ISOPERIMETRIC DOMAINS IN HOMOGENEOUS THREE-MANIFOLDS AND THE ISOPERIMETRIC CONSTANT OF THE HEISENBERG GROUP $H^1$

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Abstract. In this paper we prove that isoperimetric sets in three-dimensional homogeneous spaces diffeomorphic to $\mathbb{R}^3$ are topological balls. Due to the work in [MMPR13], this settles the Uniqueness of Isoperimetric Domains Conjecture, concerning congruence of such sets. We also prove that in three-dimensional homogeneous spheres isoperimetric sets are either two-spheres or symmetric genus-one tori. We then apply our first result to the three-dimensional Heisenberg group $H^1$, characterizing the isoperimetric sets and constants for a family of Riemannian adapted metrics. Using $\Gamma$-convergence of the perimeter functionals, we also settle an isoperimetric conjecture in $H^1$ posed by P. Pansu.

1. Introduction

Isoperimetric problems are a classical subject in mathematics, as they have both an independent interest, as well many applications from both the pure side (theory of CMC surfaces, functional inequalities like the Sobolev’s) and the applied one (phase transitions, capillarity problems, thin films). We refer the reader for example to [Mor09], [Ros05] for a general and historical treatment of the problem.

Once a volume $V > 0$ is fixed, typical questions are whether there exists a domain enclosing that volume which has minimal area (sometimes, minimal relative area), whether it is regular or unique (up to ambient isometries). It is mainly due to the first two questions that the field of geometric measure theory has developed, with an enormous impact on the mathematical literature.

Concerning instead the uniqueness problem, the first rigorous general argument was given by Steiner, [Ste42], using the notion of rearrangement, which classified isoperimetric sets as ball in the Euclidean space and was extended to several contexts such as space forms: some classical papers using these methods are for example [Alm76], [BS79], [Sch90]. Another somehow related technique which has also been widely extended is the reflection principle by Aleksandrov, see [Ale62], which applies to some partial differential equations as well, see e.g. [GNN79]. Other means to verify that spheres are the only isoperimetric sets in $\mathbb{R}^n$ rely on a clever use of the second variation formula to deduce umbilicity of the boundary, see [BdC84], or on the existence of optimal transport plans, see [Vil09].

We are interested here in the case of homogeneous three-dimensional manifolds: these represent important examples in three-dimensional geometry due to Thurston’s Geometrization Conjecture: we refer the reader to the surveys [FM10], [MP12] for some general properties and for a more detailed history of the isoperimetric problem in such spaces. Apart from the model spaces $\mathbb{R}^3$, $S^3, \mathbb{H}^3$ (see also [RR92] for the projective three-space), the isoperimetric problem is completely understood in the case of $\mathbb{H}^2 \times \mathbb{R}$ and of $S^2 \times \mathbb{R}$ due to the results in [HH89], [Ped04] and [Sou10]. Among other tools, these results rely on symmetrization or moving plane techniques, which also adapt to these cases and show that isoperimetric
sets are rotationally invariant. Indeed the isometry group of homogeneous three-spaces can be six-, four-, or three-dimensional and in general the smaller is the group the more difficult appears the classification problem. Other partial results are available in [TU12] and [MMPR14] concerning some cases of Berger spheres or large enough volumes. Due to the work in [AR04], [AR05], relying on a Hopf differential approach, the case of small volumes is also understood if the isometry group is four-dimensional, being the solutions classified as rotationally-symmetric spheres. We also refer to [DM13], [Bre13], [Dan07], [ER11], [FM07], [Hop83], [MPR11], [MR04], [MR05], [Men14], [NR06] for some recent results on the theory of constant mean curvature surfaces in homogeneous spaces.

If the topology of a homogeneous space is the same as that of \( \mathbb{R}^3 \), it has been conjectured that isoperimetric sets are topological spheres, see e.g. [MP12]. The results in [MMPR14], [MMPR13], regarding the study of index-one CMC spheres, are in strong support of this conjecture. Here a careful study of the left-invariant Gauss map is performed, which allows the authors to treat also cases where the isometry group of the space is three-dimensional (see also [AR04] for an approach using Hopf-type differentials). The first goal of this paper is to verify it in full generality, proving the following theorem.

**Theorem 1.1.** Let \( \mathcal{H} \) be a homogeneous three-space diffeomorphic to \( \mathbb{R}^3 \). Then isoperimetric sets in \( \mathcal{H} \) are topological balls.

It has also been conjectured that isoperimetric domains in the above homogeneous three-spaces are all congruent: this is known as the **Uniqueness of Isoperimetric Domains Conjecture**. Once we know that isoperimetric boundaries are spheres, using Theorem 4.1 in [MMPR13] we can indeed verify this too.

**Corollary 1.2.** The Uniqueness of Isoperimetric Domains Conjecture holds true.

Theorem 1.1 is proved using the stability condition for the second variation of the isoperimetric ratio, and indeed holds for any compact embedded stable CMC surface. Apart from the above-mentioned papers, the stability condition has been used crucially in [Her70], [RR92], [Ros06], [YY80] to deduce general genus bounds.

Our use of the stability condition is of topological flavour, and to our knowledge is of new type. Homogeneous three-spaces are known to carry a Lie group structure, see Section 2.1, and hence left-invariant vector fields (which form a three-dimensional family) induce Jacobi fields on boundaries of isoperimetric sets. It is also a standard fact that Jacobi fields on stable surfaces must have two nodal regions at most. Naively, one could think that the higher is the genus of a boundary, the more complicated the nodal regions should look like. We give indeed a precise meaning to this concept, and show that for positive genus there must be some Jacobi field with at least three nodal regions. We prove in Proposition 2.6 that the Euler characteristic of nodal regions is exactly one half of the Euler characteristic of the isoperimetric boundary. We then show in Proposition 2.8 that nodal regions are smooth: this allows to vary them continuously with respect to the left-invariant vector field generating the Jacobi field. We finally prove that this is impossible if the boundary has positive genus by a topological argument, contradicting the stability condition.

In fact our approach applies to the spherical case as well showing the following alternative, yielding either spherical topology or a symmetric genus-one situation for isoperimetric boundaries.

**Theorem 1.3.** Boundaries of isoperimetric sets in homogeneous three-spheres are either two-spheres or genus-one tori invariant under the action of a non-trivial left-invariant vector field.

In the case of Berger spheres, which have a four-dimensional isometry group, we can exploit this additional condition to extend the isoperimetric result in [TU12] to all situations, as it is conjectured there.

**Corollary 1.4.** Isoperimetric sets in Berger spheres are either rotationally invariant balls or rotationally symmetric Clifford tori.

We point out that, by the examples in [TU12], the second alternative in Corollary 1.4, and hence in Theorem 1.3, might indeed hold.

**Remark 1.5.** Theorems 1.1 and 1.3 remain valid for stable embedded CMC surfaces.
We are interested next in the isoperimetric problem in a sub-Riemannian context for the three-dimensional Heisenberg group $\mathbb{H}^1$, which is topologically $\mathbb{R}^3$ endowed with the group structure
\[(x, y, z) \circ (x', y', z') = (x + x', y + y', z + z' + yx' - x'y').\]

This structure admits two left-invariant vector fields
\[
X = \frac{\partial}{\partial x} + y \frac{\partial}{\partial z}; \quad Y = \frac{\partial}{\partial y} - x \frac{\partial}{\partial z},
\]
which generate a non-integrable contact distribution $\xi$ in $\mathbb{H}^1$. The contact distribution $\xi$ is annihilated by the contact form
\[
\Theta = dz + x dy - y dx.
\]

An open set $E$ in $\mathbb{H}^1$ is said to be of finite (sub-Riemannian) perimeter if the following supremum over smooth vector fields in the contact distribution $\xi$ is finite, see [GN96]
\[
P(E) := \sup_{\phi \in C^\infty(\mathbb{H}^1; \xi), \|\phi\| \leq 1} \int_E \text{div}_b(\phi) \Theta \wedge d\Theta.
\]

Here $\text{div}_b(\phi)$ stands for the sub-Riemannian divergence of horizontal vector field $\phi$, namely
\[
\text{div}_b(\phi) = X \langle \phi, X \rangle + Y \langle \phi, Y \rangle.
\]

The above concept of perimeter coincides with the three-dimensional spherical Hausdorff measure of the boundary, see [FSSC01], [Bal03], and also extends naturally to the curved case of pseudo-Hermitian manifolds (see [CHMY05] for a more detailed discussion): we also refer to Subsection 3.2 and to equation (27) below for an explicit formula in the case of smooth sets.

Pansu proved the following inequality for any set $E$ with smooth boundary, see [Pan82], [Pan83]
\[
P(E)^4 \geq C_0 V_\Theta(E)^3,
\]
where $V_\Theta(E)$ stands for the volume of $E$ induced by the three-form in (2). To prove such an inequality Pansu used methods from integral geometry, see also [CY09] for some extension to the case of pseudo-Hermitian manifolds. Notice that the scaling in (4) goes like in four Euclidean dimensions: this discrepancy is due to the fact that the non-contact direction is metrically degenerate, and should be counted twice from the point of view of space-homogeneity.

Pansu conjectured that the best isoperimetric constant in (4) of $\mathbb{H}^1$ is given by
\[
C_0 = 2\pi^2 \left(\frac{8}{3}\right)^3,
\]
and that a family of extremal sets for a given volume are Pansu’s spheres, whose profile is given by
\[
f_{\lambda} = \frac{1}{2\lambda^2} \left(\lambda r \sqrt{1 - \lambda^2 r^2} + \arccos(\lambda r) \right) + C; \quad C \in \mathbb{R}.
\]
Such surfaces are characterized by having constant mean curvature (in the sub-Riemannian setting), and are foliated by constant-curvature sub-Riemannian geodesics, see [LM05].

It is also believed that, up to sets of measure zero, these are the only extremals for the isoperimetric ratio. This was actually proved under some extra conditions: assuming $C^2$ regularity ([RR08]), convexity ([MR09]) or (a weak type of) radial symmetry ([Rit12]). We refer to the monograph [CDPT07] which surveys in detail the results and the history of the problem. Isoperimetric sets (realizing the largest possible constant in (4)) are known to exist by the work in [LR03], using concentration-compactness techniques (see also [GR13]): however a regularity theory for such a problem has not been fully developed so far, see e.g. [CCM09], [CHMY12]. Especially, the use of Hormander’s regularity theory on hypersurfaces of $\mathbb{H}^1$ is not possible due to the lack of non-trivial commutators for contact vector fields.

Despite the regularity issues we are able to verify Pansu’s conjecture in three dimensions, which is our next main goal.
Theorem 1.6. Inequality (4) holds with sharp constant given in (5).

To prove the above result we proceed as follows. Given $a > 0$, in the Heisenberg group we consider the adapted metric $g_a$ defined as

$$g_a = a^{4/3} \Theta^2 + \frac{1}{2} a^{-2} d\Theta(\cdot, J\cdot),$$

where $J$ is the rotation in $\xi$ sending $X$ into $Y$ and $Y$ into $-X$. Adapted metrics are a standard tool in sub-Riemannian geometry, and $g_a$ is considered an approximation of the Heisenberg contact structure for $a$ large, as the $g_a$-length of non-contact vectors diverges as $a \to +\infty$. We also chose a convenient scaling so that the volume elements of $g_a$ coincide with the Euclidean one.

The metric stays invariant after pull-backs by the vector fields $X, Y$ in (1), as well as for the vertical translations along the $z$-axis. Therefore $(H^1, g_a)$ becomes a homogeneous space, compact after taking quotient by its isometry group. For such manifolds isoperimetric sets are known to exist, see the beginning of Subsection 3.1, and their boundaries are smooth (in fact analytic).

We know by Theorem 1.1 that isoperimetric sets in $(H^1, g_a)$ are topological balls: analysing then the Jacobi fields generated by rotations around the $z$-axis we also show that, up to an isometry, isoperimetric sets are axially symmetric. Then both the isoperimetric sets and the isoperimetric constants can be studied by means of ODEs, see also [Tom93], [Tor10] for related results. It turns out that the $g_a$-area of isoperimetric sets with a given volume $V$ behaves asymptotically like $a^{4/3} C_0 V^{4/3}$ as $a \to +\infty$, where $C_0$ is a universal positive constant.

Restricting ourselves to the class of sets with fixed volume, endowed with the metric induced by the $L^1$ distance of their characteristic functions, we finally show that within this class the $g_a$-perimeter, scaled by $a^{-4/3}$, $\Gamma$-converges to the perimeter in $H^1$ as defined in (3). A standard result in $\Gamma$-convergence implies that the desired conclusion. It is still an open problem to understand whether Pansu’s spheres are the unique extremals for the isoperimetric ratio. The higher-dimensional case is currently under investigation.

Acknowledgements

J.-H.Cheng is grateful to the Ministry of Science and Technology of Taiwan for the grant NSC 101-2115-M-001-015-MY3, and would like to thank Princeton University for the kind hospitality. A.Malchiodi is supported by the PRIN project Variational and perturbative aspects of nonlinear differential problems. P. Yang is supported by NSF grant No. DMS-1104536. The authors are grateful to F.Callegaro (who also pointed out a former mistake) and B.Daniel for providing us some useful references, to A.Maffei for his help in the proof of Proposition 2.2 and to C.Procesi for his comments on Lie groups structure.

2. Isoperimetric domains in homogeneous three-spaces

After collecting some preliminary results, in this section we will prove Theorem 1.1.

2.1. Preliminaries. Let us briefly recall the structure of homogeneous three-spaces. We refer the reader to the lecture notes of Meeks III and Perez ([MP12]).

A Riemannian $n$-manifold $M$ is called homogeneous if the group of isometries of $M$ acts transitively on $M$. Let $\ell_a$ denote the left multiplication (or left translation) by $a$ in a Lie group $G$. A Riemannian metric on $G$ is called left-invariant if $\ell_a$ is an isometry for each $a \in G$. A Lie group equipped with a left-invariant metric is a homogeneous Riemannian manifold. A Lie group together with a left-invariant metric is called a metric Lie group. We have the following fact in dimension three.

Theorem 2.1. (Theorem 2.4 in [MP12]) Every simply connected, homogeneous Riemannian three-manifold is isometric to a metric Lie group except for the product manifolds $S^2(k) \times \mathbb{R}$, where $S^2(k)$ is a sphere of constant curvature $k > 0$. 
A homogeneous three-space $\mathcal{H}$ is a homogeneous Riemannian three-manifold diffeomorphic to $\mathbb{R}^3$. According to Theorem 2.1, $\mathcal{H}$ is a metric Lie group. A vector field $\mathcal{X}$ on a Lie group $G$ is called left-invariant if $\mathcal{X} = (l_a)_* \mathcal{X}$ for every $a \in G$. The left-invariant vector field $\mathcal{X}$ induces a one-parameter group of diffeomorphisms $\phi_{\mathcal{X},t}$. Denote the Lie algebra of all left-invariant vector fields by $\mathfrak{G}$. Notice that $\mathfrak{G}$ inherits naturally the metric structure from $T_{Id} \mathcal{H}$. We have next the following basic result.

**Proposition 2.2.** Given any non-zero left-invariant vector field $\mathcal{X}$ in a homogeneous three-space $\mathcal{H}$ (diffeomorphic to $\mathbb{R}^3$) and given $p \in \mathcal{H}$, the map $t \mapsto \phi_{\mathcal{X},t}(p)$ is proper.

**Proof.** It is clearly sufficient to prove it for $p = Id$. By Proposition 2.16 in [MP12], in the non-compact case this is the true provided the Lie group $G$ associated to $\mathcal{H}$ by Theorem 2.1 is not isomorphic to either $\tilde{E}(2)$ or $\tilde{SL}(2, \mathbb{R})$, as otherwise the exponential map is a homeomorphism.

From the proof of that proposition and from Remark 2.17 in [MP12], in case $G \simeq \tilde{E}(2)$ we have that $G$ is isomorphic to the semidirect product $\mathbb{R}^2 \rtimes_A \mathbb{R}$, where $A$ is the matrix

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$  

Moreover, in a suitable basis $E_1, E_2, E_3$ of the Lie algebra $\mathfrak{G}$, the exponential map is given by

$$\text{Exp}_{Id}(t(\alpha E_1 + \beta E_2 + \lambda E_3)) = (B(1, t\lambda)(t\alpha, t\beta), t\lambda),$$

where

$$B(1, s) = \frac{1}{s} \begin{pmatrix} \sin s & \cos s - 1 \\ 1 - \cos s & \sin s \end{pmatrix} \quad \text{for } s \neq 0; \quad B(1, 0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$  

If $\lambda = 0$, then $B(1, t\lambda)(t\alpha, t\beta) = (t\alpha, t\beta)$, so properness immediately follows. If $\lambda \neq 0$ and if $K$ is a compact set in $G$, from the latter formula we see immediately that if $\phi_{\mathcal{X},t}(Id) \in K$ then $t$ must be bounded, which again implies properness.

Suppose next $G$ is isomorphic to $\tilde{SL}(2, \mathbb{R})$, the universal cover of $SL(2, \mathbb{R})$. Up to conjugacy the elements of the Lie algebra $\mathfrak{G} = \mathfrak{sl}(2, \mathbb{R})$ are of the form

$$\begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} \quad a \neq 0; \quad \begin{pmatrix} a & 1 \\ 0 & -a \end{pmatrix} \quad a \neq 0; \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}; \quad \begin{pmatrix} 0 & d \\ -d & 0 \end{pmatrix}, \quad d > 0.$$  

The exponential maps of the first three matrices are given by

$$\begin{pmatrix} e^{at} & 0 \\ 0 & e^{-at} \end{pmatrix}; \quad \begin{pmatrix} e^{at} & t e^{-at} \\ 0 & e^{-at} \end{pmatrix}; \quad \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix},$$  

from which one deduces properness rather easily. In the latter case the exponential map is given by

$$\begin{pmatrix} \cos dt & \sin dt \\ -\sin dt & \cos dt \end{pmatrix}.$$  

While this map is periodic in $SL(2, \mathbb{R})$, it leaves every compact set for $t$ large in $\tilde{SL}(2, \mathbb{R})$, seen as a unit circle bundle of the hyperbolic plane. This concludes the proof. □

Given a smooth open subset $\Omega$ of $\mathcal{H}$, we denote by $\nu$ its outer unit Riemannian normal. Extending $\nu$ by push-forwards of left-translations in $\mathcal{H}$, we obtain an element $\nu$ of the Lie algebra called the left-invariant Gauss map of $\partial \Omega$.

We next discuss the structure of Jacobi fields on isoperimetric sets. Let us first recall the second variation operator for the area functional on a surface $\Sigma$ in $\mathcal{H}$

$$\mathcal{L} u = -\Delta_\Sigma u -\|II\|^2 u - \text{Ric}(\nu, \nu) u,$$

where $II$ stands for the second fundamental form of $\Sigma$ and $\nu$ for the unit outer normal to $\Sigma$. The above operator expresses the second variation of the area by mean of the formula

$$\int_{\Sigma} u \mathcal{L} u \, d\sigma,$$
where one takes a variation of $\Sigma$ with speed $u\nu$ so that infinitesimally the inner volume is preserved, which constraints $u$ to satisfy $\int_{\Sigma} u\,d\sigma = 0$.

A Jacobi field is a function $u$ which verifies $\mathcal{L} u = 0$. All the continuous ambient isometry groups (e.g. translations by left-invariant vector fields) generate Jacobi fields (note that $\mathcal{L} u$ is the variation of mean curvature). The next result is rather classical.

**Proposition 2.3.** Let $\Sigma$ be the boundary of an isoperimetric set in $\mathcal{H}$. Then each Jacobi field on $\Sigma$ has at most two nodal regions.

**Proof.** Suppose a Jacobi field $u$ has at least three nodal regions $\Omega_i$, $i = 1, 2, 3$. Then $u_i := u|_{\Omega_i}$ has constant sign in $\Omega_i$ and it satisfies $\mathcal{L} u_i = 0$ in $\Omega_i$. Therefore, 0 must be the lowest eigenvalue of $\mathcal{L}$ in $\Omega_i$, under Dirichlet boundary conditions.

Dirichlet eigenvalues are monotone decreasing in their domain. It is then possible to enlarge two nodal regions, say $\Omega_1$ and $\Omega_2$, to bigger domains $\hat{\Omega}_1$, $\hat{\Omega}_2$, and to consider the first eigenfunctions of $\mathcal{L}$ in those new domains: call them $\hat{u}_i$. As on the $\hat{\Omega}_i$’s the first eigenvalue of $\mathcal{L}$ was zero, by monotonicity the first eigenvalues in the $\hat{\Omega}_i$’s will be negative.

The sets $\hat{\Omega}_1$, $\hat{\Omega}_2$ can be taken disjoint, hence any (non trivial) linear combination $a\hat{u}_1 + b\hat{u}_2$ satisfies

$$\int_{\Sigma} (a\hat{u}_1 + b\hat{u}_2)\mathcal{L}(a\hat{u}_1 + b\hat{u}_2)\,d\sigma = a^2\int_{\hat{\Omega}_1} \hat{u}_1\mathcal{L}\hat{u}_1\,d\sigma + b^2\int_{\hat{\Omega}_2} \hat{u}_2\mathcal{L}\hat{u}_2\,d\sigma < 0.$$

We can also choose $a < 0 < b$ so that $a\hat{u}_1 + b\hat{u}_2$ integrates to zero on $\Sigma$. It would be therefore possible to perturb the interior of $\Sigma$ to a domain with the same volume but smaller area, giving a contradiction. ■

**Remark 2.4.** A similar reasoning, using linear combinations of first eigenfunctions, implies the well-known fact that boundaries of isoperimetric sets only have one component.

We next describe the structure of the nodal regions of Jacobi fields. Notice that, by the form of $\mathcal{L}$ in (6), a Jacobi field satisfies an equation of the form

$$-\Delta_\Sigma u + h(x)\,u = 0; \quad h(x) \in C^\infty(\Sigma).$$

(7)

In particular, a Jacobi field $u$ is an eigenfunction for the operator $-\Delta_\Sigma + h(x)$ with zero eigenvalue. As $u$ changes sign and as the first eigenfunction of the latter operator must have constant sign, one deduces that the first eigenfunction of this operator is negative. By the stability condition it then follows that there are no other negative eigenvalues than the first one, otherwise it would be possible to combine two linearly independent eigenfunctions to generate an unstable variation preserving the inner volume. Therefore the zero eigenvalue must be the second one. By [Che76], Theorems 2.5 and 3.2 (in the notation of [Che76] the value of the index $i$ is 1) and the stability condition, one has the following properties of nodal regions of a Jacobi field.

**Proposition 2.5.** ([Che76]) Suppose $\Sigma$ is the boundary of an isoperimetric domain in $\mathcal{H}$, and that the genus of $\Sigma$ is $g$. Then if $u$ is a Jacobi field on $\Sigma$ the nodal lines of $u$ are piecewise-smooth closed curves. Moreover, $u$ has only finitely-many degenerate zeroes, with vanishing order at most $2g + 1$. If the vanishing order of $u$ at a point $p$ is $k \geq 2$, then near $p$ each nodal region consists of $k$ asymptotically conical components with limit angles $\frac{\pi}{k}$.

### 2.2. Proof of Theorems 1.1 and 1.3.

We first prove that the Euler characteristic of a nodal region for a stable isoperimetric set with boundary of genus $g$ is $1 - g$. We also prove that nodal regions are smooth, and then derive a contradiction via a topological argument in case $g$ is positive.

As we discussed above, left-invariant vector fields in $\mathcal{H}$ naturally give rise to Jacobi fields by taking scalar products, with respect to the left-invariant metric, with the boundary normal. Given a unit-norm left-invariant vector field $\mathcal{X}$, we let $u_\mathcal{X}$ denote the function

$$u_\mathcal{X} = \langle \mathcal{X}, \nu \rangle.$$
Proposition 2.6. Suppose \( \Sigma \) is the boundary of an isoperimetric domain in \( \mathcal{H} \), and that the genus of \( \Sigma \) is \( g \). For a unit-norm left-invariant vector field \( \mathcal{X} \), let \( u_{\mathcal{X}} \not\equiv 0 \) be as in (8). Suppose \( A \subseteq \Sigma \) is an open nodal region of \( u_{\mathcal{X}} \). Then \( \chi(A) = 1 - g \).

Proof. Considering the left-invariant Gauss map \( \nu \), we claim that its degree on \( \Sigma \) is equal to \( 1 - g \). This can be seen for example using a global chart \( \Phi : \mathbb{R}^3 \to \mathcal{H} \) and considering the homotopy \( \lambda \Phi^{-1}(\Sigma) \), for \( \lambda \in (0,1] \). As \( \lambda \) decreases from 1 to 0, the degree stays invariant while the deformation of \( \Sigma \) will approach a single point in \( \mathcal{H} \). The metric of \( \mathcal{H} \) near this point will become closer and closer to the Euclidean one, so the claim follows. As \( A \) is the pre-image under \( \nu \) of the hemisphere oriented towards \( \mathcal{X} \) we deduce that, also with respect to this hemisphere

\[
\text{deg}(\nu, A) = 1 - g.
\]

We proceed next assuming that \( \partial A \) is smooth; the general case will be discussed afterwards with simple modifications of the argument. On \( \partial A \), which consists of finitely-many nodal circles for \( u_{\mathcal{X}} \), the left-invariant Gauss map \( \nu \) of \( \Sigma \) is orthogonal to \( \mathcal{X} \), and its total winding number around \( \mathcal{X} \) is given by \( 1 - g \), by (9).

If \( g_{\Sigma} \) is the metric of \( \Sigma \), we can conformally deform it on \( \overline{A} \) so that its Gaussian curvature vanishes. This can be done by considering the metric \( \tilde{g} = e^{2w}g_{\Sigma} \), where \( w \) solving the Dirichlet problem

\[
\begin{cases}
-\Delta_{g_{\Sigma}} w + K_{g_{\Sigma}} = 0 & \text{in } A; \\
w = 0 & \text{on } \partial A,
\end{cases}
\]

where \( K_{g_{\Sigma}} \) stands for the Gaussian curvature of \( g_{\Sigma} \). The above problem is uniquely solvable and if \( k_{\tilde{g}} \) stands for the geodesic curvature of \( \partial A \), recalling that \( \tilde{g} \) is flat, we have by the Gauss-Bonnet theorem

\[
\int_{\partial A} k_{\tilde{g}} ds = 2\pi \chi(A).
\]

Let \( \pi_{\mathcal{X}} \) denote the two-dimensional plane in the Lie algebra \( \mathfrak{g} \) consisting of the elements orthogonal to \( \mathcal{X} \). The number on the left-hand side equals \( 2\pi \) times the winding number around \( \mathcal{X} \) of the tangent velocity \( \mathbf{t} \) to \( \partial A \) in \( \pi_{\mathcal{X}} \). As on \( \partial A \) the vector \( \mathbf{t} \) is obtained applying a (counter-clockwise) rotation to \( \nu \) in \( \pi_{\mathcal{X}} \), the winding numbers of \( \mathbf{t} \) and \( \nu \) coincide. Then the conclusion follows from (9) and the latter formula.

If the boundary of \( A \) is not smooth, we recall that by Proposition 2.5 \( \partial A \) is piecewise smooth, with \( A \) forming finitely-many acute angles at finitely-many vertices. It is then sufficient to remove a small strip around \( \partial A \) rounding off the angles, to obtain a smooth subset \( \tilde{A} \) with the same homotopy type as \( A \). One can then repeat the above argument replacing \( \mathbf{t} \) and \( \nu \) by their projections on the plane \( \pi_{\mathcal{X}} \), which will be nearly unitary and nearly mutually orthogonal.

Remark 2.7. (a) The first example of a set with Euler characteristic \( 1 - g \) one might have in mind is a topological disk with \( g \) holes, as one would get by sectioning horizontally a symmetric \( g \)-torus. In fact, as one can see from Figure 1 on the left, there are examples of nodal regions on surfaces of genus 2 with only one boundary component (and still with Euler characteristic \( 1 - g \)).

(b) For a fixed genus \( g \), the winding number of \( \nu \) in each component of \( \partial A \) might be arbitrarily large: Figure 1 on the right shows the nodal lines of a genus-1 surface with winding numbers equal to 2. Such examples can be easily extended to higher genus tori and larger winding numbers.

We next further characterize the nodal sets of the above function \( u_{\mathcal{X}} \).

Proposition 2.8. Suppose \( \Sigma \) and \( u_{\mathcal{X}} \) are as in Proposition 2.6. Then the nodal sets of \( u_{\mathcal{X}} \) are smooth.

Proof. Let \( A \) be an open nodal region of \( u_{\mathcal{X}} \), and recall the structure result in Proposition 2.5. Then the surface \( \Sigma \) can be covered by \( A \) and an open set \( B \) obtained by a slight fattening of the (closed) set \( \Sigma \setminus A \). Notice that near a point where \( \partial A \) is not smooth we have the structure of \( A \), \( B \) and \( A \cap B \) as in Figure 2.

Notice also that \( B \) has the homotopy type of \( \Sigma \setminus A \), and that \( A \cap B \) has the homotopy type of \( l \) disjoint closed and simple curves, for some \( l \in \mathbb{N} \).
By the Mayer-Vietoris theorem (for which one can use both standard or reduced homology groups) the following sequence is exact

\[ H_2(A) \oplus H_2(B) \to H_2(\Sigma) \to H_1(A \cap B) \to H_1(A) \oplus H_1(B) \to \]

\[ \to H_1(\Sigma) \to H_0(A \cap B) \to H_0(A) \oplus H_0(B) \to H_0(\Sigma) \to 0. \]

Using the fact that both \( A, B \) are connected and have trivial second homology groups, by the above comments the exact sequence becomes

\[ 0 \to \mathbb{Z} \to \mathbb{Z} \to H_1(A) \oplus H_1(\Sigma \setminus A) \to \mathbb{Z}^{2g} \to \mathbb{Z} \to \mathbb{Z} \to 0. \]

By exactness, the alternate sums of the ranks is zero, which implies the following relation between first Betti numbers

\[ \beta_1(A) + \beta_1(\Sigma \setminus A) = 2g. \]

By Proposition 2.6 we also have

\[ \beta_1(A) = g; \quad \beta_1(\Sigma \setminus A) = g. \]

We show next that if \( \partial A \) is not smooth, then \( \beta_1(B) = \beta_1(\Sigma \setminus A) > \beta_1(\Sigma \setminus A) \), which would be a contradiction to the previous two formulas.

Call \( V \) the number of vertices of \( \partial A \), which correspond to the (finitely-many) degenerate zeroes of \( u_X \). Near each vertex \( v_i \) both \( A \) and \( \Sigma \setminus A \) consist of \( k_i \geq 2 \) circular sectors, see Proposition 2.5. We can cover
B with the sets $\Sigma \setminus \overline{A}$ and $\tilde{A}$, being $\tilde{A}$ a union of small balls around the vertices, see Figure 3. Notice that $(\Sigma \setminus \overline{A}) \cap \tilde{A}$ consists of $k_i$ contractible regions near $v_i$. Then the exact Mayer-Vietoris sequence

$$H_1((\Sigma \setminus \overline{A}) \cap \tilde{A}) \to H_1(\Sigma \setminus \overline{A}) \oplus H_1(\tilde{A}) \to H_1(B) \to H_0((\Sigma \setminus \overline{A}) \cap \tilde{A}) \to H_0(\Sigma \setminus \overline{A}) \oplus H_0(\tilde{A}) \to H_0(B) \to 0$$

becomes (again recall that $\Sigma \setminus \overline{A}$ is connected)

$$0 \to H_1(\Sigma \setminus \overline{A}) \to H_1(B) \to Z^{\sum_{i=1}^{V} k_i} \to Z^{1+V} \to Z \to 0.$$ 

Again the alternate sums of the ranks is zero, which gives

$$\beta_1(B) = \beta_1(\Sigma \setminus \overline{A}) + \sum_{i=1}^{V} k_i - V.$$ 

Since $k_i > 1$ for all $i$ and since $\beta_1(B) = \beta_1(\Sigma \setminus A)$, we obtain that $\beta_1(\Sigma \setminus A) > \beta_1(\Sigma \setminus \overline{A})$ if $\partial A$ is not smooth, which gives the desired contradiction. 

We have next the following elementary lemma: since we were unable to find a proof in the literature, we will provide a short one ourselves.

**Lemma 2.9.** Suppose $\Sigma$ is a compact orientable surface and that $\gamma$ is a closed, embedded and contractible curve on $\Sigma$. Then $\gamma$ bounds a topological disk.

**Proof.** The statement is clearly true if $\Sigma$ is a sphere by the Jordan curve theorem. For positive genus we consider the universal cover $\tilde{\Sigma}$ of $\Sigma$, which is either the plane (for $g = 1$) or a simply connected domain in $S^2$ (for $g > 1$).

The curve $\gamma$ can be lifted to a closed contractible simple curve $\tilde{\gamma}$ in $\tilde{\Sigma}$. Again by the Jordan curve theorem, $\tilde{\gamma}$ bounds a topological disk $D$: we claim that the covering projection maps $D$ injectively into the surface $\Sigma$. To see this, we write $D$ as union of disjoint closed simple curves $\tilde{\gamma}_t$ for $0 < t \leq 1$, where $\tilde{\gamma}_1 = \tilde{\gamma}$ and constant curve $\tilde{\gamma}_0$. Then we argue that the covering projection is injective on each $\tilde{\gamma}_t$: this can be seen using a continuity argument. If there is a first time $t_0 < 1$ for which there are two distinct points on $\tilde{\gamma}_{t_0}$ covering the same point on $\Sigma$, then this should already have happened for some $t > t_0$ because the annulus between $\tilde{\gamma}$ and $\tilde{\gamma}_{t_0}$ fills out a one sided neighbourhood of $\tilde{\gamma}_{t_0}$. Hence there is no such first time, and this concludes the proof.

**Proof of Theorem 1.1 Concluded.** Suppose by contradiction that there exists an isoperimetric set with boundary $\Sigma$ of positive genus $g$. Recall that by Proposition 2.2 the exponential map $t \mapsto \phi_{t, X}(p)$ of
a unit left-invariant vector field $\mathcal{X}$ is a proper map on $\mathbb{R}$. This implies that $u_\mathcal{X}$ cannot vanish identically, otherwise $\phi_{t, \mathcal{X}}$ would leave the surface invariant and force it to be non compact. Then the conclusion of Proposition 2.8 holds true independently of the choice of the left-invariant vector field $\mathcal{X}$. As nodal regions are always smooth, they will vary continuously when $\mathcal{X}(p)$, for any $p \in \mathcal{H}$, varies in the unit tangent sphere at $p$.

We claim that for any unit left-invariant vector field $\mathcal{X}$ every nodal line $\gamma_\mathcal{X}$ of $u_\mathcal{X}$ is non contractible. Otherwise by Lemma 2.9 $\gamma_\mathcal{X}$ would bound a topological disk: as by Proposition 2.3 $u_\mathcal{X}$ has at most two nodal regions, this topological disk must coincide with one of these regions. However by Proposition 2.6 each nodal region has non positive Euler characteristic if $g > 0$, which yields our claim: the non-contractibility of any nodal line.

In case there is only one nodal line, letting $\mathcal{X}(p)$ rotate continuously on the unit tangent sphere at $p$ up to $-\mathcal{X}(p)$ along a curve $\hat{\mathcal{X}}(t)$, $t \in [0, 1]$, we would find that the curve $\gamma_{\mathcal{X}}$ deforms continuously in $t$ to its inverse, and hence the homotopy classes of $\gamma_{\mathcal{X}}$ and of its inverse coincide. As orientable surfaces have homotopy group with no torsion, see for example the proposition at page 194 of [FK80], we obtain a contradiction.

In case there is more than one nodal line, the above curve of unit left-invariant vector fields $\hat{\mathcal{X}}(t)$ induces a continuous action on some disjoint (simple) nodal curves $\gamma_{\mathcal{X}}^1, \ldots, \gamma_{\mathcal{X}}^k$, permuting them when $t$ varies from 0 to 1, so that all these curves belong to the same homotopy class. We will show that in this case $\Sigma$ must be a genus-one torus and that $k = 2$. In fact, consider a smooth embedded curve $\alpha : [0, 1] \rightarrow \Sigma$ contained in the positive nodal region of $u_\mathcal{X}$ (which is connected) for $t \in (0, 1)$, and such that $\alpha(0)$ (respectively $\alpha(1)$) belongs to $\gamma_{\mathcal{X}}^1$ (respectively, to $\gamma_{\mathcal{X}}^2$).

Consider lifts $\hat{\gamma}_{\mathcal{X}}^1, \hat{\gamma}_{\mathcal{X}}^2, \hat{\alpha}$, of the curves $\gamma_{\mathcal{X}}^1, \gamma_{\mathcal{X}}^2$ and $\alpha$ to the universal cover $\hat{\Sigma}$ of $\Sigma$ so that the composition $(\hat{\gamma}_{\mathcal{X}}^2)^{-1} * \hat{\alpha} * \hat{\gamma}_{\mathcal{X}}^1$ is continuous in $\hat{\Sigma}$, and let $\hat{\alpha}'$ denote the deck transformation corresponding to the homotopy class of $\gamma_{\mathcal{X}}^1, \gamma_{\mathcal{X}}^2$ acting on $\hat{\alpha}$. Then, by Jordan’s curve theorem the closed curve obtained from the composition $(\hat{\alpha}')^{-1} (\hat{\gamma}_{\mathcal{X}}^2)^{-1} * \hat{\alpha} * \hat{\gamma}_{\mathcal{X}}^1$ bounds a topological square which projects on $\Sigma$ onto an open annulus $A$ (different from the whole $\Sigma$), bounded by $\gamma_{\mathcal{X}}^1$ and $\gamma_{\mathcal{X}}^2$, containing the curve $\alpha$.

As the interior of the image of $\alpha$ was contained in the positive nodal region of $u_\mathcal{X}$, the annulus $A$ must be union of nodal regions of $u_\mathcal{X}$, the number of which is two. As $A \neq \Sigma$, it must coincide with the positive nodal regions of $u_\mathcal{X}$. Repeating the same argument with a curve $\beta$ joining $\gamma_{\mathcal{X}}^1$ and $\gamma_{\mathcal{X}}^2$ but contained in the negative nodal region of $u_\mathcal{X}$, we obtain that also the negative nodal region of $u_\mathcal{X}$ is an annulus, that $k = 2$ and that $\Sigma$ is a torus.

Consider now the above deformation $\hat{\mathcal{X}}(t)$ of the left-invariant vector field. As $t$ varies from 0 to 1 the two curves $\gamma_{\mathcal{X}}^1$ and $\gamma_{\mathcal{X}}^2$ deform continuously, staying disjoint, embedded and exchanging each other. The pre-images of $\gamma_{\mathcal{X}}^1$ and $\gamma_{\mathcal{X}}^2$ in the universal cover of $\Sigma$, which is a plane, form two alternate sequences of disjoint non-compact curves. Define the deformation $-\hat{\mathcal{X}}(\cdot)$ as $\hat{\mathcal{X}}(\cdot)$ multiplied by $-1$ (which induces the same action as $\hat{\mathcal{X}}(\cdot)$ on nodal curves), and consider the closed loop $(-\hat{\mathcal{X}}(\cdot)) * \hat{\mathcal{X}}(\cdot)$. By the last disjointness property, under the action of $(-\hat{\mathcal{X}}(\cdot)) * \hat{\mathcal{X}}(\cdot)$ the lifts of $\gamma_{\mathcal{X}}^1$ and $\gamma_{\mathcal{X}}^2$ to the universal cover of $\Sigma$ must be mapped to different covering curves. On the other hand, as the loop $(-\hat{\mathcal{X}}(\cdot)) * \hat{\mathcal{X}}(\cdot)$ is valued into a two-sphere it must be contractible, so by continuity its action on the covering curves must be the identity. This gives a contradiction and implies that the isoperimetric set is a sphere.

**Remark 2.10.** As the picture on the left of Figure 1 shows, the homology class of a nodal line can be trivial even for positive genus, being the line in this case a boundary. Hence in the above argument it is necessary to consider the homotopy class of $\gamma_{\mathcal{X}}$, which encodes more information.

We also point out that the topological assumption on the homogeneous space $\mathcal{H}$ has only been used twice: to prove (9) and to claim that the exponential maps $\phi_{t, \mathcal{X}}$ are proper functions on $\mathbb{R}$. As we will see, in a homogeneous three-sphere the first condition still holds while the second may instead fail.

**Proof of Theorem 1.3.** Let $\Sigma$ be the boundary of an isoperimetric set in a homogeneous three-sphere: removing from the sphere a point in the complement of $\Sigma$, we can continuously retract $\Sigma$ to a point, obtaining (9) and hence also the counterparts of Propositions 2.6 and 2.8.
With a spherical ambient manifold the properness of \( t \rightarrow \phi_{t,X} \) fails, and hence we might have identically zero Jacobi fields of the type \( u,X \) as in (8). If there are none, then we can just argue as in the proof of Theorem 1.1. If there is an identically zero Jacobi field \( u,X \) then we saw that \( X \), which is everywhere non-zero, must be tangent to \( \Sigma \). Then we have by the Poincaré-Hopf index theorem that \( \Sigma \) is a genus-one torus, and that it is invariant under the action of \( X \). 

On Berger spheres we can exploit an extra invariance by rotation to always deduce rotational symmetry of isoperimetric sets.

**Proof of Corollary 1.4** If the isoperimetric set is a ball, then rotational symmetry follows from the results in [AR04], [AR05].

If the isoperimetric set is a torus, we will prove that the Gaussian curvature of its boundary vanishes identically: flat CMC tori have been classified in [TU10], [TU12], where it is shown that in such a case these have to be rotationally symmetric Clifford tori, yielding the conclusion. We have vanishing of the Gaussian curvature if for example the orbits of \( X \) fill the torus densely, as this would imply that the curvature is constant and hence zero by the Gauss-Bonnet theorem.

If instead the orbits of \( X \) on the boundary \( \Sigma \) are periodic, we first prove that this surface is locally described by an ODE. Recall that by Theorem 1.3 \( \Sigma \) is invariant under the action of a left-invariant vector field \( X \). Take \( p \in \Sigma \) near \( p \) we can take a local coordinate \( y \) such that \( X = \frac{\partial}{\partial y} \). We also take local coordinates \( x \) and \( z \) such that \( \Sigma \) is described by \( z = u(x) \). The components \( g_{\alpha\beta} \) of the left-invariant ambient metric \( g = g_{\alpha\beta} dx^\alpha dx^\beta \) depend only on \( x \) and \( z \), where \( 1 \leq \alpha, \beta \leq 3 \) and where \( x^1 = x, x^2 = y, x^3 = z \). To compute the mean curvature of \( \Sigma \) near \( p \), we take the local defining function \( \phi = u(x) - z \). The normal vector to \( \Sigma \) reads as

\[
\nabla_g \phi = g^{\alpha\beta} \frac{\partial \phi}{\partial x^\alpha} \frac{\partial}{\partial x^\beta} = (g^{1\beta} u_x - g^{3\beta}) \frac{\partial}{\partial x^\beta}.
\]

It follows that

\[
|\nabla_g \phi|^2_g := g(\nabla_g \phi, \nabla_g \phi) = (g^{1\beta} u_x - g^{3\beta})(g^{1\gamma} u_x - g^{3\gamma}) g_{\beta\gamma} = g^{11} u_x^2 - g^{31} u_x + g^{33}.
\]

The volume form \( dV_g \) has the expression

\[
dV_g = v(x, z) \, dx \wedge dy \wedge dz
\]

where \( v(x, z) = \sqrt{\det(g_{\alpha\beta})} > 0 \). We compute the area form \( dA_g \) as follows:

\[
dA_g := \frac{\nabla_g \phi}{|\nabla_g \phi|^g_g} \, dV_g
\]

\[
= \sqrt{v} \{ (g^{11} u_x - g^{31}) \, dy \wedge dz - (g^{12} u_x - g^{32}) \, dx \wedge dz + (g^{13} u_x - g^{33}) \, dx \wedge dy \}.
\]

From the formula \( d(dA_g) = H dV_g \) we compute the mean curvature \( H \) as

\[
H = \frac{1}{v} \frac{\partial}{\partial x} \left( \frac{v(g^{11} u_x - g^{31})}{|\nabla_g \phi|^g_g} \right) + \frac{1}{v} \frac{\partial}{\partial z} \left( \frac{v(g^{13} u_x - g^{33})}{|\nabla_g \phi|^g_g} \right).
\]

At a point in \( \Sigma \) near \( p \), where \( z = u(x) \), \( H \equiv c \) is then a nonlinear second-order ODE of \( u \) in view of (12). By means of the last property, it follows that if two CMC surfaces with the same mean curvature are invariant under the action of the same left-invariant vector field and have the same tangent plane at a common point \( p \), they must coincide.

At a point \( p \in \Sigma \), consider the rotations around the fibers of the Berger sphere: denote by \( Y \) a non-zero left-invariant vector field aligned along these circles. If these rotations leave \( T_p \Sigma \) invariant for some \( p \), then by the above observation \( \Sigma \) must be invariant by rotation. Notice also that if there is \( p \in \Sigma \) such that \( Y \) and \( X \) are aligned then this must happen everywhere, by left-invariance. In this case, since \( X \) is everywhere tangent to \( \Sigma \), any continuous family of pull-backs of \( \Sigma \) by group left-multiplication at a given point \( p \) will result in a rotation of the tangent plane at \( p \) around \( X \) and hence around \( Y \). By
the above ODE characterization of $\Sigma$, this implies that any left-invariant pull-back is an isometry, and therefore the Gaussian curvature must be again everywhere constant.

We can therefore assume from now on that $\mathcal{Y}$ neither preserves any tangent plane $T_p\Sigma$ nor is parallel to $\mathcal{X}$. Consider a smooth curve $\gamma : [0, 1] \to \Sigma$ with $\gamma(t) \neq 0$, which is transversal to $\mathcal{X}$ and for which $\gamma(0), \gamma(1)$ belong to the same (periodic) orbit of $\mathcal{X}$. Clearly the $\mathcal{X}$-orbits emanating from $\gamma(t), t \in [0, 1]$, will fill the whole torus $\Sigma$. Due to $\mathcal{X}$-invariance, the left-invariant Gauss map $\nu$ at $\gamma(t)$ forms a closed curve (orthogonal to $\mathcal{X}$) in the unit sphere of the Lie algebra, which coincides with the whole image of $\nu$ on $\Sigma$. We distinguish next between two cases.

**Case 1** There exists $t_0 \in [0, 1]$ such that $\frac{d}{dt}\nu(\gamma(t))|_{t=t_0} = 0$. In this case the left-invariant normal $\nu$ has zero differential at $\gamma(t_0)$, which implies that the mean curvature of $\Sigma$ is everywhere zero since the boundary is CMC. This in turn implies that $\frac{d}{dt}\nu(\gamma(t)) = 0$ for all $t$, and hence $\nu$ is constantly equal to $\nu(\gamma(t_0))$. Then the flow induced by any vector in the unit Lie algebra which is orthogonal to $\nu(\gamma(t_0))$ will leave $\Sigma$ invariant: as such vectors form a circle, we can reach any point of $\Sigma$ using a continuous family of isometries from $\gamma(t_0)$, and again obtain constant Gaussian curvature.

**Case 2** For every $t \in [0, 1]$ one has $\frac{d}{dt}\nu(\gamma(t)) \neq 0$. In this case the image of $\nu(\gamma(t)), t \in [0, 1]$ will form the whole circle $C_0$ of unit vectors in the Lie algebra orthogonal to $\mathcal{X}$. Fixing any $t_0 \in [0, 1]$, the rotation around $\gamma$ of $\nu(\gamma(t_0))$ will form a circle $C_1$ transversal to the image of $\nu(\gamma(t)), t \in [0, 1]$, at $\nu(\gamma(t_0))$, otherwise $\mathcal{X}(\gamma(t_0))$ would be parallel to $\gamma(\gamma(t_0))$.

By transversality, as both circles $C_0$ and $C_1$ lie in a unit sphere of the Lie algebra, there must be another common point, i.e. there exists $t_1 \in [0, 1]$ such that $\nu(\gamma(t_1)) \neq \nu(\gamma(t_0))$ and such that $\nu(\gamma(t_1))$ is obtained from a rotation around $\gamma$ of $\nu(\gamma(t_0))$. After left-translating $T_{\gamma(t_0)}\Sigma$ and $T_{\gamma(t_1)}\Sigma$ to the identity in the Lie group, we then obtain two planes with common vector $\mathcal{X}(Id)$ and such that the second one is obtained from the first by a rotation around $\gamma$. This then would imply $\gamma$ being parallel to $\mathcal{X}$, giving a contradiction. ■

3. The Heisenberg group

We study the isoperimetric sets and constant of the adapted metrics $g_a$ in $H^1$. Isoperimetric sets in the $g_a$-metric are axially symmetric spheres: we analyse the corresponding ODE for profiles and derive asymptotic estimates on the isoperimetric constant. We then tackle Pansu’s conjecture by studying the $\Gamma$-limit of the $g_a$-perimeters as $a \to +\infty$.

The axial symmetry of isoperimetric domains, once we know their boundaries are spherical, can be deduced from the analysis in [AR04], [AR05] (see e.g. Theorem 6 in [AR05]), which relies on the use of Hopf-type differentials. We give here a short and self-contained proof which uses the stability condition: for a related argument see [GL10].

**Proposition 3.1.** Any isoperimetric set $\Omega$ in the Heisenberg group with the adapted metric $g_a$ is, up to a left-invariant translation, an axially-symmetric topological ball.

**Proof.** We call $Z$ the $z$-axis, and we define the following open set in the Euclidean plane

$$D_\Omega = \{(a, b) \in \mathbb{R}^2 : [\phi(aX+bY), 1] \cap \Omega \neq \emptyset\}.$$ 

Here $\phi_{(aX+bY), 1} \Omega$ stands for the left-invariant translation of $\Omega$ by the vector field $aX + bY$, see the notation used in Subsection 2.1. For each $(a, b) \in D_\Omega$ we let $h(a, b)$ denote the maximal height of the translated domain

$$h(a, b) = \sup_{\phi_{(aX+bY), 1} \Omega \cap Z} z.$$ 

By the regularity of $\Sigma = \partial \Omega$ it follows that $h$ is a smooth function in $D_\Omega$, and that it attains a global maximum at some $(a_0, b_0)$ in the interior of $D_\Omega$.

We call next $\Omega_0 = \phi_{(a_0X+b_0Y), 1} \Omega$, and $\Sigma_0$ its boundary. Notice that, near the point $(0, 0, h(a_0, b_0))$, $\Sigma_0$ is locally an analytic graph over the $(x, y)$-plane which is flat at the origin. By analyticity, two
possibilities might occur. The first one is that this graph is rotationally symmetric, which by analyticity would imply that $\Sigma_0$ is axially symmetric, and hence would yield the conclusion.

The other possibility is that the graph has some leading-order (homogeneous) Taylor expansion near $(0,0,h(a_0,b_0))$. We then apply a rotation around the z-axis to the set $\Sigma_0$, which generates a Jacobi field $u_0$ on $\Sigma_0$. Given the Taylor expansion of $\Sigma_0$, there are at least four nodal lines of $u_0$ passing through the point $(0,0,h(a_0,b_0))$. As $\Sigma_0$ is a sphere, this would give at least three nodal regions for $u_0$, which is impossible by Proposition 2.3. ■

3.1. Isoperimetric constants of adapted metrics. In this subsection we study the isoperimetric ratio with respect to the metric $g_a$. First of all, recall that isoperimetric sets exist in this setting, as the ambient space is compact modulo isometries, see [Mor09], page 129: they have also regular boundaries, see [Mor09], Section 8.5. Therefore, by Theorem 1.1 and Proposition 3.1, in order to study the isoperimetric constant of $(H^1, g_a)$ it is sufficient to restrict our attention to the axially symmetric spheres enclosing a given Euclidean volume $V > 0$. We will prove that their areas, multiplied by $a^{-\frac{3}{4}}$, converge to the perimeter of a Pansu sphere in $H^1$ enclosing volume $V$.

We are going next to compute the mean curvature of an axially-symmetric surface defined by $z = u(r)$ (where $r^2 = x^2 + y^2$) with respect to $g_a$. Let $\langle \cdot, \cdot \rangle_a$ denote the inner product with respect to $g_a$. Using $\theta$ as polar angle and recalling the expression of $\Theta$ in the Introduction, write $g_a$ in polar coordinates $(r, \theta, z)$ as:

$$g_a = (a^{-\frac{1}{3}}dr)^2 + (a^{-\frac{1}{3}}r d\theta)^2 + (a^{2/3}\Theta)^2.$$  

It follows that

$$\langle dr, dr \rangle_a = a^{2/3}, \quad \langle dr, d\theta \rangle_a = 0,$$

$$\langle dr, \Theta \rangle_a = 0, \quad \langle d\theta, d\theta \rangle_a = a^{2/3}r^{-2},$$

$$\langle d\theta, \Theta \rangle_a = 0, \quad \langle \Theta, \Theta \rangle_a = a^{-4/3}.$$  

Observe that $\Theta = dz + r^2d\theta$, and hence we have

$$\langle dr, dz \rangle_a = 0, \quad \langle d\theta, dz \rangle_a = -a^{2/3}, \quad \langle dz, dz \rangle_a = a^{-4/3} + a^{2/3}r^2.$$  

Take the defining function $\varphi := z - u(r)$ (we are considering the southern hemisphere): then $d\varphi = dz - u'(r)dr$. We compute

$$\frac{|d\varphi|^2}{|d\varphi|^2} := \frac{d\varphi, d\varphi}{|d\varphi|^2},$$

$$= (u'(r))^2 (dr, dr)_a - 2u'(r) (dr, dz)_a + (dz, dz)_a$$

$$= a^{2/3}u'(r)^2 + a^{-4/3} + a^{2/3}r^2,$$  

by (13) and (14). Note that the Euclidean volume form $dV$ becomes $dV = r dr \wedge d\theta \wedge dz$ with respect to any $g_a$. Let $\cdot_a$ denote the interior product with respect to $g_a$. We can now compute the area form $dA$:

$$dA := \frac{d\varphi}{|d\varphi|^2} \cdot_a dV$$

$$= -\frac{1}{|d\varphi|^2} \left\{ u'(r)r \left[ (dr)_a^2 d\theta \wedge dz - r dr \wedge d\theta \right] + 2u'(r)(dr, dz)_a + (dz, dz)_a \right\}$$

$$= -\frac{1}{|d\varphi|^2} \left\{ a^{2/3}u'(r)r d\theta \wedge dz - (a^{-4/3} + a^{2/3}r^2)r dr \wedge d\theta \right\},$$  

by (13) and (14). Observe that $a^{1/3} \frac{\partial}{\partial r}$, $a^{1/3}\left\{r^{-1} \frac{\partial}{\partial r} - r \frac{\partial}{\partial z}\right\}$, $a^{-2/3} \frac{\partial}{\partial z}$ form an orthonormal frame with respect to $g_a$. Taking exterior differentiation of $dA$, we have

$$d(dA) = -a^{1/3} \frac{\partial}{\partial r} \left( \frac{a^{2/3}u'(r)r}{|d\varphi|^2} \right) a^{-1/3}dr \wedge d\theta \wedge dz$$

$$+ a^{-2/3} \frac{\partial}{\partial z} \left( \frac{(a^{-4/3} + a^{2/3}r^2)r}{|d\varphi|^2} \right) a^{2/3}\Theta \wedge dr \wedge d\theta.$$
The second term in the right-hand side of (17) vanishes since $|d\varphi|_a$ is a function of $r$ only in view of (15). Since mean curvature $H_a$ is given by the formula $d(dA) = H_a dV$, we obtain, from (17) and (15)

\begin{equation}
H_a = -\frac{a^{2/3}}{r} \frac{\partial}{\partial r} \left( \frac{u'(r)}{d\varphi|_a} \right) = -a^{1/3} \frac{1}{r} \frac{\partial}{\partial r} \left( \frac{u'(r)}{\sqrt{(u'(r))^2 + r^2 + a^{-2}}} \right). \tag{18}
\end{equation}

Isoperimetric sets enclosing a given volume $V$ have constant mean curvature $H_a = c_a > 0$. By (18) we have reduced the equation $H_a = c_a$ to the following ODE:

\begin{equation}
\frac{\partial}{\partial r} \left( \frac{u'(r)}{\sqrt{(u'(r))^2 + r^2 + a^{-2}}} \right) = a^{-1/3}c_a r. \tag{19}
\end{equation}

Integrating (19) from 0 to $r$ (note that $u'(0)$ exists and equals zero) gives

\begin{equation}
\frac{u'(r)}{r\sqrt{(u'(r))^2 + r^2 + a^{-2}}} = \lambda_a, \tag{20}
\end{equation}

where $\lambda_a = \frac{1}{2} a^{-1/3} c_a > 0$. We can solve (20) $u(r) := u_a(r)$ explicitly to get

\begin{equation}
\lambda_a(r) = \int_0^r s \sqrt{\frac{\lambda_a^2(s^2 + a^{-2})}{1 - \lambda_a^2 s^2}} ds + u(0). \tag{21}
\end{equation}

If we choose $u(0)$ so that $u(\lambda_a^{-1}) = 0$, using an integration by parts the volume enclosed by this profile (we have two hemispheres) is given by

\begin{equation}
V = -4\pi \int_0^{\lambda_a^{-1}} r u_a(r) dr = -4\pi \left( \frac{1}{2} r^2 u_a(r) \bigg|_{0}^{\lambda_a^{-1}} - \frac{1}{2} \int_{0}^{\lambda_a^{-1}} r^2 u_a'(r) dr \right). \tag{22}
\end{equation}

From the expression of $u_a$ one then finds

\begin{equation}
V = 2\pi \lambda_a \int_{0}^{\lambda_a^{-1}} \sqrt{\frac{r^2 + a^{-2}}{1 - \lambda_a^2 r^2}} dr. \tag{23}
\end{equation}

An explicit computation gives

\begin{equation}
V = \frac{\pi}{16 \lambda_a^4 a^4} \left[ 4 a \lambda_a (\lambda_a^2 + 3a^2) + (3a^4 + 2 \lambda_a^2 a^2 - \lambda_a^4) \left( \pi + 2 \arcsin \left( \frac{a^2 - \lambda_a^2}{a^2 + \lambda_a^2} \right) \right) \right]. \tag{24}
\end{equation}

Notice that when $\lambda_a \to 0$ then $V \to +\infty$, while as $\lambda_a \to +\infty$ $V \to 0$. Also, one has that

\begin{equation}
\frac{\partial V}{\partial \lambda_a} = -\frac{\pi}{4 a^3 \lambda_a^5} \left[ (3a^2 + \lambda_a^2) \left( \pi + 2 \arcsin \left( \frac{a^2 - \lambda_a^2}{a^2 + \lambda_a^2} \right) \right) + 12 a \lambda_a \right], \tag{25}
\end{equation}

which is negative. The latter observations imply that for any given $V > 0$ there exists a unique $\lambda_a = \lambda_a(V)$ such that the axially symmetric sphere with mean curvature $c_a = 2a^4 \lambda_a$ will enclose volume $V$. Taking $\lambda_a \to \lambda \in \mathbb{R}$, one can check that the above quantity tends to $\frac{3\pi}{8\lambda}$. Therefore, for fixed $V$, we have the asymptotic behaviour

\begin{equation}
\lambda_a(V) \to \sqrt{\frac{3\pi^2}{8V}} \quad \text{as } a \to +\infty. \tag{26}
\end{equation}

We turn next to the estimate of the boundary area. Using the fact that $dz = u'(r) dr$ and (15), on the boundary of the isoperimetric set we can write that

\begin{equation}
dA_a = |d\varphi|_a r dr \wedge d\theta = \left( a^{2/3} (u'_a(r))^2 + a^{-4/3} + a^{2/3} r^2 \right)^{1/2} r dr \wedge d\theta. \tag{27}
\end{equation}

From the explicit formula of $u_a$ we then deduce

\begin{equation}
dA_a = \left( a^{\frac{8}{3}} r^2 \lambda_a^2 \frac{r^2 + a^{-2}}{1 - \lambda_a^2 r^2} + a^{-\frac{4}{3}} + a^{\frac{2}{3}} r^2 \right)^{1/2} r dr \wedge d\theta, \tag{28}
\end{equation}

The expression of $u_a$ is given by the formula (we have two hemispheres) is given by

\begin{equation}
\left[ a^{2/3} \lambda_a^2 (a^2 + 3a^2) + (3a^4 + 2 \lambda_a^2 a^2 - \lambda_a^4) \left( \pi + 2 \arcsin \left( \frac{a^2 - \lambda_a^2}{a^2 + \lambda_a^2} \right) \right) \right]. \tag{29}
\end{equation}
and hence (recalling again that we have two hemispheres) the total area is given by
\[ A_a = 4\pi \int_0^{\lambda_a^{-1}} \left( \frac{a^2 r^2 + a^{-2}}{1 - \lambda_a^2 r^2} \right)^{\frac{3}{2}} r \, dr. \]

By an explicit computation one finds that
\[ A_a = 4\pi \int_0^{\lambda_a^{-1}} \left( \frac{a^2 r^2 + a^{-2}}{1 - \lambda_a^2 r^2} \right)^{\frac{3}{2}} r \, dr = \frac{\pi}{2a^3 \lambda_a^3} \left[ 4a\lambda_a + (\lambda_a^2 + a^2) \left( \pi + 2\arcsin \left( \frac{a^2 - \lambda_a^2}{a^2 + \lambda_a^2} \right) \right) \right]. \]

For \( \lambda_a \simeq \lambda \in \mathbb{R} \) the above quantity has the asymptotics \( A_a \simeq a^3 \frac{\pi^2}{8}. \) Therefore, if we take as \( \lambda \) the constant in the right-hand side of (21) we get that
\[ a^{-\frac{1}{4}} A_a \to \pi \frac{1}{3} \left( \frac{8V}{3} \right)^{\frac{3}{2}} \quad \text{as } a \to +\infty. \]

We finally analyse the limit profile of isoperimetric sets as \( a \to +\infty. \) Fixing \( r, u(0) \) (independent of \( a \) to be determined later) while taking \( a \to +\infty, \) we obtain the convergence of \( u_a(r) \) to \( u_V(r) \) on the compact sets of \([0, \lambda_V^{-1}]\), where
\[ u_V(r) = \int_0^r s \sqrt{\frac{\lambda_V^2 s^2}{1 - \lambda_V^2 s^2}} ds + u(0); \quad \lambda_V = \lambda = \sqrt[4]{\frac{3\pi^2}{8V}}. \]

Similarly \( u'_a(r) \) converges to \( u'_V(r) = r \sqrt{\frac{\lambda_V^2 r^2}{1 - \lambda_V^2 r^2}} \) on the compact sets of \([0, \lambda_V^{-1}]). \) Now, taking the limit \( a \to +\infty \) in (20) we get
\[ \frac{u'_V(r)}{r \sqrt{(u'_V(r))^2 + r^2}} = \lambda_V. \]

It follows from (24) that
\[ u'_V(r) = \frac{\lambda_V r^2}{\sqrt{1 - \lambda_V^2 r^2}} \quad \text{for } r \leq \frac{1}{\lambda_V}. \]

Therefore by integrating the right-hand side of (25), we finally get
\[ u_V(r) = -\frac{1}{2\lambda_V^2} \left( \lambda_V r \sqrt{1 - \lambda_V^2 r^2} + \arccos(\lambda_V r) \right) + C \]
for \( r \leq \frac{1}{\lambda_V}. \) By taking \( u(0) := u_a(0) = -\frac{r}{2\lambda_V^2} \) for all \( a, \) we have \( u_V(0) = -\frac{r}{4\lambda_V^2} \) and hence \( C = 0 \) in (26).

We have shown that the isoperimetric sets with respect to \( g_a \) up to translations converges to a Pansu sphere described by \( u_V(r) \). We can collect the above discussion in the following proposition.

**Proposition 3.2.** Given any \( a, V > 0 \) there exists a unique \( \lambda_a(V) \) such that the axially-symmetric sphere with mean curvature \( 2a^2 \lambda_a \) encloses Euclidean volume \( V. \) Moreover, as \( a \to +\infty, \) \( \lambda_a(V) \) converges to \( \lambda_V = \sqrt[4]{\frac{3\pi^2}{8V}}, \) while the isoperimetric profile \( A_a(V) \) has the asymptotic behaviour as in (22). Furthermore, \( g_a \)-isoperimetric sets converge in the \( C^2 \) sense to Pansu’s spheres, whose profiles is given by (26).

### 3.2. \( \Gamma \)-convergence of perimeters

We are going to show that the \( g_a \)-perimeter functional \( \Gamma \)-converges, once rescaled by \( a^{-\frac{1}{4}}, \) to a multiple of the sub-Riemannian perimeter as \( a \to +\infty. \) Then a standard result in \( \Gamma \)-convergence implies the convergence as well of the isoperimetric ratios. To show \( \Gamma \)-convergence of the rescaled perimeters one should prove the following two properties, concerning convergence in \( L^1 \) (with respect to Lebesgue’s measure) of characteristic functions.

**Property 1:** \( \Gamma \)-lim inf inequality. For every \( a_n \to +\infty \) and every \( E_n \) with \( \chi_{E_n} \to \chi_E \) in \( L^1 \) one has
\[ \frac{1}{2} P(E) \leq \lim \inf_{n} a_n^{-\frac{1}{4}} P_{a_n}(E_n). \]
Property 2: $\Gamma$ -- lim sup inequality. For every $E$ with $P(E) < \infty$ and every $a_n \to +\infty$ there exists $E_n$ with $\chi_{E_n} \to \chi_E$ in $L^1$ and such that

$$\frac{1}{2} P(E) \geq \limsup_n a_n^{-\frac{1}{2}} P(a_n(E_n)).$$

We have the following useful co-area formula in the Heisenberg setting, see [GN96]: for any function $u$ with compact support and bounded (sub-Riemannian) variation one has

$$\text{Var}_b(u) = \int_{-\infty}^{+\infty} P(\{u > t\}) dt,$$

where

$$\text{Var}_b(u) = \sup \left\{ \int_{\mathbb{H}^1} u \, \text{div}_b(\phi) dx : \phi \in C^1_c(\mathbb{H}^1, \xi) \right\},$$

where, recall from the Introduction, $\xi$ stands for the contact distribution of $\mathbb{H}^1$. This allows to approximate (in $L^1$) the characteristic function of every set with finite perimeter by a sequence of sets with smooth boundary whose perimeters converge to the given one, see e.g. [Mag12], Theorem II.2.8. The approximating sets can be constructed taking super-levels of mollified characteristic functions. The co-area formula shows that for a.e. $t \in [0,1]$ the $t$-super-level would satisfy the desired property. The same will also be valid of course for sets with finite perimeter in the $g_\alpha$ metric, as it is a standard Riemannian one.

We next provide an area comparison on smooth sets: we refer the reader to [CHMY05] for basic material in the following paragraph. Consider a surface $\Sigma$ in $\mathbb{H}^1$ which is the boundary of a smooth set $E$, and let $\xi = \text{span}(X, Y)$ denote the contact bundle, see the Introduction. At a non-singular point, where $\xi \cap T\Sigma$ is 1-dimensional, we choose a vector $e_1 \in \xi \cap T\Sigma$, having unit length with respect to the Levi metric given by $\frac{1}{2} d\Theta(\cdot, J\cdot)$. Here $J$ stands for the CR structure in $\xi$ sending $X$ into $Y$ and $Y$ into $-X$. Let $e_2 := J e_1$, and let $T := \frac{\partial}{\partial z}$ denote the Reeb vector field, i.e., the unique vector field satisfying $\Theta(T) = 1$ and $d\Theta(T, \cdot) = 0$. Denote the (local) co-frame field dual to $e_1$, $e_2$, $T$ by $e^1$, $e^2$, $\Theta$. Define a function $\alpha$ at non-singular points of $\Sigma$ by the condition $\alpha e_2 + T \in T\Sigma$. Letting $\Sigma_0$ denote the set of non-singular points of $\Sigma$, it turns out that (see [CHMY05])

$$P(E) = 2 \int_{\Sigma_0} |e^1 \wedge \Theta|. $$

On $\mathbb{H}^1$ the $g_\alpha$ metric becomes

$$g_\alpha := a^{4/3} \Theta^2 + \frac{1}{2} a^{-\frac{2}{3}} d\Theta(\cdot, J\cdot) = a^{4/3} \Theta^2 + a^{-2/3} (|e^1|^2 + |e^2|^2).$$

Still at a non-singular point, let $f := (a^2 + \alpha^2)^{-1/2}$ and take a $g_\alpha$-orthonormal basis $a^{1/3} e_1$, $a^{1/3} f(\alpha e_2 + T)$ in $T\Sigma$, and $f(\alpha e_2 - \alpha a^{-1} T)$, unit normal to $T\Sigma$. The dual co-frame is $a^{-1/3} e^1$, $a^{-1/3} f(\alpha e^2 + a^2 \Theta)$, and $a^{-1/3} f a (e^2 - \alpha \Theta)$. Therefore on $\Sigma$ we have that the area element in the $g_\alpha$-metric becomes

$$d\sigma_{g_\alpha} = \pm (a^{-1/3} e_1) \wedge a^{-1/3} f(\alpha e^2 + a^2 \Theta) = \pm a^{-2/3} (f \alpha e^1 \wedge e^2 + f a e^1 \wedge \Theta) = a^{-2/3} (a^2 + a^2)^{1/2} |e^1 \wedge \Theta| = a^{-2/3} (a^2 + a^2)^{1/2} |e^1 \wedge \Theta| \geq a^{1/3} |e^1 \wedge \Theta|,$$

where we have used $e^1 \wedge e^2 = \alpha e^1 \wedge \Theta$ on $T\Sigma$ [see [CHMY05]] for the third equality. In fact, $e^1 \wedge e^2$ and $\alpha e^1 \wedge \Theta$ acting on $(e_1, \alpha e_2 + T)$, a basis of $T\Sigma$, give the same value $\alpha$. 

(27)
Since the singular set has zero measure, by Lemma 3.1 in [CHMY05] (which implies that it is contained in finitely-many $C^1$-smooth curves), we have that the perimeter $P_n(E)$ of $E$ in the $g_a$-metric is given by

$$P_a(E) = \int_{\Sigma_a} d\sigma_{g_a}.$$ 

Hence from (27) and (28) we deduce that

$$P_a(E) \geq \frac{1}{2} a^{\frac{4}{3}} P(E), \quad \text{for } E \text{ smooth.}$$

**Proof of Property 1.** Let $a_n \to \infty$ and let $E_n$ be such that $\chi_{E_n} \to \chi_E$ in $L^1$. By the regularity approximation of $E_n$ in the $g_{a_n}$ metric, we can find smooth sets $\tilde{E}_n$ such that

$$\|\chi_{E_n} - \chi_{E_n}\|_{L^1} \leq \frac{1}{n}; \quad P_a(\tilde{E}_n) \leq P_a(E_n) + \frac{\tilde{a}}{n}.$$ 

The first property in the above formula implies that $\chi_{\tilde{E}_n} \to \chi_E$ in $L^1$. Therefore, as the perimeter is weakly lower-semicontinuous it follows that

$$P(E) \leq \liminf_n P(\tilde{E}_n).$$

As $\tilde{E}_n$ is smooth, using (29) and the second inequality in (30) we find

$$\frac{1}{2} P(E) \leq \liminf_n a_n^{-\frac{1}{3}} P_{a_n}(\tilde{E}_n) \leq \liminf_n a_n^{-\frac{1}{3}} \left[ P_{a_n}(E_n) + \frac{\tilde{a}}{n} \right] = \liminf_n a_n^{-\frac{1}{3}} P_{a_n}(E_n).$$

This concludes the proof of Property 1. ■

**Proof of Property 2.** Let us first prove the following claim.

**Claim.** For any smooth set $\tilde{E}$ one has

$$\lim_{n \to \infty} a_n^{-1/3} P_{a_n}(\tilde{E}) = \frac{1}{2} P(\tilde{E}).$$

From (28) one finds

$$a_n^{-1/3} P_{a_n}(\tilde{E}) = \int_{\partial \tilde{E}} \left( 1 + \frac{\alpha^2}{a_n^3} \right) \frac{1}{\alpha} |\alpha^1 \wedge \Theta|. $$

Note that the (Euclidean) two-dimensional measure of the singular set of $\partial \tilde{E}$ is zero, see the comments after (28). Near the vertical points, where $\alpha = 0$, the integrand of (32) approximates $|\alpha^1 \wedge \Theta|$, which is smooth. On the other hand near the singular points, where $\alpha$ tends to $\pm \infty$, $\partial \tilde{E}$ is a graph over the $x-y$ plane and $|\alpha^1 \wedge \Theta|$ has the form $D \, dx \wedge dy$ where $D = 1/|\alpha|$, see [CHMY05]. Note that $\alpha^1 \wedge \alpha^2 = \frac{1}{2} d\Theta = dx \wedge dy$ and hence we deduce that

$$|\alpha \wedge \alpha^1| = (1/|\alpha|) |\alpha^1 \wedge \alpha^2| = (1/|\alpha|) |dx \wedge dy|.$$ 

Therefore we can bound the integrand in (32) by some $L^1$ (with respect to the Euclidean surface measure) function uniformly in $a_n$. Then we can apply Lebesgue’s dominated convergence theorem to (32), and obtain (31).

Once the above claim is proved, by the comments before (29) one can find a sequence of smooth sets $\tilde{E}_k$ such that

$$\chi_{\tilde{E}_k} \to \chi_E \quad \text{in } L^1; \quad P(\tilde{E}_k) \to P(E).$$

By the above claim, one can then choose $E_n = \tilde{E}_k(n)$, with $k(n) \to \infty$ sufficiently slowly. This proves Property 2. ■
To conclude, after having the above two properties one can then use a general theorem on \( \Gamma \)-convergence, ensuring convergence of the minima for the sequence of functionals, see for example Corollary 7.20 in [DM93]. To incorporate the volume constraint, we need minor adaptations of the proof. Recall also that, by our choice of the adapted metric, the volume form induced by \( g_a \) coincides with the Euclidean one.

**Proof of Theorem 1.6.** We let \( \mathcal{E}_V \) denote the sets in \( H^1 \) with Euclidean volume \( V \). Given \( a \) large consider the isoperimetric set \( E_{a,V} \) for the \( g_a \) metric enclosing Euclidean volume \( V \): up to a proper translation, we can assume by Proposition 3.1 and the results in Subsection 3.1 that \( E_{a,V} \) is rotationally symmetric and symmetric with respect to the \( x - y \) plane. Then by Proposition 3.2 we have that the characteristic functions of the sets \( E_{a,V} \) converge in \( L^1 \) to the characteristic function \( \chi_{E_V^0} \) of a symmetric Pansu sphere \( E_V^0 \) enclosing Euclidean volume \( V \). If \( a_n \to +\infty \) we then have, by Property 1

\[
\frac{1}{2} \inf_{E \in \mathcal{E}_V} P(E) \leq P(E_V^0) \leq \frac{1}{2} \liminf_{n} a_n^{-\frac{1}{2}} P_{a_n}(E_{a_n,V}) = \liminf_{E \in \mathcal{E}_V} P_{a_n}(E).
\]

On the other hand let \( E_V^0 \in \mathcal{E}_V \) be an isoperimetric set for the sub-Riemannian perimeter \( P(\cdot) \), which is known to exist by the result in [LR03]. By Property 2, given \( a_n \to +\infty \) there exists a sequence \( E_n \) with \( \chi_{E_n} \to \chi_{E_V^0} \) in \( L^1 \) and such that

\[
\frac{1}{2} \inf_{E \in \mathcal{E}_V} P(E) = \frac{1}{2} P(E_V^0) \geq \limsup_{n} a_n^{-\frac{1}{2}} P_{a_n}(E_n) \geq \limsup_{E \in \mathcal{E}_n} a_n^{-\frac{1}{2}} P_{a_n}(E),
\]

where \( V_n \to V \) is the volume of \( E_n \). By the asymptotic of the isoperimetric profile for the \( g_a \) metric in Proposition 3.2, this implies

\[
\frac{1}{2} \inf_{E \in \mathcal{E}_V} P(E) \geq \limsup_{n} a_n^{-\frac{1}{2}} \inf_{E \in \mathcal{E}_V} P_{a_n}(E),
\]

Therefore (33) and the latter formula yield the equality

\[
\frac{1}{2} \inf_{E \in \mathcal{E}_V} P(E) = \liminf_{E \in \mathcal{E}_V} P_{a_n}(E),
\]

and also the fact that Pansu’s spheres realize the best constant in (4). Recall that the volume element \( dV_a \) from (2) is twice times the Euclidean volume element of \( \mathbb{R}^3 \), and hence there is a difference in the constants with respect to (22). This concludes the proof. \( \Box \)

**Remark 3.3.** The same reasoning, joint with Theorem 1.3, would allow to find the isoperimetric profile of the standard three-dimensional pseudo-hermitian sphere as well.

**References**


