NEW EXISTENCE RESULTS FOR THE MEAN FIELD EQUATION ON COMPACT SURFACES VIA DEGREE THEORY

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ABSTRACT. We consider the following class of equations with exponential non-linearities on a closed surface Σ:

$$-\Delta u = \rho_1 \left(\frac{h\, e^u}{\int_\Sigma h\, e^u \, dV_g} - \frac{1}{|\Sigma|} \right) - \rho_2 \left(\frac{h\, e^{-u}}{\int_\Sigma h\, e^{-u} \, dV_g} - \frac{1}{|\Sigma|} \right),$$

which arises as the mean field equation of the equilibrium turbulence with arbitrarily signed vortices. Here h is a smooth positive function and ρ_1, ρ_2 two positive parameters. By considering the parity of the Leray-Schauder degree associated to the problem, we prove solvability for $\rho_i \in (8\pi k, 8\pi(k+1))$, $k \in \mathbb{N}$. Our theorem provides a new existence result in the case when the underlying manifold is a sphere and gives a completely new proof for other known results.

1. INTRODUCTION

We are concerned with the equation

$$(1) \qquad -\Delta u = \rho_1 \left(\frac{h\, e^u}{\int_\Sigma h\, e^u \, dV_g} - \frac{1}{|\Sigma|} \right) - \rho_2 \left(\frac{h\, e^{-u}}{\int_\Sigma h\, e^{-u} \, dV_g} - \frac{1}{|\Sigma|} \right) \quad \text{on } \Sigma,$$

where $\Delta = \Delta_g$ is the Laplace-Beltrami operator, ρ_1, ρ_2 are two non-negative parameters, $h : \Sigma \to \mathbb{R}$ is a smooth positive function and Σ is a compact orientable surface without boundary with Riemannian metric g and total volume $|\Sigma|$. For the sake of simplicity, we will assume throughout this work that $|\Sigma| = 1$, which is no loss of generality thanks to a trivial rescaling argument.

Problem (1) plays an important role in mathematical physics as a mean field equation for the equilibrium turbulence with arbitrarily signed vortices. It was first obtained by Joyce and Montgomery [16] and by Pointin and Lundgren [29] by means of different statistical arguments. Later, several authors adopted this model; we refer for example to [9], [22], [25], [27] and the references therein. The case $\rho_1 = \rho_2$ is also related to the study of constant mean curvature surfaces, see [33], [34].

Before describing the main features of the problem and the known results, let us first consider the case $\rho_2 = 0$, namely the following standard Liouville-type equation:

$$(2) \qquad -\Delta u = \rho \left(\frac{h\, e^u}{\int_\Sigma h\, e^u \, dV_g} - 1 \right).$$

Equation (2) appears in conformal geometry in the problem of finding a conformal metric for which the Gauss curvature is a prescribed function on Σ, see [1], [6], [7],

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[20], [30]. Indeed, setting $\tilde{g} = e^{2v}g$, the Laplace-Beltrami operator for the deformed metric is given by $\Delta_{\tilde{g}} = e^{-2v}\Delta_g$ and the change of the Gauss curvature is ruled by

$$-\Delta_g v = K_{\tilde{g}} e^{2v} - K_g,$$

where K_g and $K_{\tilde{g}}$ are the Gauss curvatures of (Σ, g) and of $(\Sigma, \tilde{g})$ respectively. Problem (2) also arises in mathematical physics as a mean field equation of Euler flows, see [5], [17]. The literature on (2) is broad, and there are many results regarding existence, blow-up analysis, compactness of solutions, etc, see [10], [11], [24], [31].

As many geometric problems, also (2) carries a lack of compactness, as its solutions might blow-up. It was proved in [4], [19] and [21] that a quantization phenomenon occurs in this case. More precisely, taking a blow-up point p for a sequence $(u_n)_n$ of solutions, we have

$$(3) \qquad\qquad \lim_{r \to 0} \lim_{n \to +\infty} \int_{B_r(p)} h\, e^{u_n} dV_g = 8\pi.$$

Roughly speaking, each blow-up point carries a quantized local mass. Moreover, the limit profile of solutions becomes close to a *bubble*, namely a function $U_{\lambda,p}$ defined as

$$U_{\lambda,p}(y) = \log\left(\frac{4\lambda}{\left(1 + \lambda\, d(p,y)^2\right)^2} \right),$$

where $y \in \Sigma$, $d(p,y)$ stands for the geodesic distance and λ is a large parameter. This limit function can be viewed as the logarithm of the conformal factor of the stereographic projection from S^2 onto $\mathbb{R}^2$, composed with a dilation.

Combining the local quantization (3) with some further analysis, see for example [3], [4], we have that the set of solutions to (2) is uniformly bounded in $C^{2,\alpha}$, for any fixed $\alpha \in (0,1)$, provided $\rho \notin 8\pi\mathbb{N}$. It follows that one can define the Leray-Schauder degree associated to problem (2) with $\rho \in (8k\pi, 8(k+1)\pi), k \in \mathbb{N}$. In [19] it was shown that the degree is 1 when $\rho < 8\pi$. By the homotopic invariance of the degree, it is easy to see that the same is independent of the function h, the metric of Σ and it is constant on each interval $(8k\pi, 8(k+1)\pi)$. In fact it depends only on $k \in \mathbb{N}$ and the topological structure of Σ, as was proved in [8], where the authors provide the degree-counting formula

$$(4) \qquad\qquad \deg(\rho) = \frac{1}{k!}(-\chi(\Sigma) + 1) \cdots (-\chi(\Sigma) + k),$$

where $\chi(\Sigma)$ denotes the Euler characteristic of Σ. The proof of this result is carried out by analyzing the jump values of the degree after ρ crossing the critical thresholds. Later, this result was rephrased in [23] with a Morse theory point of view.

On the other hand, in the general case when $\rho_2 \neq 0$, namely for equation (1), there are fewer results: for example the refined blow-up analysis of solutions of (1) is not yet fully developed. Nevertheless, it was proven in [15] that the blow-up phenomenon yields a quantization property; for a blow-up point p and a sequence $(u_n)_n$ of solutions relatively to $(\rho_{1,n}, \rho_{2,n})$ the authors obtained

$$\lim_{r \to 0} \lim_{n \to +\infty} \rho_{1,n} \frac{\int_{B_r(p)} h\, e^{u_n}\, dV_g}{\int_\Sigma h\, e^{u_n}\, dV_g} \in 8\pi\mathbb{N}, \qquad \lim_{r \to 0} \lim_{n \to +\infty} \rho_{2,n} \frac{\int_{B_r(p)} h\, e^{-u_n}\, dV_g}{\int_\Sigma h\, e^{-u_n}\, dV_g} \in 8\pi\mathbb{N}.$$

Moreover, the case of positive multiples of 8π may indeed occur, see [12] and [13]. The latter volume quantization implies that the set of solutions is compact for $(\rho_1, \rho_2) \notin (8\pi\mathbb{N} \times \mathbb{R}) \cup (\mathbb{R} \cup 8\pi\mathbb{N})$.

It follows that, as before, the degree associated to (1) can still be defined outside this set of parameters. However, this strategy has not been yet investigated and the existence results mostly rely on a variational approach.

Let us now briefly discuss the history of the problem and the previous results. First of all, the case $\rho_1, \rho_2 < 8\pi$ was considered in [28]. Ohtsuka and Suzuki proved that in this regime the associated energy functional is bounded from below and coercive and therefore solutions can be found as global minima. If $\rho_i > 8\pi$ for some $i = 1, 2$, then the problem becomes subtler as the functional is not bounded and a minimization technique is not possible any more. The first result in this direction is given in [15] for $\rho_1 \in (8\pi, 16\pi)$ and $\rho_2 < 8\pi$. Via a blow-up analysis Jost et al. proved the existence of solutions on a smooth, bounded, non-simply connected domain in $\mathbb{R}^2$ with homogeneous Dirichlet boundary condition. Later, in [32], Zhou generalized this result to any compact surface without boundary by reducing the problem to the one with $\rho_2 = 0$ and applying min-max theory.

The next step was made in [14], where for the first time a doubly supercritical regime is considered, namely $\rho_i \in (8\pi, 16\pi)$ for $i = 1, 2$. The existence of solutions is proved using a min-max scheme based on a detailed description of the low sublevels of the energy functional. Finally, this was generalized in [2] for a generic choice of the parameters $\rho_1 \in (8k\pi, 8(k+1)\pi)$ and $\rho_2 \in (8l\pi, 8(l+1)\pi)$, $k, l \in \mathbb{N}$, under the assumption that Σ has positive genus.

2. The main result

We attack the problem with a different point of view and for the first time we analyze the associated Leray-Schauder degree. This is done in the spirit of [26], where a Toda system of Liouville equations arising from Chern-Simons theory was analyzed. More precisely, we study its parity and we observe that when both parameters stay in the same interval, i.e. $\rho_i \in (8k\pi, 8(k+1)\pi)$, $k \in \mathbb{N}$ for $i = 1, 2$, the degree is always odd. The main result is the following.

Theorem 2.1. *Let $h > 0$ be a smooth function and suppose $\rho_i \in (8k\pi, 8(k+1)\pi)$, $k \in \mathbb{N}$ for $i = 1, 2$. Then problem (1) has a solution.*

Observe that we recover the result of [14] and some cases of [2]: when Σ is homeomorphic to S^2 the above theorem yields a new existence result.

PROOF. For some $\alpha \in (0, 1)$ let $C_0^{2,\alpha}(\Sigma)$ be the class of $C^{2,\alpha}$ functions with zero average. Consider now the mapping $T : C_0^{2,\alpha}(\Sigma) \to C_0^{2,\alpha}(\Sigma)$ defined by

$$(5) \qquad T(u) = (-\Delta)^{-1} \left(\rho_1 \left(\frac{h\,e^u}{\int_\Sigma h\,e^u\,dV_g} - 1 \right) - \rho_2 \left(\frac{h\,e^{-u}}{\int_\Sigma h\,e^{-u}\,dV_g} - 1 \right) \right),$$

where $(-\Delta)^{-1}f, f \in C^\alpha(\Sigma)$, is intended as the solution v, with zero average, of the problem $-\Delta v = f$, which is unique. We are concerned with the map $\Psi = Id - T$ and the solutions of equation (1) will correspond to zeros of Ψ.

Clearly, by elliptic regularity theory the operator T is compact. Moreover, the set of the solutions is compact for parameters $(\rho_1, \rho_2) \notin (8\pi\mathbb{N} \times \mathbb{R}) \cup (\mathbb{R} \times 8\pi\mathbb{N})$.

Therefore, we can consider the associated degree $\deg\big(\Psi_{(\rho_1,\rho_2)}, B_r(0), 0\big)$ which is well-defined for r sufficiently large.

Consider now $\rho_i \in (8k\pi, 8(k+1)\pi)$, $k \in \mathbb{N}$ for $i = 1, 2$. Letting $\rho = \frac{1}{2}(\rho_1 + \rho_2)$, we perform the following homotopy which takes place in a connected component of $\mathbb{R}^2 \setminus ((8\pi\mathbb{N} \times \mathbb{R}) \cup (\mathbb{R} \times 8\pi\mathbb{N}))$:

$$\mathfrak{h}(t) = (1-t)(\rho_1, \rho_2) + t(\rho, \rho).$$

From the fact that the degree is constant along homotopies we obtain that

$$\deg\big(\Psi_{(\rho_1,\rho_2)}, B_r(0), 0\big) = \deg\big(\Psi_{(\rho,\rho)}, B_r(0), 0\big).$$

Observe now that by the structure of T we deduce

$$\Psi_{(\rho,\rho)}(-u) = -\Psi_{(\rho,\rho)}(u).$$

Therefore, we conclude that $\Psi_{(\rho,\rho)}$ is an odd operator. By the Borsuk theorem, see [18], it follows that the associated degree is odd an hence non zero. This guarantees us the existence of a solution to equation (1). ■

Acknowledgements

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