Abstract

In this paper, we prove $L^p$ estimates for the fractional derivatives of solutions to elliptic fractional partial differential equations whose coefficients are $VMO$. In particular, our work extends the optimal regularity known in the second order elliptic setting to a spectrum of fractional order elliptic equations.

1 Introduction

In his 1959 paper on some composition formulas for vector-valued potentials, J. Horváth introduced [7, p. 434] the differential object

$$D^s u := DI_{1-s} u.$$  \hfill (1.1)

Here, $s \in (0, 1)$ and $I_{1-s}$ is the Riesz potential of order $1-s$.

This object was subsequently termed the Riesz fractional gradient by the second and third author in [13], where it was utilized to generalize divergence form elliptic partial differential equations from the second order setting to that of differential order $2s \in (0, 2)$. In particular, assuming that $A$ is uniformly elliptic, i.e.

$$\lambda|\xi|^2 \leq A(x)\xi \cdot \xi \leq \Lambda|\xi|^2,$$  \hfill (1.2)
for all $x, \xi \in \mathbb{R}^N$ and some $0 < \lambda \leq \Lambda < +\infty$, the authors showed that given $\varphi \in H^s(\mathbb{R}^N)$ and $g \in L^2(\Omega)$ there exists $u \in H^s(\mathbb{R}^N)$ that satisfies

$$\int_{\mathbb{R}^N} A(x) D^s u(x) \cdot D^s v(x) \, dx = \int_{\mathbb{R}^N} g v$$

for all $v \in C_0^\infty(\mathbb{R}^N)$ and $u = \varphi$ in $\mathbb{R}^N \setminus \Omega$. Here, $\Omega \subset \mathbb{R}^N$ is open and bounded, $N \geq 2$, and

$$H^s(\mathbb{R}^N) := \{ u \in L^2(\mathbb{R}^N) : D^s u \in L^2(\mathbb{R}^N; \mathbb{R}^N) \},$$

which coincides with any standard definition of the fractional Sobolev space.

One observes that when $s = 1$ and the boundary of $\Omega$ is sufficiently nice, the equation (1.3) agrees with the weak formulation of a divergence form elliptic PDE, since prescribing $u$ on the complement gives rise to a trace that would be a more standard way to frame the existence. Meanwhile for $s \in (0, 1)$ one obtains a family of fractional partial differential equations with analogous structure. The interest in generalizing partial differential equations via (1.1) is two-fold. Firstly, that one should be concerned with non-integer order differential objects can be simply explained by quoting Sobolev and Nikol’skiǐ’s 1963 paper (who even implicitly consider (1.1), see [12, p. 148]) where they note that “an imbedding theory containing only derivatives of integral order is incomplete and imperfect.” Secondly, the structure of (1.1) closely resembles the gradient and therefore such a generalization preserves the structural properties of the equation, a point which we will return to later. This aspect has been important in the development of $L^1$ fractional Sobolev inequalities in terms of (1.1) in [11], as such inequalities are known to be false for the fractional Laplacian.

In this paper we continue to develop this perspective of classical equations as a part of a continuous spectrum. In particular, we take the first step in addressing for this class of equations a question of fundamental importance in the second order case, that of regularity. As there are a number of possible assumptions one can make to investigate the question of regularity of $u$ that satisfies (1.3), let us further describe the hypothesis of interest to us. In addition to the ellipticity condition (1.2), we will assume $A$ is of vanishing mean oscillation.

**Definition 1.1** We define the semi-norm (on the space of functions of bounded mean oscillation)

$$[\varphi]_{BMO} := \sup_Q \int_Q |\varphi - \int_Q \varphi|.$$  

Then we define the space of functions of vanishing mean oscillation by

$$VMO(\mathbb{R}^N) := \{ C_0^\infty(\mathbb{R}^N) \}_{BMO}.$$  

The main result of this paper is the following theorem on the regularity of such equations with VMO coefficients.

**Theorem 1.2** Suppose that $A \in VMO(\mathbb{R}^N; \mathbb{R}^{N \times N})$ satisfies (1.2), that $G \in L^p(\mathbb{R}^N; \mathbb{R}^N)$ for some $1 < p < +\infty$ and $u \in H^s(\mathbb{R}^N)$ satisfies

$$\int_{\mathbb{R}^N} A(x) D^s u(x) \cdot D^s v(x) \, dx = \int_{\mathbb{R}^N} G \cdot D^s v$$

(1.4)
for all $v \in C_c^\infty(\Omega)$. Then $D^s u \in L_p^p(\Omega)$ and for any $K \subset \subset \Omega$ there exists a constant $C = C(K, \Omega, A, s, p) > 0$ such that

$$\|D^s u\|_{L_p^p(K; \mathbb{R}^N)} \leq C \left(\|G\|_{L_p^p(\mathbb{R}^N; \mathbb{R}^N)} + \|(-\Delta)^{\frac{s}{2}} u\|_{L^2(\mathbb{R}^N)}\right).$$

Here, $(-\Delta)^{\frac{s}{2}} u$ denotes the fractional Laplacian of $u$ of order $s$, which can be defined as a Fourier multiplier with symbol $(2\pi|\xi|)^s$, see [14, p. 117]. The fractional Laplacian is related to the fractional gradient via the identity

$$D^s u \equiv R(-\Delta)^{\frac{s}{2}} u,$$  \hspace{1cm} (1.5)

for $s \in (0, 1)$ and $u$ with sufficient smoothness and integrability, and where $R = DI_1$ is the vector-valued Riesz transform. In the sequel we take (1.5) as our definition of $D^s u$, which enables us to include the classical case $s = 1$ (and more generally $s > 1$ though one loses the interpretation of a fractional gradient in this range).

Our proof is based on the beautiful technique of Iwaniec and Sbordone, introduced in [8] for $u$ satisfying (1.4) with $v \in C_c^\infty(\mathbb{R}^N)$ and $s = 1$. We recall that in this setting they had shown [8, p. 186] that (1.4) has exactly one (up to a constant) solution with the estimate

$$\|D^s u\|_{L_p^p(\mathbb{R}^N; \mathbb{R}^N)} \leq C\|G\|_{L_p^p(\mathbb{R}^N; \mathbb{R}^N)}.$$

Comparing this with our result, one sees that the preservation of structure in the equation results in regularity that is completely analogous to the well-studied elliptic theory.

As a consequence of this result we can return to the question of regularity of solutions to (1.3). In particular, one can transform equation (1.3) into (1.4) by defining $G = I_s Rg$ (where one extends $g$ by zero outside $\Omega$), since one has

$$\int_{\mathbb{R}^N} g v \, dx = \int_{\mathbb{R}^N} I_s Rg \cdot (-\Delta)^{\frac{s}{2}} v \, dx = \int_{\mathbb{R}^N} G \cdot D^s v \, dx$$

for $v \in C_c^\infty(\mathbb{R}^N)$ and $g \in L^2(\mathbb{R}^N)$. The assumption $g \in L^2(\Omega)$ then implies that $G \in L^{2N/(N-2s)}(\mathbb{R}^N; \mathbb{R}^N)$, and so our result allows us to conclude that for the solution to (1.3) we have for every $K \subset \subset \Omega$ the estimate

$$\|D^s u\|_{L^{2N/(N-2s)}(K; \mathbb{R}^N)} \leq C \left(\|g\|_{L^2(\Omega)} + \|(-\Delta)^{\frac{s}{2}} u\|_{L^2(\mathbb{R}^N)}\right).$$

When $s = 1$ this localizes the result of Iwaniec and Sbordone and can be compared with a result of Di Fazio in [5] (who in fact obtains regularity up to the boundary).

## 2 Estimates and proof of the Main Result

The main tool we utilize is the following result of Iwaniec and Sbordone [8, see p. 187, 201-206].
Theorem 2.1 (Iwaniec, Sbordone) Let $A \in VMO \cap L^\infty(\mathbb{R}^N; \mathbb{R}^{N \times N})$ satisfy (1.2). Then for all $1 < q < +\infty$, the operator

$$T := R_i A_{ij} R_j : L^q(\mathbb{R}^N) \to L^q(\mathbb{R}^N)$$

is invertible, and moreover, there exists $C = C(A, q) > 0$ such that

$$\|f\|_{L^q(\mathbb{R}^N)} \leq C \|Tf\|_{L^q(\mathbb{R}^N)}$$

(2.1)

for all $f \in L^q(\mathbb{R}^N)$.

From this we obtain the localization:

Proposition 2.2 Let $A, T$ as in Theorem 2.1. Then for any $\Omega_1, \Omega_2$ open and bounded with $\Omega_1 \subset \subset \Omega_2$, $2 < q < +\infty$, there exists $C = C(A, q, \Omega_1, \Omega_2) > 0$ such that

$$\|f\|_{L^q(\Omega_1)} \leq C (\|Tf\|_{L^q(\Omega_2)} + \|f\|_{L^2(\mathbb{R}^N)})$$

for all $f \in L^2(\mathbb{R}^N)$.

Before proving Proposition 2.2, let us recall the following commutator estimate, whose proof we provide for the convenience of the reader.

Proposition 2.3 Let $b, f : \mathbb{R}^N \to \mathbb{R}$ and define the commutator $C(b, R_i)[f]$ by

$$C(b, R_i)[f] := b R_i[f] - R_i[bf],$$

where $R_i$ is the $i$-th Riesz transform. If $b$ is Lipschitz, then

$$\|C(b, R_i)[f]\|_{L^p(\mathbb{R}^N)} \leq C[b]_{Lip(\mathbb{R}^N)} \|I_1|f|\|_{L^p(\mathbb{R}^N)}.$$ 

Proof. Since

$$R_i g(x) = c_N \int_{\mathbb{R}^N} \frac{x_i - z_i}{|x - z|^{N+1}} g(z) \, dz,$$

we have

$$C(b, R_i)[f](x) = c_N \int_{\mathbb{R}^N} \frac{x_i - z_i}{|x - z|^{N+1}} (b(x) - b(z)) f(z) \, dz,$$

and consequently,

$$|C(b, R_i)[f](x)| \leq c_N [b]_{Lip(\mathbb{R}^N)} \int_{\mathbb{R}^N} |x - z|^{-N+1} |f(z)| \, dz = C[b]_{Lip(\mathbb{R}^N)} |I_1| f|(x).$$

Proof of Proposition 2.2. Let $\eta \in C_0^\infty(\Omega_2)$ be a usual cutoff function, i.e. $\eta \geq 0$ and $\eta \equiv 1$ on a neighbourhood of $\Omega_1$. From (2.1) we have

$$\|f\|_{L^q(\Omega_1)} \leq \|\eta f\|_{L^q(\mathbb{R}^N)} \leq C \|T(\eta f)\|_{L^q(\mathbb{R}^N)}.$$

Let us now recall the definition of the commutator of an operator $T$ and two functions $b, f$ (which can be thought of as the error term to a product rule). We have

$$C(b, T)[f] := b T[f] - T[bf].$$
Then we continue the preceding estimate as follows. For \( \text{supp}\ \eta \subset K_0 \subset L_1 \subset \Omega_2 \) and denoting \( \chi_{L_1} \) the characteristic function of \( L_1 \), we estimate

\[
\|T(\eta f)\|_{L^q(\mathbb{R}^N)} = \|T(\eta \chi_{L_1} f)\|_{L^q(\mathbb{R}^N)}
\]

\[
\leq \|\eta T(\chi_{L_1} f)\|_{L^q(\mathbb{R}^N)} + \|\mathcal{C}(\eta, T)[\chi_{L_1} f]\|_{L^q(\mathbb{R}^N)}
\]

\[
\leq \|T(\chi_{L_1} f)\|_{L^q(K_0)} + \|\mathcal{C}(\eta, T)[\chi_{L_1} f]\|_{L^q(\mathbb{R}^N)}
\]

\[
\leq \|T(f)\|_{L^q(K_0)} + \mathcal{C}(\eta, T)[\chi_{L_1} f]\|_{L^q(\mathbb{R}^N)} + \|\mathcal{C}(\eta, T)[\chi_{L_1} f]\|_{L^q(\mathbb{R}^N)}
\]

\[
= \|T(f)\|_{L^q(K_0)} + I + II.
\]

Note that in the above with our \( T \) we have

\[
\mathcal{C}(\eta, T)[\chi_{L_1} f] = R_i A_{ij} \mathcal{C}(\eta, R_j)[\chi_{L_1} f] + \mathcal{C}(\eta, R_i)[A_{ij} R_j(\chi_{L_1} f)].
\]

As for \( I \), since the supports of \( L_1^c \) and \( K_0 \) are disjoint, we have the estimate

\[
\|T(\chi_{L_1^c} f)\|_{L^q(K_0)} \leq \|A\|_{\infty} \mathcal{C}_{K_0, L_1} f\|_{L^2(\mathbb{R}^N)} \tag{2.2}
\]

Indeed, let \( \tilde{K} \) be so that \( K_0 \subset \subset \tilde{K} \subset \subset L_1 \). Then by the boundedness of the Riesz transform on \( L^q(\mathbb{R}^N) \),

\[
\|T(\chi_{L_1^c} f)\|_{L^q(K_0)} \leq \|R_i(\chi_{\tilde{K}} A_{ij} R_j(\chi_{L_1^c} f))\|_{L^q(K_0)} + \|R_i(\chi_{\tilde{K}} A_{ij} R_j(\chi_{L_1^c} f))\|_{L^q(K_0)}
\]

\[
\leq \|A\|_{L^\infty(\mathbb{R}^N)} \|R_i(\chi_{L_1^c} f)\|_{L^q(\tilde{K})} + \|R_i(\chi_{\tilde{K}} R_j(\chi_{L_1^c} f))\|_{L^q(\tilde{K})}
\]

We now apply the Cauchy-Schwarz inequality to obtain

\[
\|R_j(\chi_{L_1^c} f)\|_{L^q(\tilde{K})} = \left( \int_{K_0} \left| \int_{\mathbb{R}^N \setminus L_1} f(y) \frac{x_j - y_j}{|x - y|^{N+1}} dy \right|^q dx \right)^{1/2}
\]

\[
\leq \left( \int_{K_0} \|f\|_{L^2(\mathbb{R}^N)} \left( \int_{\mathbb{R}^N \setminus L_1} \frac{1}{|x - y|^{N+1}} dy \right)^{q/2} dx \right)^{1/2}
\]

\[
\leq C|K_0|^{1/2} \|f\|_{L^2(\mathbb{R}^N)} \left( \int_{\mathbb{R}^N} \frac{1}{t^{N+1}} dt \right)^{q/2}
\]

\[
\leq C_{K_0, L_1, q}\|f\|_{L^2(\mathbb{R}^N)},
\]

where we have used the disjointness of \( K_0 \) and \( L_1^c \) (in particular that \( \text{dist}(K_0, L_1^c) = c > 0 \)). A similar argument shows that

\[
\|R_i(\chi_{\tilde{K}} A_{ij} R_j(\chi_{L_1^c} f))\|_{L^q(K_0)} \leq C_{\tilde{K}, L_1, q}\|A_{ij} R_j(\chi_{L_1^c} f))\|_{L^2(\mathbb{R}^N)},
\]

and so using the boundedness of the Riesz transform on \( L^2(\mathbb{R}^N) \), we conclude that

\[
\|R_i(\chi_{\tilde{K}} A_{ij} R_j(\chi_{L_1^c} f))\|_{L^q(K_0)} \leq C_{\tilde{K}, K_0, q}\|A\|_{L^\infty(\mathbb{R}^N)} \|f\|_{L^2(\mathbb{R}^N)}.
\]

It thus remains to estimate \( II \). Let us begin by observing that the commutator estimates with a Lipschitz continuous function (see Proposition 2.3) imply that

\[
II = \|\mathcal{C}(\eta, T)[\chi_{L_1} f]\|_{L^q(\mathbb{R}^N)}
\]

\[
\leq C_\eta (\|I_1[\chi_{L_1} f]\|_{L^q(\mathbb{R}^N)} + \|I_1[A_{ij} R_j(\chi_{L_1} f)]\|_{L^q(\mathbb{R}^N)}).
\]
In particular, \( q > 2 \) implies that \( Nq/(N + q) > 1 \) and so \( I_1 : L^{Nq/(N + q)}(\mathbb{R}^N) \to L^q(\mathbb{R}^N) \) is bounded. Moreover, \( R_j : L^r(\mathbb{R}^N) \to L^r(\mathbb{R}^N) \) is bounded for \( 1 < r < +\infty \), which combined with the fact that \( A \in L^\infty(\mathbb{R}^N; \mathbb{R}^{N \times N}) \) (recall that \( N \geq 2 \)) implies that

\[
II \leq C \|f\|_{L^{Nq/(N + q)}(L_1)}.
\]

If we let \( L_0 := \Omega_1 \), then our estimates show that

\[
\|f\|_{L^{q_0}(L_0)} \leq C \left( \|T(f)\|_{L^{q_0}(K_0)} + \|f\|_{L^2(\mathbb{R}^N)} + \|f\|_{L^{q_1}(L_1)} \right)
\]

for \( q_i := Nq/(N + iq) \). Now, if \( q_1 \leq 2 \) then an application of Hölder’s inequality implies the desired result. Otherwise we iterate the previous argument by finding

\[
K_0 \subset \subset L_1 \subset \subset K_1 \subset \subset L_2 \subset \subset \ldots K_i \subset \subset L_{i+1} \subset \subset \Omega_2
\]

to obtain the estimate

\[
\|f\|_{L^q(\Omega_1)} \leq C \left( \sum_{i=0}^{j-1} \|T(f)\|_{L^{q_i}(K_i)} + \|f\|_{L^2(\mathbb{R}^N)} + \|f\|_{L^{q_{i+1}}(L_{i+1})} \right),
\]

provided \( q_{i+1} > 1 \) (in order that \( I_1 : L^{q_{i+1}}(\mathbb{R}^N) \to L^q(\mathbb{R}^N) \)). However, \( q_1 > 2 \) implies \( q_{i+1} > 1 \), and so we continue the iteration a finite number of times until we obtain that \( q_j \leq 2 \) for some \( j \in \mathbb{N} \). Then collecting the terms our estimate reads

\[
\|f\|_{L^q(\Omega_1)} \leq C \left( \sum_{i=0}^{j-1} \|T(f)\|_{L^{q_i}(K_i)} + \|f\|_{L^2(\mathbb{R}^N)} + \|f\|_{L^{q_{i+1}}(L_{i+1})} \right),
\]

from which the inequality (2.2) is a simple consequence of Hölder’s inequality, and thus the proposition is established.

Finally, we require the following result.

**Proposition 2.4** Let \( \Omega \subset \mathbb{R}^N \) be open and bounded, \( s \in [0, N) \), and \( 2 \leq p < +\infty \). Assume that for all \( \varphi \in C^\infty_c(\Omega) \),

\[
\int f(-\Delta)^{\frac{s}{2}} \varphi = \int h(-\Delta)^{\frac{s}{2}} \varphi.
\]

Then for \( \Omega_1 \subset \subset \Omega \), there exists a constant \( C = C(\Omega_1) \) such that

\[
\|f\|_{L^p(\Omega_1)} \leq C \left( \|h\|_{L^p(\mathbb{R}^N)} + \|f\|_{L^2(\mathbb{R}^N)} \right).
\]

**Proof.** Let \( \Omega_1 \subset \subset \Omega_2 \subset \subset \Omega \) and \( \varphi \in C^\infty_c(\Omega_2) \) be such that

\[
\|f\|_{L^p(\Omega_1)} \leq 2 \int f \varphi
\]

and \( \|\varphi\|_{L^{p'}(\mathbb{R}^N)} \leq 1 \).

We argue by first reducing to the case where the support of \( \varphi \) is a ball. We can accomplish this by covering \( \Omega_2 \) with finitely many balls \( B(x_j, r_j) \) of controlled overlap such that \( B(x_j, 4r_j) \subset \subset \Omega \), where the number of balls can
be taken to depends only on the distance of \( \Omega_1 \) to \( \Omega^c \). Then by subordinating a partition of unity to balls \( B(x_j, r_j) \) we can write

\[
\varphi = \sum_{j=1}^{l} \varphi_j
\]

with \( \text{supp} \varphi_j \subset B(x_j, r_j) \) for each \( j \) and \( \| \varphi_j \| \leq \| \varphi \| \). Then for \( j \) fixed we have

\[
\int f \varphi_j = 2 \int f(-\Delta)\hat{\varphi} I_s \varphi_j = 2 \int f(-\Delta)(\eta_j I_s \varphi) + 2 \int f(-\Delta)((1 - \eta_j) I_s \varphi_j) \\
\leq 2 \left( \| h \|_{L^p(\mathbb{R}^n)} \| (-\Delta)\hat{\varphi} \|_{L^{p'}(\mathbb{R}^n)} + \| f \|_{L^2(\mathbb{R}^n)} \| (-\Delta)(1 - \eta_j) I_s \varphi \|_{L^2(\mathbb{R}^n)} \right),
\]

where \( \eta_j \in C_c^{\infty}(\Omega) \) with \( \eta \equiv 1 \) on \( B(x_j, 4r_j) \). Then if we can establish the estimates

\[
\| (-\Delta)\hat{\varphi} \|_{L^{p'}(\mathbb{R}^n)} \leq C \| \varphi \|_{L^{p'}(\mathbb{R}^n)} \quad (2.3)
\]

\[
\| (-\Delta)(1 - \eta_j) I_s \varphi \|_{L^2(\mathbb{R}^n)} \leq C \| \varphi \|_{L^{p'}(\mathbb{R}^n)}, \quad (2.4)
\]

the result will follow by summing in \( j \) and using the pointwise inequality \( |\varphi_j| \leq |\varphi| \).

Let us therefore first examine (2.3), and to save notation we drop the dependence in \( j \). If we take the three term commutator \( H_s \) introduced by Da Lio and Rivi\`ere [3]

\[
H_s(\eta, I_s \varphi) := (-\Delta)\hat{\varphi} \eta I_s \varphi - (-\Delta)\hat{\varphi} \eta I_s \varphi - \eta \varphi,
\]

we can use

\[
\| H_s(\eta, I_s \varphi) \|_{L^{p'}(\mathbb{R}^n)} \leq C \| \varphi \|_{L^{p'}(\mathbb{R}^n)}.
\]

This estimate follows via the Hardy-Littlewood decomposition in [3] or using the pointwise estimates in [10] (see [4, Theorem 1.2] for a precise version that can be applied here and also [1, 2] for various extensions). Thus, it suffices to show that

\[
\| (-\Delta)\hat{\varphi} \|_{L^{p'}(\mathbb{R}^n)} + \| \eta \varphi \|_{L^{p'}(\mathbb{R}^n)} \leq C \| \varphi \|_{L^{p'}(\mathbb{R}^n)}.
\]

The second term can be estimated in terms of the right hand side trivially since \( |\eta| \leq 1 \), while for the first term one applies H"older’s inequality with exponent \( Np'/(N - sp') \) and its H"older conjugate \( r \) when \( N - sp' > 0 \) (Note that from \( \eta \in C_c^{\infty}(\mathbb{R}^n) \) we know that \( (-\Delta)^{\frac{r}{2}} \eta \in L^r(\mathbb{R}^n) \) for any \( r \in (1, \infty) \) e.g. by interpolation), which yields

\[
\| (-\Delta)\hat{\varphi} \eta I_s \varphi \|_{L^{p'}(\mathbb{R}^n)} \leq \| (-\Delta)\hat{\varphi} \|_{L^{p'}(\mathbb{R}^n)} \| I_s \varphi \|_{L^{p'}(\mathbb{R}^n)} \| I_s \varphi \|_{L^{p'/q}(\mathbb{R}^n)} \leq C \| \varphi \|_{L^{p'}(\mathbb{R}^n)}.
\]
If \( N - sp' < 0 \), then
\[
\|(-\Delta)^{\frac{1}{2}}\eta I_s \varphi\|_{L^{p'}(\mathbb{R}^N)} \leq \|(-\Delta)^{\frac{1}{2}}\eta\|_{L^p(\mathbb{R}^N)} \|I_s \varphi\|_{L^{\infty}(\mathbb{R}^N)} \\
\leq C\|\varphi\|_{L^{p'}(\mathbb{R}^N)}
\]
follows from the fact that \( \varphi \) has compact support. When \( N - sp' = 0 \), we take \( p' < p \) and set \( \frac{1}{p'} = \frac{1}{p} - \frac{1}{p} \), then
\[
\|(-\Delta)^{\frac{1}{2}}\eta I_s \varphi\|_{L^{p'}(\mathbb{R}^N)} \leq \|(-\Delta)^{\frac{1}{2}}\eta\|_{L^p(\mathbb{R}^N)} \|I_s \varphi\|_{L^{p'}(\mathbb{R}^N)} \\
\leq C\|\varphi\|_{L^{p'}(\mathbb{R}^N)}.
\]
The estimate follows again in this case by the fact that \( \varphi \) has compact support. Finally, to establish (2.4) we write
\[
(1 - \eta) = \sum_{k=2}^{\infty} \theta_{A_k, r},
\]
where each \( \theta_{A_k, r} \) is supported on an annulus of width \( 2^kr \). Then disjoint support arguments (see, for example, Lemma 3.7 in [9]) imply the estimate
\[
\|(-\Delta)^{\frac{1}{2}}(\theta_{A_k, r} I_s \varphi)\|_{L^2(\mathbb{R}^N)} \leq C(2^k r)^{-N/2} 2^{N/p} \|\varphi\|_{L^{p'}(\mathbb{R}^N)},
\]
from which we obtain
\[
\|(-\Delta)^{\frac{1}{2}}((1 - \eta) I_s \varphi)\|_{L^2(\mathbb{R}^N)} \leq \sum_{k=2}^{\infty} \|(-\Delta)^{\frac{1}{2}}(\theta_{A_k, r} I_s \varphi)\|_{L^2(\mathbb{R}^N)} \\
\leq \left( C \sum_{k=2}^{\infty} (2^k r)^{-N/2} 2^{N/p} \right) \|\varphi\|_{L^{p'}(\mathbb{R}^N)}.
\]
As the series is summable we have established the desired inequality and therefore the theorem is proved. \( \blacksquare \)

We are now ready to prove the main result.

**Proof of Theorem 1.2.** Suppose \( G \in L^p(\mathbb{R}^N; \mathbb{R}^N) \) and \( u \in H^s(\mathbb{R}^N) \) satisfies the equation (1.4). The claim of this theorem is that for any \( K \subset \subset \Omega \), one has the estimate
\[
\|D^s u\|_{L^p(K; \mathbb{R}^N)} \leq C \left( \|G\|_{L^p(\mathbb{R}^N; \mathbb{R}^N)} + \|(-\Delta)^{\frac{1}{2}} u\|_{L^2(\mathbb{R}^N)} \right).
\]

We will see that the result is a consequence of a combination of Propositions 2.2 and 2.4, and we argue as follows. Define \( g := R^* G = -\sum_{j=1}^N R_j G_j \), so that \( g \in L^p(\mathbb{R}^N) \) and \( u \) satisfies
\[
\int_{\Omega} T (-\Delta)^{\frac{1}{2}} u (-\Delta)^{\frac{1}{2}} \varphi = \int_{\Omega} g (-\Delta)^{\frac{1}{2}} \varphi \quad \forall \varphi \in C_c^\infty(\Omega),
\]
where \( T \) is as in Proposition 2.1. Moreover, a cutoff argument similar to those previously employed implies that if \( K \subset \subset \Omega_j \), then one has
\[
\|D^s u\|_{L^p(K; \mathbb{R}^N)} = \|R(-\Delta)^{\frac{1}{2}} u\|_{L^p(K; \mathbb{R}^N)} \\
\leq C \left( \|(-\Delta)^{\frac{1}{2}} u\|_{L^p(\Omega_j)} + \|(-\Delta)^{\frac{1}{2}} u\|_{L^2(\mathbb{R}^N)} \right),
\]

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and so this and boundedness of the Riesz transforms (to obtain bounds on \( g \) in terms of \( G \) in \( L^p \)) imply that it suffices to show the estimate

\[
\|(-\Delta)^{\frac{1}{2}} u\|_{L^p(\Omega_1)} \leq C \left( \|g\|_{L^p(\mathbb{R}^N)} + \|(-\Delta)^{\frac{1}{2}} u\|_{L^2(\mathbb{R}^N)} \right),
\]

for \( \Omega_1 \subset \subset \Omega \).

We first apply Proposition 2.2 with \( f = (-\Delta)^{\frac{1}{2}} u \) and for \( \Omega_1 \subset \subset \Omega_2 \subset \subset \Omega \) yielding

\[
\|(-\Delta)^{\frac{1}{2}} u\|_{L^p(\Omega_1)} \leq C \left( \|T(-\Delta)^{\frac{1}{2}} u\|_{L^p(\Omega_2)} + \|(-\Delta)^{\frac{1}{2}} u\|_{L^2(\mathbb{R}^N)} \right).
\]

Now Proposition 2.4 and boundedness of \( T : L^2(\mathbb{R}^N) \rightarrow L^2(\mathbb{R}^N) \) gives

\[
\|T(-\Delta)^{\frac{1}{2}} u\|_{L^p(\Omega_2)} \leq C \left( \|g\|_{L^p(\mathbb{R}^N)} + \|T(-\Delta)^{\frac{1}{2}} u\|_{L^2(\mathbb{R}^N)} \right)
\leq C \left( \|g\|_{L^p(\mathbb{R}^N)} + \|(-\Delta)^{\frac{1}{2}} u\|_{L^2(\mathbb{R}^N)} \right).
\]

Therefore, we find

\[
\|(-\Delta)^{\frac{1}{2}} u\|_{L^p(\Omega_1)} \leq C \left( \|g\|_{L^p(\mathbb{R}^N)} + \|(-\Delta)^{\frac{1}{2}} u\|_{L^2(\mathbb{R}^N)} \right),
\]

which is the thesis.

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References


