A necessary condition for lower semicontinuity of line energies

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Abstract

We are interested in some energy functionals concentrated on the discontinuity lines of divergence-free 2D vector fields valued in the circle $\mathbb{S}^1$. This kind of energy has been introduced first by P. Aviles and Y. Giga in [4]. They show in particular that, with the cubic cost function $f(t) = t^3$, this energy is lower semicontinuous. In this paper, we construct a counter-example which excludes the lower semicontinuity of line energies for cost functions of the form t^p with $0 < p < 1$. We also show that, in this case, the viscosity solution corresponding to a certain convex domain is not a minimizer.

1 Introduction

1.1 Line energies

Let Ω be a Lipschitz domain in $\mathbb{R}^2$. We are interested in measurable vector fields $m\colon \Omega \to \mathbb{R}^2$ such that

$$|m| = 1 \text{ a.e. and } \nabla \cdot m = 0 \text{ on } \Omega, \tag{1}$$

where the second equation holds in a distributional sense. In the following, we will assume that m is of bounded variation so as to be able to define its *jump line*. So, we consider the set

$$A(\Omega) := \left\{ m \in BV(\Omega, \mathbb{R}^2) \,:\, |m| = 1 \text{ a.e. and } \nabla \cdot m = 0 \text{ on } \Omega \right\}.$$

Vector fields $m \in A(\Omega)$ are related to solutions of the *eikonal equation* in Ω. Let define the set

$$S(\Omega) := \{\varphi \in \mathrm{Lip}(\Omega) \,:\, |\nabla\varphi| = 1 \text{ a.e. and } \nabla\varphi \in BV(\Omega)\}.$$

Then, given $m \in A(\Omega)$, there exist a scalar function $\varphi \in S(\Omega)$ such that

$$m(x) = (\nabla\varphi(x))^{\perp} \text{ a.e.},$$

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where $(\nabla\varphi)^\perp = R\nabla\varphi$ stands for the image of $\nabla\varphi$ by the rotation R of angle $\pi/2$ centered at the origin in $\mathbb{R}^2$. Moreover, a function $\varphi \in \mathrm{Lip}(\Omega)$ satisfying $(\nabla\varphi)^\perp = m$ a.e. is unique up to a constant and is called stream function. We are now able to define line energies:

Definition 1 *Let $f\colon [0,2] \to [0,+\infty]$ be a measurable scalar function. Let $m \in A(\Omega) \subset BV(\Omega,\mathbb{R}^2)$. Then, there exists a $\mathcal{H}^1$-rectifiable jump line $J(m)$ oriented by a unit normal vector ν_x such that m has traces $m_\pm(x) \in \mathbb{S}^1$ on each side of $J(m)$ for $\mathcal{H}^1$ a.e. $x \in J(m)$ (see [3] for more details).*

Then, the energy associated to the jump cost f is denoted by $\mathcal{I}_f$ and defined for $m \in A(\Omega)$ as follows:

$$\mathcal{I}_f(m) = \int_{J(m)} f(|m_+ - m_-|)\,\mathrm{d}\mathcal{H}^1(x). \tag{2}$$

f is called the jump cost. Note that the divergence constraint on $m \in A(\Omega)$ implies that for *a.e.* $x \in J(m)$, $m_\pm(x) \in \mathbb{S}^1$ and ν_x satisfy the following condition (see figure 3):

$$m_+(x)\cdot\nu_x = m_-(x)\cdot\nu_x\ . \tag{3}$$

Then, in the orthogonal basis $(\nu_x, \nu_x^\perp)$, there exists some angle θ such that $m_\pm = (\cos\theta, \pm\sin\theta)$ and the jump size is defined as

$$t = |m_+ - m_-| = 2|\sin\theta|.$$

Similarly, $\mathcal{I}_f$ can be interpreted as a functional of the stream function on the set $S(\Omega)$: Writing $m = (\nabla\varphi)^\perp \in BV(\Omega,\mathbb{R}^2)$, then $\mathcal{I}_f(m) = \mathcal{J}_f(\varphi)$ where

$$\forall\varphi \in S(\Omega),\quad \mathcal{J}_f(\varphi) = \int_{J(\nabla\varphi)} f(|(\nabla\varphi)_+ - (\nabla\varphi)_-|)\,\mathrm{d}\mathcal{H}^1(x). \tag{4}$$

An interesting question is to find the minimizing structures of $\mathcal{I}_f$ if it exists. Remark that for this problem to be relevant, we have to consider a constraint on the boundary otherwise all constant functions are minimizers. A natural choice is to minimize $\mathcal{I}_f$ along all configurations m belonging to the set

$$A_0(\Omega) := \{m \in A(\Omega)\ :\ m\cdot n = 0 \text{ a.e. on } \partial\Omega\}\,, \tag{5}$$

where n is the exterior unit normal vector of $\partial\Omega$. In terms of the stream function φ, this is equivalent to consider the set

$$S_0(\Omega) := \{\varphi \in S(\Omega)\ :\ \varphi = 0 \text{ on } \partial\Omega\}\,. \tag{6}$$

1.2 Related models

Line energies naturally appear in micromagnetics when studying the asymptotic behavior of the magnetization in a thin ferromagnetic sample. We are going to give two simplified micromagnetic models illustrating this phenomenon.

The first example is due to P. Aviles. and Y. Giga. In [5], they have conjectured that if $f(t) = \frac{1}{3}t^3$, then $\mathcal{I}_f$ is the Γ-limit of the following Ginzburg-Landau type energy functional

$$\begin{cases} AG_\varepsilon(u) = \displaystyle\int_\Omega \varepsilon|\nabla u|^2 + \frac{1}{\varepsilon}(1-|u|^2)^2 & \text{if } u \in H^1(\Omega,\mathbb{R}^2) \text{ and } \nabla\cdot u = 0, \\ AG_\varepsilon(u) = +\infty & \text{otherwise,} \end{cases} \tag{7}$$

where Ω is a bounded open set in $\mathbb{R}^2$ and $\varepsilon > 0$ is some parameter.

It is clear that finite energy limiting configurations u, i.e. limits of finite energy sequences $(u_\varepsilon)_{\varepsilon>0}$, have to be solutions of (1). It is also straightforward to see that the energy will concentrate on the singular set of u. However, it is a challenging problem to determine rigorously the asymptotic behavior of these functionals when ε goes to zero. More precisely, given some energies E_ε depending on $\varepsilon > 0$, a fundamental question is to find some appropriate topology, given by the L^1 distance for instance, and some limiting energy E_0 such that the three following properties hold (see [6] for example):

1. Compactness: If $(u_\varepsilon)_{\varepsilon>0}$ is a finite energy sequence, i.e. $\limsup_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon) < \infty$, then $(u_\varepsilon)_{\varepsilon>0}$ is relatively compact.

2. The sequence $(E_\varepsilon)_{\varepsilon>0}$ Γ-converges to E_0, that is:

   - Γ-liminf property: For all sequence $(u_\varepsilon)_{\varepsilon>0}$ converging to some u,
   $$E_0(u) \le \liminf_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon).$$

   - Γ-limsup property: for all u, there exists a sequence $(u_\varepsilon)_{\varepsilon>0}$ converging to u such that:
   $$E_0(u) = \lim_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon).$$

For the Γ-convergence of functionals $E_\varepsilon = AG_\varepsilon$ to $E_0 = \mathcal{I}_f$ with $f(t) = \frac{1}{3}t^3$, only partial results are shown. In [5], the authors have been able to prove the Γ-liminf property for the L^1 convergence using the notion of entropies related to the problem (1) (see also [15]). Unfortunately, the Γ-limsup is still an open problem for limiting configurations $u \notin BV(\Omega,\mathbb{R}^2)$. The strong compactness of finite energy sequences has been proved by Ambrosio, De Lellis and Mantegazza in [2] and by De Simone, Kohn, Müller and Otto in [8] using a compensated compactness method based on a new notion of regular entropy on $\mathbb{R}^2$. This notion of entropy together with the kinetic equation has also been used in [13] to characterize the structure of zero-energy states of (7) and in [14] to deduce some regularity properties in the limit.

The second model we want to address comes from the Ginzburg-Landau theory in thin film micromagnetics for some asymptotical regime (see [10]). Given a bounded domain $\Omega \subset \mathbb{R}^2$ and a magnetization $m = (m_1, m_2, m_3) \colon \Omega \to \mathbb{S}^2$, where $\mathbb{S}^2$ stands for the unit sphere in $\mathbb{R}^3$, the energy of m is defined as follows:

$$E_\varepsilon(m) = \varepsilon \int_\Omega |\nabla m|^2 + \frac{1}{\varepsilon} \int_\Omega \phi(m) + \frac{1}{\beta} \int_{\mathbb{R}^2} |H|^2, \tag{8}$$

where:

- ε is a small parameter called exchange length and $0 < \beta << \varepsilon$.
- $\phi : \mathbb{S}^2 \to \mathbb{R}$ is some smooth function called anisotropy function such that:

$$\begin{cases} \phi(m) = 0 & \text{if } m \in \mathbb{S}^2 \cap \{m_3 = 0\}, \\ \phi(m) > 0 & \text{otherwise.} \end{cases} \tag{9}$$

- $H \in L^2(\mathbb{R}^2, \mathbb{R}^2)$ is the solution, called stray field, of the following problem:

$$\begin{cases} \nabla \times H = 0 & \text{on } \mathbb{R}^2, \\ \nabla \cdot H = \nabla \cdot m' & \text{on } \mathbb{R}^2, \end{cases}$$

  where $m' = (m_1, m_2)$.

A simplified model consists in adding a divergence constraint $\nabla \cdot m' = 0$ to the functional so that the last term disappears: this is equivalent to take the limit when β tends to 0.

A finite energy sequence $(m_\varepsilon)_{\varepsilon>0}$ is expected to converge to some divergence free and unit length vector field m. Some experiments show that, at least for ε very small, the magnetization is smooth out of a thin layer (very close to a line) of size ε on which it changes very quickly between two values $m_\pm$ (see [9]). The microstructures formed by the magnetization into this layer can be more or less complex. In the simplest case, it is one-dimensional, i.e. it depends only on the normal (to the jump line) variable. However more complex structures can appear as cross-tie wall ([1], [17]) or zizag-patterns ([12]) for example.

If $\phi(m) = |m_3|^\alpha$ with $0 < \alpha \leq 4$, only one-dimensional structures are expected and it is easy to compute what should be the limiting energy of functionals E_ε by a $1D$-analysis. As for the Modica-Mortola model for phase transition ([16]), E_ε is expected to Γ-converge to $c\mathcal{I}_f$ for some $c > 0$ where $f(t) = \frac{t^p}{p}$, $p = 1 + \frac{\alpha}{2}$ is the primitive of $\sqrt{\phi}$ vanishing at 0.

The case $\phi(m) = |m_3|^2$ was studied by R. Ignat and B. Merlet in [11] in which a compactness result was proved and a lower bound was found. However, the Γ-liminf property in the definition of Γ-convergence was established only for limiting $1D$ configurations of the form $m(x) = \pm\nu^\perp$ for $\pm x \cdot \nu > 0$ with $\nu \in \mathbb{S}^1$ (see figure 3 with $\theta_0 = \pi/2$).

1.3 Lower semicontinuity, Viscosity solution

As explained above, some of the line energies $\mathcal{I}_f$ are conjectured to be the Γ-limit of functionals coming from micromagnetics in the space $X = L^1$. If that is the case, $\mathcal{I}_f$ has to satisfy the following lower semicontinuity property:

Definition 2 *Let $F : X \to [0, +\infty]$ be a functional defined on some topological space X. F is said to be lower semicontinuous or l.s.c. if the following holds:*

$$\forall (x_n)_{n\geq 0} \subset X \underset{n\to+\infty}{\longrightarrow} x, \ F(x) \leq \liminf_{n\to\infty} F(x_n).$$

Since this property strongly depends on the topology of the space X, we have to specify the choice we make for the study of line energies $\mathcal{I}_f$.

First of all, due to the non convex constraint $|m| = 1$, we need strong compactness in L^1. Moreover, since all the results of the previous part (compactness and Γ-liminf property) holds for the L^1 strong topology, it seems natural to consider the line energies $\mathcal{I}_f$ in the space $X = L^1$.

However, since definition 1 uses the notion of trace of a function, another natural choice would be $X = BV$ endowed with the weak topology which is a very common choice for phase transition problems. Unfortunately, in the general case, the space BV is not adapted to our problem.

Suppose $f(t) = t^p$ with $p > 1$ for instance. Then finite energy configurations m (i.e. $m_n \underset{n\to+\infty}{\longrightarrow} m$ in L^1 with $\mathcal{I}_f(m_n) \leq C < +\infty$) are not necessarily of bounded variation since the total variation of m around its jump line can't be controlled by $\displaystyle\int_{J(m)} |m_+ - m_-|^p$ if $p > 1$. That's why we need a subspace of solutions of the problem (1) included in $L^1(\Omega)$ (and containing BV) because of the non convex constraint $|m| = 1$ such that we are still able to define a jump line $J(m)$ and traces $m_\pm$. This is done in [7] where a regularity result is shown for solution of (1) with bounded "entropy production".

Note that if X and Y are two topological spaces such that Y is continuously embeded in X and $F : X \to [0, +\infty]$ is l.s.c. in X then the restriction of F to Y is l.s.c. in Y. In this paper, we only want to prove a necessary condition for functionals $\mathcal{I}_f$ to be l.s.c.. We then prefer to restrict our analysis to BV functions (see remark 2).

In the case where $f(t) = t^p$ for some $p > 0$, only partial results are known. In [2], the following is conjectured:

Conjecture 1 *Let $\overline{\mathcal{I}_f}$ be the relaxation of $\mathcal{I}_f$ (only defined on the space BV) in L^1:*

$$\overline{\mathcal{I}_f}(m) = \mathrm{Inf}\left\{ \liminf_{n\to+\infty} \mathcal{I}_f(m_n) \ : \ m_n \in BV \ \textit{and} \ m_n \underset{n\to+\infty}{\longrightarrow} m \ \textit{in} \ L^1 \right\}. \tag{10}$$

If $f(t) = t^p$ with $1 \leq p \leq 3$ then $\overline{\mathcal{I}_f}$ is l.s.c. for the strong topology in L^1.

For $p > 3$, this conjecture is false (see [2]). The case $p = 3$ has been studied by P. Aviles and Y. Giga in [5]. More recently the case $p = 2$ has been proved by R. Ignat and B. Merlet in [11]. They also proved that for every $1 \leq p \leq 3$, there exist cost-functions $f(t) = t^p$ for $t \in [0, \sqrt{2}]$ leading to a l.s.c functional $\overline{\mathcal{I}_f}$ i.e. the conjecture is true for every $1 \leq p \leq 3$ if we resctrict the jump to angles between 0 and $\pi/4$. Here we are interested in the open case $p < 1$.

We point out that line energies associated to the cost $f(t) = t^p$ with $1 \leq p \leq 3$ correspond exactly to the expected Γ-limits of functionals (8) when $\phi(m) = |m_3|^\alpha$ with $0 < \alpha \leq 4$ where bloch walls seem to be optimal. This is quite natural since when 2D structures, as cross tie wall or zigzag wall for instance, have less energy than bloch walls, the Γ-limit of these functionals may be non lower semicontinuous. In the next part, we are going to give a 2D construction which gives some necessary condition on f for $\mathcal{I}_f$ to be l.s.c. This condition excludes cost functions of the form $f(t) = t^p$ with $p < 1$:

Theorem 1 *Let $f\colon [0,2] \to [0, +\infty]$. Let Ω be an open and non empty subset of $\mathbb{R}^2$. Assume that $\mathcal{I}_f$ is lower semicontinuous in $X = BV(\Omega, \mathbb{S}^1)$ endowed with the weak topology. Then f is lower semicontinuous and we have*

$$\limsup_{t\to 0} \frac{f(t)}{t} \leq 2 \limsup_{t\to 2} f(t). \tag{11}$$

Remark 1 *The fact that the lower semicontinuity of $\mathcal{I}_f$ implies the lower semicontinuity of f has already been proved in [11]. The main new point here is the condition* (11).

Remark 2 *Theorem 1 is stronger than an equivalent formulation in which BV is replaced by some banach space X such that BV is continuously embedded in X and where $\mathcal{I}_f$ is replaced by its relaxation in X.*

Remark 3 *The inegality* (11) *in theorem 1 is optimal in the sense that we can find a lower semicontinuous functionnal $\mathcal{I}_f$ in $BV(\Omega, \mathbb{S}^1)$ for which*

$$\limsup_{t\to 0} \frac{f(t)}{t} = 2 \limsup_{t\to 2} f(t).$$

This is based on a theorem of Ignat and Merlet in [11] stating that for a cost function f associated to an entropy, $\mathcal{I}_f$ is l.s.c.. We just have to remark that the cost function $f(t) = t^3\sqrt{4-t^2}$ is associated to the entropy $\Phi(x) = \sin 3\theta x + 3\cos 3\theta x^\perp$.

As we will see, the lower semicontinuity of functionals $\mathcal{I}_f$ is closely related to the following question: Is the viscosity solution a minimizer of $\mathcal{I}_f$? More precisely, it is expected that the following is true:

Conjecture 2 *Assume that $\overline{\mathcal{I}_f}$ is l.s.c. in L^1 and that Ω is convex. Then $(\nabla\varphi_0)^\perp$ is a global minimizer of $\mathcal{I}_f$ where $\varphi_0(x) = \operatorname{dist}(x, \partial\Omega)$.*

For a regular domain Ω the distance function $\varphi_0(x) = \text{dist}(x, \partial\Omega)$ belongs to $S_0(\Omega)$ and $(\nabla\varphi_0)^\perp$ is the viscosity solution of the problem (1). In particular, if Ω is convex, φ_0 is concave and $-D^2\varphi_0$ is a positive vectorial radon measure. In [2], the authors give a microstructure which shows that this is false for $f(t) = t^p$ if $p > 3$. As explained below, we are going to give a structure with lower energy than the viscosity solution for $p < 1$.

Proposition 1 *Let $f : [0, 2] \to [0, +\infty]$. There exists a convex domain Ω such that the following holds: let $\varphi_0 \in S_0(\Omega)$ be the distance function $\varphi_0(x) = \text{dist}(x, \partial\Omega)$. Assume that φ_0 is a minimizer of $\mathcal{J}_f$ defined by* (4)*. Then f satisfies* (11)*.*

Corrolary 1 *There exists a convex domain Ω such that the viscosity solution is not a minimizer of $\mathcal{I}_f$ if $f(t) = t^p$ with $p \in [0, 1[$.*

2 Construction of a competitor of the viscosity solution

In order to obtain inequality (11), we have to construct a domain Ω on which the jump size $t = |m_+ - m_-|$ of the viscosity solution along its singular set is very small. Then, we find a competitor whose jump size t is close to the maximal possible value $t = 2$. In other words, we want to substitute small jumps by large ones.

We will use the polar coordinates (r, θ), $r \geq 0$, $\theta \in [-\pi, \pi]$ and we will identify $\mathbb{R}^2$ and $\mathbb{C}$ with the usual bijection. Let D be the unit disk and $\mathcal{C}$ be its boundary.

Let θ_0 be a fixed angle in $]0, \pi/2[$ and define the two points $A = e^{i\theta_0}$ and $A' = e^{-i\theta_0}$ on the circle $\mathcal{C}$. Define also T_A (resp. $T_{A'}$) the tangent to the circle $\mathcal{C}$ at the point A (resp. A').

We consider the domain Ω delimited by the large arc $\mathcal{C}_{\theta_0} = \{e^{i\theta} \ : \ |\theta| > \theta_0\}$, T_A and $T_{A'}$ (see figure 1). In other words Ω is the interior or the convex envelope of $\mathcal{C} \cup \{B\}$ where $B = T_A \cap T_{A'}$. Define also $\omega = \Omega \cap \{|\theta| < \theta_0 \text{ and } r > 0\}$ and $\Gamma = \partial\Omega \cap \partial\omega = [AB] \cup [A'B]$.

We now consider two solutions φ_0 and φ in $S_0(\Omega)$ of the eikonal equation vanishing on the border :

- φ_0 is the usual distance function : $\forall x \in \Omega,\ \varphi_0(x) = \text{dist}(x, \partial\Omega)$.
- φ is the distance from the union of $\partial\Omega$ and the large arc $\mathcal{C} \setminus \mathcal{C}_{\theta_0}$:
  $\forall x \in \Omega,\ \varphi(x) = \text{dist}(x, \partial\Omega \cup \mathcal{C})$.

We also denote by $m_0 = (\nabla\varphi_0)^\perp$ and $m = (\nabla\varphi)^\perp$ the corresponding solutions of (1). Then $m_0, m \in A_0(\Omega)$.

We now compute $\mathcal{I}_f(m_0)$ and $\mathcal{I}_f(m)$ in order to prove that the function φ as lower energy than φ_0 if $f(t) = t^p$ with $p < 1$.

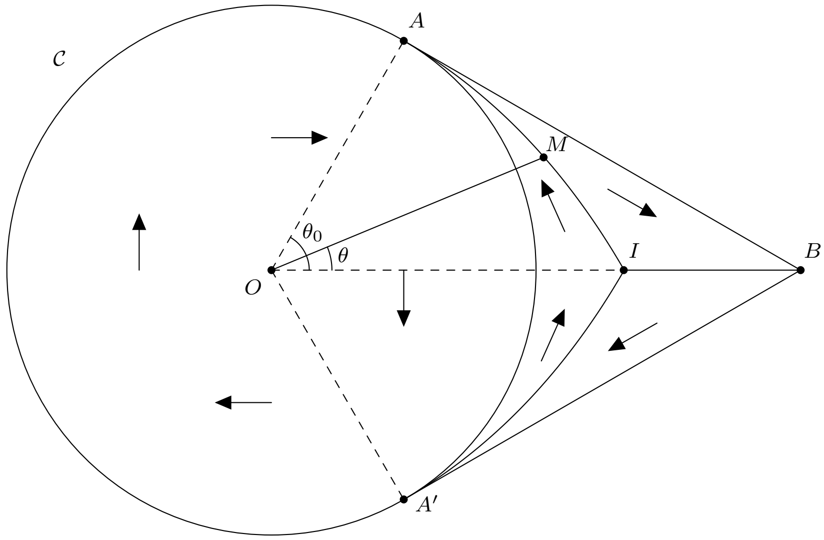

Figure 1: The domain Ω and the microstructure m

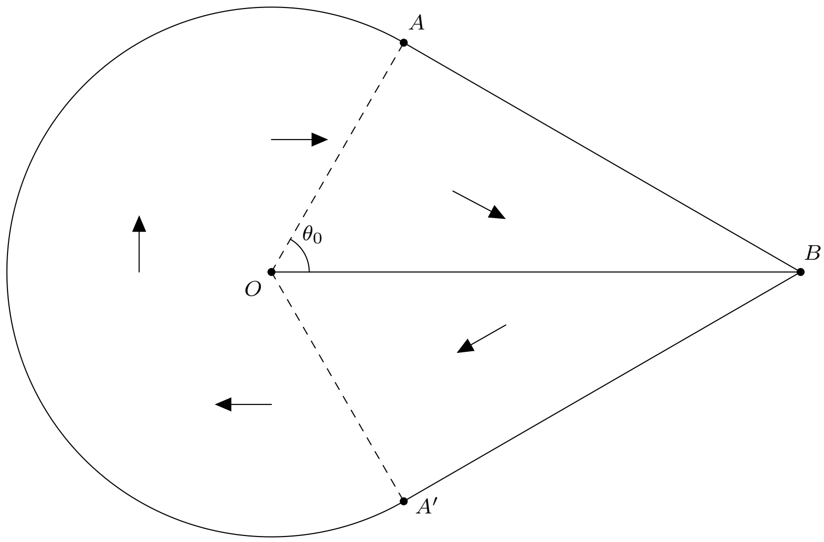

Figure 2: Viscosity solution m_0 on Ω

Heuristic: The idea is that a small jump along a fixed length is replaced by big jump on a small length : This will reduce the energy for subadditive power costs (i.e. $f(t) = t^p$ with $p < 1$) which favor "small jumps". Let us give more details.

For a small angle $\theta_0 > 0$, m_0 only presents small jumps: m_0 is $\mathcal{C}^1$ out of segment $[OB]$ on which the jump size is $|m_0^+ - m_0^-| =: t_0 = 2\sin(\theta_0)$.

On the contrary, m only presents "big" jumps: i.e. jumps whose size is close to 2. The singular set of m consists in 3 different lines : $[IB]$ whose length is equivalent to θ_0^2 and the two curves $\mathcal{C}_{\theta_0}$ and γ_{θ_0} (defined below) on which the jump size tends to 2 and the length of these lines is equivalent to $2\theta_0$.

As a result, the energy of m_0 is close to $f(2\sin\theta_0)$ while the energy of m is close to $4\theta_0 \times f(2)$. A necessary condition for m_0 to minimize $\mathcal{I}_f$ is then (see proposition 1)

$$\limsup_{t\to 0} f(t)/t \leq 2f(2).$$

This excludes sub-additive power costs. Now let us give more details on the critical angle θ_0.

Energy of m_0: The jump line of m_0 is the segment $[OB]$ and the traces of m_0 on each side of this line are given by $m_{0,\pm} = -e^{i(\pi/2 \pm \theta_0)}$. In particular,

$$\mathcal{I}_f(m_0) = f(2\sin\theta_0)|OB| = \frac{f(2\sin\theta_0)}{\cos\theta_0}. \tag{12}$$

Energy of m: The jump line of m is the union of the 3 curves:

- $\mathcal{C}_{\theta_0} = \{e^{i\theta} \ : \ |\theta| < \theta_0\}$.
- $\gamma_{\theta_0} := \{z \in \omega \ : \ d(z, \mathcal{C}_{\theta_0}) = d(z, \Gamma)\} = \{z = re^{i\theta} \ : \ |\theta| < \theta_0, \, d(z, \mathcal{C}) = d(z, \partial\Omega)\}$.
- The segment $[IB]$ where $I = \gamma_{\theta_0} \cap [OB]$.

First, let us find a polar equation for the curve γ_{θ_0}: Given $z = re^{i\theta}$ such that $|\theta| < \theta_0$ and $r > 1$ we have $d(z, \mathcal{C}_{\theta_0}) = r - 1$, it remains to compute $\lambda := d(z, \Gamma)$.

Since Ω is symmetric with respect to the axe (OB), one can restrict to the case $M = r\,e^{i\theta}$ with $0 < \theta < \theta_0$. So $\lambda := d(z, \Gamma) = |z - P|$ where P is the orthogonal projection of $M = re^{i\theta}$ on the segment $[AB]$: P should satisfy $\overrightarrow{MP} = \lambda\,\overrightarrow{OA} = \lambda\, e^{i\theta_0}$ and $\overrightarrow{MP} \cdot \overrightarrow{AP} = 0$.

We then compute

$$\begin{aligned} \overrightarrow{MP} \cdot \overrightarrow{AP} =& \overrightarrow{MP} \cdot [\overrightarrow{AO} + \overrightarrow{OM} + \overrightarrow{MP}] \\ =& \Re\{\lambda\, e^{-i\theta_0}\,(-e^{i\theta_0} + r\, e^{i\theta} + \lambda\, e^{i\theta_0})\} \\ =& \lambda[-1 + r\cos(\theta_0 - \theta) + \lambda]. \end{aligned}$$

Since $\overrightarrow{MP} \cdot \overrightarrow{AP} = 0$, this implies $\lambda = MP = 1 - r\cos(\theta_0 - \theta)$.

Then we have $z \in \gamma_{\theta_0}$ if and only if $r - 1 = 1 - r\cos(\theta_0 - \theta)$ and the polar equation of the curve γ_{θ_0} is given by

$$r(\theta) = \frac{2}{1 + \cos(\theta_0 - \theta)} \ ; \ -\theta_0 < \theta < \theta_0. \tag{13}$$

Now, we can compute the energy of m along the curve γ_{θ_0}:

- $\mathrm{d}\gamma(\theta) = \sqrt{r(\theta)^2 + r'(\theta)^2}\,\mathrm{d}\theta$ where we find $r'(\theta) = \dfrac{-2\sin(\theta_0-\theta)}{(1+\cos(\theta_0-\theta))^2}$. Introducing the notation $\alpha = \theta_0 - \theta$, we obtain

$$\mathrm{d}\gamma(\theta) = 2\frac{\sqrt{(1+\cos\alpha)^2+\sin^2\alpha}}{(1+\cos\alpha)^2}\,\mathrm{d}\theta = 2\frac{\sqrt{2(1+\cos\alpha)}}{(1+\cos\alpha)^2}\,\mathrm{d}\theta = \frac{4\cos(\alpha/2)}{(2\cos^2(\alpha/2))^2}\,\mathrm{d}\theta.$$

  So $\mathrm{d}\gamma$ writes

$$\mathrm{d}\gamma(\theta) = \cos^{-3}(\alpha/2)\,\mathrm{d}\theta.$$

- The size of the jump at the point $\gamma(\theta)$ is given by

$$t(\theta) = |m_+ - m_-| = |e^{i(\theta_0+\pi/2)} + e^{i(\theta+\pi/2)}| = |e^{i(\theta_0-\theta)} + 1|.$$

  Using once again the notation $\alpha = \theta_0 - \theta$, this gives

$$t(\theta) = \sqrt{(\cos\alpha+1)^2 + \sin^2\alpha} = \sqrt{2(1+\cos\alpha)} = 2\cos(\alpha/2).$$

- We conclude that the energy of m induced by the jump line γ_{θ_0} is given by

$$\mathcal{I}_f^1(m) = \int_{-\theta_0}^{\theta_0} \frac{f[2\cos(\alpha/2)]}{\cos^3(\alpha/2)}\,\mathrm{d}\alpha. \tag{14}$$

The energy concentrated on the arc $\mathcal{C}_{\theta_0}$ is

$$\mathcal{I}_f^2(m) = f(2)\,\mathcal{H}^1(\mathcal{C}_{\theta_0}) = 2\theta_0\, f(2). \tag{15}$$

Finally, we compute the energy on the line $[IB]$:

$$\mathcal{I}_f^3(m) = f(2\sin\theta_0)|IB|. \tag{16}$$

If the distance function is a minimizer of $\mathcal{I}_f$ we should have

$$\mathcal{I}_f(m) - \mathcal{I}_f(m_0) \geq 0.$$

Using these results, we have

$$\begin{aligned}\mathcal{I}_f(m) - \mathcal{I}_f(m_0) =& \mathcal{I}_f^1(m) + \mathcal{I}_f^2(m) + \mathcal{I}_f^3(m) - \mathcal{I}_f(m_0)\\ =& \int_{-\theta_0}^{\theta_0} \frac{f[2\cos(\alpha/2)]}{\cos^3(\alpha/2)}\,\mathrm{d}\alpha + 2\theta_0\, f(2) + (|IB| - |OB|)\, f(2\sin\theta_0).\end{aligned}$$

Since $|IB| - |OB| = -|OI| = -r(0) = -\dfrac{1}{\cos^2(\theta_0/2)}$, this gives

$$\mathcal{I}_f(m) - \mathcal{I}_f(m_0) = \int_{-\theta_0}^{\theta_0} \frac{f[2\cos(\alpha/2)]}{\cos^3(\alpha/2)}\,\mathrm{d}\alpha + 2\theta_0 f(2) - \frac{f(2\sin\theta_0)}{\cos^2(\theta_0/2)}.$$

Hence, if m_0 is a minimizer of $\mathcal{I}_f$, the following condition should be satisfied:

$$\begin{aligned}\frac{f(2\sin\theta_0)}{2\sin\theta_0} &\leq \frac{\theta_0\cos^2(\theta_0/2)}{\sin\theta_0}\left[\frac{1}{\theta_0}\int_0^{\theta_0}\frac{f[2\cos(\alpha/2)]}{\cos^3(\alpha/2)}\,\mathrm{d}\alpha + f(2)\right]\\ &\leq \frac{\theta_0}{\sin\theta_0\cos(\theta_0/2)}\times 2\ \sup\{f(t)\ :\ 2\cos(\theta_0/2)\leq t\leq 2\}.\end{aligned}$$

Taking the lim sup for $\theta_0 \to 0$ leads to (11):

$$\limsup_{t\to 0}\frac{f(t)}{t} \leq 2\limsup_{t\to 2} f(t).$$

This proves proposition 1 and corollary 1 follows from the fact that the preceding inequality holds false for $f(t) = t^p$ with $p < 1$. Note that in this case, we get something more precise that proposition 1:

Proposition 2 *There exists $\theta_0 \in]0, \pi/2[$ only depending on p such that for all $\theta \in]-\theta_0, \theta_0[$, the viscosity solution is not a minimizer of $\mathcal{I}_f$ on Ω_θ where Ω_θ is the convex set constructed in the previous part (θ being the angle $(\overrightarrow{OB}, \overrightarrow{OA})$).*

3 Lower semicontinuity of line energies, proof of theorem 1.

The fact that if $\mathcal{I}_f$ is l.s.c then f is l.s.c can be found in [11] (proposition 1). In this section we prove that (11) is a necessary condition for $\mathcal{I}_f$ to be lower semicontinuous with respect to the weak convergence in BV on bounded open subsets of $\mathbb{R}^2$.

The key is to use the construction $m \in S(\Omega)$ depending on θ_0 of the first part by restriction to ω (see figure 3.). The $1D$ transition defined by (17) corresponds to the viscosity solution m_0 of the previous part. Given a small parameter $\epsilon > 0$, it will costs less energy to substitute the $1D$ transition around its jump line by the microstructures m rescaled at the level ϵ (see figure 4).

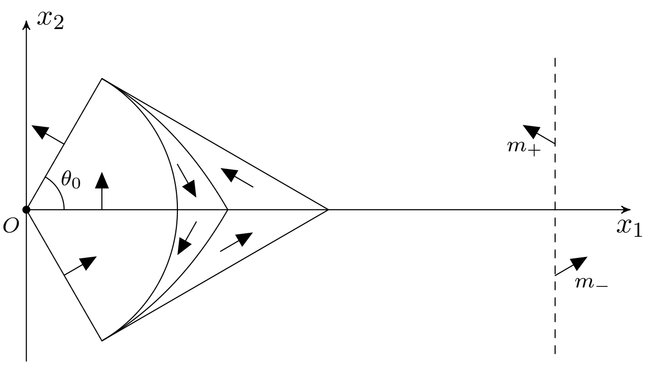

Figure 3: The vector field m on the left and the 1D-transition m_0 on the right

We are going to prove theorem 1 when $\Omega = [0,1] \times [-1,1]$. The general case follows easily.

Fix $\theta_0 \in]0, \pi/2[$ and define the $1D$ transition m_0 for $x_1 \in [0,1]$ and $x_2 \in \mathbb{R}$ by

$$m_0(x_1, x_2) = m_\pm := (\mp \sin\theta_0, \cos\theta_0) \text{ if } \pm x_2 > 0. \tag{17}$$

Then, let us consider the vector field $m = m_{\theta_0}$ of the preceding section restricted to ω and define the rescaled and prolongated vector field $\tilde{m}$ for $x_1 \in [0,1]$ and $x_2 \in \mathbb{R}$:

$$\tilde{m}(x_1, x_2) = \begin{cases} -m\left((\cos\theta_0)^{-1} x_1, (\cos\theta_0)^{-1} x_2\right) & \text{if } \left((\cos\theta_0)^{-1} x_1, (\cos\theta_0)^{-1} x_2\right) \in \omega, \\ m_0(x_1, x_2) & \text{otherwise.} \end{cases}$$

Note that $\tilde{m} \in A(\Omega)$ and is continuous on $\partial\omega$. Then, let n be a positive integer and define $m_n \in M(\Omega)$ by aligning n times the vector field $\tilde{m}$ (see figure 4). More precisely, for $0 \le i < n$ and $x = (x_1, x_2) \in \Omega$ such that $i/n \le x_1 < (i+1)/n$, define

$$m_n(x_1, x_2) = \tilde{m}(n\, x_1 - i, n\, x_2).$$

(see figure 4). We have $m_n(x_1, x_2) = m_0(x_1, x_2)$ for $|x_2| > 1/n$ and $\forall x \in \Omega$, $|m_n(x)| = 1$. Consequently, $(m_n)_{n>0}$ converge to m_0 in $L^1(\Omega)$. Moreover, $|m_n|_{BV(\Omega)} = |\tilde{m}|_{BV(\Omega)}$ so that $(m_n)_{n>0}$ is bounded in $BV(\Omega)$ and weakly converge to m_0.

Since m_n is obtain by scaling a fixed structure, it is easy to see that $\mathcal{I}_f(m_n)$ is constant. Indeed, $\mathcal{I}_f(m_n) = n \times 1/n\, \mathcal{I}_f(\tilde{m}) = \mathcal{I}_f(\tilde{m})$. That's why we obtain the following condition: assuming $\mathcal{I}_f$ is *l.s.c.*,

$$\mathcal{I}_f(m_0) = f(2\sin\theta_0) \le \liminf_{n\to\infty} \mathcal{I}_f(m_n) = \mathcal{I}_f(\tilde{m}).$$

In other words, the viscosity solution costs less energy than the construction m_{θ_0} of the preceding part. For this reason, we obtain exactly the same necessary condition (11) and theorem 1 follows.

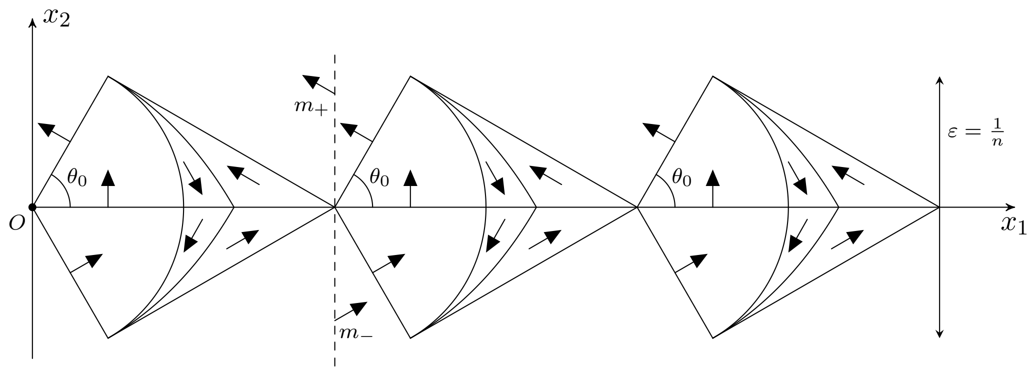

Figure 4: The microstructure m_n

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