A NOTE ON THE CONVERGENCE OF SINGULARLY PERTURBED SECOND ORDER POTENTIAL-TYPE EQUATIONS

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ABSTRACT. In this paper we study the limit as $\varepsilon \to 0$ of the singularly perturbed second order equation $\varepsilon^2 \ddot{u}_\varepsilon + \nabla_x V(t, u_\varepsilon(t)) = 0$, where $V(t, x)$ is a potential. We assume that $u_0(t)$ is one of its equilibrium points such that $\nabla_x V(t, u_0(t)) = 0$ and $\nabla_x^2 V(t, u_0(t)) > 0$. We find that, under suitable initial data, the solutions u_ε converge uniformly to u_0, by imposing mild hypotheses on V. A counterexample shows that they cannot be weakened.

INTRODUCTION

A problem of interest in various areas of applied mathematics is to find stable equilibrium points for time-dependent energies. In a simplified setting, the problem is to find an evolution $t \to u(t)$ such that

$$\begin{cases} \nabla_x V(t, u(t)) = 0, \\ \nabla_x^2 V(t, u(t)) > 0, \end{cases} \tag{1}$$

where $V(t, x)$ is a potential, ∇_x denotes the gradient with respect to x, and ∇_x^2 the corresponding Hessian. This problem can be locally solved by means of the Implicit Function Theorem, which provides a smooth solution defined in a neighborhood of $t = 0$.

Problem (1) has also been studied in finite dimension as the limit case of ε-gradient flows. A first general result was given by C. Zanini in [15], where the author studies the system

$$\varepsilon \dot{u}_\varepsilon(t) + \nabla_x V(t, u_\varepsilon(t)) = 0. \tag{2}$$

In [15] it is proved that the solutions $u_\varepsilon(t)$ to (2) converge to a solution $u(t)$ to (1), obtained by connecting smooth branches of solutions to the equilibrium equation (1) through suitable heteroclinic solutions of the ε-gradient flows (2).

In [1] V. Agostiniani analysed the second order approximation with a dissipative term:

$$\varepsilon^2 A \ddot{u}_\varepsilon(t) + \varepsilon B \dot{u}_\varepsilon(t) + \nabla_x V(t, u_\varepsilon(t)) = 0, \tag{3}$$

where A and B are positive definite and symmetric matrices. It turns out that $(u_\varepsilon, \varepsilon B \dot{u}_\varepsilon) \to (u, 0)$, where u is piecewise continuous and satisfies (1). Moreover the behaviour of the system at jump times is described by trajectories connecting the states before and after the jumps; such trajectories are given by a suitable autonomous second order system related to A, B, and $\nabla_x V$.

We remark that studying the asymptotic behaviour of solutions, as $\varepsilon \to 0$, in systems of the form (3) with $A \neq 0$ and $B = 0$ (vanishing inertia), or $A = 0$ and

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$B \neq 0$ (vanishing viscosity), or $A, B \neq 0$ (vanishing viscosity and inertia), may give a selection principle for quasistatic evolutions (namely those evolutions whose loading is assumed to be so slow that at every time the system is at equilibrium and internal oscillation can be neglected). This approach has been succesfully adopted in various situations in the case of vanishing viscosity (cf. e.g. [11, 10, 8, 2, 6, 3, 7]) and in the case of vanishing viscosity and inertia (cf. e.g. [12, 13, 5, 14, 9]). We remark that in [9] viscosity can be neglected under suitable assumptions.

The above mentioned results [1, 15] require strong smoothness assumptions on V (C^3-regularity is required). The aim of the present paper is to weaken the assumptions under which second order perturbed problems converge to (1). More precisely, we consider a second order equation of the form (3) without the dissipative term $B\dot{u}_\varepsilon$. (Notice that in general, when $B > 0$, it is easier to prove the convergence of solutions.) We therefore study the asymptotic behaviour of the solutions $u_\varepsilon(t)$ of the problem

$$\varepsilon^2 \ddot{u}_\varepsilon(t) + \nabla_x V(t, u_\varepsilon(t)) = 0 \tag{4}$$

to a continuous stable equilibrium $u_0(t)$ of (1). Our main result is that the convergence $u_\varepsilon \to u_0$ still holds under some regularity and growth conditions on V that are weaker than those required in [1, 15]. Furthermore we provide a counterexample to that convergence when such assumptions do not hold.

More precisely we require continuity for V in both variables and we assume that $V(t, \cdot) \in C^2$. We also suppose that there is a function $V_t(t, x)$ of class C^1-Carathéodory (i.e., $V_t(\cdot, x)$ is measurable and $V_t(t, \cdot)$ is of class C^1) such that

$$V(t_2, x) - V(t_1, x) = \int_{t_1}^{t_2} V_t(t, x) \, \mathrm{d}t,$$

for a.e. t_1, t_2. With some further boundedness conditions on V (listed in Section 1) we prove that $u_0(t)$ is absolutely continuous and we obtain the convergence result, see Theorem 2.3 in Section 2. Specifically, we find that solutions to (4) satisfy

$$u_\varepsilon \to u_0 \text{ uniformly and } \varepsilon \|\dot{u}_\varepsilon - \dot{u}_0\|_{L^1} \to 0 \tag{5}$$

as $\varepsilon \to 0$.

In Section 3 we show that, if we weaken the assumptions on V, we are not able to get (5). More precisely we provide a counterexample for a model case where the time-dependent energy is given by

$$V(t, x) := \frac{1}{2}|x - u_0(t)|^2.$$

We remark that, when $u_0 \in W^{1,1}(0, T)$, then V in its turn satisfies the assumptions of Section 2. In this case, solutions u_ε of (4) converge uniformly to $u_0(t)$. On the other hand we show that, if u_0 is the Cantor-Vitali function, then (5) can not be satisfied (see Example 3.3). In fact we prove that no subsequences of solutions to (4) could converge to u_0 and that the continuous functions u_0 with this property are infinitely many (see Proposition 3.1 and Remark 3.2).

This result thus shows a case in which the dynamic solutions do not converge to the expected (quasistatic) equilibrium. A similar phenomenon was observed in [4] where the authors give an example of non-convergence, in the context of a one-dimensional peeling test without viscosity: more precisely, their dynamic solutions converge to a limit that does not fulfill first order stability. Our non-convergence result in Section

3 can therefore be regarded as an example in which, in the absence of a damping viscous term, dynamic solutions do not converge to stable equilibria even in very simple situations.

1. SETTING OF THE PROBLEM

Let $V \colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ be a continuous function such that $V(t, \cdot) \in C^2(\mathbb{R}^n)$. It will play the role of a time-dependent energy. We assume that there exists a function $u_0 \in C^0([0, T]; \mathbb{R}^n)$ such that the following properties are satisfied:

$$\nabla_x V(t, u_0(t)) = 0, \quad \text{for every } t \in [0, T], \tag{1.1}$$

$$\exists\, \alpha > 0 \colon \nabla_x^2 V(t, u_0(t)) \xi \cdot \xi \geq \alpha |\xi|^2, \text{ for all } \xi \in \mathbb{R}^n. \tag{1.2}$$

Furthermore, for a.e. $t \in [0, T]$ and for every $x \in \mathbb{R}^n$, we assume that there is a constant $A > 0$ such that

$$|\nabla_x V(t, x)|, \ |\nabla_x^2 V(t, x)| \leq A. \tag{1.3}$$

We also assume that there exists a C^1-Carathéodory function $V_t \colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$, i.e., a Carathéodory function such that $V_t(t, \cdot) \in C^1(\mathbb{R}^n)$, satisfying

$$V(t_2, x) - V(t_1, x) = \int_{t_1}^{t_2} V_t(t, x) \, \mathrm{d}t, \tag{1.4}$$

for a.e. $t_1, t_2 \in \mathbb{R}$ and all $x \in \mathbb{R}^n$. Moreover, for every $R > 0$, we require that there exists $a_R \in L^1(\mathbb{R})$ such that

$$|V_t(t, x)|, \ |\nabla_x V_t(t, x)| \leq a_R(t), \tag{1.5}$$

for a.e. $t \in \mathbb{R}$ and all $x \in B_R(0)$. We notice that, by condition (1.5), it is possible to prove that $\nabla_x V$ is continuous in both variables.

We consider, for fixed $\varepsilon > 0$, the Cauchy problem

$$\begin{cases} \varepsilon^2 \ddot{u}_\varepsilon + \nabla_x V(t, u_\varepsilon(t)) = 0, \\ u_\varepsilon(0) = u_\varepsilon^0, \\ \dot{u}_\varepsilon(0) = v_\varepsilon^0, \end{cases} \tag{1.6}$$

where we assume that

$$u_\varepsilon^0 \to u_0(0) = 0 \text{ and } \varepsilon v_\varepsilon(0) \to 0. \tag{1.7}$$

Global existence and uniqueness of the solutions u_ε to (1.6) are consequences of standard theorems on ordinary differential equations thanks to the continuity of $\nabla_x V$ and to condition (1.3). Our goal is to study when convergence, as $\varepsilon \to 0$, of solutions u_ε to (1.6) satisfying conditions (1.7) to u_0 is possible.

Using (1.3) and (1.5), we now study the dependence on x of the set of Lebesgue points for a function $t \mapsto f(t, x)$ which will then play the role of V_t and $\nabla_x V_t$.

Lemma 1.1. *Let $f \colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^m$ be a Carathéodory function such that, for every $R > 0$, there exists $a_R \in L^1(\mathbb{R})$ with $f(t, x) \leq a_R(t)$ for every $x \in B_R(0)$. Fix $x \in \mathbb{R}^n$, then for a.e. $t \in \mathbb{R}$*

$$\lim_{y \to x} \lim_{h \to 0} \frac{1}{h} \int_t^{t+h} [f(\tau, y) - f(t, y)] \, \mathrm{d}\tau = 0.$$

Proof. Let $t \in \mathbb{R}$ be a right Lebesgue point for $\tau \mapsto f(\tau, x)$, i.e.,

$$\lim_{h \to 0} \frac{1}{h} \int_t^{t+h} [f(\tau, x) - f(t, x)] \, \mathrm{d}\tau = 0.$$

Let $\delta > 0$ and define

$$\omega_R^\delta(\tau) := \sup_{\substack{x,y \in B_R(0) \\ |x-y| < \delta}} |f(\tau, x) - f(\tau, y)|. \tag{1.8}$$

By assumption we have that $\omega_R^\delta(\tau) \leq 2a_R(\tau)$; moreover ω_R^δ is measurable because the supremum can be taken over all rational points and along a sequence $\delta = 1/n$. Therefore $\omega_R^\delta(\cdot) \in L^1(\mathbb{R})$.

If t is also a right Lebesgue point for $\tau \mapsto \omega_R^\delta(\tau)$ for every $\delta \in \mathbb{Q}$, $\delta > 0$ and $|x - y| < \delta$, then

$$\lim_{h \to 0} \frac{1}{h} \int_t^{t+h} |f(\tau, x) - f(t, x) - (f(\tau, y) - f(t, y))| \, \mathrm{d}\tau$$

$$\leq \lim_{h \to 0} \frac{1}{h} \int_t^{t+h} \left[\omega_R^\delta(\tau) + \omega_R^\delta(t) \right] \, \mathrm{d}\tau = 2\omega_R^\delta(t). \tag{1.9}$$

Since $f(t, \cdot)$ is uniformly continuous in $\overline{B_R(0)}$, the last term in (1.9) tends to zero as $\delta \to 0$ for a.e. $t \in \mathbb{R}$. $\square$

Remark 1.2. Given any $u \in W^{1,1}(0, T; \mathbb{R}^n)$, we are now able to get a chain rule for a.e. $t \in [0, T]$, by differentiating $z(t) := V(t, u(t))$. Indeed, if t is a Lebesgue point for $\tau \mapsto V_t(\tau, u(t))$, by (1.4) we have

$$\frac{z(t+h) - z(t)}{h} = \frac{V(t+h, u(t+h)) - V(t, u(t+h))}{h} + \frac{V(t, u(t+h)) - V(t, u(t))}{h}$$

$$= \frac{1}{h} \int_t^{t+h} V_t(\tau, u(t+h)) \, \mathrm{d}\tau + \nabla_x V(t, \xi) \frac{u(t+h) - u(t)}{h}, \tag{1.10}$$

for some point ξ belonging to the segment $[u(t), u(t+h)]$, thanks to the Mean Value Theorem. We now re-write the first summand of (1.10) in the following form:

$$\frac{1}{h} \int_t^{t+h} V_t(\tau, u(t+h)) \, \mathrm{d}\tau$$

$$= \frac{1}{h} \int_t^{t+h} [V_t(\tau, u(t+h)) - V_t(t, u(t+h))] \, \mathrm{d}\tau + V_t(t, u(t+h)). \tag{1.11}$$

In view of Lemma 1.1, the integral in (1.11) tends to zero, for a.e. $t \in [0, T]$ as $h \to 0$: this is done by a diagonal argument using the fact that $u(t+h) \to u(t)$, because u is an absolutely continuous function. Moreover, by continuity of $V_t(t, u(\cdot))$, the second summand in (1.11) tends to $V_t(t, u(t))$. Therefore, as $h \to 0$ in (1.10), we get

$$\frac{\mathrm{d}}{\mathrm{d}t} V(t, u(t)) = \dot{z}(t) = V_t(t, u(t)) + \nabla_x V(t, u(t))\dot{u}(t), \tag{1.12}$$

for a.e. $t \in [0, T]$, because $\nabla_x V(t, \cdot)$ is continuous.

We now argue similarly for $\nabla_x V(t, u(t))$ and get a chain rule again. Since $V_t(t, \cdot) \in C^1(\mathbb{R}^n)$ for a.e. t and condition (1.5) holds, by an application of the Dominated

Convergence Theorem, we have that

$$\nabla_x V(t_2, x) - \nabla_x V(t_1, x) = \int_{t_1}^{t_2} \nabla_x V_t(t, x) \, \mathrm{d}t. \tag{1.13}$$

Therefore,

$$\frac{\nabla_x V(t + h, u(t + h)) - \nabla_x V(t, u(t))}{h}$$
$$= \frac{\nabla_x V(t + h, u(t + h)) - \nabla_x V(t, u(t + h))}{h} + \frac{\nabla_x V(t, u(t + h)) - \nabla_x V(t, u(t))}{h}$$
$$= \frac{1}{h} \int_t^{t+h} \nabla_x V_t(\tau, u(t + h)) \, \mathrm{d}\tau + \frac{\nabla_x V(t, u(t + h)) - \nabla_x V(t, u(t))}{h}. \tag{1.14}$$

Since $\nabla_x V_t(t, \cdot)$ is continuous and $\nabla_x V_t(\cdot, x)$ is measurable (indeed, it can be obtained as the limit along a sequence of measurable difference quotients), then $\nabla_x V_t$ is a Carathéodory function controlled by an integrable function $a_R(t)$. Arguing as before and recalling that $V(t, \cdot) \in C^2(\mathbb{R}^n)$, we have that for a.e. $t \in [0, T]$

$$\lim_{h \to 0} \frac{\nabla_x V(t + h, u(t + h)) - \nabla_x V(t, u(t))}{h} = \nabla_x V_t(t, u(t)) + \nabla_x^2 V(t, u(t)) \dot{u}(t).$$

In particular, since $\nabla_x V(t, u_0(t)) = 0$, we have

$$\nabla_x V_t(t, u_0(t)) + \nabla_x^2 V(t, u_0(t)) \dot{u}_0(t) = 0. \tag{1.15}$$

We are now in the position to state the following result which will enable us to restrict to the case of absolutely continuous functions throughout the sequel.

Proposition 1.3. *Let* $V \colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ *be a continuous function which satisfies* $V(t, \cdot) \in C^2(\mathbb{R}^n)$ *for a.e.* $t \in \mathbb{R}$. *Let* V_t *fulfill conditions* (1.4) *and* (1.5), *and let* $u_0 \colon [0, T] \to \mathbb{R}^n$ *be a continuous function such that there exists* $\alpha > 0 :$

$$\nabla_x^2 V(t, u_0(t)) \xi \cdot \xi \geq \alpha |\xi|^2, \tag{1.16}$$

for every $\xi \in \mathbb{R}^n$ *and for a.e.* $t \in [0, T]$. *Then,* u_0 *is absolutely continuous in* $[0, T]$.

Proof. We want to show that, if ε is small enough, there exists $\delta > 0$ such that, for a.e. $t_1, t_2 \in [0, T]$ with $|t_1 - t_2| < \delta$, there exists $M_\varepsilon > 0$ and an integrable function g such that

$$|u_0(t_2) - u_0(t_1)| \leq M_\varepsilon \int_{t_1}^{t_2} g(t) \, \mathrm{d}t. \tag{1.17}$$

We know that

$$0 = \nabla_x V(t_2, u_0(t_2)) - \nabla_x V(t_1, u_0(t_1))$$
$$= \nabla_x V(t_2, u_0(t_2)) - \nabla_x V(t_1, u_0(t_2)) + \nabla_x V(t_1, u_0(t_2)) - \nabla_x V(t_1, u_0(t_1))$$
$$= \nabla_x V(t_2, u_0(t_2)) - \nabla_x V(t_1, u_0(t_2)) + \nabla_x^2 V(t_1, y)(u_0(t_2) - u_0(t_1)),$$

where y is in the segment $[u_0(t_1), u_0(t_2)]$. Therefore we have

$$u_0(t_2) - u_0(t_1) = -\nabla_x^2 V(t_1, y)^{-1} [\nabla_x V(t_2, u_0(t_2)) - \nabla_x V(t_1, u_0(t_2))]. \tag{1.18}$$

Since $\nabla_x^2 V(t_1, \cdot)$ is continuous and satisfies the coercivity assumption (1.16), we can find $\varepsilon > 0$ such that, if $y \in B_\varepsilon(u_0(t_1))$ then

$$\nabla_x^2 V(t_1, y) \xi \cdot \xi \geq \frac{\alpha}{2} |\xi|^2.$$

We can thus invert $\nabla_x^2 V(t_1, \cdot)$ in a neighborhood of $u_0(t_1)$. Let λ_y be the minimum eigenvalue of $\nabla_x^2 V(t_1, y)$. Therefore we can control the norm of $\nabla_x^2 V(t_1, y)^{-1}$ with $1/\lambda_y$. If v_y is an eigenvector of $\nabla_x^2 V(t_1, y)$ with eigenvalue λ_y, then we have

$$\lambda_y |v_y|^2 = \nabla_x^2 V(t_1, y) v_y \cdot v_y \geq \frac{\alpha}{2} |v_y|^2,$$

from which we deduce that $\lambda_y \geq \frac{\alpha}{2}$ and therefore

$$\|\nabla_x^2 V(t_1, y)^{-1}\| \leq \frac{2}{\alpha}. \tag{1.19}$$

We can now plug (1.19) in (1.18) and, arguing as in (1.13) of the previous Remark, we get

$$|u_0(t_2) - u_0(t_1)| \leq \frac{2}{\alpha} \int_{t_1}^{t_2} |\nabla_x V_t(t, u_0(t_2))\, \mathrm{d}t| \leq \frac{2}{\alpha} \int_{t_1}^{t_2} a_R(t)\, \mathrm{d}t$$

and we have thus obtained (1.17). $\qquad \square$

2. Convergence of solutions

This section is devoted to the study of the convergence for solutions u_ε of problem (1.6). We will show that u_ε uniformly converges to u_0, which is the equilibrium for the potential V introduced in the previous section, provided initial conditions (1.7) are satisfied.

We recall here the standard Gronwall lemma which will be used as a main tool in the proof of the convergence.

Lemma 2.1 (Gronwall). *Let $\varphi \in L^\infty(\mathbb{R})$, $\varphi(t) \geq 0$ for a.e. $t \in \mathbb{R}$ and $a \in L^1(\mathbb{R})$, $a(t) \geq 0$ for a.e. $t \in \mathbb{R}$. We assume that there exists a constant $C > 0$ such that*

$$\varphi(t) \leq \int_0^t a(s)\varphi(s)\, \mathrm{d}s + C, \ \text{for a.e. } t \in \mathbb{R}.$$

Then,

$$\varphi(t) \leq C \exp\left(\int_0^t a(s)\, \mathrm{d}s\right), \ \text{for a.e. } t \in \mathbb{R}.$$

Remark 2.2. From now on assume that there exists $\psi \colon [0, +\infty) \to \mathbb{R}$ such that

$$\lim_{t \to +\infty} \psi(t) = +\infty \text{ and } V(t, x) \geq \psi(|x|), \tag{2.1}$$

for all $t \in [0, T]$ and $x \in \mathbb{R}^n$, and there exist $a(\cdot), b(\cdot) \in L^1(0, T)$ such that

$$V_t(t, x) \leq a(t) + b(t)V(t, x), \tag{2.2}$$

for a.e. $t \in [0, T]$ and for all $x \in \mathbb{R}^n$. Then it is easy to deduce uniform boundedness for the sequence $\{u_\varepsilon\}$, by applying Lemma 2.1 to the following energy estimate:

$$V(t, u_\varepsilon(t)) \leq \|a\|_{L^1(0,T)} + \int_0^T b(t)V(t, u_\varepsilon(t))\, \mathrm{d}t.$$

We remark that conditions (2.1) and (2.2), which are standard in this context, are not necessary for establishing our result if we already know that the sequence $\{u_\varepsilon\}$ is uniformly bounded.

We are now in the position to state the main result of this section.

Theorem 2.3. *Let V be a function fulfilling the assumptions of Proposition 1.3 and let $u_0 \in C^0([0,T]; \mathbb{R}^n)$ be such that $\nabla_x V(t, u_0(t)) = 0$ for every $t \in [0,T]$. Assume also that conditions (2.1) and (2.2) are satisfied and that $\nabla_x^2 V(t,x)$ and $\nabla_x V_t(t,x)$ are locally equi-Lipschitz in x, uniformly in t, i.e. for every $x \in \mathbb{R}^n$ there exists $\delta > 0$ and constants $C_1, C_2 > 0$ (which may depend on x), such that, for every $|h| < \delta$*

$$|\nabla_x^2 V(t, x + h) - \nabla_x^2 V(t, x)| \leq C_1 |h|,$$
$$|\nabla_x V_t(t, x + h) - \nabla_x V_t(t, x)| \leq C_2 |h|, \tag{2.3}$$

for a.e. $t \in [0,T]$. Let u_ε be a solution of the Cauchy problem

$$\begin{cases} \varepsilon^2 \ddot{u}_\varepsilon + \nabla_x V(t, u_\varepsilon(t)) = 0, \\ u_\varepsilon(0) = u_\varepsilon^0, \\ \dot{u}_\varepsilon(0) = v_\varepsilon^0, \end{cases} \tag{2.4}$$

where the initial data u_ε^0 and v_ε^0 are such that

$$u_\varepsilon^0 \to u_0(0) = 0 \ \text{and} \ \varepsilon v_\varepsilon^0 \to 0. \tag{2.5}$$

Then, $u_\varepsilon \to u_0$ uniformly in $[0,T]$ and $\varepsilon \|\dot{u}_\varepsilon - \dot{u}_0\|_{L^1} \to 0$, as $\varepsilon \to 0$.

Proof. We fix a sequence $\varepsilon_j \to 0$ and we prove convergence for u_{ε_j}: this will show convergence for the whole family $\{u_\varepsilon\}$ to u_0, by the arbitrariness of ε_j. However we will keep writing just u_ε for the sake of simplicity of notation.

By Proposition 1.3, we have that $u_0 \in W^{1,1}(0,T; \mathbb{R}^n)$. Since $C^2([0,T]; \mathbb{R}^n)$ is dense in $W^{1,1}(0,T; \mathbb{R}^n)$, for every $k \in \mathbb{N}$ there exists a sequence $\{u_0^k\} \subset C^2([0,T]; \mathbb{R}^n)$ such that

$$\|u_0 - u_0^k\|_{W^{1,1}} < \frac{1}{k}. \tag{2.6}$$

A suitable choice of k will take place in due course. However, we can already notice that, since $W^{1,1}(0,T; \mathbb{R}^n) \subset C^0([0,T]; \mathbb{R}^n)$, then u_0^k uniformly converges to u_0 in $[0,T]$ and therefore they are all contained in a compact set containing $\{u_0(t), t \in [0,T]\}$.

We now introduce a surrogate of energy estimate, multiplying the equation in (2.4) by $\dot{u}_\varepsilon(t) - \dot{u}_0^k(t)$. After an integration we get

$$\frac{\varepsilon^2}{2} |\dot{u}_\varepsilon(t) - \dot{u}_0^k(t)|^2 + V(t, u_\varepsilon(t))$$
$$= \frac{\varepsilon^2}{2} |\dot{u}_\varepsilon(0) - \dot{u}_0^k(0)|^2 + V(0, u_\varepsilon(0)) - \int_0^t \varepsilon^2 \ddot{u}_0^k(s)(\dot{u}_\varepsilon(s) - \dot{u}_0^k(s)) \, \mathrm{d}s$$
$$+ \int_0^t \left[V_t(s, u_\varepsilon(s)) + \nabla_x V(s, u_\varepsilon(s)) \dot{u}_0^k(s) \right] \mathrm{d}s \tag{2.7}$$

Our aim is thus to infer some lower and upper bounds for (2.7) in order to get, by Lemma 2.1, convergence of $u_\varepsilon - u_0^k$ and then deduce convergence to u_0. It is thus convenient to consider the following "shifted" potential $\tilde{V}$ defined as

$$\tilde{V}(t, x) := V(t, x) - V(t, u_0^k(t)). \tag{2.8}$$

Since u_0^k is of class C^2, then all regularity assumptions on V are inherited by $\tilde{V}$. We have, in particular, that

$$\tilde{V}_t(t, x) = V_t(t, x) - V_t(t, u_0^k(t)) - \nabla_x V(t, u_0^k(t)) \dot{u}_0^k(t). \tag{2.9}$$

Moreover it is easy to show that

$$\nabla_x \tilde{V}(t, u_0(t)) = 0, \ \forall t \in [0,T]. \tag{2.10}$$

We also notice that (2.7) is equivalent to

$$\frac{\varepsilon^2}{2}|\dot{u}_\varepsilon(t) - \dot{u}_0^k(t)|^2 + \tilde{V}(t, u_\varepsilon(t))$$

$$= \frac{\varepsilon^2}{2}|\dot{u}_\varepsilon(0) - \dot{u}_0^k(0)|^2 + \tilde{V}(0, u_\varepsilon(0)) - \int_0^t \varepsilon^2 \ddot{u}_0^k(s)(\dot{u}_\varepsilon(s) - \dot{u}_0^k(s))\,\mathrm{d}s$$

$$+ \int_0^t \tilde{V}_t(s, u_\varepsilon(s)) + \nabla_x \tilde{V}(s, u_\varepsilon(s))\dot{u}_0^k(s)\,\mathrm{d}s. \tag{2.11}$$

We set $A_\varepsilon := \frac{\varepsilon^2}{2}|\dot{u}_\varepsilon(0) - \dot{u}_0^k(0)|^2 + \tilde{V}(0, u_\varepsilon(0))$, which tends to 0 as $\varepsilon \to 0$, by the initial conditions (2.5) and because $u_0^k \to u_0$ uniformly in $[0, T]$.

We now subdivide the proof into parts obtaining estimates which will then be used in the final Gronwall argument.

Lower estimate. We look for a lower bound for the summand $\tilde{V}(t, u_\varepsilon)$ in the left hand side of (2.11). We have that, by first order expansion, there exists y in the segment $[0, x]$ such that

$$V(t, x + u_0(t)) = V(t, u_0(t)) + \nabla_x V(t, u_0(t))x + \nabla_x^2 V(t, y)x \cdot x$$

$$= V(t, u_0(t)) + \nabla_x^2 V(t, y)x \cdot x, \tag{2.12}$$

because $\nabla_x V(t, u_0(t)) = 0$ for every $t \in [0, T]$. We now compute twice (2.12), once for $x = u_\varepsilon(t) - u_0(t)$ and once for $x = u_0^k(t) - u_0(t)$, and then we make the difference between the two results. Therefore, for suitable y_1 between $u_0(t)$ and $u_\varepsilon(t)$, and y_2 between $u_0(t)$ and $u_0^k(t)$, we have

$$\tilde{V}(t, u_\varepsilon(t)) = \nabla_x^2 V(t, y_1)(u_\varepsilon(t) - u_0(t)) \cdot (u_\varepsilon(t) - u_0(t))$$

$$- \nabla_x^2 V(t, y_2)(u_0^k(t) - u_0(t)) \cdot (u_0^k(t) - u_0(t)). \tag{2.13}$$

By a continuity argument and the coercivity assumption for $\nabla_x^2 V$ (1.16) we can find $\delta > 0$ such that, if $|z| < \delta$, then

$$\nabla_x^2 V(t, z + u_0(t))\xi \cdot \xi \geq \frac{\alpha}{2}|\xi|^2.$$

We apply this estimate in the first summand of the right hand side of (2.13), while for the other one we use boundedness of $\nabla_x^2 V(t, \cdot)$. We thus get

$$\tilde{V}(t, u_\varepsilon(t)) \geq \frac{\alpha}{2}|u_\varepsilon(t) - u_0(t)|^2 - c|u_0^k(t) - u_0(t)|^2, \tag{2.14}$$

for a suitable $c > 0$, provided that

$$|u_\varepsilon(t) - u_0(t)| < \delta \text{ for every } t \in [0, T] \text{ and for } \varepsilon \text{ small enough.} \tag{2.15}$$

For the moment we assume that this bound holds and postpone its proof to the end.

Since

$$\frac{\alpha}{4}|u_\varepsilon(t) - u_0^k(t)|^2 \leq \frac{\alpha}{2}|u_\varepsilon(t) - u_0(t)|^2 + \frac{\alpha}{2}|u_0(t) - u_0^k(t)|^2,$$

we deduce, from (2.14), that

$$\tilde{V}(t, u_\varepsilon(t)) \geq \frac{\alpha}{4}|u_\varepsilon(t) - u_0^k(t)|^2 - (c + \frac{\alpha}{2})|u_0^k(t) - u_0(t)|^2, \tag{2.16}$$

where the last summand on the right hand side of (2.16) is small by the uniform convergence of u_0^k to u_0.

Upper estimate. We now switch our attention to the estimate on the right hand side of (2.11), which we now re-write in the following way:

$$
A_\varepsilon - \int_0^t \varepsilon^2 \ddot{u}_0^k(s)(\dot{u}_\varepsilon(s) - \dot{u}_0^k(s))\,\mathrm{d}s + \int_0^t \left[\tilde{V}_t(s, u_\varepsilon(s)) + \nabla_x \tilde{V}(s, u_\varepsilon(s))\dot{u}_0^k(s) \right]\,\mathrm{d}s
$$
$$
=: A_\varepsilon - A_1 + A_2.
$$

Estimate of A_1. We first apply the Cauchy inequality and obtain

$$
|A_1| = \left| \int_0^t \varepsilon^2 \ddot{u}_0^k(\dot{u}_\varepsilon - \dot{u}_0^k)\,\mathrm{d}s \right| \leq \frac{\varepsilon^2}{2} \int_0^t |\ddot{u}_0^k|^2\,\mathrm{d}s + \frac{\varepsilon^2}{2} \int_0^t |\dot{u}_\varepsilon - \dot{u}_0^k|^2\,\mathrm{d}s \qquad (2.17)
$$

The second summand in (2.17) will enter the final estimate via the Gronwall lemma, while for the first one we argue in the following way. We have no information about how big $\|\ddot{u}_0^k\|_{L^2(0,T)}$ is, nevertheless we can find, for every $k \in \mathbb{N}$, an $\varepsilon > 0$ such that

$$
\|\ddot{u}_0^k\|_{L^2(0,T)}^2 \leq \frac{1}{\varepsilon}. \qquad (2.18)
$$

Then, we can invert the function which associates ε to k and get $k(\varepsilon) \to \infty$ as $\varepsilon \to 0$, though this convergence may be very slow. This is done by recalling that $\varepsilon = \varepsilon_j$ and then defining

$$
k(\varepsilon_j) := \min \left\{ k \in \mathbb{N} : \|\ddot{u}_0^k\|_{L^2(0,T)}^2 > \frac{1}{\varepsilon_j} \right\} - 1.
$$

From now on we will keep writing k intending $k(\varepsilon_j)$ with this peculiar construction and we have, combining (2.17) with (2.18),

$$
|A_1| \leq A_1^\varepsilon + \frac{\varepsilon^2}{2} \int_0^t |\dot{u}_\varepsilon - \dot{u}_0^k|^2\,\mathrm{d}s., \qquad (2.19)
$$

where $A_1^\varepsilon \to 0$ as $\varepsilon \to 0$.

Estimate of A_2. By using a variable x which will play the role of $u_\varepsilon(t) - u_0^k(t)$, we have that, for a.e. $t \in [0, T]$,

$$
|\tilde{V}_t(t, x + u_0^k) + \nabla_x \tilde{V}(t, x + u_0^k)\dot{u}_0^k|
$$
$$
\leq |\tilde{V}_t(t, x + u_0^k) - \tilde{V}_t(t, x + u_0)| + |\nabla_x \tilde{V}(t, x + u_0^k)\dot{u}_0^k - \nabla_x \tilde{V}(t, x + u_0^k)\dot{u}_0|
$$
$$
+ |\nabla_x \tilde{V}(t, x + u_0^k)\dot{u}_0 - \nabla_x \tilde{V}(t, x + u_0)\dot{u}_0| + |\tilde{V}_t(t, x + u_0) + \nabla_x \tilde{V}(t, x + u_0)\dot{u}_0|
$$
$$
(2.20)
$$

The first three summands on the right are easy to deal with, by using Lipschitz and boundedness assumptions. They are estimated, independently of x, by

$$
C(1 + |\dot{u}_0|)(|u_0^k - u_0| + |\dot{u}_0^k - \dot{u}_0|).
$$

As for the fourth summand, we call

$$
f(x) := \tilde{V}_t(t, x + u_0) + \nabla_x \tilde{V}(t, x + u_0)\dot{u}_0
$$

If we set $g(x) := f(x) - \nabla f(0)x$, then there exists y in the segment $[0, x]$ such that $g(x) - g(0) = \nabla g(y)x$. Therefore we have, for a.e. $t \in [0, T]$,

$$
\begin{aligned}
&\tilde{V}_t(t, x + u_0) + \nabla_x \tilde{V}(t, x + u_0) \cdot \dot{u}_0 - \tilde{V}_t(t, u_0) \\
&\quad - \nabla_x \tilde{V}(t, u_0) \cdot \dot{u}_0 - \nabla_x \tilde{V}_t(t, u_0) \cdot x - \nabla_x^2 \tilde{V}(t, u_0)\dot{u}_0 \cdot x \\
&\leq \left| \nabla_x \tilde{V}_t(t, y + u_0) + \nabla_x^2 \tilde{V}(t, y + u_0)\dot{u}_0 - \nabla_x \tilde{V}_t(t, u_0) - \nabla_x^2 \tilde{V}(t, u_0)\dot{u}_0 \right| |x| \\
&\leq c(1 + |\dot{u}_0|)|y||x| \\
&\leq c|x|^2(1 + |\dot{u}_0|),
\end{aligned}
\tag{2.21}
$$

since $\nabla_x \tilde{V}_t$ and $\nabla_x^2 \tilde{V}$ are locally equi-Lipschitz in x uniformly in t, by condition (2.3), and the constant $c > 0$ is independent of $x = u_\varepsilon - u_0^k$ because the functions u_ε are bounded in ε as we pointed out in Remark 2.2. Moreover, by (2.9),

$$
\begin{aligned}
|\tilde{V}_t(t, u_0)| &= |V_t(t, u_0) - V_t(t, u_0^k) - \nabla_x V(t, u_0^k) \cdot \dot{u}_0^k| \\
&\leq |V_t(t, u_0) - V_t(t, u_0^k)| + |\nabla_x V(t, u_0) \cdot \dot{u}_0^k - \nabla_x V(t, u_0^k) \cdot \dot{u}_0^k| \\
&\leq C|u_0 - u_0^k|(1 + |\dot{u}_0^k|),
\end{aligned}
\tag{2.22}
$$

for a.e. $t \in [0, T]$. As in (1.15), we have that for a.e. $t \in [0, T]$,

$$
\nabla_x \tilde{V}_t(t, u_0(t)) + \nabla_x^2 \tilde{V}(t, u_0(t))\dot{u}_0(t) = 0.
\tag{2.23}
$$

Therefore, plugging (2.10), (2.21), (2.22), and (2.23) in (2.20), we get

$$
\begin{aligned}
&|\tilde{V}_t(t, x + u_0^k) + \nabla_x \tilde{V}(t, x + u_0^k) \cdot \dot{u}_0^k| \\
&\leq c_1(1 + |\dot{u}_0|)(|u_0^k - u_0| + |\dot{u}_0^k - \dot{u}_0|) + c_2|x|^2(1 + |\dot{u}_0|) + c_3|u_0 - u_0^k|(1 + |\dot{u}_0^k|),
\end{aligned}
\tag{2.24}
$$

for a.e. $t \in [0, T]$, where c_1, c_2, and $c_3 > 0$. We may therefore compute (2.24) with $x = u_\varepsilon(t) - u_0^k(t)$ and, if we integrate between 0 and t, we find that

$$
c_1 \int_0^t (1 + |\dot{u}_0|)(|u_0^k - u_0| + |\dot{u}_0^k - \dot{u}_0|) \, \mathrm{d}s \to 0,
$$

as $k \to \infty$ by $W^{1,1}-$convergence of u_0^k to u_0. Also

$$
c_3 \int_0^t |u_0 - u_0^k|(1 + |\dot{u}_0^k|) \, \mathrm{d}s \to 0,
$$

as $k \to \infty$ using this time the uniform convergence of u_0^k to u_0 and the fact that $\dot{u}_0^k \in L^1(0, T)$. Therefore the second integral in (2.11) is estimated by

$$
c_2 \int_0^t |u_\varepsilon - u_0^k|^2(1 + |\dot{u}_0|) \, \mathrm{d}s + A_k,
\tag{2.25}
$$

where $A_k \to 0$ as $k \to \infty$.

Gronwall argument. We are now able to get, by the previous estimates, the conclusion of the proof. We set

$$
B_\varepsilon := A_\varepsilon + A_1^\varepsilon + A_{k(\varepsilon)} + (c + \frac{\alpha}{2})|u_0^{k(\varepsilon)}(t) - u_0(t)|^2,
$$

which tends to zero as $\varepsilon \to 0$, and we plug (2.16), (2.17), (2.18), and (2.25) into (2.11). We therefore have, for every $t \in [0, T]$,

$$\frac{\varepsilon^2}{2}|\dot{u}_\varepsilon(t) - \dot{u}_0^k(t)|^2 + \frac{\alpha}{4}|u_\varepsilon(t) - u_0^k(t)|^2$$

$$\leq B_\varepsilon + \frac{\varepsilon^2}{2}\int_0^t |\dot{u}_\varepsilon - \dot{u}_0^k|^2 + c_2 \int_0^t |u_\varepsilon - u_0^k|^2(1 + |\dot{u}_0|). \qquad (2.26)$$

With some further manipulations we are in position to apply Lemma 2.1. Therefore, there exists $C > 0$ such that

$$\frac{\varepsilon^2}{2}|\dot{u}_\varepsilon(t) - \dot{u}_0^k(t)|^2 + \frac{\alpha}{4}|u_\varepsilon(t) - u_0^k(t)|^2 \leq B_\varepsilon \exp\left(C\int_0^t (1 + |\dot{u}_0(s)|)\,\mathrm{d}s\right). \quad (2.27)$$

Since $u_0 \in W^{1,1}(0, T)$, we have that the right hand side of (2.27) tends to zero as $\varepsilon \to 0$, for every $t \in [0, T]$. In particular, since $|u_\varepsilon(t) - u_0(t)| \leq |u_\varepsilon(t) - u_0^k(t)| + |u_0^k(t) - u_0(t)|$, we obtain that $u_\varepsilon(t) \to u_0(t)$ uniformly in $[0, T]$ as $\varepsilon \to 0$. We also have

$$\varepsilon\|\dot{u}_\varepsilon - \dot{u}_0\|_{L^1(0,T)} \leq \varepsilon\|\dot{u}_\varepsilon - \dot{u}_0^k\|_{L^1(0,T)} + \varepsilon\|\dot{u}_0^k - \dot{u}_0\|_{L^1(0,T)}$$

$$\leq \varepsilon^2 T \int_0^T |\dot{u}_\varepsilon - \dot{u}_0^k|^2 + \varepsilon\|\dot{u}_0^k - \dot{u}_0\|_{L^1(0,T)},$$

from which we deduce

$$\varepsilon\|\dot{u}_\varepsilon - \dot{u}_0\|_{L^1} \to 0,$$

as $\varepsilon \to 0$, because $\|\dot{u}_0^k - \dot{u}_0\|_{L^1(0,T)}$ is bounded.

Proof of (2.15). In order to conclude we only need to prove that $|u_\varepsilon(t) - u_0(t)| < \delta$, for every $t \in [0, T]$ and for ε small enough. We can define, for every $\varepsilon > 0$

$$t_\varepsilon = \inf\{t \in [0, T]\colon |u_\varepsilon(t) - u_0(t)| > \delta\},$$

with the convention that $\inf \emptyset = T$. Notice that the continuity of $u_\varepsilon(\cdot) - u_0(\cdot)$ and the initial condition $u_\varepsilon(0) \to u_0(0)$ as $\varepsilon \to 0$, implies that $t_\varepsilon > 0$. We thus have that (2.15) is satisfied for every $t \in [0, t_\varepsilon)$. We now assume, by contradiction, that $t_\varepsilon < T$. Then, with the previous Gronwall argument, we can find $\bar{\varepsilon}$ so small such that $|u_\varepsilon(t) - u_0(t)| < \frac{\delta}{2}$ for every $\varepsilon \in (0, \bar{\varepsilon})$ and $t \in [0, t_\varepsilon]$. However this contradicts the continuity of $u_\varepsilon - u_0$ in $t = t_\varepsilon$. Therefore $t_\varepsilon = T$ and this concludes the proof of the theorem. $\qquad \square$

3. Counterexample

In the previous section we proved that, under certain assumptions on V, the solutions u_ε of problems (2.4) converge in $W^{1,1}(0, T)$ to u_0, whenever u_0 is continuous and the initial conditions (2.5) are satisfied. We now prove that assumptions on V can not be further relaxed in order to get the same result.

Let us consider the sample case

$$\begin{cases} \varepsilon^2 \ddot{u}_\varepsilon + u_\varepsilon - u_0 = 0, \\ u_\varepsilon(0) = u_\varepsilon^0, \\ \dot{u}_\varepsilon(0) = v_\varepsilon^0, \end{cases} \qquad (3.1)$$

where we assume that $u_\varepsilon^0 \to u_0(0) = 0$ and $\varepsilon v_\varepsilon^0 \to 0$ as $\varepsilon \to 0$. In this case the potential V is given by

$$V(t,x) := \frac{1}{2}|x - u_0(t)|^2.$$

We have that $\nabla_x V(t, u_0(t)) = 0$ for every t and $\nabla_x^2 V(t, u_0(t))$ is the identity matrix. We notice that, if we only assume continuity of u_0, then a chain rule similar to (1.12) can not be established. We can, nevertheless, find an explicit solution of (3.1) with standard methods of ordinary differential equations:

$$u_\varepsilon(t) = \left(-\frac{1}{\varepsilon}\int_0^t u_0(s)\sin\tfrac{s}{\varepsilon}\,\mathrm{d}s + u_\varepsilon^0\right)\cos\tfrac{t}{\varepsilon} + \left(\frac{1}{\varepsilon}\int_0^t u_0(s)\cos\tfrac{s}{\varepsilon}\,\mathrm{d}s + \varepsilon v_\varepsilon^0\right)\sin\tfrac{t}{\varepsilon}.$$

$$(3.2)$$

If we assume that $u_0 \in W^{1,1}(0,T)$, then assumption of Theorem 2.3 are satisfied and therefore $u_\varepsilon \to u_0$ uniformly for every $t \in [0,T]$ and that $\varepsilon \dot{u}_\varepsilon \to 0$ for a.e. $t \in \mathbb{R}$. This result can be equivalently obtained by direct computation through the explicit formula (3.2). We may remark the fact that, in the presence of a dissipative term as in [1], the convergence of the solutions to the approximated problems is satisfied with weaker assumptions on the initial conditions. More precisely if the equation is

$$\varepsilon^2 \ddot{u}_\varepsilon + \varepsilon \dot{u}_\varepsilon + u_\varepsilon - u_0 = 0,$$

then it is sufficient to assume only that

$$u_\varepsilon^0 e^{-\frac{1}{2\varepsilon}} \to 0 \text{ and } \varepsilon v_\varepsilon^0 e^{-\frac{1}{2\varepsilon}} \to 0 \text{ as } \varepsilon \to 0.$$

We now show that convergence for the problem (3.1) fails if we only assume that u_0 is continuous. This gives a counterexample to the convergence result of Theorem 2.3 when the regularity assumptions on V are not satisfied. Indeed, there is a least a continuous function that can not be approximated by solutions to second order perturbed problems, as we show in the next proposition; we will exhibit one of these functions in Example 3.3. Furthermore in $W^{1,1}$ there is a dense set of C_0^0 functions with this property, see Remark 3.2.

Proposition 3.1. *There exists $u_0 \in C_0^0([0,T])$ such that the functions u_ε, defined in (3.2), do not converge uniformly to u_0 as $\varepsilon \to 0$.*

Proof. We argue by contradiction. Assume that for every $u_0 \in C_0^0([0,T])$ u_ε uniformly converges to u_0 as $\varepsilon \to 0$. Without loss of generality we can assume that $T \geq 1$ and we show that the convergence fails at $t = 1$. Let us fix $\varepsilon_k \to 0$. Then we have, from (3.2),

$$u_{\varepsilon_k}(1) = -\frac{1}{\varepsilon_k}\int_0^1 u_0(s)[\sin\tfrac{s}{\varepsilon_k}\cos\tfrac{1}{\varepsilon_k} - \cos\tfrac{s}{\varepsilon_k}\sin\tfrac{1}{\varepsilon_k}]\,\mathrm{d}s + u_{\varepsilon_k}^0\cos\tfrac{1}{\varepsilon_k} + \varepsilon_k v_{\varepsilon_k}^0\sin\tfrac{1}{\varepsilon_k}.$$

Since $u_{\varepsilon_k}^0, \varepsilon_k v_{\varepsilon_k}^0 \to 0$ by assumption, we have convergence of $u_\varepsilon(1)$ to $u_0(1)$ if and only if the operator $F_{\varepsilon_k}: C_0^0([0,T]) \to \mathbb{R}$, defined as

$$F_{\varepsilon_k}(u_0) := -\frac{1}{\varepsilon_k}\int_0^1 u_0(s)[\sin\tfrac{s}{\varepsilon_k}\cos\tfrac{1}{\varepsilon_k} - \cos\tfrac{s}{\varepsilon_k}\sin\tfrac{1}{\varepsilon_k}]\,\mathrm{d}s,$$

converges. We thus have pointwise convergence of F_{ε_k} to F_0 defined by $F_0(u_0) = u_0(1)$. By the Banach-Steinhaus Theorem this implies uniform equiboundedness. On

the other hand we notice that

$$F_{\varepsilon_k}(u_0) = \int_0^1 u_0(s)\,\mathrm{d}\mu_{\varepsilon_k}(s),$$

where $\mathrm{d}\mu_{\varepsilon_k}(s) = -\frac{1}{\varepsilon_k}[\sin\frac{s}{\varepsilon_k}\cos\frac{1}{\varepsilon_k} - \cos\frac{s}{\varepsilon_k}\sin\frac{1}{\varepsilon_k}]\,\mathrm{d}s$. However

$$\sup_k |\mu_{\varepsilon_k}|(0,1) = \sup_k \left(\frac{1}{\varepsilon_k}\int_0^1 |\sin\frac{s-1}{\varepsilon_k}|\,\mathrm{d}s\right) = \sup_k \int_{-\frac{1}{\varepsilon_k}}^0 |\sin\tau|\,\mathrm{d}\tau = +\infty$$

which contradicts the uniform equiboundedness. $\qquad\qquad\qquad\square$

Remark 3.2. The Banach-Steinhaus Theorem also implies that the set

$$R := \{u_0 \in C_0^0([0,T])\colon \sup_\varepsilon |F_\varepsilon(u_0)| = +\infty\}$$

is dense. Therefore there are indeed infinitely many functions for which u_ε can not converge to u_0.

Example 3.3. We now give an explicit example of a continuous function that is not approximated by solutions to second order perturbed problems. We consider as u_0 the Cantor-Vitali function $\hat{u}\colon [0,1] \to [0,1]$. Plugging $u_0 = \hat{u}$ into (3.2) and through integration by parts, we get

$$u_\varepsilon(t) = \hat{u}(t) - \cos\frac{t}{\varepsilon}\int_0^t \cos\frac{s}{\varepsilon}\,\mathrm{d}\mu(s) - \sin\frac{t}{\varepsilon}\int_0^t \sin\frac{s}{\varepsilon}\,\mathrm{d}\mu(s) + u_\varepsilon^0\cos\frac{t}{\varepsilon} + \varepsilon v_\varepsilon^0\sin\frac{t}{\varepsilon}, \;\; (3.3)$$

where μ is intended to be the distributional derivative of u_0. We now choose $\varepsilon_k = \frac{1}{2k\pi}$ and remark that

$$\int \cos(2k\pi s)\,\mathrm{d}\mu(s) = \int e^{-i2k\pi s}\,\mathrm{d}\mu(s),$$

where $\hat{u}$ and μ have been extended to $\mathbb{R}$ by setting $\hat{u} = 0$ in the complement of $[0,1]$. By using the well-known expression for the Fourier Tranform of the Cantor measure we can compute (3.3) in $t = 1$ and get

$$u_{\varepsilon_k}(1) = \hat{u}(1) + u_{\varepsilon_k}^0 - \int_0^1 \cos(2k\pi s)\,\mathrm{d}\mu(s) = \hat{u}(1) + u_{\varepsilon_k}^0 - (-1)^k \prod_{h=1}^\infty \cos\frac{2k\pi}{3^h}.$$

Since $u_{\varepsilon_k}^0 \to 0$ by the assumptions on the initial conditions, we focus our attention on the term

$$(-1)^k \prod_{h=1}^\infty \cos\frac{2k\pi}{3^h} = (-1)^k f(2k\pi),$$

where we have defined $f\colon [0,+\infty) \to [-1,1]$ by

$$f(x) = \prod_{h=1}^\infty \cos\frac{x}{3^h}.$$

We now prove that there exists a sequence k_n such that $(-1)^{k_n} f(2k_n\pi)$ does not converge to 0.

By the definition f satisfies

$$f(3x) = f(x)\cos(x).$$

In particular this implies

$$f(6\pi) = f(3\cdot 2\pi) = f(2\pi).$$

Inductively one gets
$$f(3^n \cdot 2\pi) = f(2\pi)$$
and similarly
$$f(2 \cdot 3^n \cdot 2\pi) = f(4\pi).$$
Therefore we choose as k_n the sequence
$$\{3,\, 2 \cdot 3,\, 3^2,\, 2 \cdot 3^2, \dots, 3^n,\, 2 \cdot 3^n, \dots\}.$$
Along this sequence $(-1)^{k_n} f(2k_n\pi)$ tends to $-f(2\pi)$ for the odd indexes and to $f(4\pi)$ for the even ones. We now prove that $f(2\pi)$ and $f(4\pi)$ are real numbers with the same sign. This implies that $(-1)^{k_n} f(2k_n\pi)$ does not converge and therefore $u_{\varepsilon_k}(1)$ does not converge to $\hat{u}(1)$.

We have that (using the convention that $\log 0 = -\infty$)
$$\log f(x) = \sum_{h=1}^{\infty} \log \left| \cos \tfrac{x}{3^h} \right| \geq -\sum_{h=1}^{\infty} \frac{x^2}{3^{2h}},$$
if $y \in (0, 1)$, because in this interval
$$\log |\cos y| = \log \cos y \geq \frac{\cos y - 1}{\cos y}.$$
Moreover
$$1 - \frac{1}{\cos y} \geq -x^2 \iff \cos y (1 + y^2) \geq 1,$$
which is verified in $(0, 1)$ using the fact that $\cos y \geq 1 - \frac{y^2}{2}$. Since $\frac{2\pi}{3^h}$ and $\frac{4\pi}{3^h}$ are in the interval $(0, 1)$ for h large enough, then $f(2\pi)$ and $f(4\pi)$ are controlled by the geometric series and therefore $f(2\pi), f(4\pi) \neq 0$. This is enough in order to prove that along the sequences k_{2n} or k_{2n+1} convergence of $u_{\varepsilon_{k_n}}(1)$ to $\hat{u}(1)$ is not satisfied. Moreover we notice that $f(2\pi)$ and $f(4\pi)$ have the same sign because $\cos \frac{2\pi}{3} = \cos \frac{4\pi}{3} = -\frac{1}{2}$, while $\cos(\frac{2\pi}{3^n}) \geq 0$ for every $n \geq 3$. Therefore we have found more than we claimed, since $u_{\varepsilon_{k_n}}(1)$ does not convergeat all.

We have thus shown an explicit example in which convergence of (3.2) to a particular continuous function u_0 fails.

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