Differential games with incomplete information on a continuum of initial positions and without Isaacs condition *

Chloe Jimenez[1], Marc Quincampoix[1], Yuhong Xu[2,1]
[1]Laboratoire de Mathématiques de Bretagne Atlantique, CNRS-UMR 6205,
Université de Brest, 6 avenue Victor le Gorgeu, 29200 Brest, France.
[2]Mathematical center for interdiscipline research and
School of mathematical sciences, Soochow University,
Suzhou 215006, P. R. China.
E-mails: chloe.jimenez@univ-brest.fr; marc.quincampoix@univ-brest.fr;
yuhong.xu@hotmail.com

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Abstract. This article deals with a two-player zero-sum differential game with infinitely many initial positions and without Isaacs condition. The structure of information is asymmetric: The first player has a private information on the initial position while the second player knows only a probability distribution on the initial position. In the present model, we face two difficulties: First, the incomplete information structure does not reduce to a finite set (as in the famous Aumann-Maschler model for repeated games). Second, the game does not satisfy the Isaacs condition (crucially used in classical approaches to differential games). Therefore, we use tools from optimal transportation theory and stochastic control.

Our main result shows that with a suitable concept of mixed strategies, there exists a value of the game with such random strategies.

As a byproduct of our approach, we obtain the Lipschitz continuity of the random value with respect to the Wasserstein distance and we show the existence of a value in pure strategies in the specific case of an initial distribution without atoms. We also discuss an extension of our model when the asymmetric information concerns continuous scenarios.

Key words. zero-sum differential game; asymmetric information; Isaacs condition; continuous initial distribution;

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Introduction

We consider a two-player differential game with dynamics described by the following ordinary differential equation

$$\frac{dX_s}{ds} = f\left(X_s, u_s, v_s\right), \quad s \in [t, T] \tag{1}$$

parameterized by two measurable controls $u : [t, T] \mapsto U$ and $v : [t, T] \mapsto V$ chosen by the players 1 and 2, respectively. Here U and V denote two compact metric spaces. The function $f : \mathbf{R}^n \times U \times V \mapsto \mathbf{R}^n$ satisfies standard regularity assumptions in order to have a unique solution $s \mapsto X_s^{x,u,v}$ of (1) associated with an initial condition $X_0 = x$ and a pair of measurable controls (u, v).

The first player aims to minimize the payoff $g(X_T^{x,u,v})$, while the second player's objective is to maximize it (where $g : \mathbf{R}^n \mapsto \mathbf{R}$ is a given cost function).

Let us now describe the structure of information and how the game is played. For this let us fix the initial time $t = 0$.

- before the game starts, the initial position x is chosen randomly according to a probability measure μ,
- the initial state x is communicated to player 1 but not to player 2,
- the game is played on the time interval $[0, T]$,
- both players know the probability μ and observe their opponents controls.

In the game we investigate here, the role of the information is crucial. Indeed the second player does not know what the current state of the game is. However he can try to guess it at least partially by observing the actions of the first player. For this reason the first player's interest is to hide as much as possible his actions by playing randomly (choosing a random strategy), of course still trying to achieve his own goals. The second player's interest is also to conceal his actions by playing random strategies.

Such a game with asymmetric information was first introduced by Aumann and Maschler [3] in the framework of repeated games, where the first player has a private information not available for the second player. This was extended by Mertens and Zamir [21] to the case where both players have a private information not available for their opponents.

A pioneering work in the literature of differential games is the article [15] where the authors proved the existence of the value of the game under the following Isaacs condition:

$$\forall\, (x, \xi) \in \mathbf{R}^n \times \mathbf{R}^n, \ \inf_{u \in U} \sup_{v \in V} f\left(x, u, v\right) \cdot \xi = \sup_{v \in V} \inf_{u \in U} f\left(x, u, v\right) \cdot \xi. \tag{2}$$

They also show that the value function is given as the unique viscosity solution of a Hamilton–Jacobi–Isaacs equation. We refer the reader to [5] for an overview and a more complete description.

Differential games with incomplete/asymmetric information on the payoff were studied first in [9] (see also [12, 14]). The case of a continuum of types for games in continuous times has been addressed only recently in [14], in a very particular situation where there is no dynamics and where the information issue lies on the payoff.

In [10], the authors studied a differential game with *symmetric* unperfect information on the initial state. Both players are assumed to know the initial probability, but the "lack" of information treated in that paper is very specific: it is symmetric for both players and is only concerned with the current position of the game. The case with incomplete and asymmetric information on a possibly infinite set of the initial positions was studied in [11] under the supposition that the Isaacs condition (2) holds true.

Without Isaacs condition, one can easily find examples - even with complete information structure - where the value of the game does not exist in pure strategies. It is thus important to define a good concept of mixed strategies. There are only few results for differential games without Isaacs conditions. We mention [19] for differential games with positional strategies and [2] for repeated games in continuous time. Recently, the article [6] considered zero-sum differential games with complete information without Isaacs condition by imposing on the underlying controls for both players a conditional independence property (cf also [7] for an extension to stochastic differential games using backward stochastic differential equations).

The present paper investigates differential games with an incomplete information on infinitely many initial positions and without Isaacs condition. Our main result shows that it is possible to define a suitable concept of mixed strategy and to prove the existence of a value of the game associated to these strategies.

We consider a partition π of the time interval $[0, T]$. By using a suitable notion of random non-anticipative strategies with delay compatible with the partition π, we define upper and lower values (which depends on the partition). We show that both upper and lower values converge - as the mesh of π converges to zero - to the same limit which is the value of the game. The use of a partition in our approach is crucial due to the lack of Isaacs condition. Surprisingly, when the initial measure has no atoms, we show that a value exists in pure strategies. This can be somehow explained by the fact that when the initial measure has no atoms, it contains "enough randomness" to avoid the use of random strategies. This phenomenon is discussed more rigourously in Theorem 4.1.

We also explain how these results can be extended to games where the incomplete information concerns also the dynamic and the payoff. We also prove the existence of the value for these games with incomplete information on infinitely many scenarios

The organization of the paper is as follows. The next section concerns preliminaries and basic facts on probability measure spaces and the complete description of the model. Section 2 deals with strategies and value functions. Section 3 is devoted to the Lipschitz continuity of the values in random strategies along a partition π. As a consequence of this uniform continuity, the existence of the value in random strategies is obtained for arbitrary probability measure μ. In section 4, we show the existence of a value in pure strategy when the initial measure μ has no atoms. Lastly, the above model is extended to differential games with continuous scenarios.

1 Preliminaries

This section is devoted to fix the notations and assumptions. We also recall some known facts on measure theory and optimal transportation.

1.1 Assumptions on the dynamics

Throughout the paper we will consider $\mathbf{R}^n$ endowed with the Euclidian distance $|\cdot|$ and $\mathcal{B}(\mathbf{R}^n)$ the Borel σ-field.

The dynamics of the game is given by

$$\left\{ \begin{array}{l} dX_s^{x,u,v} = f\left(X_s^{x,u,v}, u_s, v_s\right) ds, \quad s \in [0,T]\,, \ (u,v) \in \mathcal{U}_{0,T} \times \mathcal{V}_{0,T}, \\ X_0^{x,u,v} = x \in \mathbf{R}^n, \end{array} \right. \tag{3}$$

The controls u and v are supposed to be admissible according to the following

Definition 1.1 *(Admissible controls) Let $t = 0$ be the initial time. We define respectively the sets of admissible controls for player 1 and player 2 by*
$\mathcal{U}_{0,T} = \{u : [0,T] \mapsto U,$ *which is Lebesque measurable* $\}$;
$\mathcal{V}_{0,T} = \{v : [0,T] \mapsto V,$ *which is Lebesque measurable* $\}$;
Both spaces are endowed with the topology generated by the convergence in $L^1([0,T])$.

Throughout the paper we assume that $f : \mathbf{R}^n \times U \times V \mapsto \mathbf{R}^n$ is bounded, continuous, and Lipschitz continuous in $x \in \mathbf{R}^n$, uniformly with respect to u and v.

Under the above assumption it is well-known that (3) has a unique solution $s \mapsto X_s^{x,u,v}$ which satisfies furthermore that there exists a constant $C > 0$ such that, for all $x, x' \in \mathbf{R}^n$ and all $s \in [0,T]$,
(i) $|X_s^{x,u,v} - x| \leq Cs$,
(ii) $\left| X_s^{x,u,v} - X_s^{x',u,v} \right| \leq C\,|x - x'|$.

Throughout the paper we assume that $g : \mathbf{R}^n \mapsto \mathbf{R}$ is a Lipschitz continuous and bounded function.

1.2 Probability distribution on the initial condition

Notation. We denote by $\mathcal{P}(\mathbf{R}^n)$ the set of Borel probability measures on $\mathbf{R}^n$ with a compact support and with finite moment of order two and by $\mathcal{P}(K)$ the set of Borel probability measures on K; $K \subset \mathbf{R}^n$ being a fixed compact set which is the closure of its interior. The space $\mathcal{C}(K)$ denotes the set of continuous functions on K.
We endow $\mathcal{P}(K)$ with the weak star topology that is:

$$\mu_n \to \mu \text{ if and only if } \int_K \varphi(x)\; d\mu_n(x) \to \int_K \varphi(x)\; d\mu(x) \;\text{ for all } \varphi \in \mathcal{C}(K).$$

Definition 1.2 *Let μ, ν in $\mathcal{P}(\mathbf{R}^n)$. We call the Wasserstein distance between μ and ν the following quantity:*

$$W_2(\mu,\nu) = \inf_{\gamma \in \mathcal{P}(\mathbf{R}^n \times \mathbf{R}^n)} \left\{ \left(\int_{\mathbf{R}^n \times \mathbf{R}^n} |x - y|^2 \; d\gamma(x,y) \right)^{1/2} \right\} \tag{4}$$

where the infimum is taken among the probability measures with μ and ν as first and second marginals[1] *.*

For any Borel measurable function $\phi : \mathbf{R}^n \to \mathbf{R}^n$ and any $\mu \in \mathcal{P}(\mathbf{R}^n)$, we define *the push forward* of μ by ϕ as the probability measure $\phi\#\mu \in \mathcal{P}(\mathbf{R}^n)$ such that:

$$\phi\#\mu(A) = \mu(\phi^{-1}(A)) \quad \forall A \subset \mathbf{R}^n \text{ Borel measurable }.$$

In other words, considering $(\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n), \mu)$ as a probability space and ϕ as a random variable, $\phi\#\mu$ is the *law* of ϕ.

We recall that the following well-known properties on the Wasserstein distance hold (see [1] and [24]):

[1] This means that for any Borel set $A \subset \mathbf{R}^n$, we have

$$\mu(A) = \gamma(A \times \mathbf{R}^n) \text{ and } \nu(A) = (\mathbf{R}^n \times A).$$

Proposition 1.1 *1) For any μ, ν in $\mathcal{P}(\mathbf{R}^n)$, the infimum defining $W_2(\mu,\nu)$ in (4) is a minimum. A γ achieving the minimum is called an optimal transport plan.*

2) W_2 is a distance on $\mathcal{P}(K)$ which is sequentially compact. Moreover the topology associated to W_2 is compatible with the weak star convergence of measure on $\mathcal{P}(K)$.

3) If $\mu \in \mathcal{P}(\mathbf{R}^n)$ is absolutely continuous with respect to the Lebesgue measure, there exists a unique Borel measurable map $S : \mathbf{R}^n \to \mathbf{R}^n$ such that:

$$W_2(\mu,\nu) = \int_{\mathbf{R}^n} |x - Sx|^2 \, d\mu(x), \quad S\#\mu = \nu.$$

The map S is called an optimal transport map.

2 Strategies and Values

Throughout the paper we will work within the following probability space:

$$(\Omega, \mathcal{F}, P) := ([0,1], \mathcal{B}([0,1]), \lambda),$$

where λ is the Lebesgue measure on $[0,1]$ endowed with Borel σ-field $\mathcal{B}([0,1])$. Let $\{\zeta_{j,l},\ l \geq 1,\ j = 1,2\}$ be a family of independent random variables following all a uniform distribution on $[0,1]$. Let us consider two compact metric spaces U and V as control state spaces used by player 1 and 2, respectively. $\mathcal{P}(U)$ and $\mathcal{P}(V)$ denote the space of all probability measures over U and V, endowed with Borel σ-field $\mathcal{B}(U)$ and $\mathcal{B}(V)$, respectively. It is an immediate consequence of Skorohod's Representation Theorem that the set $\mathcal{P}(U)$ (resp., $\mathcal{P}(V)$) coincides with the set of the laws of all U-valued (resp., V-valued) random variables defined over $([0,1], \mathcal{B}([0,1]), \lambda)$.

It is worth pointing out that even the Isaacs condition (2) is not supposed, the following condition

$$\inf_{\mu\in\mathcal{P}(U)} \sup_{\nu\in\mathcal{P}(V)} \int_{U\times V} f(x,u,v)\,\mu(du)\,\nu(dv)\cdot\xi = \sup_{\nu\in\mathcal{P}(V)} \inf_{\mu\in\mathcal{P}(U)} \int_{U\times V} f(x,u,v)\,\mu(du)\,\nu(dv)\cdot\xi,$$

- which can be viewed as an Isaacs condition on $\mathcal{P}(U)\times\mathcal{P}(V)$ - always holds true. Indeed the above relation is deduced from Sion's minimax theorem applied to the bilinear map

$$(\mu,\nu) \longrightarrow \int_{U\times V} f(x,u,v)\,\mu(du)\,\nu(dv)\cdot\xi$$

defined on the convex compact sets $\mathcal{P}(U)$ and $\mathcal{P}(V)$.

Let $\pi = \{0 = t_0 < t_1 < \ldots < t_N = T\}$ be a given partition. We now define the pure and random non-anticipative strategies with delay (NAD) along this partition.

Definition 2.1 *(Pure NAD strategies along partition π) A pure NAD strategies for player 1 is a mapping $\alpha : \mathbf{R}^n \times \mathcal{V}_{0,T} \mapsto \mathcal{U}_{0,T}$, Borel measurable and satisfying, for each $x \in \mathbf{R}^n$, $1 \leq l \leq N$,*

$v = v' \in \mathcal{V}_{0,T}$ a.e. on $[0,t_{l-1}] \Longrightarrow \forall x \in \mathbf{R}^n,\ \alpha(x,v) = \alpha(x,v')$, a.e on $[0,t_l]$.

A pure NAD strategies for player 2 is a mapping $\beta : \mathcal{U}_{0,T} \mapsto \mathcal{V}_{0,T}$, Borel measurable and satisfying, for each $1 \leq l \leq N$,

$u = u' \in \mathcal{U}_{0,T}$ a.e. on $[0,t_{l-1}] \Longrightarrow \beta(u) = \beta(u')$, a.e on $[0,t_l]$.

We denote by $\mathcal{A}^{\pi}(0,T)$ all such pure strategies for player 1, by $\mathcal{B}^{\pi}(0,T)$ all such strategies for player 2.

Since the players need to hide a part of their information, they have to play random strategies such that their counterparty cannot infer exactly their behaviors. From a technical point of view, this randomness is also the key argument to get a value of the game without Isaacs condition.

Definition 2.2 *(Random NAD strategies along partition π) A random NAD strategies for player 1 is a mapping $\alpha : \mathbf{R}^n \times \Omega \times \mathcal{V}_{0,T} \mapsto \mathcal{U}_{0,T}$ satisfying, for almost all $\omega \in \Omega$, for each $x \in \mathbf{R}^n$, $s \in [t_{l-1}, t_l)$, $1 \leq l \leq N$,*

$$\alpha(x,\omega,v)(s) = \alpha_l(x, (\zeta_{1,1}, \ldots, \zeta_{1,l})(\omega), v)(s)$$

with $\alpha_l : \mathbf{R}^n \times \mathbf{R}^l \times \mathcal{V}_{0,T} \mapsto \mathcal{U}_{0,T}$ Borel measurable and

$v = v' \in \mathcal{V}_{0,T}$ a.e. on $[0, t_{l-1}] \Longrightarrow \forall (x, \varpi) \in \mathbf{R}^n \times \mathbf{R}^l$, $\alpha_l(x,\varpi,v) = \alpha_l(x,\varpi,v')$, a.e on $[t_{l-1}, t_l]$.

A random NAD strategies for player 2 is a mapping $\beta : \Omega \times \mathcal{U}_{0,T} \mapsto \mathcal{V}_{0,T}$ satisfying, for almost all $\omega \in \Omega$, $s \in [t_{l-1}, t_l]$, $1 \leq l \leq N$,

$$\beta(\omega,u)(s) = \beta_l((\zeta_{2,1}, \ldots, \zeta_{2,l})(\omega), u)(s)$$

with $\beta_l : \mathbf{R}^l \times \mathcal{U}_{0,T} \mapsto \mathcal{V}_{0,T}$ Borel measurable and

$u = u' \in \mathcal{U}_{0,T}$ a.e. on $[0, t_{l-1}] \Longrightarrow \forall \varpi \in \mathbf{R}^l$, $\beta_l(\varpi,u) = \beta_l(\varpi,u')$, a.e on $[t_{l-1}, t_l]$.

We denote by $\mathcal{A}_r^{\pi}(0,T)$ all such random strategies for player 1, by $\mathcal{B}_r^{\pi}(0,T)$ all such strategies for player 2.

Remark 2.1 *For $s \in [0,T]$, $(u,v) \in \mathcal{U}_{0,T} \times \mathcal{V}_{0,T}$,*

$$\alpha(x,\omega,v)(s) = \sum_{l=1}^{N} \alpha_l(x, (\zeta_{1,1}, \ldots, \zeta_{1,l})(\omega), v) \cdot \mathbf{1}_{[t_{l-1}, t_l)}(s),$$
$$\beta(\omega,u)(s) = \sum_{l=1}^{N} \beta_l((\zeta_{2,1}, \ldots, \zeta_{2,l})(\omega), u) \cdot \mathbf{1}_{[t_{l-1}, t_l)}(s).$$

Due to the delay of strategies, we have the following property.

Lemma 2.1 *For any $\alpha, \beta \in \mathcal{A}_r^{\pi}(0,T) \times \mathcal{B}_r^{\pi}(0,T)$ and for almost all $\omega \in \Omega$, for each $x \in \mathbf{R}^n$, there is a unique pair $(u_{x,\omega}, v_{x,\omega}) \in \mathcal{U}_{0,T} \times \mathcal{V}_{0,T}$ such that*

$$\alpha(x,\omega,v_{x,\omega}) = u_{x,\omega},\ \beta(\omega,u_{x,\omega}) = v_{x,\omega},\ \text{a.e.,a.s. on } [0,T] \times \Omega.$$

Moreover $(x,\omega) \in \mathbf{R}^n \times \Omega \to (u_{x,\omega}, v_{x,\omega}) \in \mathcal{U}_{0,T} \times \mathcal{V}_{0,T}$ is Borel measurable.

The proof of this lemma uses standard arguments (cf for instance [10, 11]). But since the special form of the random NAD-strategies which we use here is new, we prefer to give the proof for the convenience of the reader.

Proof For each $\omega \in \Omega$ (up to a null set), $\alpha(x,\omega,v)$ and $\beta(\omega,u)$ restricted to $[t_0,t_1]$ depend only on $v \in \mathcal{V}_{0,T}$ and $u \in \mathcal{U}_{0,T}$ restricted to $[t_0,t_0]$. But $[t_0,t_0]$ is a singleton, so that $\alpha(x,\omega,v)$, $\beta(\omega,u)$ on $[t_0,t_1]$ do not depend on v and u. Thus, for any $v^0 \in \mathcal{V}_{0,T}$ and $u^0 \in \mathcal{U}_{0,T}$, we can define $u^1_{x,\omega} = \alpha\left(x,\omega,v^0\right)$, $v^1_\omega = \beta\left(\omega,u^0\right)$. We observe that this definition guarantees the measurability of the mapping $\Omega \ni \omega \mapsto \left(u^1_{x,\omega}, v^1_\omega\right) \in \mathcal{U}_{0,T} \times \mathcal{V}_{0,T}$. Moreover, from this definition we have

$$\alpha\left(x,\omega,v^1_\omega\right) = u^1_{x,\omega},\ \beta\left(\omega,u^1_{x,\omega}\right) = v^1_\omega, \text{ a.e., on } [t_0,t_1].$$

Assume that for $j \geq 2$, $\Omega \ni \omega \mapsto \left(u^{j-1}_{x,\omega}, v^{j-1}_{x,\omega}\right) \in \mathcal{U}_{0,T} \times \mathcal{V}_{0,T}$ is measurable and such that $\alpha\left(x,\omega,v^{j-1}_{x,\omega}\right) = u^{j-1}_{x,\omega}$, $\beta\left(\omega,u^{j-1}_{x,\omega}\right) = v^{j-1}_{x,\omega}$, a.e., on $[t_0,t_{j-1}]$. Then we set $u^j_{x,\omega} = \alpha\left(x,\omega,v^{j-1}_{x,\omega}\right)$, $v^j_{x,\omega} = \beta\left(\omega,u^{j-1}_{x,\omega}\right)$. Obviously, the thus defined mapping $\omega \mapsto \left(u^j_{x,\omega}, v^j_{x,\omega}\right)$ is measurable and $\left(u^j_{x,\omega}, v^j_{x,\omega}\right) = \left(u^{j-1}_{x,\omega}, v^{j-1}_{x,\omega}\right)$, a.e., on $[t_0,t_{j-1}]$. Then by the NAD property of α and β, $u^j_{x,\omega} = \alpha\left(x,\omega,v^j_{x,\omega}\right)$, $v^j_{x,\omega} = \beta\left(\omega,u^j_{x,\omega}\right)$, a.e., on $[t_0,t_j]$. Iterating the above steps, we can obtain the desired result. The uniqueness is an immediate consequence of the above construction. $\square$

Remark 2.2 *We observe that, for all $1 \leq l \leq N-1$, the processes u and v constructed in the above proof and restricted to the time interval $[0,t_l]$ are conditionally independent knowing $\zeta_1 = (\zeta_{1,1},\zeta_{2,1}),\ldots,\zeta_{l-1} = (\zeta_{1,l-1},\zeta_{2,l-1})$. Indeed, the processes u and v are of the following form:*

$$\begin{cases} u_{x,\cdot}(s) = u^1(s,x,\zeta_{1,1}) \cdot \mathbf{1}_{[0,t_1)}(s) + \sum_{l=2}^n u^l(s,x,\zeta_1,\ldots,\zeta_{l-1},\zeta_{1,l}) \cdot \mathbf{1}_{[t_{l-1},t_l)}(s), \\ v_{x,\cdot}(s) = v^1(s,\zeta_{2,1}) \cdot \mathbf{1}_{[0,t_1)}(s) + \sum_{l=2}^n v^l(s,x,\zeta_1,\ldots,\zeta_{l-1},\zeta_{2,l}) \cdot \mathbf{1}_{[t_{l-1},t_l)}(s), \end{cases}$$

where $\left(u^l,v^l\right)$ are functions of s,x and $\zeta_{j,l}$, $j=1,2$, $l \geq 1$ and $\zeta_l = (\zeta_{1,l},\zeta_{2,l})$, for $l \geq 1$.

Thanks to Lemma 2.1, to any pair $(\alpha,\beta) \in \mathcal{A}^\pi_r(0,T) \times \mathcal{B}^\pi_r(0,T)$, any $(x,\omega) \in \mathbf{R}^n \times [0,1]$ can be associated a trajectory $t \mapsto X_t^{x,\alpha(x,\omega,\cdot),\beta(\omega,\cdot)}$ defined as:

$$X_t^{x,\alpha(x,\omega,\cdot),\beta(\omega,\cdot)} = X_t^{x,u_{x,\omega},v_{x,\omega}}.$$

The controls $u_{x,\omega}$ and $v_{x,\omega}$ being as in Lemma 2.1.
We now define the lower and the upper value functions in pure and random strategies respectively.

Definition 2.3 *(Values in pure strategies) Let μ be the distribution of the initial position. For $\alpha \in \mathcal{A}^\pi(0,T)$, $\beta \in \mathcal{B}^\pi(0,T)$, we define cost function:*

$$\mathcal{J}(\mu,\alpha,\beta) = \int_{\mathbf{R}^n} g\left(X_T^{x,\alpha(x,\cdot),\beta(\cdot)}\right) d\mu(x),$$

and upper value function:

$$V^{+\pi}(\mu) = \inf_{\alpha \in \mathcal{A}^\pi(0,T)} \sup_{\beta \in \mathcal{B}^\pi(0,T)} \mathcal{J}(\mu,\alpha,\beta),$$

and lower value function:

$$V^{-\pi}(\mu) = \sup_{\beta \in \mathcal{B}^\pi(0,T)} \inf_{\alpha \in \mathcal{A}^\pi(0,T)} \mathcal{J}(\mu,\alpha,\beta).$$

As the mesh of π converges to zero, the limit (upper and lower) values of the game in pure strategies are defined as

$$V^{+}(\mu)=\limsup_{|\pi|\to 0} V^{+\pi}(\mu), \text{ and } V^{-}(\mu)=\liminf_{|\pi|\to 0} V^{-\pi}(\mu).$$

Definition 2.4 *(Values in random strategies) Let μ be the distribution of the initial position. For $\alpha \in \mathcal{A}_r^\pi(0,T)$, $\beta \in \mathcal{B}_r^\pi(0,T)$, we define cost function:*

$$\mathcal{J}(\mu,\alpha,\beta)=\int_\Omega\int_{\mathbf{R}^n} g\left(X_T^{x,\alpha(x,\omega,\cdot),\beta(\omega,\cdot)}\right) d\mu(x)dP(\omega),$$

and upper value function:

$$V_r^{+\pi}(\mu)=\inf_{\alpha\in\mathcal{A}_r^\pi(0,T)}\sup_{\beta\in\mathcal{B}_r^\pi(0,T)}\mathcal{J}(\mu,\alpha,\beta),$$

and lower value function:

$$V_r^{-\pi}(\mu)=\sup_{\beta\in\mathcal{B}_r^\pi(0,T)}\inf_{\alpha\in\mathcal{A}_r^\pi(0,T)}\mathcal{J}(\mu,\alpha,\beta).$$

As the mesh of π converges to zero, the limit (upper and lower) values of the game in random strategies are defined as

$$V_r^{+}(\mu)=\limsup_{|\pi|\to 0} V_r^{+\pi}(\mu), \text{ and } V_r^{-}(\mu)=\liminf_{|\pi|\to 0} V_r^{-\pi}(\mu).$$

The following result has been proved in [4]:

$$V^{+}(\mu)=\limsup_{|\pi|\to 0} V^{+\pi}(\mu), \text{ and } V^{-}(\mu)=\liminf_{|\pi|\to 0} V^{-\pi}(\mu).$$

Proposition 2.1 *For any $\mu\in\mathcal{P}(K)$ with finite support, we have:*

$$\lim_{|\pi|\to 0} V_r^{+\pi}(\mu)=V_r^{+}(\mu)=V_r^{-}(\mu)=\lim_{|\pi|\to 0} V_r^{-\pi}(\mu).$$

We denote by $V_r(\mu)$ the common value, which does not depend on the choice of the sequence $(\pi_n)_n$.

We associate to α the following vector

$$A(x,(\omega_{1,1},\ldots,\omega_{1,N}),v):=\left(\alpha_l(x,\omega_{1,1},\ldots,\omega_{1,l},v)\right)_{l=1,\ldots,N},$$

and to β the following vector

$$B(x,(\omega_{2,1},\ldots,\omega_{2,N}),u):=\left(\beta_l(x,\omega_{2,1},\ldots,\omega_{2,l},u)\right)_{l=1,\ldots,N}.$$

Following the proof of Lemma 2.1, we can easily show the following similar result:

Lemma 2.2 *For any $\alpha,\beta\in\mathcal{A}_r^\pi(0,T)\times\mathcal{B}_r^\pi(0,T)$ and for almost all*

$$(\varpi_1,\varpi_2)=(\omega_{1,1},...,\omega_{1,N},\omega_{2,1},...,\omega_{2,N})\in[0,1]^{2N}$$

and for each $x\in\mathbf{R}^n$, there is a unique pair $(\tilde{u}_{x,\varpi_1,\varpi_2},\tilde{v}_{x,\varpi_1,\varpi_2})\in\mathcal{U}_{0,T}\times\mathcal{V}_{0,T}$ such that, for any $l=1,...,N$, $s\in[t_{l-1},t_l]$,:

$$\alpha_l(x,\omega_{1,1},...,\omega_{1,l},\tilde{v}_{x,\varpi_1,\varpi_2})=\tilde{u}_{x,\varpi_1,\varpi_2},\quad \beta_l(\omega_{2,1},...,\omega_{2,l},\tilde{u}_{x,\varpi_1,\varpi_2})=\tilde{v}_{x,\varpi_1,\varpi_2},\ a.e.,a.s.\ on\ [0,T]\times\Omega.$$

Moreover $(x,\varpi_1,\varpi_2)\in\mathbf{R}^n\times\Omega^{2N}\to(\tilde{u}_{x,\varpi_1,\varpi_2},\tilde{v}_{x,\varpi_1,\varpi_2})\in\mathcal{U}_{0,T}\times\mathcal{V}_{0,T}$ is Borel measurable.

For any $\alpha,\beta \in \mathcal{A}_r^\pi(0,T)\times\mathcal{B}_r^\pi(0,T)$ and $(x,\varpi_1,\varpi_2)\in \mathbf{R}^n\times\Omega^{2N}$, this lemma allows to define the following trajectories:

$$t \to X_t^{x,A(x,\varpi_1,\cdot),B(\varpi_2,\cdot)} := X_t^{x,\tilde{u}_{x,\varpi_1,\varpi_2},\tilde{v}_{x,\varpi_1,\varpi_2}};$$

in the same way, for any $\alpha,\beta \in \mathcal{A}_r^\pi(0,T)\times\mathcal{B}^\pi(0,T)$ and $(x,\varpi)\in\mathbf{R}^n\times\Omega^N$, we can define $t\to X_t^{x,A(x,\varpi,\cdot),\beta(\cdot)}$.

3 Existence of the value in random strategies

To prove the main result, we first establish the following two lemmas.

Lemma 3.1 *The upper (lower) value function $V_r^{+\pi}(\mu)$ (resp. $V_r^{-\pi}(\mu)$) does not change if the second (resp. first) player use pure strategies, i.e.:*

$$V_r^{+\pi}(\mu) = \inf_{\alpha\in\mathcal{A}_r^\pi(0,T)}\ \sup_{\beta\in\mathcal{B}^\pi(0,T)} \int_\Omega\int_{\mathbf{R}^n} g\left(X_T^{x,\alpha(x,\omega,\cdot),\beta(\cdot)}\right)d\mu(x)dP(\omega),$$

$$V_r^{-\pi}(\mu) = \sup_{\beta\in\mathcal{B}_r^\pi(0,T)}\ \inf_{\alpha\in\mathcal{A}^\pi(0,T)} \int_\Omega\int_{\mathbf{R}^n} g\left(X_T^{x,\alpha(x,\cdot),\beta(\omega,\cdot)}\right)d\mu(x)dP(\omega).$$

Proof. We only prove the first equality. The second one can be obtained using similar procedure. Denote

$$\tilde{V}_r^{+\pi}(\mu) := \inf_{\alpha\in\mathcal{A}_r^\pi(0,T)}\sup_{\beta\in\mathcal{B}^\pi(0,T)} \int_\Omega\int_{\mathbf{R}^n} g\left(X_T^{x,\alpha(x,\omega\cdot),\beta(\cdot)}\right)d\mu(x)dP(\omega).$$

Since the set of pure strategies $\mathcal{B}^\pi(0,T)$ can be viewed as a subset of random strategies $\mathcal{B}_r^\pi(0,T)$, we deduce that for any $\alpha\in\mathcal{A}_r^\pi(0,T)$

$$\sup_{\beta\in\mathcal{B}^\pi(0,T)} \int_\Omega\int_{\mathbf{R}^n} g\left(X_T^{x,\alpha(x,\omega\cdot),\beta(\cdot)}\right)d\mu(x)dP(\omega) \le \sup_{\beta\in\mathcal{B}_r^\pi(0,T)} \int_\Omega\int_{\mathbf{R}^n} g\left(X_T^{x,\alpha(x,\omega\cdot),\beta(\cdot)}\right)d\mu(x)dP(\omega).$$

Hence by taking the infimum over $\alpha\in\mathcal{A}_r^\pi(0,T)$ in both sides of the above inequality we obtain that $\tilde{V}_r^{+\pi}(\mu)\le V_r^{+\pi}(\mu)$.

Now we proceed in proving $\tilde{V}_r^{+\pi}(\mu)\ge V_r^{+\pi}(\mu)$. In fact

$$\begin{aligned}
V_r^{+\pi}(\mu) &= \inf_{\alpha\in\mathcal{A}_r^\pi(0,T)}\sup_{\beta\in\mathcal{B}_r^\pi(0,T)}\int_\Omega\int_{\mathbf{R}^n} g\left(X_T^{x,\alpha(x,\omega,\cdot),\beta(\omega,\cdot)}\right)d\mu(x)dP(\omega)\\
&= \inf_{\alpha\in\mathcal{A}_r^\pi(0,T)}\sup_{\beta\in\mathcal{B}_r^\pi(0,T)}\int_\Omega\int_{\mathbf{R}^n} g\left(X_T^{x,A(x,(\zeta_{1,1},\dots,\zeta_{1,N}),\cdot),B(\zeta_{2,1},\dots,\zeta_{2,N},\cdot)}\right)d\mu(x)dP(\omega)\\
&= \inf_{\alpha\in\mathcal{A}_r^\pi(0,T)}\sup_{\beta\in\mathcal{B}_r^\pi(0,T)}\int_{\Omega^{2N}}\int_{\mathbf{R}^n} g\left(X_T^{x,A(x,\varpi_1,\cdot),B(\varpi_2,\cdot)}\right)d\mu(x)d\varpi_1 d\varpi_2\\
&\le \inf_{\alpha\in\mathcal{A}_r^\pi(0,T)}\sup_{\beta\in\mathcal{B}_r^\pi(0,T)}\sup_{\varpi_2\in\Omega^N}\int_{\Omega^N}\int_{\mathbf{R}^n} g\left(X_T^{x,A(x,\varpi_1,\cdot),B(\varpi_2,\cdot)}\right)d\mu(x)d\varpi_1\\
&= \inf_{\alpha\in\mathcal{A}_r^\pi(0,T)}\sup_{\varpi_2\in\Omega^N}\sup_{\beta\in\mathcal{B}_r^\pi(0,T)}\int_{\Omega^N}\int_{\mathbf{R}^n} g\left(X_T^{x,A(x,\varpi_1,\cdot),B(\varpi_2,\cdot)}\right)d\mu(x)d\varpi_1\\
&= \inf_{\alpha\in\mathcal{A}_r^\pi(0,T)}\sup_{\beta\in\mathcal{B}^\pi(0,T)}\int_{\Omega^N}\int_{\mathbf{R}^n} g\left(X_T^{x,A(x,\varpi_1,\cdot),B(\cdot)}\right)d\mu(x)d\varpi_1.
\end{aligned}$$

The proof is complete. $\square$

Now we prove the equi-Lipschitz continuity of $(V_r^{+\pi})$ and $(V_r^{-\pi})$, π being a a partition of $[0,T]$.

Lemma 3.2 *The families $(V_r^{+\pi})_\pi$ and $(V_r^{-\pi})_\pi$ - where π describes the set of subdivisions of $[0,T]$ - are equicontinuous (with respect to the Wasserstein distance W_2). Moreover there exists a Lipschitz constant $C>0$ such that for every partition π of $[0,T]$, the functions $V_r^{+\pi}$ and $V_r^{-\pi}$ are Lipschitz continuous with a Lipschitz constant smaller than C.*

Proof For some fixed partition π, we now show that $V_r^{+\pi}(\cdot)$ is Lipschitz continuous. Consider $\mu_0, \mu_1 \in \mathcal{P}(\mathbf{R}^n)$. For fixed $\varepsilon > 0$, let $\alpha \in \mathcal{A}_r^\pi(0,T)$ be an ε-optimal strategy for $V_r^{+\pi}(\mu_0)$. Let γ be an optimal plan for $W_2(\mu_0,\mu_1)$ (see Proposition 1.1). Then γ is disintegrated with respect to μ_1 as follows

$$d\gamma(x,y) = d\gamma_y(x)\, d\mu_1(y).$$

Let P^n be the uniform probability measure on $[0,1]^n$. By Proposition 1.1, there exists a unique map (up to a μ_1 negligible set) $\xi : \mathbf{R}^n \times [0,1]^n \to \mathbf{R}^n$ such that:

$$\xi(y,\cdot)\#(P^n) = \gamma_y \text{ for } \mu_1\text{-}a.e.\ y \in \mathbf{R}^n$$

and

$$W_2^2(P^n,\gamma_y) = \int_{[0,1]^n} |\xi(y,\varpi) - \varpi|^2 dP^n(\varpi) \quad \mu_1\text{-}a.e.\ y.$$

Arguing as in [11] (proof of Proposition 3), the map ξ is Borel measurable. We now associate to α the following vector:

$$A(x,(\omega_1,\ldots,\omega_N),v) := (\alpha_l(x,\omega_1,\ldots,\omega_n,v))_{l=1,\ldots,N}.$$

Take measurable functions $f_i : [0,1] \mapsto [0,1]$, $i = 0,...,n$ pairwise independent satisfying $f_i\#P = P$. Set also $F = (f_1,...,f_n) : [0,1] \mapsto [0,1]^n$. Then we construct a random strategy $\tilde{\alpha}$ for player 1 by setting

$$\tilde{\alpha}_l(y,\omega_1,\ldots,\omega_l,v) := \alpha_l(\xi(y,F(\omega_1)),(f_0(\omega_1),\omega_2,\ldots,\omega_l),v),$$

and

$$\begin{aligned}\tilde{A}(y,(\omega_1,\ldots,\omega_N),v) &:= (\tilde{\alpha}_l(y,\omega_1,\ldots,\omega_l,v))_{l=1,\ldots,N} \\ &= A(\xi(y,F(\omega_1)),(f_0(\omega_1),\omega_2,\ldots,\omega_N),v).\end{aligned}$$

Obviously, $\tilde{\alpha} \in \mathcal{A}_r^\pi(0,T)$.

For $\beta \in \mathcal{B}^\pi(0,T)$, since $\zeta_{1,l}$, $l = 1,\ldots,N$, are independent and uniformly distributed on $[0,1]$, we have

$$
\begin{aligned}
\mathcal{J}\left(\mu_1,\tilde{\alpha},\beta\right) &= \int_{[0,1]\times\mathbf{R}^n} g\left(X_T^{y,\tilde{\alpha}(y,\omega,\cdot),\beta(\cdot)}\right) d\mu_1(y)dP(\omega) \\
&= \int_{[0,1]\times\mathbf{R}^n} g\left(X_T^{y,\tilde{A}(y,(\zeta_{1,1},\ldots,\zeta_{1,N})(\omega),\cdot),\beta(\cdot)}\right) d\mu_1(y)dP(\omega) \\
&= \int_{[0,1]^N}\int_{\mathbf{R}^n} g\left(X_T^{y,\tilde{A}(y,\omega_1,\ldots,\omega_N,\cdot),\beta(\cdot)}\right) d\mu_1(y)d\omega_1\ldots d\omega_N \\
&= \int_{[0,1]^N}\int_{\mathbf{R}^n} g\left(X_T^{y,A(\xi(y,F(\omega_1)),(f_0(\omega_1),\omega_2,\ldots,\omega_N),\cdot),\beta(\cdot)}\right) d\mu_1(y)d\omega_1\ldots d\omega_N \\
&\qquad (\text{ by definitions of } \tilde{A}\ ) \\
&= \int_{[0,1]^N}\int_{[0,1]^n}\int_{\mathbf{R}^n} g\left(X_T^{y,A(\xi(y,\varpi),(\omega_1,\omega_2,\ldots,\omega_N),\cdot),\beta(\cdot)}\right) d\mu_1(y)d(F\#P)(\varpi)d(f_0\#P)(\omega_1)d\omega_2\ldots d\omega_N \\
&= \int_{[0,1]^N}\int_{[0,1]^n}\int_{\mathbf{R}^n} g\left(X_T^{y,A(\xi(y,\varpi),(\omega_1,\omega_2,\ldots,\omega_N),\cdot),\beta(\cdot)}\right) d\mu_1(y)d\varpi d\omega_1\ldots d\omega_N \\
&\qquad (\text{ by definitions of } f_i\ ).
\end{aligned}
$$

Now using the very definitions of ξ and γ:

$$
\begin{aligned}
\mathcal{J}\left(\mu_1,\tilde{\alpha},\beta\right) &= \int_{[0,1]^N}\int_{\mathbf{R}^n\times\mathbf{R}^n} g\left(X_T^{y,A(x,(\omega_1,\omega_2,\ldots,\omega_N),\cdot),\beta(\cdot)}\right) d\left(\xi\left(y,\cdot\right)\#P^n\right)(x)\, d\mu_1\left(y\right) d\omega_1\ldots d\omega_N \\
&= \int_{[0,1]}\int_{\mathbf{R}^n\times\mathbf{R}^n} g\left(X_T^{y,A(x,(\omega_1,\omega_2,\ldots,\omega_N),\cdot),\beta(\cdot)}\right) d\gamma_y\left(x\right) d\mu_1\left(y\right) d\omega_1\ldots d\omega_N \\
&= \int_{[0,1]}\int_{\mathbf{R}^n\times\mathbf{R}^n} g\left(X_T^{y,\alpha(x,\omega,\cdot),\beta(\cdot)}\right) d\gamma\left(x,y\right) dP\left(\omega\right) \\
&\le \int_{[0,1]}\int_{\mathbf{R}^n} g\left(X_T^{x,\alpha(x,\omega,\cdot),\beta(\cdot)}\right) d\mu_0\left(x\right) dP\left(\omega\right) + C\int_{\mathbf{R}^n\times\mathbf{R}^n} |x-y|\, d\gamma\left(x,y\right) \\
&\qquad (\text{using the Lipschitz property of } g) \\
&\le \int_{[0,1]}\int_{\mathbf{R}^n} g\left(X_T^{x,\alpha(x,\omega,\cdot),\beta(\cdot)}\right) d\mu_0\left(x\right) dP\left(\omega\right) + CW_2\left(\mu_0,\mu_1\right). \\
&\qquad (\text{by Cauchy-Schwarz inequality})
\end{aligned}
$$

Then by the ε-optimality of $\alpha\in\mathcal{A}_r^{\pi}\left(0,T\right)$ for $V_r^{+\pi}\left(\mu_0\right)$, we get

$$
\sup_{\beta\in\mathcal{B}_r^{\pi}(0,T)} \mathcal{J}\left(\mu_1,\bar{\alpha},\beta\right) \le V_r^{+\pi}\left(\mu_0\right) + CW_2\left(\mu_0,\mu_1\right) + \varepsilon.
$$

Since ε is arbitrary, it yields

$$
V_r^{+\pi}\left(\mu_1\right) \le V_r^{+\pi}\left(\mu_0\right) + CW_2\left(\mu_0,\mu_1\right).
$$

Interchange the position of μ_1 and μ_0, we get that, for a given partition π, $V_r^{+\pi}\left(\cdot\right)$ is Lipschitz with constant C. Since the constant C in the proof does not depend on the partition π, the family $(V_r^{+\pi})_\pi$ is equi-Lipschitz continuous with respect to the measure μ under the Wasserstein distance. $\square$

Theorem 3.1 *The game with initial distribution $\mu \in \mathcal{P}(K)$ has a limit value in random strategies:*

$$V_r^-(\mu) = V_r^+(\mu).$$

Moreover V_r^- and V_r^+ are the limit in the uniform convergence topology (as $|\pi| \to 0$) of $V_r^{-\pi}$ and $V_r^{+\pi}$.

Proof As stated above, the families $(V_r^{+\pi})_\pi$ and $(V_r^{-\pi})_\pi$ are equi-Lipschitz (with a Lipschitz constant which is bounded by $C > 0$ independent of the partition). Take $(\pi_n)_n$ a sequence of partitions of $[0, T]$ such that $|\pi_n| \to 0$. By Ascoli-Arzela Theorem, each of the corresponding sequence $(V_r^{+\pi_n})_n$ and $(V_r^{-\pi_n})_n$ has at least one cluster point which is also C-Lipschitz continuous. Let W^+ and W^- be such a cluster point of $(V_r^{+\pi_n})_n$ and $(V_r^{-\pi_n})_n$ respectively.

If $\mu \in \mathcal{P}(K)$ has a finite support, we may deduce from Proposition 2.1 that

$$W^+(\mu) = W^-(\mu) = V_r(\mu) \text{ for all } \mu \in \mathcal{P}(K) \text{ with finite support.} \tag{5}$$

This proves also the uniqueness of the values of any cluster point in μ.

Now we know that W^+ and W^- are Lipschitz continuous on $\mathcal{P}(K)$. The function V_r is C -Lipchitz continuous - from (5) - on the set of probability on K with finite support which is a dense set in $\mathcal{P}(K)$. Hence the function V_r has a unique continuous extension on $\mathcal{P}(K)$. Let us denote by $\tilde{V}_r$ this extension which is also C-Lipschitz.

An direct density argument enables us to deduce from (5) the following

$$W^+(\mu) = W^-(\mu) = \tilde{V}_r(\mu) \text{ for all } \mu \in \mathcal{P}(K).$$

So $W^+ = W^-$ and the set of cluster points is a singleton (equal to $\tilde{V}_r$).

Finally we get that both $(V_r^{+\pi_n})_n$ and $(V_r^{-\pi_n})_n$ has only one cluster point which does not depend on the choice of the sequence $(\pi_n)_n$. We conclude that:

$$\begin{aligned}
V_r^+(\mu) &= \limsup_{|\pi| \to 0} V_r^{+\pi}(\mu) = \lim_{|\pi| \to 0} V_r^{+\pi}(\mu) \\
&= W^+(\mu) = W^-(\mu) \\
&= \lim_{|\pi| \to 0} V_r^{-\pi}(\mu) = \liminf_{|\pi| \to 0} V_r^{-\pi}(\mu) \\
&= V_r^-(\mu).
\end{aligned}$$

The proof is complete. □

4 Existence of the limit value in pure strategies

The players can play the game in pure strategies when the initial distribution $\mu \in \mathcal{P}(\mathbf{R})$ has no atoms.

Theorem 4.1 *Consider $\mu \in \mathcal{P}(K)$ with no atoms. Then*

$$V^+(\mu) = V_r^+(\mu) = V_r^-(\mu) = V^-(\mu).$$

Proof Following Theorem 4.1 in [11], for fixed $\mu \in \mathcal{P}(\mathbf{R})$ without atoms, we can prove that, for any partition $(\pi_n)_{n\geq 1}$,

$$V_r^{+\pi_n}(\mu) = V^{+\pi_n}(\mu) \text{ and } V_r^{-\pi_n}(\mu) = V^{-\pi_n}(\mu) .. \tag{6}$$

From the proof of Theorem 3.1, for fixed $\mu \in \mathcal{P}(\mathbf{R})$ without atoms, we have

$$\left(V_r^{+\pi_n}(\mu), V_r^{-\pi_n}(\mu)\right) \to \left(V_r^{+}(\mu), V_r^{-}(\mu)\right), \text{ pointwise}$$

and the limits do not depend on the choice of the sequence $(\pi_n)_n$. Then by (6) and Theorem 3.1, we get:

$$\begin{aligned} V^+(\mu) &= \limsup_{|\pi|\to 0} V^{+\pi}(\mu) = \lim_{|\pi|\to 0} V_r^{+\pi}(\mu) \\ &= \lim_{|\pi|\to 0} V_r^{-\pi}(\mu) = \liminf_{|\pi|\to 0} V^{-\pi}(\mu) \\ &= V^-(\mu). \end{aligned}$$

□

Let $\tilde{\mathcal{P}}(K)$ denote all probability measures in $\mathcal{P}(K)$ without atoms. It is worth pointing out that $\tilde{\mathcal{P}}(K)$ is not closed for the weak star convergence of measures. Generally, V^+ and V^- are not Lipschitz continuous on $\mathcal{P}(K)$. But we have the following

Corollary 4.1 *The limit value in pure strategies V $(= V^+ = V^-)$ is Lipschitz continuous on $\tilde{\mathcal{P}}(K)$.*

Proof For $\mu_0, \mu_1 \in \tilde{\mathcal{P}}(K)$, by the equi-Lipschitz continuity of the sequence $(V_r^{+\pi_n})_{n\geq 1}$, there is a constant $C > 0$ such that

$$\left|V^{+\pi_n}(\mu_0) - V^{+\pi_n}(\mu_1)\right| = \left|V_r^{+\pi_n}(\mu_0) - V_r^{+\pi_n}(\mu_1)\right| \leq C \cdot W_2(\mu_0, \mu_1).$$

Passing to the limit as $n \to \infty$, we get that $|V(\mu_0) - V(\mu_1)| \leq C \cdot W_2(\mu_0, \mu_1)$. So the value $V(\cdot)$ in pure strategies is Lipschitz continuous on $\tilde{\mathcal{P}}(K)$. □

Remark 4.1 *In [4], in the game without Isaacs condition, both players have to use special random strategies. But here, it is surprising that the game can be played in pure strategies when the initial distribution μ has no atoms.*

Remark 4.2 *One can also extend our approach to the case where both players have private information on the initial position. See Remark 5 in [11] for details how the game is described.*

5 Extension to continuous scenarios

In this section, we consider games where the asymmetric information does not concern only the initial position but also deals with the dynamics and the payoffs. Our goal here is to discuss how our method can be applied to this case. The dynamics of the game is now given by

$$\begin{cases} dX_s^{\theta,u,v} = f\left(\theta, X_s^{\theta,u,v}, u_s, v_s\right) ds, \quad s \in [0,T], \ (u,v) \in \mathcal{U}_{0,T} \times \mathcal{V}_{0,T}, \\ X_0^{\theta,u,v} = x(\theta) \in \mathbf{R}^n, \end{cases} \tag{7}$$

where the scenario θ follows a distribution $\mu \in \mathcal{P}(I)$ where $I \subset \mathbf{R}^d$ $(d \geq 1)$ is a compact set and the initial position $x(\cdot)$ is Lipschitz continuous. For each scenario θ, the terminal payoff is given by $g(\theta, x) : \mathbf{R} \times \mathbf{R}^n \mapsto \mathbf{R}$ which is bounded and Lipschitz continuous in (θ, x). We also assume the function $f : \mathbf{R} \times \mathbf{R}^n \times U \times V \mapsto \mathbf{R}^n$ is bounded, continuous, and Lipschitz continuous in $(\theta, x) \in \mathbf{R} \times \mathbf{R}^n$, uniformly with respect to u and v. Note that the model here generalizes the one with continuous initial positions. In fact the case where f and g do not depend on θ and when $d = n$ is considered in the previous sections.

The game is played in the following way: For fixed initial time $t = 0$,

- before the game starts, the scenario θ is chosen randomly according to a probability measure μ,
- the scenario θ is communicated to player 1 but not to player 2,
- the game is played on the time interval $[0, T]$,
- both players know the probability μ and observe their opponents controls.

Random NAD strategies along partition π are defined similarly to Definition 2.2 replacing the initial position x by the scenario θ. Lemma 2.1 still holds for new defined strategies. The cost function is

$$\mathcal{J}(\mu, \alpha, \beta) = \int_{\mathbf{\Omega}} \int_{\mathbf{R}} g\left(\theta, X_T^{\theta, \alpha(\theta, \omega, \cdot), \beta(\omega, \cdot)}\right) d\mu(\theta)\, dP(\omega), \tag{8}$$

and the upper value function is $V_r^{+\pi}(\mu) = \inf_{\alpha \in \mathcal{A}_r^\pi(0,T)} \sup_{\beta \in \mathcal{B}_r^\pi(0,T)} \mathcal{J}(\mu, \alpha, \beta)$, and the lower value function is $V_r^{-\pi}(\mu) = \sup_{\beta \in \mathcal{B}_r^\pi(0,T)} \inf_{\alpha \in \mathcal{A}_r^\pi(0,T)} \mathcal{J}(\mu, \alpha, \beta)$. The limit values are defined similarly to Definition 2.4.

One can still prove the equi-Lipschitz continuity of $(V_r^{+\pi_n})_{n \geq 1}$ and $(V_r^{-\pi_n})_{n \geq 1}$ with respect to the Wasserstein distance on $\mathcal{P}(I)$ as in Lemma 3.2. Then by section 7 of [4], considering a sequence $\left(\sum_{i=1}^N p_i^N \delta_{\theta_i^N}\right)_N$ in $\mathcal{P}(I)$ converging weakly star to μ, we can get that:

$$V_r^-(\mu) = \lim_{N \to +\infty} V_r^- \left(\sum_{i=1}^N p_i^N \delta_{\theta_i^N}\right) = \lim_{N \to +\infty} V_r^+ \left(\sum_{i=1}^N p_i^N \delta_{\theta_i^N}\right) = V_r^+(\mu).$$

We can check that the game can be also played in pure strategies. In fact, by analogous proof of Theorem 4.1 in [11], we can prove that, for any partition $(\pi_n)_{n \geq 1}$, $V_r^{+\pi_n}(\mu) = V^{+\pi_n}(\mu)$ and $V_r^{-\pi_n}(\mu) = V^{-\pi_n}(\mu)$. From the proof of Theorem 3.1, for fixed $\mu \in \mathcal{P}(\mathbf{R})$ without atoms, we have $(V_r^{+\pi_n}(\mu), V_r^{-\pi_n}(\mu)) \to (V_r^+(\mu), V_r^-(\mu))$, pointwise and the limits do not depend on the choice of the sequence $(\pi_n)_n$. Then similarly to Theorem 4.1, we can get the desired result.

Remark 5.1 *There are many papers about the numerical theme of differential games. For two-persons zero-sum deterministic differential games we refer to [8] or [16] and references therein for approaches related to the Hamilton Jacobi Equation. We also quote [20] for Markov chain methods (see also [18] for other numerical works on differential games).*

To the best knowledge of the authors there is not yet any work on the computation of the value of differential game with incomplete information when the information is not in a finite dimensionnal space. Some approximation of the measure μ by discrete measure should be possible. However we think that for numerical methods we first need to study Hamilton-Jacobi-Isaacs equations on the space of measure. This is far to be understood in the present framework (we refer to [10] for related topics).

When the asymmetric information can be reduced to a finite dimensional space (for instance μ has a finite support in the context of the present paper), there are several works concerning the approximation of the value cf. [13, 17, 22].

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