat{\imath}0}$ in the notation above) such that

$$\langle \partial T_u, \varphi d\widehat{x}_i\rangle = (-1)^{i+1} \int_\Omega \frac{\partial \varphi}{\partial x_i}(x,u) + \sum_{j=1}^m \frac{\partial \varphi}{\partial y_j}(x,u) \frac{\partial u_j}{\partial x_i} \, dx = \int_{\Omega \times \mathbf{R}^m} \varphi \, d\mu_i,$$

where $u_1,\dots,u_m$ are the components of u. This is a characterization of functions in $SBV\big(\Omega,\mathbf{R}^m\big)$ with a $\mathcal{H}^{n-1}$-rectifiable jump set. Proceeding as in (3.1) we can show that the measures μ_i are given by

$$\int_{\Omega\times\mathbf{R}^m}\varphi\,d\mu_i=(-1)^i\int_{S(u)}\Big(\varphi(x,u^+(x))-\varphi(x,u^-(x))\Big)\nu_i\,d\mathcal{H}^{n-1}. \tag{5.3}$$

The measures in (5.2) can be sometimes characterized by formulas of the type (5.3), which take different forms, depending on α and β.

Let $u:\Omega\to\mathbf{R}^m$, and let Γ be the graph of u. By Theorem 2.11(i) we have that $\mathcal{H}^n(\Gamma)<+\infty$ if and only if $\partial u_\beta/\partial x_\gamma\in L^1$ for all pairs of multi-indices γ, β of the same order $k\in\{1,\dots,\min\{n,m\}\}$, where we use the notation

$$\frac{\partial u_\beta}{\partial x_\gamma}=\det\Big(((\nabla u^{\beta_i})_{\gamma_j})_{ij}\Big)\,.$$

In this case, by (2.10), the integration on the current ∂T_u can be expressed by

$$\begin{aligned}
&\langle T_u,d(\varphi dx_\alpha\wedge dy_\beta)\rangle\\
=\;&\langle T_u,\sum_{i=1}^n\frac{\partial\varphi}{\partial x_i}dx_i\wedge dx_\alpha\wedge dy_\beta\rangle+\langle T_u,\sum_{j=1}^m\frac{\partial\varphi}{\partial y_j}dy_j\wedge dx_\alpha\wedge dy_\beta\rangle\\
=\;&\int_\Omega\sum_{i=1}^n\sigma(i,\alpha)\frac{\partial\varphi}{\partial x_i}\frac{\partial u_\beta}{\partial\widehat{x}_{(i,\alpha)}}\,dx+(-1)^{|\alpha|}\sigma(\alpha)\int_\Omega\sum_{j=1}^m\frac{\partial\varphi}{\partial y_j}\frac{\partial u_{(j,\beta)}}{\partial\widehat{x}_\alpha}\,dx
\end{aligned}$$

(we define $\sigma(\alpha)\in\{-1,1\}$ by $dx_\alpha\wedge d\widehat{x}_\alpha=\sigma(\alpha)dx_1\wedge\ldots\wedge dx_n$, and $\sigma(i,\alpha)\in\{-1,0,1\}$ by $dx_i\wedge dx_\alpha\wedge d\widehat{x}_{(i,\alpha)}=\sigma(i,\alpha)dx_1\wedge\ldots\wedge dx_n$). We can reformulate then the definition of $SBV_0\big(\Omega,\mathbf{R}^m\big)$ as follows.

Proposition 5.3 *Let $u\in SBV\big(\Omega,\mathbf{R}^m\big)$ with $\partial u_\beta/\partial x_\gamma\in L^1(\Omega)$ for all pair of multi-indices γ, β of order $1,\dots,\min\{n,m\}$, and let $\mathcal{H}^{n-1}(S(u))<+\infty$. Then u belongs to $SBV_0\big(\Omega,\mathbf{R}^m\big)$ if and only if for every pair of multi-indices α, β with $|\alpha|+|\beta|=n-1$ there exists a bounded measure $\mu_{\alpha\beta}$ on $\Omega\times\mathbf{R}^m$, such that*

$$\begin{aligned}
&\int_{\Omega\times\mathbf{R}^m}\varphi\,d\mu_{\alpha\beta}\\
=\;&\int_\Omega\sum_{i=1}^n\sigma(i,\alpha)\frac{\partial\varphi}{\partial x_i}\frac{\partial u_\beta}{\partial\widehat{x}_{(i,\alpha)}}\,dx+(-1)^{|\alpha|}\sigma(\alpha)\int_\Omega\sum_{j=1}^m\frac{\partial\varphi}{\partial y_j}\frac{\partial u_{(j,\beta)}}{\partial\widehat{x}_\alpha}\,dx
\end{aligned}\tag{5.4}$$

for all $\varphi\in\mathcal{T}(\Omega\times\mathbf{R}^m)$.

Remark 5.4 Let $\lambda_h\in\mathbf{R}$ for $h=1,\dots,m$. Taking $\phi\in\mathcal{T}(\Omega\times\mathbf{R})$, and

$$\varphi(x,y)=\phi\Big(x,\sum_{h=1}^m\lambda_h y_h\Big)$$

in (5.4) it is easy to check by Definition 3.3 that $w = \sum_h \lambda_h u_h \in SBV_0\big(\Omega\big)$. In fact, taking $|\alpha| = n-2$ we have

$$\begin{aligned}
\sum_{h=1}^{m} \lambda_h \langle T_u, d(\varphi dx_\alpha \wedge dy_h)\rangle &= \sum_{h=1}^{m} \lambda_h \langle T_u, \sum_{i=1}^{n} \frac{\partial \phi}{\partial x_i} dx_i \wedge dx_\alpha \wedge dy_h\rangle \\
&\quad + \sum_{h=1}^{m}\sum_{k=1}^{m} \langle T_u, \frac{\partial \phi}{\partial y} \lambda_h \lambda_k dy_k \wedge dx_\alpha \wedge dy_h\rangle \\
&= \sum_{h=1}^{m} \lambda_h \langle T_u, \sum_{i=1}^{n} \frac{\partial \phi}{\partial x_i} dx_i \wedge dx_\alpha \wedge dy_h\rangle \\
&= \int_\Omega \sum_{i=1}^{n} \sigma(i,\alpha) \frac{\partial \phi}{\partial x_i} \frac{\partial(\sum_{h=1}^{m} \lambda_h u_h)}{\partial \widehat{x}_{(i,\alpha)}}\, dx \\
&= \langle T_w, d(\phi dx_\alpha \wedge dy)\rangle
\end{aligned}$$

(we have used the equality $dy_k \wedge dx_\alpha \wedge dy_h = -dy_h \wedge dx_\alpha \wedge dy_k$). In particular, taking $\lambda_i = \delta_{ih}$, each component u_h belongs to $SBV_0\big(\Omega\big)$.

Moreover, if u is bounded, taking $\varphi(x,y) = \phi(x) y_h$ on the range of u in (5.4), for some h and some $\phi \in C_0^1(\Omega)$, we obtain

$$\begin{aligned}
\text{(5.5)} \qquad & \int_{\Omega \times \mathbf{R}^m} \phi y_h d\mu_{\alpha\beta} \\
&= \int_\Omega \sum_{i=1}^{n} \sigma(i,\alpha) \frac{\partial \phi}{\partial x_i} u_h \frac{\partial u_\beta}{\partial \widehat{x}_{(i,\alpha)}}\, dx + (-1)^{|\alpha|} \sigma(\alpha) \int_\Omega \phi \frac{\partial u_{(h,\beta)}}{\partial \widehat{x}_\alpha}\, dx.
\end{aligned}$$

This equation implies that for every couple of multi-indices γ, δ of the same order $k \in \{1, \ldots, \min\{n,m\}\}$ the following integration by parts formula holds

$$\text{(5.6)} \qquad \langle (\mathrm{Adj}_k \nabla u)_\delta^\gamma, \phi \rangle = \int_\Omega \phi (\mathrm{adj}_k \nabla u)_\delta^\gamma\, dx + \int_{\Omega \times \mathbf{R}^m} \phi y_{\gamma_1} d\mu_{\widehat{\delta}\gamma_1'}\, ,$$

where $\gamma_1' = (\gamma_2, \ldots, \gamma_k)$, which links the distributional and the Jacobian (pointwise) adjoint matrices (of order k) of ∇u (see [12], or [13] Chapter 4 for definitions and remarks). Note that the equality

$$(\mathrm{Adj}_k \nabla u)_\delta^\gamma = (\mathrm{adj}_k \nabla u)_\delta^\gamma\, \mathcal{L}^n$$

holds if $u \in W^{1,\min\{n,m\}}(\Omega, \mathbf{R}^m)$ but it may be false even when $u \in W^{1,q}(\Omega, \mathbf{R}^m)$ with $q < \min\{n,m\}$ (see [12], [28]).

Remark 5.5 In order to better understand the meaning of the vector measures $\mu_{\alpha\beta}$ we consider the case $n = m = 2$. In this case, the orientation η of the graph Γ of u is given by

$$\eta(x,y) = \frac{1}{\sqrt{1 + |\nabla u|^2(x) + |\det \nabla u|^2(x)}} \Big(e_1 \wedge e_2 - \frac{\partial u_1}{\partial x_1}(x)\, e_2 \wedge \varepsilon_1$$

$$-\frac{\partial u_2}{\partial x_1}(x)\, e_2 \wedge \varepsilon_2 + \frac{\partial u_1}{\partial x_2}(x)\, e_1 \wedge \varepsilon_1 + \frac{\partial u_2}{\partial x_2}(x)\, e_1 \wedge \varepsilon_2 + \det\nabla u(x)\, \varepsilon_1 \wedge \varepsilon_2\Big),$$

where (e_1, e_2) and $(\varepsilon_1, \varepsilon_2)$ denote the canonical orthonormal bases on Ω and on the target space, respectively, and $\mathcal{H}^2(\Gamma) < +\infty$ if and only if $\nabla u \in L^1$ and $\det\nabla u \in L^1$. The integration of the "vertical components" of the current ∂T_u can be expressed then by

$$\langle \partial T_u, \varphi dy_1\rangle = \int_\Omega \Big(\frac{\partial\varphi}{\partial x_1}\frac{\partial u_1}{\partial x_2} - \frac{\partial\varphi}{\partial x_2}\frac{\partial u_1}{\partial x_1} - \frac{\partial\varphi}{\partial y_2}\det\nabla u\Big)\, dx \tag{5.7}$$

$$\langle \partial T_u, \varphi dy_2\rangle = \int_\Omega \Big(\frac{\partial\varphi}{\partial x_1}\frac{\partial u_2}{\partial x_2} - \frac{\partial\varphi}{\partial x_2}\frac{\partial u_2}{\partial x_1} + \frac{\partial\varphi}{\partial y_1}\det\nabla u\Big)\, dx. \tag{5.8}$$

By Proposition 5.3 we have that $u \in SBV_0\big(\Omega, \mathbf{R}^2\big)$ if and only if there exist two bounded measures μ_1 and μ_2 on $\Omega \times \mathbf{R}^2$, such that

$$\int_\Omega \Big(\frac{\partial\varphi}{\partial x_1}\frac{\partial u_1}{\partial x_2} - \frac{\partial\varphi}{\partial x_2}\frac{\partial u_1}{\partial x_1}\Big)\, dx = \int_\Omega \frac{\partial\varphi}{\partial y_2}\det\nabla u\, dx + \int_{\Omega\times\mathbf{R}^2} \varphi d\mu_1, \tag{5.9}$$

$$\int_\Omega \Big(\frac{\partial\varphi}{\partial x_2}\frac{\partial u_2}{\partial x_1} - \frac{\partial\varphi}{\partial x_1}\frac{\partial u_2}{\partial x_2}\Big)\, dx = \int_\Omega \frac{\partial\varphi}{\partial y_1}\det\nabla u\, dx + \int_{\Omega\times\mathbf{R}^2} \varphi d\mu_2. \tag{5.10}$$

for all $\varphi \in \mathcal{T}(\Omega \times \mathbf{R}^2)$. These two measures correspond to μ_{01} and μ_{02} in the notation of Definition 5.1, respectively. In particular, if u is bounded, as in (5.5) we get

$$\int_\Omega u_2 \Big(\frac{\partial\phi}{\partial x_1}\frac{\partial u_1}{\partial x_2} - \frac{\partial\phi}{\partial x_2}\frac{\partial u_1}{\partial x_1}\Big)\, dx = \int_\Omega \phi \det\nabla u\, dx + \int_{\Omega\times\mathbf{R}^2} \phi y_2 d\mu_1, \tag{5.11}$$

$$\int_\Omega u_1 \Big(\frac{\partial\phi}{\partial x_2}\frac{\partial u_2}{\partial x_1} - \frac{\partial\phi}{\partial x_1}\frac{\partial u_2}{\partial x_2}\Big)\, dx = \int_\Omega \phi \det\nabla u\, dx + \int_{\Omega\times\mathbf{R}^2} \phi y_1 d\mu_2. \tag{5.12}$$

for all $\phi \in C_0^1(\Omega)$. Formula 5.12 can be summarized in the equality, which links the distributional and the pointwise determinant,

$$\mathrm{Det}\nabla u = \det\nabla u\, \mathcal{L}^2 + \pi_\#(y_2\, \mu_1)\,. \tag{5.13}$$

Note that the equality $\mathrm{Det}\nabla u = \det\nabla u\, \mathcal{L}^n + \lambda$ may hold with non-trivial λ also when u is a Sobolev function. In the case $u : \{x \in \mathbf{R}^2 : \ |x| < 1\} \to \mathbf{R}^2$ given by $u(x) = x/|x|$, for example, $\det\nabla u = 0$, but

$$\mathrm{Det}\nabla u = \pi\delta_0.$$

Some examples by Müller ([29]) show that λ may also be a Hausdorff measure of fractional dimension restricted to a fractal set.

If $S(u)$, the restriction of u to $\Omega \setminus S(u)$, and its traces on $S(u)$ are smooth enough to justify the application of the Gauss-Green formula, the measures μ_i are easily characterized. In fact, we get

$$
\begin{aligned}
0 &= \int_\Omega \varphi(x,u)\,\mathrm{div}\Big(\frac{\partial u_1}{\partial x_2}, -\frac{\partial u_1}{\partial x_1}\Big)\,dx \\
&= \int_{S(u)} \Big(\varphi(x,u^+)\Big(\frac{\partial u_1^+}{\partial x_1}\nu_2 - \frac{\partial u_1^+}{\partial x_2}\nu_1\Big) - \varphi(x,u^-)\Big(\frac{\partial u_1^-}{\partial x_1}\nu_2 - \frac{\partial u_1^-}{\partial x_2}\nu_1\Big)\Big)\,d\mathcal{H}^1 \\
&\quad - \int_\Omega \Big(\frac{\partial\varphi}{\partial x_1}\frac{\partial u_1}{\partial x_2} - \frac{\partial\varphi}{\partial x_2}\frac{\partial u_1}{\partial x_1} - \frac{\partial\varphi}{\partial y_2}\mathrm{det}\nabla u\Big)\,dx, \\
&= -\int_{S(u)} \Big(\varphi(x,u^+)\frac{\partial u_1^+}{\partial\tau} - \varphi(x,u^-)\frac{\partial u_1^-}{\partial\tau}\Big)\,d\mathcal{H}^1 \\
&\quad - \int_\Omega \Big(\frac{\partial\varphi}{\partial x_1}\frac{\partial u_1}{\partial x_2} - \frac{\partial\varphi}{\partial x_2}\frac{\partial u_1}{\partial x_1} - \frac{\partial\varphi}{\partial y_2}\mathrm{det}\nabla u\Big)\,dx,
\end{aligned}
$$

so that

$$
\int_{\Omega\times\mathbf{R}^2} \varphi d\mu_1 = -\int_{S(u)} \Big(\varphi(x,u^+)\frac{\partial u_1^+}{\partial\tau} - \varphi(x,u^-)\frac{\partial u_1^-}{\partial\tau}\Big)\,d\mathcal{H}^1\,. \tag{5.14}
$$

In the same way we obtain

$$
\int_{\Omega\times\mathbf{R}^2} \varphi d\mu_2 = -\int_{S(u)} \Big(\varphi(x,u^+)\frac{\partial u_2^+}{\partial\tau} - \varphi(x,u^-)\frac{\partial u_2^-}{\partial\tau}\Big)\,d\mathcal{H}^1\,. \tag{5.15}
$$

In particular, the total variation of μ represents the length of the images of $S(u)$ by u^+ and u^-:

$$
|\mu|\big(\Omega\times\mathbf{R}^2\big) = \int_{S(u)} \Big(|\frac{\partial u^+}{\partial\tau}| + |\frac{\partial u^-}{\partial\tau}|\Big)\,d\mathcal{H}^1. \tag{5.16}
$$

Remark 5.6 In the physical case $n = m = 3$ the integration by parts formulas (5.6) characterize the distributional and Jacobian determinants of ∇u and its (2-dimensional) adjoint matrices. We have for all $\phi \in C_0^1(\Omega)$

$$
\langle \mathrm{Adj}\nabla u, \phi\rangle = \int_\Omega \phi\,\mathrm{adj}\nabla u\,dx + \int_\Omega \phi(x)d\lambda
$$

where λ is the matrix measure whose components are $\lambda_{ij} = \pi_\#(y_{\beta_1}\,\mu_{i\beta_2})$, $\beta = \widehat{j}$, and

$$
\langle \mathrm{Det}\nabla u, \phi\rangle = \int_\Omega \phi\,\mathrm{det}\nabla u\,dx - \sigma(\widehat{\beta})\int_{\Omega\times\mathbf{R}^3} \phi(x)y_{\widehat{\beta}}d\mu_{0\beta},
$$

for any multi-index β of order two.

Remark 5.7 Note that the inclusion $SBV_0(\Omega, \mathbf{R}^m) \subset \big(SBV_0(\Omega)\big)^m$ is strict. As an example we can take $n = m = 2$, and $u = (u_1, u_2)$, where

$$u_1(x,y) = \begin{cases} 1 & \text{if } y > 0 \\ -1 & \text{if } y < 0, \end{cases}$$

and u_2 is any function in some Sobolev space $W^{1,q}(B(0,1))$ $(q \geq 1)$, whose trace $u_2(\cdot, 0)$ is smooth but does not belong to $W^{1,1}(-1,1)$. In this case (5.16) is violated (see also Proposition 5.10).

As in the scalar case, we denote by $\partial_v T_u$ the vector of the measures $\mu_{\alpha\beta}$ as in (5.2) with $|\alpha| + |\beta| = n - 1$ and $|\alpha| < n - 1$.

Theorem 5.8 (compactness) *Let (u_h) be a sequence in $SBV_0\big(\Omega, \mathbf{R}^m\big)$ such that*

$$\sup_{h \in \mathbf{N}} \Big(\|u_h\|_\infty + \mathcal{H}^1(S(u_h)) + \int_\Omega |\nabla u|^q \, dx + \|\partial_v T_{u_h}\| \Big) < +\infty,$$

where $q \geq \min\{n,m\}$, and assume in addition that $(\partial (u_h)_\beta / \partial x_\gamma)$ is a equi-integrable sequence for every pair of multi-indices β, γ of order $\min\{n,m\}$ if $q = \min\{n,m\}$. Then, there exists a subsequence $(u_{h(k)})$ converging in $L^1_{\text{loc}}(\Omega, \mathbf{R}^m)$ to $u \in SBV_0\big(\Omega, \mathbf{R}^m\big)$, such that

$$\nabla u_{h(k)} \to \nabla u \text{ weakly in } L^q(\Omega, \mathbf{R}^{nm}),$$

$$\frac{\partial (u_{h(k)})_\beta}{\partial x_\gamma} \to \frac{\partial u_\beta}{\partial x_\gamma} \text{ weakly in } L^1(\Omega)$$

for every pair of multi-indices β, γ of equal order not greater than $\min\{n,m\}$, and $\partial T_{u_{h(k)}}$ converges weakly to ∂T_u. In particular $\partial_v T_{u_{h(k)}}$ converges weakly to $\partial_v T_u$ in the sense of measures.

Proof. Without loss of generality, by Theorem 2.3, we can suppose that (u_h) converges a.e. to $u \in SBV\big(\Omega, \mathbf{R}^m\big)$ and that ∇u_h weakly converges to ∇u in $L^p(\Omega, \mathbf{R}^{nm})$.

Fixed a pair of multi-indices β, γ of order $k \in \{2, \ldots, \min\{n,m\}\}$, by our assumptions the sequence $(\partial (u_h)_\beta / \partial x_\gamma)$ is equi-integrable. Fixed $\varepsilon > 0$ and $v \in L^1(\Omega)$, Theorem 4.3 of [6] assures the lower semicontinuity of the functional

$$F_v(w) = \int_\Omega \big(|v - \frac{\partial w_\beta}{\partial x_\gamma}| + \varepsilon |\nabla w|^q \big) \, dx$$

along the sequence (u_h) (see also [6] Corollary 4.9). Hence

$$\int_\Omega \big(|v - \frac{\partial u_\beta}{\partial x_\gamma}| + \varepsilon |\nabla u|^q \big) \, dx \leq \liminf_{h \to +\infty} \int_\Omega \big(|v - \frac{\partial (u_h)_\beta}{\partial x_\gamma}| + \varepsilon |\nabla u_h|^q \big) \, dx,$$

and, letting $\varepsilon \to 0^+$.

$$\int_\Omega |v - \frac{\partial u_\beta}{\partial x_\gamma}|\,dx \le \liminf_{h\to+\infty} \int_\Omega |v - \frac{\partial (u_h)_\beta}{\partial x_\gamma}|\,dx\,.$$

Since v is arbitrary, this inequality yields the weak convergence of $\partial(u_h)_\beta/\partial x_\gamma$ to $\partial u_\beta/\partial x_\gamma$ in $L^1(\Omega)$ (see [6] Lemma 4.4). For any form

$$\omega = \sum_{|\alpha|+|\beta|=n} \omega_{\alpha\beta} dx_\alpha \wedge dy_\beta\,,$$

the identity (see (2.10))

$$\langle T_{u_h}, \omega\rangle = \int_\Omega \sum_{|\alpha|+|\beta|=n} \sigma(\alpha)\,\omega_{\alpha\beta}(x, u_h(x)) \frac{\partial (u_h)_\beta}{\partial \widehat{x}_\alpha}\,dx$$

(if $|\beta| = 0$ we set $\partial u_\beta/\partial\widehat{x}_\alpha = 1$) easily implies the weak convergence of T_{u_h} to T_u in the sense of measures in $\Omega \times \mathbf{R}^m$ because of the strong convergence of $\omega_{\alpha\beta}(x, u_h)$ to $\omega_{\alpha\beta}(x, u)$ and the weak convergence of the gradients and of the determinants.

In the same way

$$\begin{aligned}\lim_{h\to+\infty} \langle \partial T_{u_h}, \varphi dx_\alpha \wedge dy_\beta\rangle &= \lim_{h\to+\infty} \langle T_{u_h}, d(\varphi dx_\alpha \wedge dy_\beta)\rangle \\ &= \langle T_u, d(\varphi dx_\alpha \wedge dy_\beta)\rangle = \langle \partial T_u, \varphi dx_\alpha \wedge dy_\beta\rangle\,,\end{aligned}$$

if $|\alpha| + |\beta| = n - 1$. As in Theorem 3.10 we obtain $|\langle \partial T_u, \varphi dx_\alpha \wedge dy_\beta\rangle| \le \|\varphi\|_\infty \sup_h \|\partial_v T_{u_h}\|$, so that u belongs to $SBV_0\big(\Omega, \mathbf{R}^m\big)$. □

Remark 5.9 If $n = m$ then the hypothesis of the equi-integrability of the minors of maximum order in Theorem 5.8 is simply the hypothesis of equi-integrability of $\det\nabla u_h$.

Proposition 5.10 *The traces u^+, u^- : $S(u) \to \mathbf{R}^m$ of a function u belonging to $SBV_0\big(\Omega, \mathbf{R}^m\big)$ are $\mathcal{H}^{n-1}$-approximately differentiable $\mathcal{H}^{n-1}$-a.e. on $S(u)$, and*

$$\int_{S(u)} \Big(|\nabla u^+| + |\nabla u^-|\Big)\,d\mathcal{H}^{n-1} < +\infty.$$

Proof. From Remark 5.4 we have that for every choice of $\lambda_1, \ldots, \lambda_m \in \mathbf{R}$ the function $\sum_h \lambda_h u_h$ belongs to $SBV_0(\Omega)$. In particular, by Proposition 3.11 we have that the approximate differentials $\nabla(\sum_h \lambda_h u_h)^\pm$ exist on $S(\sum_h \lambda_h u_h)$. We take care of choosing the same orientation for $S(u)$ and $S(u_h)$ on their intersection so that $(\sum_h \lambda_h u_h)^\pm = \sum_h \lambda_h u_h^\pm$.

In particular, for all i and j there exist $\nabla u_i^\pm$ on $S(u_i)$, and $\nabla(u_i^\pm + u_j^\pm)$ on $S(u_i + u_j)$. Hence, fixed i, there exist $\nabla u_i^\pm = \nabla(u_i^\pm + u_j^\pm) - \nabla u_j^\pm$ on $S(u_i + u_j) \setminus S(u_i) = S(u_j) \setminus S(u_i)$ for all j, so that $\nabla u_i^\pm$ exist on the whole $S(u) = \bigcup_j S(u_j)$.

The same argument also shows the integrability of the gradients. □

Remark 5.11 Ripeating the argument of Proposition 3.11 one can prove the stronger statement (with the notation of Theorem 2.11)

$$\int_{S(u)} \left[M_{n-1}(J\Phi^+) + M_{n-1}(J\Phi^-)\right] d\mathcal{H}^{n-1} < +\infty$$

where $\Phi^\pm(x) = (x, u^\pm(x))$. In particular, all determinants of k-minors of $\nabla u^\pm$ are summable with respect to $\mathcal{H}^{n-1} \llcorner S(u)$ for $1 \le k \le (n-1)$.

We conclude the section extending the definition of SBV_0^p to the vector-valued case.

Definition 5.12 Let $p \in [1, \infty)$. We say that a function $u \in SBV_0\big(\Omega, \mathbf{R}^m\big)$ belongs to $SBV_0^p\big(\Omega, \mathbf{R}^m\big)$ if there exist $2m$ measurable functions $w_i^\pm : S(u) \to \mathbf{R}^n$, $i = 1, \ldots, m$, such that $w_i^\pm(x)$ is orthogonal to $\nu(x)$ for $\mathcal{H}^{n-1}$-almost all $x \in S(u)$,

$$\int_{S(u)} \big(|w_i^+|^p + |w_i^-|^p\big)\, d\mathcal{H}^{n-1} < +\infty$$

for all i, and, for all pairs of multi-indices α, β with $|\alpha| + |\beta| = n-1$, $w_\beta^\pm \wedge dx_\alpha \in L^1(\Omega)$ and

$$\begin{aligned} (5.17) \quad & \langle \partial T_u, \varphi dx_\alpha \wedge dy_\beta \rangle \\ = \quad & (-1)^{n-1} \int_{S(u)} \Big(\varphi\big(x, u^+(x)\big) w_\beta^+(x) \wedge dx_\alpha - \varphi\big(x, u^-(x)\big) w_\beta^-(x) \wedge dx_\alpha\Big), \end{aligned}$$

where $w_\beta^\pm = w_{\beta_1}^\pm \wedge \ldots \wedge w_{\beta_s}^\pm$ if $\beta = (\beta_1, \ldots, \beta_s)$. In the case $p = \infty$ we require the functions $w_i^\pm$ to be essentially bounded on $S(u)$.

Proposition 5.13 *If* $u \in SBV_0^p\big(\Omega, \mathbf{R}^m\big)$ *then* $w_i^\pm = \nabla u_i^\pm$*, as defined in Proposition* 5.10*.*

Proof. Let $\lambda_h \in \mathbf{R}$, $h = 1, \ldots, m$, let $v = \sum_h \lambda_h u_h$. Since $S(v)$ is a subset of $S(u)$ we can endow it with the same orientation of $S(u)$. As in Remark 5.4 we see that for every multi-index α of order $n-2$, and $\phi \in \mathcal{T}(\Omega \times \mathbf{R})$, we have

$$\langle \partial T_v, \phi dx_\alpha \wedge dy \rangle = \sum_{h=1}^m \lambda_h \langle \partial T_u, \varphi dx_\alpha \wedge dy_h \rangle,$$

where $\varphi(x, y_1, \ldots, y_m) = \phi(x, \sum_h \lambda_h y_h)$. Hence,

$$\langle \partial T_v, \phi dx_\alpha \wedge dy \rangle$$

$$= (-1)^{n-1} \sum_{h=1}^m \lambda_h \int_{S(u)} \Big(\varphi\big(x, u^+(x)\big) w_h^+(x) \wedge dx_\alpha - \varphi\big(x, u^-(x)\big) w_h^-(x) \wedge dx_\alpha\Big).$$

This shows that $\sum_h \lambda_h u_h \in SBV_0^p(\Omega)$, and, by Proposition 4.3 also that the functions $\sum_h \lambda_h w_h^{\pm}$ coincide with the approximate differentials of $\sum_h \lambda_h u_h^{\pm}$ given by Proposition 3.11. Moreover $\partial T_v = T_{v^-} - T_{v^+}$, with $T_{v^\pm}$ as in Definition 3.12.

In particular, for all i and j, $\nabla u_i^{\pm} = w_i^{\pm}$ on $S(u_i)$, and $\nabla(u_i^{\pm}+u_j^{\pm}) = w_i^{\pm}+w_j^{\pm}$ on $S(u_i+u_j)$. Hence, fixed i there exist $w_i^{\pm} = \nabla(u_i^{\pm}+u_j^{\pm}) - \nabla u_j^{\pm} = \nabla u_i^{\pm}$ on $S(u_i+u_j) \setminus S(u_i) = S(u_j) \setminus S(u_i)$ for all j, so that $\nabla u_i^{\pm} = w_i^{\pm}$ on the whole $S(u) = \bigcup_j S(u_j)$. □

Proposition 5.14 *If $u \in SBV_0\big(\Omega, \mathbf{R}^m\big)$ then the following three conditions are equivalent:*
(i) (5.17) *holds;*
(ii) $\pi_\#(\|\partial_v T_u\|) = \Big(\sqrt{\sum_{k=1}^{n-1}(\mathbf{M}_k(|\nabla u^+|))^2} + \sqrt{\sum_{k=1}^{n-1}(\mathbf{M}_k(|\nabla u^-|))^2}\Big)\mathcal{H}^{n-1} \llcorner S(u)$;
(iii) $\pi_\#(\|\partial_v T_u\|)$ *is absolutely continuous with respect to* $\mathcal{H}^{n-1} \llcorner S(u)$.

Proof. The proof is quite similar to that one given for the scalar case (see Proposition 4.4). □

Theorem 5.15 (closure of $SBV_0^p\big(\Omega, \mathbf{R}^m\big)$) *Let $(u_h) \subset SBV_0^p\big(\Omega, \mathbf{R}^m\big)$, u as in Theorem 5.8, and assume in addition that*

$$\mathcal{H}^{n-1}(S(u)) = \lim_{h \to +\infty} \mathcal{H}^{n-1}(S(u_h)).$$

If we have

$$\sup_h \Big(\int_{S(u_h)} \Big(\sum_{k=1}^{n-1}(\mathbf{M}_k(|\nabla u^+|))^2\Big)^{p/2} + \Big(\sum_{k=1}^{n-1}(\mathbf{M}_k(|\nabla u^-|))^2\Big)^{p/2} d\mathcal{H}^{n-1}\Big) < +\infty$$

then $u \in SBV_0^p\big(\Omega, \mathbf{R}^m\big)$.

Proof. The proof can be obtained following the lines of the proof of Theorem 4.5 (closure of $SBV_0^p(\Omega)$). □

References

[1] R. Adams: *Sobolev Spaces.* Academic Press, New York, 1975

[2] G. Alberti: Rank one property for derivatives of functions with bounded variation, *Proc. Roy. Soc. Edinburgh A* **123** (1993), 239-274.

[3] L. Ambrosio: A compactness theorem for a new class of functions of bounded variation, *Boll. Un. Mat. Ital.* **3-B** (1989), 857-881.

[4] L. Ambrosio: Existence theory for a new class of variational problems, *Arch. Rational Mech. Anal.* **111** (1990), 291-322.

[5] L. Ambrosio: Variational problems in SBV, *Acta Applicandae Mathematicae* **17** (1989), 1-40.

[6] L. Ambrosio: On the lower semicontinuity of quasi-convex integrals in $SBV(\Omega, \mathbf{R}^k)$, *Nonlinear Anal.* **23** (1994), 404-425.

[7] L. Ambrosio: A new proof of the SBV compactness theorem, *Calc. Var.* **3** (1995), 127-137.

[8] L. Ambrosio and D. Pallara: Partial regularity of free discontinuity sets I, (1994), (to appear).

[9] L. Ambrosio, N. Fusco and D. Pallara: Partial regularity of free discontinuity sets II, (1994), (to appear).

[10] G. Anzellotti, S. Delladio and G. Scianna: BV functions over rectifiable currents, *Ann. Mat. Pura Appl.* (to appear).

[11] C. Atkinson, Stress singularities and fracture mechanics, *Appl. Mech. Rev.* **32** (1979), 123-135.

[12] J. M. Ball: Convexity conditions and existence theorems in nonlinear elasticity. *Arch. Rational Mech. Anal.* **63** (1977), 337-403.

[13] B. Dacorogna: *Direct Methods in the Calculus of Variations,* Springer Verlag, Berlin, 1989.

[14] G. Dal Maso, J. M. Morel and S. Solimini: A variational method in image segmentation: existence and approximation results, *Acta Math.* **168** (1992), 89–151.

[15] E. De Giorgi: Free Discontinuity Problems in Calculus of Variations, in: *Frontiers in pure and applied Mathematics,* a collection of papers dedicated to J. L. Lions on the occasion of his 60^{th} birthday, R. Dautray ed., North Holland, 1991.

[16] E. De Giorgi and L. Ambrosio: Un nuovo tipo di funzionale del Calcolo delle Variazioni, *Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Nat.* **(8) 82** (1988), 199-210.

[17] E. De Giorgi, M. Carriero and A. Leaci: Existence theorem for a minimum problem with free discontinuity set, *Arch. Rational Mech. Anal.* **108** (1989), 195-218.

[18] L. C. EVANS AND R. F. GARIEPY: *Lecture Notes on Measure Theory and Fine Properties of Functions.* CRC Press, Ann Harbor, 1992.

[19] H. FEDERER: *Geometric Measure Theory.* Springer Verlag, Berlin, 1969.

[20] E. GAGLIARDO: Caratterizzazione delle tracce sulla frontiera relative ad alcune classi di funzioni in più variabili. *Rend. Sem. Mat. Padova* **27** (1957) 284–305.

[21] M. GIAQUINTA, G. MODICA AND J. SOUCEK: Cartesian currents, weak diffeomorphism, and existence theorems in nonlinear elasticity, *Arch. Rational Mech. Anal.* **106** (1989), 97-159. Erratum and addendum, *Arch. Rational Mech. Anal.* **109** (1990), 385-392.

[22] M. GIAQUINTA, G. MODICA AND J. SOUCEK: Area and the area formula, . *Rend Semin. Mat Fis. Milano* **62** (1992), 53-87.

[23] A. A. GRIFFITH, The phenomenon of rupture and flow in solids, *Phil. Trans. Royal Soc. London A* **221** (1920), 163-198.

[24] J. E. HUTCHINSON: Fractals and self similarity, *Indiana Univ. Math. J.* **30** (1981), 713-747.

[25] H. LIEBOWITZ, *Fracture: an Advanced Treatise*, Academic Press, 1969.

[26] F. MORGAN: *Geometric Measure Theory – A Beginner's Guide.* Academic Press, Boston, 1988.

[27] J. M. MOREL AND S. SOLIMINI: *Variational Models in Image Segmentation,* Birkhäuser, Boston, 1995.

[28] S. MÜLLER: Det=det – A remark on the distributional determinant, *C. R. Acad. Sci. Paris* **311** (1990), 13-17.

[29] S. MÜLLER: On the singular support of the distributional determinant, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **10** (1993), 657-696.

[30] D. MUMFORD AND J. SHAH: Optimal approximation by piecewise smooth functions and associated variational problems, *Comm. Pure Appl. Math.* **17** (1989) 577-685.

[31] E. OSSANNA: *SBV* functions defined on currents, *J. Math. Pures Appl.*, (to appear).

[32] K. PALANISWAMY AND G. KNAUSS, On the problem of crack extension in brittle solids under general loading, in *Mechanics Today Vol. 4* (S. Nemat-Nasser ed.), Pergamon Press, London, 1978, 87-147.

[33] J. R. Rice, Mathematical analysis in the mechanics of fracture, in *Fracture: an Advanced Treatise, Vol. 2* (H. Liebowitz ed.), Academic Press, 1969, 191-311.

[34] L. Simon: *Lectures on Geometric Measure Theory,* Proceedings of the Centre for Mathematical Analysis, Australian National University, Canberra 1983.

[35] A. I. Vol'pert: Spaces BV and quasi-linear equations. *Math. USSR Sb.* **17** (1969), 225-267.

[36] A. I. Vol'pert and S. I. Hudjaev: *Analysis in Classes of Discontinuous Functions and Equations of Mathematical Physics.* Kluwer Academic Publishers, Dordrecht, 1985.

[37] W. P. Ziemer: *Weakly Differentiable Functions.* Springer-Verlag, Berlin, 1989.