

A DIFFUSIVE MODEL FOR MACROSCOPIC CROWD MOTION WITH DENSITY CONSTRAINTS

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ABSTRACT. In the spirit of the macroscopic crowd motion models with hard congestion (i.e. a strong density constraint $\rho \leq 1$) introduced by Maury *et al.* some years ago, we analyze a variant of the same models where diffusion of the agents is also taken into account. From the modeling point of view, this means that individuals try to follow a given spontaneous velocity, but are subject to a Brownian diffusion, and have to adapt to a density constraint which introduces a pressure term affecting the movement. From the PDE point of view, this corresponds to a modified Fokker-Planck equation, where with an additional gradient of a pressure (only living in the saturated zone $\{\rho = 1\}$) in the drift. The paper proves existence and some estimates, based on optimal transport techniques.

1. INTRODUCTION

In the past few years modeling crowd behavior has become a very active field of applied mathematics. Beyond their importance in real life applications, these modeling problems serve as basic ideas to understand many other phenomena coming for example from biology (cell migration, tumor growth, pattern formations in animal populations, etc.), particle physics and economics. A first non-exhaustive list of references for these problems is [6, 7, 8, 9, 12, 14, 15, 17, 18, 24]. A very natural question in all these models is the problem of congestion phenomena: in many practical situations, very high quantities of individuals could try to occupy the same spot, which could be impossible, or lead to strong negative effects on the motion, because of natural limitations on the crowd density.

These phenomenas have been studied by using different models, which could be either “microscopic” (based on ODEs on the motion of a high number of agents) or “macroscopic” (describing the agents via their density and velocity, typically with Eulerian formalism). Let us concentrate on the macroscopic models, where the density ρ plays a crucial role. These very same models can be characterized either by “soft congestion” effects (i.e. the higher the density the slower the motion), or by “hard congestion” (i.e. an abrupt threshold effect: if the density touches a certain maximal value, the motion is strongly affected, while nothing happens for smaller values of the density). See [23] for comparison between the different classes of models. This last class of models, due to the discontinuity in the congestion effects, presents new mathematical difficulties, which cannot be analyzed with the usual techniques from conservation laws (or, more generally, evolution PDEs) used for soft congestion.

A very powerful tool to attack macroscopic hard-congestion problems is the theory of optimal transportation (see [32, 29]), as we can see in [22, 23, 28, 30]. In this framework, the density of the

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agents solves a continuity equation (with velocity field taking into account the congestion effects), and can be seen as a curve in the Wasserstein space.

Our aim in this paper is to endow the macroscopic hard congestion models of [22, 23, 28, 30] with diffusion effects. In other words, we want to model a crowd, where every agent has a spontaneous velocity, a non-degenerate diffusion (i.e. a stochastic component in its motion), driven by a Brownian motion, and is subject to a density constraint. Implementing this new element into these models could give a better approximation of reality, when dealing with large populations. We also underline that one of the goals of this analysis¹ is to better “prepare” these hard congestion crowd motion models for a possible analysis in the framework of Mean Field Games (see [19, 20, 21], and also [31]). These MFG models usually involve a stochastic term, also implying regularizing effects, which are useful in the mathematical analysis of the corresponding PDEs.

1.1. The existing first order models in the light of [22, 23]. Some macroscopic models for crowd motion with density constraints and “hard congestion” effects were studied in [23] and [22]. We briefly present them as follows:

- The density of the population in a bounded (convex) domain $\Omega \subset \mathbb{R}^d$ is described by a probability measure $\rho \in \mathcal{P}(\Omega)$. The initial density $\rho_0 \in \mathcal{P}(\Omega)$ evolves in time, and ρ_t denotes its value at each time $t \in [0, T]$.
- The spontaneous velocity field of the population is a given time-dependent field, denoted by u_t . It represents the velocity that each individual would like to follow in the absence of the others. Ignoring the density constraint, this would give rise to the continuity equation $\partial_t \rho_t + \nabla \cdot (\rho_t u_t) = 0$. We observe that in the original work [22] the vector field $u_t(x)$ was taken of the form $-\nabla D(x)$ (independent of time and of gradient form) but we try here to be more general (see [28] for the general case, which requires some extra regularity assumption).
- The set of admissible densities will be denoted by $\mathcal{K} := \{\rho \in \mathcal{P}(\Omega) : \rho \leq 1\}$. In order to guarantee that \mathcal{K} is neither empty nor trivial, we suppose $|\Omega| > 1$.
- The set of admissible velocity fields with respect to the density ρ is characterized by the sign of the divergence of the velocity field on the saturated zone; formally we set

$$\text{adm}(\rho) := \left\{ v : \Omega \rightarrow \mathbb{R}^d : \nabla \cdot v \geq 0 \text{ on } \{\rho = 1\} \right\}.$$

- We consider the projection operator P in the $L^2(\mathcal{L}^d)$ or in the $L^2(\rho)$ sense (which will be the same, because the only relevant zone is $\{\rho = 1\}$).
- Finally we solve the following modified continuity equation for ρ

$$(1.1) \quad \partial_t \rho_t + \nabla \cdot (\rho_t P_{\text{adm}(\rho_t)}[u_t]) = 0,$$

where the main point is that ρ is advected by a vector field, compatible with the constraints, which is the closest to the spontaneous one.

The problem in solving Equation (1.1) is that the projected field has very low regularity: it is a priori only L^2 in x , and it does not depend smoothly on ρ neither (since a density 1 and a density $1 - \varepsilon$ give very different projection operators). By the way, its divergence is not well-defined neither.

¹The insertion of a diffusive behavior in the model also leads to easier and more general uniqueness results: this is the object of an on-going work of the first author in collaboration with S. Di Marino, see [11].

To handle this issue we need to redefine the set of admissible velocities by duality:

$$\text{adm}(\rho) = \left\{ v \in L^2(\rho) : \int_{\Omega} v \cdot \nabla p \leq 0, \forall p \in H^1(\Omega), p \geq 0, p(1 - \rho) = 0 \right\}.$$

With the help of this formulation we always have the orthogonal decomposition

$$u = P_{\text{adm}(\rho)}[u] + \nabla p,$$

where $p \in \text{press}(\rho) := \{p \in H^1(\rho) : p \geq 0, p(1 - \rho) = 0\}$. Indeed, the cones $\text{adm}(\rho)$ and $\nabla \text{press}(\rho)$ are dual to each other. Via this approach the continuity equation (1.1) can be rewritten as a system for (ρ, p) which is

$$(1.2) \quad \begin{cases} \partial_t \rho_t + \nabla \cdot (\rho_t (u_t - \nabla p_t)) = 0 \\ p \geq 0, \rho \leq 1, p(1 - \rho) = 0. \end{cases}$$

We can naturally endow this equation/system with initial condition $\rho(0, x) = \rho_0(x)$ ($\rho_0 \in \mathcal{K}$) and with Neumann boundary conditions.

1.2. A diffusive counterpart. The goal of our work is to study a second order model of crowd movements with hard congestion effect where beside the transport factor a non-degenerate diffusion is present as well. The diffusion is the consequence of a randomness (a Brownian motion) in the movement of the crowd.

With the ingredients that we introduced so far, we would like to modify the Fokker-Planck equation $\partial_t \rho_t - \Delta \rho_t + \nabla \cdot (\rho_t u_t) = 0$ (always equipped with the natural Neumann boundary conditions on $\partial\Omega$) in order to take into account the density constraint $\rho_t \leq 1$. We observe that the Fokker Planck equation is derived from a motion $dX_t = u_t(X_t)dt + \sqrt{2}dB_t$, but is macroscopically represented by the advection of the density ρ_t by the vector field $-\frac{\nabla \rho_t}{\rho_t} + u_t$. Projecting onto the set of admissible velocities raises a natural question: should we project only u_t , and *then* apply the diffusion, or project the whole vector field, including $-\frac{\nabla \rho_t}{\rho_t}$? Indeed, this is not a real issue, as $\frac{\nabla \rho_t}{\rho_t} = 0$ on the saturated set $\{\rho_t = 1\}$. This corresponds to the fact that the Heat Kernel preserves the constraint $\rho \leq 1$. As a consequence, we consider the modified Fokker-Planck type equation

$$(1.3) \quad \begin{cases} \partial_t \rho_t - \Delta \rho_t + \nabla \cdot (\rho_t P_{\text{adm}(\rho_t)}[u_t]) = 0, \\ \rho(0, x) = \rho_0(x), \text{ in } \Omega, \end{cases}$$

which can also be written equivalently as

$$(1.4) \quad \begin{cases} \partial_t \rho_t - \Delta \rho_t + \nabla \cdot (\rho_t (u_t - \nabla p_t)) = 0 \\ p \geq 0, \rho \leq 1, p(1 - \rho) = 0, \rho(0, x) = \rho_0(x), \text{ in } \Omega. \end{cases}$$

This equation describes the law of a motion where each particle solves the stochastic differential equation

$$dX_t = (u_t(X_t) - \nabla p_t(X_t))dt + \sqrt{2}dB_t,$$

where B_t is the standard d -dimensional Brownian motion. More precisely, if we know the solution X_t of the SODE above, and we define $\rho_t := \mathcal{L}(X_t)$, the law of the random variable X_t , this will solve the Fokker-Planck equation (1.4).

1.3. Structure of the paper and main results. The main goal of the paper is to provide an existence result, with some extra estimates, for the Fokker-Planck equation (1.4) via time discretization, using the so-called splitting method (the two main ingredients of the equation, i.e. the advection with diffusion on one hand, and the density constraint on the other hand, are treated one after the other). In Section 2 we will collect some preliminary results, including what we need from optimal transport and from the previous works about density-constrained crowd motion, in particular on the projection operator onto the set \mathcal{K} . In Section 3 we will provide the existence result we aim at, by a splitting scheme and some entropy bounds; the solution will be a curve of measures in $H^1([0, T]; \mathbb{W}_2(\Omega))$. In Section 4 we will make use of BV estimates to justify that the solution we just built is also $\text{Lip}([0, T]; \mathbb{W}_1(\Omega))$ and satisfies a global BV bound $\|\rho_t\|_{BV} \leq C$: this requires to combine BV estimates on the Fokker-Planck equation (which are available depending on the regularity of the vector field u) with BV estimates on the projection operator on \mathcal{K} (which have been recently proven in [10]). Section 5 presents a short review of alternative approaches, all discretized in time, but based either on gradient-flow techniques (the JKO scheme, see [13]) or on different splitting methods. Finally, in the Appendix we detail the BV estimates on the Fokker-Planck equation (without any density constraint) that we could find; this seems to be a delicate matter, interesting in itself, and we are not aware of the sharp assumptions on the vector field u to guarantee the BV estimate that we need.

2. PRELIMINARIES

2.1. Basic definitions and general facts on optimal transport. Here we collect some tools from the theory of optimal transportation, Wasserstein spaces, its dynamical formulation, etc. which will be used later on. We set our problem either in a compact convex domain $\Omega \subset \mathbb{R}^d$ with smooth boundary (or in the d -dimensional flat torus $\Omega := \mathbb{T}^d$, even if we will not adapt all our notations to the torus case). We refer to [32, 29] for more details. Given two probability measures $\mu, \nu \in \mathcal{P}(\Omega)$ and for $p \geq 1$ we define the usual Wasserstein metric by means of the Monge-Kantorovich optimal transportation problem

$$W_p(\mu, \nu) := \inf \left\{ \int_{\Omega \times \Omega} |x - y|^p d\gamma(x, y) : \gamma \in \Pi(\mu, \nu) \right\}^{\frac{1}{p}},$$

where $\Pi(\mu, \nu) := \{\gamma \in \mathcal{P}(\Omega \times \Omega) : (\pi^x)_\# \gamma = \mu, (\pi^y)_\# \gamma = \nu\}$ and π^x and π^y denote the canonical projections from $\Omega \times \Omega$ onto Ω . This quantity happens to be a distance on $\mathcal{P}(\Omega)$ which metrizes the weak-* convergence of probability measures; we denote by $\mathbb{W}_p(\Omega)$ the space of probabilities on Ω endowed with this distance.

Moreover, in the quadratic case $p = 2$ and under the assumption $\mu \ll \mathcal{L}^d$ (the d -dimensional Lebesgue measure on Ω) in the late 80's Y. Brenier showed (see [4, 5]) that actually the optimal γ in the above problem is induced by a map, which is the gradient of a convex function, i.e. there exists $T : \Omega \rightarrow \Omega$ and $\psi : \Omega \rightarrow \mathbb{R}$ convex such that $T = \nabla \psi$ and $\bar{\gamma} := (\text{id}, T)_\# \mu$. The function ψ is obtained as $\psi(x) = \frac{1}{2}|x|^2 - \varphi(x)$, where φ is the so-called Kantorovich potential for the transport from μ to ν , and is characterized as the solution of a dual problem that we will not develop here. In this way, the optimal transport map T can also be written as $T(x) = x - \nabla \varphi(x)$. Later in the 90's R. McCann (see [25]) introduced a notion of interpolation between probability measures: the curve $\mu_t := ((1-t)x + ty)_\# \bar{\gamma}$, for $t \in [0, 1]$, gives a constant speed geodesic in the Wasserstein space connecting $\mu_0 := \mu$ and $\mu_1 := \nu$.

Based on this notion of interpolation in 2000 J.-D. Benamou and Y. Brenier used some ideas from fluid mechanics to give a dynamical formulation to the Monge-Kantorovich problem (see [3]). They showed that

$$\frac{1}{p}W_p^p(\mu, \nu) = \inf \{ \mathcal{B}_p(E_t, \mu_t) : \partial_t \mu_t + \nabla \cdot (E_t) = 0, \mu_0 = \mu, \mu_1 = \nu \},$$

where $\mathcal{B}_p : \mathfrak{M}([0, 1] \times \Omega)^d \times L^\infty([0, 1]; \mathbb{W}_p(\Omega)) \rightarrow \overline{\mathbb{R}^2}$ is given by

$$\mathcal{B}_p(E, \mu) := \begin{cases} \int_0^1 \int_\Omega \frac{1}{p} \left| \frac{dE_t}{d\mu_t} \right|^p d\mu_t(x) dt, & \text{if } E_t \ll \mu_t, \text{ a.e. } t \in [0, 1], \\ +\infty, & \text{otherwise.} \end{cases}$$

It is well-known that \mathcal{B}_p is jointly convex and l.s.c. w.r.t the weak-* convergence of measures (see Section 5.3.1 in [29]) and that, if $\partial_t \mu_t + \nabla \cdot E_t = 0$, then $\mathcal{B}_p(E, \mu) < +\infty$ implies that $t \mapsto \mu_t$ is a curve in $W^{1,p}([0, 1]; \mathbb{W}_p(\Omega))$. Coming back to curves in Wasserstein spaces, it is well known (see [1] or Section 5.3 in [29]) that for any distributional solution μ_t (being a narrowly continuous curve in $(\mathcal{P}(\Omega), W_p)$) of the continuity equation $\partial_t \mu_t + \nabla \cdot (v_t \mu_t) = 0$ we have the relations

$$|\mu'|_{W_p}(t) \leq \|v_t\|_{L^p_{\mu_t}} \quad \text{and} \quad W_p(\varrho_t, \varrho_s) \leq \int_s^t |\mu'|_{W_p}(\tau) d\tau,$$

where we denoted by $|\mu'|_{W_p}(t)$ the metric derivative w.r.t. W_p of the curve μ_t (see for instance [2] for general notions about curves in metric spaces and their metric derivative). For curves μ_t which are geodesic in \mathbb{W}_p we have the equality

$$W_p(\mu, \nu) = \int_0^1 |\mu'|_{W_p}(t) dt = \int_0^1 \|v_t\|_{L^p_{\mu_t}} dt.$$

The last equality is in fact the Benamou-Brenier formula with the optimal velocity field v_t being the density of the optimal E_t w.r.t. the optimal μ_t . This optimal velocity field v_t can be computed as $v_t := (T - \text{id}) \circ (T_t)^{-1}$, where $T_t := (1 - t)\text{id} + tT$ is the transport in McCann's interpolation (we assume here that the initial measure μ_0 is absolutely continuous, so that we can use transport maps instead of plans). This vector field is easily obtained if we consider that in this interpolation particles move with constant speed $T(x) - x$, but x represents here a Lagrangian coordinate, and not an Eulerian one: if we want to know the velocity at time t at a given point, we have to find out first the original position of the particle passing through that point at that time.

In the sequel we will also need the notion of entropy of a probability density, and for any probability measure $\varrho \in \mathcal{P}(\Omega)$ we define it as

$$\mathcal{E}(\varrho) := \begin{cases} \int_\Omega \varrho(x) \log \varrho(x) dx, & \text{if } \varrho \ll \mathcal{L}^d, \\ +\infty, & \text{otherwise.} \end{cases}$$

We recall that this functional is l.s.c. and geodesically convex in the \mathbb{W}_2 topology.

As we will be mainly working with absolutely continuous probability measures (w.r.t. Lebesgue), we often identify measures with their densities.

²We denote by $\mathfrak{M}(X)$ the signed Radon measures on X . Observe that $\mu \in L^\infty([0, 1]; \mathbb{W}_p(\Omega))$ only means that $\mu = (m\mu_t)_t$ is a time-dependent family of probability measures.

2.2. Projection problems in Wasserstein spaces. Our analysis relies a lot on the projection operator $P_{\mathcal{K}}$ in the sense of W_2 . Here $\mathcal{K} := \{\rho \in \mathcal{P}(\Omega) : \rho \leq 1\}$ and

$$P_{\mathcal{K}}[\mu] := \operatorname{argmin}_{\rho \in \mathcal{K}} \frac{1}{2} W_2^2(\mu, \rho).$$

We recall (see [22, 30] and [10]) the main properties of the projection $P_{\mathcal{K}}$ operator.

- As far as Ω is compact, for any probability measure μ , the minimizer in $\min_{\rho \in \mathcal{K}} \frac{1}{2} W_2^2(\mu, \rho)$ exists and is unique, and the operator $P_{\mathcal{K}}$ is continuous (it is even $C^{0,1/2}$ for the W_2 distance).
- The projection $P_{\mathcal{K}}[\mu]$ saturates the constraint $\rho \leq 1$ in the sense that for any $\mu \in \mathcal{P}(\Omega)$ there exists a measurable set $B \subseteq \Omega$ such that $P_{\mathcal{K}}[\mu] = \mathbb{1}_B + \mu^{\text{ac}} \mathbb{1}_{B^c}$, where μ^{ac} is the absolutely continuous part of μ .
- The projection is characterized in terms of a pressure field, in the sense that $\rho = P_{\mathcal{K}}[\mu]$ if and only if there exists a Lipschitz function $p \geq 0$, with $p(1 - \rho) = 0$, and such that the optimal transport map T from ρ to μ is given by $T := \text{id} - \nabla \varphi = \text{id} + \nabla p$.
- There is (as proven in [10]) a quantified BV estimate: if $\mu \in BV$ (in the sense that it is absolutely continuous and that its density belongs to $BV(\Omega)$), then $P_{\mathcal{K}}[\mu]$ is also BV and

$$TV(P_{\mathcal{K}}[\mu], \Omega) \leq TV(\mu, \Omega).$$

This last BV estimate will be crucial in Section 4, and it is important to have it in this very form (other estimates of the form $TV(P_{\mathcal{K}}[\mu], \Omega) \leq aTV(\mu, \Omega) + b$ would not be as useful as this one, as they cannot be easily iterated).

3. EXISTENCE VIA A SPLITTING-UP TYPE ALGORITHM (*Main Scheme*)

Similarly to the approach in [23] (see the algorithm (13) and Theorem 3.5) for a general, non-gradient, vector field, we will build a theoretical algorithm, after time-discretization, to produce a solution of (1.4). In this section the spontaneous velocity field is a general vector field $u_t : \Omega \rightarrow \mathbb{R}^d$ (not necessarily a gradient), which depends also on time. We will work on a time interval $[0, T]$ and in a bounded convex domain $\Omega \subset \mathbb{R}^d$ (the case of the flat torus is even simpler and we will not discuss it in details). We consider $\rho_0 \in \mathcal{P}^{\text{ac}}(\Omega)$ to be given, which represents the initial density of the population, and we suppose $\rho_0 \in \mathcal{K}$.

3.1. Splitting using the Fokker-Planck equation. We assume here that $u_t \in L^\infty(\Omega)^d$ for all $t \in [0, T]$. Let us consider the following scheme.

Main scheme: Let $\tau > 0$ be a small time step with $N := \lceil T/\tau \rceil$. Let us set $\rho_0^\tau := \rho_0$ and for every $k \in \{1, \dots, N\}$ we define ρ_{k+1}^τ from ρ_k^τ in the following way. First we solve

$$(3.1) \quad \begin{cases} \partial_t \varrho_t - \Delta \varrho_t + \nabla \cdot (\varrho_t u_{t+k\tau}) = 0, & t \in [0, \tau], \\ \varrho_0 = \rho_k^\tau, \end{cases}$$

equipped with the natural Neumann boundary condition ($(\nabla \varrho_t - u_t) \cdot n = 0$ a.e. on $\partial\Omega$) and set $\rho_{k+1}^\tau = P_{\mathcal{K}}[\tilde{\rho}_{k+1}^\tau]$, where $\tilde{\rho}_{k+1}^\tau = \varrho_\tau$. See Figure 1 on the right.

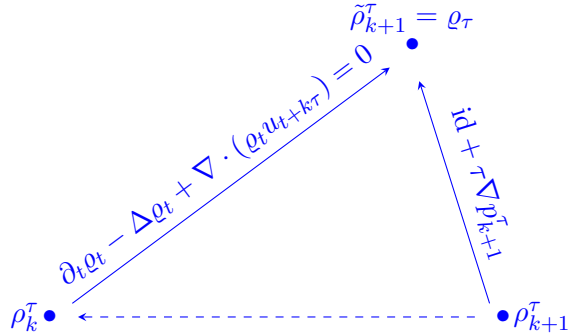


FIGURE 1. One time step

This means: first follow the Fokker-Planck equation, ignoring the density constraint, for a time τ , then project. In order to state and prove the convergence of the scheme, we need to define some suitable interpolations of the discrete sequence of densities that we have just introduced.

First interpolation. We define the following curves of densities, velocities and momentums constructed with the help of the ρ_k^τ 's. First set

$$\rho_t^\tau := \begin{cases} \varrho_{2(t-k\tau)}, & \text{if } t \in [k\tau, (k+1/2)\tau[, \\ (\text{id} + 2((k+1)\tau - t)\nabla p_{k+1}^\tau)_\# \rho_{k+1}^\tau, & \text{if } t \in [(k+1/2)\tau, (k+1)\tau[, \end{cases}$$

where ϱ_t is the solution of the Fokker-Planck equation (3.1) with initial datum ρ_k^τ and ∇p_{k+1}^τ arises from the projection of $\tilde{\rho}_{k+1}^\tau$, more precisely $(\text{id} + \tau\nabla p_{k+1}^\tau)$ is the optimal transport from ρ_{k+1}^τ to $\tilde{\rho}_{k+1}^\tau$. What are we doing? We are fitting into a time interval of length τ the two steps of our algorithm. First we follow the FP equation (3.1) at double speed, then we interpolate between the measure we reached and its projection following the geodesic between them. This geodesic is easily described as an image measure of ρ_{k+1}^τ through McCann's interpolation. By the construction it is clear that ρ_t^τ is a continuous curve in $\mathcal{P}(\Omega)$ for $t \in [0, T]$. We now define a family of time-dependent vector fields though

$$v_t^\tau := \begin{cases} -2\frac{\nabla\varrho_{2(t-k\tau)}}{\varrho_{2(t-k\tau)}} + 2u_t, & \text{if } t \in [k\tau, (k+1/2)\tau[, \\ -2\nabla p_{k+1}^\tau \circ (\text{id} + 2((k+1)\tau - t)\nabla p_{k+1}^\tau)^{-1}, & \text{if } t \in [(k+1/2)\tau, (k+1)\tau[, \end{cases}$$

and finally let us define the curve of momentums simply as $E_t^\tau := \rho_t^\tau v_t^\tau$.

Second interpolation. We define another interpolation as follows. Set

$$\tilde{\rho}_t^\tau := \varrho_{t-k\tau}, \quad \text{if } t \in [k\tau, (k+1)\tau[,$$

where ϱ_t is (again) the solution of the Fokker-Planck equation (3.1) on the time interval $[0, \tau]$ with initial datum ρ_k^τ . Here we do not double its speed. We define the curve of velocities

$$\tilde{v}_t^\tau := -\frac{\nabla\varrho_{t-k\tau}}{\varrho_{t-k\tau}} + u_t, \quad \text{if } t \in [k\tau, (k+1)\tau[,$$

and we build the curve of momentums by $\tilde{E}_t^\tau := \tilde{\rho}_t^\tau \tilde{v}_t^\tau$.

Mind the two differences in the construction of ρ_t^τ and $\tilde{\rho}_t^\tau$ (hence in the construction of v_t^τ and \tilde{v}_t^τ and E_t^τ and \tilde{E}_t^τ): 1) the first one is continuous in time, while the second one is not; 2) in the first construction we have taken into account the projection operator explicitly, while in the second one we see it just in an indirect manner (via the 'jumps' occurring at every time of the form $t = k\tau$).

Third interpolation. For each τ , we also define piecewise constant curves,

$$\begin{aligned} \hat{\rho}_t^\tau &:= \rho_{k+1}^\tau, & \text{if } t \in [k\tau, (k+1)\tau[, \\ \hat{v}_t^\tau &:= \nabla p_{k+1}^\tau, & \text{if } t \in [k\tau, (k+1)\tau[, \end{aligned}$$

and $\hat{E}_t^\tau := \hat{\rho}_t^\tau \hat{v}_t^\tau$. We remark that $p_{k+1}^\tau(1 - \rho_{k+1}^\tau) = 0$, hence the curve of momentums is just

$$\hat{E}_t^\tau := \nabla p_{k+1}^\tau, \quad \text{if } t \in [k\tau, (k+1)\tau[.$$

In order to prove the convergence of the scheme above, we will obtain uniform $H^1([0, T]; \mathbb{W}_2(\Omega))$ bounds for the curves ρ^τ . A key observation here is that the metric derivative (w.r.t. W_2) of the solution of the Fokker-Planck equation is comparable with the time differential of the entropy functional along the same solution (see Lemma 3.2). Now we state the main theorem of this section.

Theorem 3.1. *There exists a continuous curve $[0, T] \ni t \mapsto \rho_t \in \mathbb{W}_2(\Omega)$ and some measures $E, \tilde{E}, \hat{E} \in \mathfrak{M}([0, T] \times \Omega)$ such that the curves $\rho^\tau, \tilde{\rho}^\tau, \hat{\rho}^\tau$ converge uniformly in $\mathbb{W}_2(\Omega)$ to ρ and*

$$E^\tau \xrightarrow{*} E, \quad \tilde{E}^\tau \xrightarrow{*} \tilde{E}, \quad \hat{E}^\tau \xrightarrow{*} \hat{E}, \quad \text{in } \mathfrak{M}([0, T] \times \Omega)^d, \quad \text{as } \tau \rightarrow 0.$$

Moreover $E = \tilde{E} - \hat{E}$ and for a.e. t there exist $v_t, \tilde{v}_t, \hat{v}_t \in L^2_{\rho_t}(\Omega)^d$ such that $E = \rho v, \tilde{E} = \rho \tilde{v}, \hat{E} = \rho \hat{v}, \int_0^T (\|v_t\|_{L^2_{\rho_t}}^2 + \|\tilde{v}_t\|_{L^2_{\rho_t}}^2 + \|\hat{v}_t\|_{L^2_{\rho_t}}^2) dt < +\infty, v = \tilde{v} - \hat{v}, \tilde{E}_t = \rho_t u_t - \nabla \rho_t$ and $\hat{v}_t = \nabla p_t$ with $p_t \geq 0$ and $p_t(1 - \rho_t) = 0$. As a consequence, ρ_t is a weak solution of the “modified” Fokker-Planck equation

$$(3.2) \quad \begin{cases} \partial_t \rho_t - \Delta \rho_t + \nabla \cdot (\rho_t (u_t - \nabla p_t)) = 0, & \text{in }]0, T] \times \Omega, \\ p_t \geq 0, \rho_t \leq 1, p_t(1 - \rho_t) = 0, & \text{in } [0, T] \times \Omega, \\ (\nabla \rho_t - u_t + \nabla p_t) \cdot n = 0, & \text{on } [0, T] \times \partial\Omega, \\ \rho(t = 0, \cdot) = \rho_0, & \end{cases}$$

with the natural Neumann boundary conditions on $\partial\Omega$.

To prove this theorem we will use the following tools.

Lemma 3.2. *Let us consider a solution ϱ_t of the Fokker-Planck equation with the velocity field u_t . Then for any time interval $]a, b[$ we have the following estimate*

$$(3.3) \quad \frac{1}{2} \int_a^b \int_\Omega \left| -\frac{\nabla \varrho_t}{\varrho_t} + u_t \right|^2 \varrho_t dx dt \leq \mathcal{E}(\varrho_a) - \mathcal{E}(\varrho_b) + \frac{1}{2} \int_a^b \int_\Omega |u_t|^2 \varrho_t dx dt$$

In particular this implies

$$(3.4) \quad \frac{1}{2} \int_a^b |\varrho'_t|_{W_2}^2 dt \leq \mathcal{E}(\varrho_a) - \mathcal{E}(\varrho_b) + \frac{1}{2} \int_a^b \int_\Omega |u_t|^2 \varrho_t dx dt,$$

where $|\varrho'_t|_{W_2}$ denotes the metric derivative of the curve $t \mapsto \varrho_t \in \mathbb{W}_2(\Omega)$.

Proof. To prove this inequality, let us compute first

$$\begin{aligned} \frac{d}{dt} \mathcal{E}(\varrho_t) &= \int_\Omega (\log \varrho_t + 1) \partial_t \varrho_t dx = \int_\Omega \log \varrho_t (\Delta \varrho_t - \nabla \cdot (\varrho_t u_t)) dx \\ &= \int_\Omega \left(-\frac{|\nabla \varrho_t|^2}{\varrho_t} + u_t \cdot \nabla \varrho_t \right) dx, \end{aligned}$$

where we used the conservation of mass (hence $\int_\Omega \partial_t \varrho_t dx = 0$) and the Neumann boundary conditions in the integration by parts. We now compare this with

$$\begin{aligned} \frac{1}{2} \int_\Omega \left| -\frac{\nabla \varrho_t}{\varrho_t} + u_t \right|^2 \varrho_t dx - \frac{1}{2} \int_\Omega |u_t|^2 \varrho_t dx &= \int_\Omega \left(\frac{1}{2} \frac{|\nabla \varrho_t|^2}{\varrho_t} - \nabla \varrho_t \cdot u_t \right) dx \\ &\leq \int_\Omega \left(\frac{|\nabla \varrho_t|^2}{\varrho_t} - \nabla \varrho_t \cdot u_t \right) dx = -\frac{d}{dt} \mathcal{E}(\varrho_t). \end{aligned}$$

This provides the first part of the statement, i.e. (3.3). If we combine this with the fact that the metric derivative of the curve $t \mapsto \varrho_t$ is always less or equal than the $L^2_{\varrho_t}$ norm of the velocity field in the continuity equation, we also get

$$\frac{1}{2} |\varrho'_t|_{W_2}^2 - \frac{1}{2} \int_\Omega |u_t|^2 \varrho_t \leq -\frac{d}{dt} \mathcal{E}(\varrho_t),$$

and hence (3.4). □

Corollary 3.3. *From the inequality (3.4) we deduce that*

$$\mathcal{E}(\varrho_b) - \mathcal{E}(\varrho_a) \leq \frac{1}{2} \int_a^b \int_{\Omega} |u_t|^2 \varrho_t \, dx dt,$$

hence in particular for $u_t \in L^\infty(\Omega)^d$ we have that

$$\mathcal{E}(\varrho_b) - \mathcal{E}(\varrho_a) \leq \frac{1}{2} \|u\|_{L^\infty}^2 (b - a).$$

In particular, if $\varrho_a \leq 1$, then we have that

$$\mathcal{E}(\varrho_b) \leq \frac{1}{2} \|u\|_{L^\infty}^2 (b - a).$$

The same estimate can be applied to the curve $\tilde{\rho}^\tau$, with $a = k\tau$ and $b \in]k\tau, (k+1)\tau[$, thus obtaining $\mathcal{E}(\tilde{\rho}_t^\tau) \leq C\tau$ for every t .

Lemma 3.4. *For any $\rho \in \mathcal{P}(\Omega)$ we have $\mathcal{E}(P_{\mathcal{K}}[\rho]) \leq \mathcal{E}(\rho)$.*

Proof. We can assume $\rho \ll \mathcal{L}^d$ otherwise the claim is straightforward. As we pointed out in Section 2.2, we know that there exists a measurable set $B \subseteq \Omega$ such that

$$P_{\mathcal{K}}[\rho] = \mathbb{1}_B + \rho \mathbb{1}_{B^c}.$$

Hence it is enough to prove that

$$\int_B \rho \log \rho \, dx \geq 0 = \int_B P_{\mathcal{K}}[\rho] \log P_{\mathcal{K}}[\rho] \, dx,$$

as the entropies on B^c coincide. As the mass of ρ and $P_{\mathcal{K}}[\rho]$ is the same on the whole Ω , and they coincide on B^c , we have $\int_B \rho(x) \, dx = \int_B P_{\mathcal{K}}[\rho] \, dx = |B|$.

Then, by Jensen's inequality we have

$$\frac{1}{|B|} \int_B \rho \log \rho \, dx \geq \left(\frac{1}{|B|} \int_B \rho \, dx \right) \log \left(\frac{1}{|B|} \int_B \rho \, dx \right) = 0.$$

The entropy decay follows. □

To analyse the pressure field we will need the following result.

Lemma 3.5. *Let $\{p^\tau\}_{\tau>0}$ be a bounded sequence in $L^2([0, T]; H^1(\Omega))$ and $\{\rho^\tau\}_{\tau>0}$ a sequence of piecewise constant curves valued in $\mathbb{W}_2(\Omega)$, which satisfy $W_2(\rho^\tau(a), \rho^\tau(b)) \leq C\sqrt{b-a} + \tau$ for all $a < b \in [0, T]$ and $\rho^\tau \leq C$ for a fixed constant C . Suppose that*

$$p^\tau \geq 0, \quad p^\tau(1 - \rho^\tau) = 0, \quad \rho^\tau \leq 1,$$

and that

$$p^\tau \rightharpoonup p \text{ weakly in } L^2([0, T]; H^1(\Omega)) \quad \text{and} \quad \rho^\tau \rightarrow \rho \text{ uniformly in } \mathbb{W}_2(\Omega).$$

Then $p(1 - \rho) = 0$ a.e.

Proof. The proof of this result is the same as in Step 3 of Section 3.2 of [22] (see also [28]), hence we omit it. □

Lemma 3.6. (i) For every $\tau > 0$ and k we have

$$W_2^2(\rho_k^\tau, \tilde{\rho}_{k+1}^\tau), W_2^2(\rho_k^\tau, \rho_{k+1}^\tau) \leq \tau C (\mathcal{E}(\rho_k^\tau) - \mathcal{E}(\rho_{k+1}^\tau)) + C\tau^2,$$

where $C > 0$ only depends on $\|u\|_{L^\infty}$.

(ii) There exists a constant C , only depending on ρ_0 and $\|u\|_{L^\infty}$, such that $\mathcal{B}_2(E^\tau, \rho^\tau) \leq C$, $\mathcal{B}_2(\tilde{E}^\tau, \tilde{\rho}^\tau) \leq C$ and $\mathcal{B}_2(\hat{E}^\tau, \hat{\rho}^\tau) \leq C$.

(iii) For the curve $[0, T] \ni t \mapsto \rho_t^\tau$ we have that

$$\int_0^T |(\rho_t^\tau)'|_{W_2}^2 dt \leq C,$$

for a $C > 0$ independent of τ . In particular, we have a uniform Hölder bound on ρ_t : $W_2(\rho_t(a), \rho_t(b)) \leq C\sqrt{b-a}$ for every $b > a$.

(iv) $E^\tau, \tilde{E}^\tau, \hat{E}^\tau$ are uniformly bounded sequences in $\mathfrak{M}([0, T] \times \Omega)^d$.

Proof. (i) First by the triangle inequality and by the fact that $\rho_{k+1}^\tau = P_{\mathcal{K}}[\tilde{\rho}_{k+1}^\tau]$ we have that

$$(3.5) \quad W_2(\rho_k^\tau, \rho_{k+1}^\tau) \leq W_2(\rho_k^\tau, \tilde{\rho}_{k+1}^\tau) + W_2(\tilde{\rho}_{k+1}^\tau, \rho_{k+1}^\tau) \leq 2W_2(\rho_k^\tau, \tilde{\rho}_{k+1}^\tau).$$

We use (as before) the notation ϱ_t , $t \in [0, \tau]$ for the solution of the Fokker-Planck equation (3.1) with initial datum ρ_k^τ , in particular we have $\varrho_\tau = \tilde{\rho}_{k+1}^\tau$. Using Lemma 3.2 and since $\varrho_0 = \rho_k^\tau$ and $\varrho_\tau = \tilde{\rho}_{k+1}^\tau$ we have by (3.4) and using $W_2(\rho_k^\tau, \tilde{\rho}_{k+1}^\tau) \leq \int_0^\tau |\varrho_t'|_{W_2} dt$

$$\begin{aligned} W_2^2(\rho_k^\tau, \tilde{\rho}_{k+1}^\tau) &\leq \left(\tau^{\frac{1}{2}} \left(\int_0^\tau |\varrho_t'|_{W_2}^2 dt \right)^{\frac{1}{2}} \right)^2 \leq 2\tau (\mathcal{E}(\varrho_0) - \mathcal{E}(\varrho_\tau)) + \tau \int_0^\tau \int_\Omega |u_{k\tau+t}|^2 \varrho_t dx dt \\ &\leq 2\tau (\mathcal{E}(\rho_k^\tau) - \mathcal{E}(\tilde{\rho}_{k+1}^\tau)) + C\tau^2 \leq 2\tau (\mathcal{E}(\rho_k^\tau) - \mathcal{E}(\rho_{k+1}^\tau)) + C\tau^2, \end{aligned}$$

where $C > 0$ is a constant depending just on $\|u\|_{L^\infty}$. We also used the fact that $\mathcal{E}(\rho_{k+1}^\tau) \leq \mathcal{E}(\tilde{\rho}_{k+1}^\tau)$, a consequence of Lemma 3.4.

Now by the means of (3.5) we obtain

$$W_2^2(\rho_k^\tau, \rho_{k+1}^\tau) \leq \tau C (\mathcal{E}(\rho_k^\tau) - \mathcal{E}(\rho_{k+1}^\tau)) + C\tau^2.$$

(ii) We use Lemma 3.2 on the intervals of type $[k\tau, (k+1/2)\tau[$ and the fact that on each interval of type $[(k+1/2)\tau, (k+1)\tau[$ the curve ρ_t^τ is a constant speed geodesic. In particular, on these intervals we have

$$|(\rho^\tau)'|_{W_2} = \|v_t^\tau\|_{L^2_{\rho_t^\tau}} = 2\tau \|\nabla p_{k+1}^\tau\|_{L^2_{\rho_{k+1}^\tau}} = 2W_2(\rho_{k+1}^\tau, \tilde{\rho}_{k+1}^\tau).$$

On the other hand we also have

$$\tau^2 \|\nabla p_{k+1}^\tau\|_{L^2_{\rho_{k+1}^\tau}}^2 = W_2^2(\rho_{k+1}^\tau, \tilde{\rho}_{k+1}^\tau) \leq W_2^2(\rho_k^\tau, \tilde{\rho}_{k+1}^\tau) \leq \tau C (\mathcal{E}(\rho_k^\tau) - \mathcal{E}(\rho_{k+1}^\tau)) + C\tau^2.$$

Hence we obtain

$$\begin{aligned} &\int_{k\tau}^{(k+1)\tau} \|v_t^\tau\|_{L^2(\rho_t^\tau)}^2 dt \\ &= \int_{k\tau}^{(k+1/2)\tau} \int_\Omega 4 \left| -\frac{\nabla \varrho_{2(t-k\tau)}}{\varrho_{2(t-k\tau)}} + u_{2t-k\tau} \right|^2 \varrho_{2(t-k\tau)}(x) dx dt + 4 \int_{(k+1/2)\tau}^{(k+1)\tau} \int_\Omega |\nabla p_{k+1}^\tau|^2 \rho_{k+1}^\tau dx dt \\ &\leq C (\mathcal{E}(\rho_k^\tau) - \mathcal{E}(\rho_{k+1}^\tau)) + C\tau + 2\tau \|\nabla p_{k+1}^\tau\|_{L^2_{\rho_{k+1}^\tau}}^2 \end{aligned}$$

$$\leq C (\mathcal{E}(\rho_k^\tau) - \mathcal{E}(\rho_{k+1}^\tau)) + C\tau.$$

Hence by adding up we obtain

$$\mathcal{B}_2(E^\tau, \rho^\tau) \leq \sum_k \{C (\mathcal{E}(\rho_k^\tau) - \mathcal{E}(\rho_{k+1}^\tau)) + C\tau\} = C (\mathcal{E}(\rho_0^\tau) - \mathcal{E}(\rho_{N+1}^\tau)) + CT \leq C.$$

The estimate on $\mathcal{B}_2(\tilde{E}^\tau, \tilde{\rho}^\tau)$ and $\mathcal{B}_2(\hat{E}^\tau, \hat{\rho}^\tau)$ are completely analogous and descend from the previous computations.

(iii) The estimate on $\mathcal{B}_2(E^\tau, \rho^\tau)$ implies a bound on $\int_0^T |(\rho_t^\tau)'|_{W_2}^2 dt$ because v^τ is a velocity field for ρ^τ (i.e., the pair (E^τ, ρ^τ) solves the continuity equation).

(iv) In order to estimate the total mass of E we write

$$\begin{aligned} |E^\tau|([0, T] \times \Omega) &= \int_0^T \int_\Omega |v_t^\tau| \rho_t^\tau dx dt \leq \int_0^T \left(\int_\Omega |v_t^\tau|^2 \rho_t^\tau dx \right)^{\frac{1}{2}} \left(\int_\Omega \rho_t^\tau dx \right)^{\frac{1}{2}} dt \\ &\leq \sqrt{T} \left(\int_0^T \int_\Omega |v_t^\tau|^2 \rho_t^\tau dx dt \right)^{\frac{1}{2}} \leq C. \end{aligned}$$

The bounds on \tilde{E}^τ and \hat{E}^τ rely on the same argument. \square

Proof of Theorem 3.1. We use the tools from Lemma 3.6.

Step 1. By the bounds on the metric derivative of the curves ρ_t^τ we get compactness, i.e. there exists a curve $[0, T] \ni t \mapsto \rho_t \in \mathcal{P}(\Omega)$ such that ρ_t^τ converges uniformly in $[0, T]$ w.r.t. W_2 , in particular weakly-* in $\mathcal{P}(\Omega)$ for all $t \in [0, T]$. It is easy to see that $\tilde{\rho}^\tau$ and $\hat{\rho}^\tau$ are converging to the same curve. Indeed we have $\tilde{\rho}_t^\tau = \rho_{\tilde{s}(t)}^\tau$ and $\hat{\rho}_t^\tau = \rho_{\hat{s}(t)}^\tau$ for $|\tilde{s}(t) - t| \leq \tau$ and $|\hat{s}(t) - t| \leq \tau$, which implies $W_2(\rho_t^\tau, \tilde{\rho}_t^\tau), W_2(\rho_t^\tau, \hat{\rho}_t^\tau) \leq C\tau^{\frac{1}{2}}$. This provides the convergence to the same limit.

Step 2. By the boundedness of E^τ, \tilde{E}^τ and \hat{E}^τ in $\mathfrak{M}([0, T] \times \Omega)^d$ we have the existence of $E, \tilde{E}, \hat{E} \in \mathfrak{M}([0, T] \times \Omega)^d$ such that $E^\tau \xrightarrow{*} E, \tilde{E}^\tau \xrightarrow{*} \tilde{E}, \hat{E}^\tau \xrightarrow{*} \hat{E}$ as $\tau \rightarrow 0$. Now we show that $E = \tilde{E} - \hat{E}$. Indeed, let us show that for any $f \in \text{Lip}([0, T] \times \Omega)^d$ test function we have that

$$\left| \int_0^T \int_\Omega f_t \cdot (E_t^\tau - (\tilde{E}_t^\tau + \hat{E}_t^\tau)) (dx, dt) \right| \rightarrow 0,$$

as $\tau \rightarrow 0$. First for each $k \in \{0, \dots, N\}$ we have that

$$\begin{aligned} \int_{k\tau}^{(k+1/2)\tau} \int_\Omega f_t \cdot E_t^\tau (dx, dt) &= \int_{k\tau}^{(k+1)\tau} \int_\Omega f_{(t+k\tau)/2} \cdot (-\nabla \varrho_{t-k\tau} + u_t \varrho_{t-k\tau}) (dx, dt) \\ &= \int_{k\tau}^{(k+1)\tau} \int_\Omega f_t \cdot \tilde{E}_t^\tau (dx, dt) + \int_{k\tau}^{(k+1)\tau} \int_\Omega (f_{(t+k\tau)/2} - f_t) \cdot \tilde{E}_t^\tau (dx, dt) \end{aligned}$$

and

$$\begin{aligned} \int_{(k+1/2)\tau}^{(k+1)\tau} \int_\Omega f_t \cdot E_t^\tau (dx, dt) &= \int_{k\tau}^{(k+1)\tau} \int_\Omega -f_{(t+(k+1)\tau)/2} \circ (\text{id} + ((k+1)\tau - t)\nabla p_{k+1}^\tau) \cdot \nabla p_{k+1}^\tau \rho_{k+1}^\tau (dx, dt) \\ &= - \int_{k\tau}^{(k+1)\tau} \int_\Omega f_t \cdot \hat{E}_t^\tau (dx, dt) \\ &\quad + \int_{k\tau}^{(k+1)\tau} \int_\Omega (f_t - f_{(t+(k+1)\tau)/2} \circ (\text{id} + ((k+1)\tau - t))) \cdot \hat{v}_t^\tau \hat{\rho}_t^\tau (dx, dt) \end{aligned}$$

This implies that

$$\begin{aligned}
\left| \int_0^T \int_{\Omega} f_t \cdot (E_t^\tau - \tilde{E}_t^\tau + \hat{E}_t^\tau)(dx, dt) \right| &\leq \sum_k \int_{k\tau}^{(k+1)\tau} \text{Lip}(f)\tau \int_{\Omega} |\tilde{E}_t^\tau|(dx, dt) \\
&+ \sum_k \int_{k\tau}^{(k+1)\tau} \text{Lip}(f)\tau \int_{\Omega} (1 + |\hat{v}_t^\tau|) |\hat{E}_t^\tau|(dx, dt) \\
&\leq \tau C \text{Lip}(f) \left(|\tilde{E}^\tau|([0, T] \times \Omega) + |\hat{E}^\tau|([0, T] \times \Omega) + \mathcal{B}_2(\hat{E}, \hat{\rho}) \right) \\
&\leq \tau C \text{Lip}(f),
\end{aligned}$$

for a uniform constant $C > 0$. Letting $\tau \rightarrow 0$ we prove the claim.

Step 3. The bounds on $\mathcal{B}_2(E^\tau, \rho^\tau)$, $\mathcal{B}_2(\tilde{E}^\tau, \tilde{\rho}^\tau)$ and $\mathcal{B}_2(\hat{E}^\tau, \hat{\rho}^\tau)$ pass to the limit by semicontinuity and allow to conclude that E, \tilde{E} and \hat{E} are vector valued Radon measures absolutely continuous w.r.t. ρ . Hence there exist $v_t, \tilde{v}_t, \hat{v}_t \in L^2_{\rho_t}(\Omega)$ such that $E = \rho v$, $\tilde{E} = \rho \tilde{v}$ and $\hat{E} = \rho \hat{v}$.

Step 4. We now look at the equations satisfied by E, \tilde{E} and \hat{E} . First we use $\partial_t \rho^\tau + \nabla \cdot E^\tau = 0$, we pass to the limit as $\tau \rightarrow 0$, and we get

$$\partial_t \rho + \nabla \cdot E = 0.$$

Then, we use $\tilde{E}^\tau = -\nabla \tilde{\rho}^\tau + u_t \tilde{\rho}^\tau$, we pass to the limit again as $\tau \rightarrow 0$, and we get

$$\tilde{E} = -\nabla \rho + u_t \rho.$$

To justify the above limit, the only delicate point is passing to the limit the term $u_t \tilde{\rho}^\tau$, since u is only L^∞ , and $\tilde{\rho}^\tau$ converges weakly as measures, and we are a priori only allowed to multiply it by continuous functions. Yet, we remark that by Corollary 3.3 we have that $\mathcal{E}(\tilde{\rho}_t^\tau) \leq C\tau$ for all $t \in [0, T]$. In particular, this provides, for each t , uniform integrability for $\tilde{\rho}_t^\tau$ and turns the weak convergence as measures into weak convergence in L^1 . This allows to multiply by u_t in the weak limit.

Finally, we look at \hat{E}^τ . There exists a piecewise constant (in time) function p^τ (defined as p_{k+1}^τ on every interval $]k\tau, (k+1)\tau[$) such that $p^\tau \geq 0$, $p^\tau(1 - \hat{\rho}^\tau) = 0$,

$$(3.6) \quad \int_0^T \int_{\Omega} |\nabla p^\tau|^2(dx, dt) = \int_0^T \int_{\Omega} |\nabla p^\tau|^2 \hat{\rho}^\tau(dx, dt) = \int_0^T \int_{\Omega} |\hat{v}^\tau|^2 \hat{\rho}^\tau(dx, dt) \leq C$$

and $\hat{E}^\tau = \nabla p^\tau \hat{\rho}^\tau = \nabla p^\tau$. The bound (3.6) implies that p^τ is uniformly bounded in $L^2(0, T; H^1(\Omega))$. Since for every t we have $|\{p_t^\tau = 0\}| \geq |\{\hat{\rho}_t^\tau < 1\}| \geq |\Omega| - 1$, we can use a version of Poincaré's inequality, and get a uniform bound in $L^2([0, T]; L^2(\Omega)) = L^2([0, T] \times \Omega)$. Hence there exists $p \in L^2([0, T] \times \Omega)$ such that $p^\tau \rightharpoonup p$ weakly in L^2 as $\tau \rightarrow 0$. In particular we have $\hat{E} = \nabla p$. Moreover it is clear that $p \geq 0$ and by Lemma 3.5 we obtain $p(1 - \rho) = 0$ a.e. as well. Indeed, the assumptions of the Lemma are easily checked: we only need to estimate $W_2(\hat{\rho}^\tau(a), \hat{\rho}^\tau(b))$ for $b > a$, but we have

$$W_2(\hat{\rho}^\tau(a), \hat{\rho}^\tau(b)) = W_2(\rho^\tau(k_a\tau), \rho^\tau(k_b\tau)) \leq C \sqrt{k_b - k_a}, \quad \text{for } k_b\tau \leq b + \tau \text{ and } k_a \geq a.$$

Once we have $\hat{E} = \nabla p$ with $p(1 - \rho) = 0$, $p \in L^2([0, T]; H^1(\Omega))$ and $\rho \in L^\infty$, we can also write

$$\hat{E} = \nabla p = \rho \nabla p.$$

If we sum up our results, using $E = \tilde{E} - \hat{E}$, we have

$$\partial_t \rho - \Delta \rho + \nabla \cdot (\rho(u - \nabla p)) = 0 \quad \text{together with } p \geq 0, \rho \leq 1, p(1 - \rho) = 0.$$

As usual, this equation is satisfied in a weak sense, with Neumann boundary conditions. \square

4. UNIFORM $\text{Lip}([0, T]; \mathbb{W}_1)$ AND BV ESTIMATES

In this section we provide uniform estimates for the curves ρ^τ , $\tilde{\rho}^\tau$ and $\hat{\rho}^\tau$ of the following form: we prove uniform BV (in space) bounds on $\tilde{\rho}^\tau$ (which implies the same bound for $\hat{\rho}^\tau$) and uniform Lipschitz bounds in time for the W_1 distance for ρ^τ . This is a small improvement compared to the previous section, both in time regularity (Lipschitz instead of H^1 , but for W_1 instead of W_2) and in space (any higher order regularity of ρ was absent from the previous results). Nevertheless there is a price to pay for this improvement: we have to assume higher regularity for the velocity field. These uniform Lipschitz in time bounds are based both on BV estimates for the Fokker-Planck equation (see Lemma A.1 from Appendix A) and for the projection operator $P_{\mathcal{K}}$ (see [10]). The assumption on u is essentially the following: we need to control the growth of the total variation of the solutions of the Fokker-Planck equation (3.1), and we need to iterate this bound along time steps.

We will discuss in the Appendix the different BV estimates on the Fokker-Planck equation that we were able to find. The desired estimate is true for $u_t \in C^{1,1}(\Omega)$ and $u_t(x) \cdot n(x) = 0$ for $x \in \partial\Omega$, and seems to be an open problem if u is only Lipschitz continuous. We will also assume $\rho_0 \in BV(\Omega)$. Despite these extra regularity assumptions, we think these estimates have their own interest, exploiting some finer properties of the solutions of the Fokker-Planck equation and of the Wasserstein projection operator.

Before entering into the details of the estimates, we want to discuss why we concentrate on BV estimates (instead of Sobolev ones) and on W_1 (instead of W_p , $p > 1$). The main reason is the role of the projection operator: indeed, even if $\rho \in W^{1,p}(\Omega)$, we do not have in general $P_{\mathcal{K}}[\rho] \in W^{1,p}$ because the projection creates some jumps at the boundary of $\{P_{\mathcal{K}}[\rho] = 1\}$. This prevents from obtaining any $W^{1,p}$ estimate for $p > 1$. On the other hand, [10] exactly proves a BV estimate on $P_{\mathcal{K}}[\rho]$ and paves the way to BV bounds for our equation. Concerning the regularity in time, we observe that the velocity field in the Fokker-Planck equation contains a term in $\nabla\rho/\rho$. Since the metric derivative in \mathbb{W}_p is given by the L^p norm (w.r.t. ρ_t) of the velocity field, it is clear that estimates in \mathbb{W}_p for $p > 1$ would require spatial $W^{1,p}$ estimates on the solution itself, which are impossible for $p > 1$. The precise result that we prove is

Theorem 4.1. *Supposing $\|u_t\|_{C^{1,1}} \leq C$ and $\rho_0 \in BV(\Omega)$, then we $\|\tilde{\rho}_t^\tau\|_{BV} \leq C$ and $W_1(\rho_k^\tau, \rho_{k+1}^\tau) \leq C\tau$. As a consequence we also have $\rho \in \text{Lip}([0, T]; \mathbb{W}_1) \cap L^\infty([0, T]; BV(\Omega))$.*

To prove this theorem we need the following lemmas.

Lemma 4.2. *Suppose $\|u_t\|_{\text{Lip}} \leq C$ and $u_t \cdot n = 0$ on $\partial\Omega$. Then for the solution ϱ_t of (A.1) we have the estimate*

$$\|\varrho_t\|_{L^\infty} \leq \|\varrho_0\|_{L^\infty} e^{Ct},$$

where $C = \|\nabla \cdot u_t\|_{L^\infty}$.

Proof. Let us set $f : [0, +\infty[\times \Omega \rightarrow \mathbb{R}$, $f_t := \varrho_t e^{-Ct} \geq 0$ with a fixed constant $C > 0$. We have

$$\partial_t f_t = \partial_t \varrho_t e^{-Ct} - C f_t = e^{-Ct} (\Delta \varrho_t - \nabla \cdot (\varrho_t u_t)) - C f_t,$$

which means that f_t is a solution of

$$\partial_t f_t = \Delta f_t - \nabla \cdot (f_t u_t) - C f_t.$$

Now for a $p > 1$ let us denote $\mathcal{I}_p(t) := \|f_t\|_{L^p}^p$ and compute

$$\begin{aligned} \frac{d}{dt}\mathcal{I}_p(t) &= p \int_{\Omega} |f_t|^{p-1} \partial_t f_t \, dx = p \int_{\Omega} |f_t|^{p-1} (\Delta f_t - \nabla \cdot (f_t u_t) - C f_t) \, dx \\ &= -pC\mathcal{I}_p(t) - p(p-1) \int_{\Omega} |f_t|^{p-2} |\nabla f_t|^2 \, dx + p(p-1) \int_{\Omega} |f_t|^{p-1} \nabla f_t \cdot u_t \, dx \\ &\leq -pC\mathcal{I}_p(t) - (p-1) \int_{\Omega} |f_t|^p \nabla \cdot u_t \, dx \\ &\leq ((p-1)\|\nabla \cdot u_t\|_{L^\infty} - pC)\mathcal{I}_p(t) \end{aligned}$$

By Grönwall's lemma we obtain that

$$\mathcal{I}_p(t) \leq e^{t((p-1)\|\nabla \cdot u\|_{L^\infty} - pC)} \mathcal{I}_p(0),$$

hence

$$\mathcal{I}_p^{1/p}(t) \leq e^{t((p-1)/p\|\nabla \cdot u\|_{L^\infty} - C)} \mathcal{I}_p^{1/p}(0),$$

and sending $p \rightarrow +\infty$ we get that

$$\|f_t\|_{L^\infty} \leq e^{t(\|\nabla \cdot u\|_{L^\infty} - C)} \|f_0\|_{L^\infty}.$$

Using the definition of f_t we get the estimation

$$\|\varrho_t\|_{L^\infty} \leq e^{t\|\nabla \cdot u\|_{L^\infty}} \|\varrho_0\|_{L^\infty},$$

which proves the claim. \square

We remark that the above lemma implies in particular that after every step in the *Main scheme* we have $\tilde{\rho}_{k+1}^\tau \leq e^{c\tau} \leq 1 + C\tau$, where $c := \|\nabla \cdot u\|_{L^\infty}$. Let us now present the following lemma as well.

Corollary 4.3. *Along the iterations of our Main scheme, for every k we have $W_1(\tilde{\rho}_{k+1}^\tau, \rho_{k+1}^\tau) \leq \tau C$ for a constant $C > 0$ independent of τ .*

Proof. With the saturation property of the projection (see Section 2.2 or [10]), we know that there exists a measurable set $B \subseteq \Omega$ such that $\rho_{k+1}^\tau = \tilde{\rho}_{k+1}^\tau \mathbb{1}_B + \mathbb{1}_{\Omega \setminus B}$. On the other hand we know that

$$\begin{aligned} W_1(\tilde{\rho}_{k+1}^\tau, \rho_{k+1}^\tau) &= \sup_{f \in \text{Lip}_1(\Omega), 0 \leq f \leq \text{diam}(\Omega)} \int_{\Omega} f(\tilde{\rho}_{k+1}^\tau - \rho_{k+1}^\tau) \, dx \\ &= \sup_{f \in \text{Lip}_1(\Omega), 0 \leq f \leq \text{diam}(\Omega)} \int_{\Omega \setminus B} f(\tilde{\rho}_{k+1}^\tau - 1) \, dx \leq \tau C |\Omega| \text{diam}(\Omega). \end{aligned}$$

We used the fact that the competitors f in the dual formula can be taken positive and bounded by the diameter of Ω , just by adding a suitable constant. This implies as well that C is depending on $c, |\Omega|$ and $\text{diam}(\Omega)$. \square

Proof of Theorem 4.1. First we take care of the BV estimate. Lemma A.1 in the Appendix guarantees, for $t \in]k\tau, (k+1)\tau[$, that we have $TV(\tilde{\rho}_t^\tau) \leq C\tau + e^{C\tau} TV(\rho_k^\tau)$. Together with the BV bound on the projection that we presented in Section 2.2 (taken from [10]), this can be iterated, providing a uniform bound (depending on $TV(\rho_0)$, T and $\sup_t \|u_t\|_{C^{1,1}}$) on $\|\tilde{\rho}_t^\tau\|_{BV}$. Passing this estimate to the limit as $\tau \rightarrow 0$ we get $\rho \in L^\infty([0, T]; BV(\Omega))$.

Then we estimate the behavior in terms of W_1 . We estimate

$$\begin{aligned} W_1(\rho_k^\tau, \tilde{\rho}_{k+1}^\tau) &\leq \int_{k\tau}^{(k+1)\tau} |(\tilde{\rho}_t^\tau)'|_{W_1} dt \leq \int_{k\tau}^{(k+1)\tau} \int_{\Omega} \left(\frac{|\nabla \tilde{\rho}_t^\tau|}{\tilde{\rho}_t^\tau} + |u_t| \right) \tilde{\rho}_t^\tau dx dt \\ &\leq \int_{k\tau}^{(k+1)\tau} \|\tilde{\rho}_t^\tau\|_{BV} dt + C\tau \leq C\tau. \end{aligned}$$

Hence, we obtain

$$W_1(\rho_k^\tau, \rho_{k+1}^\tau) \leq W_1(\rho_k^\tau, \tilde{\rho}_{k+1}^\tau) + W_1(\tilde{\rho}_{k+1}^\tau, \rho_{k+1}^\tau) \leq \tau C.$$

This in particular means, for $b > a$,

$$W_1(\hat{\rho}^\tau(a), \hat{\rho}^\tau(b)) \leq C(b - a + \tau).$$

We can pass this relation to the limit, using that, for every t , we have $\hat{\rho}_t^\tau \rightarrow \rho_t$ in $\mathbb{W}_2(\Omega)$ (and hence also in $\mathbb{W}_1(\Omega)$, since $W_1 \leq W_2$), we get

$$W_1(\rho(a), \rho(b)) \leq C(b - a),$$

which means that ρ is Lipschitz continuous in $\mathbb{W}_1(\Omega)$. \square

5. VARIATIONS ON A THEME: SOME REFORMULATIONS OF THE *Main scheme*

In this section we propose some alternative approaches to study the problem (1.4). The general idea is to discretize in time, and give a way to produce a measure ρ_{k+1}^τ starting from ρ_k^τ . Observe that the interpolations that we proposed in the previous sections $\rho^\tau, \tilde{\rho}^\tau$ and $\hat{\rho}^\tau$ are only technical tools to state and prove a convergence result, and the most important point is exactly the definition of ρ_{k+1}^τ .

The alternative approaches proposed here explore different ideas, more difficult to implement than then one that we presented in Section 3, and/or restricted to some particular cases (for instance when u is a gradient). They have their own modeling interest and this is the main reason justifying their sketchy presentation.

5.1. Variant 1: transport, diffusion then projection. We recall that the original splitting approach for the equation without diffusion ([23, 28]) exhibited an important difference compared to what we did in Section 3. Indeed, in the first phase of each time step (i.e. before the projection) the particles follow the vector field u and $\tilde{\rho}_{k+1}^\tau$ was not defined as the solution of a continuity equation with advection velocity given by u_t , but as the image of ρ_k^τ via a straight-line transport: $\tilde{\rho}_{k+1}^\tau := (\text{id} + \tau u_{k\tau})\# \rho_k^\tau$. One can wonder whether it is possible to follow a similar approach here.

A possible way to proceed is the following: take a random variable X distributed according to ρ_k^τ , and define $\tilde{\rho}_{k+1}^\tau$ as the law of $X + \tau u_{k\tau}(X) + B_\tau$, where B is a Brownian motion, independent of X . This exactly means that every particle moves starting from its initial position X , following a displacement ruled by u , but adding a stochastic effect in the form of the value at time τ of a Brownian motion. We can check that this means

$$\tilde{\rho}_{k+1}^\tau := \eta_\tau * ((\text{id} + \tau u_{k\tau})\# \rho_k^\tau),$$

where η_τ is a Gaussian kernel with zero-mean and variance τ , i.e. $\eta_\tau(x) := \frac{1}{\sqrt{4\tau\pi}} e^{-\frac{|x|^2}{4\tau}}$.

Then we define

$$\rho_{k+1}^\tau := P_{\mathcal{K}}[\tilde{\rho}_{k+1}^\tau].$$

Despite the fact that this scheme is very natural and essentially not that different from the *Main scheme*, we have to be careful with the analysis. First we have to quantify somehow the distance $W_p(\rho_k^\tau, \tilde{\rho}_{k+1}^\tau)$ for some $p \geq 1$ and show that this is of order τ in some sense. Second, we need to be careful when performing the convolution with the heat kernel (or adding the Brownian motion, which is the same): this requires either to work in the whole space (which was not our framework) or in a periodic setting ($\Omega = \mathbb{T}^d$, the flat torus, which is quite restrictive). Otherwise, the “explicit” convolution step should be replaced with some other construction, such as following the Heat equation (with Neumann boundary conditions) for a time τ . But this brings back to a situation very similar to the *Main scheme*, with the additional difficulty that we do not really have estimates on $(\text{id} + \tau u_{k\tau})\#\rho_k^\tau$.

5.2. Variant 2: gradient flow techniques for gradient velocity fields. In this section we assume that the velocity field of the population is given by the opposite of the gradient of a function, $u_t = -\nabla V_t$ a typical example is given when we take for V the distance function to the exit (see the discussions in [22] about this type of question). We start from the case where V does not depend on time, and we suppose $V \in W^{1,1}(\Omega)$. In this particular case – beside the splitting approach – the problem has a variational structure, hence it is possible to show the existence by the means of gradient flows in Wasserstein spaces.

Since the celebrated paper of Jordan, Kinderlehrer and Otto ([13]) we know that the solutions of the Fokker-Planck equation (with a gradient vector field) can be obtained with the help of the gradient flow of a perturbed entropy functional with respect to the Wasserstein distance W_2 . This formulation of JKO scheme was also used in [22] for the first order model with density constraints. It is easy to combine the JKO scheme with density constraints to study the second order/diffusive model. As a slight modification of the model from [22], we can consider the following discrete implicate Euler (or JKO) scheme. As usual, we fix a time step $\tau > 0$, $\rho_0^\tau = \rho_0$ and for all $k \in \mathbb{N}$ we just need to define ρ_{k+1}^τ . We take

$$(5.1) \quad \rho_{k+1}^\tau = \operatorname{argmin}_{\rho \in \mathcal{P}(\Omega)} \left\{ \int_{\Omega} V(x)\rho(x) \, dx + \mathcal{E}(\rho) + I_{\mathcal{K}}(\rho) + \frac{1}{2\tau} W_2^2(\rho, \rho_k^\tau) \right\},$$

where $I_{\mathcal{K}}$ is the indicator function of \mathcal{K} , which is

$$I_{\mathcal{K}}(x) := \begin{cases} 0, & \text{if } x \in \mathcal{K}, \\ +\infty, & \text{otherwise.} \end{cases}$$

The usual techniques from [13, 22] can be used to identify that the problem (1.4) is the gradient flow of the functional $\rho \mapsto J(\rho) := \int_{\Omega} V(x)\rho(x) \, dx + \mathcal{E}(\rho) + I_{\mathcal{K}}(\rho)$ and that the above discrete scheme converges (up to a subsequence) to a solution of (1.4), thus proving existence. The key estimate for compactness is

$$\frac{1}{2\tau} W_2^2(\rho_{k+1}^\tau, \rho_k^\tau) \leq J(\rho_k^\tau) - J(\rho_{k+1}^\tau),$$

which can be summed up (as on the r.h.s. we have a telescopic series), thus obtaining the same bounds on \mathcal{B}_2 that we used in Section 3.

It is also possible to study a variant where V depends on time. We assume for simplicity that $V \in \text{Lip}([0, T] \times \Omega)$ (this is a simplification, less regularity in space, such as $W^{1,1}$, could be sufficient). In this case we define

$$J_t(\rho) := \int_{\Omega} V_t(x)\rho(x) \, dx + \mathcal{E}(\rho) + I_{\mathcal{K}}(\rho)$$

and

$$(5.2) \quad \rho_{k+1}^\tau = \operatorname{argmin}_{\rho \in \mathcal{P}(\Omega)} \left\{ J_{k\tau}(\rho) + \frac{1}{2\tau} W_2^2(\rho, \rho_k^\tau) \right\},$$

The analysis proceeds similarly, with the only exception that the we get

$$\frac{1}{2\tau} W_2^2(\rho_{k+1}^\tau, \rho_k^\tau) \leq J_{k\tau}(\rho_k^\tau) - J_{k\tau}(\rho_{k+1}^\tau),$$

which is no more a a telescopic series. Yet, we have $J_{k\tau}(\rho_{k+1}^\tau) \geq J_{(k+1)\tau}(\rho_{k+1}^\tau) + \operatorname{Lip}(V)\tau$, and we can go on with a telescopic sum plus a remainder of the order of τ .

5.3. Variant 3: transport then gradient flow-like step with the penalized entropy functional. We present now a different scheme, which combines some of the previous approaches. It could formally provide a solution of the same equation, but presents some extra difficulties.

We define now $\tilde{\rho}_{k+1}^\tau := (\operatorname{id} + \tau u_{k\tau})\# \rho_k^\tau$ and with the help of this we define

$$\rho_{k+1}^\tau := \operatorname{argmin}_{\rho \in \mathcal{K}} \mathcal{E}(\rho) + \frac{1}{2\tau} W_2^2(\rho, \tilde{\rho}_{k+1}^\tau).$$

In the last optimization problem we minimize a strictly convex and l.s.c. functionals, and hence we have existence and uniqueness of the solution. The formal reason for this scheme being adapted to the equation is that we perform a step of a JKO scheme in the spirit of [13] (without the density constraint) or of [22] (without the entropy term). This should let a term $-\Delta\rho - \nabla \cdot (\rho \nabla p)$ appear in the evolution equation. The term $\nabla \cdot (\rho u)$ is due to the first step (the definition of $\tilde{\rho}_{k+1}^\tau$). To explain a little bit more for the unexperienced reader, we consider the optimality conditions for the above minimization problem. Following [22], we can say that $\rho \in \mathcal{K}$ is optimal if and only if there exists a constant $\ell \in \mathbb{R}$ and a Kantorovich potential φ for the transport from ρ to ρ_k^τ such that

$$\rho = \begin{cases} 1 & \text{on } (\ln \rho + \frac{\varphi}{\tau}) < \ell, \\ 0 & \text{on } (\ln \rho + \frac{\varphi}{\tau}) > \ell, \\ \in [0, 1] & \text{on } (\ln \rho + \frac{\varphi}{\tau}) = \ell. \end{cases}$$

We then define $p = (\ell - \ln \rho - \frac{\varphi}{\tau})_+$ and we get $p \in \operatorname{press}(\rho)$. Moreover, ρ -a.e. $\nabla p = -\frac{\nabla \rho}{\rho} - \frac{\nabla \varphi}{\tau}$. We then use the fact that the optimal transport is of the form $T = \operatorname{id} - \nabla \varphi$ and obtain a situation as is sketched in Figure 2.

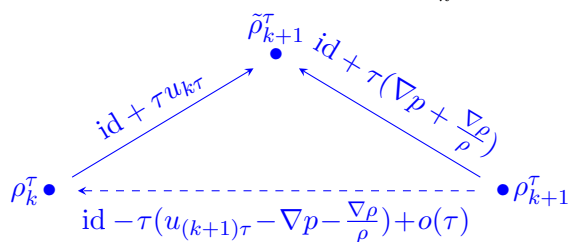


FIGURE 2. One time step

Notice that $(\operatorname{id} + \tau u_{k\tau})^{-1}(\operatorname{id} + \tau p) = \operatorname{id} - \tau(u_{(k+1)\tau} - \nabla p) + o(\tau)$ provided u is regular enough. Formally we can pass to the limit $\tau \rightarrow 0$ and have

$$\partial_t \rho - \Delta \rho + \nabla \cdot (\rho(u - \nabla p)) = 0.$$

Yet, this turns out to be quite naïve, because we cannot get proper estimates on $W_2(\rho_k^\tau, \rho_{k+1}^\tau)$. Indeed, this is mainly due to the hybrid nature of the scheme, i.e. a gradient flow for the diffusion and the projection part on one hand and a free transport on the other hand. The typical estimate in the JKO scheme comes from the fact that one can bound $W_2(\rho_k^\tau, \rho_{k+1}^\tau)^2/\tau$ with the opposite of the increment of the energy, and that this gives rise to a telescopic sum. Yet, this is not the case whenever the base point for a new time step is not equal to the previous minimizer. These kinds of difficulties are matter of current study, in particular for mixed systems and/or multiple populations.

APPENDIX A. *BV*-TYPE ESTIMATES FOR THE FOKKER-PLANCK EQUATION

Here we present some Total Variation (*TV*) decay results (in time) for the solutions of the Fokker-Planck equation. Some are very easy, some trickier. The goal is to look at those estimates which can be easily iterated in time and combined with the decay of the *TV* via the projection operator, as we did in Section 4.

Let us take a vector field $v : [0, +\infty[\times \Omega \rightarrow \mathbb{R}^d$ (we will choose later which regularity we need) and consider in Ω the problem

$$(A.1) \quad \begin{cases} \partial_t \rho_t - \Delta \rho_t + \nabla \cdot (\rho_t v_t) = 0, & \text{in }]0, +\infty[\times \Omega, \\ (\nabla \rho_t - v_t) \cdot n = 0, & \text{on }]0, +\infty[\times \partial\Omega, \\ \rho(0, \cdot) = \rho_0, & \text{in } \Omega, \end{cases}$$

for $\rho_0 \in BV(\Omega) \cap \mathcal{P}(\Omega)$.

Lemma A.1. *Suppose $\|v_t\|_{C^{1,1}} \leq C$ for a.e. $t \in [0, +\infty[$. Suppose that either $\Omega = \mathbb{T}^d$, or that Ω is convex and $v \cdot n = 0$ on $\partial\Omega$. Then, we have the following total variation decay estimate*

$$(A.2) \quad \int_{\Omega} |\nabla \rho_t| dx \leq C(t-s) + e^{C(t-s)} \int_{\Omega} |\nabla \rho_s| dx, \quad \forall 0 \leq s \leq t,$$

where $C > 0$ is a constant depending just on the $C^{1,1}$ norm of v .

Proof. First we remark that by the regularity of v the quantity $\|v\|_{L^\infty} + \|Dv\|_{L^\infty} + \|\nabla(\nabla \cdot v)\|_{L^\infty}$ is uniformly bounded. Let us drop now the dependence on t in our notation and calculate in coordinates

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} |\nabla \rho| dx &= \int_{\Omega} \frac{\nabla \rho}{|\nabla \rho|} \cdot \nabla(\partial_t \rho) dx = \int_{\Omega} \frac{\nabla \rho}{|\nabla \rho|} \cdot \nabla(\Delta \rho - \nabla \cdot (v\rho)) dx = \int_{\Omega} \sum_j \frac{\rho_j}{|\nabla \rho|} \left(\sum_i \rho_{ij} - (\nabla \cdot (v\rho))_j \right) dx \\ &= - \int_{\Omega} \sum_{i,j,k} \left(\frac{\rho_{ij}^2}{|\nabla \rho|} - \frac{\rho_j \rho_k \rho_{ki} \rho_{ij}}{|\nabla \rho|^3} \right) dx + B_1 - \int_{\Omega} \sum_{j,i} \frac{\rho_j}{|\nabla \rho|} (v_{ij}^i \rho + v_i^i \rho_j + v_j^i \rho_i + v^i \rho_{ij}) dx \\ &\leq B_1 + C + C \int_{\Omega} |\nabla \rho| dx + \int_{\Omega} |\nabla \rho| |\nabla \cdot v| dx + B_2 \\ &\leq B_1 + B_2 + C + C \int_{\Omega} |\nabla \rho| dx. \end{aligned}$$

Here the B_i are the boundary terms, i.e.

$$B_1 := \int_{\partial\Omega} \sum_{i,j} \frac{\rho_j n^i \rho_{ij}}{|\nabla \rho|} d\mathcal{H}^{d-1} \quad \text{and} \quad B_2 := - \int_{\partial\Omega} v \cdot n |\nabla \rho| d\mathcal{H}^{d-1}.$$

The constant $C > 0$ only depends on $\|v\|_{L^\infty} + \|\nabla \cdot v\|_{L^\infty} + \|\nabla(\nabla \cdot v)\|_{L^\infty}$. We used as well the fact that $-\int_{\Omega} \sum_{i,j,k} \left(\frac{\rho_{ij}^2}{|\nabla \rho|} - \frac{\rho_j \rho_k \rho_{ki} \rho_{ij}}{|\nabla \rho|^3} \right) dx \leq 0$.

Now, it is clear that in the case of the torus the boundary terms B_1 and B_2 do not exist, hence we conclude by Grönwall's lemma. In the case of the convex domain we have $B_2 = 0$ (because of the assumption $v \cdot n = 0$) and $B_1 \leq 0$ because of the next Lemma A.2. \square

Lemma A.2. *Suppose that $u : \Omega \rightarrow \mathbb{R}^d$ is a smooth vector field with $u \cdot n = 0$ on $\partial\Omega$, ρ is a smooth function with $\nabla \rho \cdot n = 0$ on $\partial\Omega$, and that $\Omega \subset \mathbb{R}^d$ is a smooth convex set parametrized as*

$\Omega = \{h < 0\}$ for a smooth convex function h with $|\nabla h| = 1$ on $\partial\Omega$ (so that $n = \nabla h$ on $\partial\Omega$). Then we have, on the whole boundary $\partial\Omega$, $\sum_{i,j} u_j^i \rho_j n^i = - \sum_{i,j} u^i h_{ij} \rho_j$.

In particular, we have $\sum_{i,j} \rho_{ij} \rho_j n^i \leq 0$.

Proof. The Neumann boundary assumption on u means $u(\gamma(t)) \cdot \nabla h(\gamma(t)) = 0$ for every curve γ valued in $\partial\Omega$ and for all t . Differentiating in t , we get

$$\sum_{i,j} u_j^i(\gamma(t)) (\gamma'(t))^j h_i(\gamma(t)) + \sum_{i,j} u^i(\gamma(t)) h_{ij}(\gamma(t)) (\gamma'(t))^j = 0.$$

Take a point $x_0 \in \partial\Omega$ and choose a curve γ with $\gamma(t_0) = x_0$ and $\gamma'(t_0) = \nabla \rho(x_0)$ (which is possible, since this vector is tangent to $\partial\Omega$ by assumption). This gives the first part of the statement. The second part, i.e. $\sum_{i,j} \rho_{ij} \rho_j n^i \leq 0$, is obtained by taking $u = \nabla \rho$ and using that $D^2 h(x_0)$ is a positive definite matrix. \square

Remark A.3. If we look attentively at the proof of Lemma A.1, we can see that we did not really exploit the regularizing effects of the diffusion term in the equation. This means that the regularity estimate that we provide are the same that we would have without diffusion: in this case, the density ρ_t is obtained from the initial density as the image through the flow of v . Thus, the density depends on the determinant of the Jacobian of the flow, hence on the derivatives of v . It is normal that, if we want BV bounds on ρ_t , we need assumptions on two derivatives of v .

We would like to prove some form of BV estimates under weaker regularity assumptions on v , trying to exploit the diffusion effects. In particular, we would like to treat the case where v is only $C^{0,1}$. As we will see in the following lemma, this degenerates in some sense.

Lemma A.4. *Suppose that Ω is either the torus or a smooth convex set $\Omega = \{h < 0\}$ parameterized as a level set of a smooth convex function h . Let $v_t : \Omega \rightarrow \mathbb{R}^d$ be a vector field for $t \in [0, T]$, Lipschitz and bounded in space, uniformly in time. In the case of a convex domain, suppose $v \cdot n = 0$ on $\partial\Omega$. Let $H : \mathbb{R}^d \rightarrow \mathbb{R}$ be given by $H(z) := \sqrt{\varepsilon^2 + |z|^2}$. Now let ρ_t (sufficiently smooth) be the solution of the Fokker-Planck equation with homogeneous Neumann boundary condition.*

Then there exists a constant $C > 0$ (depending on v and Ω) such that

$$(A.3) \quad \int_{\Omega} H(\nabla \rho_t) dx \leq \int_{\Omega} H(\nabla \rho_0) dx + C\varepsilon t + \frac{C}{\varepsilon} \int_0^t \|\rho_s\|_{L^\infty}^2 ds.$$

Proof. First let us discuss about some properties of H . It is smooth, its gradient is $\nabla H(z) = \frac{z}{H(z)}$ and it satisfies $\nabla H(z) \cdot z \leq H(z)$, $\forall z \in \mathbb{R}^d$. Moreover its Hessian matrix is given by

$$[H_{ij}(z)]_{i,j \in \{1, \dots, d\}} = \left[\frac{\delta^{ij} H^2(z) - z^i z^j}{H^3(z)} \right]_{i,j \in \{1, \dots, d\}} = \frac{1}{H(z)} I_d - \frac{1}{H^3(z)} z \otimes z, \quad \forall z \in \mathbb{R}^d,$$

where $\delta^{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } j \neq i, \end{cases}$ is the Kronecker symbol. Note that, from this computation, the matrix $D^2 H \geq 0$ is bounded from above by $\frac{1}{H}$, and hence by ε^{-1} . Moreover we introduce a uniform constant $C > 0$ such that $\|v\|_{L^\infty}^2 |\Omega| + \|\nabla \cdot v\|_{L^\infty} + \|Dv\|_{L^\infty} \leq C$.

Now to show the estimate of this lemma we calculate the quantity $\frac{d}{dt} \int H(\nabla \rho_t) dx$.

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} H(\nabla \rho_t) dx &= \int_{\Omega} \nabla H(\nabla \rho_t) \cdot \partial_t \nabla \rho_t dx = \int_{\Omega} \nabla H(\nabla \rho_t) \cdot \nabla (\Delta \rho_t - \nabla \cdot (v_t \rho_t)) dx \\ &= \int_{\Omega} \nabla H(\nabla \rho_t) \cdot \nabla \Delta \rho_t dx - \int_{\Omega} \nabla H(\nabla \rho_t) \cdot \nabla (\nabla \cdot (v_t \rho_t)) dx \\ &=: (I) + (II) \end{aligned}$$

Now we study each term separately and for the simplicity we drop the t subscripts in the followings. We start from the case of the torus, where there is no boundary term in the integration by parts.

$$\begin{aligned} (I) &= \int_{\Omega} \nabla H(\nabla \rho) \cdot \nabla \Delta \rho dx = \int_{\Omega} \sum_{j,i} H_j(\nabla \rho) \rho_{jii} dx = - \int_{\Omega} \sum_{j,i,k} H_{kj}(\nabla \rho) \rho_{ik} \rho_{ji} dx \\ (II) &= - \int_{\Omega} \nabla H(\nabla \rho) \cdot \nabla (\nabla \cdot (v \rho)) dx = - \int_{\Omega} \sum_{i,j} H_j(\nabla \rho) (v^i \rho)_{ij} dx \\ &= \int_{\Omega} \sum_{i,j,k} H_{jk}(\nabla \rho) \rho_{ki} v_j^i \rho dx + \int_{\Omega} \sum_{i,j,k} H_{jk}(\nabla \rho) \rho_{ki} v^i \rho_j dx \\ &=: (II_a) + (II_b). \end{aligned}$$

First look at the term (II_a) . Since the matrix H_{jk} is positive definite, we can apply a Young inequality for each index i and obtain

$$\begin{aligned} (II_a) = \int_{\Omega} \sum_{i,j,k} H_{jk}(\nabla \rho) \rho_{ki} v_j^i \rho dx &\leq \frac{1}{2} \int_{\Omega} \sum_{i,j,k} H_{jk}(\nabla \rho) \rho_{ki} \rho_{ij} dx + \frac{1}{2} \int_{\Omega} \sum_{i,j,k} H_{jk}(\nabla \rho) v_j^i v_k^i \rho^2 dx \\ &\leq \frac{1}{2} |(I)| + C \|\rho\|_{L^2}^2 \|D^2 H\|_{L^\infty}. \end{aligned}$$

The L^2 norm in the second term will be estimated by the L^∞ norm for the sake of simplicity (see Remark A.5 below).

For the term (II_b) we first make a point-wise computation

$$\begin{aligned} \sum_{i,j,k} H_{jk}(\nabla \rho) \rho_{ki} v^i \rho_j &= \frac{1}{H^3(\nabla \rho)} \sum_i [D_i^2 \rho \cdot (\varepsilon^2 I_d + |\nabla \rho|^2 I_d - \nabla \rho \otimes \nabla \rho) \cdot \nabla \rho] v^i \\ &= \frac{\varepsilon^2}{H^3(\nabla \rho)} \sum_i v^i D_i^2 \rho \cdot \nabla \rho = -\varepsilon^2 \sum_i v^i \partial_i \left(\frac{1}{H(\nabla \rho)} \right). \end{aligned}$$

where $D_i^2 \rho$ denotes the i^{th} row in the Hessian matrix of ρ and we used $(|\nabla \rho|^2 I_d - \nabla \rho \otimes \nabla \rho) \cdot \nabla \rho = 0$.

Integrating by parts we obtain

$$(II_b) = \varepsilon^2 \int_{\Omega} (\nabla \cdot v) \frac{1}{H(\nabla \rho)} dx \leq C \varepsilon^2 \|1/H\|_{L^\infty} \leq C \varepsilon,$$

where we used $H(z) \geq \varepsilon$.

Summing up all the terms we get and using $\|D^2H\| \leq \varepsilon^{-1}$ we get

$$\frac{d}{dt} \int_{\Omega} H(\nabla \rho_t) dx \leq -\frac{1}{2}|(I)| + C\|\rho_t\|_{L^\infty}^2 \|D^2H\|_{L^\infty} + C\varepsilon \leq C\varepsilon + C\|\rho_t\|_{L^\infty}^2 \varepsilon^{-1},$$

which proves the claim.

If we switch to the case of a smooth bounded convex domain Ω , we have to handle boundary terms. These terms are

$$\int_{\partial\Omega} \sum_{i,j} H_j(\nabla \rho) \rho_{ij} n^i - \int_{\partial\Omega} \sum_{i,j} H_j(\nabla \rho) \rho v_j^i n^i,$$

where we ignored those terms involving $n^i v^i$ (i.e., the integration by parts in (II_b)), and the term $H_j(\nabla \rho) \rho_j n^i v^i$ in the integration by parts of (II_a) , since we already supposed $v \cdot n = 0$. We use here Lemma A.2, which provides

$$\sum_{i,j} H_j(\nabla \rho) \rho_{ij} n^i - \rho H_j(\nabla \rho) v_j^i n^i = \frac{1}{H(\nabla \rho)} \sum_{i,j} (\rho_j \rho_{ij} n^i - \rho \rho_j v_j^i n^i) = -\frac{1}{H(\nabla \rho)} \sum_{i,j} (\rho_j h_{ij} \rho_i - \rho \rho_j h_{ij} v^i).$$

If we use the fact that the matrix D^2h is positive definite and a Young inequality, we get $\sum_{i,j} \rho_j h_{ij} \rho_i \geq 0$ and

$$\rho \sum_{i,j} |\rho_j h_{ij} v^i| \leq \frac{1}{2} \sum_{i,j} \rho_j h_{ij} \rho_i + \frac{1}{2} \sum_{i,j} \rho^2 v^j h_{ij} v^i,$$

which implies

$$\frac{1}{H(\nabla \rho)} \sum_{i,j} (\rho_j \rho_{ij} n^i - \rho \rho_j v_j^i n^i) \leq \frac{\rho^2}{H(\nabla \rho)} \|D^2h\|_{L^\infty} |v|^2 \leq \frac{C\|\rho\|_{L^\infty}^2}{\varepsilon}.$$

This provides the desired estimate on the boundary term. \square

Remark A.5. In the above proof, we needed to use the L^∞ norm of ρ only in the boundary term. When there is no boundary term, the L^2 norm is enough, in order to handle the term (II_a) . In both cases, the norm of ρ can be bounded in terms of the initial norm multiplied by e^{Ct} , where C bounds the divergence of v . On the other hand, in the torus case, one only needs to suppose $\rho_0 \in L^2$ and in the convex case $\rho_0 \in L^\infty$. Both assumptions are satisfied in the applications to crowd motion with density constraints.

We have seen that the constants in the above inequality depend on ε and explode as $\varepsilon \rightarrow 0$. This prevents us to obtain a clean estimate on the BV norm in this context, but at least proves that $\rho_0 \in BV \Rightarrow \rho_t \in BV$ for all $t > 0$ (to achieve this result, we just need to take $\varepsilon = 1$). Unfortunately, the quantity which is estimated is not the BV norm, but the integral $\int H(\nabla \rho)$. This is not enough for the purpose of the applications to Section 4, as it is unfortunately not true that the projection operator decreases the value of this other functional³.

If we stay interested to the value of the BV norm, we can provide the following estimate.

³Here is a simple counter-example: consider $\mu = g(x)dx$ a BV density on $[0, 2] \subset \mathbb{R}$, with g defined as follows. Divide the interval $[0, 2]$ into $2K$ intervals J_i of length $2r$ (with $2rK = 1$); call t_i the center of each interval J_i (i.e. $t_i = i2r + r$, for $i = 0, \dots, 2K - 1$) and set $g(x) = L + \sqrt{r^2 - (x - t_i)^2}$ on each J_i with i odd, and $g(x) = 0$ on J_i for i even, taking $L = 1 - \pi r/4$. It is not difficult to check that the projection of μ is equal to the indicator function of the union of all the intervals J_i with i odd, and that the value of $\int H(\nabla \rho)$ has increased by $K(2 - \pi/2)r = 1 - \pi/4$, i.e. by a positive constant (see Figure 3).

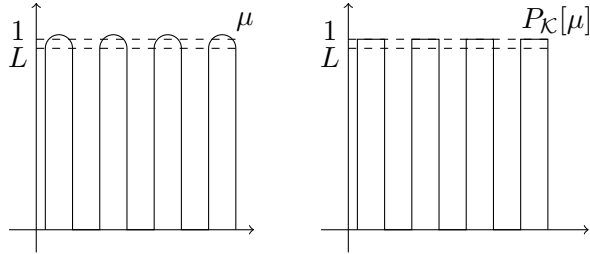


FIGURE 3. The counter-example to the decay of $\int H(\nabla\rho)$, which corresponds to the total length of the graph

Lemma A.6. *Under the assumptions of Lemma A.4, if we suppose $\rho_0 \in BV(\Omega) \cap L^\infty(\Omega)$, then, for $t \leq T$, we have*

$$(A.4) \quad \int_{\Omega} |\nabla\rho_t| \, dx \leq \int_{\Omega} |\nabla\rho_0| \, dx + C\sqrt{t},$$

where the constant C depends on v , on T and on $\|\rho_0\|_{L^\infty}$.

Proof. Using the L^∞ estimate of Lemma 4.2, we will assume that $\|\rho_t\|_{L^\infty}$ is bounded by a constant (which depends on v , on T and on $\|\rho_0\|_{L^\infty}$). Then, we can write

$$\int_{\Omega} |\nabla\rho_t| \, dx \leq \int_{\Omega} H(\nabla\rho_t) \, dx \leq \int_{\Omega} H(\nabla\rho_0) \, dx + C\varepsilon t + \frac{Ct}{\varepsilon} \leq \int_{\Omega} (|\nabla\rho_0| + \varepsilon) \, dx + C\varepsilon t + \frac{Ct}{\varepsilon}.$$

It is sufficient to choose, for fixed t , $\varepsilon = \sqrt{t}$, in order to prove the claim. \square

Unfortunately, this \sqrt{t} behavior is not suitable to be iterated, and the above estimate is useless for the sake of Section 4. The existence of an estimate (for v Lipschitz) of the form $TV(\rho_t) \leq TV(\rho_0) + Ct$, or $TV(\rho_t) \leq TV(\rho_0)e^{Ct}$, or even $f(TV(\rho_t)) \leq f(TV(\rho_0))e^{Ct}$, for any increasing function $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, seems to be an open question.

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